① $$I = \int_0^{\pi} dx \int_1^{2} dy\ \delta(\sin x)\,\delta(x^2-y^2)$$

First $$\int_1^2 dy\ \delta(x^2-y^2) = \frac{1}{\left|\frac{\partial}{\partial y}(x^2-y^2)\right|_{y=|x|}} = \frac{1}{2|x|}$$

This is true if $1<|x|<2$, otherwise the zero of the δ-function is outside the range of the y integration and so we would get 0.

$$I = \int_1^2 \frac{dx}{2x}\ \delta(\sin x)\,\underbrace{\theta(x-1)\,\theta(2-x)}_{\substack{1 \text{ if } 1<x<2 \\ 0 \text{ otherwise}}}$$

Now $\sin x \neq 0$ for $1<x<2$

Thus $\boxed{I = 0}$

(2) $$x(t) = \int_{-\infty}^{\infty} G(t-t') F(t') dt'$$

$$\tilde{x}(\omega) = \tilde{G}(\omega) \tilde{F}(\omega)$$

Input $F(t) = e^{-\lambda t} \theta(t)$ so that

$$\tilde{F}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} e^{-\lambda t} \theta(t) dt = \int_{0}^{\infty} e^{-(\lambda + i\omega)t} dt = \frac{1}{\lambda + i\omega}$$

Output $x(t) = e^{-\lambda t} - e^{-(\alpha+\lambda)t}$ so that

$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \left[e^{-\lambda t} - e^{-(\alpha+\lambda)t} \right] \theta(t) dt =$$

no output before input

$$= \frac{1}{\lambda + i\omega} - \frac{1}{\alpha + \lambda + i\omega} = \frac{\alpha}{(\lambda + i\omega)(\alpha + \lambda + i\omega)} = \tilde{G}(\omega) \underbrace{\tilde{F}(\omega)}_{\frac{1}{\lambda + i\omega}}$$

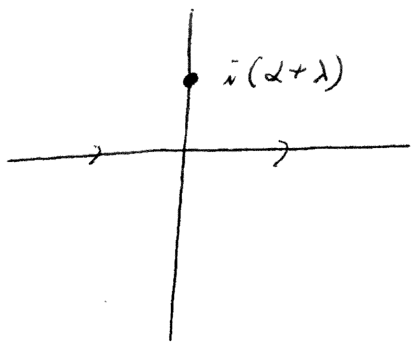
$$\boxed{\tilde{G}(\omega) = \frac{\alpha}{\alpha + \lambda + i\omega}}$$

When $F(t) = F_0 \, \delta(t)$ we have

$$\tilde{F}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} F_0 \, \delta(t) dt = F_0$$

$$\Rightarrow \quad x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \tilde{G}(\omega) \tilde{F}(\omega) = \frac{\alpha F_0}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega \, e^{i\omega t}}{\lambda + \alpha + i\omega}$$

This has a pole at $\omega = i(\alpha+\lambda)$



If $t<0$ we can add a semi-circle in the lower half plane. No pole is enclosed so $x(t<0) = 0$.

If $t>0$ we can add a semi-circle in the upper half plane. Residue theorem gives

$$\int_C \frac{d\omega\, e^{i\omega t}}{i\left[\omega - i(\alpha+\lambda)\right]} = \frac{2\pi i}{i}\, e^{i\left[i(\alpha+\lambda)\right]t} = 2\pi\, e^{-(\alpha+\lambda)t}$$

Then $\boxed{x(t) = \alpha F_0\, e^{-(\alpha+\lambda)t}\, \theta(t)}$

(3) $$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(x\cos\theta)\, d\theta$$

Laplace transform $$\hat{J}_0(s) = \int_0^{\infty} J_0(x)\, e^{-xs}\, dx =$$

$$= \int_0^{\infty} dx\, e^{-xs}\, \mathrm{Re}\int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{ix\cos\theta} = \mathrm{Re}\int_0^{2\pi}\frac{d\theta}{2\pi} \underbrace{\int_0^{\infty} dx\, e^{-(s-i\cos\theta)x}}_{\frac{1}{s-i\cos\theta}}$$

$$= \mathrm{Re}\int_0^{2\pi} \frac{d\theta}{2\pi}\,\frac{1}{s-i\cos\theta}$$

Use $z = e^{i\theta}$ $\quad dz = iz\,d\theta \quad \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$$\hat{J}_0(s) = \frac{1}{2\pi}\,\mathrm{Re}\int_C \frac{dz}{iz}\,\frac{1}{s-\frac{i}{2}\left(z+\frac{1}{z}\right)} = \frac{1}{2\pi}\,\mathrm{Re}\int_C \frac{2\,dz}{z^2+2isz+1}$$

$C \uparrow$ unit circle

$$= \frac{1}{2\pi}\,\mathrm{Re}\int_C \frac{2\,dz}{(z-z_+)(z-z_-)} \qquad z_\pm = -is \pm i\sqrt{s^2+1}$$

One can check that z_+ lies inside the unit circle while z_- lies outside. Use residue theorem.

$$\hat{J}_0(s) = \frac{1}{\pi}\,\mathrm{Re}\,\frac{2\pi i}{z_+-z_-} = \frac{1}{\pi}\,\mathrm{Re}\,\frac{2\pi i}{2i\sqrt{s^2+1}}$$

$$\boxed{\hat{J}_0(s) = \frac{1}{\sqrt{s^2+1}}}$$

(4) $$\epsilon(\omega) - 1 = \omega_p^2 \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - 2i\gamma_j \omega}$$

$$\mathrm{Im}\,\epsilon(\omega) = 2\omega_p^2 \sum_j \frac{f_j \gamma_j \omega}{(\omega^2-\omega_j^2)^2 + 4\gamma_j^2\omega^2}$$

$$\mathrm{Re}\,\epsilon(\omega) = 1 + \omega_p^2 \sum_j \frac{f_j(\omega_j^2-\omega^2)}{(\omega^2-\omega_j^2)^2 + 4\gamma_j^2\omega^2}$$

The dispersion relation is

$$\mathrm{Re}\,\epsilon(\omega) = 1 + \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\mathrm{Im}\,\epsilon(\omega')\,d\omega'}{\omega' - \omega}$$

The poles are located in both upper & lower half planes.

When $\gamma_j < \omega_j$ they are located at

$\pm \nu_j \pm i\gamma_j$ where $\nu_j = \sqrt{\omega_j^2 - \gamma_j^2}$.

Because $\mathrm{Im}\,\epsilon(-\omega) = -\mathrm{Im}\,\epsilon(\omega)$ we can rewrite the formula as

$$\mathrm{Re}\,\epsilon(\omega) = 1 + \frac{1}{\pi} P\int_0^{\infty} d\omega'\,\mathrm{Im}\,\epsilon(\omega')\left[\frac{1}{\omega'-\omega} + \frac{1}{\omega'+\omega}\right]$$

which shows that $\mathrm{Re}\,\epsilon(\omega)$ is even in ω.

If we had $\operatorname{Im} \epsilon(\omega)$ for $\omega \geq 0$ in numerical form we would just evaluate the integral numerically. But we have an analytical expression for $\operatorname{Im} \epsilon(\omega)$. One way to proceed would be to use the method of partial fractions. The denominator is a fifth order polynomial (we know all the roots) so that method would be extremely tedious. Instead we proceed as follows. In class we derived

$$\frac{1}{\omega' - \omega - i\varepsilon} = P\frac{1}{\omega' - \omega} + i\pi\,\delta(\omega' - \omega)$$

assuming that ω' and ω are real. We can take the complex conjugate to get

$$\frac{1}{\omega' - \omega + i\varepsilon} = P\frac{1}{\omega' - \omega} - i\pi\,\delta(\omega' - \omega).$$

This formula can also be derived similar to what we did in class. Add them together

$$P\frac{1}{\omega' - \omega} = \frac{1}{2}\left[\frac{1}{\omega' - \omega + i\varepsilon} + \frac{1}{\omega' - \omega - i\varepsilon}\right]$$

The RHS can now be analytically continued in ω'.

Then we get

$$\mathrm{Re}\, \epsilon(\omega) - 1 = \frac{\omega_p^2}{\pi} \sum_j f_j \gamma_j \int_{-\infty}^{\infty} \frac{d\omega' \omega'}{(\omega'^2 - \omega_j^2)^2 + 4\gamma_j^2 \omega'^2} \cdot$$

$$\cdot \left[\frac{1}{\omega' - \omega + i\varepsilon} + \frac{1}{\omega' - \omega - i\varepsilon} \right]$$

close contour in upper half plane for convenience

close contour in lower half plane for convenience

Then application of the residue theorem gives

$$\int_C \frac{d\omega' \, \omega'}{(\omega' - \nu_j - i\gamma_j)(\omega' + \nu_j - i\gamma_j)(\omega' - \nu_j + i\gamma_j)(\omega' + \nu_j + i\gamma_j)(\omega' - \omega + i\varepsilon)}$$

$$= \frac{-\pi}{2\gamma_j} \frac{1}{\omega^2 - \omega_j^2 - i2\gamma_j\omega}$$ while the term with

$\dfrac{1}{\omega' - \omega - i\varepsilon}$ gives $\dfrac{-\pi}{2\gamma_j} \dfrac{1}{\omega^2 - \omega_j^2 + i2\gamma_j\omega}$.

Their sum is $\dfrac{-\pi}{\gamma_j} \dfrac{\omega^2 - \omega_j^2}{(\omega^2 - \omega_j^2)^2 + 4\gamma_j^2\omega^2}$

Putting this back into the dispersion relation gives

$$\mathrm{Re}\,\epsilon(\omega) = 1 + \omega_p^2 \sum_j \frac{f_j\,(\omega_j^2 - \omega^2)}{(\omega^2 - \omega_j^2)^2 + 4\gamma_j^2 \omega^2}$$

which agrees with the original expression for $\epsilon(\omega)$.