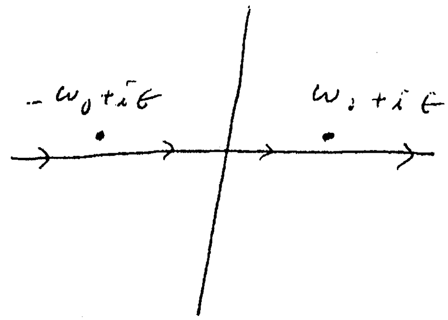
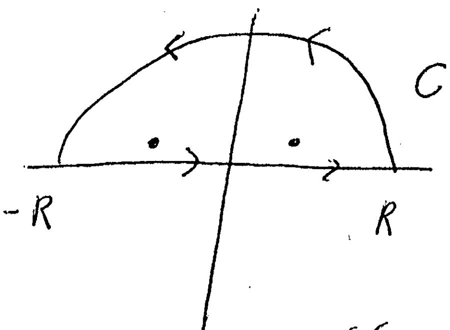
(1) $I = \int_{-\infty}^{\infty} \frac{e^{i\omega t}\, d\omega}{(\omega - i\epsilon)^2 - \omega_0^2}$



Poles $\omega = \pm\omega_0 + i\epsilon$

When $t > 0$ we can add a semi-circle in the upper half-plane.



That contribution is zero as $R \to \infty$.

$$I = \int_C \frac{e^{i\omega t}\, dt}{(\omega - \omega_0 - i\epsilon)(\omega + \omega_0 - i\epsilon)} = 2\pi i \left[\frac{e^{i(\omega_0 + i\epsilon)t}}{2\omega_0} + \frac{e^{i(-\omega_0 + i\epsilon)t}}{-2\omega_0} \right]$$

$$= 2\pi i \cdot \frac{2i \sin(\omega_0 t)}{2\omega_0} = -\frac{2\pi}{\omega_0} \sin(\omega_0 t)$$

When $t < 0$ we can add a semi-circle in the lower half-plane. No poles are enclosed so then $I = 0$.

$$\boxed{I = -\frac{2\pi}{\omega_0} \sin(\omega_0 t)\, \theta(t)}$$

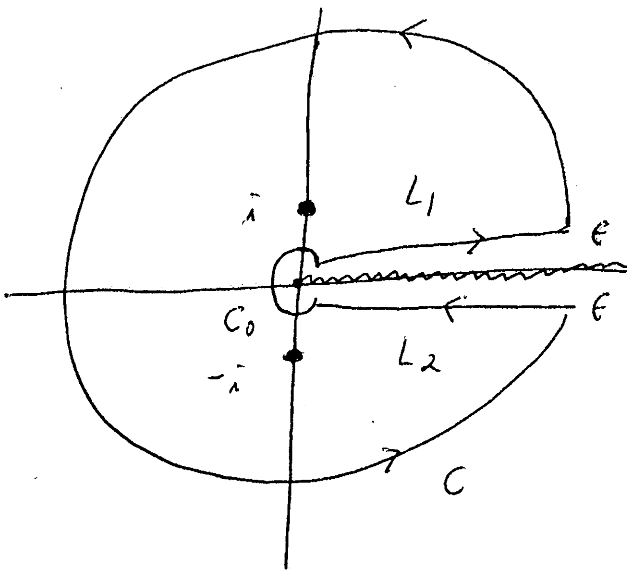
② The integral $\int_0^\infty \frac{\ln x\, dx}{1+x^2}$ is elementary.

$$\int_0^\infty \frac{\ln x\, dx}{1+x^2} \underset{\uparrow}{=} \int_\infty^0 \frac{\ln y^{-1}}{1+\frac{1}{y^2}}\left(\frac{-dy}{y^2}\right) = -\int_0^\infty \frac{\ln y\, dy}{1+y^2}$$

$$x = \frac{1}{y} \quad dx = \frac{-dy}{y^2}$$

Hence $\boxed{\int_0^\infty \frac{\ln x\, dx}{1+x^2} = 0}$

Next $\int_0^\infty \frac{(\ln x)^2 dx}{1+x^2} = \int_0^\infty \frac{(\ln z)^2 dz}{(z+i)(z-i)} = \int_0^\infty f(z)\, dz$



branch cut

Just as in lecture.

$$\int_C f\,dz + \int_{C_0} f\,dz + \int_{L_1} f\,dz + \int_{L_2} f\,dz = 2\pi i\left[\frac{\left(\ln e^{\frac{i\pi}{2}}\right)^2}{2i} + \frac{\left(\ln e^{\frac{i3\pi}{2}}\right)^2}{-2i}\right]$$

$$= \pi\left[\left(i\frac{\pi}{2}\right)^2 - \left(i\frac{3\pi}{2}\right)^2\right] = 2\pi^3$$

$$\int_C f\,dz = 0 \quad \text{when} \quad R \to \infty$$

$$\int_{C_0} f\,dz = 0 \quad \text{when} \quad r_0 \to 0$$

$$\int_{l_1} f\,dz + \int_{l_2} f\,dz = \int_0^\infty \frac{(\ln re^{i\epsilon})^2}{(re^{i\epsilon})^2+1}\,dr + \int_\infty^0 \frac{(\ln re^{i(2\pi-\epsilon)})^2\,dr}{(re^{i(2\pi-\epsilon)})^2+1}$$

$$= \int_0^\infty \frac{(\ln r + i\epsilon)^2\,dr}{r^2+1} - \int_0^\infty \frac{(\ln r + i(2\pi-\epsilon))^2\,dr}{r^2+1}$$

$$= -2\pi i \int_0^\infty \frac{\ln r\,dr}{r^2+1} + 4\pi^2 \underbrace{\int_0^\infty \frac{dr}{r^2+1}}_{\frac{\pi}{2}}$$

This gives once again $\displaystyle\int_0^\infty \frac{\ln r\,dr}{r^2+1} = 0$

which is nothing new. Note the cancelation of the $(\ln r)^2$ terms.

Repeat the calculation for $\displaystyle\int_0^\infty \frac{(\ln z)^3 dz}{z^2+1} = \int_0^\infty g\,dz$

$$\int_C g\,dz + \int_{C_0} g\,dz + \int_{L_1} g\,dz + \int_{L_2} g\,dz = 2\pi i\left[\frac{\left(i\frac{\pi}{2}\right)^3}{2i} + \frac{\left(i\frac{3\pi}{2}\right)^3}{-2i}\right] =$$

$$= \frac{13}{4}\pi^4 i$$

$$\int_{L_1} g\,dz + \int_{L_2} g\,dz = \int_0^\infty \frac{dx}{x^2+1}\left[(\ln x)^3 - (\ln x + 2\pi i)^3\right] =$$

$$= -2\pi i \int_0^\infty \frac{3(\ln x)^2 dx}{x^2+1} - (2\pi i)^3 \underbrace{\int_0^\infty \frac{dx}{x^2+1}}_{\frac{\pi}{2}}$$

Putting it all together gives

$$\boxed{\int_0^\infty \frac{(\ln x)^2 dx}{x^2+1} = \frac{\pi^3}{8}}$$