

HW 5
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Nasser M. Abbasi

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1 Problem 1

Evaluate the following integral for $t > 0$ and for $t < 0$ when $\omega_0 > 0$ and $\epsilon \rightarrow 0^+$

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2} d\omega$$

Solution

Case $t > 0$

We select the upper half for contour C since when $t > 0$ the integral on upper half will vanish as will be shown below.

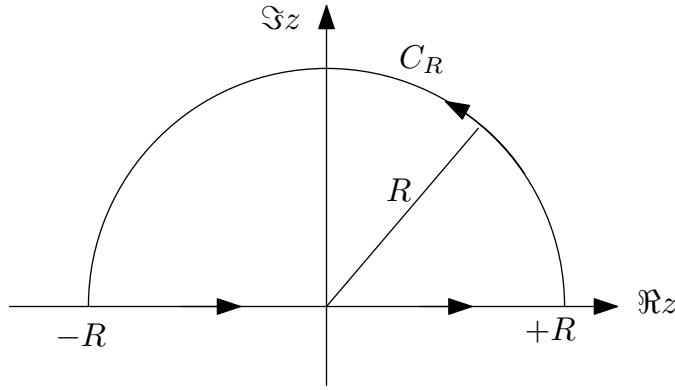


Figure 1: Contour used for $t > 0$

Hence

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2} d\omega &= \oint_C \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz \\ &= \lim_{R \rightarrow \infty} (P.V.) \int_{-R}^R \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz + \int_{C_R} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz \\ &= 2\pi i \sum \text{Residue} \end{aligned}$$

Therefore, if we can show that $\lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz = 0$, then the above implies that

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2} d\omega = 2\pi i \sum \text{Residue} \quad (1)$$

Now we need to find the residues inside the contour shown. There is a pole when $(\omega - i\epsilon)^2 = \omega_0^2$ or $\omega - i\epsilon = \pm\omega_0$ or $\omega = i\epsilon \pm \omega_0$. Hence there are two simple poles, they are

$$z_1 = i\epsilon + \omega_0$$

$$z_2 = i\epsilon - \omega_0$$

They are both in upper half, inside the contour (since $\omega_0 > 0$ and ϵ is positive).

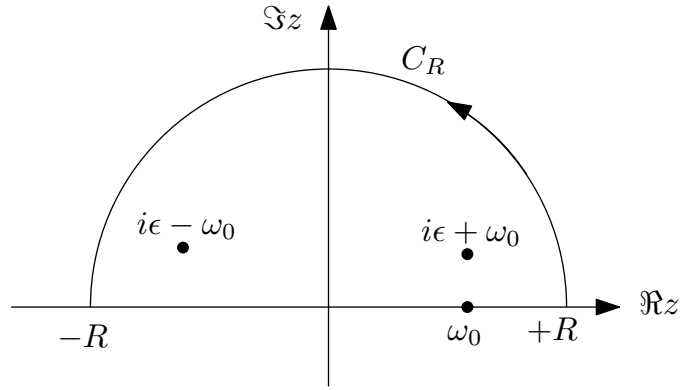


Figure 2: Locations of poles

Now we find the residues

$$\begin{aligned}
 \text{Residue}(z_1) &= \lim_{z \rightarrow z_1} (z - z_1) \frac{e^{izt}}{(z - z_1)(z - z_2)} \\
 &= \lim_{z \rightarrow z_1} \frac{e^{izt}}{(z - z_2)} \\
 &= \frac{e^{it(i\epsilon + \omega_0)}}{(i\epsilon + \omega_0) - (i\epsilon - \omega_0)} \\
 &= \frac{e^{-t\epsilon} e^{it\omega_0}}{2\omega_0}
 \end{aligned} \tag{2}$$

And

$$\begin{aligned}
 \text{Residue}(z_2) &= \lim_{z \rightarrow z_2} (z - z_2) \frac{e^{izt}}{(z - z_1)(z - z_2)} \\
 &= \lim_{z \rightarrow z_2} \frac{e^{izt}}{(z - z_1)} \\
 &= \frac{e^{it(i\epsilon - \omega_0)}}{(i\epsilon - \omega_0) - (i\epsilon + \omega_0)} \\
 &= \frac{e^{-t\epsilon} e^{-it\omega_0}}{-2\omega_0}
 \end{aligned} \tag{3}$$

Substituting (2,3) into (1) gives

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2} d\omega &= 2\pi i \left(\frac{e^{-t\epsilon} e^{it\omega_0}}{2\omega_0} + \frac{e^{-t\epsilon} e^{-it\omega_0}}{-2\omega_0} \right) \\
&= \frac{2\pi i}{2\omega_0} e^{-t\epsilon} (e^{it\omega_0} - e^{-it\omega_0}) \\
&= \frac{2\pi}{\omega_0} e^{-t\epsilon} \left(\frac{e^{it\omega_0} - e^{-it\omega_0}}{-2i} \right) \\
&= -\frac{2\pi}{\omega_0} e^{-t\epsilon} \left(\frac{e^{it\omega_0} - e^{-it\omega_0}}{2i} \right) \\
&= -\frac{2\pi}{\omega_0} e^{-t\epsilon} \sin(t\omega_0)
\end{aligned}$$

Now, to finish the solution, we must show that $\lim_{R \rightarrow \infty} \int_{CR} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz = 0$. But

$$\begin{aligned}
\int_{CR} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz &\leq \left| \int_{CR} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz \right|_{\max} \\
&\leq \int_{CR} \left| \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} \right|_{\max} dz \\
&= \left| \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} \right|_{\max} \int_{CR} dz \\
&= \left| \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} \right|_{\max} \int_0^\pi R d\theta \\
&= R\pi \left| \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} \right|_{\max}
\end{aligned} \tag{4}$$

But

$$\begin{aligned}
\left| \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} \right|_{\max} &\leq \frac{|e^{izt}|_{\max}}{|(z - z_1)(z - z_2)|_{\min}} \\
&\leq \frac{|e^{izt}|_{\max}}{|(z - z_1)|_{\min} |(z - z_2)|_{\min}} \\
&= \frac{|e^{it(x+iy)}|_{\max}}{|(z - z_1)|_{\min} |(z - z_2)|_{\min}} \\
&= \frac{|e^{itx-ty}|_{\max}}{|(z - z_1)|_{\min} |(z - z_2)|_{\min}} \\
&= \frac{|e^{itx}|_{\max} |e^{-ty}|_{\max}}{|(z - z_1)|_{\min} |(z - z_2)|_{\min}} \\
&\leq \frac{|e^{-ty}|_{\max}}{|(z - z_1)|_{\min} |(z - z_2)|_{\min}}
\end{aligned}$$

Now, since $y > 0$ (we are in the upper half) and also since $t > 0$, then $|e^{-ty}|_{\max} = 1$, which occurs when $y = 0$. Hence the above becomes

$$\left| \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} \right|_{\max} \leq \frac{1}{|(z - z_1)|_{\min} |(z - z_2)|_{\min}}$$

By inverse triangle inequality $|z - z_1|_{\min} \geq |z|^2 + |z_1|^2 = R^2 + |\epsilon^2 + \omega_0^2|^2$ and $|z - z_2|_{\min} \geq |z|^2 + |z_2|^2 = R^2 + |\epsilon^2 + \omega_0^2|^2$. The above becomes

$$\left| \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} \right|_{\max} \leq \frac{1}{2R^2 + 2|\epsilon^2 + \omega_0^2|^2}$$

Substituting the above in (4) gives

$$\begin{aligned}
\lim_{R \rightarrow \infty} \int_{CR} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz &\leq \lim_{R \rightarrow \infty} R\pi \left(\frac{1}{2R^2 + 2|\epsilon^2 + \omega_0^2|^2} \right) \\
&= \pi \lim_{R \rightarrow \infty} \frac{R}{2R^2 + 2|\epsilon^2 + \omega_0^2|^2}
\end{aligned}$$

But $2|\epsilon^2 + \omega_0^2|^2$ is a finite value, say β so the above is

$$\lim_{R \rightarrow \infty} \int_{CR} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz \leq \pi \lim_{R \rightarrow \infty} \frac{R}{2R^2 + \beta}$$

And it is clear now that the above limit goes to zero. In other words, $\lim_{R \rightarrow \infty} \frac{R}{2R^2 + \beta} =$

$$\lim_{R \rightarrow \infty} \frac{\frac{1}{R}}{2 + \frac{\beta}{R^2}} = \frac{0}{2} = 0.$$

Hence The final solution is

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2} d\omega = -\frac{2\pi}{\omega_0} e^{-t\epsilon} \sin(t\omega_0)$$

Case $t < 0$

Here, we must use the lower half for the contour in order for the half circle contour integral to vanish.

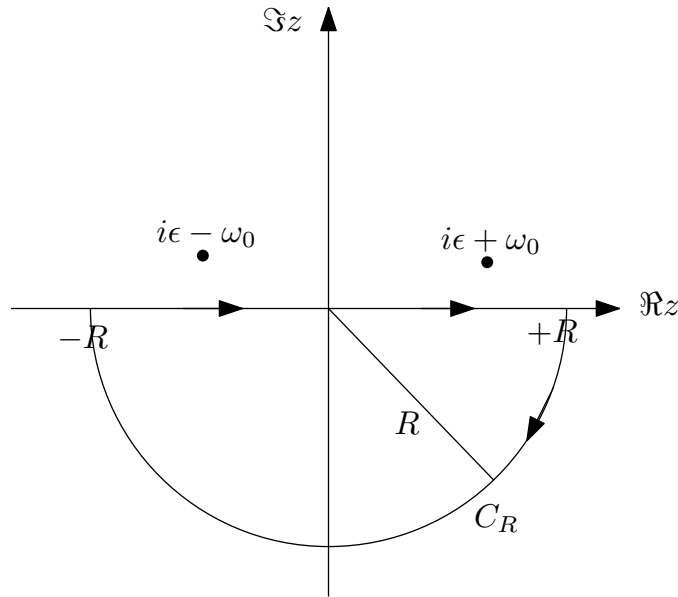


Figure 3: Contour for $t < 0$

In this case the sum of residues is zero (since both poles are in the upper half), then we see right away that

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2} d\omega = 0 \quad t < 0$$

But we must show that $\lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz = 0$ here as well for the above result to be

valid. Similar to what was done earlier:

$$\int_{CR} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz \leq \left| \int_{CR} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz \right|_{\max} \leq \int_{CR} \left| \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} \right|_{\max} dz \quad (1)$$

$$= R\pi \left| \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} \right|_{\max} \quad (4)$$

But

$$\begin{aligned} \left| \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} \right|_{\max} &\leq \frac{|e^{izt}|_{\max}}{|(z - z_1)|_{\min} |(z - z_2)|_{\min}} \\ &\leq \frac{|e^{it(x+iy)}|_{\max}}{|(z - z_1)|_{\min} |(z - z_2)|_{\min}} \\ &= \frac{|e^{itx-ty}|_{\max}}{|(z - z_1)|_{\min} |(z - z_2)|_{\min}} \\ &= \frac{|e^{itx}|_{\max} |e^{-ty}|_{\max}}{|(z - z_1)|_{\min} |(z - z_2)|_{\min}} \\ &\leq \frac{|e^{-ty}|_{\max}}{|(z - z_1)|_{\min} |(z - z_2)|_{\min}} \end{aligned}$$

Since $y < 0$ (we are now in lower half) and also since $t < 0$, then $|e^{-ty}|_{\max} = 1$, which occurs when $y = 0$. Hence

$$\left| \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} \right|_{\max} \leq \frac{1}{|(z - z_1)|_{\min} |(z - z_2)|_{\min}}$$

But by inverse triangle inequality $|(z - z_1)|_{\min} \geq |z|^2 + |z_1|^2 = R^2 + |\epsilon^2 + \omega_0^2|^2$ and $|(z - z_2)|_{\min} \geq |z|^2 + |z_2|^2 = R^2 + |\epsilon^2 + \omega_0^2|^2$. Hence the above becomes

$$\left| \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} \right|_{\max} \leq \frac{1}{2R^2 + 2|\epsilon^2 + \omega_0^2|^2}$$

The rest follows what was done in first part. Therefore

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{CR} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz &\leq \lim_{R \rightarrow \infty} R\pi \left(\frac{1}{2R^2 + 2|\epsilon^2 + \omega_0^2|^2} \right) \\ &= \pi \lim_{R \rightarrow \infty} \frac{R}{2R^2 + 2|\epsilon^2 + \omega_0^2|^2} \end{aligned}$$

But $2|\epsilon^2 + \omega_0^2|^2$ is finite number, say β so the above is

$$\lim_{R \rightarrow \infty} \int_{CR} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz \leq \pi \lim_{R \rightarrow \infty} \frac{R}{2R^2 + \beta}$$

And it is clear now that the above limit goes to zero.

The final solution is

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2} d\omega = 0 \quad t < 0$$

2 Problem 2

Evaluate the following integrals $\int_0^{\infty} \frac{\ln x}{1+x^2} dx$ and $\int_0^{\infty} \frac{\ln^2 x}{1+x^2} dx$. In order to find the second one you need to consider the integral $\int_0^{\infty} \frac{\ln^3 x}{1+x^2} dx$

Solution

2.1 Part (a)

There are two ways to find $\int_0^{\infty} \frac{\ln x}{1+x^2} dx$. One uses a substitution method and requires no complex contour integration and the second method uses $\int_0^{\infty} \frac{\ln^2 x}{1+x^2} dx$ with complex integration to find $\int_0^{\infty} \frac{\ln x}{1+x^2} dx$.

Method one

Let $x = \frac{1}{y}$. Hence $dx = -\frac{1}{y^2} dy$. When $x = 0 \rightarrow y = \infty$ and when $x = \infty \rightarrow y = 0$. Hence the integral $\int_0^{\infty} \frac{\ln x}{1+x^2} dx$ becomes

$$\begin{aligned} \int_0^{\infty} \frac{\ln x}{1+x^2} dx &= \int_{\infty}^0 \frac{\ln\left(\frac{1}{y}\right)}{1+\frac{1}{y^2}} \left(-\frac{1}{y^2} dy\right) \\ &= - \int_{\infty}^0 \frac{\ln\left(\frac{1}{y}\right)}{\frac{y^2+1}{y^2}} \left(\frac{1}{y^2} dy\right) \\ &= - \int_{\infty}^0 \frac{\ln\left(\frac{1}{y}\right)}{y^2+1} dy \\ &= \int_{\infty}^0 \frac{\ln(y)}{y^2+1} dy \\ &= - \int_0^{\infty} \frac{\ln(y)}{y^2+1} dy \end{aligned}$$

Since on the RHS y is arbitrary integration variable, we can rename it back to x . Hence the above becomes

$$\begin{aligned} \int_0^{\infty} \frac{\ln x}{1+x^2} dx &= - \int_0^{\infty} \frac{\ln(x)}{x^2+1} dy \\ 2 \int_0^{\infty} \frac{\ln x}{1+x^2} dx &= 0 \end{aligned}$$

Therefore

$$\int_0^{\infty} \frac{\ln x}{1+x^2} dx = 0$$

Method two

In this method will use complex integration on $\int_0^{\infty} \frac{\ln^2 z}{1+z^2} dz$ to show that $\int_0^{\infty} \frac{\ln z}{1+z^2} dz = 0$. The following contour will be used.

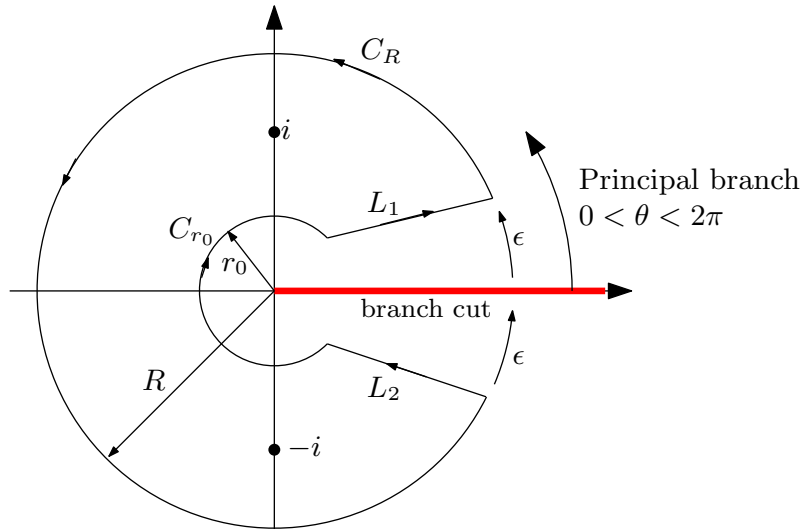


Figure 4: Contour for problem 2, showing location of poles at $\pm i$

$$\begin{aligned} \oint \frac{\ln^2 z}{1+z^2} dz &= \oint f(z) dz \\ &= \int_{L_2} f(z) dz + \int_{C_{r_0}} f(z) dz + \int_{L_1} f(z) dz + \int_{C_R} f(z) dz \\ &= 2\pi i \sum \text{Residue} \end{aligned}$$

Hence

$$\int_{L_2} f(z) dz + \int_{C_{r_0}} f(z) dz + \int_{L_1} f(z) dz + \int_{C_R} f(z) dz = 2\pi i \sum \text{Residue} \quad (1)$$

There are two poles in $\frac{\ln^2 z}{(z-i)(z+i)}$. Residue at $z_1 = i$ is

$$\begin{aligned}
 \text{Residue}(i) &= \lim_{z \rightarrow i} (z-i) \frac{\ln^2 z}{(z-i)(z+i)} \\
 &= \lim_{z \rightarrow i} \frac{\ln^2 z}{z+i} \\
 &= \frac{\ln^2 i}{2i} \\
 &= \frac{(\ln(1) + i\frac{\pi}{2})^2}{2i} \\
 &= \frac{(i\frac{\pi}{2})^2}{2i} \\
 &= \frac{-\frac{\pi^2}{4}}{2i} \\
 &= \frac{-\pi^2}{8i}
 \end{aligned} \tag{2}$$

And

$$\begin{aligned}
 \text{Residue}(-i) &= \lim_{z \rightarrow -i} (z+i) \frac{\ln^2 z}{(z-i)(z+i)} \\
 &= \lim_{z \rightarrow -i} \frac{\ln^2 z}{z-i} \\
 &= \frac{\ln^2(-i)}{-2i}
 \end{aligned}$$

But $\ln(-i) = \ln(1) + i\frac{3}{2}\pi$. Notice that the phase is $\frac{3}{2}\pi$ and not $-\frac{\pi}{2}$ since we are using principle branch defined as $0 < \theta < 2\pi$. Therefore the above becomes

$$\begin{aligned}
 \text{Residue}(-i) &= \frac{(\ln(1) + i\frac{3}{2}\pi)^2}{-2i} \\
 &= \frac{-\frac{9}{4}\pi^2}{-2i} \\
 &= \frac{9\pi^2}{8i}
 \end{aligned} \tag{3}$$

Adding (2+3) and substituting in (1) gives

$$\begin{aligned}
 \int_{L_2} f(z) dz + \int_{C_{r_0}} f(z) dz + \int_{L_1} f(z) dz + \int_{C_R} f(z) dz &= 2\pi i \left(\frac{-\pi^2}{8i} + \frac{9\pi^2}{8i} \right) \\
 \int_{L_2} f(z) dz + \int_{C_{r_0}} f(z) dz + \int_{L_1} f(z) dz + \int_{C_R} f(z) dz &= 2\pi^3
 \end{aligned}$$

We will show at the end that $\lim_{r_0 \rightarrow 0} \int_{C_{r_0}} f(z) dz = 0$ and that $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$. Given this, the above simplifies to only two integrals to evaluate

$$\int_{L_2} f(z) dz + \int_{L_1} f(z) dz = 2\pi^3 \quad (3A)$$

We will now work on finding $\int_{L_1} f(z) dz$. Let $z = re^{i\epsilon}$, hence $dz = dre^{i\epsilon}$ and the integral becomes

$$\begin{aligned} \int_{L_1} \frac{\ln^2 z}{1+z^2} dz &= \int_0^\infty \frac{\ln^2(re^{i\epsilon})}{1+(re^{i\epsilon})^2} dre^{i\epsilon} \\ &= e^{i\epsilon} \int_0^\infty \frac{(\ln r + i\epsilon)^2}{1+r^2 e^{2i\epsilon}} dr \\ &= e^{i\epsilon} \int_0^\infty \frac{\ln^2 r + i^2 \epsilon^2 + 2i\epsilon \ln r}{1+r^2 e^{2i\epsilon}} dr \end{aligned}$$

Now taking the limit as $\epsilon \rightarrow 0$ the above becomes

$$\int_{L_1} \frac{\ln^2 z}{1+z^2} dz = \int_0^\infty \frac{\ln^2 r}{1+r^2} dr \quad (4)$$

We will now work on finding $\int_{L_2} f(z) dz$. Let $z = re^{i(2\pi-\epsilon)}$, hence $dz = dre^{i(2\pi-\epsilon)}$ and the integral becomes

$$\begin{aligned} \int_{L_2} \frac{\ln^2 z}{1+z^2} dz &= \int_\infty^0 \frac{\ln^2(re^{i(2\pi-\epsilon)})}{1+(re^{i(2\pi-\epsilon)})^2} dre^{i(2\pi-\epsilon)} \\ &= e^{i(2\pi-\epsilon)} \int_\infty^0 \frac{(\ln(r) + i(2\pi-\epsilon))^2}{1+r^2 e^{2i(2\pi-\epsilon)}} dr \\ &= e^{i(2\pi-\epsilon)} \int_\infty^0 \frac{\ln^2(r) - (2\pi-\epsilon)^2 + 2i(2\pi-\epsilon) \ln r}{1+r^2 e^{2i(2\pi-\epsilon)}} dr \\ &= e^{i(2\pi-\epsilon)} \int_\infty^0 \frac{\ln^2(r) - (4\pi^2 + \epsilon^2 - 4\pi\epsilon) + 2i(2\pi-\epsilon) \ln r}{1+r^2 e^{2i(2\pi-\epsilon)}} dr \end{aligned}$$

Taking the limit as $\epsilon \rightarrow 0$ the above becomes

$$\int_{L_2} \frac{\ln^2 z}{1+z^2} dz = e^{i2\pi} \int_\infty^0 \frac{\ln^2(r) - 4\pi^2 + 4\pi i \ln r}{1+r^2 e^{i4\pi}} dr$$

But $e^{i2\pi} = 1$ and $e^{i4\pi} = 1$ then the above becomes

$$\begin{aligned}
\int_{L_2} \frac{\ln^2 z}{1+z^2} dz &= \int_{\infty}^0 \frac{\ln^2(r) - 4\pi^2 + 4\pi i \ln r}{1+r^2} dr \\
&= \int_{\infty}^0 \frac{\ln^2 r}{1+r^2} dr - \int_{\infty}^0 \frac{4\pi^2}{1+r^2} dr + 4\pi i \int_{\infty}^0 \frac{\ln r}{1+r^2} dr \\
&= - \int_0^{\infty} \frac{\ln^2 r}{1+r^2} dr + \int_0^{\infty} \frac{4\pi^2}{1+r^2} dr - 4\pi i \int_0^{\infty} \frac{\ln r}{1+r^2} dr
\end{aligned} \tag{5}$$

Using (4,5) in (3A) gives

$$\begin{aligned}
- \int_0^{\infty} \frac{\ln^2 r}{1+r^2} dr + \int_0^{\infty} \frac{4\pi^2}{1+r^2} dr - 4\pi i \int_0^{\infty} \frac{\ln r}{1+r^2} dr + \int_0^{\infty} \frac{\ln^2 r}{1+r^2} dr &= 2\pi^3 \\
4\pi^2 \int_0^{\infty} \frac{1}{1+r^2} dr - 4\pi i \int_0^{\infty} \frac{\ln r}{1+r^2} dr &= 2\pi^3
\end{aligned}$$

But $\int_0^{\infty} \frac{1}{1+r^2} dr = \arctan(r)_0^{\infty} = \arctan(\infty) - \arctan(0) = \frac{\pi}{2}$, hence the above becomes

$$\begin{aligned}
4\pi^2 \left(\frac{\pi}{2}\right) - 4\pi i \int_0^{\infty} \frac{\ln r}{1+r^2} dr &= 2\pi^3 \\
-4\pi i \int_0^{\infty} \frac{\ln r}{1+r^2} dr &= 0
\end{aligned}$$

Which implies

$$\boxed{\int_0^{\infty} \frac{\ln r}{1+r^2} dr = 0}$$

Which is the same result obtained using method one above.

Appendix Here we will show that $\lim_{r_0 \rightarrow 0} \int_{C_{r_0}} f(z) dz = 0$ and $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$.

For $\lim_{r_0 \rightarrow 0} \int_{C_{r_0}} f(z) dz$, let $z = r_0 e^{i\theta}$. Hence $dz = r_0 i e^{i\theta} d\theta$ and the integral becomes

$$\lim_{r_0 \rightarrow 0} \int_{2\pi-\epsilon}^{\epsilon} \frac{\ln^2(r_0 e^{i\theta})}{1+r_0^2 e^{2i\theta}} r_0 i e^{i\theta} d\theta = \lim_{r_0 \rightarrow 0} i \int_{2\pi-\epsilon}^{\epsilon} \frac{\ln^2(r_0 e^{i\theta})}{1+r_0^2 e^{2i\theta}} r_0 d\theta$$

As $\epsilon \rightarrow 0$ the above becomes

$$\begin{aligned}
\lim_{r_0 \rightarrow 0} \int_{2\pi-\epsilon}^{\epsilon} \frac{\ln^2(r_0 e^{i\theta})}{1+r_0^2 e^{2i\theta}} r_0 i e^{i\theta} d\theta &= \lim_{r_0 \rightarrow 0} i \int_{2\pi}^0 \frac{\ln^2(r_0 e^{i\theta})}{1+r_0^2 e^{i\theta}} r_0 d\theta \\
&\leq \lim_{r_0 \rightarrow 0} \left| i \int_{2\pi}^0 \frac{\ln^2(r_0 e^{i\theta})}{1+r_0^2 e^{i\theta}} r_0 d\theta \right|_{\max} \\
&\leq \lim_{r_0 \rightarrow 0} \int_{2\pi}^0 \left| \frac{\ln^2(r_0 e^{i\theta})}{1+r_0^2 e^{i\theta}} \right|_{\max} r_0 d\theta \\
&\leq \lim_{r_0 \rightarrow 0} \left| \frac{\ln^2(r_0 e^{i\theta})}{1+r_0^2 e^{i\theta}} \right|_{\max} \int_{2\pi}^0 r_0 d\theta \\
&= 2\pi r_0 \lim_{r_0 \rightarrow 0} \left| \frac{\ln^2(r_0 e^{i\theta})}{1+r_0^2 e^{i\theta}} \right|_{\max} \\
&\leq 2\pi r_0 \lim_{r_0 \rightarrow 0} \frac{|\ln^2(r_0 e^{i\theta})|_{\max}}{|1+r_0^2 e^{i\theta}|_{\min}} \\
&= 2\pi r_0 \lim_{r_0 \rightarrow 0} \frac{|\ln r_0 + i\theta|_{\max}^2}{|1+r_0^2 e^{i\theta}|_{\min}} \\
&= 2\pi r_0 \lim_{r_0 \rightarrow 0} \frac{|\ln^2 r_0 + (i\theta)^2 + 2i\theta \ln r_0|_{\max}}{1-r_0^2} \\
&= 2\pi \lim_{r_0 \rightarrow 0} \frac{r_0 \ln^2 r_0 - 4\pi^2 r_0 + 4\pi r_0 \ln r_0}{1-r_0^2} \\
&= 2\pi \lim_{r_0 \rightarrow 0} \left(\frac{r_0 \ln^2 r_0}{1-r_0^2} - 4\pi^2 \frac{r_0}{1-r_0^2} + 4\pi \frac{r_0 \ln r_0}{1-r_0^2} \right)
\end{aligned}$$

But $\lim_{r_0 \rightarrow 0} \frac{r_0 \ln^2 r_0}{1-r_0^2} = 0$ and $\lim_{r_0 \rightarrow 0} \frac{r_0}{1-r_0^2} = 0$ and $\lim_{r_0 \rightarrow 0} \frac{r_0 \ln r_0}{1-r_0^2} = 0$ Hence all terms on the RHS above become zero in the limit. Therefore

$$\begin{aligned}
\lim_{r_0 \rightarrow 0} \int_{2\pi-\epsilon}^{\epsilon} \frac{\ln^2(r_0 e^{i\theta})}{1+r_0^2 e^{2i\theta}} r_0 i e^{i\theta} d\theta &= \lim_{r_0 \rightarrow 0} \int_{C_{r_0}} \frac{\ln^2 z}{1+z^2} dz \\
&= 0
\end{aligned}$$

Now we will do the same $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz$, let $z = Re^{i\theta}$. Hence $dz = Rie^{i\theta} d\theta$ and the integral becomes

$$\lim_{R \rightarrow \infty} \int_{\epsilon}^{2\pi-\epsilon} \frac{\ln^2(Re^{i\theta})}{1+R^2 e^{2i\theta}} Rie^{i\theta} d\theta = \lim_{R \rightarrow \infty} i \int_{\epsilon}^{2\pi-\epsilon} \frac{\ln^2(Re^{i\theta})}{1+R^2 e^{2i\theta}} R d\theta$$

As $\epsilon \rightarrow 0$ the above becomes

$$\begin{aligned}
\lim_{R \rightarrow \infty} \int_{\epsilon}^{2\pi-\epsilon} \frac{\ln^2(Re^{i\theta})}{1+R^2e^{2i\theta}} Rie^{i\theta} d\theta &= \lim_{R \rightarrow \infty} i \int_0^{2\pi} \frac{\ln^2(Re^{i\theta})}{1+R^2e^{2i\theta}} R d\theta \\
&\leq \lim_{R \rightarrow \infty} \left| i \int_0^{2\pi} \frac{\ln^2(Re^{i\theta})}{1+R^2e^{2i\theta}} R d\theta \right|_{\max} \\
&\leq \lim_{R \rightarrow \infty} \int_0^{2\pi} \left| \frac{\ln^2(Re^{i\theta})}{1+R^2e^{2i\theta}} R \right|_{\max} d\theta \\
&\leq \lim_{R \rightarrow \infty} \left| \frac{\ln^2(Re^{i\theta})}{1+R^2e^{2i\theta}} \right| \int_0^{2\pi} R d\theta \\
&= 2\pi \lim_{R \rightarrow \infty} R \frac{|\ln^2(Re^{i\theta})|_{\max}}{|1+R^2e^{2i\theta}|_{\min}} \\
&= 2\pi \lim_{R \rightarrow \infty} R \frac{|\ln R + i\theta|_{\max}^2}{1-R^2} \\
&= 2\pi \lim_{R \rightarrow \infty} R \frac{|\ln^2 R - \theta^2 + 2i\theta \ln R|_{\max}}{1-R^2} \\
&\leq 2\pi \lim_{R \rightarrow \infty} \frac{R \ln^2 R - 4\pi^2 R + 4\pi R \ln R}{1-R^2} \\
&= 2\pi \lim_{R \rightarrow \infty} \left(\frac{R \ln^2 R}{1-R^2} - 4\pi^2 \frac{R}{1-R^2} + 4\pi \frac{R \ln R}{1-R^2} \right)
\end{aligned}$$

But $\lim_{R \rightarrow \infty} \frac{R \ln^2 R}{1-R^2} = 0$ and $\lim_{R \rightarrow \infty} \frac{R}{1-R^2} = 0$ and $\lim_{R \rightarrow \infty} \frac{R \ln R}{1-R^2} = 0$ Hence all terms on the RHS above become zero in the limit. Therefore

$$\begin{aligned}
\lim_{R \rightarrow \infty} \int_{\epsilon}^{2\pi-\epsilon} \frac{\ln^2(Re^{i\theta})}{1+R^2e^{2i\theta}} Rie^{i\theta} d\theta &= \lim_{R \rightarrow \infty} \int_{C_R} \frac{\ln^2 z}{1+z^2} dz = 0 \\
&= 0
\end{aligned}$$

2.2 Part (b)

We will now find $\int_0^{\infty} \frac{\ln^3 z}{1+z^2} dz$ in order to determine $\int_0^{\infty} \frac{\ln^2 z}{1+z^2} dz$. We will use the same contour integration as part (a) above.

$$\int_{L_2} f(z) dz + \int_{C_{r_0}} f(z) dz + \int_{L_1} f(z) dz + \int_{C_R} f(z) dz = 2\pi i \sum \text{Residue} \quad (1)$$

There are two poles in $\frac{\ln^3 z}{(z-i)(z+i)}$. Residue at $z_1 = i$ is

$$\begin{aligned}
 \text{Residue}(i) &= \lim_{z \rightarrow i} (z - i) \frac{\ln^3 z}{(z - i)(z + i)} \\
 &= \lim_{z \rightarrow i} \frac{\ln^3 z}{z + i} \\
 &= \frac{\ln^3 i}{2i} \\
 &= \frac{(\ln(1) + i\frac{\pi}{2})^3}{2i} \\
 &= \frac{(i\frac{\pi}{2})^3}{2i} \\
 &= \frac{-i\frac{\pi^3}{8}}{2i} \\
 &= \frac{-\pi^3}{16}
 \end{aligned} \tag{2}$$

And

$$\begin{aligned}
 \text{Residue}(-i) &= \lim_{z \rightarrow -i} (z + i) \frac{\ln^3 z}{(z - i)(z + i)} \\
 &= \lim_{z \rightarrow -i} \frac{\ln^3 z}{z - i} \\
 &= \frac{\ln^3(-i)}{-2i}
 \end{aligned}$$

But $\ln(-i) = \ln(1) + i\frac{3}{2}\pi$. Notice that the phase is $\frac{3}{2}\pi$ and not $-\frac{\pi}{2}$ since we are using principle branch defined as $0 < \theta < 2\pi$. Therefore the above becomes

$$\begin{aligned}
 \text{Residue}(-i) &= \frac{(\ln(1) + i\frac{3}{2}\pi)^3}{-2i} \\
 &= \frac{(i\frac{3}{2}\pi)^3}{-2i} \\
 &= \frac{-i\frac{27}{8}\pi^3}{-2i} \\
 &= \frac{27\pi^3}{16}
 \end{aligned} \tag{3}$$

Adding (2+3) and substituting in (1) gives

$$\begin{aligned} \int_{L_2} f(z) dz + \int_{C_{r_0}} f(z) dz + \int_{L_1} f(z) dz + \int_{C_R} f(z) dz &= 2\pi i \left(\frac{-\pi^3}{16} + \frac{27\pi^3}{16} \right) \\ \int_{L_2} f(z) dz + \int_{C_{r_0}} f(z) dz + \int_{L_1} f(z) dz + \int_{C_R} f(z) dz &= \frac{13}{4} \pi^4 i \end{aligned}$$

We will show below that $\lim_{r_0 \rightarrow 0} \int_{C_{r_0}} f(z) dz = 0$ and that $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$, which simplifies the above to

$$\int_{L_2} f(z) dz + \int_{L_1} f(z) dz = \frac{13}{4} \pi^4 i \quad (3A)$$

We will now work on finding $\int_{L_1} f(z) dz$. Let $z = re^{i\epsilon}$, hence $dz = dre^{i\epsilon}$ and the integral becomes

$$\begin{aligned} \int_{L_1} \frac{\ln^3 z}{1+z^2} dz &= \int_0^\infty \frac{\ln^3(re^{i\epsilon})}{1+(re^{i\epsilon})^2} dre^{i\epsilon} \\ &= e^{i\epsilon} \int_0^\infty \frac{(\ln r + i\epsilon)^3}{1+r^2 e^{2i\epsilon}} dr \\ &= e^{i\epsilon} \int_0^\infty \frac{(\ln^2 r + i^2 \epsilon^2 + 2i\epsilon \ln r)(\ln r + i\epsilon)}{1+r^2 e^{2i\epsilon}} dr \\ &= e^{i\epsilon} \int_0^\infty \frac{(\ln^3 r + i^2 \epsilon^2 \ln r + 2i\epsilon \ln^2 r) + (i\epsilon \ln^2 r + i^3 \epsilon^3 + 2i^2 \epsilon^2 \ln r)}{1+r^2 e^{2i\epsilon}} dr \end{aligned}$$

Now taking the limit as $\epsilon \rightarrow 0$ the above becomes

$$\int_{L_1} \frac{\ln^3 z}{1+z^2} dz = \int_0^\infty \frac{\ln^3 r}{1+r^2} dr \quad (4)$$

We will now work on finding $\int_{L_2} f(z) dz$. Let $z = re^{i(2\pi-\epsilon)}$, hence $dz = dre^{i(2\pi-\epsilon)}$ and the integral becomes

$$\begin{aligned} \int_{L_2} \frac{\ln^3 z}{1+z^2} dz &= \int_\infty^0 \frac{\ln^3(re^{i(2\pi-\epsilon)})}{1+(re^{i(2\pi-\epsilon)})^2} dre^{i(2\pi-\epsilon)} \\ &= e^{i(2\pi-\epsilon)} \int_\infty^0 \frac{(\ln(r) + i(2\pi-\epsilon))^3}{1+r^2 e^{2i(2\pi-\epsilon)}} dr \end{aligned}$$

But $\lim_{\epsilon \rightarrow 0} e^{i(2\pi-\epsilon)} = e^{2\pi i} = 1$ and the above becomes

$$\begin{aligned} \int_{L_2} \frac{\ln^3 z}{1+z^2} dz &= \int_\infty^0 \frac{(\ln^2 r - (2\pi-\epsilon)^2 + 2i(2\pi-\epsilon)\ln r)(\ln(r) + i(2\pi-\epsilon))}{1+r^2 e^{2i(2\pi-\epsilon)}} dr \\ &= \int_\infty^0 \frac{\ln^3 r - \ln r(2\pi-\epsilon)^2 + 2i(2\pi-\epsilon)\ln^2 r + i(2\pi-\epsilon)\ln^2 r - i(2\pi-\epsilon)^3 + 2i^2(2\pi-\epsilon)^2 \ln r}{1+r^2 e^{2i(2\pi-\epsilon)}} dr \end{aligned}$$

Taking the limit as $\epsilon \rightarrow 0$ the above becomes

$$\begin{aligned} \int_{L_2} \frac{\ln^3 z}{1+z^2} dz &= \int_{\infty}^0 \frac{\ln^3 r - 4\pi^2 \ln r + 4\pi i \ln^2 r + 2\pi i \ln^2 r - i(2\pi - \epsilon)^2 (2\pi - \epsilon) + 2i^2 (4\pi^2 + \epsilon^2 - 4\pi\epsilon) \ln r}{1+r^2 e^{4\pi i}} dr \\ &= \int_{\infty}^0 \frac{\ln^3 r - 4\pi^2 \ln r + 4\pi i \ln^2 r + 2\pi i \ln^2 r - i(4\pi^2 + \epsilon^2 - 4\pi\epsilon) (2\pi - \epsilon) - 8\pi^2 \ln r}{1+r^2} dr \\ &= \int_{\infty}^0 \frac{\ln^3 r - 4\pi^2 \ln r + 6\pi i \ln^2 r - i(8\pi^3 + 2\pi\epsilon^2 - 8\pi^2\epsilon) - (4\pi^2\epsilon + \epsilon^3 - 4\pi\epsilon^2) - 8\pi^2 \ln r}{1+r^2} dr \end{aligned}$$

Taking the limit as $\epsilon \rightarrow 0$ the above becomes

$$\begin{aligned} \int_{L_2} \frac{\ln^3 z}{1+z^2} dz &= \int_{\infty}^0 \frac{\ln^3(r) - 4\pi^2 \ln r + 6\pi i \ln^2 r - 8i\pi^3 - 8\pi^2 \ln r}{1+r^2} dr \\ &= \int_{\infty}^0 \frac{\ln^3(r) - 12\pi^2 \ln r + 6\pi i \ln^2 r - 8i\pi^3}{1+r^2} dr \end{aligned}$$

Hence the above becomes

$$\begin{aligned} \int_{L_2} \frac{\ln^3 z}{1+z^2} dz &= \int_{\infty}^0 \frac{\ln^3(r)}{1+r^2} dr - 12\pi^2 \int_{\infty}^0 \frac{\ln r}{1+r^2} dr + 6\pi i \int_{\infty}^0 \frac{\ln^2 r}{1+r^2} dr - 8i\pi^3 \int_{\infty}^0 \frac{1}{1+r^2} dr \\ &= - \int_0^{\infty} \frac{\ln^3 r}{1+r^2} dr + 12\pi^2 \int_0^{\infty} \frac{\ln r}{1+r^2} dr - 6\pi i \int_0^{\infty} \frac{\ln^2 r}{1+r^2} dr + 8i\pi^3 \int_0^{\infty} \frac{1}{1+r^2} dr \end{aligned}$$

But $\int_0^{\infty} \frac{\ln r}{1+r^2} dr = 0$ from part (a) and $\int_0^{\infty} \frac{1}{1+r^2} dr = \frac{\pi}{2}$, hence the above becomes

$$\int_{L_2} \frac{\ln^3 z}{1+z^2} dz = - \int_0^{\infty} \frac{\ln^3 r}{1+r^2} dr - 6\pi i \int_0^{\infty} \frac{\ln^2 r}{1+r^2} dr + 4i\pi^4 \quad (5)$$

Using (4,5) in (3A) gives

$$\begin{aligned} \int_{L_2} f(z) dz + \int_{L_1} f(z) dz &= \frac{13}{4} \pi^4 i \\ \left(- \int_0^{\infty} \frac{\ln^3 r}{1+r^2} dr - 6\pi i \int_0^{\infty} \frac{\ln^2 r}{1+r^2} dr + 4i\pi^4 \right) + \left(\int_0^{\infty} \frac{\ln^3 r}{1+r^2} dr \right) &= \frac{13}{4} \pi^4 i \\ -6\pi i \int_0^{\infty} \frac{\ln^2 r}{1+r^2} dr + 4i\pi^4 &= \frac{13}{4} \pi^4 i \\ \int_0^{\infty} \frac{\ln^2 r}{1+r^2} dr &= \frac{\frac{13}{4} \pi^4 i - 4i\pi^4}{-6\pi i} \\ \int_0^{\infty} \frac{\ln^2 r}{1+r^2} dr &= \frac{13\pi^4 i - 16i\pi^4}{-24\pi i} \\ &= \frac{-3\pi^4 i}{-24\pi i} \\ &= \frac{\pi^3}{8} \end{aligned}$$

Which implies

$$\boxed{\int_0^\infty \frac{\ln^2 x}{1+x^2} dx = \frac{\pi^3}{8}}$$

Appendix Here we will show that $\lim_{r_0 \rightarrow 0} \int_{C_{r_0}} f(z) dz = 0$ and $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$.

For $\lim_{r_0 \rightarrow 0} \int_{C_{r_0}} f(z) dz$, let $z = r_0 e^{i\theta}$. Hence $dz = r_0 i e^{i\theta} d\theta$ and the integral becomes

$$\lim_{r_0 \rightarrow 0} \int_{2\pi-\epsilon}^\epsilon \frac{\ln^3(r_0 e^{i\theta})}{1+r_0^2 e^{2i\theta}} r_0 i e^{i\theta} d\theta = \lim_{r_0 \rightarrow 0} i \int_{2\pi-\epsilon}^\epsilon \frac{\ln^3(r_0 e^{i\theta})}{1+r_0^2 e^{2i\theta}} r_0 d\theta$$

As $\epsilon \rightarrow 0$ the above becomes

$$\begin{aligned} \lim_{r_0 \rightarrow 0} \int_{2\pi-\epsilon}^\epsilon \frac{\ln^3(r_0 e^{i\theta})}{1+r_0^2 e^{2i\theta}} r_0 i e^{i\theta} d\theta &= \lim_{r_0 \rightarrow 0} i \int_{2\pi}^0 \frac{\ln^3(r_0 e^{i\theta})}{1+r_0^2 e^{2i\theta}} r_0 d\theta \\ &\leq \lim_{r_0 \rightarrow 0} \left| i \int_{2\pi}^0 \frac{\ln^3(r_0 e^{i\theta})}{1+r_0^2 e^{2i\theta}} r_0 d\theta \right|_{\max} \\ &\leq \lim_{r_0 \rightarrow 0} \int_{2\pi}^0 \left| \frac{\ln^3(r_0 e^{i\theta})}{1+r_0^2 e^{2i\theta}} \right|_{\max} r_0 d\theta \\ &\leq \lim_{r_0 \rightarrow 0} \left| \frac{\ln^3(r_0 e^{i\theta})}{1+r_0^2 e^{2i\theta}} \right|_{\max} \int_{2\pi}^0 r_0 d\theta \\ &= 2\pi r_0 \lim_{r_0 \rightarrow 0} \left| \frac{\ln^3(r_0 e^{i\theta})}{1+r_0^2 e^{2i\theta}} \right|_{\max} \\ &\leq 2\pi r_0 \lim_{r_0 \rightarrow 0} \frac{\left| \ln^3(r_0 e^{i\theta}) \right|_{\max}}{\left| 1+r_0^2 e^{2i\theta} \right|_{\min}} \\ &\leq 2\pi r_0 \lim_{r_0 \rightarrow 0} \frac{\left| \ln r_0 + i\theta \right|_{\max}^2 \left| \ln r_0 + i\theta \right|_{\max}}{\left| 1+r_0^2 e^{2i\theta} \right|_{\min}} \end{aligned}$$

But from part (a) we showed that $2\pi r_0 \lim_{r_0 \rightarrow 0} \frac{\left| \ln r_0 + i\theta \right|_{\max}^2}{\left| 1+r_0^2 e^{2i\theta} \right|_{\min}} = 0$, hence it follows that the RHS above goes to zero. Therefore

$$\begin{aligned} \lim_{r_0 \rightarrow 0} \int_{2\pi-\epsilon}^\epsilon \frac{\ln^3(r_0 e^{i\theta})}{1+r_0^2 e^{2i\theta}} r_0 i e^{i\theta} d\theta &= \lim_{r_0 \rightarrow 0} \int_{C_{r_0}} \frac{\ln^3 z}{1+z^2} dz \\ &= 0 \end{aligned}$$

Now we will do the same $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz$, let $z = R e^{i\theta}$. Hence $dz = R i e^{i\theta} d\theta$ and the

integral becomes

$$\lim_{R \rightarrow \infty} \int_{\epsilon}^{2\pi - \epsilon} \frac{\ln^3(Re^{i\theta})}{1 + R^2 e^{2i\theta}} Rie^{i\theta} d\theta = \lim_{R \rightarrow \infty} i \int_{\epsilon}^{2\pi - \epsilon} \frac{\ln^3(Re^{i\theta})}{1 + R^2 e^{2i\theta}} R d\theta$$

As $\epsilon \rightarrow 0$ the above becomes

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{\epsilon}^{2\pi - \epsilon} \frac{\ln^3(Re^{i\theta})}{1 + R^2 e^{2i\theta}} Rie^{i\theta} d\theta &= \lim_{R \rightarrow \infty} i \int_0^{2\pi} \frac{\ln^3(Re^{i\theta})}{1 + R^2 e^{2i\theta}} R d\theta \\ &\leq \lim_{R \rightarrow \infty} \left| i \int_0^{2\pi} \frac{\ln^3(Re^{i\theta})}{1 + R^2 e^{2i\theta}} R d\theta \right|_{\max} \\ &\leq \lim_{R \rightarrow \infty} \int_0^{2\pi} \left| \frac{\ln^3(Re^{i\theta})}{1 + R^2 e^{2i\theta}} R \right|_{\max} d\theta \\ &\leq \lim_{R \rightarrow \infty} \left| \frac{\ln^3(Re^{i\theta})}{1 + R^2 e^{2i\theta}} \right| \int_0^{2\pi} R d\theta \\ &= 2\pi \lim_{R \rightarrow \infty} R \frac{|\ln^3(Re^{i\theta})|_{\max}}{|1 + R^2 e^{2i\theta}|_{\min}} \\ &\leq 2\pi \lim_{R \rightarrow \infty} R \frac{|\ln R + i\theta|_{\max}^2 |\ln(Re^{i\theta})|_{\max}}{1 - R^2} \end{aligned}$$

But from part (a) we showed that $2\pi \lim_{R \rightarrow \infty} R \frac{|\ln R + i\theta|_{\max}^2}{1 - R^2} = 0$, hence it follows that the RHS above goes to zero. Therefore

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{\epsilon}^{2\pi - \epsilon} \frac{\ln^3(Re^{i\theta})}{1 + R^2 e^{2i\theta}} Rie^{i\theta} d\theta &= \lim_{R \rightarrow \infty} \int_{C_R} \frac{\ln^3 z}{1 + z^2} dz = 0 \\ &= 0 \end{aligned}$$