HW 4
Physics 5041 Mathematical Methods for Physics
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Contents

1 Problem 1

Let C denote the square contour with corners at $\pm 2, \pm 2i$ and which is taken in counter clockwise direction. Use the residue theorem to evaluate the integral $\int_C f(x)\,dz$ for the following functions

(a) $\frac{e^{-z}}{z-i\frac{\pi}{2}}$, (b) $\frac{\cos z}{z\left(z^2+8\right)}$ (c) $\frac{z}{z+1}$

Solution

1.1 Part (a)

The function $f(z) = \frac{e^{-z}}{z-i\frac{\pi}{2}}$ has a simple pole at $z = i\frac{\pi}{2} \approx 1.57i$, hence it is inside the contour.

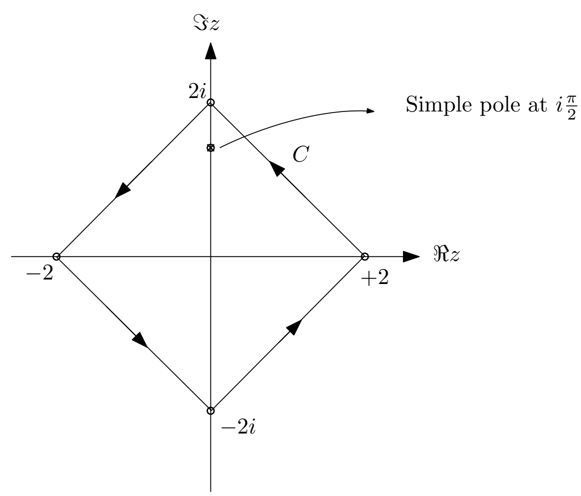

Figure 1: Location of pole relative to contour

Hence by residue theorem

$$\oint_C f(z)\,dz = 2\pi i \ \text{Residue}\left(f(z)\right)_{z=z_0}$$

So we just need to find the residue of $f(z)$ at $z = z_0 = i\frac{\pi}{2}$. Since this is a simple plot, then the residue is given by

$$\begin{aligned}
\text{Residue}\left(f(z)\right)_{z=z_0} &= \lim_{z\to z_0}(z-z_0)f(z) \\
&= \lim_{z\to i\frac{\pi}{2}}\left(z-i\frac{\pi}{2}\right)\frac{e^{-z}}{z-i\frac{\pi}{2}} \\
&= \lim_{z\to i\frac{\pi}{2}} e^{-z} \\
&= e^{-i\frac{\pi}{2}} \\
&= \cos\left(\frac{\pi}{2}\right)-i\sin\left(\frac{\pi}{2}\right) \\
&= -i
\end{aligned}$$

Therefore

$$\begin{aligned}
\oint_C \frac{e^{-z}}{z-i\frac{\pi}{2}}dz &= 2\pi i(-i) \\
&= 2\pi
\end{aligned}$$

1.2 Part (b)

The function $f(z)=\frac{\cos z}{z\left(z^2+8\right)}$ has one simple pole at $z=0$ which is inside the contour, and a poles at $z=\pm i\sqrt{8}=\pm 2i\sqrt{2}\approx\pm 2.83i$ but these are outside the contour.

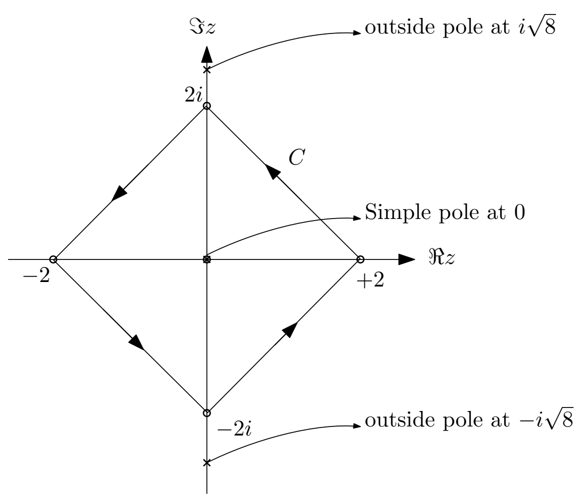

Figure 2: Location of pole relative to contour

Therefore by residue theorem, only the pole inside the contour which is at $z=0$ will contribute to the integral. So we just need to find residue at $z=0$

$$
\begin{aligned}
\operatorname{Residue}\left(f(z)\right)_{z=z_0} &= \lim_{z\to z_0}(z-z_0)\,f(z)\\
&= \lim_{z\to 0}(z)\,\frac{\cos z}{z\left(z^2+8\right)}\\
&= \lim_{z\to 0}\frac{\cos z}{\left(z^2+8\right)}\\
&= \frac{1}{8}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\oint_C \frac{\cos z}{z\left(z^2+8\right)}dz &= 2\pi i\left(\frac{1}{8}\right)\\
&= i\frac{\pi}{4}
\end{aligned}
$$

1.3 Part (c)

The function $f(z)=\frac{z}{2(z+1)}$ has one simple pole at $z=-1$ which is inside the contour.

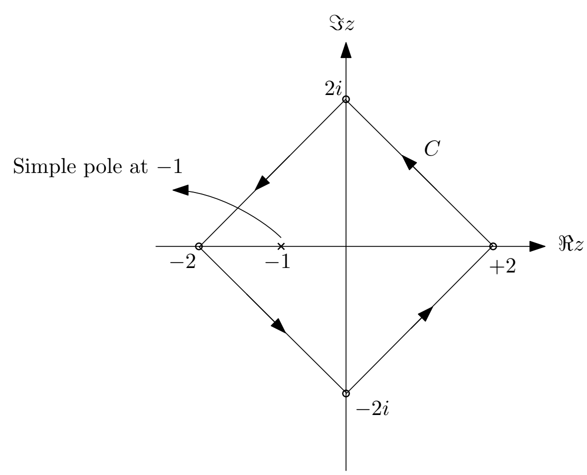

Figure 3: Location of pole relative to contour

So we just need to find residue at $z = -1$

$$\begin{aligned}
\text{Residue}\left(f(z)\right)_{z=z_0} &= \lim_{z\to z_0} (z - z_0) f(z) \\
&= \lim_{z\to -1} (z+1) \frac{z}{2(z+1)} \\
&= \lim_{z\to -1} \frac{z}{2} \\
&= \frac{-1}{2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\oint_C \frac{z}{2(z+1)} dz &= 2\pi i \left(\frac{-1}{2}\right) \\
&= -i\pi
\end{aligned}$$

2 Problem 2

Assume that $f(z)$ is analytic on and interior to a closed contour C and that the point z_0 lies inside C. Show that

$$\oint_C \frac{f'(z)}{z-z_0}dz = \oint_C \frac{f(z)}{(z-z_0)^2}dz$$

Solution

We see that $g(z) = \frac{f'(z)}{z-z_0}$ has a simple pole at $z = z_0$. Therefore

$$\oint_C g(z)\,dz = 2\pi i\,(b_1) \tag{1}$$

Where b_1 is the Residue of $g(z)$ at z_0. By definition the residue of a simple pole is found as follows

$$\begin{aligned} b_1 &= \lim_{z\to z_0}(z-z_0)\,g(z) \\ &= \lim_{z\to z_0}(z-z_0)\frac{f'(z)}{z-z_0} \\ &= \lim_{z\to z_0} f'(z) \\ &= f'(z_0) \end{aligned}$$

Hence (1) becomes

$$\oint_C g(z)\,dz = (2\pi i)\,f'(z_0)$$

$$\oint_C \frac{f'(z)}{z-z_0}dz = (2\pi i)\,f'(z_0) \tag{2}$$

But per lecture notes, page 46 on complex analysis, it shows that

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$$

Substituting the above back into RHS of (2) results in

$$\oint_C \frac{f'(z)}{z-z_0}dz = 2\pi i\left(\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz\right)$$

Therefore

$$\oint_C \frac{f'(z)}{z-z_0}dz = \oint_C \frac{f(z)}{(z-z_0)^2}dz$$

QED.

3 Problem 3

Give the Laurent series expansion both in powers of z and in powers of $(z-1)$ for the function $\frac{1}{z^2(1-z)}$

Solution

There is a pole of order 2 at $z=0$ and a pole of order one at $z=1$. Therefore, there is a Laurent series expansion about $z=0$ which is valid in inside a disk or radius 1 centered at $z=0$. Around $z=1$ there is another Laurent series expansion of the function, which is valid inside a disk centered at $z=0$ of radius 1.

Laurent series expansion around $z=0$

$$
\begin{aligned}
\frac{1}{z^2(1-z)} &= \frac{1}{z^2}\frac{1}{(1-z)} \\
&= \frac{1}{z^2}\left(1+z+z^2+z^3+\cdots\right) \qquad |z|<1 \\
&= \frac{1}{z^2}+\frac{1}{z}+1+z+z^2+z^3+\cdots
\end{aligned}
$$

We see from the above that the residue at $z=0$ is 1 which is the coefficient of $\frac{1}{z}$ term.

Laurent series expansion around $z=1$

Let $u=z-1$, hence $z=u+1$ and the function $\frac{1}{z^2(1-z)}$ in terms of u becomes

$$
\frac{1}{(1+u)^2(-u)} = \frac{-1}{u}\frac{1}{(1+u)^2} \tag{1}
$$

But $\frac{1}{(1+u)^2}=(1+u)^{-2}$. Applying Binomial expansion $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$ which is valid for $|x|<1$ then we see that for $n=-2$ we obtain

$$
(1+u)^{-2}=1+(-2)u+\frac{(-2)(-2-1)}{2!}u^2+\frac{(-2)(-2-1)(-2-2)}{3!}u^3+\cdots
$$

The above is valid for $|u|<1$ or $|z-1|<1$ or $0<z<2$. Simplifying the above gives

$$
\frac{1}{(1-u)^2}=1-2u+3u^2-4u^3+\cdots
$$

Substituting the above back into (1) gives

$$
\begin{aligned}
\frac{-1}{u}\frac{1}{(1+u)^2} &= \frac{-1}{u}\left(1-2u+3u^2-4u^3+\cdots\right) \\
&= \frac{-1}{u}+2-3u+4u^2-\cdots
\end{aligned}
$$

But since $u=z-1$ then the above becomes

$$
\frac{1}{z^2(1-z)}=\frac{-1}{z-1}+2-3(z-1)+4(z-1)^2-5(z-1)^3+\cdots
$$

We see from the above that the residue of $f(z)$ is -1 at $z = 1$.

In summary

1. Laurent series around $z = 0$ is $\frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots$ which is valid inside disk centered at $z = 0$ of radius 1
2. Laurent series around $z = 1$ is $\frac{-1}{z-1} + 2 - 3(z-1) + 4(z-1)^2 - 5(z-1)^3 + \cdots$ which is valid inside disk centered at $z = 1$ of radius 1

Note that there is another Laurent series expansions that can be found, which is for the region $1 < |z| < \infty$, which is outside a disk of radius 1 centered at $z = 0$. But the problem is asking for the above two expansions only.

4 Problem 4

Evaluate the integral $\int_0^\infty \frac{dx}{1+x^4} dx$

solution

Since the integrand is even, then

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx$$

Now we consider the following contour

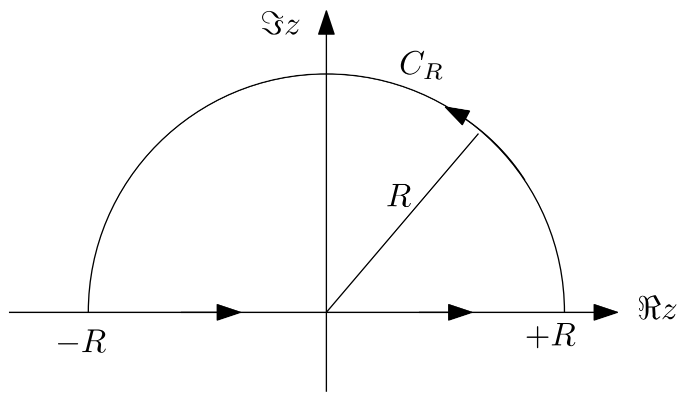

Figure 4: contour used for problem 4

Therefore

$$\oint_C f(z)\,dz = \left(\lim_{R\to\infty}\int_{-R}^{0} f(x)\,dx + \lim_{\tilde{R}\to\infty}\int_0^{\tilde{R}} f(x)\,dx\right) + \lim_{R\to\infty}\int_{C_R} f(z)\,dz$$

Using Cauchy principal value the integral above can be written as

$$\begin{aligned}\oint_C f(z)\,dz &= \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty}\int_{C_R} f(z)\,d \\ &= 2\pi i \sum \text{Residue}\end{aligned}$$

Where $\sum$ Residue is sum of residues of $\frac{1}{z^4+1}$ for poles that are inside the contour C. Therefore the above becomes

$$\begin{aligned}\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx &= 2\pi i \sum \text{Residue} - \lim_{R\to\infty}\int_{C_R} f(z)\,dz \\ \int_{-\infty}^{\infty} \frac{1}{x^4+1} dx &= 2\pi i \sum \text{Residue} - \lim_{R\to\infty}\int_{C_R} \frac{1}{z^4+1} dz \qquad (1)\end{aligned}$$

Now we will show that $\lim_{R\to\infty} \int_{C_R} \frac{1}{z^4+1} dz = 0$. Since

$$\left| \int_{C_R} \frac{1}{z^4+1} dz \right| \leq ML$$
$$= \left| f(z) \right|_{\max} (\pi R) \tag{2}$$

But

$$f(z) = \frac{1}{\left(z^2 - i\right)\left(z^2 + i\right)}$$

Hence, and since $z = R\, e^{i\theta}$ then

$$\left| f(z) \right|_{\max} \leq \frac{1}{\left| z^2 - i \right|_{\min} \left| z^2 + i \right|_{\min}}$$

Using the inverse triangle inequality then $\left|z^2 - i\right| \geq |z|^2 + 1$ and $\left|z^2 + i\right| \geq |z|^2 - 1$, and because $|z| = R$ then the above becomes

$$\left| f(z) \right|_{\max} \leq \frac{1}{\left(R^2 + 1\right)\left(R^2 - 1\right)}$$
$$= \frac{1}{R^4 - 1}$$

Therefore (2) becomes

$$\left| \int_{C_R} \frac{1}{z^4+1} dz \right| \leq \frac{\pi R}{R^4 - 1}$$

Then it is clear that as $R \to \infty$ the above goes to zero since $\lim_{R\to\infty} \frac{\pi R}{R^4-1} = \lim_{R\to\infty} \frac{\frac{\pi}{R^3}}{1-\frac{1}{R^4}} = \frac{0}{1} = 0$. Equation (1) now simplifies to

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = 2\pi i \sum \text{Residue} \tag{2A}$$

We just now need to find the residues of $\frac{1}{z^4+1}$ located in upper half plane. The zeros of the denominator $z^4 + 1 = 0$ are at $z = -1^{\frac{1}{4}} = \left(e^{i\pi}\right)^{\frac{1}{4}}$, then the first zero is at $e^{i\frac{\pi}{4}}$, and the second zero at $e^{i\left(\frac{\pi}{4}+\frac{\pi}{2}\right)} = e^{i\left(\frac{3}{4}\pi\right)}$ and the third zero at $e^{i\left(\frac{3}{4}\pi+\frac{\pi}{2}\right)} = e^{i\left(\frac{5}{4}\pi\right)}$ and the fourth zero at $e^{i\left(\frac{5}{4}\pi+\frac{\pi}{2}\right)} = e^{i\frac{7}{4}\pi}$. Hence poles are at

$$z_1 = e^{i\frac{\pi}{4}}$$
$$z_2 = e^{i\frac{3}{4}\pi}$$
$$z_3 = e^{i\frac{5}{4}\pi}$$
$$z_4 = e^{i\frac{7}{4}\pi}$$

Out of these only the first two are in upper half plane. Hence since these are simple poles, we can use the following to find the residues

$$\begin{aligned}
\text{Residue}\left(z_1\right) &= \lim_{z\to z_1}\left(z-z_1\right)f\left(z\right)\\
&= \lim_{z\to z_1}\left(z-z_1\right)\frac{1}{z^4-1}
\end{aligned}$$

Applying L'Hopitals rule, the above becomes

$$\begin{aligned}
\text{Residue}\left(z_1\right) &= \lim_{z\to z_1}\frac{\frac{d}{dz}\left(z-z_1\right)}{\frac{d}{dz}\left(z^4-1\right)}\\
&= \lim_{z\to e^{i\frac{\pi}{4}}}\frac{1}{4z^3}\\
&= \frac{1}{4\left(e^{i\frac{\pi}{4}}\right)^3}\\
&= \frac{1}{4e^{i\frac{3\pi}{4}}}
\end{aligned}$$

Similarly for the other residue

$$\begin{aligned}
\text{Residue}\left(z_2\right) &= \lim_{z\to z_2}\left(z-z_2\right)f\left(z\right)\\
&= \lim_{z\to z_2}\left(z-z_2\right)\frac{1}{z^4-1}
\end{aligned}$$

Applying L'Hopitals

$$\begin{aligned}
\text{Residue}\left(z_2\right) &= \lim_{z\to e^{i\frac{3}{4}\pi}}\frac{1}{4z^3}\\
&= \frac{1}{4\left(e^{i\frac{3}{4}\pi}\right)^3}\\
&= \frac{1}{4e^{i\frac{9\pi}{4}}}\\
&= \frac{1}{4e^{i\frac{\pi}{4}}}
\end{aligned}$$

Now that we found all the residues, then (2A) becomes

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = 2\pi i \left(\frac{1}{4e^{i\frac{3\pi}{4}}} + \frac{1}{4e^{i\frac{\pi}{4}}} \right)$$

$$= 2\pi i \left(\frac{\sqrt{2}}{4i} \right)$$

$$= \frac{1}{2}\sqrt{2}\pi$$

But $\int_0^{\infty} \frac{1}{x^4+1} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{x^4+1} dx$, therefore

$$\int_0^{\infty} \frac{1}{x^4+1} dx = \frac{1}{2} \left(\frac{1}{2}\sqrt{2}\pi \right)$$

$$= \frac{1}{4}\sqrt{2}\pi$$

$$= \frac{2}{4\sqrt{2}}\pi$$

$$= \frac{1}{2\sqrt{2}}\pi$$

5 Problem 5

Evaluate the following integral $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta} d\theta$ when $a > |b|$

solution

This is converted to complex integration by using $z = re^{i\theta} = e^{i\theta}$ since $r = 1$. Therefore $dz = ie^{i\theta} d\theta$ or

$$dz = izd\theta$$

In addition,

$$\begin{aligned}\cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ &= \frac{z + z^{-1}}{2}\end{aligned}$$

And

$$\begin{aligned}\sin^2\theta &= \frac{1}{2} - \frac{1}{2}\cos 2\theta \\ &= \frac{1}{2} - \frac{1}{2}\left(\frac{e^{i2\theta} + e^{-i2\theta}}{2}\right) \\ &= \frac{1}{2} - \frac{1}{2}\left(\frac{z^2 + z^{-2}}{2}\right) \\ &= \frac{1}{2} - \frac{1}{4}\left(z^2 + z^{-2}\right)\end{aligned}$$

Using all of the above back in the original integral gives

$$\begin{aligned}I &= \int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta \\ &= \oint_C \frac{\frac{1}{2} - \frac{1}{4}\left(z^2 + z^{-2}\right)}{a + b\left(\frac{z+z^{-1}}{2}\right)} \frac{dz}{iz}\end{aligned}$$

Where the contour C is around the unit circle in counter clockwise direction. Therefore

$$
\begin{aligned}
I &= \frac{1}{i}\oint_C \frac{\frac{1}{2}-\frac{1}{4}\left(z^2+\frac{1}{z^2}\right)}{a+\frac{b}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{z} \\
&= \frac{1}{i}\oint_C \frac{\frac{1}{2}-\frac{1}{4}\left(\frac{z^4+1}{z^2}\right)}{a+\frac{b}{2}\left(\frac{z^2+1}{z}\right)}\frac{dz}{z} \\
&= \frac{1}{i}\oint_C \frac{\frac{\frac{z^2}{2}-\frac{1}{4}\left(z^4+1\right)}{z^2}}{\frac{az+\frac{b}{2}\left(z^2+1\right)}{z}}\frac{dz}{z} \\
&= \frac{1}{i}\oint_C \frac{\frac{z^2}{2}-\frac{1}{4}\left(z^4+1\right)}{az+\frac{b}{2}\left(z^2+1\right)}\frac{dz}{z^2} \\
&= \frac{1}{i}\oint_C \frac{\frac{2z^2}{4}-\frac{1}{4}\left(z^4+1\right)}{\frac{2az}{2}+\frac{b}{2}\left(z^2+1\right)}\frac{dz}{z^2} \\
&= \frac{1}{i}\oint_C \frac{2z^2-z^4-1}{4az+2bz^2+2b}\frac{dz}{z^2} \\
&= \frac{1}{i}\oint_C \frac{1}{z^2}\frac{2z^2-z^4-1}{2bz^2+4az+2b}dz \\
&= \frac{1}{i}\oint_C \frac{1}{z^2}\frac{\frac{1}{b}z^2-\frac{1}{2b}z^4-\frac{1}{2b}}{z^2+\frac{2a}{b}z+1}dz \\
&= \frac{1}{2bi}\oint_C \frac{1}{z^2}\frac{2z^2-z^4-1}{\left(z^2+\frac{2a}{b}z+1\right)}dz
\end{aligned}
$$

Now we can use the residue theorem. There is a pole at $z=0$ of order 2 and two poles which are the roots of $z^2+\frac{2a}{b}z+1=0$. Hence

$$I = 2\pi i \sum \text{Residue}$$

First we find the roots of $z^2+\frac{2a}{b}z+1=0$ to see the location of the poles and if there are inside the unit circle or not. These are

$$
\begin{aligned}
-\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} &= -\frac{\frac{2a}{b}}{2} \pm \frac{1}{2}\sqrt{\left(\frac{2a}{b}\right)^2 - 4} \\
&= -\frac{a}{b} \pm \frac{1}{2}\sqrt{4\frac{a^2}{b^2} - 4} \\
&= -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1}
\end{aligned}
$$

Since $a > |b|$ then $\frac{a^2}{b^2} > 1$ and the value under the square root is real. Hence both roots are real. Roots are

$$
\begin{aligned}
z_1 &= -\frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1} \\
z_2 &= -\frac{a}{b} - \sqrt{\frac{a^2}{b^2} - 1}
\end{aligned}
$$

Now we need to decide the location of these poles. Let $\frac{a}{b} = x$. Where $x > 1$ since $a > |b|$. Then the roots can be written as

$$
\begin{aligned}
z_1 &= -x + \sqrt{x^2 - 1} \\
z_2 &= -x - \sqrt{x^2 - 1}
\end{aligned}
$$

Now $\sqrt{x^2 - 1}$ is always smaller than x but $\left(\sqrt{x^2 - 1} - x\right)$ can not be larger than 1 in magnitude. Hence z_1 will always be inside the unit disk. On the other hand, $\left(\sqrt{x^2 - 1} + x\right)$ will always be larger than 1 in magnitude (the sign is not important, we just wanted to know which pole is smaller or larger than 1 only. Therefore we conclude that z_1 is inside the unit disk and z_2 is outside.

Therefore, we need to find residue at $z = 0$ and $z = z_1$ and not at $z = z_2$. The function $f(z)$ is from above is

$$
\begin{aligned}
f(z) &= \frac{1}{z^2} \frac{2z^2 - z^4 - 1}{\left(z^2 + \frac{2a}{b}z + 1\right)} \\
&= \frac{1}{z^2} \frac{2z^2 - z^4 - 1}{(z - z_1)(z - z_2)}
\end{aligned}
$$

Residue of $f(z)$ at $z = 0$

Since this pole is of order $n = 2$, then

$$\begin{aligned}
\text{Residue} &= \lim_{z\to 0} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{(z-z_0)^n f(z)}{(n-1)!} \right) \\
&= \lim_{z\to 0} \frac{d}{dz} \left((z-z_0)^2 \frac{1}{z^2} \frac{2z^2 - z^4 - 1}{\left(z^2 + \frac{2a}{b} z + 1\right)} \right) \\
&= \lim_{z\to 0} \frac{d}{dz} \left(z^2 \frac{1}{z^2} \frac{2z^2 - z^4 - 1}{\left(z^2 + \frac{2a}{b} z + 1\right)} \right) \\
&= \lim_{z\to 0} \frac{d}{dz} \left(\frac{2z^2 - z^4 - 1}{z^2 + \frac{2a}{b} z + 1} \right) \\
&= \lim_{z\to 0} \frac{\left(4z - 4z^3\right)\left(z^2 + \frac{2a}{b} z + 1\right) - \left(2z^2 - z^4 - 1\right)\left(2z + \frac{2a}{b}\right)}{\left(z^2 + \frac{2a}{b} z + 1\right)^2} \\
&= -(-1)\left(\frac{2a}{b}\right) \\
&= \frac{2a}{b}
\end{aligned}$$

<u>Residue at $z_1 = -\frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1}$</u>

Since this pole is of order 1, then the reside is

$$\begin{aligned}
\text{Residue} &= \lim_{z\to z_1} \left((z - z_1) f(z) \right) \\
&= \lim_{z\to z_1} \left((z - z_1) \frac{1}{z^2} \frac{2z^2 - z^4 - 1}{(z - z_1)(z - z_2)} \right) \\
&= \lim_{z\to z_1} \left(\frac{1}{z^2} \frac{2z^2 - z^4 - 1}{z - z_2} \right) \\
&= \frac{1}{z_1^2} \frac{2z_1^2 - z_1^4 - 1}{z_1 - z_2} \\
&= \frac{-1}{z_1^2} \frac{z_1^4 - 2z_1^2 + 1}{z_1 - z_2} \\
&= \frac{-1}{z_1^2} \frac{\left(z_1^2 - 1\right)^2}{z_1 - z_2}
\end{aligned}$$

Let $\frac{a}{b} = x$, hence $\sqrt{\frac{a^2}{b^2} - 1} = \sqrt{x^2 - 1}$. Therefore we can write $z_1 = -x + \sqrt{x^2 - 1}$ and $z_2 =$

$-x-\sqrt{x^2-1}$ and now the above becomes

$$\text{Residue} = \frac{-1}{\left(-x+\sqrt{x^2-1}\right)^2}\frac{\left(\left(-x+\sqrt{x^2-1}\right)^2-1\right)^2}{\left(-x+\sqrt{x^2-1}\right)-\left(-x-\sqrt{x^2-1}\right)}$$

$$= \frac{-1}{2}\frac{\left(\left(-x+\sqrt{x^2-1}\right)^2-1\right)^2}{\left(-x+\sqrt{x^2-1}\right)^2\sqrt{x^2-1}}$$

But $\left(-x+\sqrt{x^2-1}\right)^2 = x^2+\left(x^2-1\right)-2x\sqrt{x^2-1} = 2x^2-2x\sqrt{x^2-1}-1$ and the above becomes

$$\text{Residue} = \frac{-1}{2}\frac{\left(2x^2-2x\sqrt{x^2-1}-1-1\right)^2}{\left(2x^2-2x\sqrt{x^2-1}-1\right)\sqrt{x^2-1}}$$

$$= \frac{-1}{2}\frac{\left(2x^2-2x\sqrt{x^2-1}-2\right)^2}{\left(2x^2-2x\sqrt{x^2-1}-1\right)\sqrt{x^2-1}}$$

$$= \frac{-1}{2}\frac{4\left(x^2-x\sqrt{x^2-1}-1\right)^2}{\left(2x^2-1-2x\sqrt{x^2-1}\right)\sqrt{x^2-1}}$$

$$= -2\frac{\left(\left(x^2-1\right)-x\sqrt{x^2-1}\right)^2}{\left(2x^2-1-2x\sqrt{x^2-1}\right)\sqrt{x^2-1}}$$

Expanding gives

$$\begin{aligned}
\text{Residue} &= -2\frac{\left(x^2-1\right)^2 + \left(x\sqrt{x^2-1}\right)^2 - 2\left(x^2-1\right)x\sqrt{x^2-1}}{\left(2x^2-1-2x\sqrt{x^2-1}\right)\sqrt{x^2-1}} \\
&= -2\frac{\left(x^2-1\right)^2 + x^2\left(x^2-1\right) - 2\left(x^2-1\right)x\sqrt{x^2-1}}{\left(2x^2-1-2x\sqrt{x^2-1}\right)\sqrt{x^2-1}} \\
&= -2\frac{\left(x^2-1\right)\left(x^2-1+x^2-2x\sqrt{x^2-1}\right)}{\left(2x^2-1-2x\sqrt{x^2-1}\right)\sqrt{x^2-1}} \\
&= -2\frac{\left(x^2-1\right)\left(2x^2-1-2x\sqrt{x^2-1}\right)}{\left(2x^2-1-2x\sqrt{x^2-1}\right)\sqrt{x^2-1}}
\end{aligned}$$

Dividing numerator and denominator by $\left(x^2-1\right)$

$$\begin{aligned}
\text{Residue} &= -2\frac{\sqrt{x^2-1}\left(2x^2-1-2x\sqrt{x^2-1}\right)}{\left(2x^2-1-2x\sqrt{x^2-1}\right)} \\
&= -2\sqrt{x^2-1}
\end{aligned}$$

Since $x=\frac{a}{b}$ then the above becomes

$$\text{Residue} = -2\sqrt{\frac{a^2}{b^2}-1}$$

We found all residues. The sum is

$$\sum \text{Residue} = \frac{2a}{b} - 2\sqrt{\frac{a^2}{b^2}-1}$$

From the above we see now that

$$\begin{aligned}
I &= \frac{1}{2bi}\oint\limits_C \frac{1}{z^2}\frac{2z^2 - z^4 - 1}{\left(z^2 + \frac{2a}{b}z + 1\right)}dz \\
&= \frac{1}{2bi}\left(2\pi i \sum \text{Residue}\right) \\
&= \frac{1}{2bi}\left(2\pi i\left(\frac{2a}{b} - 2\sqrt{\frac{a^2}{b^2} - 1}\right)\right) \\
&= \frac{\pi}{b}\left(\frac{2a}{b} - 2\sqrt{\frac{a^2}{b^2} - 1}\right) \\
&= \frac{\pi}{b}\left(\frac{2a}{b} - \frac{2}{b}\sqrt{a^2 - b^2}\right) \\
&= \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right)
\end{aligned}$$

Hence the final result is

$$\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}d\theta = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right)$$