University Course

MATH 4567
Applied Fourier Analysis

University of Minnesota, Twin Cities
Spring 2019

My Class Notes

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Spring 2019

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Chapter 1

Introduction

1.1 syllabus

MATH 4567, Section 002, Spring 2019, MWF 3:35-4:25, Vincent Hall 2

Instructor: Jiaping Wang; Office: Vincent Hall 230; web page: www.math.umn.edu/~jiaping

Office hours: MWF 2:30-3:20 (subject to change)

Course title and a brief description: Fourier Analysis

Fourier series and Fourier transform. Convergence. Fourier series, transform in complex form. Solution of wave, heat, Laplace equations by separation of variables. Sturm-Liouville systems. Applications.

Prerequisites: 2243 or 2373 or 2573

Text and material: Fourier Series and Boundary Value Problems, 8th edition, by Brown and Churchill, McGraw Hill Publisher. The course will cover Chapters 1-8, and selected material from Chapter 11.

Course work: The class time will be devoted to lectures where you should gain understanding of the basic concepts and methods, realize connections to other parts of mathematics you have learned (linear algebra), and eventually build a global picture of the theory of (generalized) Fourier series. You will broaden your knowledge and develop solving routines out of class: you are expected to carefully study the text and solve a number of exercises. Assigned homework is the minimum you can do for your practice.

Assignments: Homework assignments will be posted on my web page and collected in class on Wednesday. One homework (the worst grade or a homework missed for any reason) will be dropped at the end. No late homework will be accepted. You may discuss homework problems with other students, however, you are supposed to work out and write down the solutions yourself. Please write complete solutions clearly on one side of letter-size sheets. Questions or objections to grading must be brought up within a week after the graded work is returned to you.

Exams and grading policy: There will be three one-hour exams covering appropriate parts of the material. No books, notes or technology are allowed for the exams. Make-up exams are discouraged, but can only be given for legitimate reasons such as illness or university sponsored events (written documentation and, except for medical emergencies, prior approval are required).

Grading scheme: homework 25%, 3 midterm exams 75% (25% each).

Exam dates: Monday, February 25; Monday, April 1; Monday, May 6.

Incomplete will only be assigned at extraordinary circumstances (such as hospitalization), and only if a major part of the class work has been completed. Academic dishonesty in any portion of the course shall be grounds for assigning a grade of F or N for the entire course.

1.2 Links

1. Instructor web page

Chapter 2

HWs

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2.1 HW 1

Local contents

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2.1.1 Section 5, Problem 3

Problem Find (a) the Fourier cosine series and (b) the Fourier sine series on the interval $0 < x < \pi$ for $f(x) = x^2$

Solution

Part a

The function x^2 over $0 < x < \pi$ is

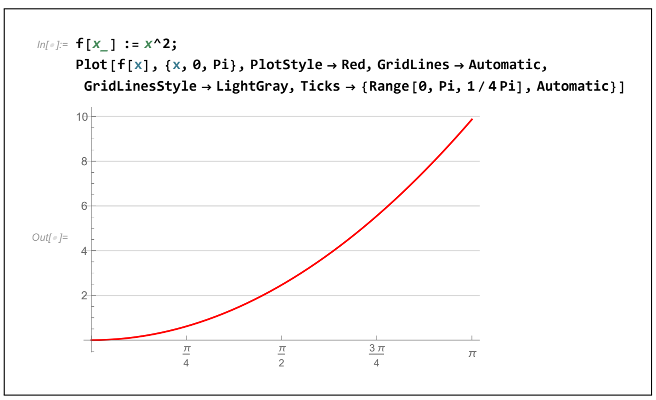

Figure 2.1: Original function

The first step is to do an even extension of x^2 from $0 < x < \pi$ to $-\pi < x < \pi$ which means its period becomes $T = 2\pi$. The even extension of $f(x)$ is given by

$$f_e(x) = \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}$$

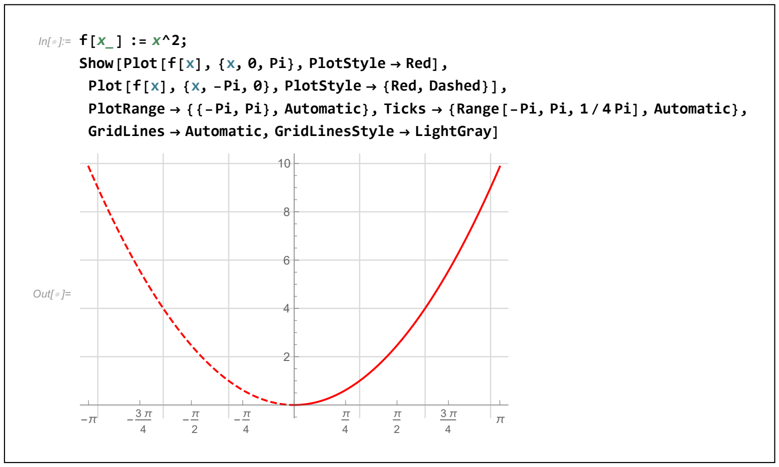

Figure 2.2: Even extension of original function

The next step is to make the above function periodic with period $T = 2\pi$ by repeating it each 2π as shown below

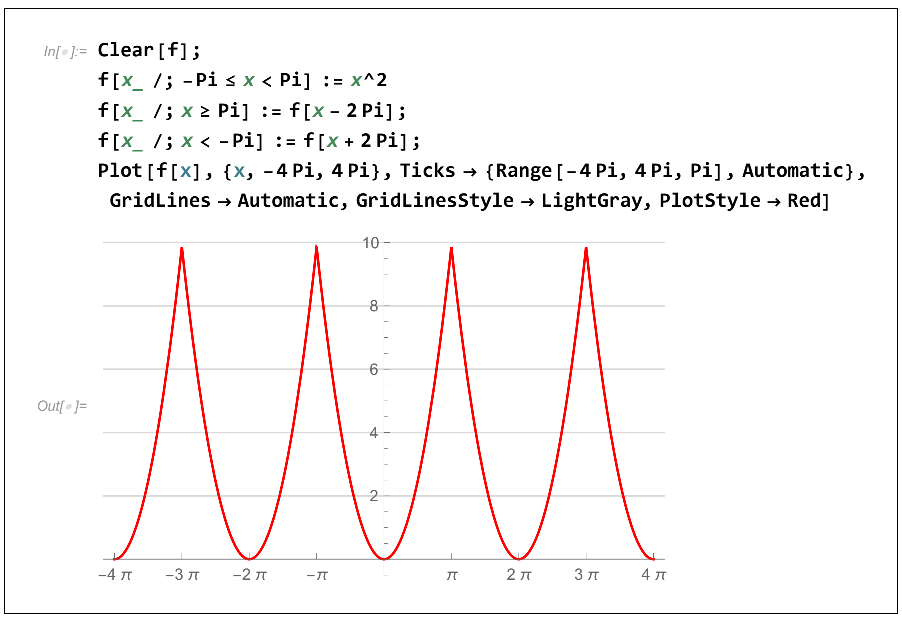

Figure 2.3: Even extension of original function

Now that we have a periodic function above with period $T = 2\pi$ then we can find its Fourier cosine series. Which is just the cosine series part of its Fourier series given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T} n x\right)$$

Since $T = 2\pi$, the above becomes

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \tag{1}$$

Where

$$\begin{aligned} a_0 &= \frac{1}{\left(\frac{T}{2}\right)} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx \\ &= \frac{2}{2\pi} \int_{-\frac{2\pi}{2}}^{\frac{2\pi}{2}} f(x)\,dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx \end{aligned}$$

Because $f(x)$ is an even function (we did an even extension to force this), then the above can be written as

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2}{\pi}\left(\frac{\pi^3}{3}\right) = \frac{2}{3}\pi^2 \tag{2}$$

And for $n > 0$ then

$$a_n = \frac{1}{\left(\frac{T}{2}\right)} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \cos\left(\frac{2\pi}{T} nx\right) dx$$

But $T = 2\pi$ and the above becomes

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx$$

But $f(x)$ is even functiuon and $\cos$ is even, hence the product is even and the above simplifies to

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx)\,dx$$

Integration by parts. $udv = uv - \int vdu$. Let $u = x^2, dv = \cos nx$, therefore $du = 2x, v = \frac{\sin nx}{n}$. The above becomes

$$\begin{aligned} a_n &= \frac{2}{\pi} \left([uv] - \int vdu \right) \\ &= \frac{2}{\pi} \left(\left[x^2 \frac{\sin nx}{n} \right]_0^{\pi} - \int_0^{\pi} 2x \frac{\sin nx}{n} dx \right) \end{aligned}$$

Since n is integer, the term $\left[x^2 \frac{\sin nx}{n}\right]_0^{\pi} \to 0$ and the above simplifies to

$$\begin{aligned} a_n &= \frac{2}{\pi} \left(-\frac{2}{n} \int_0^{\pi} x \sin nx dx \right) \\ &= \frac{-4}{n\pi} \int_0^{\pi} x \sin nx dx \end{aligned}$$

The integral $\int_0^{\pi} x \sin nx dx$ is evaluated by parts again. Let $u = x, dv = \sin nx \to du = 1, v =$

$-\frac{\cos nx}{n}$ and the above becomes

$$\begin{aligned}
a_n &= \frac{-4}{n\pi}\left([uv] - \int vdu\right) \\
&= \frac{-4}{n\pi}\left(-\left[x\frac{\cos nx}{n}\right]_0^\pi + \frac{1}{n}\int_0^\pi \cos nx dx\right) \\
&= \frac{-4}{n\pi}\left(-\frac{1}{n}\pi\cos(n\pi) + \frac{1}{n^2}\overbrace{[\sin nx]_0^\pi}^{0}\right) \\
&= \frac{4}{n^2}\cos(n\pi) \\
&= \frac{4}{n^2}(-1)^n \qquad (3)
\end{aligned}$$

Substituting (2,3) into (1) gives

$$\begin{aligned}
f(x) &\sim \frac{\frac{2}{3}\pi^2}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos(nx) \\
&= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)
\end{aligned}$$

The convergence is fast due to the term $\frac{1}{n^2}$. This plot show the approximation as the number of terms increases. After only 4 terms we see the approximation is very close to original function x^2 shown in dashed lines in the plot below.

```
In[•]:= fApprox[x_, nTerms_] := π^2/3 + 4 Sum[(-1)^n/n^2 Cos[n x], {n, 1, nTerms}];
        Grid@
         Partition[
          Table[Plot[{x^2, fApprox[x, n]}, {x, -Pi, Pi}, PlotStyle → {Dashed, Red},
            PlotLabel → Row[{"Using ", n, " terms"}]], {n, 1, 4}], 2]
```

Out[•]=

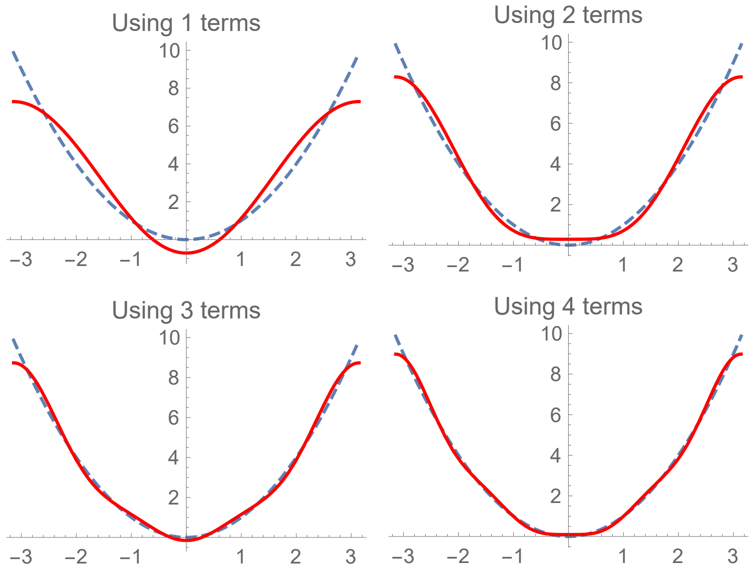

Figure 2.4: Fourier approximation as more terms are added

Part b

Because we want to find the Fourier sine series now, then the first step is to do an odd extension of x^2 from $0 < x < \pi$ to $-\pi < x < \pi$ which means its period is $T = 2\pi$. Odd extension of $f(x)$ is given by

$$f_o(x) = \begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \end{cases}$$

```
In[]:= f[x_] := x^2;
Show[Plot[f[x], {x, 0, Pi}, PlotStyle → Red],
 Plot[-f[-x], {x, -Pi, 0}, PlotStyle → {Red, Dashed}],
 PlotRange → {{-Pi, Pi}, {-10, 10}}, Ticks → {Range[-Pi, Pi, 1/4 Pi], Automatic},
 GridLines → Automatic, GridLinesStyle → LightGray]
```

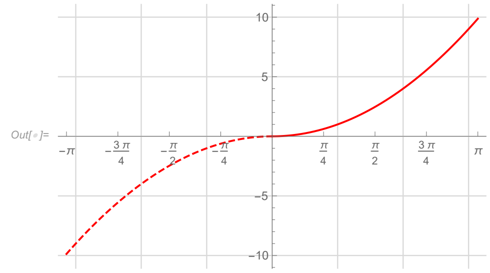

Figure 2.5: Odd extension of x^2

The next step is to make the function function periodic with period $T = 2\pi$ by repeating it each 2π as follows

```
In[]:= Clear[f];
f[x_ /; -Pi < x < Pi] := If[x < 0, -x^2, x^2];
f[x_ /; x > Pi] := f[x - 2 Pi];
f[x_ /; x < -Pi] := f[x + 2 Pi];
Plot[f[x], {x, -4 Pi, 4 Pi}, Ticks → {Range[-4 Pi, 4 Pi, Pi], Automatic},
 PlotStyle → Red, Exclusions → {x == -3 Pi, x == -Pi, x == Pi, x == 3 Pi}, Mesh → None,
 GridLines → Automatic, GridLinesStyle → LightGray]
```

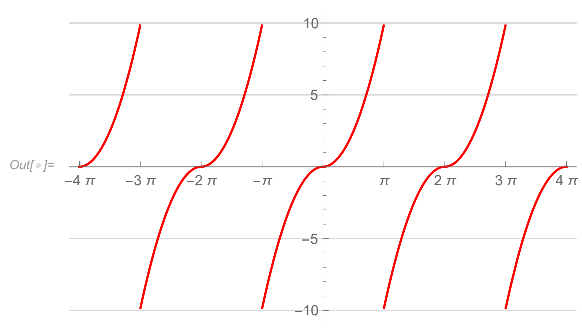

Figure 2.6: Making the odd extension periodic

Now that we have a periodic function with period $T = 2\pi$ we can find its Fourier sine series, which is just the sin part of its Fourier series, given by

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi}{T} nx\right)$$

But $T = 2\pi$, and the above becomes

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx) \tag{1}$$

Where

$$b_n = \frac{1}{\left(\frac{T}{2}\right)} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \sin\left(\frac{2\pi}{T} nx\right) dx$$

But $T = 2\pi$, and the above becomes

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

But now $f(x)$ is odd function (we did an odd extension) and $\sin$ is odd. Hence product is even. Therefore the above simplifies to

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx \\ &= \frac{2}{\pi} \int_0^{\pi} x^2 \sin(nx)\, dx \end{aligned}$$

Integration by parts. $udv = uv - \int vdu$. Let $u = x^2, dv = \sin nx$, therefore $du = 2x, v = \frac{-\cos nx}{n}$. The above becomes

$$\begin{aligned} b_n &= \frac{2}{\pi}\left([uv] - \int vdu\right) \\ &= \frac{2}{\pi}\left(-\left[x^2 \frac{\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi} 2x \frac{\cos nx}{n} dx\right) \\ &= \frac{2}{\pi}\left(-\frac{1}{n}\left[\pi^2 \cos n\pi\right] + \frac{2}{n} \int_0^{\pi} x \cos nx dx\right) \\ &= -\frac{2\pi}{n} \cos n\pi + \frac{4}{n\pi} \int_0^{\pi} x \cos nx dx \end{aligned}$$

The integral $\int_0^{\pi} x \cos nx dx$ is evaluated by parts again. Let $u = x, dv = \cos nx \rightarrow du = 1, v =$

$\frac{\sin nx}{n}$ and the above becomes

$$
\begin{aligned}
b_n &= -\frac{2\pi}{n}\cos n\pi + \frac{4}{n\pi}\left([uv] - \int v du\right)\\
&= -\frac{2\pi}{n}\cos n\pi + \frac{4}{n\pi}\left(\overbrace{\left[x\frac{\sin nx}{n}\right]_0^{\pi}}^{0} - \int \frac{\sin nx}{n} dx\right)\\
&= -\frac{2\pi}{n}\cos n\pi - \frac{4}{n^2\pi}\int \sin nx dx\\
&= -\frac{2\pi}{n}\cos n\pi - \frac{4}{n^2\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}\\
&= -\frac{2\pi}{n}\cos n\pi + \frac{4}{n^3\pi}\left[\cos nx\right]_0^{\pi}\\
&= -\frac{2\pi}{n}\cos n\pi + \frac{4}{n^3\pi}\left[\cos n\pi - 1\right]\\
&= -\frac{2\pi}{n}(-1)^n + \frac{4}{n^3\pi}\left((-1)^n - 1\right)\\
&= -\frac{2\pi}{n}(-1)^n - \frac{4}{n^3\pi}\left(1 - (-1)^n\right)\\
&= \frac{2\pi}{n}(-1)^{n+1} - \frac{4}{n^3\pi}\left(1 - (-1)^n\right) \qquad (2)
\end{aligned}
$$

Substituting (2) into (1) gives

$$
\begin{aligned}
f(x) &\sim \sum_{n=1}^{\infty}\left(\frac{2\pi}{n}(-1)^{n+1} - \frac{4}{n^3\pi}\left(1 - (-1)^n\right)\right)\sin(nx)\\
&= 2\pi^2\sum_{n=1}^{\infty}\left(\frac{1}{n\pi}(-1)^{n+1} - \frac{2}{(n\pi)^3}\left(1 - (-1)^n\right)\right)\sin(nx)
\end{aligned}
$$

In this case, we needed more terms to obtain good convergence. Because the periodic extension is now discontinuous at $x = n\pi$ where n is odd. In part (a), the periodic extension was continuous over the whole domain. The following plot shows we needed more terms compared to part (a) to start seeing good convergence. This shows the result for one period from $-\pi$ to π. The blue color is for the original odd extended function and the red color is its Fourier seriesapproximation.

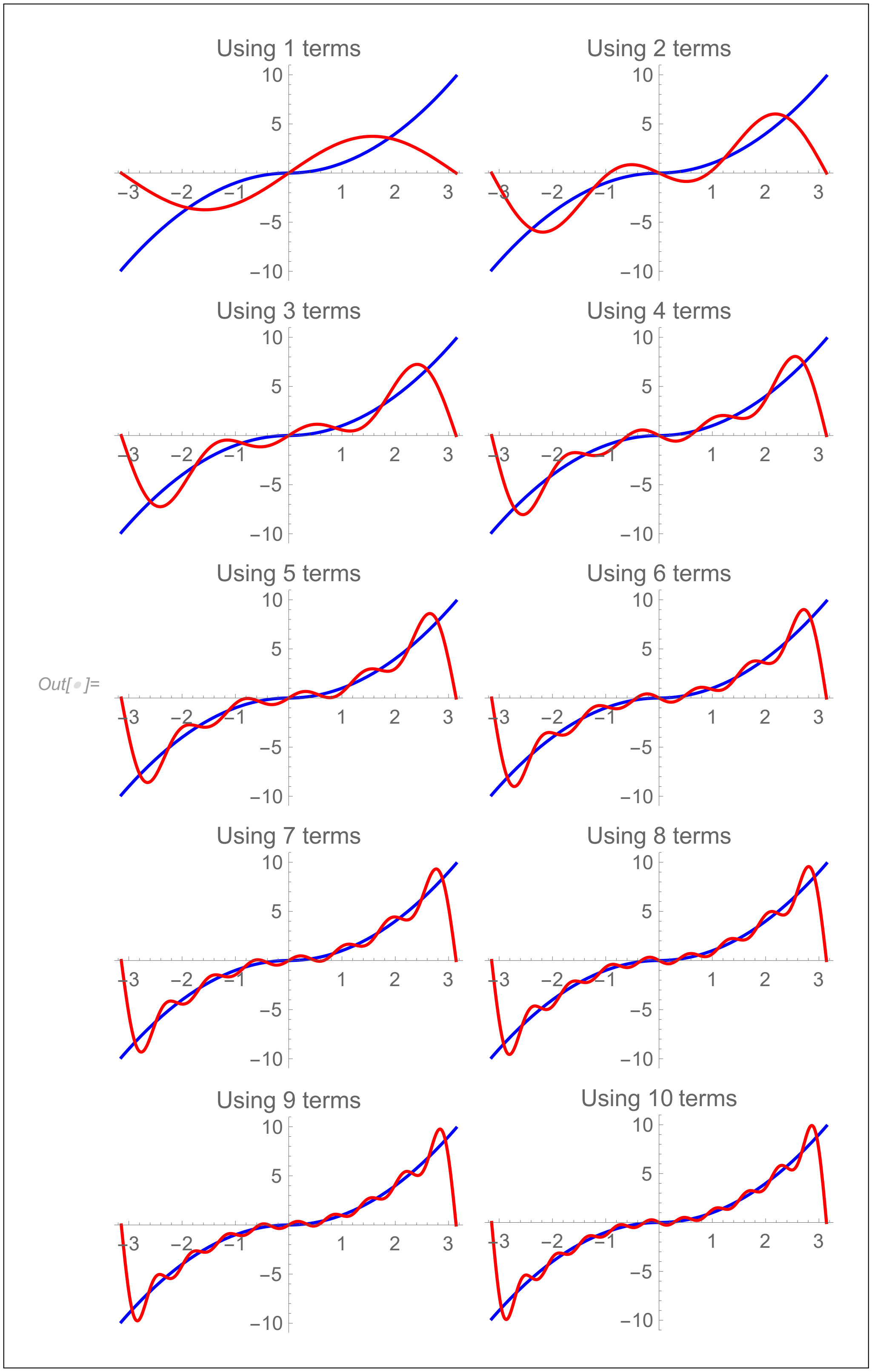

Figure 2.7: Fourier approximation of odd extension of x^2 over one period

```
In[•]:= fApprox[x_, nTerms_] :=
          2 π^2 Sum[(1/(n π) (-1)^(n+1) - 2/(n π)^3 (1 - (-1)^n)) Sin[n x], {n, 1, nTerms}];
        f[x_] := If[x < 0, -x^2, x^2];
        Grid@
         Partition[
          Table[Plot[{f[x], fApprox[x, n]}, {x, -Pi, Pi}, PlotStyle → {Blue, Red},
            PlotLabel → Row[{"Using ", n, " terms"}]], {n, 1, 10}], 2]
```

Figure 2.8: Code used to draw Fourier approximation for odd extension for one period

Due to discontinuous in the periodic extended function, there will be a Gibbs effect at the points of discontinuities $x = n\pi$ where n is odd, where the approximation converges to the average of the function at those point. To see this, here is a plot showing the result for the case of 16 terms over 3 periods instead of one period as the above plot showed.

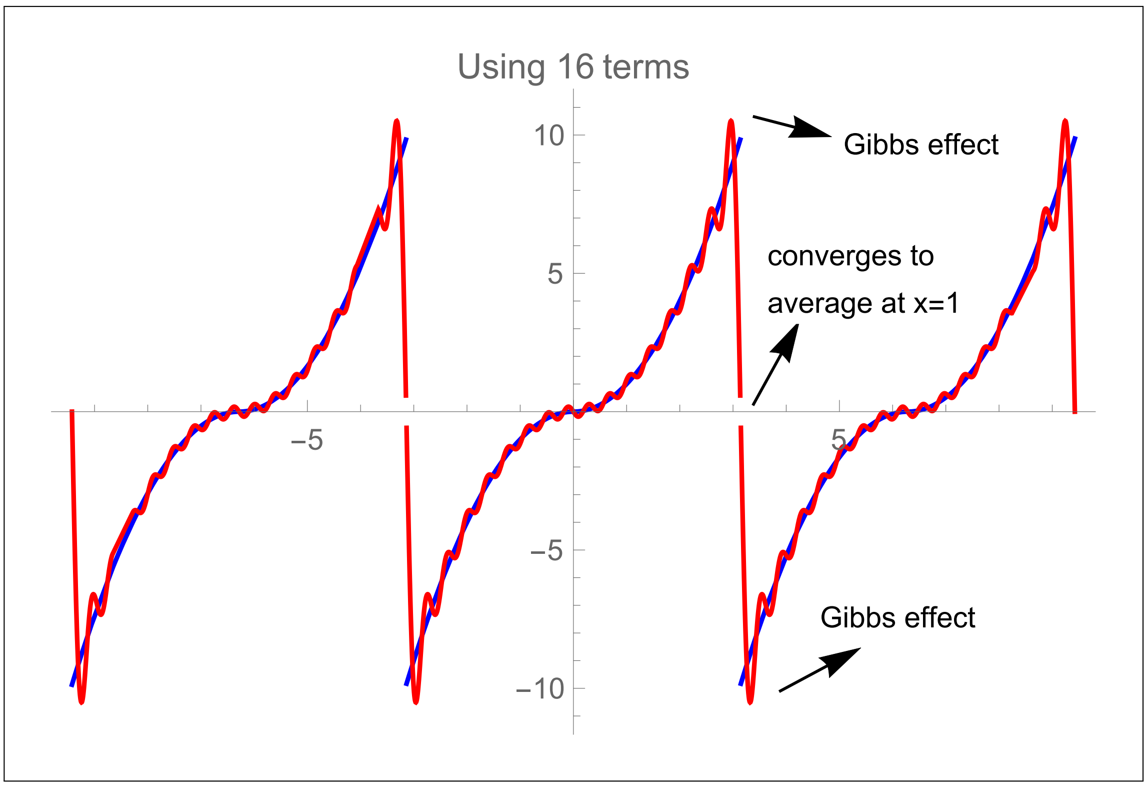

Figure 2.9: Fourier approximation of odd extension of x^2 over 3 periods to see Gibbs effect

```
In[ ]:= fApprox[x_, nTerms_] := 2 π^2 Sum[(1/(n π) (-1)^(n+1) - 2/(n π)^3 (1 - (-1)^n)) Sin[n x], {n, 1, nTerms}];
        Clear[f];
        f[x_ /; -Pi < x < Pi] := If[x < 0, -x^2, x^2];
        f[x_ /; x > Pi] := f[x - 2 Pi];
        f[x_ /; x < -Pi] := f[x + 2 Pi];
        Plot[{f[x], fApprox[x, 16]}, {x, -3 Pi, 3 Pi}, PlotStyle → {Blue, Red},
         PlotLabel → Row[{"Using ", 16, " terms"}], Exclusions → {x == -3 Pi, x == -Pi, x == Pi, x == 3 Pi}]
```

Figure 2.10: Code used to draw the above plot

2.1.2 Section 5, Problem 5

Problem By referring to the sine series for x in example 1 and one found for x^2 in above problem show that

$$x(\pi - x) \sim \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^3} \qquad 0 < x < \pi$$

Solution

From example 1, the Fourier sine series for x defined on $0 < x < \pi$, was found to be

$$x \sim 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin x \qquad 0 < x < \pi$$

By writing $x(\pi - x) = \pi x - x^2$ then we see that

$$\begin{aligned}
\pi x - x^2 &\sim \pi\left(2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin x\right) - \left(2\pi^2\sum_{n=1}^{\infty}\left(\frac{1}{n\pi}(-1)^{n+1} - \frac{2}{(n\pi)^3}\left(1-(-1)^n\right)\right)\sin(nx)\right)\\
&= \sum_{n=1}^{\infty}2\pi\frac{(-1)^{n+1}}{n}\sin x - \sum_{n=1}^{\infty}2\pi^2\left(\frac{1}{n\pi}(-1)^{n+1} - \frac{2}{(n\pi)^3}\left(1-(-1)^n\right)\right)\sin(nx)\\
&= \sum_{n=1}^{\infty}\left[2\pi\frac{(-1)^{n+1}}{n} - 2\pi^2\left(\frac{1}{n\pi}(-1)^{n+1} - \frac{2}{(n\pi)^3}\left(1-(-1)^n\right)\right)\right]\sin(nx)\\
&= \sum_{n=1}^{\infty}\left[2\pi\frac{(-1)^{n+1}}{n} - \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{n^3\pi}\left(1-(-1)^n\right)\right]\sin(nx)\\
&= \sum_{n=1}^{\infty}\frac{4}{n^3\pi}\left(1-(-1)^n\right)\sin(nx)
\end{aligned}$$

Now when $n = 2,4,6,\cdots$ then $\left(1-(-1)^n\right) = 0$ and when $n = 1,3,5,\cdots$ then $\left(1-(-1)^n\right) = 2$. Hence the above sum becomes

$$\begin{aligned}
\pi x - x^2 &\sim \sum_{n=1,3,5,\cdots}^{\infty}\frac{8}{n^3\pi}\sin(nx)\\
&\sim \frac{8}{\pi}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^3}\sin(nx)
\end{aligned}$$

Let $n = 2m-1$. Then when $n=1 \rightarrow m=1$, $n=3 \rightarrow m=2$, $n=5 \rightarrow m=3$ and so on. Hence the above sum can be written using m as summation index as follows

$$\pi x - x^2 \sim \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\left((2m-1)x\right)$$

Since summation index can be named anything, then renaming summation index from m back to n gives the form required

$$\pi x - x^2 \sim \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\left((2n-1)x\right)$$

2.1.3 Section 7, Problem 1

Problem Find the Fourier series on interval $-\pi < x < \pi$ that corresponds to

$$f(x) = \begin{cases} -\frac{\pi}{2} & -\pi < x < 0 \\ \frac{\pi}{2} & 0 < x < \pi \end{cases}$$

Solution

A plot of the function $f(x)$ over $-\pi < x < \pi$ is

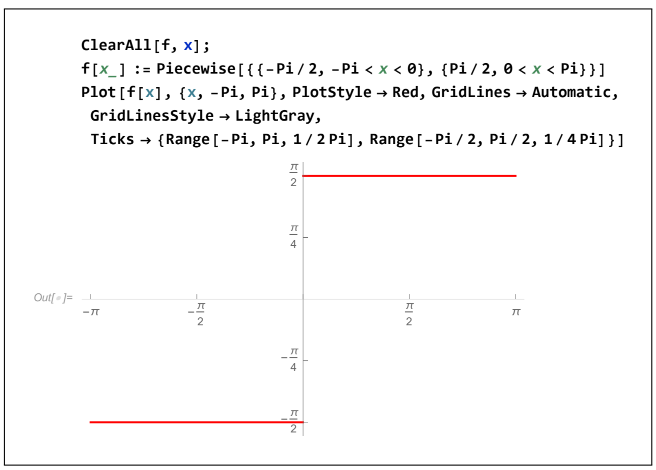

Figure 2.11: Plot of $f(x)$ for problem section 7.1

The periodic extension (with period $T = 2\pi$) becomes (shown for $-3\pi < x < 3\pi$)

```
Clear[f];
f[x_ /; -Pi < x < Pi] := If[x < 0, -Pi / 2, Pi / 2];
f[x_ /; x > Pi] := f[x - 2 Pi];
f[x_ /; x < -Pi] := f[x + 2 Pi];
Plot[f[x], {x, -3 Pi, 3 Pi}, Ticks → {Range[-4 Pi, 4 Pi, Pi], Automatic},
 PlotStyle → Red,
 Exclusions → {x == -Pi, x == -2 Pi, x == -3 Pi, x == 0, x == Pi, x == 2 Pi, x == 3 Pi},
 ExclusionsStyle → Dashed, Mesh → None, GridLines → Automatic,
 GridLinesStyle → LightGray]
```

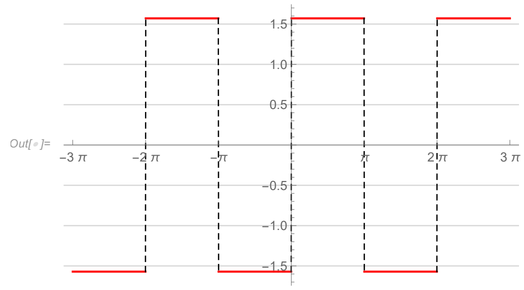

Figure 2.12: Plot of $f(x)$ for problem section 7.1 after periodic extension

Since the function $f(x)$ is now periodic then its Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi}{T}x\right) + b_n \sin\left(\frac{2n\pi}{T}x\right)$$

Where T is the period of the function being approximated which is $T = 2\pi$ in this case. Hence the above simplifies to

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Since the function $f(x)$ is an odd function then only b_n terms exist and the above reduces to

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx) \tag{1}$$

Where

$$\begin{aligned} b_n &= \frac{1}{\left(\frac{T}{2}\right)} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \sin\left(\frac{2n\pi}{T}x\right) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \end{aligned}$$

Since $f(x)$ is odd and $\sin$ is odd, then the product is even, and the above simplifies to the

Fourier sine series

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \\
&= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}\right)\sin(nx)\,dx \\
&= \int_0^{\pi}\sin(nx)\,dx \\
&= \left[\frac{-\cos nx}{n}\right]_0^{\pi} \\
&= -\frac{1}{n}\left[\cos n\pi - 1\right] \\
&= \frac{1}{n}\left[1+(-1)^{n+1}\right]
\end{aligned}$$

Therefore (1) becomes

$$f(x)\sim\sum_{n=1}^{\infty}\left(\frac{1}{n}\left(1+(-1)^{n+1}\right)\right)\sin(nx)$$

When $n=2,4,6,\cdots$ then $b_n=0$ and when $n=1,3,5,\cdots$ then $b_n=\frac{2}{n}$. Therefore the above can be written as

$$f(x)\sim\sum_{n=1,3,5,\cdots}^{\infty}\frac{2}{n}\sin(nx)$$

Let $n=2m-1$. Then when $n=1\to m=1$, $n=3\to m=2$, $n=5\to m=3$ and so on. Hence the above sum can be written using m as summation index as follows

$$f(x)\sim\sum_{m=1}^{\infty}\frac{2}{2m-1}\sin((2m-1)x)$$

Since summation index can be named anything, then renaming summation index from m to n gives

$$f(x)\sim\sum_{n=1}^{\infty}\frac{2}{2n-1}\sin((2n-1)x)$$

Since the periodic extension of the original function $f(x)$ is discontinuous at points $x=n\pi$, then the Fourier approximation will converge to the average of $f(x)$ at these points and Gibbs effect will result at these points as well. The following plot shows the result

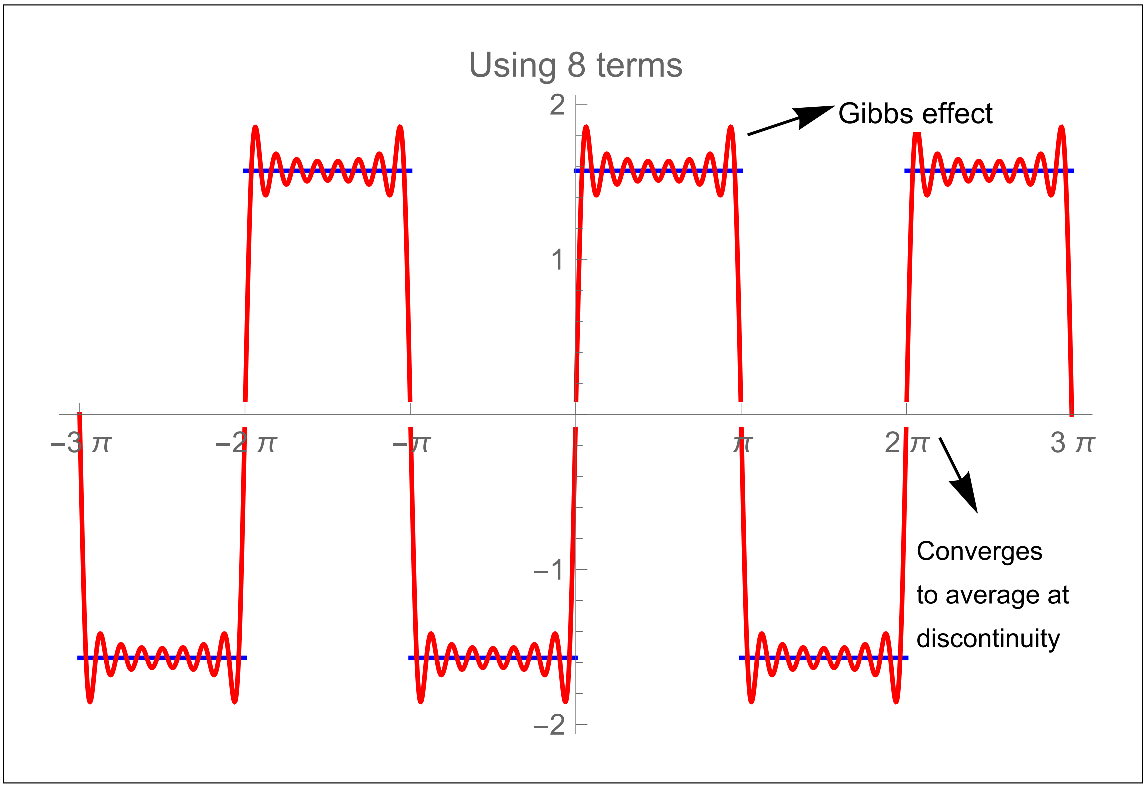

Figure 2.13: Fourier approximations using 8 terms

```
In[∘]:= fApprox[x_, nTerms_] := Sum[2/(2 n - 1) Sin[(2 n - 1) x], {n, 1, nTerms}];
        Clear[f];
        f[x_ /; -Pi < x < Pi] := If[x < 0, -Pi / 2, Pi / 2];
        f[x_ /; x > Pi] := f[x - 2 Pi];
        f[x_ /; x < -Pi] := f[x + 2 Pi];
        Plot[{f[x], fApprox[x, 8]}, {x, -3 Pi, 3 Pi}, PlotStyle → {Blue, Red},
         PlotLabel → Row[{"Using ", 8, " terms"}],
         Exclusions → {x == -Pi, x == -2 Pi, x == -3 Pi, x == 0, x == Pi, x == 2 Pi, x == 3 Pi},
         Ticks → {Range[-4 Pi, 4 Pi, Pi], Automatic}]
```

Figure 2.14: Code used to generate the above plot

2.1.4 Chapter 1, Section 7, Problem 3

Problem Find the Fourier series on interval $-\pi < x < \pi$ that corresponds to $f(x) = x + \frac{1}{4}x^2$. suggestions: Use the series for x in example 2, section 7 and the one for x^2 found above in problem Section 5, Problem 3(a).

Solution

Since x is odd, then we can from example 2 use the Fourier sine series for x defined on $-\pi < x < \pi$

$$x \sim 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx) \qquad (-\pi < x < \pi) \tag{1}$$

And since x^2 is even, then we can use the Fourier cosine series found in problem Section 5, Problem 3(a) solved above

$$x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^2}\cos(nx) \qquad (-\pi < x < \pi) \tag{2}$$

Using (1,2), then we can write $x + \frac{1}{4}x^2$ Fourier series as

$$\begin{aligned} x + \frac{1}{4}x^2 &\sim \left(2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\right) + \frac{1}{4}\left(\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^2}\cos(nx)\right) \\ &\sim \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^2}\cos(nx) + \frac{2(-1)^{n+1}}{n}\sin nx \\ &\sim \frac{\pi^2}{12} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{\cos(nx)}{n^2} - \frac{2\sin(nx)}{n}\right) \end{aligned}$$

2.1.5 Section 7, Problem 4

Problem Find the Fourier series on interval $-\pi < x < \pi$ that corresponds to $f(x) = e^{ax}$ where $a \neq 0$. suggestion: Use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ to write $a_n + ib_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,e^{inx}dx$ for $n = 1, 2, 3, \cdots$. Then after evaluating this single integral, equate real and imaginary parts.

Solution

$$e^{ax} \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{2\pi}{T}nx\right) + b_n\sin\left(\frac{2\pi}{T}nx\right)$$

But $T = 2\pi$ and the above becomes

$$e^{ax} \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

Where

$$\begin{aligned} a_0 &= \frac{1}{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}dx \\ &= \frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi a}\left(e^{a\pi} - e^{-a\pi}\right) \end{aligned}$$

But $\frac{e^{a\pi}-e^{-a\pi}}{2} = \sinh(a\pi)$ hence the above simplifies to

$$a_0 = \frac{2}{\pi a}\sinh(a\pi)$$

And for $n > 0$

$$\begin{aligned} a_n &= \frac{1}{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos\left(\frac{2\pi}{T}nx\right)dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos(nx)\,dx \qquad (1) \end{aligned}$$

Let $I = \int_{-\pi}^{\pi} e^{ax}\cos(nx)\,dx$. Using integration by parts, $\int u dv = uv - \int v du$. Let $u = \cos nx, dv = e^{ax}$ then $v = \frac{e^{ax}}{a}, du = -n\sin(nx)$. Hence

$$\begin{aligned} I &= uv - \int v du \\ &= \left[\cos(nx)\frac{e^{ax}}{a}\right]_{-\pi}^{\pi} + \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx \\ &= \left[\cos(n\pi)\frac{e^{a\pi}}{a} - \cos(n\pi)\frac{e^{-a\pi}}{a}\right] + \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx \\ &= (-1)^n\left[\frac{e^{a\pi}-e^{-a\pi}}{a}\right] + \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx \\ &= \frac{2(-1)^n}{a}\left[\frac{e^{a\pi}-e^{-a\pi}}{2}\right] + \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx \\ &= \frac{2(-1)^n}{a}\sinh(a\pi) + \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx \end{aligned}$$

Applying integration by parts again on the integral above. Let $u = \sin nx, dv = e^{ax}$ then $v = \frac{e^{ax}}{a}, du = n\cos(nx)$ and the above becomes

$$\begin{aligned} I &= \frac{2(-1)^n}{a}\sinh(a\pi) + \frac{n}{a}\left(\left(\sin nx\frac{e^{ax}}{a}\right)_{-\pi}^{\pi} - \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}\cos(nx)\,dx\right) \\ &= \frac{2(-1)^n}{a}\sinh(a\pi) + \frac{n}{a}\left(\overbrace{\frac{1}{a}\left(\sin(n\pi)e^{a\pi} + \sin(n\pi)e^{-a\pi}\right)}^{0} - \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}\cos(nx)\,dx\right) \\ &= \frac{2(-1)^n}{a}\sinh(a\pi) - \frac{n^2}{a^2}\int_{-\pi}^{\pi} e^{ax}\cos(nx)\,dx \end{aligned}$$

But $\int_{-\pi}^{\pi} e^{ax}\cos(nx)\,dx = I$, the original integral we are solving for. Hence solving for I from the above gives gives

$$
\begin{aligned}
I &= \frac{2(-1)^n}{a}\sinh(a\pi) - \frac{n^2}{a^2}I \\
I + \frac{n^2}{a^2}I &= \frac{2(-1)^n}{a}\sinh(a\pi) \\
I\left(1+\frac{n^2}{a^2}\right) &= \frac{2(-1)^n}{a}\sinh(a\pi) \\
I &= \frac{\frac{2(-1)^n}{a}\sinh(a\pi)}{1+\frac{n^2}{a^2}} \\
&= \frac{2a(-1)^n\sinh(a\pi)}{a^2+n^2} \qquad (2)
\end{aligned}
$$

Using (2) in (1) gives

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos(nx)\,dx \\
&= \frac{a}{\pi}\frac{2(-1)^n\sinh(a\pi)}{a^2+n^2} \qquad (3)
\end{aligned}
$$

Now we will do the same to find b_n

$$
\begin{aligned}
b_n &= \frac{1}{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin\left(\frac{2\pi}{T}nx\right)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx \qquad (4)
\end{aligned}
$$

Let $I = \int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx$. Using integration by parts, $\int udv = uv - \int vdu$. Let $u = \sin(nx), dv = e^{ax}$ then $v = \frac{e^{ax}}{a}, du = n\cos(nx)$. Hence

$$
\begin{aligned}
I &= uv - \int vdu \\
&= \left[\sin(nx)\frac{e^{ax}}{a}\right]_{-\pi}^{\pi} - \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}\cos(nx)\,dx \\
&= \overbrace{\left[\sin(n\pi)\frac{e^{a\pi}}{a} - \sin(n\pi)\frac{e^{-a\pi}}{a}\right]}^{0} - \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}\cos(nx)\,dx \\
&= -\frac{n}{a}\int_{-\pi}^{\pi} e^{ax}\cos(nx)\,dx
\end{aligned}
$$

Now we apply integration by parts again on the integral above. Let $u = \cos nx, dv = e^{ax}$ then

$v=\frac{e^{ax}}{a}, du=-n\sin(nx)$ and the above becomes

$$\begin{aligned}
I &= -\frac{n}{a}\left(\left(\cos(nx)\frac{e^{ax}}{a}\right)_{-\pi}^{\pi}+\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx\right)\\
&= -\frac{n}{a}\left(\frac{1}{a}\left(\cos(n\pi)e^{a\pi}-\cos(n\pi)e^{-a\pi}\right)+\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx\right)\\
&= -\frac{n}{a}\left(\frac{1}{a}\cos(n\pi)\left(e^{a\pi}-e^{-a\pi}\right)+\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx\right)\\
&= -\frac{n}{a}\left(\frac{2}{a}\cos(n\pi)\left(\frac{e^{a\pi}-e^{-a\pi}}{2}\right)+\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx\right)\\
&= -\frac{n}{a}\left(\frac{2}{a}\cos(n\pi)\sinh(a\pi)+\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx\right)\\
&= -\frac{2n}{a^2}(-1)^n\sinh(a\pi)-\frac{n^2}{a^2}\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx
\end{aligned}$$

But $\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx=I$. Hence solving for I gives

$$\begin{aligned}
I &= -\frac{2n}{a^2}(-1)^n\sinh(a\pi)-\frac{n^2}{a^2}I\\
I+\frac{n^2}{a^2}I &= -\frac{2n}{a^2}(-1)^n\sinh(a\pi)\\
I\left(1+\frac{n^2}{a^2}\right) &= -\frac{2n}{a^2}(-1)^n\sinh(a\pi)\\
I &= -\frac{\frac{2n}{a^2}(-1)^n\sinh(a\pi)}{1+\frac{n^2}{a^2}}\\
I &= -\frac{2n(-1)^n}{a^2+n^2}\sinh(a\pi) \qquad (5)
\end{aligned}$$

Using (5) in (4) gives

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx\\
&= -\frac{1}{\pi}\frac{2n(-1)^n}{a^2+n^2}\sinh(a\pi)
\end{aligned}$$

Now that we found a_0, a_n, b_n then the Fourier series is

$$
\begin{aligned}
e^{ax} &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \\
&\sim \frac{\frac{2}{\pi a}\sinh(a\pi)}{2} + \sum_{n=1}^{\infty} \frac{a}{\pi}\frac{2(-1)^n \sinh(a\pi)}{a^2+n^2}\cos(nx) - \frac{1}{\pi}\frac{2n(-1)^n}{a^2+n^2}\sinh(a\pi)\sin(nx) \\
&\sim \frac{\sinh(a\pi)}{\pi a} + \frac{1}{\pi}\sinh(a\pi)\sum_{n=1}^{\infty}\frac{2(-1)^n}{a^2+n^2}(a\cos(nx) - n\sin(nx)) \\
&\sim \sinh(a\pi)\left(\frac{1}{\pi a} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{2(-1)^n}{a^2+n^2}(a\cos(nx) - n\sin(nx))\right) \\
&\sim \frac{2\sinh(a\pi)}{\pi}\left(\frac{1}{2a} + \sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}(a\cos(nx) - n\sin(nx))\right)
\end{aligned}
$$

Which is what we are required to show.

The following plots shows the approximation as more terms are added. We also notice the Gibbs effect at the points of discontinuities after the original function was periodic extended. The value $a = 1$ was used. Hence this is approximation of e^x using $-\pi < x < \pi$ as original period.

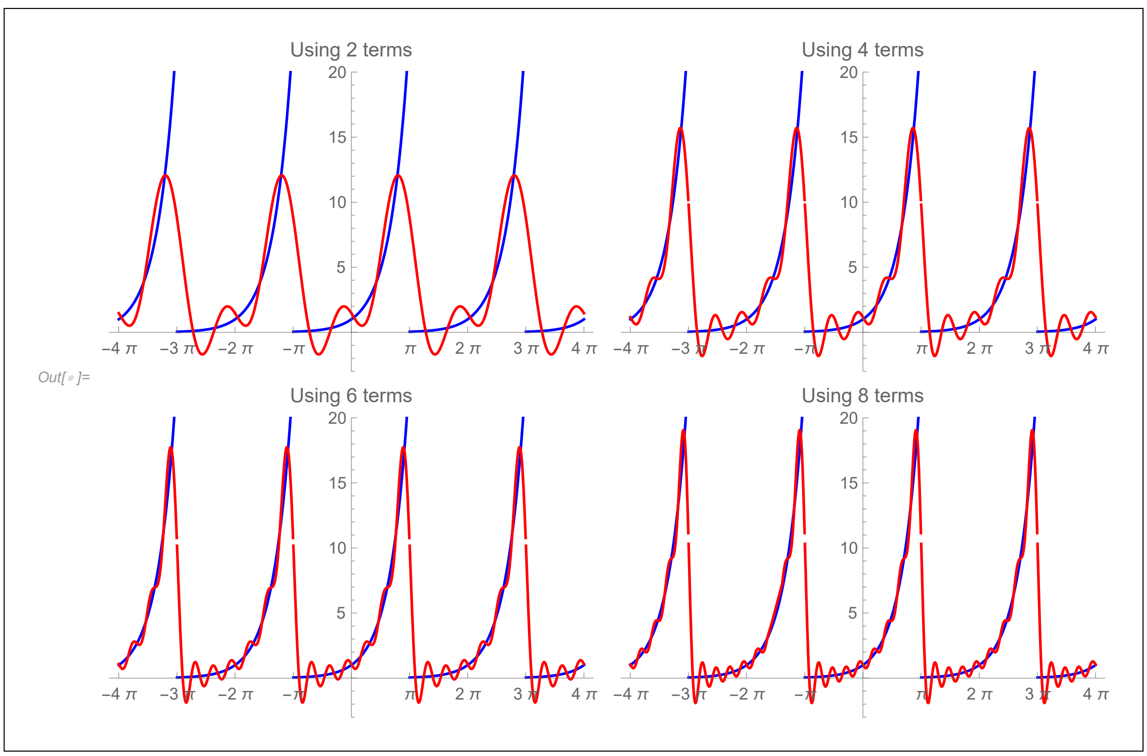

Figure 2.15: Fourier approximations using with increasing terms

```
In[•]:= a = 1;
fApprox[x_, nTerms_] := (2 Sinh[a Pi])/π (1/(2 a) + Sum[(-1)^n/(a^2 + n^2) (a Cos[n x] - n Sin[n x]), {n, 1, nTerms}]);
Clear[f];
f[x_ /; -Pi < x < Pi] := Exp[a x];
f[x_ /; x > Pi] := f[x - 2 Pi];
f[x_ /; x < -Pi] := f[x + 2 Pi];
Grid@Partition[Table[
   Plot[{f[x], fApprox[x, nTerms]}, {x, -4 Pi, 4 Pi}, PlotStyle → {Blue, Red},
    PlotLabel → Row[{"Using ", nTerms, " terms"}],
    Exclusions → {x == -Pi, x == -2 Pi, x == -3 Pi, x == 0, x == Pi, x == 2 Pi, x == 3 Pi},
    Ticks → {Range[-4 Pi, 4 Pi, Pi], Automatic},
    PlotRange → {Automatic, {-3, 20}}, ImageSize → 300], {nTerms, 2, 8, 2}], 2]
```

Figure 2.16: Code used to generate the above plot

2.1.6 Chapter 1, Section 8, Problem 1

Problem (a) Use the Fourier sine series found in example 1, section 5 for $f(x) = x$ for $0 < x < \pi$, to show that

$$x \sim \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi x \qquad (-1 < x < 1) \tag{1}$$

(b) Obtain the correspondence in part (a) by using expression (11) in section 9 for the coefficient in a Fourier sine series on $0 < x < c$

Part a

The Fourier sine series found in example 1, section 5 for $f(x) = x$ for $0 < x < \pi$ is

$$x \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \qquad (0 < x < \pi) \tag{2}$$

Which has period $T_2 = 2\pi$ after odd extension. To convert the above to the range $-1 < x < 1$, then by looking at this diagram

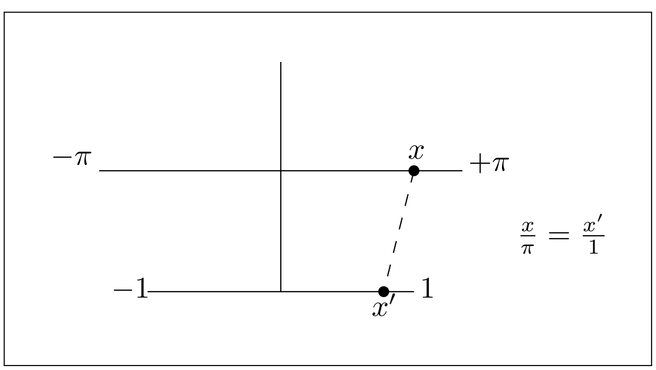

Figure 2.17: Finding scale for correspondence

We see that by symmetry $\frac{x}{\pi} = \frac{x'}{1}$. Hence $x = \pi x'$. Therefore we want $x \to \pi x'$ but x' is just

x in the new domain. Hence $x \to \pi x$ in the new Fourier series. Therefore replacing x by πx in (2) gives

$$x \sim 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x \qquad (0<x<1) \tag{3}$$

Equation (3) is now scaled by multiplying it by $\frac{x'}{x}=\frac{1}{\pi}$ giving

$$x \sim \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x \qquad (0<x<1) \tag{4}$$

Part b

Expression (11) in section 8 is

$$b_n = \frac{2}{c}\int_0^c f(x)\sin\left(\frac{n\pi x}{c}\right)dx$$

Let $c=1$ and since $f(x)=x$, then above becomes

$$b_n = 2\int_0^1 x\sin(n\pi x)\,dx$$

Let $u=x, dv=\sin(n\pi x)$ then $du=1, v=\frac{-\cos(n\pi x)}{n\pi}$. Hence $udv=uv-\int vdu$ and the integral above becomes

$$\begin{aligned}
b_n &= 2\left(\frac{-1}{n\pi}\left[x\cos(n\pi x)\right]_0^1+\frac{1}{n\pi}\int_0^1\cos(n\pi x)\,dx\right)\\
&= 2\left(\frac{-1}{n\pi}\left[\cos(n\pi)\right]+\frac{1}{n\pi}\left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1\right)\\
&= 2\left(\frac{-1}{n\pi}\left[(-1)^n\right]+\frac{1}{(n\pi)^2}\overbrace{\left[\sin(n\pi x)\right]_0^1}^{0}\right)\\
&= \frac{2}{n\pi}(-1)^{n+1}
\end{aligned}$$

Hence

$$\begin{aligned}
x &\sim \sum_{n=1}^{\infty} b_n\sin n\pi x\\
&\sim \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^{n+1}\sin n\pi x
\end{aligned}$$

Which is the same as (1) in part (a)

2.1.7 Chapter 1, Section 8, Problem 6

Problem Use method in example 2 section 8 to show that

$$e^x \sim \frac{\sinh c}{c} + 2\sinh c \sum_{n=1}^{\infty} \frac{(-1)^n}{c^2+(n\pi)^2}\left(c\cos\left(\frac{n\pi x}{c}\right) - n\pi\sin\left(\frac{n\pi x}{c}\right)\right) \qquad -c<x<c$$

Solution

From problem 4 section 7, we know that

$$e^{ax} \sim \frac{\sinh a\pi}{a\pi} + 2\frac{\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\left(a\cos(nx) - n\sin(nx)\right) \qquad -\pi<x<\pi \tag{1}$$

To convert the above to the range $-c<x<c$, then by looking at this diagram

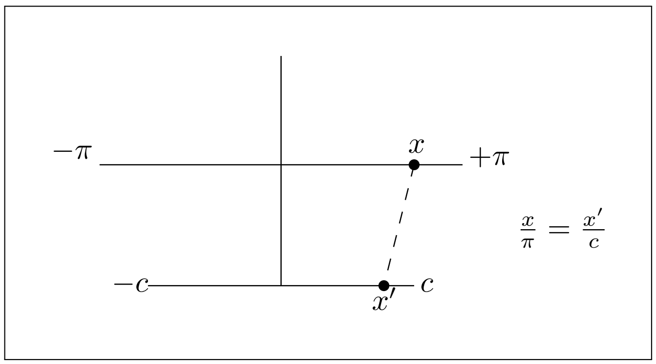

Figure 2.18: Finding scale for correspondence

We see that by symmetry, $\frac{x}{\pi} = \frac{x'}{c}$ where x' is the x in the new range we want, which is $-c<x<c$. Hence $x = \frac{x'\pi}{c}$ or since x' is just x in the new domain, then this implies $x \to \frac{x\pi}{c}$. Then replacing x by $\frac{x\pi}{c}$ in (1) gives

$$e^{\frac{a\pi x}{c}} \sim \frac{\sinh a\pi}{a\pi} + 2\frac{\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\left(a\cos\left(\frac{n\pi x}{c}\right) - n\sin\left(\frac{n\pi x}{c}\right)\right) \qquad -x<x<c \tag{2}$$

We see that the trigonometric terms inside the sum is multiplied by a, hence we replace that by $\frac{c}{\pi}$ in the above. This is the same as $\frac{x'}{x} = \frac{c}{\pi}$. Hence letting $a = \frac{c}{\pi}$ in (2) gives

$$\begin{aligned}
e^x &\sim \frac{\sinh c}{c} + 2\frac{\sinh c}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(\frac{c}{\pi}\right)^2+n^2}\left(\frac{c}{\pi}\cos\left(\frac{n\pi x}{c}\right) - n\sin\left(\frac{n\pi x}{c}\right)\right)\\
&\sim \frac{\sinh c}{c} + 2\frac{\sinh c}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{\frac{c^2}{\pi}+\pi n^2}\left(c\cos\left(\frac{n\pi x}{c}\right) - n\pi\sin\left(\frac{n\pi x}{c}\right)\right)\\
&\sim \frac{\sinh c}{c} + 2\sinh c\sum_{n=1}^{\infty}\frac{(-1)^n}{c^2+\pi^2 n^2}\left(c\cos\left(\frac{n\pi x}{c}\right) - n\pi\sin\left(\frac{n\pi x}{c}\right)\right)
\end{aligned}$$

Which is what we asked to show.

2.2 HW 2

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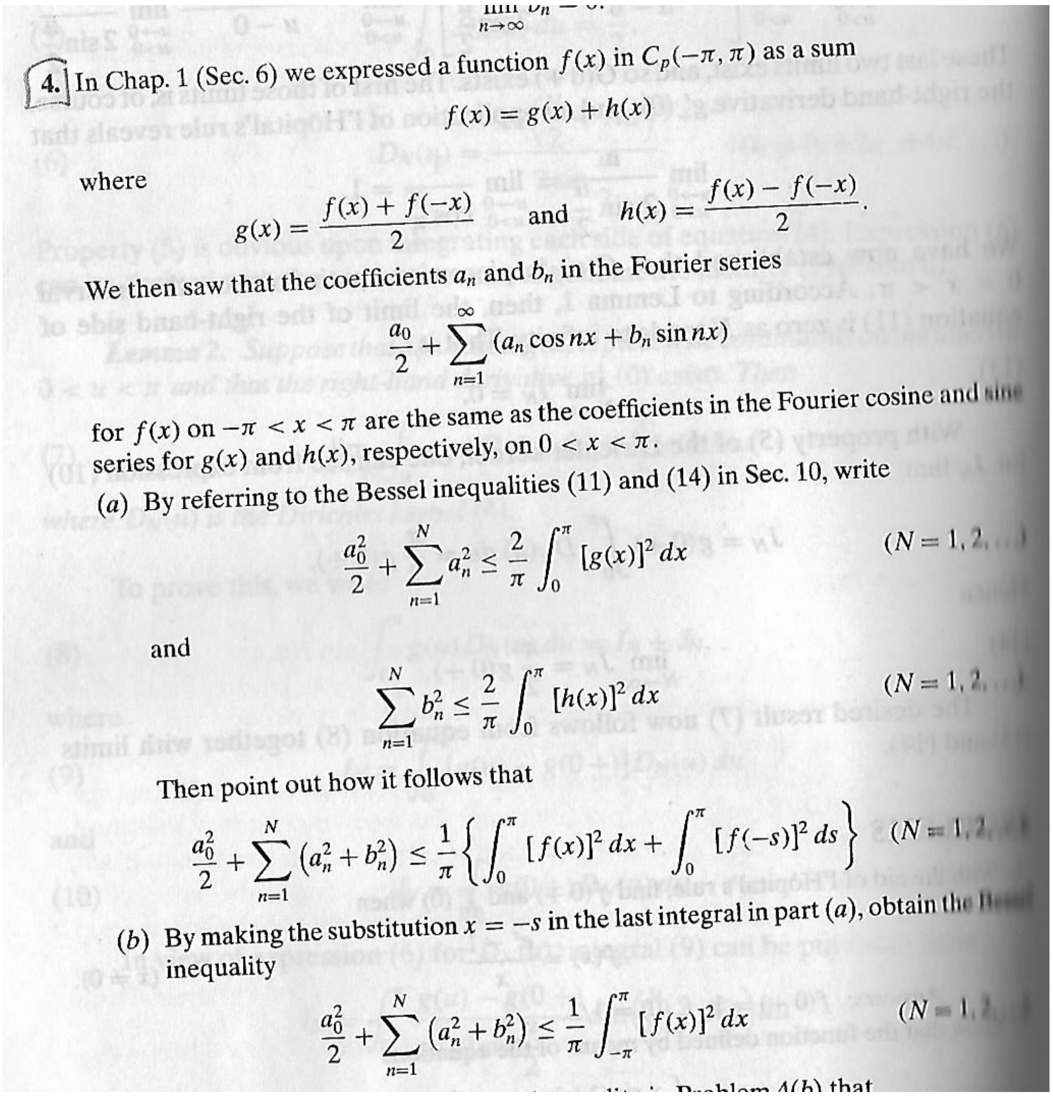
2.2.1 Section 11, Problem 4



4. In Chap. 1 (Sec. 6) we expressed a function $f(x)$ in $C_p(-\pi,\pi)$ as a sum

$$f(x) = g(x) + h(x)$$

where

$$g(x) = \frac{f(x)+f(-x)}{2} \quad \text{and} \quad h(x) = \frac{f(x)-f(-x)}{2}.$$

We then saw that the coefficients a_n and b_n in the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

for $f(x)$ on $-\pi < x < \pi$ are the same as the coefficients in the Fourier cosine and sine series for $g(x)$ and $h(x)$, respectively, on $0 < x < \pi$.

(*a*) By referring to the Bessel inequalities (11) and (14) in Sec. 10, write

$$\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 \le \frac{2}{\pi}\int_0^{\pi} [g(x)]^2\,dx \qquad (N = 1, 2, \ldots)$$

and

$$\sum_{n=1}^{N} b_n^2 \le \frac{2}{\pi}\int_0^{\pi} [h(x)]^2\,dx \qquad (N = 1, 2, \ldots)$$

Then point out how it follows that

$$\frac{a_0^2}{2} + \sum_{n=1}^{N} (a_n^2 + b_n^2) \le \frac{1}{\pi}\left\{\int_0^{\pi} [f(x)]^2\,dx + \int_0^{\pi} [f(-s)]^2\,ds\right\} \qquad (N = 1, 2, \ldots)$$

(*b*) By making the substitution $x = -s$ in the last integral in part (*a*), obtain the Bessel inequality

$$\frac{a_0^2}{2} + \sum_{n=1}^{N} (a_n^2 + b_n^2) \le \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2\,dx \qquad (N = 1, 2, \ldots)$$

Figure 2.19: Problem statement

Part (a)

Writing

$$\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 \le \frac{2}{\pi}\int_0^{\pi} \left[g(x)\right]^2 dx \tag{1}$$

$$\sum_{n=1}^{N} b_n^2 \le \frac{2}{\pi}\int_0^{\pi} [h(x)]^2\,dx \tag{2}$$

Adding (1)+(2) gives

$$\begin{aligned}
\frac{a_0^2}{2}+\sum_{n=1}^{N}\left(a_n^2+b_n^2\right) &\leq \frac{2}{\pi}\int_0^{\pi}\left[g(x)\right]^2+\left[h(x)\right]^2 dx\\
&= \frac{2}{\pi}\int_0^{\pi}\left[\frac{f(x)+f(-x)}{2}\right]^2+\left[\frac{f(x)-f(-x)}{2}\right]^2 dx\\
&= \frac{2}{\pi}\int_0^{\pi}\frac{f^2(x)+f^2(-x)+2f(x)f(-x)}{4}+\frac{f^2(x)+f^2(-x)-2f(x)f(-x)}{4}dx\\
&= \frac{1}{2\pi}\int_0^{\pi}f^2(x)+f^2(-x)+f^2(x)+f^2(-x)\,dx\\
&= \frac{1}{2\pi}\int_0^{\pi}2f^2(x)+2f^2(-x)\,dx\\
&= \frac{1}{\pi}\left(\int_0^{\pi}f^2(x)+f^2(-x)\,dx\right)\\
&= \frac{1}{\pi}\left(\int_0^{\pi}\left[f(x)\right]^2 dx+\int_0^{\pi}\left[f(-s)\right]^2 ds\right) \qquad (3)
\end{aligned}$$

Part (b)

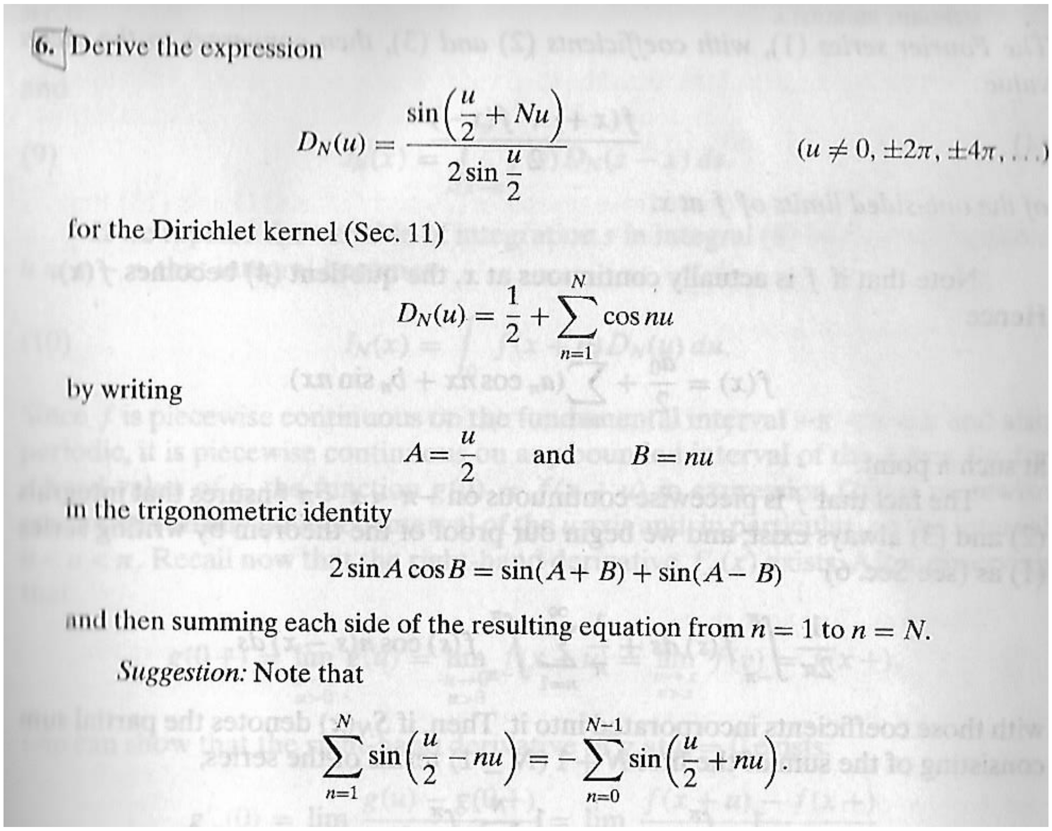
Let $x=-s$ in the last integral. Therefore $dx=-ds$. When $s=0$ then $x=0$ and when $s=\pi$ then $x=-\pi$, then (3) becomes

$$\begin{aligned}
\frac{a_0^2}{2}+\sum_{n=1}^{N}\left(a_n^2+b_n^2\right) &\leq \frac{1}{\pi}\left(\int_0^{\pi}\left[f(x)\right]^2 dx+\int_0^{-\pi}\left[f(x)\right]^2(-dx)\right)\\
&= \frac{1}{\pi}\left(\int_0^{\pi}\left[f(x)\right]^2 dx-\int_0^{-\pi}\left[f(x)\right]^2 dx\right)
\end{aligned}$$

But $\int_0^{-\pi}=-\int_{-\pi}^{0}$ and the above becomes

$$\begin{aligned}
\frac{a_0^2}{2}+\sum_{n=1}^{N}\left(a_n^2+b_n^2\right) &\leq \frac{1}{\pi}\left(\int_0^{\pi}\left[f(x)\right]^2 dx+\int_{-\pi}^{0}\left[f(x)\right]^2 dx\right)\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(x)\right]^2 dx
\end{aligned}$$

2.2.2 Section 11, Problem 6



6. Derive the expression

$$D_N(u) = \frac{\sin\left(\frac{u}{2} + Nu\right)}{2\sin\frac{u}{2}} \qquad (u \neq 0, \pm 2\pi, \pm 4\pi, \dots)$$

for the Dirichlet kernel (Sec. 11)

$$D_N(u) = \frac{1}{2} + \sum_{n=1}^{N} \cos nu$$

by writing

$$A = \frac{u}{2} \quad \text{and} \quad B = nu$$

in the trigonometric identity

$$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$

and then summing each side of the resulting equation from $n = 1$ to $n = N$.

Suggestion: Note that

$$\sum_{n=1}^{N} \sin\left(\frac{u}{2} - nu\right) = -\sum_{n=0}^{N-1} \sin\left(\frac{u}{2} + nu\right).$$

Figure 2.20: Problem statement

We want to show the following (I've used x instead of u as it is more natural).

$$\frac{1}{2} + \sum_{n=1}^{N} \cos nx = \frac{\sin\left(\left(N + \frac{1}{2}\right)x\right)}{2\sin\frac{x}{2}} \tag{1}$$

Or, similarly, we want to show the following

$$\sin\frac{x}{2} + \sum_{n=1}^{N} 2\sin\frac{x}{2}\cos nx = \sin\left(\left(N + \frac{1}{2}\right)x\right) \tag{2}$$

We will now work on the left side of (2) only and see if we can simplify it to obtain the right side of (2). Writing the LHS of (2) as

$$\sin\frac{x}{2} + \sum_{n=1}^{N} 2\sin\frac{x}{2}\cos nx = \sin\frac{x}{2} + \sum_{n=1}^{N} 2\sin A\cos B \tag{3}$$

Where $A = \frac{x}{2}, B = nx$. But $\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$. Hence (3) becomes

$$\begin{aligned}
\sin\frac{x}{2} + \sum_{n=1}^{N} 2\sin\frac{x}{2}\cos nx &= \sin\frac{x}{2} + \sum_{n=1}^{N}\sin(A+B) + \sin(A-B)\\
&= \sin\frac{x}{2} + \sum_{n=1}^{N}\sin\left(\frac{x}{2} + nx\right) + \sin\left(\frac{x}{2} - nx\right)\\
&= \sin\frac{x}{2} + \sum_{n=1}^{N}\sin\left(\left(n+\frac{1}{2}\right)x\right) + \sin\left(\left(\frac{1}{2}-n\right)x\right)\\
&= \sin\frac{x}{2} + \sum_{n=1}^{N}\sin\left(\left(n+\frac{1}{2}\right)x\right) - \sin\left(\left(n-\frac{1}{2}\right)x\right)
\end{aligned}$$

Expanding few terms to see the pattern shows

$$\begin{aligned}
\sin\frac{x}{2} + \sum_{n=1}^{N}\sin\left(\left(n+\frac{1}{2}\right)x\right) - \sin\left(\left(n-\frac{1}{2}\right)x\right) &= \sin\frac{x}{2} + \left[\sin\left(\left(1+\frac{1}{2}\right)x\right) - \sin\left(\left(1-\frac{1}{2}\right)x\right)\right]\\
&+ \left[\sin\left(\left(2+\frac{1}{2}\right)x\right) - \sin\left(\left(2-\frac{1}{2}\right)x\right)\right]\\
&+ \left[\sin\left(\left(3+\frac{1}{2}\right)x\right) - \sin\left(\left(3-\frac{1}{2}\right)x\right)\right] + \cdots
\end{aligned}$$

Or

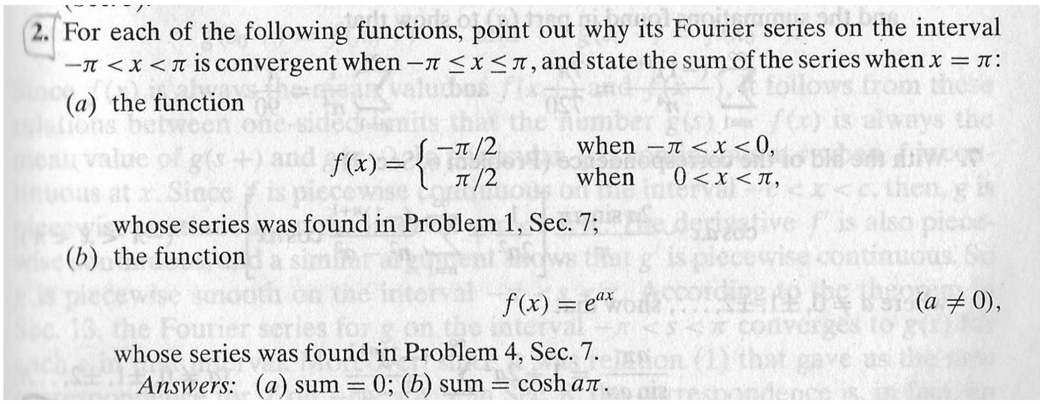
$$\begin{aligned}
\sum_{n=1}^{N}\sin\left(\left(n+\frac{1}{2}\right)x\right) - \sin\left(\left(n-\frac{1}{2}\right)x\right) &= \sin\frac{x}{2} + \left[\sin\left(\frac{3}{2}x\right) - \sin\left(\frac{1}{2}x\right)\right]\\
&+ \left[\sin\left(\frac{5}{2}x\right) - \sin\left(\frac{3}{2}x\right)\right]\\
&+ \left[\sin\left(\frac{7}{2}x\right) - \sin\left(\frac{5}{2}x\right)\right] + \cdots
\end{aligned}$$

We see that all terms cancel except for the term before the last term, which is $\sin\left(\left(N+\frac{1}{2}\right)x\right)$. (In the above limited expansion of terms, this will be the term $\sin\left(\frac{7}{2}x\right)$ which remains.) Hence as $n \to N$, the above simplifies to

$$\sin\frac{x}{2} + \sum_{n=1}^{N}\sin\left(\frac{x}{2} + nx\right) + \sin\left(\frac{x}{2} - nx\right) = \sin\left(\left(N+\frac{1}{2}\right)x\right)$$

Which is (2) which was obtained from (1). Hence (1) was verified to be valid.

2.2.3 Section 14, Problem 2



2. For each of the following functions, point out why its Fourier series on the interval $-\pi < x < \pi$ is convergent when $-\pi \le x \le \pi$, and state the sum of the series when $x = \pi$:

(*a*) the function

$$f(x) = \begin{cases} -\pi/2 & \text{when } -\pi < x < 0, \\ \pi/2 & \text{when } 0 < x < \pi, \end{cases}$$

whose series was found in Problem 1, Sec. 7;

(*b*) the function

$$f(x) = e^{ax} \qquad (a \neq 0),$$

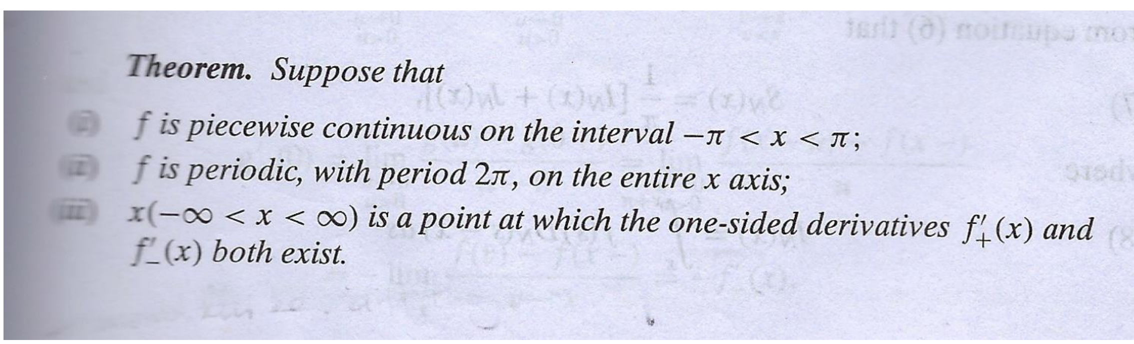
whose series was found in Problem 4, Sec. 7.

Answers: (*a*) sum $= 0$; (*b*) sum $= \cosh a\pi$.

Figure 2.21: Problem statement

Part (a)

The Fourier series for $f(x)$ is convergent since $f(x)$, after periodic extension, satisfies the 3 points of the Fourier theorem in the textbook at page 35



Theorem. *Suppose that*

(i) *f is piecewise continuous on the interval $-\pi < x < \pi$;*

(ii) *f is periodic, with period 2π, on the entire x axis;*

(iii) *$x(-\infty < x < \infty)$ is a point at which the one-sided derivatives $f'_+(x)$ and $f'_-(x)$ both exist.*

Figure 2.22: Fourier theorem

Point (i) is satisfied since $f(x)$ is piecewise continuous and also point (ii) when doing periodic extension. Also point (iii) is satisfied, since the left sided and right sides limit exist at each x.

```
In[ ]:= Clear[f];
f[x_ /; -Pi < x < Pi] := Piecewise[{{-Pi / 2, -Pi < x < 0}, {Pi / 2, 0 < x < Pi}}]
f[x_ /; x > Pi] := f[x - 2 Pi];
f[x_ /; x < -Pi] := f[x + 2 Pi];
Plot[f[x], {x, -4 Pi, 4 Pi}, PlotStyle → {Thick, Red}, Mesh → None,
 GridLines → Automatic, GridLinesStyle → LightGray,
 Ticks → {Range[-4 Pi, 4 Pi, Pi], Automatic},
 Exclusions → {x == -3 Pi, x == -Pi, x == 0, x == Pi, x == 3 Pi},
 Mesh → None, ExclusionsStyle → Dashed]
```

Out[ ]=

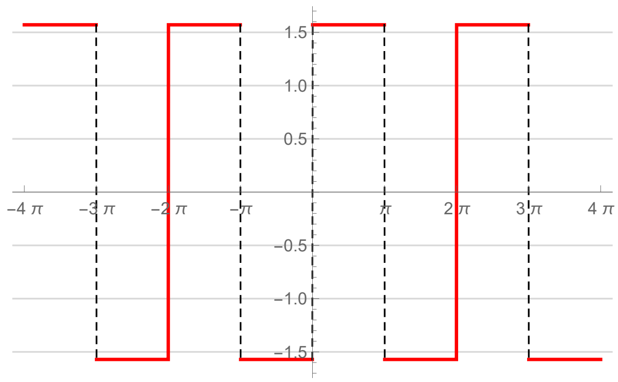


Figure 2.23: $f(x)$ after periodic extension

Therefore the Fourier series will converge to the average of the function $f(x)$ at $x = \pi$. This average is

$$\frac{f(\pi^-) + f(\pi^+)}{2} = \frac{\frac{\pi}{2} - \frac{\pi}{2}}{2} = 0$$

Part (b)

The Fourier series for $f(x) = e^{ax}$ is convergent since $f(x)$, after periodic extension, satisfies the 3 points of the Fourier theorem in the textbook at page 35. Point (i) is satisfied is piecewise continuous and also point (ii) when doing periodic extension. Also point (iii) is satisfied, since the left sided and right sides limit exist at each x. Here is a plot, using $a = \frac{1}{4}$ for illustration

In[∘]:=
```
Clear[f];
a = 1/4;
f[x_ /; -Pi < x < Pi] := Exp[a x];
f[x_ /; x > Pi] := f[x - 2 Pi];
f[x_ /; x < -Pi] := f[x + 2 Pi];
Plot[f[x], {x, -4 Pi, 4 Pi}, PlotStyle → {Thick, Red}, Mesh → None,
 GridLines → Automatic, GridLinesStyle → LightGray,
 Ticks → {Range[-4 Pi, 4 Pi, Pi], Automatic},
 Exclusions → {x == -3 Pi, x == -Pi, x == 0, x == Pi, x == 3 Pi},
 Mesh → None, ExclusionsStyle → Dashed,
 AxesOrigin → {0, 0}]
```

Out[∘]=

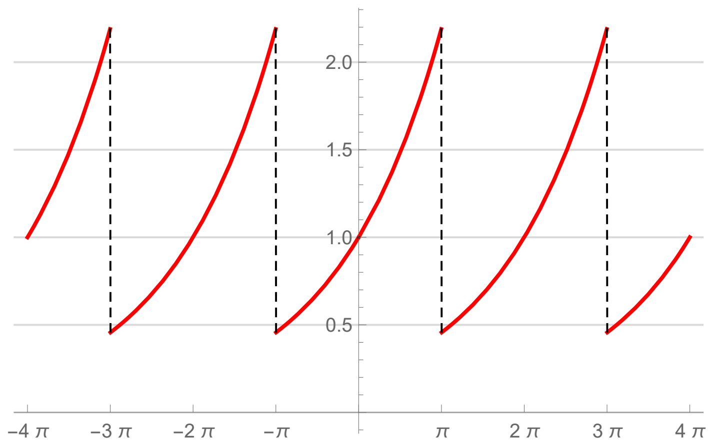
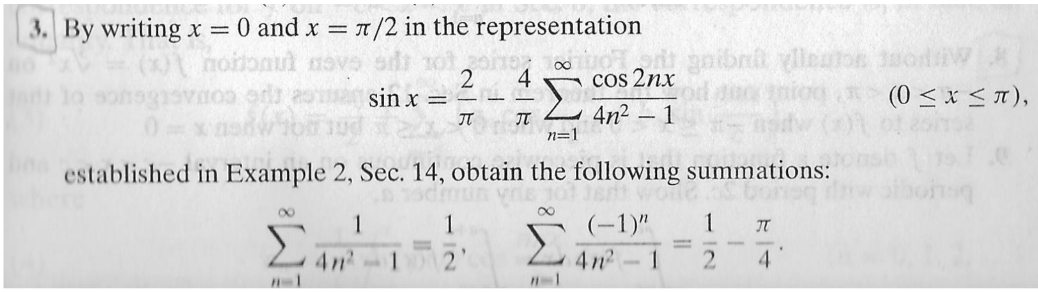


Figure 2.24: $f(x) = e^{ax}$ after periodic extension (Using $a = \frac{1}{4}$

Therefore the Fourier series will converge to the average of the function $f(x)$ at $x = \pi$. This average is

$$\frac{f(\pi^-) + f(\pi^+)}{2} = \frac{e^{a\pi} + e^{-a\pi}}{2} = \cosh(a\pi)$$

2.2.4 Section 14, Problem 3



3. By writing $x = 0$ and $x = \pi/2$ in the representation

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1} \qquad (0 \le x \le \pi),$$

established in Example 2, Sec. 14, obtain the following summations:

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}, \qquad \sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = \frac{1}{2} - \frac{\pi}{4}.$$

Figure 2.25: Problem statement

Substituting $x = 0$ in the given representation gives

$$\begin{aligned} 0 &= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} \\ -2 &= -4\sum_{n=1}^{\infty}\frac{1}{4n^2-1} \\ \frac{1}{2} &= \sum_{n=1}^{\infty}\frac{1}{4n^2-1} \end{aligned}$$

And substituting $x = \frac{\pi}{2}$ in the given representation gives

$$\begin{aligned} 1 &= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{4n^2-1} \\ 1 - \frac{2}{\pi} &= -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} \\ \pi - 2 &= -4\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} \\ \frac{1}{2} - \frac{\pi}{4} &= \sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} \end{aligned}$$

2.2.5 Section 14, Problem 6

6. (*a*) Use the correspondence

$$x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \qquad (0 < x < \pi),$$

found in Problem 3(*a*), Sec. 5, to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(*b*) Use the correspondence (Problem 6, Sec. 5)

$$x^4 \sim \frac{\pi^4}{5} + 8\sum_{n=1}^{\infty} (-1)^n \frac{(n\pi)^2 - 6}{n^4} \cos nx \qquad (0 < x < \pi)$$

and the summations found in part (*a*) to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Figure 2.26: Problem statement

Part (a)

$$x^2 \sim \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \tag{1}$$

Letting $x = 0$ in (1) gives (After doing periodic extension, then $x = 0$ is now in the domain).

$$\begin{aligned} 0 &= \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \\ -\frac{1}{3}\pi^2 &= 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \\ -\frac{\pi^2}{12} &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \end{aligned}$$

Multiplying both sides by -1 gives the result needed

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

Now we need to obtain the second result $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$. Let $x=\pi$ in (1) (After doing periodic extension, then $x=\pi$ is now in the domain) gives

$$
\begin{aligned}
\pi^2 &= \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n\\
\pi^2-\frac{1}{3}\pi^2 &= 4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}\\
\frac{1}{6}\pi^2 &= \sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}
\end{aligned}
$$

But $\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}$ since the power $2n$ is always even. This gives the result needed

$$\frac{1}{6}\pi^2=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Part (b)

$$x^4 \sim \frac{\pi^4}{5}+8\sum_{n=1}^{\infty}(-1)^n\frac{(n\pi)^2-6}{n^4}\cos nx \tag{2}$$

Letting $x=0$ in (2) gives

$$
\begin{aligned}
0 &= \frac{\pi^4}{5}+8\sum_{n=1}^{\infty}(-1)^n\frac{(n\pi)^2-6}{n^4}\\
-\frac{\pi^4}{5} &= 8\left(\sum_{n=1}^{\infty}(-1)^n\frac{(n\pi)^2}{n^4}-6\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\right)\\
\frac{\pi^4}{5} &= 8\left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(n\pi)^2}{n^4}+6\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\right)\\
\frac{\pi^4}{5} &= 8\left(\pi^2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}+6\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\right)
\end{aligned}
$$

But from part (a), we found that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$. Using this in the above results in

$$\begin{aligned}
\frac{\pi^4}{5} &= 8\left(\pi^2\left(\frac{\pi^2}{12}\right)+6\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\right)\\
\frac{\pi^4}{5} &= \frac{8}{12}\pi^4+48\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\\
\frac{\pi^4}{5}-\frac{8\pi^4}{12} &= 48\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\\
-\frac{7}{15}\pi^4 &= 48\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\\
-\frac{7}{720}\pi^4 &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}
\end{aligned}$$

Multiplying both sides by -1 gives the result needed

$$\frac{7}{720}\pi^4=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}$$

Now we need to obtain the second result $\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^2}{90}$. Let $x=\pi$ in (2) gives

$$\begin{aligned}
\pi^4 &= \frac{\pi^4}{5}+8\sum_{n=1}^{\infty}(-1)^n\frac{(n\pi)^2-6}{n^4}(-1)^n\\
&= \frac{\pi^4}{5}+8\sum_{n=1}^{\infty}(-1)^{2n}\frac{(n\pi)^2-6}{n^4}
\end{aligned}$$

But $(-1)^{2n}=1$ for all n. The above simplifies to

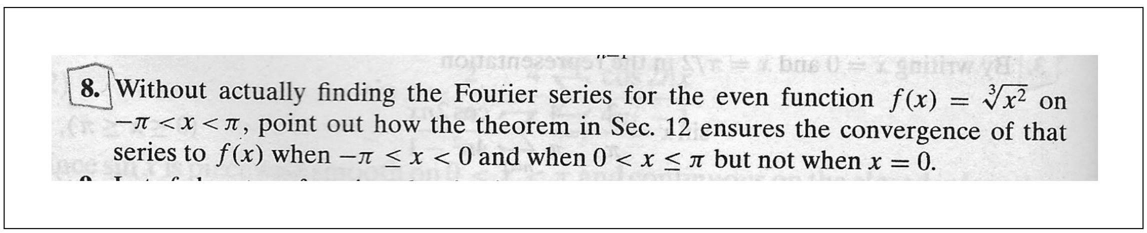
$$\begin{aligned}
\pi^4 &= \frac{\pi^4}{5}+8\sum_{n=1}^{\infty}\frac{(n\pi)^2-6}{n^4}\\
\pi^4-\frac{\pi^4}{5} &= 8\left(\sum_{n=1}^{\infty}\frac{(n\pi)^2}{n^4}-6\sum_{n=1}^{\infty}\frac{1}{n^4}\right)\\
\frac{4\pi^4}{5} &= 8\left(\pi^2\sum_{n=1}^{\infty}\frac{1}{n^2}-6\sum_{n=1}^{\infty}\frac{1}{n^4}\right)
\end{aligned}$$

But from part(a) we found that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ hence the above simplifies to

$$\begin{aligned}\frac{4\pi^4}{5} &= 8\left(\pi^2\left(\frac{\pi^2}{6}\right)-6\sum_{n=1}^{\infty}\frac{1}{n^4}\right)\\ \frac{4\pi^4}{40} &= \frac{\pi^4}{6}-6\sum_{n=1}^{\infty}\frac{1}{n^4}\\ -\frac{1}{15}\pi^4 &= -6\sum_{n=1}^{\infty}\frac{1}{n^4}\\ \frac{1}{90}\pi^4 &= \sum_{n=1}^{\infty}\frac{1}{n^4}\end{aligned}$$

Which is the result we are asked to show.

2.2.6 Section 14, Problem 8



8. Without actually finding the Fourier series for the even function $f(x)=\sqrt[3]{x^2}$ on $-\pi<x<\pi$, point out how the theorem in Sec. 12 ensures the convergence of that series to $f(x)$ when $-\pi\le x<0$ and when $0<x\le\pi$ but not when $x=0$.

Figure 2.27: Problem statement

We first notice that the function $f(x)$ is not differentiable at $x=0$

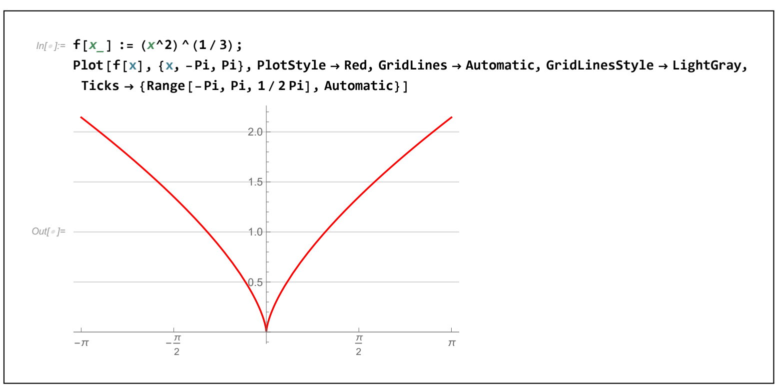


Figure 2.28: plot of $(x^2)^{1/3}$

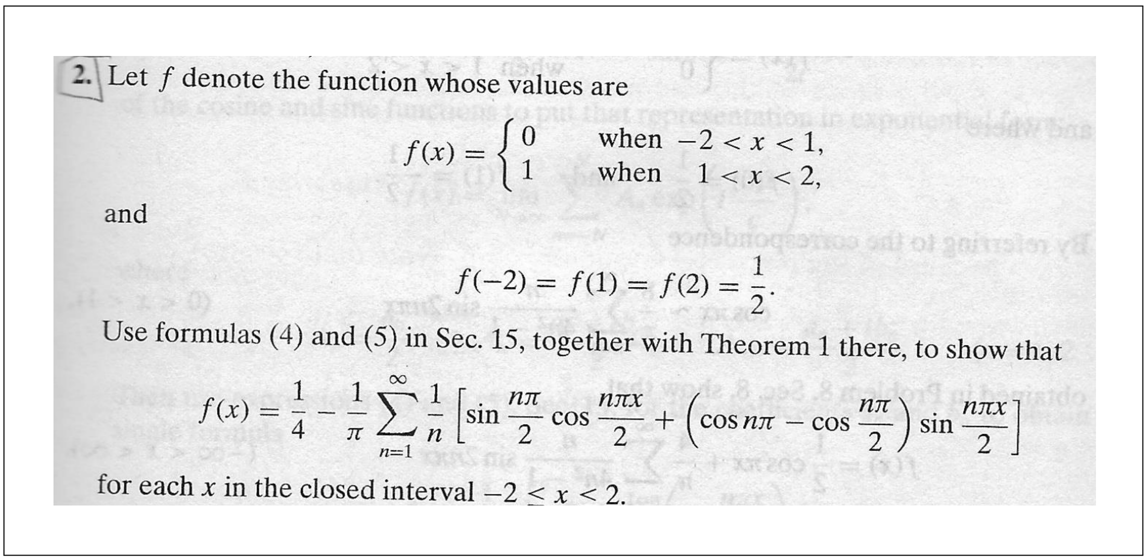
This is because, when $x_0=0$ the left sided derivative is equal to the right sided derivative

$$\lim_{\substack{x\to x_0\\ x<x_0}} f(x)=f'_-(x_0)\neq\lim_{\substack{x\to x_0\\ x>x_0}} f(x)=f'_+(x_0)$$

Since $f'_{-}(0) = -\infty$ while $f'_{+}(0) = +\infty$. The function is therefore piecewise continuous on each $-\pi < x < \pi$ but it is not differentiable at $x = 0$. But Fourier theorem, looking at point (iii) in the book, only says that if $f'_{-}(x_0)$ exist and if $f'_{+}(x_0)$ exist, then the Fourier series converges to the average of $f(x)$ at point x_0. In this example $f'_{-}(0) = -\infty$ and $f'_{+}(0) = +\infty$, which means these limits do not exist.

Hence we see that point (i) and (ii) in the Fourier theorem in the book are satisfied, but it is point (iii) which not satisfied at $x = 0$. Therefore Fourier series does not converge to $f(x)$ at $x = 0$ only while on other x in the domain it does.

2.2.7 Section 15, Problem 2



2. Let f denote the function whose values are

$$f(x) = \begin{cases} 0 & \text{when } -2 < x < 1, \\ 1 & \text{when } \quad 1 < x < 2, \end{cases}$$

and

$$f(-2) = f(1) = f(2) = \frac{1}{2}.$$

Use formulas (4) and (5) in Sec. 15, together with Theorem 1 there, to show that

$$f(x) = \frac{1}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\sin\frac{n\pi}{2}\cos\frac{n\pi x}{2} + \left(\cos n\pi - \cos\frac{n\pi}{2}\right)\sin\frac{n\pi x}{2}\right]$$

for each x in the closed interval $-2 \leq x < 2$.

Figure 2.29: Problem statement

A plot of the function $f(x)$ and its periodic extension is given below

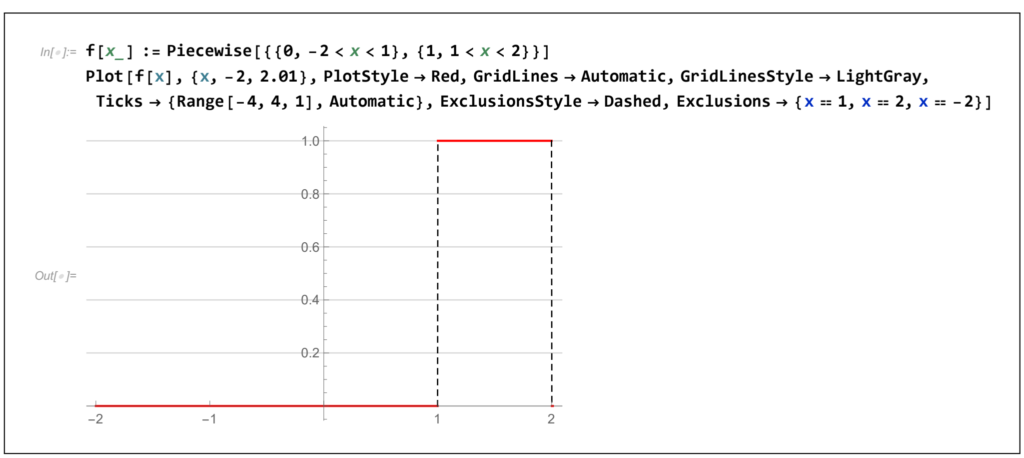


Figure 2.30: plot of $f(x)$ over one period

```
Clear[f];
f[x_ /; -2 < x < 2] := Piecewise[{{0, -2 < x < 1}, {1, 1 < x < 2}}]
f[x_ /; x > 2] := f[x - 4];
f[x_ /; x < -2] := f[x + 4];
Plot[f[x], {x, -8, 8}, PlotStyle → {Thick, Red}, Mesh → None,
 GridLines → Automatic, GridLinesStyle → LightGray,
 Ticks → {Range[-8, 8, 1], Automatic},
 Exclusions → {x == -7, x == -6, x == -3, x == -2, x == 1, x == 2, x == 5, x == 6},
 Mesh → None, ExclusionsStyle → Dashed]
```

Out[∘]=

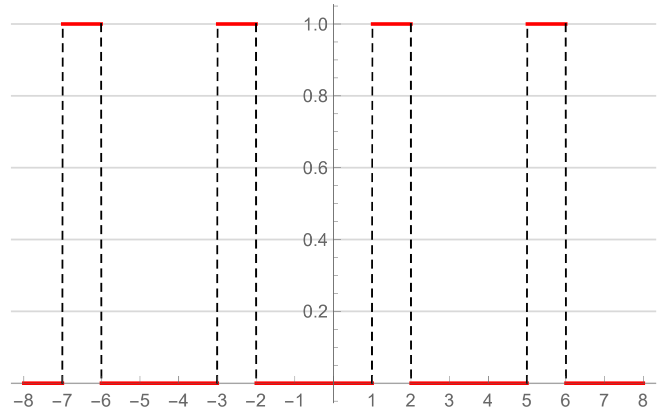


Figure 2.31: plot of $f(x)$ extended to become periodic. Showing 3 periods

The Fourier transform of $f(x)$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T} nx\right) + b_n \sin\left(\frac{2\pi}{T} nx\right)$$

Where T is the period of the function (after periodic extension) which is 4. Hence the above becomes

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{2} nx\right) + b_n \sin\left(\frac{\pi}{2} nx\right)$$

Since $f(x)$ meets the requirements of the Fourier theorem on page 35 of the text (at points of discontinues, the function is $\frac{1}{2}$ which is the average at those points), then $\sim$ can be replaced by $=$ above

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{2} nx\right) + b_n \sin\left(\frac{\pi}{2} nx\right) \tag{1}$$

Where

$$a_0 = \frac{1}{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx = \frac{1}{\frac{4}{2}} \int_{-\frac{4}{2}}^{\frac{4}{2}} f(x)\,dx = \frac{1}{2} \int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left(\int_{-2}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx\right)$$
$$= \frac{1}{2}\left(\int_{1}^{2} dx\right) = \frac{1}{2}(x)_1^2 = \frac{1}{2}$$

And

$$\begin{aligned}
a_n &= \frac{1}{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \cos\left(\frac{2\pi}{T} nx\right) dx = \frac{1}{2} \int_{-2}^{2} f(x) \cos\left(\frac{\pi}{2} nx\right) dx \\
&= \frac{1}{2} \left(\int_{-2}^{1} f(x) \cos\left(\frac{\pi}{2} nx\right) dx + \int_{1}^{2} f(x) \cos\left(\frac{\pi}{2} nx\right) dx \right) \\
&= \frac{1}{2} \int_{1}^{2} f(x) \cos\left(\frac{\pi}{2} nx\right) dx \\
&= \frac{1}{2} \int_{1}^{2} \cos\left(\frac{\pi}{2} nx\right) dx \\
&= \frac{1}{2} \left[\frac{\sin\left(\frac{\pi}{2} nx\right)}{\frac{\pi n}{2}} \right]_1^2 \\
&= \frac{1}{\pi n} \left(\sin(\pi n) - \sin\left(\frac{\pi n}{2}\right) \right) \\
&= \frac{-1}{\pi n} \sin\left(\frac{\pi n}{2}\right)
\end{aligned}$$

And

$$\begin{aligned}
b_n &= \frac{1}{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \sin\left(\frac{2\pi}{T} nx\right) dx = \frac{1}{2} \int_{-2}^{2} f(x) \sin\left(\frac{\pi}{2} nx\right) dx \\
&= \frac{1}{2} \left(\int_{-2}^{1} f(x) \sin\left(\frac{\pi}{2} nx\right) dx + \int_{1}^{2} f(x) \sin\left(\frac{\pi}{2} nx\right) dx \right) \\
&= \frac{1}{2} \int_{1}^{2} f(x) \sin\left(\frac{\pi}{2} nx\right) dx \\
&= \frac{1}{2} \int_{1}^{2} \sin\left(\frac{\pi}{2} nx\right) dx \\
&= \frac{-1}{2} \left[\frac{\cos\left(\frac{\pi}{2} nx\right)}{\frac{\pi n}{2}} \right]_1^2 \\
&= \frac{-1}{\pi n} \left[\cos(\pi n) - \cos\left(\frac{\pi}{2} n\right) \right] \\
&= \frac{-1}{\pi n} \left[\cos(\pi n) - \cos\left(\frac{\pi n}{2}\right) \right]
\end{aligned}$$

Using these results in (1) gives

$$
\begin{aligned}
f(x) &= \frac{1}{4} + \sum_{n=1}^{\infty} \left(\frac{-1}{\pi n}\sin\left(\frac{\pi n}{2}\right)\right)\cos\left(\frac{\pi}{2}nx\right) + \left(\frac{-1}{\pi n}\left[\cos(\pi n) - \cos\left(\frac{\pi n}{2}\right)\right]\right)\sin\left(\frac{\pi}{2}nx\right) \\
&= \frac{1}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sin\left(\frac{\pi n}{2}\right)\right)\cos\left(\frac{\pi}{2}nx\right) + \frac{1}{n}\left(\cos(\pi n) - \cos\left(\frac{\pi n}{2}\right)\right)\sin\left(\frac{\pi}{2}nx\right) \\
&= \frac{1}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\sin\left(\frac{\pi n}{2}\right)\cos\left(\frac{\pi}{2}nx\right) + \left(\cos(\pi n) - \cos\left(\frac{\pi n}{2}\right)\right)\sin\left(\frac{\pi}{2}nx\right)\right]
\end{aligned}
$$

Which is the result we are asked to show. To verify this, the following shows the convergence to $f(x)$ when using more and more terms in the series.

```
fApprox[x_, nTerms_] := 1/4 - 1/π Sum[1/n (Sin[π n/2] Cos[π n x/2] + (Cos[π n] - Cos[π n/2]) Sin[π n x/2]), {n, 1, nTerms}];
Clear[f];
f[x_ /; -2 < x < 2] := Piecewise[{{0, -2 < x < 1}, {1, 1 < x < 2}}]
f[x_ /; x > 2] := f[x - 4];
f[x_ /; x < -2] := f[x + 4];
Grid[Partition[Table[Plot[{f[x], fApprox[x, n]}, {x, -Pi, Pi},
    PlotStyle → {Blue, Red},
    PlotLabel → Style[Row[{"Using ", n, " terms"}], Bold],
    ImageSize → 250],
   {n, 1, 10}], 2], Frame → All, FrameStyle → Gray]
```

Figure 2.32: Code used to draw the plot

Out[∘]=

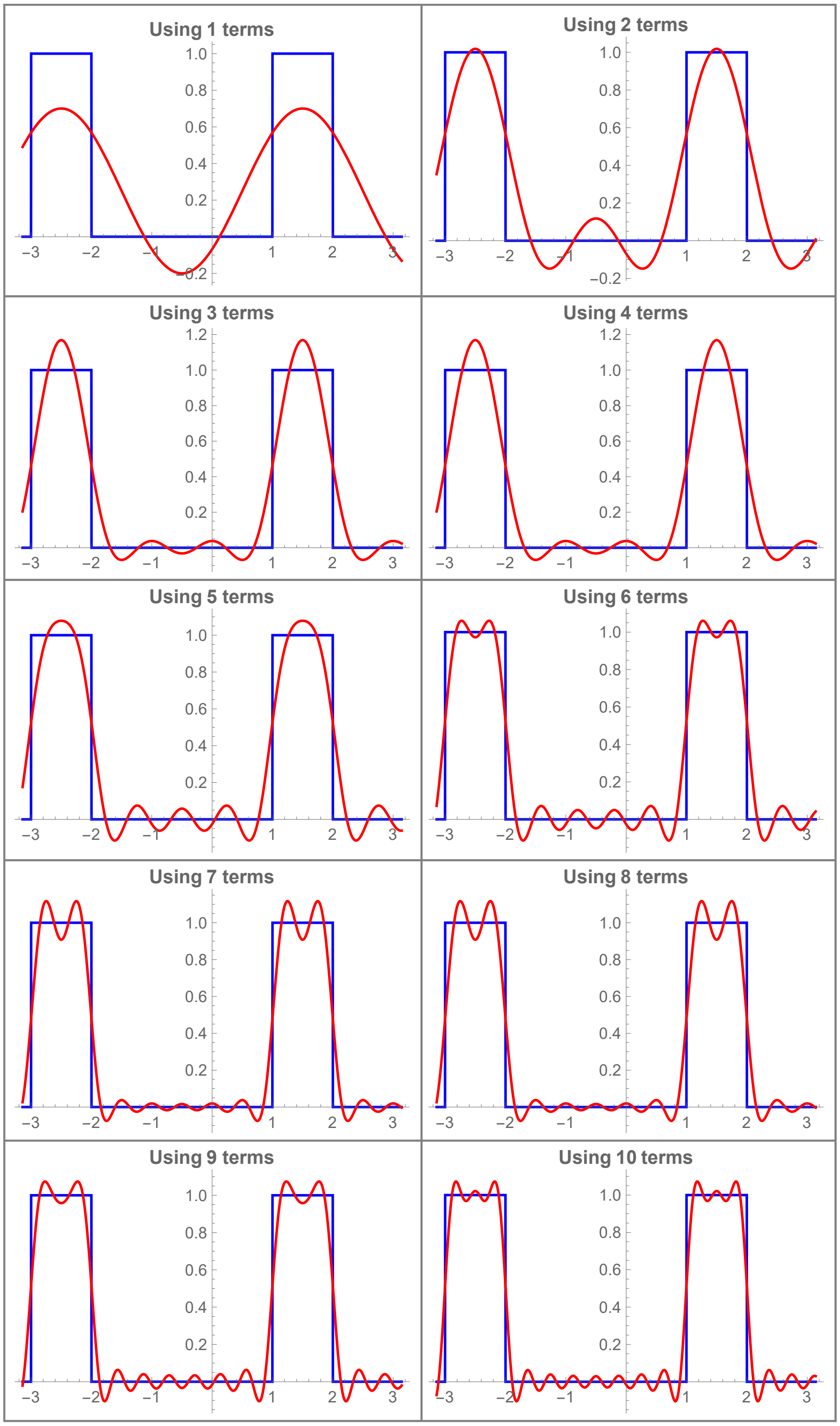
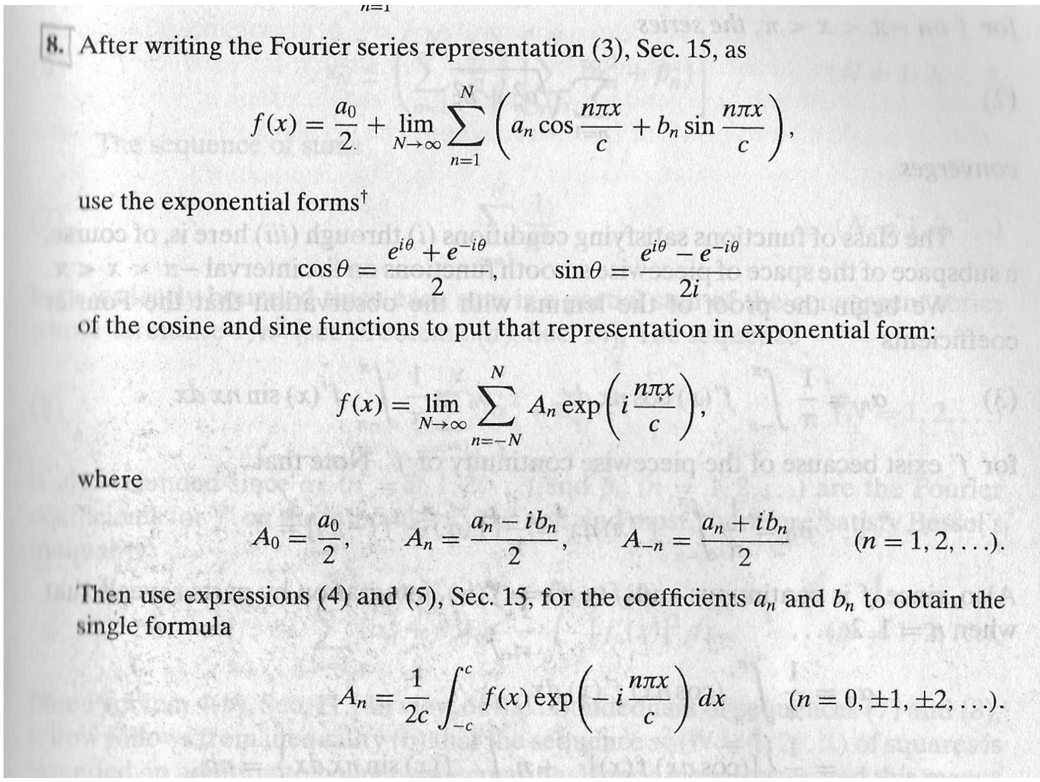


Figure 2.33: Fourier series approximation as more terms added

We notice that the Fourier series approximation converges to $\frac{1}{2}$ at the points of discontinuities.

But these are the actual values of $f(x)$ at those points.

2.2.8 Section 15, Problem 8



8. After writing the Fourier series representation (3), Sec. 15, as

$$f(x) = \frac{a_0}{2} + \lim_{N\to\infty} \sum_{n=1}^{N} \left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c} \right),$$

use the exponential forms†

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

of the cosine and sine functions to put that representation in exponential form:

$$f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} A_n \exp\left(i\frac{n\pi x}{c} \right),$$

where

$$A_0 = \frac{a_0}{2}, \qquad A_n = \frac{a_n - ib_n}{2}, \qquad A_{-n} = \frac{a_n + ib_n}{2} \qquad (n = 1, 2, \ldots).$$

Then use expressions (4) and (5), Sec. 15, for the coefficients a_n and b_n to obtain the single formula

$$A_n = \frac{1}{2c} \int_{-c}^{c} f(x) \exp\left(-i\frac{n\pi x}{c} \right) dx \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Figure 2.34: Problem statement

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + \lim_{N\to\infty} \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi}{c}x\right) + b_n \sin\left(\frac{n\pi}{c}x\right) \\
&= \frac{a_0}{2} + \lim_{N\to\infty} \sum_{n=1}^{N} a_n \left(\frac{e^{i\frac{n\pi}{c}x} + e^{-i\frac{n\pi}{c}x}}{2} \right) + b_n \left(\frac{e^{i\frac{n\pi}{c}x} - e^{-i\frac{n\pi}{c}x}}{2i} \right) \\
&= \frac{a_0}{2} + \lim_{N\to\infty} \sum_{n=1}^{N} a_n \left(\frac{e^{i\frac{n\pi}{c}x} + e^{-i\frac{n\pi}{c}x}}{2} \right) - ib_n \left(\frac{e^{i\frac{n\pi}{c}x} - e^{-i\frac{n\pi}{c}x}}{2} \right) \\
&= \frac{a_0}{2} + \lim_{N\to\infty} \sum_{n=1}^{N} e^{i\frac{n\pi}{c}x} \left(\frac{a_n - ib_n}{2} \right) + e^{-i\frac{n\pi}{c}x} \left(\frac{a_n + ib_n}{2} \right) \\
&= \frac{a_0}{2} + \lim_{N\to\infty} \sum_{n=1}^{N} e^{i\frac{n\pi}{c}x} \left(\frac{a_n - ib_n}{2} \right) + \sum_{n=-N}^{-1} e^{i\frac{n\pi}{c}x} \left(\frac{a_n + ib_n}{2} \right) \qquad (1)
\end{aligned}$$

Let

$$A_n = \begin{cases} \left(\frac{a_n - ib_n}{2}\right) & n > 0 \\ \frac{a_0}{2} & n = 0 \\ \left(\frac{a_n + ib_n}{2}\right) & n < 0 \end{cases}$$

Then (1) can be written as

$$f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} A_n e^{i\frac{n\pi}{c}x}$$

Since

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\left(\frac{n\pi}{c}x\right)dx \qquad n = 0, 1, 2, \cdots$$
$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\left(\frac{n\pi}{c}x\right)dx \qquad n = 1, 2, \cdots$$

Then $a_n + ib_n$ gives

$$\begin{aligned} a_n - ib_n &= \frac{1}{c}\int_{-c}^{c} f(x)\cos\left(\frac{n\pi}{c}x\right)dx - i\frac{1}{c}\int_{-c}^{c} f(x)\sin\left(\frac{n\pi}{c}x\right)dx \\ &= \frac{1}{c}\left(\int_{-c}^{c} f(x)\cos\left(\frac{n\pi}{c}x\right)dx + \int_{-c}^{c} f(x)\left(-i\sin\left(\frac{n\pi}{c}x\right)\right)dx\right) \\ &= \frac{1}{c}\int_{-c}^{c} f(x)\left[\cos\left(\frac{n\pi}{c}x\right) - i\sin\left(\frac{n\pi}{c}x\right)\right]dx \\ &= \frac{1}{c}\int_{-c}^{c} f(x)\, e^{-i\frac{n\pi}{c}x}dx \end{aligned}$$

But $a_n - ib_n = 2A_n$ from first part of this problem. Hence the above becomes

$$A_n = \frac{1}{2c}\int_{-c}^{c} f(x)\, e^{-i\frac{n\pi}{c}x}dx \qquad n = 0, \pm 1, \pm 2, \cdots$$

2.3 HW 3

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2.3.1 Section 20, Problem 1

1. Show that the function

$$f(x) = \begin{cases} 0 & \text{when } -\pi \le x \le 0, \\ \sin x & \text{when } \quad 0 < x \le \pi \end{cases}$$

satisfies all the conditions in the theorem in Sec. 17. Then, with the aid of the Weierstrass M-test in Sec. 17, verify that the Fourier series

$$\frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1} \qquad (-\pi < x < \pi)$$

for f, found in Problem 7, Sec. 7, converges uniformly on the interval $-\pi \le x \le \pi$, as the theorem in Sec. 17 tells us. Also, state why this series is differentiable in the interval $-\pi < x < \pi$, except at the point $x = 0$, and describe graphically the function that is represented by the differentiated series for all x.

Figure 2.35: Problem statement

The function $f(x)$ is

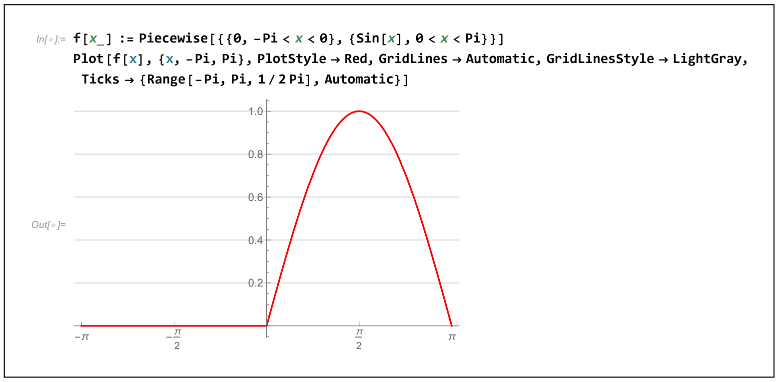


Figure 2.36: Plot of $f(x)$

The function $f(x)$ is continuous on $-\pi \leq x \leq \pi$. Also $f(-\pi) = f(\pi) = 0$. We now need to show that $f'(x)$ is piecewise continuous. But

$$f'(x) = \begin{cases} 0 & -\pi \leq x \leq 0 \\ \cos x & 0 < x \leq \pi \end{cases} \tag{1}$$

Therefore $f'(x)$ exist and is piecewise continuous on $-\pi < x < \pi$. From the above, we see that $f(x)$ meets the 3 conditions in theorem of section 17, hence we know that the Fourier series of $f(x)$ is absolutely and uniformly convergent. (Here we need to use the M test to confirm this).

The Fourier series of $f(x)$ is

$$\frac{a_0}{2} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{4n^2-1}$$

Now, to apply the M test, consider the two series

$$\sum_{n=1}^{\infty}\overbrace{\frac{\cos(2nx)}{4n^2-1}}^{f_n}, \sum_{n=1}^{\infty}\overbrace{\frac{1}{4n^2-1}}^{M_n}$$

To show Fourier series is uniformly convergent to $f(x)$, using the M test, then we need to show that $|f_n| \leq M_n$ for each n. The series M_n qualifies to use for the Weierstrass series, since each term in it is positive constant and it is convergent series. To show that M_n is convergent, we can compare it to $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Since each term $\frac{1}{4n^2-1} < \frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent since any $\sum_{n=1}^{\infty}\frac{1}{n^s}$ for $s > 1$ is convergent (we can show this if needed using the integral test). Hence we can go ahead and use M_n series. Now we just need to show that

$$\left|\frac{\cos(2nx)}{4n^2-1}\right| \leq \frac{1}{4n^2-1}$$

For each n. But $\cos(2nx) \leq 1$ for each n. Hence the above is true for each n and it follows that the above Fourier series is indeed uniformly convergent to $f(x)$.

From (1), At $x = 0$ we have

$$f'_+(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x\to 0^+} \frac{\sin(x)}{x} = 1$$

And

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x\to 0^+} \frac{0}{x} = 0$$

Since $f'_+(0) \neq f'_-(0)$ then $f(x)$ is not differentiable at $x = 0$. This is plot of $f'(x)$ and we see graphically that due to jump discontinuity, that $f'(x)$ is not differentiable at $x = 0$

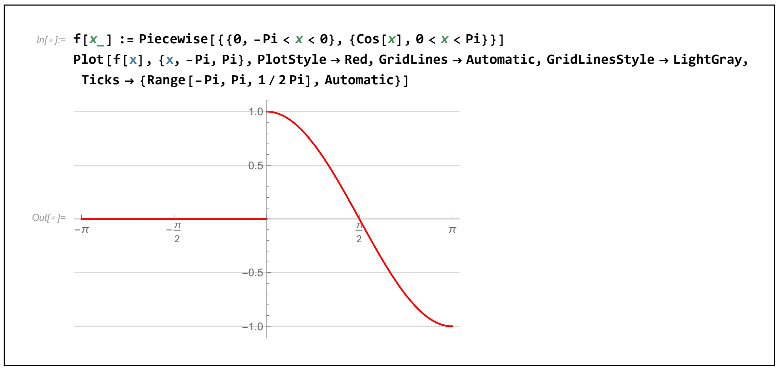


Figure 2.37: Plot of $f'(x)$ shown for one period

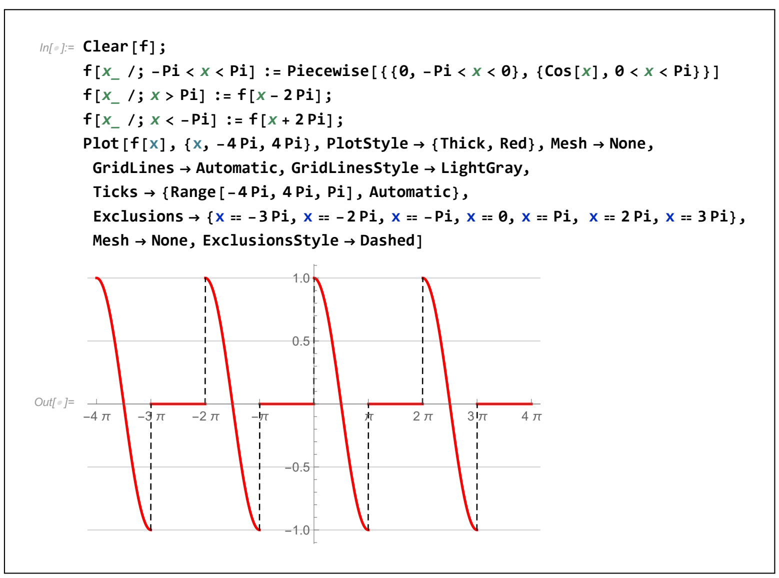


Figure 2.38: Plot of $f'(x)$ for all x, shown for 3 periods

2.3.2 Section 20, Problem 2

2. We know from Example 1, Sec. 3, that the series

$$\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

is the Fourier cosine series for the function $f(x)=x$ on the interval $0<x<\pi$. Differentiate this series term by term to obtain a representation for the derivative $f'(x)=1$ on that interval. State why the procedure is reliable here.

Figure 2.39: Problem statement

Solution

After doing an even extension of $f(x)=x$ on $0<x<\pi$ to $-\pi\le x\le\pi$, we see that $f(x)$ satisfies the conditions of Theorem section 20 for differentiating the Fourier series term by term. Since

1. $f(x)$ is continuous on the interval $-\pi\le x\le\pi$
2. $f(-\pi)=f(\pi)$
3. $f'(x)$ is piecewise continuous on $-\pi<x<\pi$

The only point that $f(x)$ is not differentiable is $x=0$ which implies $f'(x)$ is piecewise continuous. But that is OK. It is $f(x)$ which must be continuous. Hence differentiating the series term by term to obtain representation of $f(x)$ on $0<x<\pi$ is reliable.

2.3.3 Section 20, Problem 5

5. Integrate from $s = 0$ to $s = x$ $(-\pi \leq x \leq \pi)$ the Fourier series

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin ns$$

in Example 1, Sec. 19, and the one

$$2\sum_{n=1}^{\infty}\frac{\sin(2n-1)s}{2n-1}$$

appearing in Sec. 18. In each case, describe graphically the function that is represented by the new series.

Figure 2.40: Problem statement

Part 1

$$S = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(ns)$$

The above is the Fourier sine series for $f(x) = x$, on $0 < x < \pi$. Integrating gives

$$\int_0^x\left(2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(ns)\right)ds = 2\sum_{n=1}^{\infty}\int_0^x\frac{(-1)^{n+1}}{n}\sin(ns)\,ds$$

We did integration term by term, since that is always allowed (not like with differentiation term by term, where we have to check). Hence the above becomes

$$\begin{aligned}
2\sum_{n=1}^{\infty}\int_0^x\frac{(-1)^{n+1}}{n}\sin(ns)\,ds &= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\int_0^x\sin(ns)\,ds\right)\\
&= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(-\frac{\cos ns}{n}\right)_0^x\\
&= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+2}}{n^2}(\cos ns)_0^x
\end{aligned}$$

But $(-1)^{n+2} = (-1)^n$ and the above becomes

$$2\sum_{n=1}^{\infty}\int_0^x\frac{(-1)^{n+1}}{n}\sin(ns)\,ds = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(\cos nx - 1)$$

But $\int_0^x s\,ds = \frac{1}{2}x^2$. So the above is the Fourier series of $\frac{1}{2}x^2$. A plot of the above is

```
f[x_] := 1/2 x^2;
Plot[f[x], {x, 0, Pi}, PlotStyle → Red,
 GridLines → Automatic,
 GridLinesStyle → LightGray,
 Ticks → {Range[0, Pi, 1/2 Pi], Automatic}]
```

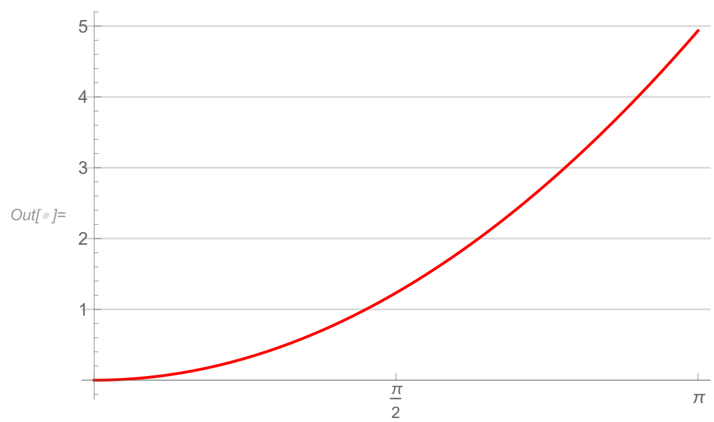


Figure 2.41: The function represented by the above series $f(x) = \frac{1}{2}x^2$

Part 2

$$S = 2\sum_{n=1}^{\infty} \frac{\sin((2n-1)s)}{2n-1}$$

The above is the Fourier sine series for $f(x) = \frac{\pi}{2}$, on $0 < x < \pi$. Integrating gives

$$\int_0^x \left(2\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)s)\right) ds = 2\sum_{n=1}^{\infty} \int_0^x \frac{1}{2n-1} \sin((2n-1)s)\, ds$$

We did integration term by term, since that is always allowed (not like with differentiation term by term, where we have to check). Hence the above becomes

$$\begin{aligned} 2\sum_{n=1}^{\infty} \int_0^x \frac{1}{2n-1} \sin((2n-1)s)\, ds &= 2\sum_{n=1}^{\infty} \frac{1}{2n-1} \int_0^x \sin((2n-1)s)\, ds \\ &= 2\sum_{n=1}^{\infty} \frac{1}{2n-1} \left(\frac{-\cos(2n-1)s}{(2n-1)}\right)_0^x \\ &= 2\sum_{n=1}^{\infty} -\frac{(\cos((2n-1)x)-1)}{(2n-1)^2} \end{aligned}$$

Since $\int_0^x \frac{\pi}{2} ds = \frac{\pi}{2}x$, then the above is the representation of this function. Here is a plot to confirm this, showing the above series expansion as more terms are added, showing it converges to $\frac{\pi}{2}x$

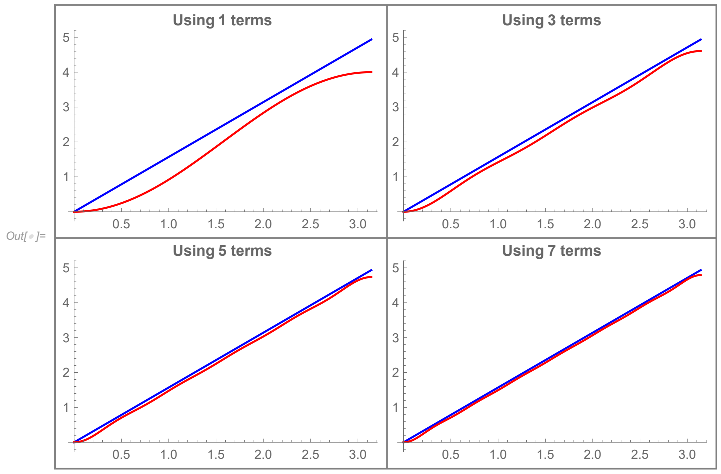


Figure 2.42: The function represented by the above series $f(x) = \frac{\pi}{2}x$ against its Fourier series

```
fApprox[x_, nTerms_] := 2 Sum[-(Cos[(2 n - 1) x] - 1)/(2 n - 1)^2, {n, 1, nTerms}];
Clear[f];
f[x_ /; 0 < x < Pi] := x*Pi/2;
Grid[Partition[Table[Plot[{f[x], fApprox[x, n]}, {x, 0, Pi},
    PlotStyle → {Blue, Red},
    PlotLabel → Style[Row[{"Using ", n, " terms"}], Bold],
    ImageSize → 250],
   {n, 1, 10, 2}], 2], Frame → All, FrameStyle → Gray]
```

Figure 2.43: Code used to plot the above

2.3.4 Section 27, Problem 1

1. Let $u(x)$ denote the steady-state temperatures in a slab bounded by the planes $x = 0$ and $x = c$ when those faces are kept at fixed temperatures $u = 0$ and $u = u_0$, respectively. Set up the boundary value problem for $u(x)$ and solve it to show that

$$u(x) = \frac{u_0}{c}x \quad \text{and} \quad \Phi_0 = K\frac{u_0}{c},$$

where Φ_0 is the flux of heat to the left across each plane $x = x_0$ $(0 \le x_0 \le c)$.

Figure 2.44: Problem statement

The heat PDE is $u_t = u_{xx}$. At steady state, $u_t = 0$ leading to $u_{xx} = 0$. So at steady state, the solution depends on x only. This has the solution

$$u(x) = Ax + B \tag{1}$$

With boundary conditions

$$u(0) = 0$$
$$u(c) = u_0$$

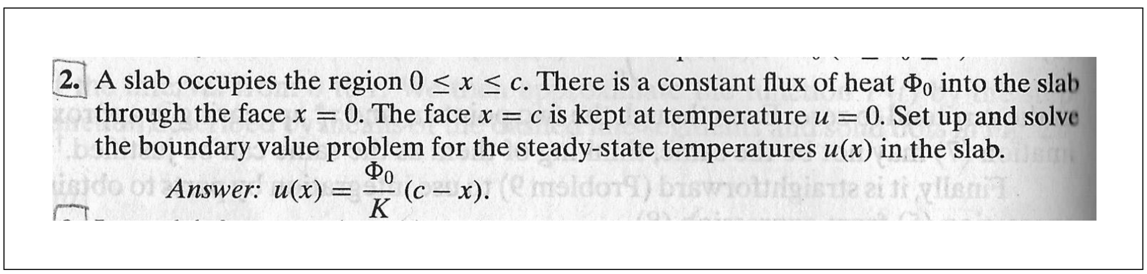
When $x = 0$ then $0 = B$. Hence the solution becomes $u(x) = Ax$. To find A, we apply the second boundary conditions. At $x = c$ this gives $u_0 = cA$ or $A = \frac{u_0}{c}$. Hence the solution (1) now becomes

$$u(x) = \frac{u_0}{c} x$$

Now the flux is defined as $\Phi_0 = K\frac{du}{dx}$ at each edge surface. But $\frac{du}{dx} = \frac{u_0}{c}$ from above. Therefore

$$\Phi_0 = K\frac{u_0}{c}$$

2.3.5 Section 27, Problem 2


2. A slab occupies the region $0 \le x \le c$. There is a constant flux of heat Φ_0 into the slab through the face $x = 0$. The face $x = c$ is kept at temperature $u = 0$. Set up and solve the boundary value problem for the steady-state temperatures $u(x)$ in the slab.

Answer: $u(x) = \frac{\Phi_0}{K}(c - x)$.

Figure 2.45: Problem statement

note: When looking for solution, assume it is a function of x only.

The heat PDE is $u_t = u_{xx}$. At steady state, $u_t = 0$ leading to $u_{xx} = 0$. So at steady state, the solution depends on x only. This has the solution

$$u(x) = Ax + B \tag{1}$$

Since there is constant flux at $x = 0$, then this means $K \left.\frac{du}{dx}\right|_{x=0} = -\Phi_0$. The reason for the minus sign, is that flux is always pointing to the outside of the surface. Hence on the left surface, it will be in the negative x direction and on the right side, it will be on the positive x direction.

Using this, the boundary conditions can be written as

$$\left.\frac{du}{dx}\right|_{x=0} = -K\Phi_0$$
$$u(c) = 0$$

Applying the left boundary condition gives

$$A = -K\Phi_0$$

Hence the solution becomes $u(x) = -K\Phi_0 x + B$.

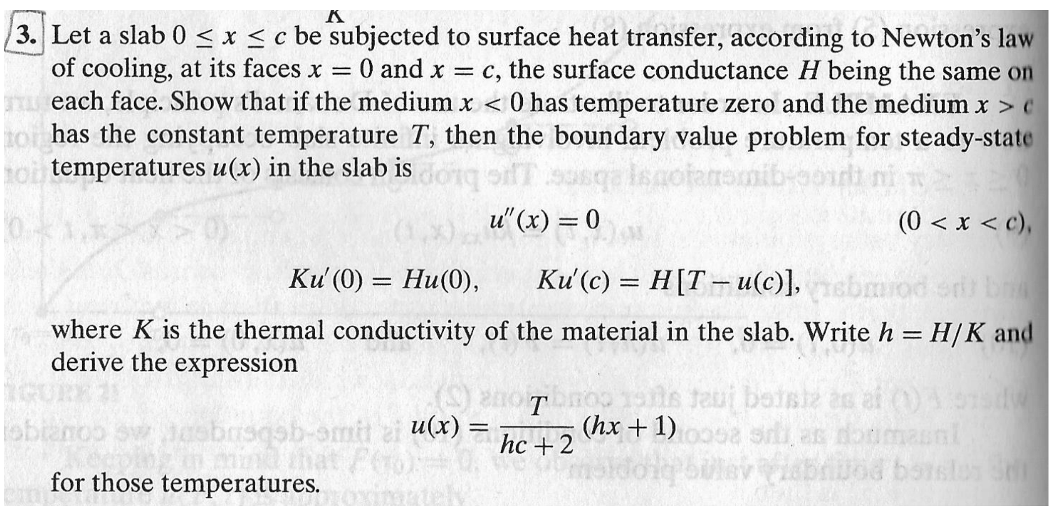
At $x = c$ the second B.C. leads to $0 = -K\Phi_0 c + B$ or

$$B = K\Phi_0 c$$

Hence the solution (1) becomes

$$\begin{aligned} u(x) &= -K\Phi_0 x + K\Phi_0 c \\ &= K\Phi_0 (c - x) \end{aligned}$$

2.3.6 Section 27, Problem 3



3. Let a slab $0 \le x \le c$ be subjected to surface heat transfer, according to Newton's law of cooling, at its faces $x = 0$ and $x = c$, the surface conductance H being the same on each face. Show that if the medium $x < 0$ has temperature zero and the medium $x > c$ has the constant temperature T, then the boundary value problem for steady-state temperatures $u(x)$ in the slab is

$$u''(x) = 0 \qquad (0 < x < c),$$

$$Ku'(0) = Hu(0), \qquad Ku'(c) = H[T - u(c)],$$

where K is the thermal conductivity of the material in the slab. Write $h = H/K$ and derive the expression

$$u(x) = \frac{T}{hc+2}(hx+1)$$

for those temperatures.

Figure 2.46: Problem statement

We start with

$$\Phi = H(T_{\text{outside}} - u) \tag{1}$$

Where T is the temperature on the outside and u is the temperature on the surface and Φ is the flux at the surface and H is surface conductance. Let us look at the left surface, at $x = 0$. The flux there is negative, since it points to the negative x direction. Therefore

$$\Phi = -K \left.\frac{du}{dx}\right|_{x=0} \tag{2}$$

From (1,2) we obtain

$$-K \left.\frac{du}{dx}\right|_{x=0} = H(T_{\text{outside}} - u(0))$$

But $T_{\text{outside}} = 0$ outside the left surface and the above becomes

$$-K \left.\frac{du}{dx}\right|_{x=0} = H(0 - u(0))$$

The minus signs cancel, giving

$$\left.\frac{du}{dx}\right|_{x=0} = \frac{H}{K}u(0)$$

$$u'(0) = hu(0) \tag{3}$$

Now, let us look at the right side. There the flux is positive. Hence at $x = c$ we have

$$K \left.\frac{du}{dx}\right|_{x=c} = H(T_{\text{outside}} - u(c))$$

But $T_{\text{outside}} = T$ on the right side. Hence the above reduces to

$$\left.\frac{du}{dx}\right|_{x=c} = \frac{H}{K}(T - u(c))$$

$$u'(c) = h(T - u(c)) \tag{4}$$

Now that we found the boundary conditions, we look at the solution. As before, at steady state we have

$$u''(x) = 0$$

$$u(x) = Ax + B \tag{5}$$

Hence $u'(x) = A$. Therefore

$$u'(0) = A = hu(0) \tag{6}$$

$$u'(c) = A = h(T - u(c)) \tag{7}$$

But we also know that, from (5) that

$$u(0) = B \tag{8}$$

$$u(c) = Ac + B \tag{9}$$

Substituting (8,9) into (6,7) in order to eliminate $u(0), u(c)$ from (6,7) gives

$$A = hB \tag{6A}$$

$$A = h(T - (Ac + B)) \tag{7A}$$

Now from (6A,7A) we solve for A, B. Substituting (7A) into (6A) gives

$$\begin{aligned}
hB &= h(T - (hBc + B)) \\
hB &= hT - h^2Bc - hB \\
2hB + h^2Bc &= hT \\
B &= \frac{hT}{h(2 + hc)} \\
&= \frac{T}{2 + hc}
\end{aligned}$$

Hence

$$A = hB$$
$$= \frac{hT}{2+hc}$$

Now that we found A, B then since $u(x) = Ax + B$, then

$$u(x) = \frac{hT}{2+hc}x + \frac{T}{2+hc}$$
$$= \frac{hTx + T}{2+hc}$$
$$= \frac{T}{2+hc}(1+hx)$$

Which is the result we are asked to show.

2.3.7 Section 27, Problem 7

6. A slender wire lies along the x axis, and surface heat transfer takes place along the wire into the surrounding medium at a fixed temperature T. Modify the procedure in Sec. 22 to show that if $u = u(x,t)$ denotes temperatures in the wire, then

$$u_t = ku_{xx} + b(T - u),$$

where b is a positive constant.

Suggestion: Let r denote the radius of the wire, and apply Newton's law of cooling to see that the quantity of heat entering the element in Fig. 22 through its cylindrical surface per unit time is approximately $H[T - u(x,t)]2\pi r\,\Delta x$.

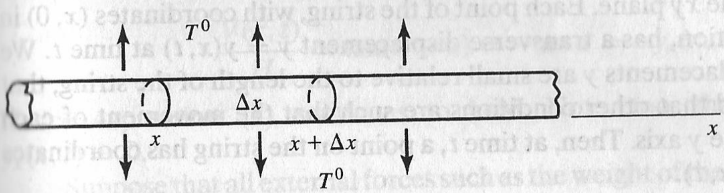


FIGURE 22

7. Show that the special case

$$u_t = ku_{xx} - bu$$

of the differential equation derived in Problem 6 can be transformed into the one-dimensional heat equation (Sec. 22)

$$v_t = kv_{xx}$$

with the substitution $u(x,t) = e^{-bt}v(x,t)$.

Figure 2.47: Problem statement

$$u_t = ku_{xx} - bu \tag{1}$$

Let $u(x,t) = e^{-bt}v(x,t)$ then

$$\begin{aligned} u_t &= -be^{-bt}v + e^{-bt}v_t \\ u_x &= e^{-bt}v_x \\ u_{xx} &= e^{-bt}v_{xx} \end{aligned}$$

Substituting the above back into (1) gives

$$-be^{-bt}v + e^{-bt}v_t = ke^{-bt}v_{xx} - be^{-bt}v$$

Since $e^{-bt} \neq 0$, then the above simplifies to

$$\begin{aligned} -bv + v_t &= kv_{xx} - bv \\ v_t &= kv_{xx} \end{aligned}$$

QED.

2.4 HW 4

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2.4.1 Section 27, Problem 8

8. Suppose that temperatures u in a solid hemisphere $r \le 1, 0 \le \theta \le \pi/2$ are independent of the spherical coordinate ϕ, so that $u = u(r,\theta)$, and that the base of the hemisphere is insulated (Fig. 23). Use transformation (13), Sec. 25, which relates spherical and cylindrical coordinates, to show that

$$\frac{\partial u}{\partial \theta} = -\rho\frac{\partial u}{\partial z} + z\frac{\partial u}{\partial \rho}.$$

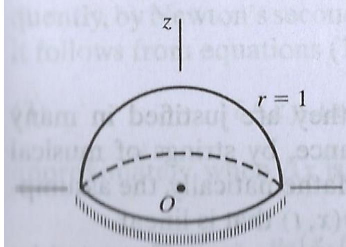


FIGURE 23

Thus show that u must satisfy the boundary condition

$$u_\theta\left(r, \frac{\pi}{2}\right) = 0.$$

Figure 2.48: Problem statement

<u>Solution</u>

The cylindrical and spherical coordinates are defined as given in the textbook figures shown below

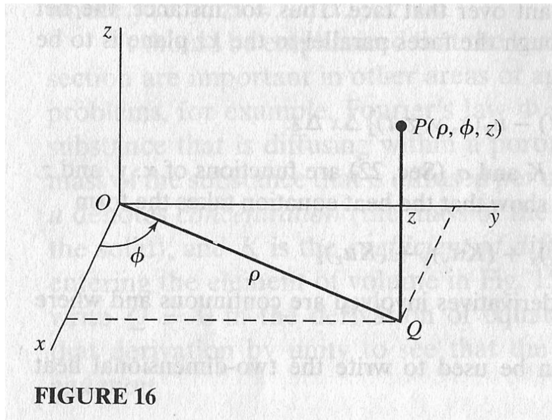


Figure 2.49: Cylinderical coordinates

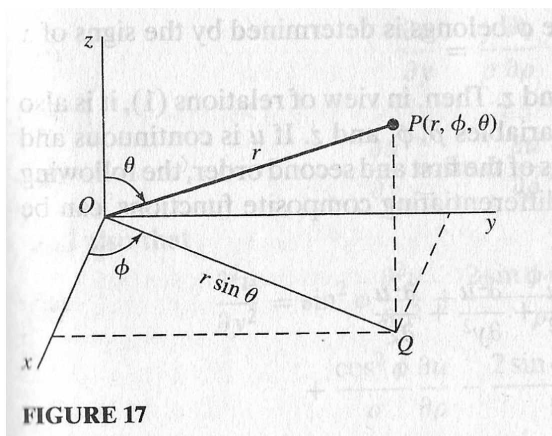


Figure 2.50: Spherical coordinates

The relation between these is given by (13) in the book

$$z = r\cos\theta \tag{1}$$

$$\rho = r\sin\theta \tag{2}$$

$$\phi = \phi \tag{3}$$

To obtain the required formula, we will use the chain rule. Since in spherical we have $u \equiv u(r,\theta)$ and in cylindrical we have $u \equiv u\left(\rho, z\right)$, then by chain rule

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta}$$

But from (2) $\frac{\partial \rho}{\partial \theta} = r\cos\theta$ and from (1) $\frac{\partial z}{\partial \theta} = -r\sin\theta$, hence the above becomes

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial \rho}(r\cos\theta) + \frac{\partial u}{\partial z}(-r\sin\theta)$$

But $r\cos\theta = z$ and $-r\sin\theta = \rho$, hence the above simplifies to

$$\frac{\partial u}{\partial \theta} = z\frac{\partial u}{\partial \rho} - \rho\frac{\partial u}{\partial z} \tag{4}$$

Which is the result required to show. Now we need to show that $\frac{\partial u}{\partial \theta}$ evaluated at boundary $r = 1, \theta = \frac{\pi}{2}$ is zero. But $\theta = \frac{\pi}{2}$ implies that $z = 0$, since $z = r\cos\theta$. Hence (4) now reduces to

$$\frac{\partial u}{\partial \theta} = -\rho\frac{\partial u}{\partial z} \tag{4}$$

Since $\theta = \frac{\pi}{2}$, then $\frac{\partial u}{\partial z}$ is the directional derivative normal to the base surface. But we are told it is insulated. This implies that $\frac{\partial u}{\partial z} = 0$, since by definition this is what insulated means. Therefore $\frac{\partial u}{\partial \theta} = 0$ at $r = 1, \theta = \frac{\pi}{2}$, which is what we are asked to show.

2.4.2 Section 28, Problem 1

1. A stretched string, with its ends fixed at the points 0 and $2c$ on the x axis, hangs at rest under its own weight. The y axis is directed vertically upward. Point out how it follows from the nonhomogeneous wave equation (6), Sec. 28, that the static displacements $y(x)$ of points on the string must satisfy the differential equation

$$a^2 y''(x) = g \qquad \left(a^2 = \frac{H}{\delta}\right)$$

on the interval $0 < x < 2c$, in addition to the boundary conditions

$$y(0) = 0, \qquad y(2c) = 0.$$

By solving this boundary value problem, show that the string hangs in the parabolic arc

$$(x - c)^2 = \frac{2a^2}{g}\left(y + \frac{gc^2}{2a^2}\right) \qquad (0 \le x \le 2c)$$

and that the depth of the vertex of the arc varies directly with c^2 and δ and inversely with H.

Figure 2.51: Problem statement

Eq (6) in section 28 is

$$y_{tt}(x,t) = a^2 y_{xx}(x,t) - g$$

At static displacement, by definition, there is no time dependency, hence $y_{tt} = 0$ and the above becomes

$$0 = a^2 y_{xx}(x,t) - g$$

Therefore now this becomes an ODE instead of a PDE since it does not depend on time, and we can write the above as

$$a^2 y''(x) = g \tag{1}$$

The boundary conditions $y(0,t) = 0$ and $y(2x,t) = 0$ now become $y(0) = 0, y(2x) = 0$. Now we need to solve (1) with these boundary conditions. This is an boundary value ODE.

$$y''(x) = \frac{g}{a^2}$$

The RHS is constant. The solution to the homogeneous ODE $y'' = 0$ is $y_h = Ax + B$. Let the particular solution be $y_p = C_3 x^2$, then $y_p' = 2C_3 x$ and $y_p'' = 2C_3$. Substituting this in the above ODE gives

$$\begin{aligned} 2C_3 &= \frac{g}{a^2} \\ C_3 &= \frac{g}{2a^2} \end{aligned}$$

Hence $y_p(x) = \frac{g}{2a^2}x^2$. Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= Ax + B + \frac{g}{2a^2}x^2 \end{aligned} \tag{2}$$

Now we will use the boundary conditions to find A, B above. At $x = 0$, (2) becomes

$$0 = B$$

Hence solution (2) reduces to

$$y(x) = Ax + \frac{g}{2a^2}x^2 \tag{3}$$

At $x = 2c$, the second boundary condition gives

$$\begin{aligned} 0 &= 2cA + \frac{g}{2a^2}\left(4c^2\right) \\ A &= \frac{-g}{2a^2}\frac{\left(4c^2\right)}{2c} \\ &= \frac{-gc}{a^2} \end{aligned}$$

Hence the solution (3) becomes

$$\begin{aligned} y &= \frac{-gc}{a^2}x + \frac{g}{2a^2}x^2 \\ y &= \frac{gx^2 - 2gcx}{2a^2} \end{aligned} \tag{4}$$

To get the result needed, we can manipulate this more as follows. From (4)

$$\begin{aligned} 2a^2y &= gx^2 - 2gcx \\ &= g\left(x^2 - 2cx\right) \\ &= g\left(x-c\right)^2 - gc^2 \end{aligned}$$

Hence

$$\begin{aligned} g\left(x-c\right)^2 &= 2a^2y + gc^2 \\ \left(x-c\right)^2 &= \frac{2a^2y}{g} + c^2 \\ &= \frac{2a^2}{g}\left(y + \frac{gc^2}{2a^2}\right) \end{aligned}$$

Now since $a^2 = \frac{H}{\delta}$ then the above becomes

$$\begin{aligned} \frac{g}{2a^2}\left(x-c\right)^2 &= y + \frac{gc^2}{2a^2} \\ y &= \frac{1}{2a^2}\left(g\left(x-c\right)^2 - gc^2\right) \\ &= \frac{g}{2\frac{H}{\delta}}\left(\left(x-c\right)^2 - c^2\right) \\ &= \frac{\delta}{H}\frac{g}{2}\left(\left(x-c\right)^2 - c^2\right) \end{aligned}$$

We see now that y is directly proportional to δ and c^2 and inversely proportional to H.

2.4.3 Section 28, Problem 5

5. A strand of wire 1 ft long, stretched between the origin and the point 1 on the x axis, weighs 0.032 lb ($\delta g = 0.032$, $g = 32$ ft/s^2) and $H = 10$ lb. At the instant $t = 0$, the strand lies along the x axis but has a velocity of 1 ft/s in the direction of the y axis, perhaps because the supports were in motion and were brought to rest at that instant. Assuming that no external forces act along the wire, state why the displacements $y(x,t)$ should satisfy this boundary value problem:

$$y_{tt}(x,t) = 10^4 y_{xx}(x,t) \qquad (0 < x < 1, t > 0),$$

$$y(0,t) = y(1,t) = 0, \qquad y(x,0) = 0, \qquad y_t(x,0) = 1.$$

Figure 2.52: Problem statement

solution

The wave PDE in 1D is given by

$$y_{tt}(x,t) = a^2 y_{xx}(x,t) \tag{1}$$

Where

$$a^2 = \frac{H}{\delta}$$

Where H is the tension in the strand and δ is the mass per unit length of the strand. But $weight = (mass)\,g$. hence $\delta = \frac{weight}{g}$. We are given that $weight = 0.032$ lb, and that $g = 32$ ft/s^2. This implies that

$$\delta = \frac{0.032}{32} = \frac{1}{1000}$$

Hence

$$a^2 = \frac{10}{\frac{1}{1000}} = 10^4$$

Therefore (1) becomes

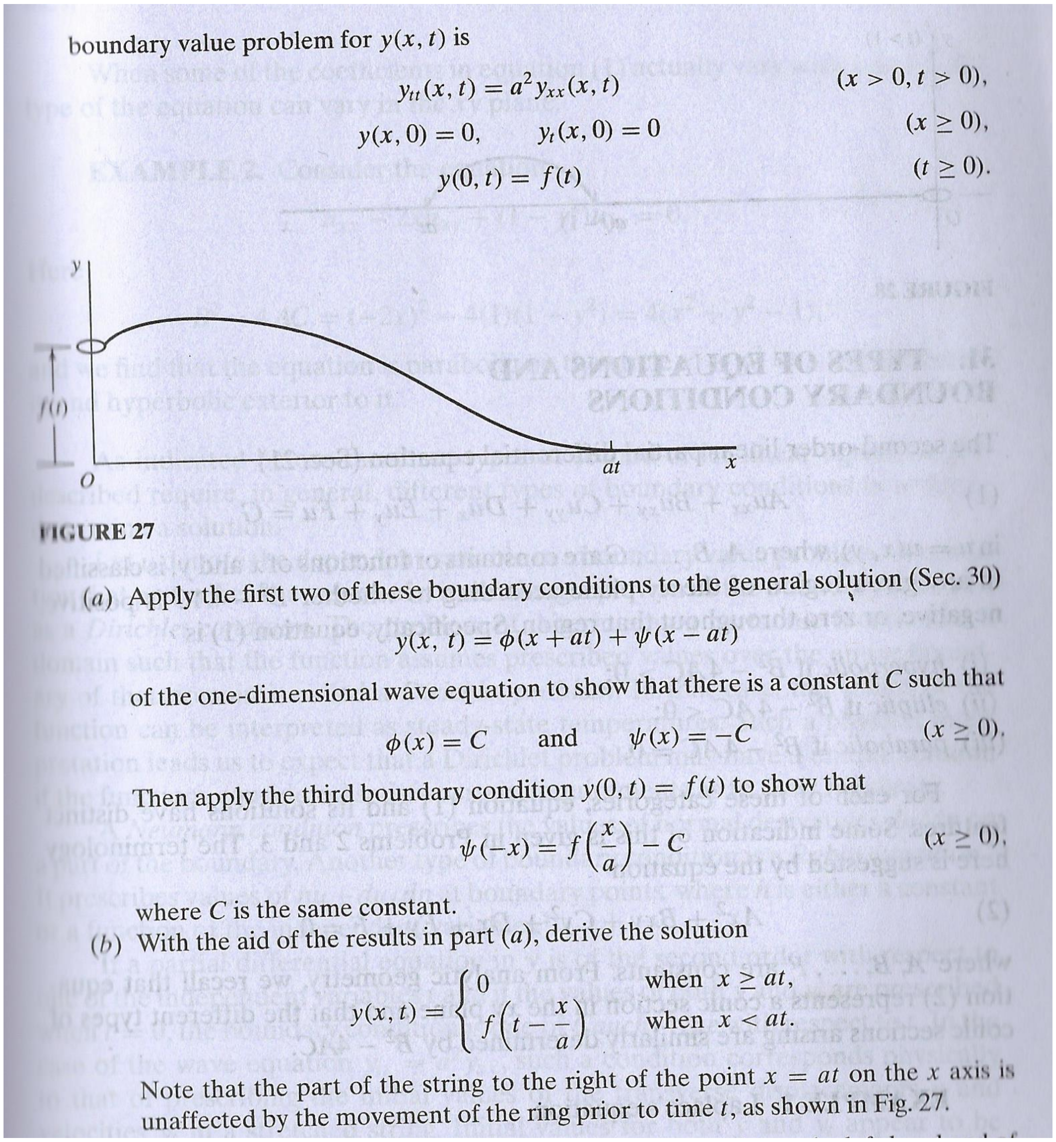
$$y_{tt}(x,t) = 10^4 y_{xx}(x,t) \tag{2}$$

Since at $t = 0$ we are told that strand lies along the $x - axis$, then $y(x,0) = 0$ and problem says $y_t(x,0) = 1$. For boundary conditions, since strand fixed at $x = 0$ and $x = 1$, then this implies $y(0,t) = 0$ and $y(1,t) = 0$. Therefore the PDE is

$$\begin{aligned}
y_{tt}(x,t) &= 10^4 y_{xx}(x,t) \qquad 0 < x < 1, t > 0\\
y(x,0) &= 0\\
y_t(x,0) &= 1\\
y(0,t) &= 0\\
y(1,t) &= 0
\end{aligned}$$

2.4.4 Section 30, Problem 3

3. Let $y(x,t)$ represent transverse displacements in a long stretched string one end of which is attached to a ring that can slide along the y axis. The other end is so far out on the positive x axis that it may be considered to be infinitely far from the origin. The ring is initially at the origin and is then moved along the y axis (Fig. 27) so that $y = f(t)$ when $x = 0$ and $t \geq 0$, where f is a prescribed continuous function and $f(0) = 0$. We assume that the string is initially at rest on the x axis; thus $y(x,t) \to 0$ as $x \to \infty$. The



boundary value problem for $y(x,t)$ is

$$y_{tt}(x,t) = a^2 y_{xx}(x,t) \qquad (x>0, t>0),$$
$$y(x,0) = 0, \qquad y_t(x,0) = 0 \qquad (x \geq 0),$$
$$y(0,t) = f(t) \qquad (t \geq 0).$$

FIGURE 27

(*a*) Apply the first two of these boundary conditions to the general solution (Sec. 30)

$$y(x,t) = \phi(x+at) + \psi(x-at)$$

of the one-dimensional wave equation to show that there is a constant C such that

$$\phi(x) = C \quad \text{and} \quad \psi(x) = -C \qquad (x \geq 0).$$

Then apply the third boundary condition $y(0,t) = f(t)$ to show that

$$\psi(-x) = f\left(\frac{x}{a}\right) - C \qquad (x \geq 0),$$

where C is the same constant.

(*b*) With the aid of the results in part (*a*), derive the solution

$$y(x,t) = \begin{cases} 0 & \text{when } x \geq at, \\ f\left(t - \frac{x}{a}\right) & \text{when } x < at. \end{cases}$$

Note that the part of the string to the right of the point $x = at$ on the x axis is unaffected by the movement of the ring prior to time t, as shown in Fig. 27.

Figure 2.53: Problem statement

Part a

Applying the first initial conditions $y(x,0) = 0$ to the solution

$$y(x,t) = \phi(x+at) + \psi(x-at) \tag{1}$$

Gives

$$0 = \phi(x) + \psi(x) \tag{2}$$

But $y_t = a\phi' - a\psi'$. Hence the second initial conditions at $t = 0$ gives

$$0 = a\phi'(x) - a\psi'(x) \tag{3}$$

Taking derivative of (2) and multiplying the resulting equation by a gives

$$0 = a\phi'(x) + a\psi'(x) \tag{2A}$$

Adding (3,2A) gives

$$\begin{aligned} 2a\phi'(x) &= 0 \\ \phi'(x) &= 0 \end{aligned}$$

Therefore

$$\phi(x) = C \tag{4}$$

Where C is an arbitrary constant. Substituting the above result back in (2) gives

$$0 = C + \psi(x)$$

$$\psi(x) = -C \tag{5}$$

From (4,5) we see that

$$\begin{aligned} \phi(x) &= C \\ \psi(x) &= -C \end{aligned}$$

Now applying boundary condition $y(0,t) = f(t)$ to (1) gives

$$f(t) = \phi(at) + \psi(-at)$$

But a is the speed of the wave given by $a = \frac{x}{t}$ or $t = \frac{x}{a}$. Hence the above becomes

$$\begin{aligned} f\left(\frac{x}{a}\right) &= \phi(x) + \psi(-x) \\ \psi(-x) &= f\left(\frac{x}{a}\right) - \phi(x) \end{aligned}$$

Since $\phi(x) = C$ from equation (4), then the final result is obtained

$$\psi(-x) = f\left(\frac{x}{a}\right) - C \qquad x \geq 0 \tag{6}$$

Part b

Since the part to the right of $x = at$ is unaffected by the movement of the right, then

$$y(x,t) = 0 \qquad x \geq at \tag{1}$$

So now we need to find the solution for $x < at$ and $x \geq 0$. From

$$y(x,t) = \phi(x+at) + \psi(x-at)$$

And using (6) in part (a), we see that $\psi(x-at) = f\left(\frac{-(x-at)}{a}\right) - C$. Therefore the above becomes

$$y(x,t) = \phi(x+at) + f\left(\frac{-(x-at)}{a}\right) - C$$

But also from part (a) $\phi(x+at)=C$. Hence the above simplifies to

$$\begin{aligned} y(x,t) &= c + f\left(\frac{-(x-at)}{a}\right) - C \\ &= f\left(\frac{-x+at}{a}\right) \\ &= f\left(t-\frac{x}{a}\right) \qquad x < at \end{aligned} \tag{2}$$

Combining (1) and (2) shows that

$$y(x,t) = \begin{cases} 0 & x \geq at \\ f\left(t-\frac{x}{a}\right) & x < at \end{cases}$$

2.4.5 Section 30, Problem 4

4. Use the solution obtained in Problem 3 to show that if the ring at the left-hand end of the string in that problem is moved according to the function

$$f(t) = \begin{cases} \sin \pi t & \text{when } 0 \leq t \leq 1, \\ 0 & \text{when } \quad t > 1, \end{cases}$$

then

$$y(x,t) = \begin{cases} 0 & \text{when } x \leq a(t-1) \text{ or } x \geq at, \\ \sin\left[\pi\left(t-\dfrac{x}{a}\right)\right] & \text{when } \quad a(t-1) < x < at. \end{cases}$$

Observe that the ring is lifted up 1 unit and then returned to the origin, where it remains after time $t=1$. The expression for $y(x,t)$ here shows that when $t>1$, the string coincides with the x axis except on an interval of length a, where it forms one arch of a sine curve (Fig. 28). Furthermore, as t increases, the arch moves to the right with speed a.

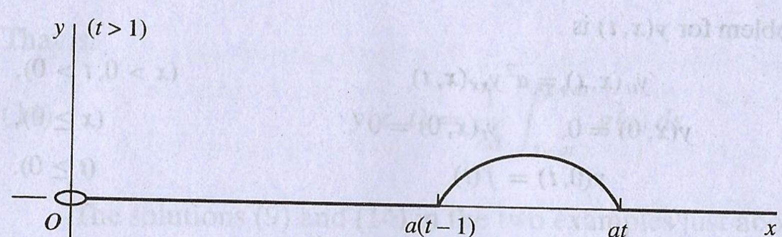


FIGURE 28

Figure 2.54: Problem statement

This requires just substitution of the function $f(t)$ given into the solution found above which is

$$y(x,t) = \begin{cases} 0 & x \geq at \\ f\left(t - \frac{x}{a}\right) & x < at \end{cases} \tag{1}$$

But

$$f(t) = \begin{cases} \sin \pi t & 0 \leq t \leq 1 \\ 0 & t > 1 \end{cases} \tag{2}$$

Substituting (2) into (1) gives, after replacing each t in (2) by $t - \frac{x}{a}$ the result needed

$$y(x,t) = \begin{cases} 0 & x \geq at \\ \sin\left(\pi\left(t - \frac{x}{a}\right)\right) & a(t-1) < x < at \end{cases}$$

2.4.6 Section 31, Problem 2

2. Consider the partial differential equation

$$Ay_{xx} + By_{xt} + Cy_{tt} = 0 \qquad (A \neq 0, C \neq 0),$$

where A, B, and C are constants, and assume that it is *hyperbolic*, so that $B^2 - 4AC > 0$.

(*a*) Use the transformation

$$u = x + \alpha t, \qquad v = x + \beta t \qquad (\alpha \neq \beta)$$

to obtain the new differential equation

$$(A + B\alpha + C\alpha^2)y_{uu} + [2A + B(\alpha + \beta) + 2C\alpha\beta]y_{uv} + (A + B\beta + C\beta^2)y_{vv} = 0.$$

(*b*) Show that when α and β have the values

$$\alpha_0 = \frac{-B + \sqrt{B^2 - 4AC}}{2C} \quad \text{and} \quad \beta_0 = \frac{-B - \sqrt{B^2 - 4AC}}{2C},$$

respectively, the differential equation in part (*a*) reduces to $y_{uv} = 0$.

(*c*) Conclude from the result in part (*b*) that the general solution of the original differential equation is

$$y = \phi(x + \alpha_0 t) + \psi(x + \beta_0 t),$$

where ϕ and ψ are arbitrary functions that are twice differentiable. Then show how the general solution (7), Sec. 30, of the wave equation

$$-a^2 y_{xx} + y_{tt} = 0$$

follows as a special case.

Figure 2.55: Problem Statement

Part a

We want to do the transformation from $y(x,t)$ to $y(u,v)$. Therefore

$$\frac{\partial y}{\partial x}=\frac{\partial y}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial y}{\partial v}\frac{\partial v}{\partial x}$$

But $\frac{\partial u}{\partial x}=1$ and $\frac{\partial v}{\partial x}=1$, hence the above becomes

$$\frac{\partial y}{\partial x}=\frac{\partial y}{\partial u}+\frac{\partial y}{\partial v}$$

And

$$\begin{aligned}\frac{\partial^2 y}{\partial x^2}&=\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\\&=\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u}+\frac{\partial y}{\partial v}\right)\\&=\frac{\partial}{\partial x}\frac{\partial y}{\partial u}+\frac{\partial}{\partial x}\frac{\partial y}{\partial v}\\&=\left(\frac{\partial^2 y}{\partial u^2}\frac{\partial u}{\partial x}+\frac{\partial^2 y}{\partial uv}\frac{\partial v}{\partial x}\right)+\left(\frac{\partial^2 y}{\partial v^2}\frac{\partial v}{\partial x}+\frac{\partial^2 y}{\partial vu}\frac{\partial u}{\partial x}\right)\end{aligned}$$

But $\frac{\partial u}{\partial x}=1, \frac{\partial v}{\partial x}=1$, hence the above becomes

$$\begin{aligned}\frac{\partial^2 y}{\partial x^2}&=\frac{\partial^2 y}{\partial u^2}+2\frac{\partial^2 y}{\partial uv}+\frac{\partial^2 y}{\partial v^2}\\y_{xx}&=y_{uu}+y_{vv}+2y_{uv}\end{aligned}\tag{1}$$

Similarly,

$$\frac{\partial y}{\partial t}=\frac{\partial y}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial y}{\partial v}\frac{\partial v}{\partial t}$$

But $\frac{\partial u}{\partial t}=\alpha$ and $\frac{\partial v}{\partial t}=\beta$, hence the above becomes

$$\frac{\partial y}{\partial t}=\alpha\frac{\partial y}{\partial u}+\beta\frac{\partial y}{\partial v}$$

And

$$
\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) \\
&= \frac{\partial}{\partial t}\left(\alpha\frac{\partial y}{\partial u} + \beta\frac{\partial y}{\partial v}\right) \\
&= \alpha\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial u}\right) + \beta\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial v}\right) \\
&= \alpha\left(\frac{\partial^2 y}{\partial u^2}\frac{\partial u}{\partial t} + \frac{\partial^2 y}{\partial uv}\frac{\partial v}{\partial t}\right) + \beta\left(\frac{\partial^2 y}{\partial v^2}\frac{\partial v}{\partial t} + \frac{\partial^2 y}{\partial uv}\frac{\partial u}{\partial t}\right)
\end{aligned}
$$

But $\frac{\partial u}{\partial t} = \alpha$ and $\frac{\partial v}{\partial t} = \beta$, hence the above becomes

$$
\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= \alpha\left(\alpha\frac{\partial^2 y}{\partial u^2} + \beta\frac{\partial^2 y}{\partial uv}\right) + \beta\left(\beta\frac{\partial^2 y}{\partial v^2} + \alpha\frac{\partial^2 y}{\partial uv}\right) \\
&= \alpha^2\frac{\partial^2 y}{\partial u^2} + \alpha\beta\frac{\partial^2 y}{\partial uv} + \beta^2\frac{\partial^2 y}{\partial v^2} + \alpha\beta\frac{\partial^2 y}{\partial uv} \\
y_{tt} &= \alpha^2 y_{uu} + \beta^2 y_{vv} + 2\alpha\beta y_{uv} \qquad (2)
\end{aligned}
$$

And to obtain y_{xt}, then starting from above result obtained

$$
\frac{\partial y}{\partial t} = \alpha\frac{\partial y}{\partial u} + \beta\frac{\partial y}{\partial v}
$$

Now taking partial derivative w.r.t. x gives

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial t}\right) &= \frac{\partial}{\partial x}\left(\alpha\frac{\partial y}{\partial u} + \beta\frac{\partial y}{\partial v}\right) \\
&= \alpha\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u}\right) + \beta\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial v}\right) \\
&= \alpha\left(\frac{\partial^2 y}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 y}{\partial uv}\frac{\partial v}{\partial x}\right) + \beta\left(\frac{\partial^2 y}{\partial v^2}\frac{\partial v}{\partial x} + \frac{\partial^2 y}{\partial uv}\frac{\partial u}{\partial x}\right)
\end{aligned}
$$

But $\frac{\partial u}{\partial x} = 1, \frac{\partial v}{\partial x} = 1$, hence the above becomes

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial t}\right) &= \alpha\left(\frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial uv}\right) + \beta\left(\frac{\partial^2 y}{\partial v^2} + \frac{\partial^2 y}{\partial uv}\right) \\
y_{xt} &= \alpha y_{uu} + \left(\alpha + \beta\right) y_{vu} + \beta y_{vv} \qquad (3)
\end{aligned}
$$

Substituting (1,2,3) into $Ay_{xx} + By_{xt} + Cy_{tt} = 0$ results in

$$
A\left(y_{uu} + y_{vv} + 2y_{uv}\right) + B\left(\alpha y_{uu} + \left(\alpha+\beta\right)y_{vu} + \beta y_{vv}\right) + C\left(\alpha^2 y_{uu} + \beta^2 y_{vv} + 2\alpha\beta y_{uv}\right) = 0
$$

Or

$$
y_{uu}\left(A + B\alpha + C\alpha^2\right) + y_{uv}\left(2A + B\left(\alpha+\beta\right) + 2C\alpha\beta\right) + y_{vv}\left(A + B\beta + C\beta^2\right) = 0
$$

Part b

Looking at the term above for y_{uu} we see it is $A+B\alpha+C\alpha^2$ which has the root

$$\begin{aligned}\alpha &= -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2-4ac}\\ &= -\frac{B}{2C} \pm \frac{1}{2C}\sqrt{B^2-4AC}\end{aligned}$$

Hence if we pick the root $\alpha=\alpha_0=-\frac{B}{2C}+\frac{1}{2C}\sqrt{B^2-4AC}$ then the term y_{uu} vanishes. Similarly for the term multiplied by y_{vv} which is $A+B\beta+C\beta^2$. The root is

$$\beta = -\frac{B}{2C} \pm \frac{1}{2C}\sqrt{B^2-4AC}$$

And if we pick $\beta=\beta_0=-\frac{B}{2C}-\frac{1}{2C}\sqrt{B^2-4AC}$ then the term y_{vv} vanishes also in the PDE obtained in part (a), and now the PDE becomes

$$y_{uv}\left(2A+B\left(\alpha+\beta\right)+2C\alpha\beta\right)=0$$

Substituting the above selected roots α_0,β_0 into the above in place of α,β since these are the values we picked, then the above becomes

$$\begin{aligned}y_{uv}\left(2A+B\left(-\frac{B}{2C}+\frac{1}{2C}\sqrt{B^2-4AC}-\frac{B}{2C}-\frac{1}{2C}\sqrt{B^2-4AC}\right)+2C\alpha\beta\right)&=0\\ y_{uv}\left(2A-\frac{2B^2}{2C}+2C\alpha\beta\right)&=0\end{aligned}$$

And again replacing $\alpha\beta$ above with α_0,β_0 results in

$$\begin{aligned}y_{uv}\left(2A-\frac{2B^2}{2C}+2C\left(-\frac{B}{2C}+\frac{1}{2C}\sqrt{B^2-4AC}\right)\left(-\frac{B}{2C}-\frac{1}{2C}\sqrt{B^2-4AC}\right)\right)&=0\\ y_{uv}\left(2A-\frac{2B^2}{2C}+2C\left(\frac{B^2}{4C^2}+\frac{1}{4C^2}\left(B^2-4AC\right)\right)\right)&=0\\ y_{uv}\left(2A-\frac{2B^2}{2C}+\frac{B^2}{2C}+\frac{1}{2C}\left(B^2-4AC\right)\right)&=0\\ y_{uv}\left(2A-\frac{2B^2}{2C}+\frac{B^2}{2C}+\frac{B^2}{2C}-2A\right)&=0\\ \frac{B^2}{2C}y_{uv}&=0\end{aligned}$$

Since $B\neq 0, C\neq 0$ then the above simplifies to

$$y_{uv}=0$$

Part c

Since

$$y_{uv}=0$$

Or

$$\frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u}\right)=0$$

The implies that

$$\frac{\partial y}{\partial u}=\Phi(u)$$

Integrating w.r.t. u gives

$$y(u,v)=\int\Phi(u)\,du+\psi(v)$$

Where $\psi(v)$ is the constant of integration which is a function.

Let $\int\Phi(u)\,du=\phi(u)$ then the above can be written as

$$y(u,v)=\phi(u)+\psi(v)$$

Or in terms of x,t, since $u=x+\alpha t$ and $v=x+\beta t$ the above solution becomes

$$y(x,t)=\phi(x+\alpha t)+\psi\left(x+\beta t\right)$$

Where ϕ,ψ are arbitrary functions twice differentiable. When $\alpha=+a,\beta=-a$, then the above becomes

$$y(x,t)=\phi(x+at)+\psi(x-at)$$

Which is the general solution (7) in section (30). QED

2.4.7 Section 31, Problem 3

3. Show that under the transformation

$$u=x,\qquad v=\alpha x+\beta t\qquad(\beta\neq 0),$$

the given differential equation in Problem 2 becomes

$$Ay_{uu}+(2A\alpha+B\beta)y_{uv}+(A\alpha^2+B\alpha\beta+C\beta^2)y_{vv}=0.$$

Then show that this new equation reduces to

(*a*) $y_{uu}+y_{vv}=0$ when the original equation is *elliptic* $(B^2-4AC<0)$ and

$$\alpha=\frac{-B}{\sqrt{4AC-B^2}},\qquad\beta=\frac{2A}{\sqrt{4AC-B^2}};$$

(*b*) $y_{uu}=0$ when the original equation is *parabolic* $(B^2-4AC=0)$ and

$$\alpha=-B,\qquad\beta=2A.$$

Figure 2.56: Problem Statement

The differential equation in problem 2 is

$$Ay_{xx} + By_{xt} + Cy_{tt} = 0$$

We want to do the transformation from $y(x,t)$ to $y(u,v)$ with

$$\begin{aligned} u &= x \\ v &= \alpha x + \beta t \end{aligned}$$

Now

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x}$$

But $\frac{\partial u}{\partial x} = 1$ and $\frac{\partial v}{\partial x} = \alpha$, hence the above becomes

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \alpha\frac{\partial y}{\partial v}$$

And

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t}$$

But $\frac{\partial u}{\partial t} = 0$ and $\frac{\partial v}{\partial t} = \beta$, hence the above becomes

$$\frac{\partial y}{\partial t} = \beta\frac{\partial y}{\partial v}$$

Therefore

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u} + \alpha\frac{\partial y}{\partial v}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u}\right) + \alpha\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial v}\right) \\ &= \left(\frac{\partial^2 y}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 y}{\partial uv}\frac{\partial v}{\partial x}\right) + \alpha\left(\frac{\partial^2 y}{\partial v^2}\frac{\partial v}{\partial x} + \frac{\partial^2 y}{\partial vu}\frac{\partial u}{\partial x}\right) \\ &= \left(\frac{\partial^2 y}{\partial u^2} + \alpha\frac{\partial^2 y}{\partial uv}\right) + \alpha\left(\alpha\frac{\partial^2 y}{\partial v^2} + \frac{\partial^2 y}{\partial vu}\right) \\ &= \frac{\partial^2 y}{\partial u^2} + \alpha\frac{\partial^2 y}{\partial uv} + \alpha^2\frac{\partial^2 y}{\partial v^2} + \alpha\frac{\partial^2 y}{\partial vu} \\ y_{xx} &= y_{uu} + \alpha^2 y_{vv} + 2\alpha y_{uv} \end{aligned} \tag{1}$$

Similarly,

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial t}\right) \\
&= \frac{\partial}{\partial x}\left(\beta\frac{\partial y}{\partial v}\right) \\
&= \beta\left(\frac{\partial^2 y}{\partial v^2}\frac{\partial v}{\partial t} + \frac{\partial^2 y}{\partial vu}\frac{\partial u}{\partial t}\right) \\
&= \beta\left(\beta\frac{\partial^2 y}{\partial v^2}\right) \\
y_{tt} &= \beta^2 y_{vv} \qquad (2)
\end{aligned}$$

And to obtain y_{xt}, then starting from above result obtained

$$\frac{\partial y}{\partial t} = \beta\frac{\partial y}{\partial v}$$

Now taking partial derivative w.r.t. x gives

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial t}\right) &= \frac{\partial}{\partial x}\left(\beta\frac{\partial y}{\partial v}\right) \\
&= \beta\left(\frac{\partial^2 y}{\partial v^2}\frac{\partial v}{\partial x} + \frac{\partial^2 y}{\partial vu}\frac{\partial u}{\partial x}\right) \\
&= \beta\left(\alpha\frac{\partial^2 y}{\partial v^2} + \frac{\partial^2 y}{\partial vu}\right) \\
y_{xt} &= \alpha\beta y_{vv} + \beta y_{vu} \qquad (3)
\end{aligned}$$

Substituting (1,2,3) into $Ay_{xx} + By_{xt} + Cy_{tt} = 0$ results in

$$A\left(y_{uu} + \alpha^2 y_{vv} + 2\alpha y_{uv}\right) + B\left(\alpha\beta y_{vv} + \beta y_{vu}\right) + C\left(\beta^2 y_{vv}\right) = 0$$

Or

$$Ay_{uu} + y_{uv}\left(2A\alpha + B\beta\right) + y_{vv}\left(A\alpha^2 + B\alpha\beta + C\beta^2\right) = 0 \qquad (4)$$

Which is what asked to show.

Part a

Setting $\alpha = \frac{-B}{\sqrt{4AC-B^2}}, \beta = \frac{2A}{\sqrt{4AC-B^2}}$ in (4) above results in

$$\begin{aligned}
Ay_{uu} + y_{uv}\left(2A\left(\frac{-B}{\sqrt{4AC-B^2}}\right) + B\left(\frac{2A}{\sqrt{4AC-B^2}}\right)\right) + y_{vv}\left(A\alpha^2 + B\alpha\beta + C\beta^2\right) &= 0 \\
Ay_{uu} + y_{vv}\left(A\alpha^2 + B\alpha\beta + C\beta^2\right) &= 0
\end{aligned}$$

And the above now becomes

$$
\begin{aligned}
Ay_{uu} + y_{vv}\left(A\left(\frac{-B}{\sqrt{4AC-B^2}}\right)^2 + B\left(\frac{-B}{\sqrt{4AC-B^2}}\right)\left(\frac{2A}{\sqrt{4AC-B^2}}\right) + C\left(\frac{2A}{\sqrt{4AC-B^2}}\right)^2\right) &= 0\\
Ay_{uu} + y_{vv}\left(\frac{AB^2}{4AC-B^2} - \frac{2B^2A}{4AC-B^2} + \frac{4CA^2}{4AC-B^2}\right) &= 0\\
Ay_{uu} + y_{vv}\left(\frac{AB^2-2B^2A+4CA^2}{4AC-B^2}\right) &= 0\\
Ay_{uu} + Ay_{vv}\left(\frac{-B^2+4CA}{4AC-B^2}\right) &= 0\\
Ay_{uu} + Ay_{vv} &= 0\\
A\left(y_{uu}+y_{vv}\right) &= 0
\end{aligned}
$$

Therefore, since $A \neq 0$ the above becomes

$$y_{uu} + y_{vv} = 0$$

Part b

Setting $\alpha = -B, \beta = 2A$ in (4) above results in

$$
\begin{aligned}
Ay_{uu} + y_{uv}(-2AB+2AB) + y_{vv}\left(AB^2-2B^2A+4CA^2\right) &= 0\\
Ay_{uu} + y_{vv}\left(4CA^2-B^2A\right) &= 0\\
Ay_{uu} - Ay_{vv}\left(B^2-4CA\right) &= 0
\end{aligned}
$$

But $B^2 - 4CA = 0$, therefore the above becomes

$$y_{uu} = 0$$

2.5 HW 5

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2.5.1 Section 34, Problem 3

3. Verify that each of the functions

$$u_0 = y, \qquad u_n = \sinh ny \cos nx \qquad (n = 1, 2, \ldots)$$

satisfies Laplace's equation (Sec. 23)

$$u_{xx}(x, y) + u_{yy}(x, y) = 0 \qquad (0 < x < \pi,\ 0 < y < 2)$$

and the boundary conditions

$$u_x(0, y) = u_x(\pi, y) = 0, \qquad u(x, 0) = 0.$$

Then use the superposition principle in Sec. 33 to show formally, without considering questions of convergence, differentiability, or continuity, that the series

$$u(x, y) = A_0\, y + \sum_{n=1}^{\infty} A_n \sinh ny \cos nx$$

satisfies the same differential equation and boundary conditions.

Figure 2.57: Problem statement

<u>Solution</u>

The boundary conditions are

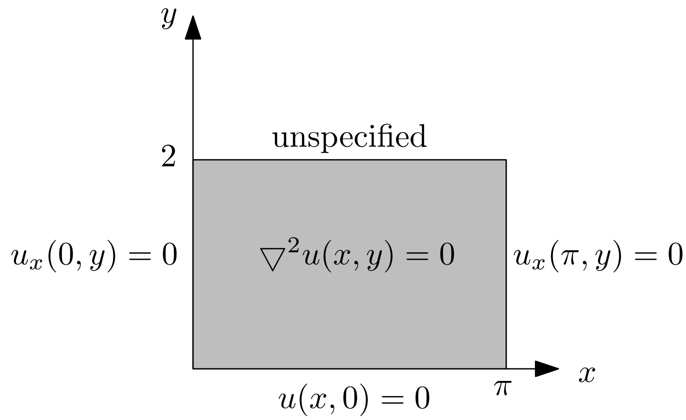


Figure 2.58: Boundary conditions

Let

$$u(x,y) = X(x)\,Y(y)$$

Substitution in the PDE $u_{xx} + y_{yy} = 0$ leads to

$$X''Y + Y''Y = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

Where λ is the separation constant. We obtain two ODE's

$$X'' + \lambda X = 0 \tag{1}$$

$$Y'' - \lambda Y = 0 \tag{2}$$

We use the $X(x)$ ODE (1) to determine the eigenvalues, since that ODE has both boundary conditions specified:

$$\begin{aligned} X'' + \lambda X &= 0 \\ X'(0) &= 0 \\ X'(\pi) &= 0 \end{aligned}$$

<u>Case $\lambda < 0$</u>

Solution is

$$X(x) = A\cosh\left(\sqrt{-\lambda}x\right) + B\sinh\left(\sqrt{-\lambda}x\right)$$

$$X'(x) = A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}x\right) + B\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}x\right)$$

At $x = 0$ the above gives

$$\begin{aligned} 0 &= B\sqrt{-\lambda}\cosh(0) \\ &= B\sqrt{-\lambda} \end{aligned}$$

Hence $B=0$ and the solution (3) reduces to

$$X(x) = A\cosh\left(\sqrt{-\lambda}x\right)$$
$$X'(x) = A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}x\right)$$

At $x=\pi$ the above becomes

$$0 = A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\pi\right)$$

For non-trivial solution we want $\sinh\left(\sqrt{-\lambda}\pi\right)=0$, but $\sinh$ is only zero when its argument is zero, which is not possible here, since $\lambda\neq 0$. Therefore $\lambda<0$ is not possible.

Case $\lambda=0$

Solution becomes $X=Ax+B$. Hence $X'=A$. At $x=0$ this leads to $A=0$. Therefore the solution now becomes $X=B$. Hence $X'=0$. Therefore the second boundary conditions at $x=\pi$ is automatically satisfied. Hence the solution is $X(x)=B$, a constant. We pick $X(x)=1$. Therefore $\lambda=0$ is eigenvalue with associated eigenfunction $X_0(x)=1$.

Case $\lambda>0$

The solution becomes

$$X(x) = A\cos\left(\sqrt{\lambda}x\right)+B\sin\left(\sqrt{\lambda}x\right)$$
$$X'(x) = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)+B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)$$

At $x=0$ the above becomes

$$0 = B\sqrt{\lambda}$$

Hence $B=0$ and the solution reduces to

$$X(x) = A\cos\left(\sqrt{\lambda}x\right)$$
$$X'(x) = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)$$

At $x=\pi$ the above gives

$$0 = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right)$$
$$\sin\left(\sqrt{\lambda}\pi\right) = 0$$

Therefore $\sqrt{\lambda}\pi=n\pi$ for $n=1,2,3,\cdots$. Hence

$$\lambda_n = n^2 \qquad n=1,2,3,\cdots$$

And the solution (corresponding eigenfunctions) is

$$X_n(x) = \cos\left(\sqrt{\lambda_n}x\right) = \cos(nx)$$

In summary, the solution to the X ODE resulted in

$$X_0(x) = 1 \qquad n = 0 \tag{3}$$
$$X_n(x) = \cos(nx) \qquad n = 1,2,3,\cdots$$

Now we solve for the Y ODE

$$Y'' - \lambda Y = 0$$
$$Y(0) = 0$$

We are only given boundary conditions on bottom edge.

<u>case $\lambda = 0$</u>

$$Y = Ay + B$$

When $y = 0$ the above leads to $0 = B$. Hence the corresponding eigenfunction is $Y_0(y) = y$.

<u>case $\lambda > 0$</u>

The solution becomes

$$Y(y) = A\cosh\left(\sqrt{\lambda}y\right) + B\sinh\left(\sqrt{\lambda}y\right)$$

At $y = 0$ the above gives

$$0 = A\cosh(0)$$
$$= A$$

Hence the solution reduces to

$$Y(y) = B\sinh\left(\sqrt{\lambda}y\right)$$

Therefore the eigenfunctions for $n = 1,2,3,\cdots$ are $Y_n(y) = \sinh(ny)$ since $\lambda_n = n^2$ for $n = 1,2,3,\cdots$.

In summary, the solution to the Y ODE resulted in

$$Y_0(y) = y \qquad n = 0 \tag{4}$$
$$Y_n(x) = \sinh(ny) \qquad n = 1,2,3,\cdots$$

From (3,4) we see that

$$u_n(x,y) = X_n(x)\,Y_n(y)$$

For $n = 0$ the above becomes

$$u_0(x,y) = (1)(y)$$
$$= y$$

And for $n = 1,2,3,\cdots$

$$u_n(x,y) = \sinh(ny)$$
$$= \cos(nx)\sinh(ny)$$

Using superposition, then

$$\begin{aligned} u\left(x,y\right) &= X(x)\,Y\left(y\right) \\ &= A_0 u_0 + \sum_{n=1}^{\infty} A_n u_n \\ &= A_0 y + \sum_{n=1}^{\infty} A_n \cos(nx)\sinh\left(ny\right) \end{aligned}$$

QED.

2.5.2 Section 37, Problem 1

1. In Problem 3, Sec. 34, the functions

$$u_0 = y, \qquad u_n = \sinh ny \cos nx \qquad (n = 1, 2, \ldots)$$

were shown to satisfy Laplace's equation

$$u_{xx}(x, y) + u_{yy}(x, y) = 0 \qquad (0 < x < \pi,\ 0 < y < 2)$$

and the homogeneous boundary conditions

$$u_x(0, y) = u_x(\pi, y) = 0, \qquad u(x, 0) = 0.$$

After writing $u = X(x)Y(y)$ and separating variables, use the solutions of the Sturm-Liouville problem (1) in Sec. 35 to show how the functions u_0 and u_n $(n = 1, 2, \ldots)$ can be discovered. Then, by proceeding formally, derive the following solution of the boundary value problem that results when the nonhomogeneous condition $u(x, 2) = f(x)$ is included:

$$u(x, y) = A_0\, y + \sum_{n=1}^{\infty} A_n \sinh ny \cos nx,$$

where

$$A_0 = \frac{1}{2\pi}\int_0^{\pi} f(x)\,dx, \qquad A_n = \frac{2}{\pi \sinh 2n}\int_0^{\pi} f(x) \cos nx\,dx \quad (n = 1, 2, \ldots).$$

Figure 2.59: Problem statement

Solution

The boundary conditions now become as follows

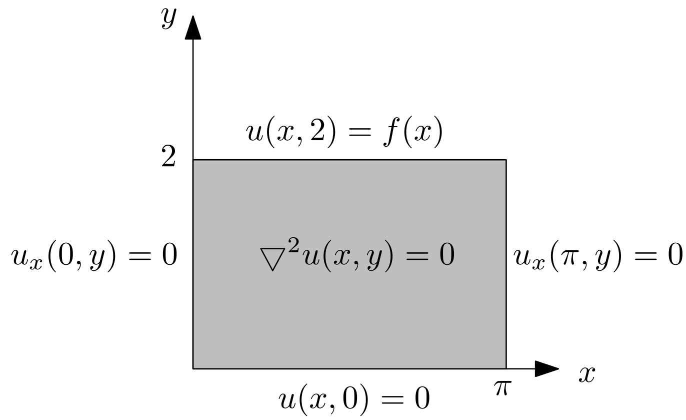


Figure 2.60: Boundary conditions

From the above problem we know the general solution is

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos(nx)\sinh(ny) \tag{1}$$

Now we impose the remaining boundary condition $u(x,2) = f(x)$. Therefore the above becomes

$$f(x) = 2A_0 + \sum_{n=1}^{\infty} A_n \cos(nx)\sinh(2n)$$

Multiplying both sides by $\cos(mx)$ integrating w.r.t. x from $x = 0$ to $x = \pi$ results in

$$\int_0^\pi f(x)\cos(mx)\,dx = \int_0^\pi 2A_0\cos(mx)\,dx + \left[\int_0^\pi \sum_{n=1}^{\infty} A_n \cos(nx)\cos(mx)\sinh(2n)\,dx\right]$$

$$\int_0^\pi f(x)\cos(mx)\,dx = \int_0^\pi 2A_0\cos(mx)\,dx + \left[\sum_{n=1}^{\infty} A_n \sinh(2n)\left(\int_0^\pi \cos(nx)\cos(mx)\,dx\right)\right]$$

<u>case $m = 0$</u>

$$\begin{aligned}\int_0^\pi f(x)\,dx &= \int_0^\pi 2A_0 dx\\ &= 2A_0\pi\\ A_0 &= \frac{1}{2\pi}\int_0^\pi f(x)\,dx \end{aligned} \tag{2}$$

<u>case $m = 1, 2, \cdots$</u>

$$\int_0^\pi f(x)\cos(mx)\,dx = \sum_{n=1}^{\infty} A_n \sinh(2n)\left(\int_0^\pi \cos(nx)\cos(mx)\,dx\right)$$

But $\int_0^\pi \cos(nx)\cos(mx)\,dx = 0$ for all $m \neq n$ and $\frac{\pi}{2}$ when $m = n$. Hence the above simplifies to

$$\int_0^{\pi} f(x)\cos(mx)\,dx = \frac{\pi}{2}A_m \sinh(2m)$$
$$A_m = \frac{2}{\pi\sinh(2m)}\int_0^{\pi} f(x)\cos(mx)\,dx$$

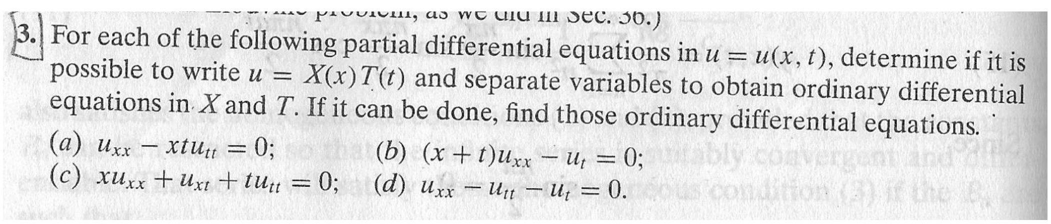
Since m is summation index, we can rename it to n and the above becomes

$$A_n = \frac{2}{\pi\sinh(2n)}\int_0^{\pi} f(x)\cos(nx)\,dx \tag{3}$$

Using (2,3) in (1) gives the final solution

$$u(x,y) = \left(\frac{1}{2\pi}\int_0^{\pi} f(x)\,dx\right)y + \sum_{n=1}^{\infty}\left(\frac{2}{\pi\sinh(2n)}\int_0^{\pi} f(x)\cos(nx)\,dx\right)\cos(nx)\sinh(ny)$$

2.5.3 Section 37, Problem 3



3. For each of the following partial differential equations in $u = u(x,t)$, determine if it is possible to write $u = X(x)T(t)$ and separate variables to obtain ordinary differential equations in X and T. If it can be done, find those ordinary differential equations.
(a) $u_{xx} - xtu_{tt} = 0$; (b) $(x+t)u_{xx} - u_t = 0$;
(c) $xu_{xx} + u_{xt} + tu_{tt} = 0$; (d) $u_{xx} - u_{tt} - u_t = 0$.

Figure 2.61: Problem statement

Part (a)

$$u_{xx} - xtu_{tt} = 0$$

Let $u = X(x)T(t)$. Substituting this into the above PDE gives

$$X''T - xtT''X = 0$$

Dividing by $XT \neq 0$ gives

$$\frac{X''}{X} - xt\frac{T''}{T} = 0$$

Diving by x gives

$$\frac{1}{x}\frac{X''}{X} - t\frac{T''}{T} = 0$$
$$\frac{1}{x}\frac{X''}{X} = t\frac{T''}{T} = -\lambda$$

Hence it possible to separate them. The generated ODE's are

$$X'' + \lambda x X = 0$$
$$T'' + \lambda \frac{T}{t} = 0$$

Part (b)

$$(x + t)\, u_{xx} - u_t = 0$$

Let $u = X(x)\, T(t)$. Substituting this into the above PDE gives

$$(x + t)\, X''T - T'X = 0$$

Dividing by $XT \neq 0$ gives

$$x\frac{X''}{X} + t\frac{X''}{X} - \frac{T'}{T} = 0$$

It is <u>not possible to separate them</u>.

Part (c)

$$xu_{xx} + u_{xt} + tu_{tt} = 0$$

Let $u = X(x)\, T(t)$. Substituting this into the above PDE gives

$$xX''T - \frac{\partial}{\partial t}(X'T) + tT''X = 0$$
$$xX''T - X'T'X + tT''X = 0$$

Dividing by $XT \neq 0$ gives

$$x\frac{X''}{X} - X'T' + t\frac{T''}{T} = 0$$

It is <u>not possible to separate them</u>.

Part (d)

$$u_{xx} - u_{tt} - u_t = 0$$

Let $u = X(x)\, T(t)$. Substituting this into the above PDE gives

$$X''T - T''X - T'X = 0$$

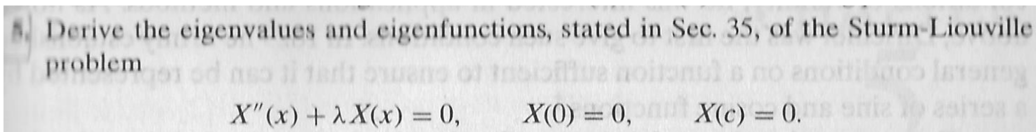
Dividing by $XT \neq 0$ gives

$$\frac{X''}{X} - \frac{T''}{T} - \frac{T'}{T} = 0$$
$$\frac{X''}{X} = \frac{T''}{T} + \frac{T'}{T} = -\lambda$$

It is possible to separate them. The ODE's are

$$X'' + \lambda X = 0$$
$$T'' + T' + \lambda T = 0$$

2.5.4 Section 37, Problem 5



5. Derive the eigenvalues and eigenfunctions, stated in Sec. 35, of the Sturm-Liouville problem

$$X''(x) + \lambda X(x) = 0, \qquad X(0) = 0, \qquad X(c) = 0.$$

Figure 2.62: Problem statement

Case $\lambda < 0$

Solution is

$$X(x) = A\cosh\left(\sqrt{-\lambda}x\right) + B\sinh\left(\sqrt{-\lambda}x\right)$$

At $x = 0$ the above gives

$$0 = A$$

Hence the solution becomes

$$X(x) = B\sinh\left(\sqrt{-\lambda}x\right)$$

At $x = c$ the above becomes

$$0 = B\sinh\left(\sqrt{-\lambda}c\right)$$

For non-trivial solution we want $\sinh\left(\sqrt{-\lambda}c\right) = 0$. But $\sinh$ is zero only when its argument is zero. Which means $\sqrt{-\lambda}c = 0$ which is not possible. Hence $\lambda < 0$ is not possible.

Case $\lambda = 0$

Solution is

$$X(x) = Ax + B$$

At $x = 0$ the above gives

$$0 = B$$

Hence the solution becomes

$$X(x) = B$$

At $x = c$ the above becomes

$$0 = B$$

Which gives trivial solution. Hence $\lambda = 0$ is not possible.

Case $\lambda > 0$

Solution is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

At $x = 0$ the above gives

$$0 = A$$

Hence the solution becomes

$$X(x) = B\sin\left(\sqrt{\lambda}x\right)$$

At $x = c$ the above becomes

$$0 = B\sin\left(\sqrt{\lambda}c\right)$$

For non trivial solution we want $\sin\left(\sqrt{\lambda}c\right) = 0$ which implies

$$\sqrt{\lambda}c = n\pi \qquad n = 1,2,3,\cdots$$
$$\lambda_n = \left(\frac{n\pi}{c}\right)^2$$

Therefore the eigenvalues are $\lambda_n = \left(\frac{n\pi}{c}\right)^2$ for $n = 1,2,3,\cdots$ and the eigenfunctions are $X_n(x) = \sin\left(\frac{n\pi}{c}x\right)$ for $n = 1,2,3,\cdots$.

2.5.5 Section 39, Problem 2

center plane $x = \pi/2$.

2. Suppose that $f(x) = \sin x$ in Example 1, Sec. 39. Find $u(x,t)$ and verify the result fully.

Suggestion: Use the integration formula obtained in Problem 9, Sec. 5.

Answer: $u(x,t) = e^{-kt}\sin x$.

Figure 2.63: Problem statement

Solution

Example 1 is: Solve $u_t = ku_{xx}$ with $u(0,t) = 0$ and $u(\pi,t) = 0$. We now use initial conditions $u(x,0) = \sin(x)$. The eigenvalues are $\lambda_n = n^2$ for $n = 1,2,3,\cdots$ and eigenfunctions are $\sin(nx)$. The general solution for this example is given in the book as

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-kn^2 t}\sin(nx)$$

At $t = 0$ the above becomes

$$\sin x = \sum_{n=1}^{\infty} B_n \sin(nx) \tag{1}$$

By comparing sides, we see that only $n = 1$ term exist. Hence $B_1 = 1$ and all other terms are zero. Hence the solution is, for $n = 1$

$$u(x,t) = e^{-kt} \sin(x)$$

To verify this, we start with (1) and multiply both sides by $\sin(mx)$ and integrate which gives

$$\begin{aligned} \int_0^{\pi} \sin x \sin(mx)\, dx &= \int_0^{\pi} \sum_{n=1}^{\infty} B_n \sin(nx) \sin(mx)\, dx \\ &= \sum_{n=1}^{\infty} B_n \left(\int_0^{\pi} \sin(nx) \sin(mx)\, dx \right) \end{aligned}$$

But $\int_0^{\pi} \sin(nx) \sin(mx)\, dx = 0$ for $m \neq n$ and $\frac{\pi}{2}$ for $n = m$. Hence the above gives

$$\int_0^{\pi} \sin x \sin(mx)\, dx = B_m \frac{\pi}{2}$$

Similarly, $\int_0^{\pi} \sin x \sin(mx)\, dx = 0$ for $m \neq 1$ and $\frac{\pi}{2}$ when $m = 1$, therefore the above becomes

$$\begin{aligned} \frac{\pi}{2} &= B_1 \frac{\pi}{2} \\ B_1 &= 1 \end{aligned}$$

And all other $B_n = 0$. Which gives the same result obtain above, which is $u(x,t) = e^{-kt} \sin(x)$

2.5.6 Section 39, Problem 4

temperatures 0 and u_0, respectively, and whose initial ...

4. Suppose that the conditions on the faces of the slab in Example 2, Sec. 39, are reversed, so that

$$u(0,t) = u_0 \quad \text{and} \quad u(\pi, t) = 0.$$

By replacing x with $\pi - x$ in solution (15) in that example, show that the solution of this new boundary value problem is

$$u(x,t) = u_0 \left[1 - \frac{x}{\pi} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-n^2 kt} \sin nx \right].$$

Figure 2.64: Problem statement

Solution

We need to solve

$$u_t = ku_{xx} \qquad t > 0, 0 < x < \pi$$

With boundary conditions

$$\begin{aligned} u(0,t) &= u_0 \\ u(\pi,t) &= 0 \end{aligned}$$

And initial conditions

$$u(x,0) = 0$$

Solution (15) is

$$u(x,t) = \frac{u_0}{\pi}\left[x + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2kt}\sin(nx)\right] \tag{15}$$

Replacing x by $\pi - x$ in (15) gives

$$\begin{aligned} u(x,t) &= \frac{u_0}{\pi}\left[(\pi - x) + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2kt}\sin(n(\pi - x))\right] \\ &= \frac{u_0}{\pi}(\pi - x) + 2\frac{u_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2kt}\sin(n\pi - nx) \end{aligned} \tag{2}$$

Using $\sin(A - B) = \sin A\cos B + \cos A\sin B$, then

$$\sin(n\pi - nx) = \sin(n\pi)\cos(nx) + \cos(n\pi)\sin(nx)$$

But $\sin(n\pi) = 0$ since n is integer and $\cos(n\pi) = (-1)^n$, then $\sin(n\pi - nx) = (-1)^n\sin(nx)$. Substituting this in (2) gives

$$\begin{aligned} u(x,t) &= u_0 - u_0\frac{x}{\pi} + 2\frac{u_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2kt}(-1)^n\sin(nx) \\ &= u_0\left[1 - \frac{x}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n}e^{-n^2kt}\sin(nx)\right] \\ &= u_0\left[1 - \frac{x}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-n^2kt}\sin(nx)\right] \end{aligned}$$

Which is the result required.

2.6 HW 6

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2.6.1 Section 40, Problem 1

1 The initial temperature of a slab $0 \le x \le \pi$ is zero throughout, and the face $x = 0$ is kept at that temperature. Heat is supplied through the face $x = \pi$ at a constant rate A $(A > 0)$ per unit area, so that $Ku_x(\pi, t) = A$ (see Sec. 26). Write

$$u(x,t) = U(x,t) + \Phi(x)$$

and use the solution of the problem in Example 2, Sec. 40, to derive the expression

$$u(x,t) = \frac{A}{K}\left\{x + \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\exp\left[-\frac{(2n-1)^2 k}{4}t\right]\sin\frac{(2n-1)x}{2}\right\}$$

for the temperatures in this slab.

Figure 2.65: Problem statement

Solution

The PDE to solve is

$$u_{tt} = ku_{xx}$$

With boundary conditions

$$\begin{aligned} u(0,t) &= 0 \\ Ku_x(\pi,t) &= A \end{aligned} \tag{1}$$

And initial conditions

$$u(x,0) = 0$$

The solution to example 2 section 40 is

$$U(x,t)=\sum_{n=1}^{\infty}B_{2n-1}\exp\left(\frac{-(2n-1)^2k}{4}t\right)\sin\left(\frac{(2n-1)x}{2}\right) \tag{2}$$

With

$$B_{2n-1}=\frac{2}{\pi}\int_0^{\pi}f(x)\sin\left(\frac{(2n-1)x}{2}\right)dx$$

Now, in this problem, we start by writing

$$u(x,t)=U(x,t)+\Phi(x) \tag{3}$$

The function $\Phi(x)$ needs to satisfy the nonhomogeneous B.C. (1). Let

$$\Phi(x)=c_1x+c_2$$

When $x=0$ this gives $0=c_2$. Hence $\Phi(x)=c_1x$. Taking derivative gives $\Phi'(x)=c_1$. But from (1) $K\Phi'(\pi)=A$. Hence $c_1=\frac{A}{K}$. Therefore

$$\Phi(x)=\frac{A}{K}x$$

Substituting the above back into (3) gives

$$u(x,t)=U(x,t)+\frac{A}{K}x$$

But $U(x,t)$ is given by (2), hence the above becomes

$$u(x,t)=\frac{A}{K}x+\sum_{n=1}^{\infty}B_{2n-1}\exp\left(\frac{-(2n-1)^2k}{4}t\right)\sin\left(\frac{(2n-1)x}{2}\right) \tag{4}$$

At $t=0$, the initial conditions is 0. Hence the above becomes

$$-\frac{A}{K}x=\sum_{n=1}^{\infty}B_{2n-1}\sin\left(\frac{(2n-1)x}{2}\right)$$

Hence B_{2n-1} is the Fourier sine series of $-\frac{A}{K}x$ given by

$$\begin{aligned}B_{2n-1}&=\frac{2}{\pi}\int_0^{\pi}\left(-\frac{A}{K}x\right)\sin\left(\frac{(2n-1)x}{2}\right)dx\\&=-\frac{2A}{\pi K}\int_0^{\pi}x\sin\left(\frac{(2n-1)x}{2}\right)dx\end{aligned}$$

Integration by parts. Let $u=x, dv=\sin\left(\frac{(2n-1)x}{2}\right)$, hence $du=1$ and $v=-\frac{2}{(2n-1)}\cos\left(\frac{(2n-1)x}{2}\right)$

and the above becomes

$$
\begin{aligned}
B_{2n-1} &= -\frac{2A}{\pi K}\left(\left[-\frac{2x}{(2n-1)}\cos\left(\frac{(2n-1)\,x}{2}\right)\right]_0^{\pi} + \int_0^{\pi}\frac{2}{(2n-1)}\cos\left(\frac{(2n-1)\,x}{2}\right)dx\right)\\
&= -\frac{2A}{\pi K}\left(-\frac{2}{(2n-1)}\left[x\cos\left(\frac{(2n-1)\,x}{2}\right)\right]_0^{\pi} + \frac{4}{(2n-1)^2}\left[\sin\left(\frac{(2n-1)\,x}{2}\right)\right]_0^{\pi}\right)\\
&= -\frac{2A}{\pi K}\left(-\frac{2\pi}{(2n-1)}\cos\left(\frac{(2n-1)\,\pi}{2}\right) + \frac{4}{(2n-1)^2}\sin\left(\frac{(2n-1)\,\pi}{2}\right)\right)
\end{aligned}
$$

Since $2n-1$ is odd, then the cosine terms above vanish and the above simplifies to

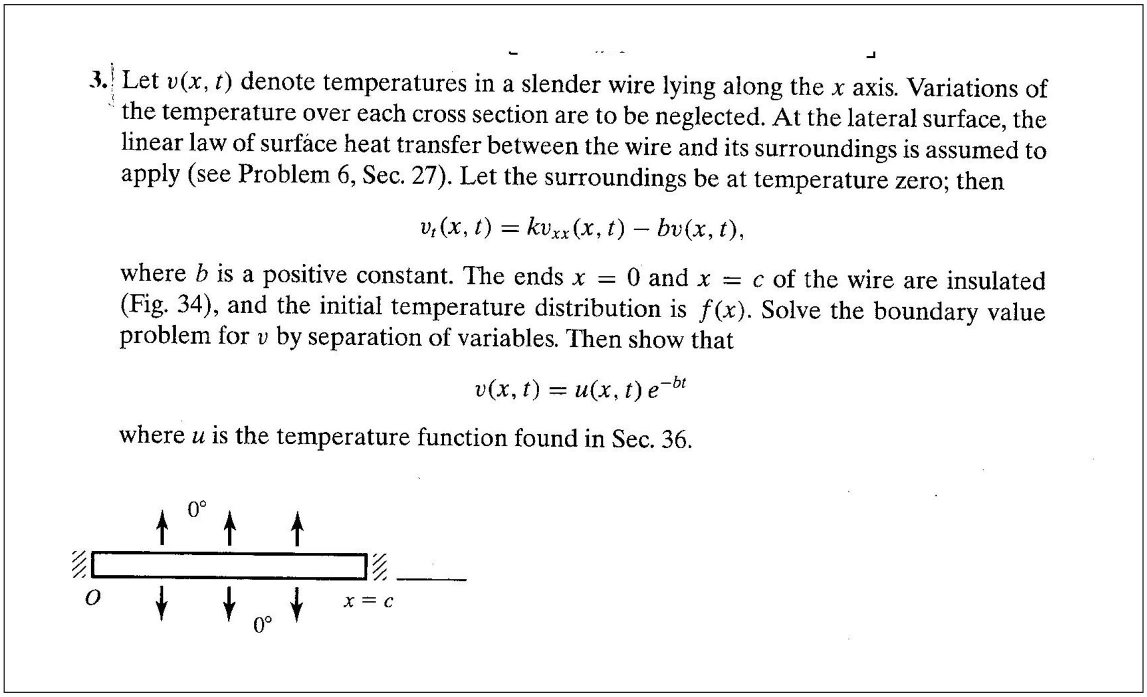
$$
\begin{aligned}
B_{2n-1} &= -\frac{A}{\pi K}\frac{8\,(-1)^{n+1}}{(2n-1)^2}\\
&= \frac{A}{\pi K}\frac{8\,(-1)^{n+2}}{(2n-1)^2}\\
&= \frac{A}{\pi K}\frac{8\,(-1)^{n}}{(2n-1)^2}
\end{aligned}
$$

Substituting the above in (4) gives

$$
\begin{aligned}
u\,(x,t) &= \frac{A}{K}x + \sum_{n=1}^{\infty}\frac{A}{\pi K}\frac{8\,(-1)^{n}}{(2n-1)^2}\exp\left(\frac{-\,(2n-1)^2\,k}{4}t\right)\sin\left(\frac{(2n-1)\,x}{2}\right)\\
&= \frac{A}{K}\left\{x + \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n}}{(2n-1)^2}\exp\left(\frac{-\,(2n-1)^2\,k}{4}t\right)\sin\left(\frac{(2n-1)\,x}{2}\right)\right\}
\end{aligned}
$$

Which is the result required.

2.6.2 Section 40, Problem 3



3. Let $v(x,t)$ denote temperatures in a slender wire lying along the x axis. Variations of the temperature over each cross section are to be neglected. At the lateral surface, the linear law of surface heat transfer between the wire and its surroundings is assumed to apply (see Problem 6, Sec. 27). Let the surroundings be at temperature zero; then

$$v_t(x,t) = kv_{xx}(x,t) - bv(x,t),$$

where b is a positive constant. The ends $x=0$ and $x=c$ of the wire are insulated (Fig. 34), and the initial temperature distribution is $f(x)$. Solve the boundary value problem for v by separation of variables. Then show that

$$v(x,t) = u(x,t)\,e^{-bt}$$

where u is the temperature function found in Sec. 36.

Figure 2.66: Problem statement

Solution

The PDE is

$$v_t = kv_{xx} - bv$$

With boundary conditions

$$v_x(0,t) = 0$$
$$v_x(c,t) = 0$$

And initial conditions

$$v(x,0) = f(x)$$

Let $v(x,t) = X(x)T(t)$. Substituting into the PDE gives

$$T'X = kX''T - bXT$$

Dividing by $XT \neq 0$ gives

$$\frac{T'}{T} = k\frac{X''}{X} - b$$
$$\frac{T'}{T} + b = k\frac{X''}{X}$$
$$\frac{T'}{kT} + \frac{b}{k} = \frac{X''}{X} = -\lambda$$

Where λ is the separation constant. We obtain the boundary value eigenvalue ODE as

$$\begin{aligned} X'' + \lambda X &= 0 \qquad (1)\\ X'(0) &= 0\\ X'(c) &= 0 \end{aligned}$$

And the time ODE as

$$\begin{aligned} \frac{T'}{kT} + \frac{b}{k} &= -\lambda\\ T' + \frac{b}{k}kT &= -\lambda kT\\ T' + \frac{b}{k}kT + \lambda kT &= 0\\ T' + T(b + \lambda k) &= 0 \end{aligned}$$

Now we solve the space ODE (1) in order to determine the eigenvalues λ.

<u>Case $\lambda < 0$</u>

The solution to (1) becomes

$$\begin{aligned} X(x) &= A\cosh\left(\sqrt{-\lambda}x\right) + B\sinh\left(\sqrt{-\lambda}x\right)\\ X' &= A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}x\right) + B\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}x\right) \end{aligned}$$

Satisfying $X'(0) = 0$ gives

$$0 = B\sqrt{-\lambda}$$

Hence $B = 0$ and the solution becomes $X(x) = A\cosh\left(\sqrt{-\lambda}x\right)$. Therefore $X' = A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}x\right)$. Satisfying $X'(c) = 0$ gives

$$0 = A\sqrt{\lambda}\sinh\left(\sqrt{-\lambda}c\right)$$

But sinh is zero only when its argument is zero, which is not the case here since $\lambda \neq 0$. This implies $A = 0$, leading to trivial solution. Therefore $\lambda < 0$ is not possible.

<u>Case $\lambda = 0$</u>

The solution to (1) becomes

$$\begin{aligned} X(x) &= Ax + B\\ X' &= A \end{aligned}$$

Satisfying $X'(0) = 0$ gives

$$0 = A$$

And the solution becomes $X(x) = B$. Therefore $X' = 0$. Satisfying $X'(c) = 0$ gives

$$0 = 0$$

Which is valid for any B. Hence choosing $B = 1$ shows that $\lambda = 0$ is valid eigenvalue with corresponding eigenfunction $X_0(x) = 1$.

Case $\lambda > 0$

The solution to (1) becomes

$$\begin{aligned} X(x) &= A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right) \\ X' &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) \end{aligned}$$

Satisfying $X'(0) = 0$ gives

$$0 = B\sqrt{\lambda}$$

Hence $B = 0$ and the solution becomes $X(x) = A\cos\left(\sqrt{\lambda}x\right)$. Therefore $X' = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)$. Satisfying $X'(c) = 0$ gives

$$0 = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}c\right)$$

For nontrivial solution we want

$$\begin{aligned} \sin\left(\sqrt{\lambda}c\right) &= 0 \\ \sqrt{\lambda}c &= n\pi \qquad n = 1, 2, 3, \cdots \\ \lambda_n &= \left(\frac{n\pi}{c}\right)^2 \end{aligned} \tag{2}$$

And the corresponding eigenfunctions

$$X_n(x) = \cos\left(\sqrt{\lambda_n}x\right) \tag{3}$$

Now that we found λ_n, we can solve the time ODE $T' + T(b + \lambda k) = 0$. The solution is

$$T_n(t) = e^{-(b+\lambda_n k)t} \tag{4}$$

Hence the fundamental solution is

$$\begin{aligned} v_n(x,t) &= X_n(x)\,T_n(t) \\ &= \cos\left(\sqrt{\lambda_n}x\right)e^{-(b+\lambda_n k)t} \end{aligned}$$

And the general solution is the superposition of all these solutions

$$\begin{aligned} v(x,t) &= A_0 X_0 T_0 + \sum_{n=1}^{\infty} A_n X_n(x)\,T_n(t) \\ &= A_0 e^{-bt} + \sum_{n=1}^{\infty} A_n \cos\left(\sqrt{\lambda_n}x\right)e^{-(b+\lambda_n k)t} \end{aligned}$$

Which can be written as

$$v(x,t) = u(x,t)\,e^{-bt}$$

Where $u(x,t)$ is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\sqrt{\lambda_n}x\right)e^{-\lambda_n kt}$$

Which is the same as given in section 36, page 106. In the above

$$\lambda_0 = 0$$
$$\lambda_n = \left(\frac{n\pi}{c}\right)^2 \qquad n = 1,2,3,\cdots$$

2.6.3 Section 41, Problem 3

3. A *hollow* sphere $1 \le r \le 2$ is initially at temperature zero. The interior surface is kept at that temperature, and the outer one is maintained at a constant temperature u_0. Set up the boundary value problem for the temperatures

$$u = u(r,t) \qquad (1 < r < 2, t > 0)$$

and follow these steps to solve it:

(*a*) Write $v(r,t) = ru(r,t)$ to obtain a new boundary value problem for $v(r,t)$. Then put $s = r - 1$ to obtain the problem

$$v_t = kv_{ss} \qquad (0 < s < 1, t > 0),$$
$$v = 0 \text{ when } s = 0, \qquad v = 2u_0 \text{ when } s = 1,$$
$$v = 0 \text{ when } t = 0.$$

(*b*) Use the result in Problem 2, Sec. 40, to write a solution of the boundary value problem reached in part (*a*). Then show how it follows from the substitutions made in part (*a*) that

$$u(r,t) = 2u_0\left[1 - \frac{1}{r} + \frac{2}{\pi r}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 kt}\sin n\pi(r-1)\right].$$

Figure 2.67: Problem statement

Solution

The heat PDE in spherical coordinates, assuming no dependency on ϕ nor on θ is given by

$$u_t = k\nabla^2 u \tag{1}$$
$$= k\frac{1}{r}(ru)_{rr}$$

Where $1 < r < 2$ and $t > 0$. With the boundary conditions

$$u(1,t) = 0$$
$$u(2,0) = u_0$$

And initial conditions

$$u(r,0) = 0$$

Part (a)

Let $v(r,t) = ru(r,t)$. Hence $v_t = ru_t$ and $\frac{1}{r}(ru)_{rr} = \frac{1}{r}v_{rr}$. Substituting these in(1), the PDE simplifies to

$$v_t = kv_{rr} \tag{2}$$

And the boundary conditions $u(1,t) = 0$ becomes $v(1,t) = 0$ and $u(2,0) = u_0$ becomes $v(2,t) = 2u_0$. And initial conditions $u(r,0) = 0$ becomes $v(r,0) = 0$. Hence the new boundary conditions

$$\begin{aligned} v(1,t) &= 0 \\ v(2,t) &= 2u_0 \end{aligned}$$

And new initial conditions

$$v(r,0) = 0$$

Now let $s = r - 1$. Since $\frac{\partial r}{\partial s} = 1$, then the PDE becomes $v_t = kv_{ss}$. When $r = 1$, then $s = 0$ and the boundary conditions $v(1,t) = 0$ becomes $v(0,t) = 0$ and the boundary conditions $v(2,t) = 2u_0$ becomes $v(1,t) = 2u_0$. And initial conditions do not change. Hence the new problem is to solve for $v(s,t)$ in

$$\begin{aligned} v_t &= kv_{ss} \\ v(1,t) &= 0 \\ v(1,t) &= 2u_0 \\ v(s,0) &= 0 \end{aligned} \tag{3}$$

With $0 < s < 1$ and $t > 0$.

Part (b)

The PDE (3) in part(a) is now the same as result of problem 2 section 40. Hence we can use that solution for (3) which gives

$$v(s,t) = 2u_0 \left[x + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2\pi^2 kt} \sin(n\pi s) \right]$$

Replacing s by $r - 1$ in the above gives

$$v(r,t) = 2u_0 \left[(r-1) + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2\pi^2 kt} \sin(n\pi(r-1)) \right]$$

But $v(r,t) = ru(r,t)$, hence $u(r,t) = \frac{v}{r}$ and therefore

$$\begin{aligned} u(r,t) &= 2u_0\left[\frac{(r-1)}{r} + \frac{2}{\pi r}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 kt}\sin(n\pi(r-1))\right] \\ &= 2u_0\left[\left(1-\frac{1}{r}\right) + \frac{2}{\pi r}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 kt}\sin(n\pi(r-1))\right] \end{aligned}$$

Which is the result required.

2.6.4 Section 42, Problem 4

4. A bar, with its lateral surface insulated, is initially at temperature zero, and its ends $x = 0$ and $x = c$ are kept at that temperature. Because of internally generated heat, the temperatures in the bar satisfy the differential equation

$$u_t(x,t) = ku_{xx}(x,t) + q(x,t) \qquad (0 < x < c,\ t > 0).$$

Use the method of variation of parameters to derive the temperature formula

$$u(x,t) = \frac{2}{c}\sum_{n=1}^{\infty} I_n(t)\sin\frac{n\pi x}{c},$$

where $I_n(t)$ denotes the iterated integrals

$$I_n(t) = \int_0^t \exp\left[-\frac{n^2\pi^2 k}{c^2}(t-\tau)\right]\int_0^c q(x,\tau)\sin\frac{n\pi x}{c}\,dx\,d\tau \qquad (n = 1, 2, \ldots).$$

Suggestion: Write

$$q(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{c} \qquad \text{where} \qquad b_n(t) = \frac{2}{c}\int_0^c q(x,t)\sin\frac{n\pi x}{c}\,dx.$$

Figure 2.68: Problem statement

Solution

Using method of eigenfunction expansion (or method of variation of parameters as the book calls it), we start by assuming the solution to the PDE $u_t = ku_{xx} + q(x,t)$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\Phi_n(x) \tag{1}$$

Where $\Phi_n(x)$ are the eigenfunctions associated with the homogeneous PDE $u_t = ku_{xx}$ with the homogeneous boundary conditions $u(0,t) = 0$ and $u(c,t) = 0$. But we solved this homogeneous PDE before. It has eigenvalues and corresponding eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{c}\right)^2 \qquad n = 1, 2, 3, \cdots$$

$$\Phi_n(x) = \sin\left(\sqrt{\lambda_n}x\right)$$

Substituting (1) into the original PDE $u_t = ku_{xx} + q(x,t)$ results in

$$\frac{\partial}{\partial t}\sum_{n=1}^{\infty} a_n(t)\Phi_n(x) = k\frac{\partial^2}{\partial x^2}\sum_{n=1}^{\infty} a_n(t)\Phi_n(x) + q(x,t)$$
$$\sum_{n=1}^{\infty} a_n'(t)\Phi_n(x) = k\sum_{n=1}^{\infty} a_n(t)\Phi_n''(x) + q(x,t)$$

But from the Sturm-Liouville ODE, we know that $\Phi_n''(x) + \lambda_n\Phi_n(x) = 0$. Hence $\Phi_n''(x) = -\lambda_n\Phi_n(x)$ and the above reduces to

$$\sum_{n=1}^{\infty} a_n'(t)\Phi_n(x) = -k\sum_{n=1}^{\infty} a_n(t)\lambda_n\Phi_n(x) + q(x,t) \tag{2}$$

Since the eigenfunctions $\Phi_n(x)$ are complete, we can expand $q(x,t)$ using them. Therefore

$$q(x,t) = \sum_{n=1}^{\infty} b_n(t)\Phi_n(x)$$

Substituting the above back in (2) gives

$$\sum_{n=1}^{\infty} a_n'(t)\Phi_n(x) = -k\sum_{n=1}^{\infty} a_n(t)\lambda_n\Phi_n(x) + \sum_{n=1}^{\infty} b_n(t)\Phi_n(x)$$

Since $\Phi_n(x)$ are never zero, we can simplify the above to

$$a_n'(t) = -ka_n(t)\lambda_n + b_n(t)$$
$$a_n'(t) + ka_n(t)\lambda_n = b_n(t)$$

The above is first order ODE in $I_n(t)$. It is linear ODE. The integrating factor is $\mu = e^{\int k\lambda_n dt} = e^{k\lambda_n t}$. Multiplying the above ODE by this integrating factor gives

$$\frac{d}{dt}\left(a_n(t)e^{k\lambda_n t}\right) = b_n(t)e^{k\lambda_n t}$$

Integrating both sides

$$a_n(t)e^{k\lambda_n t} = \int_0^t b_n(\tau)e^{k\lambda_n\tau}d\tau$$
$$a_n(t) = \int_0^t b_n(\tau)e^{-k\lambda_n(t-\tau)}d\tau$$

Now that we found $a_n(t)$, we substitute it back into (1) which gives

$$u(x,t) = \sum_{n=1}^{\infty}\left(\int_0^t b_n(\tau)e^{-k\lambda_n(t-\tau)}d\tau\right)\Phi_n(x) \tag{3}$$

What is left is to find $b_n(t)$. Since $q(x,t) = \sum_{n=1}^{\infty} b_n(t)\Phi_n(x)$, then by orthogonality we obtain

$$\begin{aligned}
\int_0^c q(x,t)\Phi_m(x)\,dx &= \int_0^c \sum_{n=1}^{\infty} b_n(t)\Phi_n(x)\Phi_m(x)\,dx \\
&= \sum_{n=1}^{\infty} b_n(t)\int_0^c \Phi_n(x)\Phi_m(x)\,dx \\
&= b_m(t)\int_0^c \Phi_m^2(x)\,dx \\
&= b_m(t)\frac{c}{2}
\end{aligned}$$

Hence

$$b_n(t) = \frac{2}{c}\int_0^c q(x,t)\Phi_m(x)\,dx$$

Substituting this back into (3) gives

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}\left(\int_0^t e^{-k\lambda_n(t-\tau)}\frac{2}{c}\left(\int_0^c q(x,\tau)\Phi_m(x)\,dx\right)d\tau\right)\Phi_n(x) \\
&= \frac{2}{c}\sum_{n=1}^{\infty}\left(\int_0^t e^{-k\lambda_n(t-\tau)}\left(\int_0^c q(x,\tau)\Phi_m(x)\,dx\right)d\tau\right)\Phi_n(x) \qquad (4)
\end{aligned}$$

If we let

$$I_n(t) = \int_0^t e^{-k\lambda_n(t-\tau)}\left(\int_0^c q(x,\tau)\Phi_m(x)\,dx\right)d\tau$$

Then (4) becomes

$$u(x,t) = \frac{2}{c}\sum_{n=1}^{\infty} I_n(t)\Phi_n(x)$$

Since $\Phi_n(x) = \sin\left(\frac{n\pi}{c}x\right)$ then the above is

$$u(x,t) = \frac{2}{c}\sum_{n=1}^{\infty} I_n(t)\sin\left(\frac{n\pi}{c}x\right)$$

Which is what required to show.

2.6.5 Section 42, Problem 5

5. By writing $c = 1$, $k = 1$, and $q(x,t) = xp(t)$ in the solution found in Problem 4, obtain the solution already found in Problem 1.

Figure 2.69: Problem statement

Solution

The solution in problem 4 above us

$$u(x,t) = \frac{2}{c}\sum_{n=1}^{\infty} I_n(t)\sin\left(\frac{n\pi}{c}x\right) \tag{1}$$

Where

$$I_n(t) = \int_0^t e^{-k\lambda_n(t-\tau)}\left(\int_0^c q(x,\tau)\sin\left(\frac{n\pi}{c}x\right)dx\right)d\tau$$

And $\lambda_n = \left(\frac{n\pi}{c}\right)^2$. Let $c=1, k=1$ and $q(x,t) = xp(t)$, then the above becomes

$$I_n(t) = \int_0^t e^{-n^2\pi^2(t-\tau)}\left(\int_0^1 xp(\tau)\sin(n\pi x)\,dx\right)d\tau$$

Substituting this in (1), using $c=1$, then (1) becomes

$$\begin{aligned}
u(x,t) &= 2\sum_{n=1}^{\infty}\left(\int_0^t e^{-n^2\pi^2(t-\tau)}\left(\int_0^1 xp(\tau)\sin(n\pi x)\,dx\right)d\tau\right)\sin(n\pi x)\\
&= 2\sum_{n=1}^{\infty}\left(\int_0^t p(\tau)\,e^{-n^2\pi^2(t-\tau)}\left(\int_0^1 x\sin(n\pi x)\,dx\right)d\tau\right)\sin(n\pi x) \qquad (2)
\end{aligned}$$

But $\int_0^1 x\sin(n\pi x)\,dx$ can now be integrated by parts. Let $u = x, dv = \sin(n\pi x)$, hence $du = 1, v = -\frac{\cos(n\pi x)}{n\pi}$ and therefore

$$\begin{aligned}
\int_0^1 x\sin(n\pi x)\,dx &= -\frac{1}{n\pi}\left[x\cos(n\pi x)\right]_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx\\
&= -\frac{1}{n\pi}\cos(n\pi) + \frac{1}{n\pi}\left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1\\
&= -\frac{1}{n\pi}(-1)^n + \frac{1}{n^2\pi^2}\left[\sin(n\pi)\right]\\
&= \frac{(-1)^{n+1}}{n\pi}
\end{aligned}$$

Substituting this back in (2) gives

$$\begin{aligned}
u(x,t) &= 2\sum_{n=1}^{\infty}\left(\int_0^t p(\tau)\,e^{-n^2\pi^2(t-\tau)}\left(\frac{(-1)^{n+1}}{n\pi}\right)d\tau\right)\sin(n\pi x)\\
&= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x)\left(\int_0^t p(\tau)\,e^{-n^2\pi^2(t-\tau)}d\tau\right)
\end{aligned}$$

Which is the solution for problem 1.

2.6.6 Section 42, Problem 8

8. Using a series of the form

$$u(x,t) = A_0(t) + \sum_{n=1}^{\infty} A_n(t)\cos\frac{n\pi x}{c}$$

and the expansion (see Example 1 in Sec. 8)

$$x^2 = \frac{c^2}{3} + \frac{4c^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{c} \qquad (0 < x < c),$$

solve the following temperature problem for a slab $0 \le x \le c$ with insulated faces:

$$u_t(x,t) = ku_{xx}(x,t) + ax^2 \qquad (0 < x < c, t > 0),$$
$$u_x(0,t) = 0, \qquad u_x(c,t) = 0, \qquad u(x,0) = 0,$$

where a is a constant. Thus, show that

$$u(x,t) = ac^2\left\{\frac{t}{3} + \frac{4c^2}{\pi^4 k}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\left[1 - \exp\left(-\frac{n^2\pi^2 k}{c^2}t\right)\right]\cos\frac{n\pi x}{c}\right\}.$$

Figure 2.70: Problem statement

Solution

The PDE to solve is

$$u_t = ku_{xx} + ax^2$$

With boundary conditions

$$u_x(0,t) = 0$$
$$u_x(c,t) = 0$$

And initial conditions

$$u(x,0) = 0$$

Using method of eigenfunction expansion, we start by assuming the solution to the PDE $u_t = ku_{xx} + ax^2$ is given by

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t)\Phi_n(x) \tag{1}$$

Where $\Phi_n(x)$ are the eigenfunctions associated with the homogeneous PDE $u_t = ku_{xx}$ with the homogeneous boundary conditions $u_x(0,t) = 0$ and $u_x(c,t) = 0$. But we solved this homogeneous PDE before. It has eigenvalues and corresponding eigenfunctions

$$\begin{aligned}\lambda_0 &= 0\\ \Phi_0(x) &= 1\\ \lambda_n &= \frac{n^2\pi^2}{c^2} \qquad n = 1,2,3,\cdots\\ \Phi_n(x) &= \cos\left(\frac{n\pi}{c}x\right)\end{aligned}$$

Substituting (1) into the original PDE $u_t = ku_{xx} + ax^2$ results in

$$\begin{aligned}\frac{\partial}{\partial t}\sum_{n=0}^{\infty} a_n(t)\Phi_n(x) &= k\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} a_n(t)\Phi_n(x) + ax^2\\ \sum_{n=0}^{\infty} a_n'(t)\Phi_n(x) &= k\sum_{n=0}^{\infty} a_n(t)\Phi_n''(x) + ax^2\end{aligned}$$

But from the Sturm-Liouville ODE, we know that $\Phi_n''(x) + \lambda_n\Phi_n(x) = 0$. Hence $\Phi_n''(x) = -\lambda_n\Phi_n(x)$ and the above reduces to

$$\sum_{n=0}^{\infty} a_n'(t)\Phi_n(x) = -k\sum_{n=0}^{\infty} a_n(t)\lambda_n\Phi_n(x) + ax^2 \tag{2}$$

Since the eigenfunctions $\Phi_n(x)$ are complete, we can expand ax^2 using them. Therefore

$$ax^2 = \sum_{n=0}^{\infty} b_n(x)\Phi_n(x)$$

Substituting the above back in (2) gives

$$\sum_{n=0}^{\infty} a_n'(t)\Phi_n(x) = -k\sum_{n=0}^{\infty} a_n(t)\lambda_n\Phi_n(x) + \sum_{n=0}^{\infty} b_n(x)\Phi_n(x)$$

Since $\Phi_n(x)$ are never zero, we can simplify the above to

$$\begin{aligned}a_n'(t) &= -ka_n(t)\lambda_n + b_n(x)\\ a_n'(t) + ka_n(t)\lambda_n &= b_n(x)\end{aligned}$$

The above is first order ODE in $I_n(t)$. It is linear ODE. The integrating factor is $\mu = e^{\int k\lambda_n dt} = e^{k\lambda_n t}$. Multiplying the above ODE by this integrating factor gives

$$\frac{d}{dt}\left(a_n(t)e^{k\lambda_n t}\right) = b_n(x)e^{k\lambda_n t}$$

Integrating both sides

$$\begin{aligned}a_n(t)e^{k\lambda_n t} &= b_n(x)\int_0^t e^{k\lambda_n\tau}d\tau\\ a_n(t) &= b_n(x)\int_0^t e^{-k\lambda_n(t-\tau)}d\tau \end{aligned}\tag{3}$$

What is left is to find $b_n(x)$. Since $ax^2 = \sum_{n=0}^{\infty} b_n(x)\Phi_n(x)$, and from example 1 section 8, we found that

$$b_0(x) = a\frac{c^2}{3}$$
$$b_n(x) = a\frac{4c^2}{\pi^2}\frac{(-1)^n}{n^2} \qquad n = 1,2,3,\cdots$$

Hence when $n = 0$, then (3) becomes (since $\lambda_0 = 0$)

$$\begin{aligned}
a_0(t) &= a\frac{c^2}{3}\int_0^t d\tau \\
&= \frac{ac^2}{3}t
\end{aligned}$$

When $n > 0$ then (3) becomes

$$\begin{aligned}
a_n(t) &= \left(a\frac{4c^2}{\pi^2}\frac{(-1)^n}{n^2}\right)\int_0^t e^{-k\lambda_n(t-\tau)}d\tau \\
&= \frac{(-1)^n}{n^2}\frac{4ac^2}{\pi^2}\int_0^t e^{-k\left(\frac{n\pi}{c}\right)^2(t-\tau)}d\tau \\
&= \frac{(-1)^n}{n^2}\frac{4ac^2}{\pi^2}e^{-k\left(\frac{n\pi}{c}\right)^2 t}\int_0^t e^{k\left(\frac{n\pi}{c}\right)^2\tau}d\tau \\
&= \frac{(-1)^n}{n^2}\frac{4ac^2}{\pi^2}e^{-k\left(\frac{n\pi}{c}\right)^2 t}\left[\frac{e^{k\left(\frac{n\pi}{c}\right)^2\tau}}{k\left(\frac{n\pi}{c}\right)^2}\right]_0^t \\
&= \frac{(-1)^n}{n^2}\frac{4ac^2}{\pi^2}\frac{e^{-k\left(\frac{n\pi}{c}\right)^2 t}}{k\left(\frac{n\pi}{c}\right)^2}\left[e^{k\left(\frac{n\pi}{c}\right)^2 t} - 1\right] \\
&= \frac{(-1)^n}{n^2}\frac{4ac^2}{\pi^2}\frac{1 - e^{-k\left(\frac{n\pi}{c}\right)^2 t}}{k\frac{n^2\pi^2}{c^2}} \\
&= \frac{(-1)^n}{n^4}\frac{4ac^4}{k\pi^4}\left(1 - e^{-k\left(\frac{n\pi}{c}\right)^2 t}\right)
\end{aligned}$$

Now that we found $a_n(t)$, we substitute it back into (1) which gives

$$\begin{aligned}
u(x,t) &= a_0(t) + \sum_{n=1}^{\infty} a_n(t)\Phi_n(x) \\
u(x,t) &= \frac{ac^2}{3}t + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\frac{4ac^4}{k\pi^4}\left(1 - e^{-k\left(\frac{n\pi}{c}\right)^2 t}\right)\cos\left(\frac{n\pi}{c}x\right) \\
&= \frac{ac^2}{3}t + \frac{4ac^4}{k\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\left(1 - e^{-k\left(\frac{n\pi}{c}\right)^2 t}\right)\cos\left(\frac{n\pi}{c}x\right) \\
&= ac^2\left\{\frac{t}{3} + \frac{4c^2}{k\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\left(1 - e^{-k\left(\frac{n\pi}{c}\right)^2 t}\right)\cos\left(\frac{n\pi}{c}x\right)\right\}
\end{aligned}$$

Which is the result required to show.

2.6.7 Section 43, Problem 1

1. The faces and edges $x=0$ and $x=\pi$ $(0<y<\pi)$ of a square plate $0\leq x\leq\pi$, $0\leq y\leq\pi$ are insulated. The edges $y=0$ and $y=\pi$ $(0<x<\pi)$ are kept at temperatures 0 and $f(x)$, respectively. Let $u(x,y)$ denote steady temperatures in the plate and derive the expression

$$u(x,y)=A_0\,y+\sum_{n=1}^{\infty}A_n\sinh ny\cos nx,$$

where

$$A_0=\frac{1}{\pi^2}\int_0^{\pi}f(x)\,dx \qquad \text{and} \qquad A_n=\frac{2}{\pi\sinh n\pi}\int_0^{\pi}f(x)\cos nx\,dx$$

$$(n=1,2,\ldots).$$

Find $u(x,y)$ when $f(x)=u_0$, where u_0 is a constant.

Figure 2.71: Problem statement

Solution

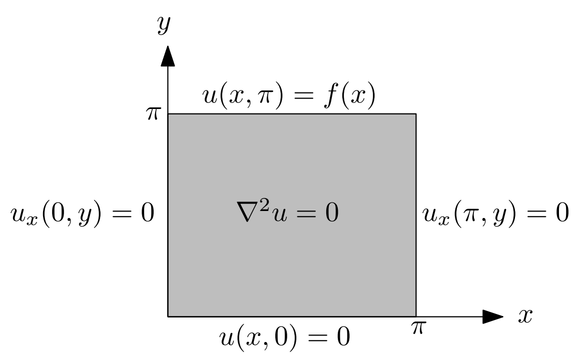


Figure 2.72: PDE and boundary conditions

Let $u(x,y)=X(x)\,Y(y)$. The PDE becomes

$$\begin{aligned}X''Y+Y''X&=0\\ \frac{X''}{X}&=-\frac{Y''}{Y}=-\lambda\end{aligned}$$

Hence the eigenvalue problem is

$$\begin{aligned} X'' + \lambda X &= 0 \qquad (1)\\ X'(0) &= 0\\ X'(\pi) &= 0 \end{aligned}$$

And the ODE for $Y(y)$ is

$$Y'' - \lambda Y = 0$$

We start by solving (1) to find the eigenvalues and eigenfunctions.

Case $\lambda < 0$ The solution is

$$\begin{aligned} X &= A\cosh\left(\sqrt{-\lambda}x\right) + B\sinh\left(\sqrt{-\lambda}x\right)\\ X' &= A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}x\right) + B\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}x\right) \end{aligned}$$

At $x = 0$ the above becomes

$$0 = B\sqrt{-\lambda}$$

Hence $B = 0$ and the solution becomes

$$\begin{aligned} X &= A\cosh\left(\sqrt{-\lambda}x\right)\\ X' &= A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}x\right) \end{aligned}$$

At $x = \pi$ the above gives

$$0 = A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\pi\right)$$

For nontrivial solution $\sinh\left(\sqrt{-\lambda}\pi\right) = 0$ but this is not possible since sinh is zero only when its argument is zero and this is not the case here. Hence $\lambda < 0$ is not eigenvalue.

Case $\lambda = 0$ The solution is

$$\begin{aligned} X &= Ax + B\\ X' &= A \end{aligned}$$

At $x = 0$ the above becomes

$$0 = A$$

Hence the solution becomes

$$\begin{aligned} X &= B\\ X' &= 0 \end{aligned}$$

At $x = \pi$ the above gives

$$0 = 0$$

Therefore $\lambda = 0$ is eigenvalue with $X_0(x) = 1$.

Case $\lambda > 0$ The solution is

$$X = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$
$$X' = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)$$

At $x = 0$ the above becomes

$$0 = B\sqrt{\lambda}$$

Hence $B = 0$ and the solution becomes

$$X = A\cos\left(\sqrt{\lambda}x\right)$$
$$X' = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)$$

At $x = \pi$ the above gives

$$0 = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right)$$

For nontrivial solution

$$\sin\left(\sqrt{\lambda}\pi\right) = 0$$
$$\sqrt{\lambda}\pi = n\pi \qquad n = 1, 2, 3, \cdots$$
$$\lambda_n = n^2$$

And the corresponding eigenfunctions $X_n(x) = \cos(nx)$. Therefore in summary we have

eigenvalue		eigenfunction
$\lambda_0 = 0$		1
$\lambda_n = n^2$	$n = 1, 2, 3, \cdots$	$\cos(nx)$

Hence the $Y(y)$ ode becomes

$$Y'' - \lambda_n Y = 0$$
$$Y'' - n^2 Y = 0$$

The solution to the above is, when $n = 0$

$$Y_0 = A_0 y + B_0$$

When $y = 0$ the above gives $0 = B_0$. Hence $Y_0 = A_0 y$.

When $n > 0$

$$Y_n(y) = B_n\cosh(ny) + A_n\sinh(ny)$$

When $y = 0$ the above gives $0 = B_n$, Hence

$$Y_n(y) = A_n\sinh(ny)$$

Hence the fundamental solution is

$$u(x, y) = X_n Y_n$$

And the general solution is the superposition of these solutions

$$u\left(x,y\right) = A_0 X_0 Y_0 + \sum_{n=1}^{\infty} A_n Y_n X_n$$

Therefore

$$u\left(x,y\right) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh\left(ny\right)\cos\left(nx\right) \tag{A}$$

What is left is to determine A_0 and A_n. At $y=\pi$ the above gives

$$f\left(x\right) = A_0\pi + \sum_{n=1}^{\infty} A_n \sinh\left(n\pi\right)\cos\left(nx\right)$$

Multiplying both sides by $\cos\left(mx\right)$ and integrating gives

$$\int_0^\pi f\left(x\right)\cos\left(mx\right)dx = \int_0^\pi A_0\pi\cos\left(mx\right)dx + \int_0^\pi \sum_{n=1}^{\infty} A_n\sinh\left(n\pi\right)\cos\left(nx\right)\cos\left(mx\right)dx \tag{1}$$

For $m=0$, (1) becomes

$$\begin{aligned}\int_0^\pi f\left(x\right)dx &= \int_0^\pi A_0\pi dx\\ \int_0^\pi f\left(x\right)dx &= A_0\pi^2\\ A_0 &= \frac{1}{\pi^2}\int_0^\pi f\left(x\right)dx \end{aligned} \tag{2}$$

For $m>0$, (1) becomes

$$\begin{aligned}\int_0^\pi f\left(x\right)\cos\left(mx\right)dx &= \int_0^\pi \sum_{n=1}^{\infty} A_n\sinh\left(n\pi\right)\cos\left(nx\right)\cos\left(mx\right)dx\\ \int_0^\pi f\left(x\right)\cos\left(mx\right)dx &= A_m\sinh\left(m\pi\right)\int_0^\pi \cos^2\left(nx\right)dx\\ &= A_m\sinh\left(m\pi\right)\frac{\pi}{2}\end{aligned}$$

Hence

$$A_n = \frac{2}{\pi\sinh\left(n\pi\right)}\int_0^\pi f\left(x\right)\cos\left(nx\right)dx \tag{3}$$

When $f\left(x\right)=u_0$ a constant, then (2) becomes

$$\begin{aligned}A_0 &= \frac{1}{\pi^2}\int_0^\pi u_0 dx\\ &= \frac{u_0}{\pi}\end{aligned}$$

And (3) becomes

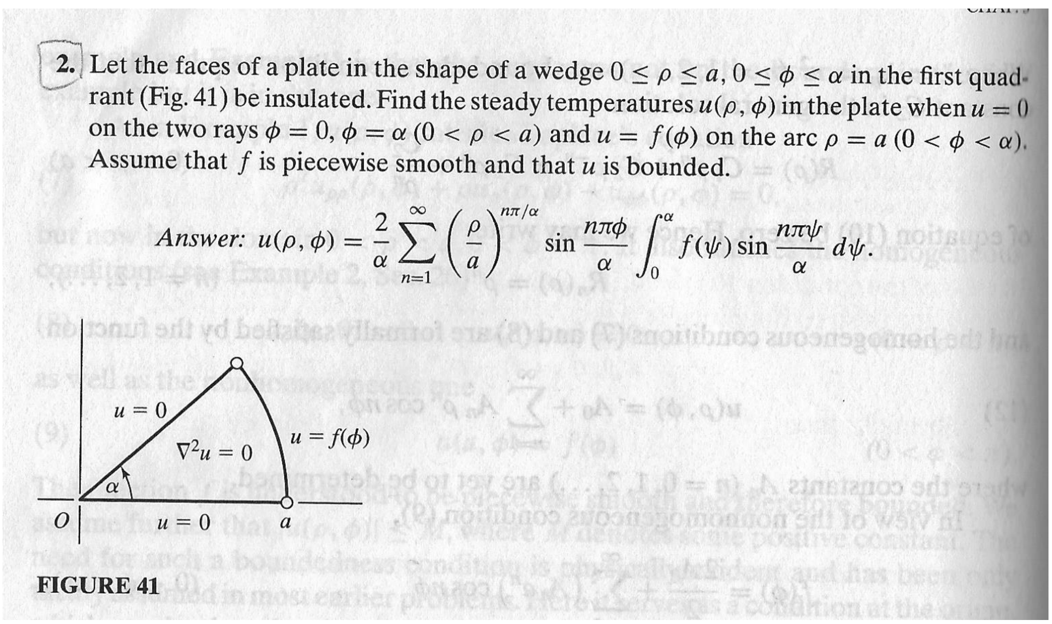
$$
\begin{aligned}
A_n &= \frac{2}{\pi \sinh(n\pi)} \int_0^{\pi} u_0 \cos(nx)\, dx \\
&= \frac{2u_0}{\pi \sinh(n\pi)} \left[\frac{\sin(nx)}{n} \right]_0^{\pi} \\
&= 0
\end{aligned}
$$

Hence the solution (A) becomes

$$
\boxed{u(x,y) = u_0 \frac{y}{\pi}}
$$

This shows the final solution changes linearly in y. When $y = 0$ then $u(x,0) = 0$ and when $y = \pi$, then $u(x,\pi) = u_0$.

2.6.8 Section 44, Problem 2



2. Let the faces of a plate in the shape of a wedge $0 \le \rho \le a$, $0 \le \phi \le \alpha$ in the first quadrant (Fig. 41) be insulated. Find the steady temperatures $u(\rho, \phi)$ in the plate when $u = 0$ on the two rays $\phi = 0$, $\phi = \alpha$ $(0 < \rho < a)$ and $u = f(\phi)$ on the arc $\rho = a$ $(0 < \phi < \alpha)$. Assume that f is piecewise smooth and that u is bounded.

Answer: $u(\rho, \phi) = \frac{2}{\alpha} \sum_{n=1}^{\infty} \left(\frac{\rho}{a}\right)^{n\pi/\alpha} \sin \frac{n\pi\phi}{\alpha} \int_0^{\alpha} f(\psi) \sin \frac{n\pi\psi}{\alpha}\, d\psi.$

FIGURE 41

Figure 2.73: Problem statement

<u>Solution</u>

The PDE $\nabla^2 u(\rho, \phi) = 0$ in polar coordinates is

$$
u_{\rho\rho} + \frac{1}{\rho} u_{\rho} + \frac{1}{\rho^2} u_{\phi\phi} = 0
$$

For $0<\rho<a$ and $0<\phi<\alpha$. With boundary conditions

$$\begin{aligned}u\left(\rho,0\right)&=0\\u\left(\rho,\alpha\right)&=0\\u\left(a,\phi\right)&=f\left(\phi\right)\end{aligned}$$

And since u is bounded, then we have an extra condition $u\left(0,\phi\right)<\infty$.

Let $u\left(\rho,\phi\right)=R\left(\rho\right)\Phi\left(\phi\right)$. Substituting into the above PDE gives

$$\begin{aligned}R''\Phi+\frac{1}{\rho}R'\Phi+\frac{1}{\rho^2}\Phi''R&=0\\\frac{R''}{R}+\frac{1}{\rho}\frac{R'}{R}+\frac{1}{\rho^2}\frac{\Phi''}{\Phi}&=0\\\frac{\Phi''}{\Phi}&=-\left(\rho^2\frac{R''}{R}+\rho\frac{R'}{R}\right)=-\lambda\end{aligned}$$

Where λ is the separation constant. The above gives the boundary values problem to solve for λ

$$\begin{aligned}\Phi''+\lambda\Phi&=0 \qquad (1)\\\Phi(0)&=0\\\Phi(\alpha)&=0\end{aligned}$$

And

$$\begin{aligned}\rho^2\frac{R''}{R}+\rho\frac{R'}{R}&=\lambda\\\rho^2R''+\rho R'-\lambda R&=0 \qquad (2)\end{aligned}$$

We start with (1) to find λ then use the result to solve (2). The ODE (1) we solved before, it has the eigenvalues

$$\lambda_n=\left(\frac{n\pi}{\alpha}\right)^2\qquad n=1,2,3,\cdots$$

And corresponding eigenfunctions

$$\Phi_n\left(\phi\right)=\sin\left(\frac{n\pi}{\alpha}\phi\right) \qquad (3)$$

Now (2) can be solved. This is a Euler ODE. Using $R\left(\rho\right)=\rho^m$ and substituting into (2) gives

$$\begin{aligned}
\rho^2 m(m-1)\rho^{m-2} + \rho m \rho^{m-1} - \left(\frac{n\pi}{\alpha}\right)^2 \rho^m &= 0\\
m(m-1)\rho^m + m\rho^m - \left(\frac{n\pi}{\alpha}\right)^2 \rho^m &= 0\\
m(m-1) + m - \left(\frac{n\pi}{\alpha}\right)^2 &= 0\\
m^2 &= \left(\frac{n\pi}{\alpha}\right)^2
\end{aligned}$$

Hence

$$m = \pm\frac{n\pi}{\alpha}$$

Therefore the solution to (2) is

$$R_n(\rho) = A_n \rho^{\frac{n\pi}{\alpha}} + B_n \rho^{\frac{-n\pi}{\alpha}}$$

We immediately reject the solution $\rho^{\frac{-n\pi}{\alpha}}$ since this blows up at origin where $\rho \rightarrow 0$. Hence the above becomes

$$R_n(\rho) = A_n \rho^{\frac{n\pi}{\alpha}} \tag{4}$$

Now that we found $\Phi_n(\phi)$ and $R_n(\rho)$, then we use superposition to obtain the general solution

$$\begin{aligned}
u(\rho,\phi) &= \sum_{n=1}^{\infty} R_n(\rho)\Phi_n(\phi)\\
&= \sum_{n=1}^{\infty} A_n \rho^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi}{\alpha}\phi\right) \qquad (5)
\end{aligned}$$

At $\rho = a$, $u(a,\phi) = f(\phi)$, hence the above becomes

$$f(\phi) = \sum_{n=1}^{\infty} A_n a^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi}{\alpha}\phi\right)$$

By orthogonality we obtain

$$\begin{aligned}
\int_0^{\alpha} f(\phi)\sin\left(\frac{m\pi}{\alpha}\phi\right)d\phi &= \int_0^{\alpha}\sum_{n=1}^{\infty} A_n a^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi}{\alpha}\phi\right)\sin\left(\frac{m\pi}{\alpha}\phi\right)d\phi\\
&= A_m a^{\frac{m\pi}{\alpha}} \int_0^{\alpha} \sin^2\left(\frac{m\pi}{\alpha}\phi\right)d\phi\\
&= A_m a^{\frac{m\pi}{\alpha}} \frac{\alpha}{2}
\end{aligned}$$

Solving for A_n from the above gives

$$A_n = \frac{2}{\alpha} a^{\frac{-n\pi}{\alpha}} \int_0^{\alpha} f(\phi)\sin\left(\frac{n\pi}{\alpha}\phi\right)d\phi$$

Substituting the above in (5) gives the final solution

$$
\begin{aligned}
u(\rho,\phi) &= \sum_{n=1}^{\infty}\left(\frac{2}{\alpha}a^{\frac{-n\pi}{\alpha}}\int_0^\alpha f(\psi)\sin\left(\frac{n\pi}{\alpha}\psi\right)d\psi\right)\rho^{\frac{n\pi}{\alpha}}\sin\left(\frac{n\pi}{\alpha}\phi\right)\\
&= \frac{2}{\alpha}\sum_{n=1}^{\infty}\left(\frac{\rho}{a}\right)^{\frac{n\pi}{\alpha}}\sin\left(\frac{n\pi}{\alpha}\phi\right)\left(\int_0^\alpha f(\psi)\sin\left(\frac{n\pi}{\alpha}\psi\right)d\psi\right)
\end{aligned}
$$

2.6.9 Section 49, Problem 2

2. Solve the boundary value problem

$$u_t(x,t) = ku_{xx}(x,t) \qquad (-\pi < x < \pi, t > 0),$$

$$u(-\pi,t) = u(\pi,t), \qquad u_x(-\pi,t) = u_x(\pi,t), \qquad u(x,0) = f(x).$$

The solution $u(x,t)$ represents, for example, temperatures in an insulated wire of length 2π that is bent into a unit circle and has a given temperature distribution along it. For

convenience, the wire is thought of as being cut at one point and laid on the x axis between $x = -\pi$ and $x = \pi$. The variable x then measures the distance along the wire, starting at the point $x = -\pi$; and the points $x = -\pi$ and $x = \pi$ denote the same point on the circle. The first two boundary conditions in the problem state that the temperatures and the flux must be the same for each of those values of x. This problem was of considerable interest to Fourier himself, and the wire has come to be known as *Fourier's ring*.

Answer: $u(x,t) = A_0 + \sum_{n=1}^{\infty} e^{-n^2kt}(A_n\cos nx + B_n\sin nx)$,

where

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

and

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad (n = 1,2,\ldots).$$

Figure 2.74: Problem statement

Solution

$$u_t = ku_{xx}$$

With $-\pi < x < \pi, t > 0$ and periodic boundary conditions

$$u(-\pi,t) = u(\pi,t)$$
$$u_x(-\pi,t) = u_x(\pi,t)$$

And initial conditions

$$u(x,0) = f(x)$$

Normal process of separation of variables leads to eigenvalue problem

$$X'' + \lambda X = 0 \tag{1}$$
$$X(-\pi) = X(\pi)$$
$$X'(-\pi) = X'(\pi)$$

And the time ODE

$$T' + k\lambda T = 0 \tag{2}$$

We start by solving (1) to find the eigenvalues and eigenfunctions.

Case $\lambda < 0$

Solution is

$$X(x) = A\cosh\left(\sqrt{-\lambda}x\right) + B\sinh\left(\sqrt{-\lambda}x\right)$$
$$X'(x) = A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}x\right) + B\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}x\right)$$

The boundary conditions $X(-\pi) = X(\pi)$ results in (using the fact that $\cosh$ is even and $\sinh$ is odd)

$$A\cosh\left(\sqrt{-\lambda}\pi\right) + B\sinh\left(\sqrt{-\lambda}\pi\right) = A\cosh\left(\sqrt{-\lambda}\pi\right) - B\sinh\left(\sqrt{-\lambda}\pi\right)$$
$$B\sinh\left(\sqrt{-\lambda}\pi\right) = -B\sinh\left(\sqrt{-\lambda}\pi\right)$$
$$B\sinh\left(\sqrt{-\lambda}\pi\right) = 0 \tag{3}$$

The boundary conditions $X'(-\pi) = X'(\pi)$ results in (using the fact that $\cosh$ is even and $\sinh$ is odd)

$$A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\pi\right) + B\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\pi\right) = -A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\pi\right) + B\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\pi\right)$$
$$A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\pi\right) = -A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\pi\right)$$
$$A\sinh\left(\sqrt{-\lambda}\pi\right) = 0 \tag{4}$$

So we obtain (3,4) equations, here they are again

$$B\sinh\left(\sqrt{-\lambda}\pi\right) = 0$$
$$A\sinh\left(\sqrt{-\lambda}\pi\right) = 0$$

There are two possibility, either $\sinh\left(\sqrt{-\lambda}\pi\right)=0$ or $\sinh\left(\sqrt{-\lambda}\pi\right)\neq 0$. If $\sinh\left(\sqrt{-\lambda}\pi\right)\neq 0$ then this leads to trivial solution, as it implies that both $A=0$ and $B=0$. On the other hand, if $\sinh\left(\sqrt{-\lambda}\pi\right)=0$ then this implies that $\sqrt{-\lambda}\pi=0$ since $\sinh$ is only zero when its argument is zero which is not the case here. This implies that $\lambda<0$ is not possible.

Case $\lambda=0$

The solution now becomes $X(x)=Ax+B$. Satisfying the boundary conditions $X(-\pi)=X(\pi)$ gives

$$\begin{aligned} A\pi+B &= -A\pi+B \\ 2A\pi &= 0 \\ A &= 0 \end{aligned}$$

Hence the solution becomes

$$\begin{aligned} X(x) &= B \\ X' &= 0 \end{aligned}$$

Satisfying the boundary conditions $X'(-\pi)=X'(\pi)$ gives $0=0$. Hence $\lambda=0$ is possible eigenvalue, with corresponding eigenfunction as constant, say 1.

Case $\lambda>0$

Solution is

$$\begin{aligned} X(x) &= A\cos\left(\sqrt{\lambda}x\right)+B\sin\left(\sqrt{\lambda}x\right) \\ X'(x) &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)+B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) \end{aligned}$$

The boundary conditions $X(-\pi)=X(\pi)$ results in (using the fact that $\cos$ is even and $\sin$ is odd)

$$\begin{aligned} A\cos\left(\sqrt{\lambda}\pi\right)+B\sin\left(\sqrt{\lambda}\pi\right) &= A\cos\left(\sqrt{\lambda}\pi\right)-B\sin\left(\sqrt{\lambda}\pi\right) \\ B\sin\left(\sqrt{\lambda}\pi\right) &= -B\sin\left(\sqrt{\lambda}\pi\right) \\ B\sin\left(\sqrt{\lambda}\pi\right) &= 0 \end{aligned} \tag{5}$$

The boundary conditions $X'(-\pi)=X'(\pi)$ results in (using the fact that $\cosh$ is even and $\sinh$ is odd)

$$\begin{aligned} -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right)+B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) &= A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right)+B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) \\ -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) &= A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) \\ A\sin\left(\sqrt{\lambda}\pi\right) &= 0 \end{aligned} \tag{6}$$

So we obtain (5,6) equations, here they are again

$$B\sin\left(\sqrt{\lambda}\pi\right) = 0$$
$$A\sin\left(\sqrt{\lambda}\pi\right) = 0$$

There are two possibility, either $\sin\left(\sqrt{\lambda}\pi\right) = 0$ or $\sin\left(\sqrt{\lambda}\pi\right) \neq 0$. If $\sin\left(\sqrt{\lambda}\pi\right) \neq 0$ then this leads to trivial solution, as it implies that both $A = 0$ and $B = 0$. If $\sin\left(\sqrt{\lambda}\pi\right) = 0$ then this implies that $\sqrt{\lambda}\pi = n\pi$ where $n = 1, 2, 3, \cdots$. Hence $\lambda > 0$ is possible with eigenvalues and corresponding eigenfunctions given by

$$\lambda_n = n^2 \qquad n = 1, 2, 3, \cdots$$
$$X_n(x) = A_n\cos(nx) + B_n\sin(nx)$$

Now that we solved the eigenvalue problem (1), we use the eigenvalues found to solve the time ODE (2)

$$T' + k\lambda_n T = 0$$

When $\lambda = 0$, this becomes $T' = 0$ or $T_0(t)$ is constant. When $\lambda > 0$ the solution is

$$\begin{aligned} T_n(t) &= e^{-k\lambda_n t} \\ &= e^{-kn^2 t} \end{aligned}$$

Hence the fundamental solution is

$$u_n(x,t) = X_n(x)\,T_n(t)$$

And by superposition, the general solution is

$$u(x,t) = A_0 X_0(x)\,T_0(t) + \sum_{n=1}^{\infty}\left(A_n\cos(nx) + B_n\sin(nx)\right)e^{-kn^2 t}$$

But $X_0(x) = 1$ and $T_0(t)$ is constant. Hence the above simplifies to

$$u(x,t) = A_0 + \sum_{n=1}^{\infty}\left(A_n\cos(nx) + B_n\sin(nx)\right)e^{-kn^2 t}$$

What is left is to find A_0, A_n, B_n. At $t = 0$ the above gives

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n\cos(nx) + B_n\sin(nx) \tag{7}$$

For $n = 0$, by orthogonality we obtain

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\,dx &= \int_{-\pi}^{\pi} A_0 dx \\ \int_{-\pi}^{\pi} f(x)\,dx &= A_0(2\pi) \\ A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \end{aligned}$$

For $n > 0$. We start by multiplying both sides of (7) by $\cos(mx)$ and integrating both sides.

This gives

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx &= \int_{-\pi}^{\pi}\left(\sum_{n=1}^{\infty} A_n\cos(nx)\cos(mx) + B_n\sin(nx)\cos(mx)\right)dx\\ &= \sum_{n=1}^{\infty} A_n\int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx + \sum_{n=1}^{\infty} B_n\int_{-\pi}^{\pi}\sin(nx)\cos(mx)\,dx\end{aligned}$$

But $\int_{-\pi}^{\pi}\sin(nx)\cos(mx)\,dx = 0$ for all n,m. And $\int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx = \int_{-\pi}^{\pi}\cos^2(mx)\,dx$ and zero for all other $n \neq m$. Hence the above simplifies to

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx &= A_m\int_{-\pi}^{\pi}\cos^2(mx)\,dx\\ &= A_m\pi\end{aligned}$$

Therefore

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

To find B_n we do the same, but now we multiply both sides of (7) by $\sin(mx)$ and this leads to

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx &= \int_{-\pi}^{\pi}\left(\sum_{n=1}^{\infty} A_n\cos(nx)\sin(mx) + B_n\sin(nx)\sin(mx)\right)dx\\ &= \sum_{n=1}^{\infty} A_n\int_{-\pi}^{\pi}\cos(nx)\sin(mx)\,dx + \sum_{n=1}^{\infty} B_n\int_{-\pi}^{\pi}\sin(nx)\sin(mx)\,dx\end{aligned}$$

But $\int_{-\pi}^{\pi}\cos(nx)\sin(mx)\,dx = 0$ for all n,m. And $\int_{-\pi}^{\pi}\sin(nx)\sin(mx)\,dx = \int_{-\pi}^{\pi}\sin^2(mx)\,dx$ and zero for all other $n \neq m$. Hence the above simplifies to

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx &= B_m\int_{-\pi}^{\pi}\sin^2(mx)\,dx\\ &= B_m\pi\end{aligned}$$

Therefore

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

This completes the solution. The final solution is

$$\begin{aligned}u(x,t) &= A_0 + \sum_{n=1}^{\infty}\left(A_n\cos(nx) + B_n\sin(nx)\right)e^{-kn^2t}\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \sum_{n=1}^{\infty} e^{-kn^2t}\left[\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx\right)\cos(nx) + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx\right)\sin(nx)\right]\end{aligned}$$

2.7 HW 7

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2.7.1 Section 45, Problem 4

4. A string, stretched between the points 0 and π on the x axis and initially at rest, is released from the position $y = f(x)$. Its motion is opposed by air resistance, which is proportional to the velocity at each point (Sec. 28). Let the unit of time be chosen so that the equation of motion becomes

$$y_{tt}(x,t) = y_{xx}(x,t) - 2\beta y_t(x,t) \qquad (0 < x < \pi,\ t > 0),$$

where β is a positive constant. Assuming that $0 < \beta < 1$, derive the expression

$$y(x,t) = e^{-\beta t}\sum_{n=1}^{\infty} B_n\left(\cos\alpha_n t + \frac{\beta}{\alpha_n}\sin\alpha_n t\right)\sin nx,$$

where

$$\alpha_n = \sqrt{n^2-\beta^2}, \qquad B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \qquad (n = 1, 2, \ldots),$$

for the transverse displacements.

Figure 2.75: Problem statement

Solution

Solve for $y(x,t)$ in

$$y_{tt} = y_{xx} - 2\beta y_t \qquad (t > 0, 0 < x < \pi) \tag{1}$$

Boundary conditions

$$y(0,t) = 0$$
$$y(\pi,t) = 0$$

Initial conditions

$$\begin{aligned} y(x,0) &= f(x) \\ y_t(x,0) &= 0 \end{aligned}$$

Let $y = XT$. Substituting in (1) gives

$$T''X = X''T - 2\beta T'X$$

Dividing by $XT \neq 0$

$$\begin{aligned} \frac{T''}{T} &= \frac{X''}{X} - 2\beta\frac{T'}{T} \\ \frac{T''}{T} + 2\beta\frac{T'}{T} &= \frac{X''}{X} = -\lambda \end{aligned}$$

Where λ is separation constant. Due to nature of boundary conditions being both homogeneous, then we know $\lambda > 0$ is only possible case from earlier HW's. The eigenvalue problem is

$$X'' + \lambda X = 0$$

Which we know has eigenvalues $\lambda = n^2$ for $n = 1, 2, \cdots$ with corresponding eigenfunctions

$$X_n = \sin(nx) \tag{1}$$

Now we solve the time ODE using these eigenvalues.

$$\begin{aligned} \frac{T''}{T} + 2\beta\frac{T'}{T} &= -n^2 \\ T'' + 2\beta T' + n^2 T &= 0 \end{aligned}$$

This is standard second order ODE with positive damping β and since n^2 is positive. The characteristic equation is

$$r^2 + 2\beta r + n^2 = 0$$

The roots are

$$\begin{aligned} r &= -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} \\ &= -\frac{2\beta}{2} \pm \frac{1}{2}\sqrt{4\beta^2 - 4n^2} \\ &= -\beta \pm \sqrt{\beta^2 - n^2} \\ &= -\beta \pm i\sqrt{n^2 - \beta^2} \end{aligned}$$

Hence the solution is

$$\begin{aligned} T_n(t) &= A_n e^{r_1 t} + B_n e^{r_2 t} \\ &= A_n e^{\left(-\beta + i\sqrt{n^2-\beta^2}\right)t} + B_n e^{\left(-\beta - i\sqrt{n^2-\beta^2}\right)t} \\ &= e^{-\beta t}\left(A_n e^{i\sqrt{n^2-\beta^2}t} + B_n e^{-i\sqrt{n^2-\beta^2}t}\right) \end{aligned}$$

But the above can be rewritten using Euler relation as (the constants A_n, B_n will be different, but kept them the same names for simplicity)

$$T_n(t) = e^{-\beta t}\left(A_n \cos\left(\sqrt{n^2-\beta^2}t\right) + B_n \sin\left(\sqrt{n^2-\beta^2}t\right)\right)$$

Let $\alpha_n = \sqrt{n^2-\beta^2}$, then the above becomes

$$T_n(t) = e^{-\beta t}\left(A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t)\right) \tag{2}$$

Since the PDE is linear and homogenous, then by superposition we obtain the final solution as

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} X_n T_n \\ &= \sum_{n=1}^{\infty} e^{-\beta t}\left(A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t)\right)\sin(nx) \end{aligned} \tag{3}$$

Now initial conditions are applied to determine A_n, B_n. At $t=0$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(nx)$$

Hence A_n are the Fourier sine coefficient of the representation of $f(x)$ which implies

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \tag{4}$$

Taking time derivative of (3) gives

$$y_t(x,t) = \sum_{n=1}^{\infty}\left[-\beta e^{-\beta t}\left(A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t)\right) + e^{-\beta t}\left(-\alpha_n A_n \sin(\alpha_n t) + \alpha_n B_n \cos(\alpha_n t)\right)\right]\sin(nx)$$

At $t=0$ the above becomes (since released from rest)

$$0 = \sum_{n=1}^{\infty}\left(-\beta A_n + \alpha_n B_n\right)\sin(nx)$$

Therefore

$$-\beta A_n + \alpha_n B_n = 0$$

Hence $B_n = \frac{\beta A_n}{\alpha_n}$. Therefore (3) becomes

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} e^{-\beta t}\left(A_n \cos(\alpha_n t) + \frac{\beta A_n}{\alpha_n}\sin(\alpha_n t)\right)\sin(nx) \\ &= e^{-\beta t}\sum_{n=1}^{\infty} A_n\left(\cos(\alpha_n t) + \frac{\beta}{\alpha_n}\sin(\alpha_n t)\right)\sin(nx) \end{aligned}$$

Where $A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$ and $\alpha_n = \sqrt{n^2-\beta^2}$. Which is the result required to show (Book used B in place A, but it is the same thing, just different name for a constant).

2.7.2 Section 46, Problem 2

2. Let a, b, and ω denote nonzero constants. The general solution of the ordinary differential equation

$$y''(t) + a^2 y(t) = b \sin \omega t$$

is of the form $y = y_c + y_p$, where y_c is the general solution of the complementary equation $y''(t) + a^2 y(t) = 0$ and y_p is any particular solution of the original nonhomogeneous equation.†

(*a*) Suppose that $\omega \neq a$. After substituting

$$y_p = A \cos \omega t + B \sin \omega t,$$

where A and B are constants, into the given differential equation, determine values of A and B such that y_p is a solution. Thus, derive the general solution

$$y(t) = C_1 \cos at + C_2 \sin at + \frac{b}{a^2 - \omega^2} \sin \omega t$$

of that equation.

(*b*) Suppose that $\omega = a$ and find constants A and B such that

$$y_p = At \cos \omega t + Bt \sin \omega t$$

is a particular solution of the given differential equation. Thus obtain the general solution

$$y(t) = C_1 \cos at + C_2 \sin at - \frac{b}{2a} t \cos at.$$

Figure 2.76: Problem statement

Solution

Part a

suppose $\omega \neq a$. Let

$$y_p = A \cos \omega t + B \sin \omega t \tag{1}$$

Then

$$\begin{aligned} y_p' &= -A\omega \sin \omega t + B\omega \cos \omega t \\ y_p'' &= -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t \end{aligned}$$

Substituting the above back into the given ODE gives

$$
\begin{aligned}
y_p''(t) + a^2 y_p(t) &= b\sin\omega t\\
\left(-A\omega^2\cos\omega t - B\omega^2\sin\omega t\right) + a^2\left(A\cos\omega t + B\sin\omega t\right) &= b\sin\omega t\\
\cos\omega t\left(-A\omega^2 + a^2 A\right) + \sin\omega t\left(-B\omega^2 + a^2 B\right) &= b\sin\omega t \qquad (2)
\end{aligned}
$$

By comparing coefficients, we see that

$$
\begin{aligned}
-A\omega^2 + a^2 A &= 0\\
A\left(a^2 - \omega^2\right) &= 0
\end{aligned}
$$

Since $\omega \neq a$ then this implies that $A = 0$. And from (2), we see that

$$
\begin{aligned}
-B\omega^2 + a^2 B &= b\\
B &= \frac{b}{a^2 - \omega^2}
\end{aligned}
$$

Therefore (1) becomes

$$
y_p = \frac{b}{a^2 - \omega^2}\sin\omega t \qquad (3)
$$

Now we need to find the complementary solution to

$$
y_c'' + a^2 y = 0
$$

Since $a^2 > 0$, then the solution is the standard one given by

$$
y_c(t) = C_1\cos at + C_2\sin at \qquad (4)
$$

Adding (3,4) gives the general solution

$$
y(t) = C_1\cos at + C_2\sin at + \frac{b}{a^2 - \omega^2}\sin\omega t
$$

Part (b)

Let

$$
y_p = At\cos\omega t + Bt\sin\omega t \qquad (1)
$$

Then

$$
\begin{aligned}
y_p' &= A\cos\omega t - At\omega\sin\omega t + B\sin\omega t + Bt\omega\cos\omega t\\
y_p'' &= -A\omega\sin\omega t - \left(A\omega\sin\omega t + At\omega^2\cos\omega t\right) + B\omega\cos\omega t + \left(B\omega\cos\omega t - Bt\omega^2\sin\omega t\right)\\
&= \left(-At\omega^2 + 2B\omega\right)\cos\omega t + \left(-2A\omega - Bt\omega^2\right)\sin\omega t
\end{aligned}
$$

Substituting the above back into the given ODE gives

$$
\begin{aligned}
y_p''(t) + a^2 y_p(t) &= b\sin\omega t\\
\left(\left(-At\omega^2 + 2B\omega\right)\cos\omega t + \left(-2A\omega - Bt\omega^2\right)\sin\omega t\right) + a^2\left(At\cos\omega t + Bt\sin\omega t\right) &= b\sin\omega t\\
\cos\omega t\left(-At\omega^2 + 2B\omega + a^2 At\right) + \sin\omega t\left(-2A\omega - Bt\omega^2 + a^2 Bt\right) &= b\sin\omega t \qquad (2)
\end{aligned}
$$

By comparing coefficients, we see that

$$-At\omega^2 + 2B\omega + a^2At = 0$$
$$At\left(-\omega^2 + a^2\right) + B(2\omega) = 0 \tag{3}$$

And from (2), we see also that

$$-2A\omega - Bt\omega^2 + a^2Bt = b$$
$$A(-2\omega) + Bt\left(-\omega^2 + a^2\right) = b \tag{4}$$

But since $\omega = a$, then (3) becomes

$$B(2\omega) = 0$$
$$B = 0$$

And (4) becomes

$$A(-2\omega) = b$$
$$A = \frac{-b}{2a}$$

Substituting these values we found for A, B, in (1) gives

$$y_p = \frac{-b}{2a} t \cos \omega t$$

But $\omega = a$, therefore

$$y_p = \frac{-b}{2a} t \cos at \tag{5}$$

The complementary solution do not change from part (a). Hence the general solution is

$$y(t) = C_1 \cos at + C_2 \sin at - \frac{b}{2a} t \cos at$$

Which is the result required to show.

2.7.3 Section 46, Problem 3

3. Use the general solutions derived in Problem 2 to obtain the following solutions of the initial value problem

$$y''(t) + a^2 y(t) = b \sin \omega t, \qquad y(0) = 0, \qquad y'(0) = 0:$$

$$y(t) = \begin{cases} \dfrac{b}{\omega^2 - a^2}\left(\dfrac{\omega}{a}\sin at - \sin \omega t\right) & \text{when } \omega \neq a, \\ \dfrac{b}{2a}\left(\dfrac{1}{a}\sin at - t\cos at\right) & \text{when } \omega = a. \end{cases}$$

Figure 2.77: Problem statement

Solution

The general solution from problem 2 is

$$y(t) = \begin{cases} C_1 \cos at + C_2 \sin at + \frac{b}{a^2-\omega^2} \sin \omega t & \omega \neq a \\ C_1 \cos at + C_2 \sin at - \frac{b}{2a} t \cos at & \omega = a \end{cases}$$

We need to find C_1, C_2 when initial conditions are $y(0) = 0, y'(0) = 0$ for each of the above cases.

case $\omega \neq a$

$y(0) = 0$ gives

$$0 = C_1$$

Hence solution now becomes

$$y(t) = C_2 \sin at + \frac{b}{a^2 - \omega^2} \sin \omega t$$

Taking time derivative gives

$$y'(t) = aC_2 \cos at + \frac{\omega b}{a^2 - \omega^2} \cos \omega t$$

At $t = 0$ the above gives

$$\begin{aligned} 0 &= aC_2 + \frac{\omega b}{a^2 - \omega^2} \\ C_2 &= \frac{1}{a} \frac{\omega b}{\omega^2 - a^2} \end{aligned}$$

Using C_1, C_2 found above, the solution becomes

$$\begin{aligned} y(t) &= \frac{1}{a} \frac{\omega b}{\omega^2 - a^2} \sin at + \frac{b}{a^2 - \omega^2} \sin \omega t \\ &= \frac{b}{a^2 - \omega^2} \left(\frac{\omega}{a} \sin at - \sin \omega t \right) \qquad (1) \end{aligned}$$

case $\omega = a$

$y(0) = 0$ gives

$$0 = C_1$$

Hence solution now becomes

$$y(t) = C_2 \sin at - \frac{b}{2a} t \cos at$$

Taking time derivative gives

$$y'(t) = aC_2 \cos at - \left(\frac{b}{2a} \cos at - \frac{b}{2a} t^2 \sin at \right)$$

At $t = 0$ the above gives

$$
\begin{aligned}
0 &= aC_2 - \frac{b}{2a} \\
C_2 &= \frac{1}{a}\frac{b}{2a}
\end{aligned}
$$

Using C_1, C_2 found above, the solution becomes

$$
\begin{aligned}
y(t) &= \frac{1}{a}\frac{b}{2a}\sin at - \frac{b}{2a} t\cos at \\
&= \frac{b}{2a}\left(\frac{1}{a}\sin at - t\cos at\right) \qquad (2)
\end{aligned}
$$

From (1,2) we see that

$$
y(t) = \begin{cases} \frac{b}{a^2-\omega^2}\left(\frac{\omega}{a}\sin at - \sin \omega t\right) & \omega \neq a \\ \frac{b}{2a}\left(\frac{1}{a}\sin at - t\cos at\right) & \omega = a \end{cases}
$$

Which is the result required to show.

2.7.4 Section 52, Problem 3

3. Assume that a function $f(x)$ has the Fourier integral representation (8), Sec. 50, which can be written

$$
f(x) = \lim_{c\to\infty} \int_0^c [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\, d\alpha.
$$

Use the exponential forms (compare with Problem 8, Sec. 15)

$$
\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}
$$

of the cosine and sine functions to show formally that

$$
f(x) = \lim_{c\to\infty}\int_{-c}^{c} C(\alpha)\, e^{i\alpha x}\, d\alpha,
$$

where

$$
C(\alpha) = \frac{A(\alpha) - i\,B(\alpha)}{2}, \qquad C(-\alpha) = \frac{A(\alpha)+i\,B(\alpha)}{2} \qquad (\alpha > 0).
$$

Then use expressions (9), Sec. 50, for $A(\alpha)$ and $B(\alpha)$ to obtain the single formula†

$$
C(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx \qquad (-\infty < \alpha < \infty).
$$

Figure 2.78: Problem statement

Solution

$$
\begin{aligned}
f(x) &= \int_0^\infty \left(A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)\right) d\alpha \\
&= \int_0^\infty \left(A(\alpha)\left(\frac{e^{i\alpha x}+e^{-i\alpha x}}{2}\right) - iB(\alpha)\left(\frac{e^{i\alpha x}-e^{-i\alpha x}}{2}\right)\right) d\alpha \\
&= \int_0^\infty \left(e^{i\alpha x}\left(\frac{A(\alpha)-iB(\alpha)}{2}\right) + e^{-i\alpha x}\left(\frac{A(\alpha)+iB(\alpha)}{2}\right)\right) d\alpha \\
&= \int_0^\infty e^{i\alpha x}\frac{A(\alpha)-iB(\alpha)}{2} d\alpha + \int_0^\infty e^{-i\alpha x}\frac{A(\alpha)+iB(\alpha)}{2} d\alpha \\
&= \int_0^\infty e^{i\alpha x}\frac{A(\alpha)-iB(\alpha)}{2} d\alpha + \int_0^\infty e^{-i\alpha x}\frac{A(\alpha)+iB(\alpha)}{2} d\alpha \\
&= \int_0^\infty e^{i\alpha x}\frac{A(\alpha)-iB(\alpha)}{2} d\alpha + \int_{-\infty}^0 e^{i\alpha x}\frac{A(\alpha)+iB(\alpha)}{2} d\alpha \\
&= \int_{-\infty}^\infty C(\alpha) e^{i\alpha x} d\alpha
\end{aligned}
$$

Where

$$
C(\alpha) = \frac{A(\alpha)-iB(\alpha)}{2} \quad , \quad C(-\alpha) = \frac{A(\alpha)+iB(\alpha)}{2} \qquad \alpha > 0
$$

Expression (9) section (5) is

$$
\begin{aligned}
A(\alpha) &= \frac{1}{\pi}\int_{-\infty}^\infty f(x)\cos(\alpha x)\, dx \\
B(\alpha) &= \frac{1}{\pi}\int_{-\infty}^\infty f(x)\sin(\alpha x)\, dx
\end{aligned}
$$

Substituting the above in $C(\alpha) = \frac{A(\alpha)-iB(\alpha)}{2}$ gives

$$
\begin{aligned}
C(\alpha) &= \frac{1}{2}\left(\frac{1}{\pi}\int_{-\infty}^\infty f(x)\cos(\alpha x)\, dx - i\frac{1}{\pi}\int_{-\infty}^\infty f(x)\sin(\alpha x)\, dx\right) \\
&= \frac{1}{2\pi}\left(\int_{-\infty}^\infty f(x)\cos(\alpha x)\, dx - \int_{-\infty}^\infty f(x)\, i\sin(\alpha x)\, dx\right) \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty f(x)\left(\cos(\alpha x) - i\sin(\alpha x)\right) dx
\end{aligned}
$$

But using Euler relation $\cos(\alpha x) - i\sin(\alpha x) = e^{i\alpha x}$ then the above reduces to

$$
C(\alpha) = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)\, e^{i\alpha x} dx \qquad -\infty < \alpha < \infty
$$

Which is what required to show.

2.7.5 Section 53, Problem 4

4. Use the theorem in Sec. 53 to show that if

$$f(x)=\begin{cases}0 & \text{when } x<0 \text{ or } x>\pi,\\ \sin x & \text{when } 0\leq x\leq \pi,\end{cases}$$

then

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\frac{\cos\alpha x+\cos\alpha(\pi-x)}{1-\alpha^2}\,d\alpha \qquad (-\infty<x<\infty).$$

In particular, write $x=\pi/2$ to show that

$$\int_0^{\infty}\frac{\cos(\alpha\pi/2)}{1-\alpha^2}\,d\alpha=\frac{\pi}{2}.$$

Figure 2.79: Problem statement

Solution

Since $f(x)$ is piecewise continuous and absolutely integrable (sine function), then

$$\frac{f(x^+)+f(x^-)}{2}=\frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty}f(s)\cos(\alpha(s-x))\,ds\right)d\alpha$$

Substituting for $f(s)$ inside the integral for the function given gives

$$\frac{f(x^+)+f(x^-)}{2}=\frac{1}{\pi}\int_0^{\infty}\left(\int_0^{\pi}\sin(s)\cos(\alpha s-\alpha x)\,ds\right)d\alpha$$

Where we used $\int_0^{\pi}$ only, since the function is zero everywhere else. Using $2\sin A\cos B=\sin(A+B)+\sin(A-B)$ then the above can be written as

$$\begin{aligned}\frac{f(x^+)+f(x^-)}{2}&=\frac{1}{\pi}\int_0^{\infty}\left(\frac{1}{2}\int_0^{\pi}\sin(s+\alpha s-\alpha x)+\sin(s-(\alpha s-\alpha x))\,ds\right)d\alpha\\&=\frac{1}{2\pi}\int_0^{\infty}\left(\int_0^{\pi}\sin(s+\alpha s-\alpha x)+\sin(s-\alpha s+\alpha x)\,ds\right)d\alpha \qquad (1)\end{aligned}$$

But

$$\begin{aligned}\int_0^{\pi}\sin(s+\alpha s-\alpha x)\,ds&=\left[\frac{-\cos(s+\alpha s-\alpha x)}{1+\alpha}\right]_0^{\pi}\\&=\frac{-1}{1+\alpha}(\cos(\pi+\alpha\pi-\alpha x)-\cos(-\alpha x))\\&=\frac{-1}{1+\alpha}(\cos(\pi+\alpha(\pi-x))-\cos(\alpha x))\end{aligned}$$

But $\cos(\pi+\alpha(\pi-x)) = -\cos(\alpha(\pi-x))$, and the above becomes

$$\int_0^{\pi} \sin(s+\alpha s-\alpha x)\,ds = \frac{1}{1+\alpha}\left(\cos(\alpha(\pi-x))+\cos(\alpha x)\right) \tag{2}$$

Similarly

$$\begin{aligned}\int_0^{\pi} \sin(s-\alpha s+\alpha x)\,ds &= \left[\frac{-\cos(s-\alpha s+\alpha x)}{1-\alpha}\right]_0^{\pi}\\ &= \frac{-1}{1-\alpha}\left(\cos(\pi-\alpha\pi+\alpha x)-\cos(\alpha x)\right)\\ &= \frac{-1}{1-\alpha}\left(\cos(\pi-\alpha(\pi+x))-\cos(\alpha x)\right)\\ &= \frac{-1}{1-\alpha}\left(-\cos(-\alpha(\pi+x))-\cos(\alpha x)\right)\\ &= \frac{1}{1-\alpha}\left(\cos(\alpha(\pi+x))+\cos(\alpha x)\right) \qquad (3)\end{aligned}$$

Substituting (2,3) back in (1) gives

$$\begin{aligned}\frac{f(x^+)+f(x^-)}{2} &= \frac{1}{2\pi}\int_0^{\infty}\left(\frac{1}{1+\alpha}\left(\cos(\alpha(\pi-x))+\cos(\alpha x)\right)+\frac{1}{1-\alpha}\left(\cos(\alpha(\pi+x))+\cos(\alpha x)\right)\right)d\alpha\\ &= \frac{1}{2\pi}\int_0^{\infty}\left(\cos(\alpha(\pi-x))\left(\frac{1}{1+\alpha}+\frac{1}{1-\alpha}\right)+\cos(\alpha x)\left(\frac{1}{1+\alpha}+\frac{1}{1-\alpha}\right)\right)d\alpha\\ &= \frac{1}{2\pi}\int_0^{\infty}\left(\cos(\alpha(\pi-x))\left(\frac{2}{1-\alpha^2}\right)+\cos(\alpha x)\left(\frac{2}{1-\alpha^2}\right)\right)d\alpha\\ &= \frac{1}{\pi}\int_0^{\infty}\frac{\cos(\alpha(\pi-x))+\cos(\alpha x)}{1-\alpha^2}d\alpha\end{aligned}$$

But $f(x)$ is continuous then $\frac{f(x^+)+f(x^-)}{2} = f(x)$ and the above becomes

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\frac{\cos(\alpha(\pi-x))+\cos(\alpha x)}{1-\alpha^2}d\alpha$$

When $x=\frac{\pi}{2}$ the above gives

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi}\int_0^{\infty}\frac{\cos\left(\alpha\left(\pi-\frac{\pi}{2}\right)\right)+\cos\left(\alpha\frac{\pi}{2}\right)}{1-\alpha^2}d\alpha$$

But $f\left(\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{2}\right)=1$, hence

$$\begin{aligned}1 &= \frac{1}{\pi}\int_0^{\infty}\frac{\cos\left(\alpha\frac{\pi}{2}\right)+\cos\left(\alpha\frac{\pi}{2}\right)}{1-\alpha^2}d\alpha\\ &= \frac{1}{\pi}\int_0^{\infty}\frac{2\cos\left(\alpha\frac{\pi}{2}\right)}{1-\alpha^2}d\alpha\end{aligned}$$

Therefore

$$\frac{\pi}{2} = \int_0^\infty \frac{\cos\left(\alpha\frac{\pi}{2}\right)}{1-\alpha^2} d\alpha$$

2.8 HW 8

Local contents

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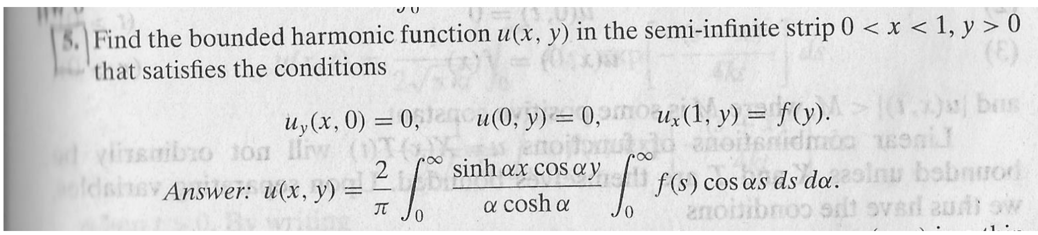
2.8.1 Section 57, Problem 5



5. Find the bounded harmonic function $u(x, y)$ in the semi-infinite strip $0 < x < 1$, $y > 0$ that satisfies the conditions

$$u_y(x,0) = 0, \quad u(0,y) = 0, \quad u_x(1,y) = f(y).$$

Answer: $u(x,y) = \frac{2}{\pi}\int_0^\infty \frac{\sinh \alpha x \cos \alpha y}{\alpha \cosh \alpha} \int_0^\infty f(s) \cos \alpha s \, ds \, d\alpha.$

Figure 2.80: Problem statement

<u>Solution</u>

$$\begin{aligned}
\nabla^2 u\left(x,y\right) &= 0 \qquad \left(0<x<1,y>0\right)\\
u_y\left(x,0\right) &= 0\\
u\left(0,y\right) &= 0\\
u_x\left(1,y\right) &= f\left(y\right)
\end{aligned}$$

As normal, we use separation of variables, ending in $\frac{X''}{X}+\frac{Y''}{Y}=-\lambda$. We will take the eigenvalue problem along the Y direction. This leads to

$$\begin{aligned}
Y''+\lambda Y &= 0\\
Y'\left(0\right) &= 0
\end{aligned}$$

Where $\lambda=\alpha^2,\alpha>0$. The steps that led to this were done before. Therefore the solution is

$$\begin{aligned}
Y\left(y\right) &= c_1\cos\left(\alpha y\right)+c_2\sin\left(\alpha y\right)\\
Y'\left(y\right) &= -c_1\alpha\sin\left(\alpha y\right)+c_2\alpha\cos\left(\alpha y\right)
\end{aligned}$$

At $y=0$ the above gives

$$0=c_2\alpha$$

Which implies $c_2 = 0$. Hence the eigenfunctions are

$$Y_\alpha\left(y\right) = \cos\left(\alpha y\right)$$

With the eigenvalues being $\lambda = \alpha^2$ for all real positive values of α. The corresponding $X(x)$ ode is

$$\begin{aligned} X'' - \lambda X &= 0 \\ X(0) &= 0 \end{aligned}$$

The solution to this is $X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$, which at $x = 0$ gives

$$0 = c_1 + c_2$$

Which makes the solution as $X(x) = c_1 e^{\alpha x} - c_1 e^{-\alpha x} = c_1\left(e^{\alpha x} - e^{-\alpha x}\right) = 2c_1 \sinh(\alpha x) = c_3 \sinh(\alpha x)$. Therefore the general solution is given by the real form of the Fourier integral

$$u\left(x,y\right) = \int_0^\infty A(\alpha)\sinh(\alpha x)\cos\left(\alpha y\right) d\alpha \tag{1}$$

Taking derivative w.r.t. x gives

$$u_x\left(x,y\right) = \int_0^\infty A(\alpha)\,\alpha\cosh(\alpha x)\cos\left(\alpha y\right) d\alpha$$

At $x = 1$ the above becomes

$$f\left(y\right) = \int_0^\infty \left(A(\alpha)\,\alpha\cosh(\alpha)\right)\cos\left(\alpha y\right) d\alpha$$

Therefore

$$\begin{aligned} A(\alpha)\,\alpha\cosh(\alpha) &= \frac{2}{\pi}\int_0^\infty f\left(y\right)\cos\left(\alpha y\right) d\alpha \\ A(\alpha) &= \frac{2}{\pi\alpha\cosh(\alpha)}\int_0^\infty f\left(y\right)\cos\left(\alpha y\right) d\alpha \end{aligned}$$

Substituting the above in (1) gives the solution

$$\begin{aligned} u\left(x,y\right) &= \int_0^\infty \left(\frac{2}{\pi\alpha\cosh(\alpha)}\int_0^\infty f(s)\cos(\alpha s)\,ds\right)\sinh(\alpha x)\cos\left(\alpha y\right) d\alpha \\ &= \frac{2}{\pi}\int_0^\infty \frac{\sinh(\alpha x)\cos\left(\alpha y\right)}{\alpha\cosh(\alpha)}\left(\int_0^\infty f(s)\cos(\alpha s)\,ds\right) d\alpha \end{aligned}$$

Which is the result required to show.

2.8.2 Section 58, Problem 5

5. (*a*) The face $x = 0$ of a semi-infinite solid $x \geq 0$ is insulated, and the initial temperature distribution is $f(x)$. Derive the temperature formula

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-x/(2\sqrt{kt})}^{\infty} f(x + 2\sigma\sqrt{kt})\, e^{-\sigma^2}\, d\sigma + \frac{1}{\sqrt{\pi}} \int_{x/(2\sqrt{kt})}^{\infty} f(-x + 2\sigma\sqrt{kt})\, e^{-\sigma^2}\, d\sigma.$$

(*b*) Show that if the function f in part (*a*) is defined by means of the equations

$$f(x) = \begin{cases} 1 & \text{when } 0 < x < c, \\ 0 & \text{when } \quad x > c, \end{cases}$$

then

$$u(x,t) = \frac{1}{2}\operatorname{erf}\left(\frac{c+x}{2\sqrt{kt}}\right) + \frac{1}{2}\operatorname{erf}\left(\frac{c-x}{2\sqrt{kt}}\right).$$

Figure 2.81: Problem statement

Solution

Part (a)

$$\begin{aligned} u_t(x,t) &= k u_{xx}(x,t) \qquad (0 < x < \infty, t > 0) \\ u(x,0) &= f(x) \\ u_x(0,t) &= 0 \end{aligned}$$

Applying separation of variables leads to

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda$$

Hence

$$\begin{aligned} X'' + \lambda X &= 0 \\ X'(0) &= 0 \\ |X(x)| &< M \end{aligned}$$

Since on semi-infinite domain, then only $\lambda > 0$ are possible eigenvalues. Let $\lambda = \alpha^2, \alpha > 0$, Where α takes on all positive real values. Then the solution to the eigenvalue ODE is

$$\begin{aligned} X_\alpha(x) &= c_1 \cos(\alpha x) + c_2 \sin(\alpha x) \\ X'_\alpha(x) &= -c_1 \alpha \sin(\alpha x) + c_2 \alpha \cos(\alpha x) \end{aligned}$$

At $x = 0$

$$0 = c_2 \alpha$$

Hence $c_2 = 0$ and the eigenfunctions are

$$X_\alpha(x) = \cos(\alpha x)$$

The time ODE is therefore $T' + \alpha^2 k T = 0$ which has solution $T = e^{-k\alpha^2 t}$. Hence the solution is given by the real Fourier integral

$$u(x,t) = \int_0^\infty A(\alpha) e^{-k\alpha^2 t} \cos(\alpha x) \, d\alpha \tag{1}$$

At $t = 0$, using initial conditions, then the above becomes

$$\begin{aligned} f(x) &= \int_0^\infty A(\alpha) \cos \alpha x d\alpha \\ A(\alpha) &= \frac{2}{\pi} \int_0^\infty f(s) \cos(\alpha s) \, ds \end{aligned} \tag{2}$$

Using (2) in (1) gives

$$u(x,t) = \int_0^\infty \left(\frac{2}{\pi} \int_0^\infty f(s) \cos(\alpha s) \, ds \right) e^{-k\alpha^2 t} \cos(\alpha x) \, d\alpha$$

Changing the order of integration

$$u(x,t) = \frac{1}{\pi} \int_0^\infty \int_0^\infty \left(e^{-k\alpha^2 t} \left[2\cos(\alpha x) \cos(\alpha s) \right] d\alpha \right) f(s) \, ds \tag{3}$$

Using trig identity $\cos(A)\cos(B) = \frac{\cos(A+B)+\cos(A-B)}{2}$, then

$$\begin{aligned} 2\cos(\alpha x)\cos(\alpha s) &= \cos(\alpha x + \alpha s) + \cos(\alpha x - \alpha s) \\ &= \cos(\alpha(x+s)) + \cos(\alpha(x-s)) \end{aligned}$$

Substituting the above in (3) gives

$$\begin{aligned} u(x,t) &= \frac{1}{\pi} \int_0^\infty \int_0^\infty \left(e^{-k\alpha^2 t} \left[\cos(\alpha(x+s)) + \cos(\alpha(x-s)) \right] d\alpha \right) f(s) \, ds \\ &= \frac{1}{\pi} \int_0^\infty \left(\int_0^\infty e^{-k\alpha^2 t} \cos(\alpha(x+s)) \, d\alpha + \int_0^\infty e^{-k\alpha^2 t} \cos(\alpha(x-s)) \, d\alpha \right) f(s) \, ds \end{aligned}$$

Using the formula

$$\int_0^\infty e^{-\alpha^2 c} \cos(\alpha b) \, d\alpha = \frac{1}{2} \sqrt{\frac{\pi}{c}} \exp\left(-\frac{b^2}{4c} \right)$$

Where in our case $c = kt$ and $b = (x+s)$ for the first integral, and $b = (x-s)$ for the second integral. Using the above formula in (4) results in

$$u(x,t) = \frac{1}{\pi} \int_0^\infty \left(\frac{1}{2} \sqrt{\frac{\pi}{kt}} \exp\left(-\frac{(x+s)^2}{4kt} \right) + \frac{1}{2} \sqrt{\frac{\pi}{kt}} \exp\left(-\frac{(x-s)^2}{4kt} \right) \right) f(s) \, ds$$

For $t > 0$. Hence the above becomes

$$u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_0^\infty f(s) \exp\left(-\frac{(x+s)^2}{4kt}\right) ds + \frac{1}{2\sqrt{\pi k t}} \int_0^\infty f(s) \exp\left(-\frac{(x-s)^2}{4kt}\right) ds$$

By writing $s = -x + 2\sigma\sqrt{kt}$ for the first integral above, then $\frac{ds}{d\sigma} = 2\sqrt{kt}$. When $s = 0$ then $\sigma = \frac{x}{2\sqrt{kt}}$ and when $s = \infty$ then $\sigma = \infty$. And by writing $s = x + 2\sigma\sqrt{kt}$ for the second integral above, then $\frac{ds}{d\sigma} = 2\sqrt{kt}$. When $s = 0$ then $\sigma = -\frac{x}{2\sqrt{kt}}$ Hence the above integral becomes

$$u(x,t) = \frac{2\sqrt{kt}}{2\sqrt{\pi k t}} \int_{\frac{x}{2\sqrt{kt}}}^\infty f\left(-x + 2\sigma\sqrt{kt}\right) \exp\left(-\frac{\left(-x + \left(x + 2\sigma\sqrt{kt}\right)\right)^2}{4kt}\right) d\sigma$$
$$+ \frac{2\sqrt{kt}}{2\sqrt{\pi k t}} \int_{-\frac{x}{2\sqrt{kt}}}^\infty f\left(x + 2\sigma\sqrt{kt}\right) \exp\left(-\frac{\left(x - \left(x + 2\sigma\sqrt{kt}\right)\right)^2}{4kt}\right) d\sigma$$

Simplifying gives

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{\pi}} \int_{-\frac{x}{2\sqrt{kt}}}^\infty f\left(x + 2\sigma\sqrt{kt}\right) e^{-\frac{\left(-2\sigma\sqrt{kt}\right)^2}{4kt}} d\sigma + \frac{1}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{kt}}}^\infty f\left(-x + 2\sigma\sqrt{kt}\right) e^{-\frac{\left(2\sigma\sqrt{kt}\right)^2}{4kt}} d\sigma \\ &= \frac{1}{\sqrt{\pi}} \int_{-\frac{x}{2\sqrt{kt}}}^\infty f\left(x + 2\sigma\sqrt{kt}\right) e^{-\sigma^2} d\sigma + \frac{1}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{kt}}}^\infty f\left(-x + 2\sigma\sqrt{kt}\right) e^{-\sigma^2} d\sigma + \end{aligned} \tag{4}$$

Which is the result required to show.

Part b

$$f(x) = \begin{cases} 1 & 0 < x < c \\ 0 & x > 0 \end{cases}$$

Considering the first function in (4), where in the following $f(x) \equiv f\left(x + 2\sigma\sqrt{kt}\right)$ then (4) becomes

$$u(x,t) = \frac{1}{\sqrt{\pi}} \left(\int_0^{\frac{c+x}{2\sqrt{kt}}} e^{-\sigma^2} d\sigma + \int_0^{\frac{c-x}{2\sqrt{kt}}} e^{-\sigma^2} d\sigma \right)$$

But $\frac{2}{\sqrt{\pi}} \int_0^{\frac{c+x}{2\sqrt{kt}}} e^{-\sigma^2} d\sigma = \operatorname{erf}\left(\frac{c+x}{2\sqrt{kt}}\right)$ and $\frac{2}{\sqrt{\pi}} \int_0^{\frac{c-x}{2\sqrt{kt}}} e^{-\sigma^2} d\sigma = \operatorname{erf}\left(\frac{c-x}{2\sqrt{kt}}\right)$, hence the above becomes

$$u(x,t) = \frac{1}{2} \operatorname{erf}\left(\frac{c+x}{2\sqrt{kt}}\right) + \frac{1}{2} \operatorname{erf}\left(\frac{c-x}{2\sqrt{kt}}\right)$$

2.8.3 Section 58, Problem 7

7. Verify that for any constant C, the function

$$v(x,t) = Cxt^{-3/2}\exp\left(-\frac{x^2}{4kt}\right)$$

satisfies the heat equation $v_t = kv_{xx}$ when $x > 0$ and $t > 0$. Also, verify that for those values of x and t,

Thus show that $v(x,t)$ can be added to the solution (9) found in Sec. 58 to form other solutions of the problem there if the temperature function is not required to be bounded. Note that v is unbounded as x and t tend to zero (this can be seen by letting x vanish while $t = x^2$).

Figure 2.82: Problem statement

<u>Solution</u>

We need to substitute the solution $v(x,t) = Cxt^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}}$ into the PDE $v_t = kv_{xx}$ and see if it satisfies it.

$$\begin{aligned} v_t &= \frac{-3}{2}Cxt^{\frac{-5}{2}}e^{\frac{-x^2}{4kt}} + Cxt^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}}\left(\frac{x^2}{4kt^2}\right) \\ &= \frac{-3}{2}Cxt^{\frac{-5}{2}}e^{\frac{-x^2}{4kt}} + C\frac{x^3}{4kt^2}t^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}} \end{aligned}$$

And

$$\begin{aligned} v_x &= Ct^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}} - \frac{x^2}{2kt}Ct^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}} \\ v_{xx} &= \frac{-x}{2kt}Ct^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}} - \left(\frac{x}{kt}Ct^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}} - \frac{4x^3}{(4kt)^2}Ct^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}}\right) \\ &= \frac{-2x}{4kt}Ct^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}} - \left(\frac{x}{kt}Ct^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}} - \frac{x^3}{4k^2t^2}Ct^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}}\right) \\ &= \frac{-x}{2kt}Ct^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}} - \frac{x}{kt}Ct^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}} + \frac{4x^3}{(4kt)^2}Ct^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}} \\ &= -\frac{3}{2}\frac{x}{k}Ct^{\frac{-5}{2}}e^{\frac{-x^2}{4kt}} + C\frac{x^3}{4k^2t^2}t^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}} \end{aligned}$$

Hence $v_t = kv_{xx}$ becomes

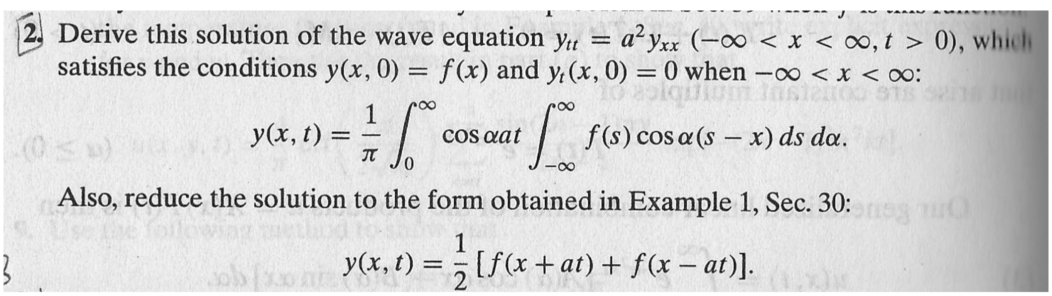
$$\frac{-3}{2}Cxt^{\frac{-5}{2}}e^{\frac{-x^2}{4kt}} + C\frac{x^3}{4kt^2}t^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}} = k\left(-\frac{3}{2}\frac{x}{k}Ct^{\frac{-5}{2}}e^{\frac{-x^2}{4kt}} + C\frac{x^3}{4k^2t^2}t^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}}\right)$$
$$\frac{-3}{2}Cxt^{\frac{-5}{2}}e^{\frac{-x^2}{4kt}} + C\frac{x^3}{4kt^2}t^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}} = -\frac{3}{2}xCt^{\frac{-5}{2}}e^{\frac{-x^2}{4kt}} + C\frac{x^3}{4kt^2}t^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}}$$
$$0 = 0$$

Hence it is satisfied for any constant C.

Using $v(x,t) = Cxt^{\frac{-3}{2}}e^{\frac{-x^2}{4kt}}$, we see that $\lim_{x\to 0^+} v(x,t) = 0$. Also $\lim_{t\to 0^+} v(x,t) = 0$.

Since the solution to the heat PDE is now not required to be bounded and since $v(x,t)$ has zero initial conditions, then because the PDE is linear and homogeneous, then solution as $v(x,t)$ can be added to the solution in (9) using superposition.

2.8.4 Section 59, Problem 2



2. Derive this solution of the wave equation $y_{tt} = a^2 y_{xx}$ $(-\infty < x < \infty, t > 0)$, which satisfies the conditions $y(x,0) = f(x)$ and $y_t(x,0) = 0$ when $-\infty < x < \infty$:

$$y(x,t) = \frac{1}{\pi}\int_0^\infty \cos\alpha a t \int_{-\infty}^{\infty} f(s)\cos\alpha(s-x)\,ds\,d\alpha.$$

Also, reduce the solution to the form obtained in Example 1, Sec. 30:

$$y(x,t) = \frac{1}{2}[f(x+at) + f(x-at)].$$

Figure 2.83: Problem description

<u>solution</u>

Let $y(x,t) = X(x)T(t)$, then the PDE becomes

$$T''X = a^2X''T$$
$$\frac{1}{a^2}\frac{T''}{T} = \frac{X''}{X} = -\lambda$$

We take the $X(x)$ ode as the eigenvalue problem. Since the domain is infinite, then only positive eigenvalue are valid as was shown before. Let $\lambda = \alpha^2, \alpha > 0$. Hence the eigenfunctions are

$$X_\alpha(x) = A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)$$

The time ODE becomes

$$\frac{1}{a^2}\frac{T''}{T} = -\alpha^2$$
$$T'' + a^2\alpha^2 T = 0$$

Which has the solution

$$T_\alpha(t) = C(\alpha)\cos(a\alpha t) + D(\alpha)\sin(a\alpha t)$$

Hence the solution is given by the Fourier real integral

$$\begin{aligned}
y(x,t) &= \int_0^\infty T_\alpha(t) X_\alpha(x)\,d\alpha \qquad (1)\\
&= \int_0^\infty \left(C(\alpha)\cos(a\alpha t) + D(\alpha)\sin(a\alpha t)\right)\left(A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)\right)d\alpha\\
&= \int_0^\infty C(\alpha)A(\alpha)\cos(a\alpha t)\cos(\alpha x)\,d\alpha + \int_0^\infty C(\alpha)B(\alpha)\cos(a\alpha t)\sin(\alpha x)\,d\alpha\\
&\quad + \int_0^\infty D(\alpha)A(\alpha)\sin(a\alpha t)\cos(\alpha x)\,d\alpha + \int_0^\infty D(\alpha)B(\alpha)\sin(a\alpha t)\sin(\alpha x)\,d\alpha \qquad (2)
\end{aligned}$$

Taking time derivative

$$\begin{aligned}
y_t(x,t) &= \int_0^\infty -a\alpha C(\alpha)A(\alpha)\sin(a\alpha t)\cos(\alpha x)\,d\alpha + \int_0^\infty a\alpha C(\alpha)B(\alpha)\sin(a\alpha t)\sin(\alpha x)\,d\alpha\\
&\quad + \int_0^\infty a\alpha D(\alpha)A(\alpha)\cos(a\alpha t)\cos(\alpha x)\,d\alpha + \int_0^\infty a\alpha D(\alpha)B(\alpha)\cos(a\alpha t)\sin(\alpha x)\,d\alpha
\end{aligned}$$

At $t=0$ the above becomes

$$0 = \int_0^\infty a\alpha D(\alpha)A(\alpha)\cos(\alpha x)\,d\alpha + \int_0^\infty a\alpha D(\alpha)B(\alpha)\sin(\alpha x)\,d\alpha$$

Which simplifies to

$$\begin{aligned}
0 &= \int_0^\infty D(\alpha)A(\alpha)\cos(\alpha x)\,d\alpha + \int_0^\infty D(\alpha)B(\alpha)\sin(\alpha x)\,d\alpha\\
&= \int_0^\infty D(\alpha)\left(A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)\right)d\alpha
\end{aligned}$$

Therefore, since $A(\alpha), B(\alpha)$ can not be both zero, else eigenfunction is zero, then it must be that $D(\alpha)=0$. Hence the solution in (2) becomes

$$y(x,t) = \int_0^\infty C(\alpha)A(\alpha)\cos(a\alpha t)\cos(\alpha x)\,d\alpha + \int_0^\infty C(\alpha)B(\alpha)\cos(a\alpha t)\sin(\alpha x)\,d\alpha \qquad (3)$$

Let $C(\alpha)A(\alpha) = C_1(\alpha)$ and let $C(\alpha)B(\alpha) = C_2(\alpha)$ as two new constants, and the above becomes

$$y(x,t) = \int_0^\infty C_1(\alpha)\cos(a\alpha t)\cos(\alpha x)\,d\alpha + \int_0^\infty C_2(\alpha)\cos(a\alpha t)\sin(\alpha x)\,d\alpha$$

At $t=0$ the above becomes

$$f(x) = \int_0^\infty C_1(\alpha)\cos(\alpha x)\,d\alpha + \int_0^\infty C_2(\alpha)\sin(\alpha x)\,d\alpha$$

Hence

$$C_1(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(s)\cos(\alpha s)\,ds$$
$$C_2(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(s)\sin(\alpha s)\,ds$$

Therefore (3) becomes

$$\begin{aligned} y(x,t) &= \frac{1}{\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} f(s)\cos(\alpha s)\,ds\right)\cos(a\alpha t)\cos(\alpha x)\,d\alpha \\ &+ \frac{1}{\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} f(s)\sin(\alpha s)\,ds\right)\cos(a\alpha t)\sin(\alpha x)\,d\alpha \end{aligned}$$

Changing order of integrations in the above for both integrals results in

$$\begin{aligned} y(x,t) &= \frac{1}{\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} \cos(a\alpha t)\cos(\alpha s)\cos(\alpha x)\,d\alpha\right) f(s)\,ds \\ &+ \frac{1}{\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} \cos(a\alpha t)\sin(\alpha s)\sin(\alpha x)\,d\alpha\right) f(s)\,ds \end{aligned} \tag{4}$$

But

$$\begin{aligned} \cos(\alpha s)\cos(\alpha x) &= \frac{1}{2}\left(\cos(\alpha s + \alpha x) + \cos(\alpha s - \alpha x)\right) \\ &= \frac{1}{2}\left(\cos(\alpha(s+x)) + \cos(\alpha(s-x))\right) \end{aligned}$$

and

$$\begin{aligned} \sin(\alpha s)\sin(\alpha x) &= \frac{1}{2}\left(\cos(\alpha s - \alpha x) - \cos(\alpha s + \alpha x)\right) \\ &= \frac{1}{2}\left(\cos(\alpha(s-x)) - \cos(\alpha(s+x))\right) \end{aligned}$$

Substituting the above two relations back in (4) gives

$$\begin{aligned} y(x,t) &= \frac{1}{2\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} \cos(a\alpha t)\left(\cos(\alpha(s+x)) + \cos(\alpha(s-x))\right)d\alpha\right) f(s)\,ds \\ &+ \frac{1}{2\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} \cos(a\alpha t)\left(\cos(\alpha(s-x)) - \cos(\alpha(s+x))\right)d\alpha\right) f(s)\,ds \end{aligned}$$

Simplifying, terms cancel giving

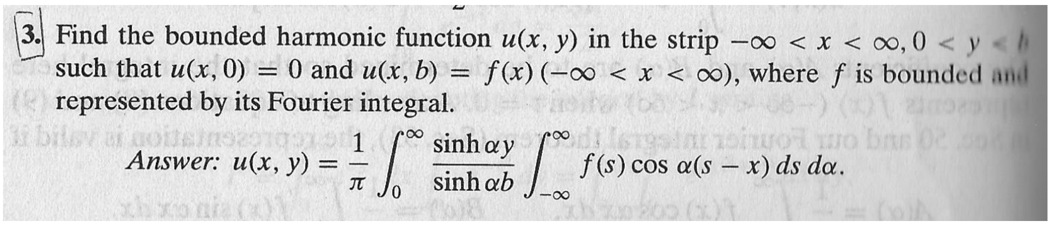
$$\begin{aligned} y(x,t) &= \frac{1}{2\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} \cos(a\alpha t)\left[\cos(\alpha(s-x)) + \cos(\alpha(s-x))\right]d\alpha\right) f(s)\,ds \\ &= \frac{1}{\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} \cos(a\alpha t)\cos(\alpha(s-x))\,d\alpha\right) f(s)\,ds \end{aligned}$$

Changing order of integration

$$y(x,t) = \frac{1}{\pi}\int_{0}^{\infty}\cos(a\alpha t)\int_{-\infty}^{\infty} f(s)\cos(\alpha(s-x))\,ds\,d\alpha$$

Which is the result required to show.

2.8.5 Section 59, Problem 3



3. Find the bounded harmonic function $u(x, y)$ in the strip $-\infty < x < \infty, 0 < y < b$ such that $u(x, 0) = 0$ and $u(x, b) = f(x)$ $(-\infty < x < \infty)$, where f is bounded and represented by its Fourier integral.

Answer: $u(x, y) = \frac{1}{\pi} \int_0^\infty \frac{\sinh \alpha y}{\sinh \alpha b} \int_{-\infty}^{\infty} f(s) \cos \alpha(s - x)\, ds\, d\alpha.$

Figure 2.84: Problem description

<u>solution</u>

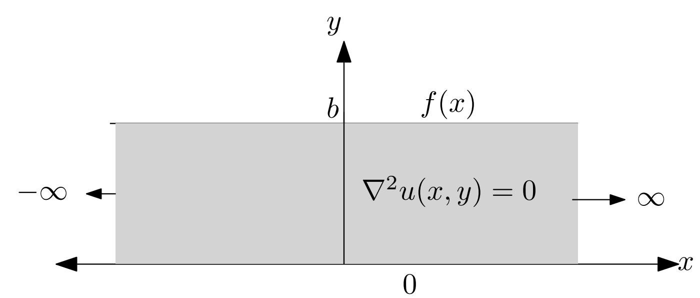


Figure 2.85: Solution domain for PDE

Let $u = X(x)\, Y(y)$, then $u_{xx} + y_{xx} = 0$ becomes

$$X''X + Y''X = 0$$
$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

Taking the eigenvalue ODE to be on the x axis, then

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

Hence

$$X'' + \lambda X = 0$$
$$|X(x)| < \infty$$

Hence λ can only be positive real. Let $\lambda = \alpha^2, \alpha > 0$. Therefore the eigenfunctions are

$$X_\alpha(x) = A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x \tag{1}$$

For the ODE $Y'' - Y\alpha^2 = 0$ the solution is

$$Y_\alpha(y) = C(\alpha)\cosh(\alpha y) + D(\alpha)\sinh(\alpha y) \tag{2}$$

Hence the solution is

$$\begin{aligned} u(x,y) &= \int_0^\infty X_\alpha(x)\,Y_\alpha(y)\,d\alpha \\ &= \int_0^\infty (A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x)\left(C(\alpha)\cosh(\alpha y) + D(\alpha)\sinh(\alpha y)\right)d\alpha \end{aligned} \tag{3}$$

When $y = 0$, the above becomes

$$0 = \int_0^\infty (A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x)\,C(\alpha)\,d\alpha$$

Which implies that $C(\alpha) = 0$. Therefore the solution (3) simplifies to

$$\begin{aligned} u(x,y) &= \int_0^\infty (A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x))\,D(\alpha)\sinh(\alpha y)\,d\alpha \\ &= \int_0^\infty A(\alpha)\,D(\alpha)\sinh(\alpha y)\cos\alpha x + B(\alpha)\,D(\alpha)\sinh(\alpha y)\sin(\alpha x)\,d\alpha \end{aligned}$$

Let $A(\alpha)D(\alpha) = C_1(\alpha)$ and let $B(\alpha)D(\alpha) = C_2(\alpha)$, hence the above solution becomes

$$u(x,y) = \int_0^\infty C_1(\alpha)\sinh(\alpha y)\cos\alpha x + C_2(\alpha)\sinh(\alpha y)\sin(\alpha x)\,d\alpha \tag{4}$$

When $y = b$ the above becomes

$$f(x) = \int_0^\infty C_1(\alpha)\sinh(\alpha b)\cos\alpha x + C_2(\alpha)\sinh(\alpha b)\sin(\alpha x)\,d\alpha$$

Therefore

$$\begin{aligned} C_1(\alpha)\sinh(\alpha b) &= \frac{1}{\pi}\int_{-\infty}^\infty f(s)\cos(\alpha s)\,ds \\ C_1(\alpha) &= \frac{1}{\pi\sinh(\alpha b)}\int_{-\infty}^\infty f(s)\cos(\alpha s)\,ds \end{aligned} \tag{5}$$

And

$$\begin{aligned} C_2(\alpha)\sinh(\alpha b) &= \frac{1}{\pi}\int_{-\infty}^\infty f(s)\sin(\alpha s)\,ds \\ C_2(\alpha) &= \frac{1}{\pi\sinh(\alpha b)}\int_{-\infty}^\infty f(s)\sin(\alpha s)\,ds \end{aligned} \tag{6}$$

Using (5,6) in (4) gives

$$
\begin{aligned}
u\left(x,y\right) &= \int_0^\infty \left(\frac{1}{\pi \sinh\left(\alpha b\right)} \int_{-\infty}^\infty f\left(s\right)\cos\left(\alpha s\right) ds\right) \sinh\left(\alpha y\right)\cos\left(\alpha x\right) + \left(\frac{1}{\pi \sinh\left(\alpha b\right)} \int_{-\infty}^\infty f\left(s\right)\sin\left(\alpha s\right) ds\right) \sinh\left(\alpha
\\
&= \int_0^\infty \left(\frac{\sinh\left(\alpha y\right)}{\pi \sinh\left(\alpha b\right)} \int_{-\infty}^\infty f\left(s\right)\cos\left(\alpha s\right)\cos\alpha x ds\right) + \left(\frac{\sinh\left(\alpha y\right)}{\pi \sinh\left(\alpha b\right)} \int_{-\infty}^\infty f\left(s\right)\sin\left(\alpha s\right)\sin\left(\alpha x\right) ds\right) d\alpha \\
&= \frac{1}{\pi}\int_0^\infty \frac{\sinh\left(\alpha y\right)}{\sinh\left(\alpha b\right)} \left(\int_{-\infty}^\infty f\left(s\right)\cos\left(\alpha s\right)\cos\alpha x + f\left(s\right)\sin\left(\alpha s\right)\sin\left(\alpha x\right) ds\right) d\alpha \\
&= \frac{1}{\pi}\int_0^\infty \frac{\sinh\left(\alpha y\right)}{\sinh\left(\alpha b\right)} \left(\int_{-\infty}^\infty f\left(s\right)\left[\cos\left(\alpha s\right)\cos\alpha x + \sin\left(\alpha s\right)\sin\left(\alpha x\right)\right] ds\right) d\alpha \\
&= \frac{1}{\pi}\int_0^\infty \frac{\sinh\left(\alpha y\right)}{\sinh\left(\alpha b\right)} \left(\int_{-\infty}^\infty f\left(s\right)\cos\alpha\left(s-x\right) ds\right) d\alpha
\end{aligned}
$$

Which is the result required to show.

2.9 HW 9

Local contents

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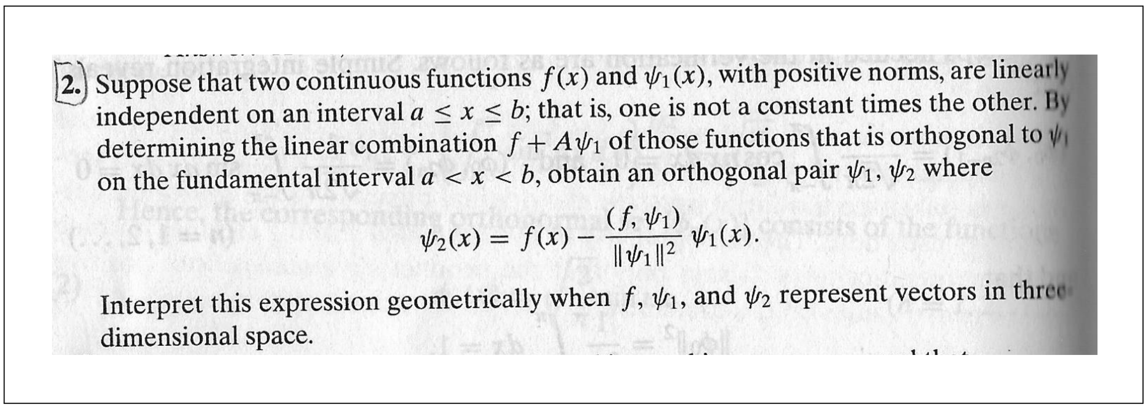
2.9.1 Section 61, Problem 2



2. Suppose that two continuous functions $f(x)$ and $\psi_1(x)$, with positive norms, are linearly independent on an interval $a \le x \le b$; that is, one is not a constant times the other. By determining the linear combination $f + A\psi_1$ of those functions that is orthogonal to ψ_1 on the fundamental interval $a < x < b$, obtain an orthogonal pair ψ_1, ψ_2 where

$$\psi_2(x) = f(x) - \frac{(f, \psi_1)}{\|\psi_1\|^2}\psi_1(x).$$

Interpret this expression geometrically when f, ψ_1, and ψ_2 represent vectors in three dimensional space.

Figure 2.86: Problem statement

Solution

Let $\psi_2 = f + A\psi_1$ such that $\langle \psi_2, \psi_1 \rangle = 0$. Hence

$$\begin{aligned}
\langle f + A\psi_1, \psi_1 \rangle &= 0\\
\langle f, \psi_1 \rangle + \langle A\psi_1, \psi_1 \rangle &= 0\\
\langle f, \psi_1 \rangle + A\langle \psi_1, \psi_1 \rangle &= 0\\
\langle f, \psi_1 \rangle + A\|\psi_1\|^2 &= 0\\
A &= -\frac{\langle f, \psi_1 \rangle}{\|\psi_1\|^2}
\end{aligned}$$

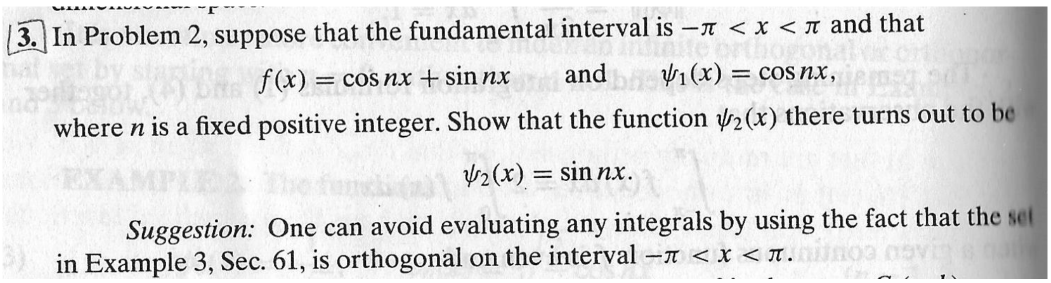
Therefore, since $\psi_2 = f + A\psi_1$ then

$$\psi_2 = f - \frac{\langle f, \psi_1 \rangle}{\|\psi_1\|^2}\psi_1$$

Geometrically, the term $\frac{\langle \psi_1, f \rangle}{\|\psi_1\|^2}\psi_1$ represents the projection of f on ψ_1. The term $\frac{\psi_1}{\|\psi_1\|}$ makes a unit vector in the direction of ψ_1 and the term $\frac{\langle f, \psi_1 \rangle}{\|\psi_1\|}$ is the magnitude of projection

$\left\|\psi_1\right\| \cos(\theta)$ where θ is the inner angle between f, ψ_1. The result of $-\frac{\langle f,\psi_1\rangle}{\left\|\psi_1\right\|^2}\psi_1$ is a vector in the opposite direction of ψ_1. Adding this to f gives ψ_2 which is now orthogonal to f. This process is called Gram Schmidt.

2.9.2 Section 61, Problem 3



3. In Problem 2, suppose that the fundamental interval is $-\pi < x < \pi$ and that

$$f(x) = \cos nx + \sin nx \quad \text{and} \quad \psi_1(x) = \cos nx,$$

where n is a fixed positive integer. Show that the function $\psi_2(x)$ there turns out to be

$$\psi_2(x) = \sin nx.$$

Suggestion: One can avoid evaluating any integrals by using the fact that the set in Example 3, Sec. 61, is orthogonal on the interval $-\pi < x < \pi$.

Figure 2.87: Problem statement

<u>Solution</u>

Let

$$\begin{aligned} f &= \cos nx + \sin nx \\ \psi_1 &= \cos nx \end{aligned}$$

Then by Gram Schmidt process from problem 2 we know that

$$\psi_2 = f - \frac{\langle f,\psi_1\rangle}{\left\|\psi_1\right\|^2}\psi_1$$

Hence

$$\begin{aligned} \psi_2 &= (\cos nx + \sin nx) - \frac{\int_{-\pi}^{\pi}(\cos nx + \sin nx)\cos nx dx}{\int_{-\pi}^{\pi}\cos^2(nx)\,dx}\cos nx \\ &= (\cos nx + \sin nx) - \frac{\int_{-\pi}^{\pi}\cos nx\cos nx dx + \int_{-\pi}^{\pi}\sin nx\cos nx dx}{\pi}\cos nx \end{aligned}$$

But $\int_{-\pi}^{\pi}\cos nx\cos nx dx = \int_{-\pi}^{\pi}\cos^2 nx dx = \pi$ and $\int_{-\pi}^{\pi}\sin nx\cos nx dx = 0$ since these are orthogonal. Hence the above simplifies to

$$\begin{aligned} \psi_2 &= (\cos nx + \sin nx) - \cos nx \\ &= \sin nx \end{aligned}$$

2.9.3 Section 63, Problem 3

3. In the space of *continuous* functions on the interval $a \le x \le b$, prove that if two functions f and g have the same Fourier constants with respect to a *closed* (Sec. 62) orthonormal set $\{\phi_n(x)\}$, then f and g must be identical. Thus show that f is uniquely determined by its Fourier constants.

Suggestion: Note that $(f - g, \phi_n) = 0$ for all values of n when

$$(f, \phi_n) = (g, \phi_n)$$

for all n. Then use the definition of a closed orthonormal set to show that $\|f - g\| = 0$. Finally, refer to the suggestion with Problem 4, Sec. 61.

Figure 2.88: Problem statement

Solution

The Fourier coefficients of $f - g$ are given by $\langle f - g, \phi_n \rangle$ by definition. But due to linearity of inner product, this can be written as

$$\langle f - g, \phi_n \rangle = \langle f, \phi_n \rangle - \langle g, \phi_n \rangle$$

But $\langle f, \phi_n \rangle$ are the Fourier coefficients of f and $\langle g, \phi_n \rangle$ are the Fourier coefficients of g, and we are told these are the same. Therefore

$$\langle f - g, \phi_n \rangle = 0$$

Which implies that $\|f - g\| = 0$. Using part(b) in problem 4, section 61, which says that if $\|f\| = 0$ then $f(x) = 0$ except at possibly finite number of points in the interval, then applying this to $\|f - g\| = 0$ leads to

$$f - g = 0$$

Which implies $f = g$ which is what required to show.

2.9.4 Section 63, Problem 4

4. Let $\{\phi_n(x)\}$ be an orthonormal set in the space of *continuous* functions on the interval $a \le x \le b$, and suppose that the generalized Fourier series for a function $f(x)$ in that space converges *uniformly* (Sec. 17) to a sum $s(x)$ on that interval.

(*a*) Show that $s(x)$ and $f(x)$ have the same Fourier constants with respect to $\{\phi_n(x)\}$.
(*b*) Use results in part (*a*) and Problem 3 to show that if $\{\phi_n(x)\}$ is *closed* (Sec. 62), then $s(x) = f(x)$ on the interval $a \leq x \leq b$.

Suggestion: Recall from Sec. 17 that the sum of a uniformly convergent series of continuous functions is continuous and that such a series can be integrated term by term.

Figure 2.89: Problem description

solution

Part (a)

Let the generalized Fourier series of $f(x)$ be

$$f(x) = \sum_{n=1}^{\infty} \langle f(x), \phi_n \rangle \phi_n$$

Let the sum the above converges uniformly to be $s(x)$. Therefore we have, per problem statement the following equality

$$\sum_{n=1}^{\infty} \langle f(x), \phi_n \rangle \phi_n = s(x)$$

Taking the inner product of both sides with respect to ϕ_m gives

$$\begin{aligned} \int_a^b \left(\sum_{n=1}^{\infty} \langle f(x), \phi_n \rangle \phi_n \right) \phi_m dx &= \int_a^b s(x) \phi_m dx \\ &= \langle s(x), \phi_m \rangle \end{aligned}$$

Since the sum converges uniformly, then we are allowed to integrate the left side term by term while keeping the equality with the right side. Hence moving the integration inside the sum gives

$$\sum_{n=1}^{\infty} \langle f(x), \phi_n \rangle \int_a^b \phi_n \phi_m dx = \langle s(x), \phi_m \rangle$$

But due to orthogonality of ϕ_n and ϕ_m and since they are normalized, then $\int_a^b \phi_n \phi_m dx = \langle \phi_n, \phi_m \rangle = 1$ if $n = m$ and zero otherwise. Hence the above simplifies to

$$\langle f(x), \phi_m \rangle = \langle s(x), \phi_m \rangle$$

And since the above is valid for any arbitrary $m = 1 \cdots \infty$, then it shows that $f(x)$ and $s(x)$ have the same generalized Fourier coefficients.

Part (b)

From part (a), we found

$$\langle f, \phi_n \rangle = \langle s, \phi_n \rangle$$

By linearity of inner product, the above is the same as

$$\begin{aligned} \langle f, \phi_n \rangle - \langle s, \phi_n \rangle &= 0 \\ \langle f - s, \phi_n \rangle &= 0 \end{aligned}$$

But from problem 3, we know that $\langle f - s, \phi_n \rangle = 0$ implies $\|f - s\| = 0$.

Next, using part(b) in problem 4, section 61, which says that if $\|f\| = 0$ then $f(x) = 0$ except at possibly finite number of points in the interval, then applying this to our case here that $\|f - s\| = 0$ leads to

$$\begin{aligned} f - s &= 0 \\ f &= s \end{aligned}$$

Which is the result required to show.

2.9.5 Section 66, Problem 4

4. (*a*) Use the same steps as in Example 3, Sec. 61, to verify that the set of functions

$$\phi_0(x) = \frac{1}{\sqrt{2c}}, \qquad \phi_{2n-1}(x) = \frac{1}{\sqrt{c}} \cos \frac{n\pi x}{c}, \qquad \phi_{2n}(x) = \frac{1}{\sqrt{c}} \sin \frac{n\pi x}{c} \qquad (n = 1, 2, \ldots)$$

is orthonormal on the interval $-c < x < c$. (This set becomes the one in that example when $c = \pi$.)

(*b*) By proceeding as in Example 3, Sec. 63, show that the generalized Fourier series corresponding to a function $f(x)$ in $C_p(-c, c)$ with respect to the orthonormal set in part (*a*) can be written as an ordinary Fourier series on $-c < x < c$ (Sec. 15), with the usual coefficients a_n and b_n.

(*c*) Derive Bessel's inequality

$$\frac{a_0^2}{2} + \sum_{n=1}^{N} \left(a_n^2 + b_n^2\right) \le \frac{1}{c} \int_{-c}^{c} [f(x)]^2 \, dx \qquad (N = 1, 2, \ldots)$$

for the coefficients a_n and b_n in part (*b*) from the general form (1), Sec. 65, of that inequality for Fourier constants. [Compare with inequality (6), Sec. 66.]

Suggestion: In part (*a*), some integrals to be used can be evaluated by writing

$$x = \frac{\pi}{c} s$$

in integrals (1) and (4), Sec. 61.

Figure 2.90: Problem description

<u>solution</u>

Part (a)

We need to find

$$\langle \phi_0, \phi_{2n} \rangle$$
$$\langle \phi_0, \phi_{2n-1} \rangle$$
$$\langle \phi_{2n}, \phi_{2m} \rangle$$
$$\langle \phi_{2n-1}, \phi_{2m-1} \rangle$$
$$\langle \phi_{2m-1}, \phi_{2n} \rangle$$

And also show that

$$\begin{aligned}
\langle \phi_0, \phi_0 \rangle &= \|\phi_0\|^2 = 1 \\
\langle \phi_{2n}, \phi_{2n} \rangle &= \|\phi_{2n}\|^2 = 1 \\
\langle \phi_{2n-1}, \phi_{2n-1} \rangle &= \|\phi_{2n-1}\|^2 = 1
\end{aligned}$$

$\underline{\langle \phi_0, \phi_{2n} \rangle}$

$$\begin{aligned}
\langle \phi_0, \phi_{2n} \rangle &= \int_{-c}^{c} \frac{1}{\sqrt{2c}} \frac{1}{\sqrt{c}} \cos\left(\frac{n\pi}{c}x\right) dx \\
&= \frac{1}{c\sqrt{2}} \left[\frac{\sin\left(\frac{n\pi}{c}x\right)}{\frac{n\pi}{c}} \right]_{-c}^{c} \\
&= \frac{c}{n\pi c\sqrt{2}} \left[\sin\left(\frac{n\pi}{c}x\right) \right]_{-c}^{c} \\
&= \frac{1}{n\pi\sqrt{2}} \left[\sin(n\pi) + \sin(n\pi) \right] \\
&= 0
\end{aligned}$$

Since n is integer.

$\underline{\langle \phi_0, \phi_{2n-1} \rangle}$

$$\begin{aligned}
\langle \phi_0, \phi_{2n-1} \rangle &= \int_{-c}^{c} \frac{1}{\sqrt{2c}} \frac{1}{\sqrt{c}} \sin\left(\frac{n\pi}{c}x\right) dx \\
&= \frac{1}{c\sqrt{2}} \left[\frac{-\cos\left(\frac{n\pi}{c}x\right)}{\frac{n\pi}{c}} \right]_{-c}^{c} \\
&= \frac{-c}{n\pi c\sqrt{2}} \left[\cos\left(\frac{n\pi}{c}x\right) \right]_{-c}^{c} \\
&= \frac{-1}{n\pi\sqrt{2}} \left[\cos(n\pi) - \cos(n\pi) \right] \\
&= 0
\end{aligned}$$

$\underline{\langle \phi_{2n}, \phi_{2m} \rangle}$

$$\begin{aligned}
\langle \phi_{2n}, \phi_{2m} \rangle &= \int_{-c}^{c} \frac{1}{\sqrt{c}} \sin\left(\frac{n\pi}{c}x\right) \frac{1}{\sqrt{c}} \sin\left(\frac{m\pi}{c}x\right) dx \\
&= \frac{1}{c} \int_{-c}^{c} \sin\left(\frac{n\pi}{c}x\right) \sin\left(\frac{m\pi}{c}x\right) dx
\end{aligned}$$

Let $\frac{c}{\pi}s = x$, then $dx = \frac{c}{\pi}ds$. When $x = -c$ then $s = -\pi$ and when $x = c$ then $s = \pi$ and the above becomes

$$\begin{aligned}\langle \phi_{2n}, \phi_{2m} \rangle &= \frac{1}{c}\int_{-\pi}^{\pi} \sin(ns)\sin(ms)\frac{c}{\pi}ds\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(ns)\sin(ms)\,ds\end{aligned}$$

Since the integrand is even, then

$$\langle \phi_{2n}, \phi_{2m} \rangle = \frac{2}{\pi}\int_{0}^{\pi} \sin(ns)\sin(ms)\,ds$$

From equation (1), page 192 we see that

$$\langle \phi_{2n}, \phi_{2m} \rangle = 0$$

Since n, m are different.

<u>$\langle \phi_{2n-1}, \phi_{2m-1} \rangle$</u>

$$\begin{aligned}\langle \phi_{2n-1}, \phi_{2m-1} \rangle &= \int_{-c}^{c} \frac{1}{\sqrt{c}}\cos\left(\frac{n\pi}{c}x\right)\frac{1}{\sqrt{c}}\cos\left(\frac{m\pi}{c}x\right)dx\\ &= \frac{1}{c}\int_{-c}^{c} \cos\left(\frac{n\pi}{c}x\right)\cos\left(\frac{m\pi}{c}x\right)dx\end{aligned}$$

Let $\frac{c}{\pi}s = x$, then $dx = \frac{c}{\pi}ds$. When $x = -c$ then $s = -\pi$ and when $x = c$ then $s = \pi$ and the above becomes

$$\begin{aligned}\langle \phi_{2n-1}, \phi_{2m-1} \rangle &= \frac{1}{c}\int_{-\pi}^{\pi} \cos(ns)\cos(ms)\frac{c}{\pi}ds\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(ns)\cos(ms)\,ds\end{aligned}$$

Since the integrand is even, then

$$\langle \phi_{2n-1}, \phi_{2m-1} \rangle = \frac{2}{\pi}\int_{0}^{\pi} \cos(ns)\cos(ms)\,ds$$

From equation (4), page 192 we see that

$$\langle \phi_{2n-1}, \phi_{2m-1} \rangle = 0$$

Since n, m are different.

<u>$\langle \phi_{2m-1}, \phi_{2n} \rangle$</u>

$$\begin{aligned}\langle \phi_{2m-1}, \phi_{2n} \rangle &= \int_{-c}^{c} \frac{1}{\sqrt{c}}\cos\left(\frac{m\pi}{c}x\right)\frac{1}{\sqrt{c}}\sin\left(\frac{n\pi}{c}x\right)dx\\ &= \frac{1}{c}\int_{-c}^{c} \cos\left(\frac{m\pi}{c}x\right)\sin\left(\frac{n\pi}{c}x\right)dx\end{aligned}$$

Let $\frac{c}{\pi}s = x$, then $dx = \frac{c}{\pi}ds$. When $x = -c$ then $s = -\pi$ and when $x = c$ then $s = \pi$ and the above becomes

$$\begin{aligned}\langle \phi_{2m-1}, \phi_{2n} \rangle &= \frac{1}{c} \int_{-\pi}^{\pi} \cos(ms)\sin(ns)\frac{c}{\pi} ds \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(ms)\sin(ns)\,ds\end{aligned}$$

Using $\cos(ms)\sin(ns) = \frac{1}{2}(\cos(s(m+n)) + \cos(s(m-n)))$. Hence the above becomes

$$\langle \phi_{2m-1}, \phi_{2n} \rangle = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi}\cos(s(m+n))\,ds + \int_{-\pi}^{\pi}\cos(s(m-n))\,ds\right)$$

Since the integration is over one full period, then each is zero. Hence

$$\langle \phi_{2m-1}, \phi_{2n} \rangle = 0$$

$\langle \phi_0, \phi_0 \rangle$

$$\begin{aligned}\langle \phi_0, \phi_0 \rangle &= \int_{-c}^{c} \frac{1}{\sqrt{2c}}\frac{1}{\sqrt{2c}} dx \\ \|\phi_0\|^2 &= \frac{1}{2c}\int_{-c}^{c} dx \\ &= 1\end{aligned}$$

Hence $\|\phi_0\| = 1$.

$\langle \phi_{2n}, \phi_{2n} \rangle$

$$\begin{aligned}\langle \phi_{2n}, \phi_{2n} \rangle &= \int_{-c}^{c} \frac{1}{\sqrt{c}}\sin\left(\frac{n\pi}{c}x\right)\frac{1}{\sqrt{c}}\sin\left(\frac{n\pi}{c}x\right)dx \\ &= \frac{1}{c}\int_{-c}^{c}\sin^2\left(\frac{n\pi}{c}x\right)dx \\ &= \frac{1}{c}\int_{-c}^{c}\frac{1}{2} - \frac{1}{2}\cos\left(2\frac{n\pi}{c}x\right)dx \\ &= \frac{1}{2c}\left(\int_{-c}^{c}dx - \int_{-c}^{c}\cos\left(2\frac{n\pi}{c}x\right)dx\right) \\ &= \frac{1}{2c}\left(2c - \left[\frac{\sin\left(2\frac{n\pi}{c}x\right)}{2\frac{n\pi}{c}}\right]_{-c}^{c}\right) \\ &= \frac{1}{2c}\left(2c - \frac{c}{2n\pi}\left[\sin\left(2\frac{n\pi}{c}x\right)\right]_{-c}^{c}\right) \\ &= \frac{1}{2c}(2c) \\ &= 1\end{aligned}$$

Hence $\|\phi_{2n}\| = 1$.

$\langle \phi_{2n-1}, \phi_{2n-1} \rangle$

$$
\begin{aligned}
\langle \phi_{2n-1}, \phi_{2n-1} \rangle &= \int_{-c}^{c} \frac{1}{\sqrt{c}} \cos\left(\frac{n\pi}{c}x\right) \frac{1}{\sqrt{c}} \cos\left(\frac{n\pi}{c}x\right) dx \\
\left\|\phi_{2n-1}\right\|^2 &= \frac{1}{c}\int_{-c}^{c} \cos^2\left(\frac{n\pi}{c}x\right) dx \\
&= \frac{1}{c}\int_{-c}^{c} \frac{1}{2} + \frac{1}{2}\sin\left(2\frac{n\pi}{c}x\right) dx \\
&= \frac{1}{2c}\left(\int_{-c}^{c} dx + \int_{-c}^{c} \sin\left(2\frac{n\pi}{c}x\right) dx\right) \\
&= \frac{1}{2c}\left(2c - \left[\frac{\cos\left(2\frac{n\pi}{c}x\right)}{2\frac{n\pi}{c}}\right]_{-c}^{c}\right) \\
&= \frac{1}{2c}\left(2c - \frac{c}{2n\pi}\left[\cos\left(2\frac{n\pi}{c}x\right)\right]_{-c}^{c}\right) \\
&= \frac{1}{2c}\left(2c - \frac{c}{2n\pi}\left[\cos(2n\pi) - \cos(2n\pi)\right]\right) \\
&= \frac{1}{2c}2c \\
&= 1
\end{aligned}
$$

Hence $\left\|\phi_{2n-1}\right\| = 1$.

Part (b)

$$
\begin{aligned}
\phi_0(x) &= \frac{1}{\sqrt{2c}} \\
\phi_{2n-1}(x) &= \frac{1}{\sqrt{c}}\cos\left(\frac{n\pi x}{c}\right) \\
\phi_{2n}(x) &= \frac{1}{\sqrt{c}}\sin\left(\frac{n\pi x}{c}\right)
\end{aligned}
$$

On $-c < x < c$. The generalized Fourier series for $f(x)$ in $C_p(-c, c)$ is

$$
\sum_{n=0}^{\infty} c_n\phi_n(x) = c_0\phi_0(x) + \sum_{n=1}^{\infty}\left(c_{2n-1}\phi_{2n-1}(x) + c_{2n}\phi_{2n}(x)\right)
$$

That is

$$
f(x) \sim c_0\frac{1}{\sqrt{2c}} + \sum_{n=1}^{\infty}\left(\frac{c_{2n-1}}{\sqrt{c}}\cos\left(\frac{n\pi x}{c}\right) + \frac{c_{2n}}{\sqrt{c}}\sin\left(\frac{n\pi x}{c}\right)\right) \tag{1}
$$

Where

$$
c_0 = \langle f, \phi_0(x) \rangle = \frac{1}{\sqrt{2c}}\int_{-c}^{c} f(x)\, dx
$$

And

$$c_{2n-1} = \langle f, \phi_{2n-1}(x) \rangle = \frac{1}{\sqrt{c}} \int_{-c}^{c} f(x) \cos\left(\frac{n\pi x}{c}\right) dx \qquad n = 1, 2, \cdots$$

$$c_{2n} = \langle f, \phi_{2n}(x) \rangle = \frac{1}{\sqrt{c}} \int_{-c}^{c} f(x) \sin\left(\frac{n\pi x}{c}\right) dx \qquad n = 1, 2, \cdots$$

If we write

$$a_0 = 2\frac{c_0}{\sqrt{2c}}, a_n = \frac{c_{2n-1}}{\sqrt{c}}, b_n = \frac{c_{2n}}{\sqrt{c}} \qquad n = 1, 2, \cdots$$

Then (1) becomes

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{c}\right) + b_n \sin\left(\frac{n\pi x}{c}\right)$$

Where

$$a_n = \frac{1}{c} \int_{-c}^{c} f(x) \cos\left(\frac{n\pi x}{c}\right) dx \qquad n = 1, 2, \cdots$$

$$b_n = \frac{1}{c} \int_{-c}^{c} f(x) \sin\left(\frac{n\pi x}{c}\right) dx \qquad n = 1, 2, \cdots$$

This is the ordinary Fourier series on $-c < x < c$.

Part (c)

From (1) section 65

$$\sum_{n=0}^{N} c_n^2 \leq \|f\|^2 \tag{1}$$

But from part (b) we found that

$$a_0 = 2\frac{c_0}{\sqrt{2c}}, a_n = \frac{c_{2n-1}}{\sqrt{c}}, b_n = \frac{c_{2n}}{\sqrt{c}} \qquad n = 1, 2, \cdots$$

Hence

$$c_0 = \frac{a_0}{2}\sqrt{2c}$$

$$c_{2n-1} = a_n \sqrt{c}$$

$$c_{2n} = b_n \sqrt{c}$$

Substituting the above into (1) gives

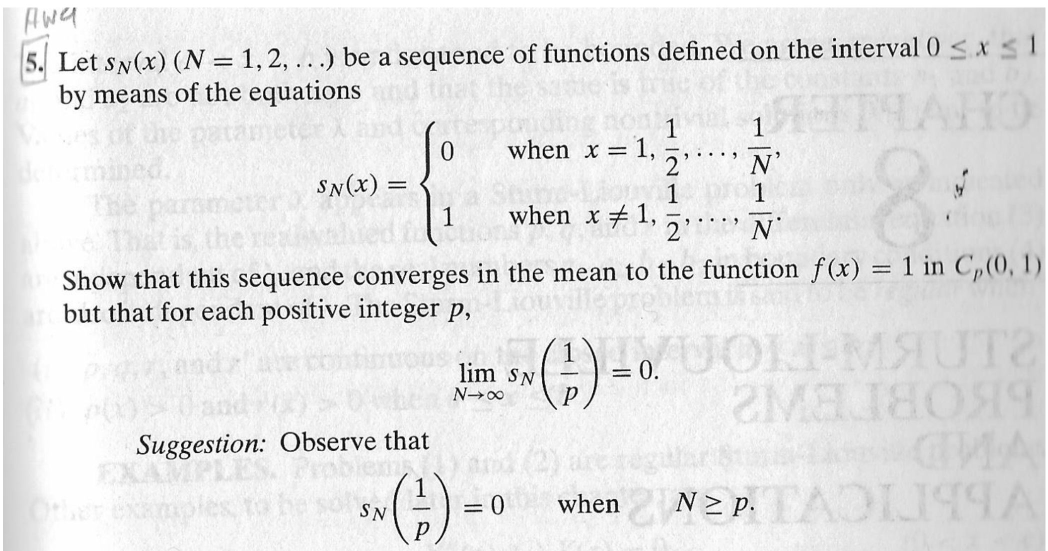
$$c_0^2 + \sum_{n=1}^{N} c_{2n-1}^2 + \sum_{n=1}^{N} c_{2n}^2 \leq \|f\|^2$$

$$\left(\frac{a_0}{2}\sqrt{2c}\right)^2 + \sum_{n=1}^{N}\left(a_n\sqrt{c}\right)^2 + \sum_{n=1}^{N}\left(b_n\sqrt{c}\right)^2 \leq \int \left[f(x)\right]^2 dx$$

$$\left(\frac{a_0^2}{4}2c\right) + \sum_{n=1}^{N} a_n^2 c + \sum_{n=1}^{N} b_n^2 c \leq \int \left[f(x)\right]^2 dx$$

$$\frac{a_0^2}{2} + \sum_{n=1}^{N}\left(a_n^2 + b_n^2\right) \leq \frac{1}{c}\int \left[f(x)\right]^2 dx$$

2.9.6 Section 66, Problem 5



HW9

5. Let $s_N(x)$ $(N = 1, 2, \ldots)$ be a sequence of functions defined on the interval $0 \leq x \leq 1$ by means of the equations

$$s_N(x) = \begin{cases} 0 & \text{when } x = 1, \frac{1}{2}, \ldots, \frac{1}{N}, \\ 1 & \text{when } x \neq 1, \frac{1}{2}, \ldots, \frac{1}{N}. \end{cases}$$

Show that this sequence converges in the mean to the function $f(x) = 1$ in $C_p(0, 1)$ but that for each positive integer p,

$$\lim_{N\to\infty} s_N\left(\frac{1}{p}\right) = 0.$$

Suggestion: Observe that

$$s_N\left(\frac{1}{p}\right) = 0 \quad \text{when} \quad N \geq p.$$

Figure 2.91: Problem description

<u>solution</u>

The function $S_N(x)$ is almost 1 everywhere as can be seen from this diagram

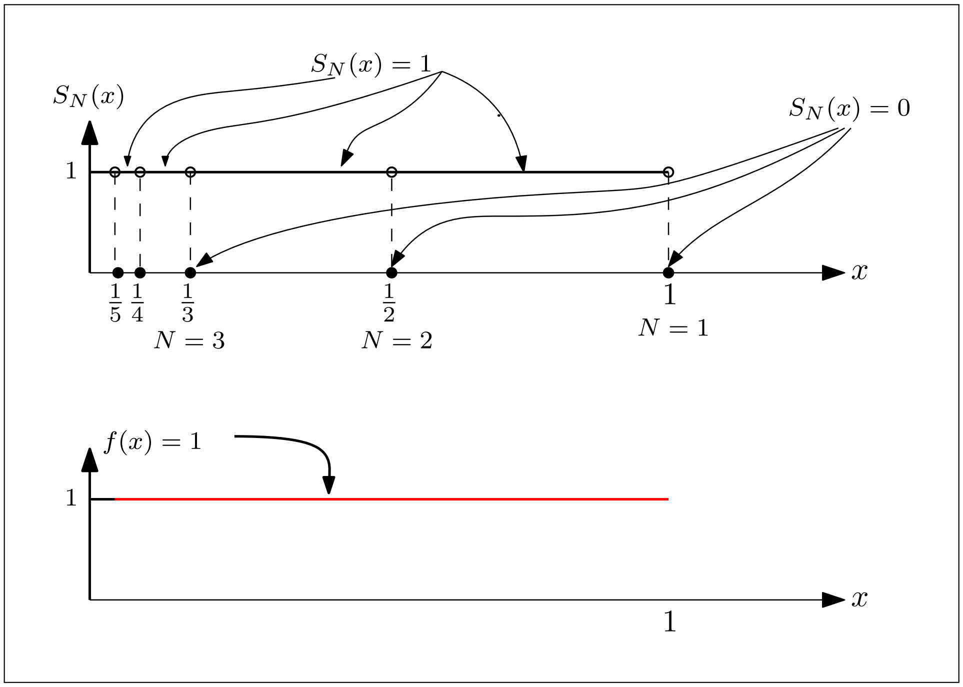


Figure 2.92: Showing the function $S_N(x)$ and $f(x)$

And the problem is asking us to show that $S_N(x) \to f(x)$ in the mean. This means we need to show the following is true

$$\lim_{N\to\infty} \left\| S_N(x) - f(x) \right\| = 0$$

Except at possibly finite number of points x. But this is the case here. Looking at $S_N(x)$ we see it is equal to $f(x) = 1$ everywhere except at the points $x = 1, \frac{1}{2}, \frac{1}{3}, \cdots$ and compared to all the points between 0 and 1, then $S_N(x) = f(x) = 1$ almost everywhere. Even though as $N \to \infty$ the number of points where $S_N(x) \neq 1$ increases, it is still finitely many compared to the number of points where $S_N(x) = f(x) = 1$.

To answer the second part: Since $S_N(x) = 0$ at any x value which can written as $\frac{1}{p}$ where p is an integer (this by definition given), then $S_N\left(\frac{1}{p}\right) = 0$. Then it clearly follows that $\lim_{N\to\infty} S_N\left(\frac{1}{p}\right) = 0$.

2.10 HW 10

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2.10.1 Section 69, Problem 1

1. (*a*) After writing the differential equation in the regular Sturm-Liouville problem

$$[xX'(x)]' + \frac{\lambda}{x}X(x) = 0 \qquad (1 < x < b),$$

$$X(1) = 0, \qquad X(b) = 0$$

in Cauchy-Euler form (see Problem 1, Sec. 44), use the substitution $x = \exp s$ to transform the problem into one consisting of the differential equation

$$\frac{d^2X}{ds^2} + \lambda X = 0 \qquad (0 < s < \ln b)$$

and the boundary conditions

$$X = 0 \quad \text{when} \quad s = 0 \qquad \text{and} \qquad X = 0 \quad \text{when} \quad s = \ln b.$$

Then, by simply referring to the solutions of the Sturm-Liouville problem (4) in Sec. 35, show that the eigenvalues and eigenfunctions of the original problem here are

$$\lambda_n = \alpha_n^2, \qquad X_n(x) = \sin(\alpha_n \ln x) \qquad (n = 1, 2, \ldots),$$

where $\alpha_n = n\pi / \ln b$.

(*b*) By making the substitution

$$s = \pi \frac{\ln x}{\ln b}$$

in the integral involved and then referring to Problem 9, Sec. 5, give a direct verification that the set of eigenfunctions $X_n(x)$ obtained in part (*a*) is orthogonal on the interval $1 < x < b$ with weight function $p(x) = 1/x$, as ensured by Theorem 1 in Sec. 69.

Figure 2.93: Problem statement

Solution

Part (a)

$$
\begin{aligned}
X'(x) + xX''(x) + \frac{\lambda}{x}X(x) &= 0 \\
x^2X''(x) + xX'(x) + \lambda X(x) &= 0 \qquad (1)
\end{aligned}
$$

To transform the above to $X''(s) + \lambda X(s) = 0$, let $x = e^s$. Therefore $\frac{dx}{ds} = e^s$ or $\frac{ds}{dx} = e^{-s}$. Now

$$
\begin{aligned}
\frac{dX}{dx} &= \frac{dX}{ds}\frac{ds}{dx} \\
&= \frac{dX}{ds}e^{-s} \qquad (2)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{d^2X}{dx^2} &= \frac{d}{dx}\left(\frac{dX}{dx}\right) \\
&= \frac{d}{dx}\left(\frac{dX}{ds}e^{-s}\right)
\end{aligned}
$$

Hence, by product rule

$$
\begin{aligned}
\frac{d^2X}{dx^2} &= \frac{d^2X}{ds^2}\frac{ds}{dx}e^{-s} + \frac{dX}{ds}\frac{d}{dx}\left(e^{-s}\right) \\
&= \frac{d^2X}{ds^2}e^{-s}e^{-s} + \frac{dX}{ds}\frac{d}{ds}\left(e^{-s}\right)\frac{ds}{dx} \\
&= \frac{d^2X}{ds^2}e^{-2s} + \frac{dX}{ds}\left(-e^{-s}\right)\left(e^{-s}\right) \\
&= e^{-2s}\frac{d^2X}{ds^2} - e^{-2s}\frac{dX}{ds} \qquad (3)
\end{aligned}
$$

Substituting (2,3) back into (1) gives

$$
x^2\left(e^{-2s}\frac{d^2X}{ds^2} - e^{-2s}\frac{dX}{ds}\right) + x\left(\frac{dX}{ds}e^{-s}\right) + \lambda X = 0
$$

But $x = e^s$ and the above simplifies to

$$
\begin{aligned}
e^{2s}\left(e^{-2s}\frac{d^2X}{ds^2} - e^{-2s}\frac{dX}{ds}\right) + e^s\left(\frac{dX}{ds}e^{-s}\right) + \lambda X &= 0 \\
\frac{d^2X}{ds^2} - \frac{dX}{ds} + \frac{dX}{ds} + \lambda X &= 0 \\
\frac{d^2X(s)}{ds^2} + \lambda X(s) &= 0
\end{aligned}
$$

When $X(1) = 0$, which means when $x = 1$, and since $x = e^s$, then when $s = 0$. Hence $X(1) = 0$ becomes $X(0) = 0$. And when $x = b$, then $s = \ln(b)$. Hence the second condition becomes $X(\ln(b)) = 0$. Therefore the new B.C. are

$$X(0) = 0$$
$$X(\ln(b)) = 0$$

By referring to problem (4) in section 35 we see that the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{c}\right)^2$$

Where here $c = \ln(b)$. Hence

$$\begin{aligned}\lambda_n &= \left(\frac{n\pi}{\ln(b)}\right)^2 \qquad n = 1,2,3,\cdots \\ &= \alpha_n^2\end{aligned}$$

Where $\alpha_n = \frac{n\pi}{\ln(b)}$. And the eigenfunctions are, per section 35

$$X_n(s) = \sin(\alpha_n s)$$

In terms of x, the eigenfunctions become

$$X_n(s) = \sin(\alpha_n \ln x)$$

Part (b)

$$\langle X_n(x), X_m(x)\rangle = \int_1^b \sin(\alpha_n \ln x)\sin(\alpha_m \ln x)\, p(x)\, dx$$

But from $(xX'(x))' + \frac{\lambda}{x}X(x) = 0$ and comparing this to $(rX')' + (\lambda p + q)X = 0$, we see that $r(x) = x$ and $q = 0$ and $p = \frac{1}{x}$. Hence the above integral becomes

$$\langle X_n(x), X_m(x)\rangle = \int_1^b \frac{1}{x}\sin(\alpha_n \ln x)\sin(\alpha_m \ln x)\, dx$$

Let $s = \frac{\ln x}{\ln b}\pi$. Then $\frac{ds}{dx} = \frac{1}{x}\frac{\pi}{\ln b}$ or $dx = \frac{x}{\pi}\ln(b)\, ds$. When $x = 1$ then $s = 0$ and when $x = b$ then $s = \pi$. Hence the above integral becomes

$$\begin{aligned}\langle X_n(x), X_m(x)\rangle &= \int_{s=0}^{s=\pi} \frac{1}{x}\sin\left(\alpha_n \frac{s\ln b}{\pi}\right)\sin\left(\alpha_m \frac{s\ln b}{\pi}\right)\left(\frac{x}{\pi}\ln(b)\, ds\right) \\ &= \frac{1}{\pi}\ln(b)\int_0^\pi \sin\left(\alpha_n \frac{s\ln b}{\pi}\right)\sin\left(\alpha_m \frac{s\ln b}{\pi}\right) ds\end{aligned}$$

But $\alpha_n = \frac{n\pi}{\ln(b)}$ and $\alpha_m = \frac{m\pi}{\ln(b)}$, therefore the above becomes

$$\begin{aligned}\langle X_n(x), X_m(x)\rangle &= \frac{1}{\pi}\ln(b)\int_0^\pi \sin\left(\frac{n\pi}{\ln(b)}\frac{s\ln b}{\pi}\right)\sin\left(\frac{m\pi}{\ln(b)}\frac{s\ln b}{\pi}\right) ds \\ &= \frac{1}{\pi}\ln(b)\int_0^\pi \sin(ns)\sin(ms)\, ds \qquad (1)\end{aligned}$$

Referring to Problem 9., section 5 which says that

$$\int_0^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = 0 \end{cases}$$

Applying this to (1) shows that

$$\langle X_n(x), X_m(x)\rangle = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = 0 \end{cases}$$

Hence $X_n(x)$ and $X_m(x)$ are orthogonal, since this is the definition of orthogonality.

2.10.2 Section 72, Problem 3

3. $X'' + \lambda X = 0, \quad X'(0) = 0, \quad X(c) = 0.$

Answer: $\lambda_n = \alpha_n^2, \quad \phi_n(x) = \sqrt{\frac{2}{c}}\cos\alpha_n x \quad (n = 1, 2, \ldots); \quad \alpha_n = \frac{(2n-1)\pi}{2c}.$

Figure 2.94: Problem statement

Solution

Solve for eigenvalues and normalized eigenfunctions.

$$\begin{aligned} X'' + \lambda X &= 0 \\ X'(0) &= 0 \\ X(c) &= 0 \end{aligned}$$

Writing the boundary conditions in SL standard form

$$\begin{aligned} a_1 X(0) + a_2 X'(0) &= 0 \\ b_1 X(c) + b_2 X'(c) &= 0 \end{aligned}$$

Shows that $a_1 = 0, a_2 = 1$ and $b_1 = 1, b_2 = 0$. Therefore $a_1 a_2 = 0$ and $b_1 b_2 = 0$. But we know that if $a_1 a_2 \geq 0$ and $b_1 b_2 \geq 0$, then $\lambda > 0$ is only possible eigenvalues. Let $\lambda_n = \alpha_n^2$. $\alpha > 0$. Hence the solution to the ODE is

$$\begin{aligned} X_n(x) &= A\cos(\alpha_n x) + B\sin(\alpha_n x) \\ X_n'(x) &= -A\alpha_n \sin(\alpha_n x) + B\alpha_n \cos(\alpha_n x) \end{aligned}$$

First B.C $X'(0) = 0$ gives

$$0 = B\alpha_n$$

Which implies $B = 0$. Hence the solution now becomes $X_n(x) = A\cos(\alpha_n x)$. For the second

BC

$$\begin{aligned}0 &= A\cos(\alpha_n c)\\ 0 &= \cos(\alpha_n c)\end{aligned}$$

Which implies

$$\begin{aligned}\alpha_n c &= \frac{\pi}{2}, 3\frac{\pi}{2}, 5\frac{\pi}{2}, \cdots\\ &= (2n-1)\frac{\pi}{2} \qquad n = 1,2,3,\cdots\end{aligned}$$

Hence

$$\alpha_n = \frac{(2n-1)}{c}\frac{\pi}{2} \qquad n = 1,2,3,\cdots$$

And the corresponding eigenfunctions are

$$\begin{aligned}X_n(x) &= \cos(\alpha_n x)\\ &= \cos\left(\frac{(2n-1)}{c}\frac{\pi}{2}x\right)\end{aligned}$$

To find the normalized $X_n(x)$ which we call it $\phi_n(x)$, then by definition

$$\phi_n(x) = \frac{X_n(x)}{\|X_n(x)\|}$$

But

$$\|X_n(x)\|^2 = \int_0^c p(x)\, X_n^2(x)\, dx$$

Comparing the ODE $X'' + \lambda X = 0$ to $(rX')' + \left(\lambda p + q\right)X = 0$, we see that $r(x) = 1$ and $q = 0$ and $p = 1$. Hence the above becomes

$$\begin{aligned}\|X_n(x)\|^2 &= \int_0^c \cos^2(\alpha_n x)\, dx\\ &= \frac{c}{2}\end{aligned}$$

Therefore $\|X_n(x)\| = \sqrt{\frac{c}{2}}$ which shows that

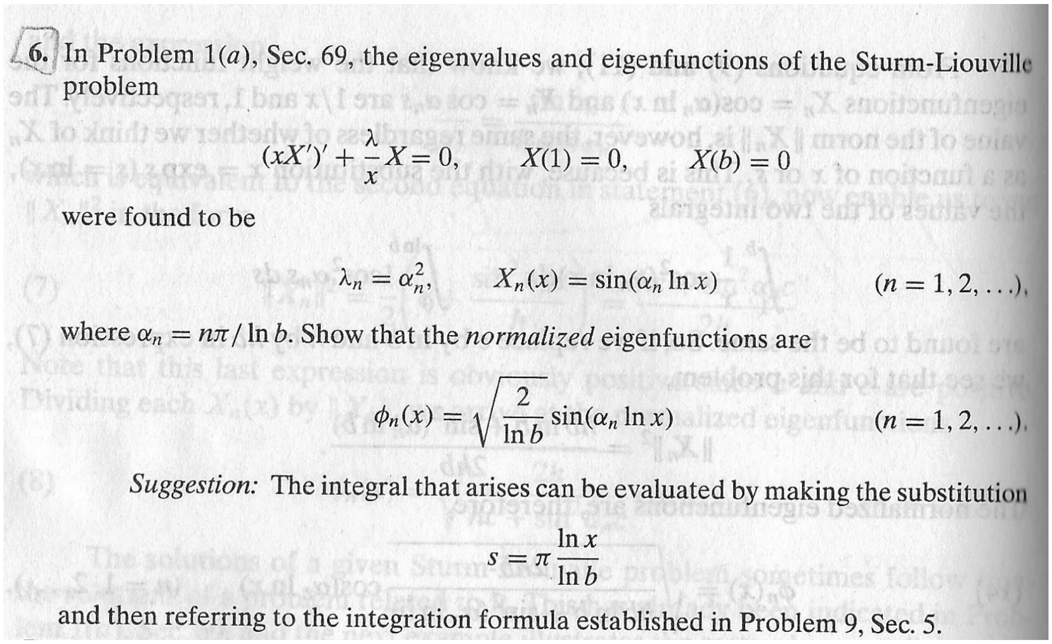
$$\begin{aligned}\phi_n(x) &= \frac{X_n(x)}{\sqrt{\frac{c}{2}}}\\ &= \sqrt{\frac{2}{c}}\cos(\alpha_n x)\end{aligned}$$

where

$$\alpha_n = \frac{(2n-1)}{c}\frac{\pi}{2} \qquad n = 1,2,3,\cdots$$

Which is what required to show.

2.10.3 Section 72, Problem 6



6. In Problem 1(a), Sec. 69, the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$(xX')' + \frac{\lambda}{x}X = 0, \qquad X(1) = 0, \qquad X(b) = 0$$

were found to be

$$\lambda_n = \alpha_n^2, \qquad X_n(x) = \sin(\alpha_n \ln x) \qquad (n = 1, 2, \ldots),$$

where $\alpha_n = n\pi/\ln b$. Show that the *normalized* eigenfunctions are

$$\phi_n(x) = \sqrt{\frac{2}{\ln b}}\sin(\alpha_n \ln x) \qquad (n = 1, 2, \ldots).$$

Suggestion: The integral that arises can be evaluated by making the substitution

$$s = \pi\frac{\ln x}{\ln b}$$

and then referring to the integration formula established in Problem 9, Sec. 5.

Figure 2.95: Problem statement

Solution

$$X_n(x) = \sin(\alpha_n \ln x)$$
$$\alpha_n = \frac{n\pi}{\ln b} \qquad n = 1, 2, 3, \cdots$$

The normalized eigenfunction is given by

$$\phi_n(x) = \frac{X_n(x)}{\|X_n(x)\|}$$

But

$$\|X_n(x)\|^2 = \int_1^b p(x) X_n^2(x)\,dx$$

Comparing the ODE $(xX')' + \frac{\lambda}{x}X = 0$ to $(rX')' + (\lambda p + q)X = 0$, we see that $r(x) = x$ and $q = 0$ and $p = \frac{1}{x}$. Hence the above becomes

$$\|X_n(x)\|^2 = \int_1^b \frac{1}{x}\sin^2(\alpha_n \ln x)\,dx$$

Let $s=\frac{\ln x}{\ln b}\pi$. Then $\frac{ds}{dx}=\frac{1}{x}\frac{\pi}{\ln b}$ or $dx=\frac{x}{\pi}\ln(b)\,ds$. When $x=1$ then $s=0$ and when $x=b$ then $s=\pi$. Hence the above integral becomes

$$\begin{aligned}\|X_n(x)\|^2 &= \int_{s=0}^{s=\pi}\frac{1}{x}\sin^2\left(\alpha_n\frac{s\ln b}{\pi}\right)\left(\frac{x}{\pi}\ln(b)\,ds\right)\\ &= \frac{1}{\pi}\ln(b)\int_0^{\pi}\sin^2\left(\alpha_n\frac{s\ln b}{\pi}\right)ds\end{aligned}$$

But $\alpha_n=\frac{n\pi}{\ln(b)}$ therefore the above becomes

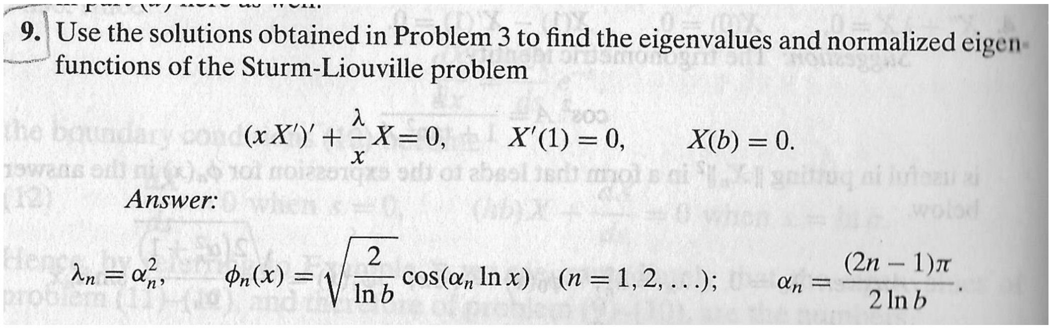
$$\begin{aligned}\|X_n(x)\|^2 &= \frac{1}{\pi}\ln(b)\int_0^{\pi}\sin^2\left(\frac{n\pi}{\ln(b)}\frac{s\ln b}{\pi}\right)ds\\ &= \frac{1}{\pi}\ln(b)\int_0^{\pi}\sin^2(ns)\,ds\\ &= \frac{1}{\pi}\ln(b)\int_0^{\pi}\frac{1}{2}-\frac{1}{2}\cos(2ns)\,ds\\ &= \frac{1}{\pi}\ln(b)\left(\frac{\pi}{2}-\frac{1}{2}\sin\left(\frac{2ns}{2n}\right)_0^{\pi}\right)\\ &= \frac{1}{\pi}\ln(b)\left(\frac{\pi}{2}-\frac{1}{2}\sin(s)_0^{\pi}\right)\\ &= \frac{1}{2}\ln(b)\end{aligned}$$

Hence

$$\begin{aligned}\phi_n(x) &= \frac{\sin(\alpha_n\ln x)}{\sqrt{\frac{1}{2}\ln(b)}}\\ &= \sqrt{\frac{2}{\ln(b)}}\sin(\alpha_n\ln x)\end{aligned}$$

Which is what required to show.

2.10.4 Section 72, Problem 9



9. Use the solutions obtained in Problem 3 to find the eigenvalues and normalized eigenfunctions of the Sturm-Liouville problem

$$(xX')' + \frac{\lambda}{x}X = 0, \qquad X'(1) = 0, \qquad X(b) = 0.$$

Answer:

$$\lambda_n = \alpha_n^2, \qquad \phi_n(x) = \sqrt{\frac{2}{\ln b}}\cos(\alpha_n \ln x) \quad (n = 1, 2, \ldots); \qquad \alpha_n = \frac{(2n-1)\pi}{2\ln b}.$$

Figure 2.96: Problem description

<u>solution</u>

From problem section 69 problem 1, we know that $(xX'(x))' + \frac{\lambda}{x}X(x) = 0$ can be transformed to $X''(s) + \lambda X(s) = 0$ using $x = e^s$. With boundary conditions in s found as follows. When $x = 1$ then $s = 0$ and when $x = b$ then $s = \ln b$. Hence we obtain the SL problem

$$\begin{aligned} X''(s) + \lambda X(s) &= 0 \\ X'(0) &= 0 \\ X(\ln b) &= 0 \end{aligned} \tag{1}$$

But problem 3 is

$$\begin{aligned} X'' + \lambda X &= 0 \\ X'(0) &= 0 \\ X(c) &= 0 \end{aligned} \tag{2}$$

And it had the solution

$$\phi_n(x) = \sqrt{\frac{2}{c}}\cos(\alpha_n x)$$

where

$$\alpha_n = \frac{(2n-1)}{c}\frac{\pi}{2} \qquad n = 1, 2, 3, \cdots$$

By comparing (2) and (1) we see it is the same problem, except $c \to \ln b$. Hence the solution to (2) is the same as the solution in (1) but with c replaced by $\ln b$. Hence the solution is

$$\begin{aligned} \phi_n(s) &= \sqrt{\frac{2}{\ln b}}\cos(\alpha_n s) \\ \alpha_n &= \frac{(2n-1)}{\ln b}\frac{\pi}{2} \qquad n = 1, 2, 3, \cdots \end{aligned}$$

But $s=\ln x$, hence the above becomes

$$\begin{aligned}\phi_n(x) &= \sqrt{\frac{2}{\ln b}}\cos(\alpha_n \ln x)\\ \alpha_n &= \frac{(2n-1)}{\ln b}\frac{\pi}{2} \qquad n=1,2,3,\cdots\end{aligned}$$

Which is what required to show.

2.11 HW 11

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2.11.1 Section 73, Problem 8

8. Find the Fourier constants c_n for the function $f(x) = x$ $(1 < x < b)$ with respect to the normalized eigenfunctions in Problem 6, Sec. 72, and reduce those constants to the form

$$c_n = \sqrt{2\ln b}\,\frac{n\pi[1+(-1)^{n+1}b]}{(\ln b)^2+(n\pi)^2} \qquad (n = 1, 2, \ldots).$$

Suggestion: The integration formula

$$\int e^x \sin ax\, dx = \frac{e^x(\sin ax - a\cos ax)}{1+a^2},$$

derived in calculus, is useful here.

Figure 2.97: Problem statement

Solution

$$\begin{aligned} c_n &= \langle f(x), \phi_n(x)\rangle \\ &= \int_1^b p(x) f(x) \phi_n(x)\, dx \end{aligned}$$

But $p(x) = \frac{1}{x}$ and $\phi_n(x) = \sqrt{\frac{2}{\ln b}} \sin(\alpha_n \ln x)$ and $f(x) = x$ therefore the above becomes

$$\begin{aligned} c_n &= \int_1^b \frac{1}{x} x \sqrt{\frac{2}{\ln b}} \sin(\alpha_n \ln x)\, dx \\ &= \sqrt{\frac{2}{\ln b}} \int_1^b \sin(\alpha_n \ln x)\, dx \end{aligned}$$

But $\alpha_n = \frac{n\pi}{\ln b}$, therefore

$$c_n = \sqrt{\frac{2}{\ln b}} \int_1^b \sin\left(\frac{n\pi}{\ln b}\ln x\right) dx$$

Let $s = \pi\frac{\ln x}{\ln b}$, hence $\frac{ds}{dx} = \frac{\pi}{\ln b}\frac{1}{x}$. When $x = 1 \rightarrow s = 0$ and when $x = b \rightarrow s = \pi$. The above becomes

$$c_n = \sqrt{\frac{2}{\ln b}} \int_0^\pi \sin(ns) \frac{\ln(b)}{\pi} x ds$$

But $\ln x = \frac{s}{\pi}\ln b$, hence $x = e^{s\frac{\ln b}{\pi}}$, and the above becomes

$$c_n = \frac{\sqrt{2\ln(b)}}{\pi} \int_0^\pi e^{s\frac{\ln b}{\pi}} \sin(ns) ds \tag{1}$$

Using

$$\int e^{ax}\sin(bx) ds = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

Where in our case $a = \frac{\ln b}{\pi}$ and $b = n$. Applying the above gives

$$\begin{aligned}
\int_0^\pi e^{s\frac{\ln b}{\pi}} \sin(ns) ds &= \left[\frac{e^{\frac{\ln b}{\pi}x}}{\left(\frac{\ln b}{\pi}\right)^2 + n^2}\left(\frac{\ln b}{\pi}\sin nx - n\cos nx\right)\right]_0^\pi \\
&= \frac{1}{\left(\frac{\ln b}{\pi}\right)^2 + n^2}\left[e^{\frac{\ln b}{\pi}\pi}\left(\frac{\ln b}{\pi}\sin n\pi - n\cos n\pi\right) - (0-n)\right]
\end{aligned}$$

But $\sin n\pi = 0$ since n integer, giving

$$\begin{aligned}
\int_0^\pi e^{s\frac{\ln b}{\pi}} \sin(ns) ds &= \frac{1}{\left(\frac{\ln b}{\pi}\right)^2 + n^2}[-bn\cos n\pi + n] \\
&= \frac{\pi^2}{(\ln b)^2 + \pi^2 n^2}\left[-bn(-1)^n + n\right] \\
&= \frac{\pi^2\left(bn(-1)^{n+1} + n\right)}{(\ln b)^2 + \pi^2 n^2}
\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}
c_n &= \frac{\sqrt{2\ln(b)}}{\pi}\frac{n\pi^2\left(1 + (-1)^{n+1} b\right)}{(\ln b)^2 + (\pi n)^2} \\
&= \sqrt{2\ln(b)}\frac{n\pi\left(1 + (-1)^{n+1} b\right)}{(\ln b)^2 + (\pi n)^2}
\end{aligned}$$

Where $n = 1, 2, 3, \cdots$, which is the result required to show.

2.11.2 Section 73, Problem 10

10. Suppose that a function f, defined on the interval $0 < x < c$, is piecewise smooth there.

(*a*) Use the normalized eigenfunctions (Problem 7, Sec. 72)

$$\phi_n(x) = \sqrt{\frac{2}{c}} \sin \alpha_n x \qquad (n = 1, 2, \ldots),$$

where

$$\alpha_n = \frac{(2n-1)\pi}{2c},$$

to show formally that

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \alpha_n x \qquad (0 < x < c),$$

where

$$B_n = \frac{2}{c} \int_0^c f(x) \sin \alpha_n x \, dx \qquad (n = 1, 2, \ldots).$$

(*b*) Note that according to Problem 6, Sec. 15, the series in part (*a*) is actually a Fourier sine series for an extension of f on the interval $0 < x < 2c$. Then, with the aid of Theorem 2 in Sec. 15, state why the representation in part (*a*) is valid for each point x $(0 < x < c)$ at which f is continuous.

Figure 2.98: Problem statement

Solution

Part (a)

$$\phi_n(x) = \sqrt{\frac{2}{c}} \sin(\alpha_n x) \qquad n = 1, 2, 3, \cdots$$
$$\alpha_n = \pi \frac{2n-1}{2x}$$

Since $\phi_n(x)$ are complete, then we can represent $f(x)$ using $\phi_n(x)$ as generalized Fourier series using

$$f(x) = \sum_{n=1}^{\infty} B_n \phi_n(x) \qquad 0 < x < c$$

To find B_n, since $\phi_n(x)$ are orthonormal eigenfunctions then

$$\begin{aligned} B_n &= \langle f(x), \phi_n(x) \rangle \\ &= \int_0^c p(x) f(x) \phi_n dx \end{aligned}$$

But problem (7) section 72 is $X'' + \lambda X = 0$ which implies that $p(x) = 1$. Hence the above becomes

$$\begin{aligned} B_n &= \int_0^c f(x) \sqrt{\frac{2}{c}} \sin(\alpha_n x) dx \\ &= \sqrt{\frac{2}{c}} \int_0^c f(x) \sin(\alpha_n x) dx \end{aligned}$$

Which is the result required to show.

Part (b)

Theorem 2 section 15 gives the conditions on $f(x)$ for it to have a Fourier sine series which converges to $f(x)$ where $f(x)$ is continuous and converges to mean value of $f(x)$ where $f(x)$ have a jump discontinuity.

Since $f(x)$ is piecewise continuous in this problem, then for those regions where $f(x)$ is continuous between $0 < x < c$, the series found in part(a) converges to $f(x)$ and is valid Fourier sine series representation of $f(x)$ there.

2.11.3 Section 74, Problem 1

1. Show that when $f(x) = 1$ $(0 < x < 1)$ in the boundary value problem (1)–(2) in Sec. 74, the solution (6)–(7) there reduces to

$$u(x,t) = 2h \sum_{n=1}^{\infty} \frac{\sin \alpha_n}{\alpha_n (h + \sin^2 \alpha_n)} \exp\left(-\alpha_n^2 kt\right) \cos \alpha_n x,$$

where $\tan \alpha_n = h/\alpha_n$ $(\alpha_n > 0)$.

Figure 2.99: Problem statement

Solution

Solution (6) is given by

$$u(x,t) = \sum_{n=1}^{\infty} A_n \exp\left(-\alpha_n^2 kt\right) \cos(\alpha_n x) \tag{6}$$

Where

$$A_n = \frac{2h}{h+\sin^2\alpha_n}\int_0^1 f(x)\cos(\alpha_n x)\,dx$$

But $f(x) = 1$ which reduces the above to

$$\begin{aligned} A_n &= \frac{2h}{h+\sin^2\alpha_n}\int_0^1 \cos(\alpha_n x)\,dx \\ &= \frac{2h}{h+\sin^2\alpha_n}\left[\sin(\alpha_n x)\right]_0^1 \\ &= \frac{2h}{h+\sin^2\alpha_n}\sin(\alpha_n) \end{aligned}$$

Hence (6) becomes

$$u(x,t) = 2h\sum_{n=1}^{\infty}\frac{\sin(\alpha_n)}{h+\sin^2\alpha_n}\exp\left(-\alpha_n^2 kt\right)\cos(\alpha_n x)$$

But from example 1, section 72 we are given that $\tan(\alpha_n c) = \frac{h}{\alpha_n}$. But $c = 1$ in this problem, hence

$$\tan(\alpha_n) = \frac{h}{\alpha_n}$$

Which is what required to show.

2.11.4 Section 74, Problem 4

4. (*a*) Give a physical interpretation of the boundary value problem

$$u_t(x,t) = ku_{xx}(x,t) \qquad (0 < x < 1,$$

$$u(0,t) = 0, \qquad u_x(1,t) = -hu(1,t), \qquad u(x,0) = f(x),$$

where h is a positive constant. Then derive the solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n \exp\left(-\alpha_n^2 kt\right) \sin \alpha_n x,$$

where $\tan \alpha_n = -\alpha_n / h$ $(\alpha_n > 0)$ and

$$B_n = \frac{2h}{h + \cos^2 \alpha_n} \int_0^1 f(x) \sin \alpha_n x \, dx \qquad (n = 1,$$

(*b*) Use an argument similar to the one at the end of Sec. 74 to show that the s found in part (*a*) formally satisfies the boundary value problem (8)–(10) section when the function f there is *odd*, or when

$$f(-x) = -f(x) \qquad (-1 <$$

Figure 2.100: Problem statement

Solution

Part (a)

$u(0,t) = 0$ means that the left surface is kept at fixed temperature which is zero. And $u_x(1,t) + hu(1,t) = 0$ means that the surface heat transfer takes place at face $x = 1$ into the medium at temperature zero. To solve the PDE, we first check the boundary conditions by writing them as

$$a_1 u(0,t) + a_2 u_x(0,t) = 0$$
$$b_1 u(1,t) + b_2 u_x(1,t) = 0$$

Then $a_1 = 0, a_2 = 0$. Hence $a_1 a_2 = 0$. And $b_1 = 1, b_2 = h$. Then since it is assumed that $h > 0$ per section 26, then $b_1 b_2 \geq 0$. And since $q(x) = 0$ from the PDE itself, then we know that eigenvalues are $\lambda \geq 0$.

Let $u = X(x)T(t)$ then the PDE becomes

$$T'X = X''T$$
$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

Hence the Sturm Liouville problem is

$$\begin{aligned} X'' + \lambda X &= 0 \\ X(0) &= 0 \\ X'(1) + hX(1) &= 0 \end{aligned}$$

Where $p(x) = 1$.

<u>Case $\lambda = 0$</u>

Solution is

$$X(x) = Ax + B$$

At $x = 0$

$$0 = B$$

Hence solution becomes

$$X(x) = Ax$$

At $x = 1$ the second boundary conditions gives

$$\begin{aligned} A + hA &= 0 \\ A(1 + h) &= 0 \end{aligned}$$

For non trivial solution $1 + h = 0$ or $h = -1$. But we assumed that $h > 0$. Therefore $\lambda = 0$ is not eigenvalue.

<u>Case $\lambda > 0$</u>

Let $\lambda = \alpha^2, \alpha > 0$. Hence solution is

$$X(x) = A\cos(\alpha x) + B\sin(\alpha x)$$

At $X(0) = 0$

$$0 = A$$

The solution becomes

$$X(x) = B\sin(\alpha x)$$

At $x = 1$ the second boundary conditions gives

$$\begin{aligned} B\alpha\cos(\alpha) + hB\sin(\alpha) &= 0 \\ \alpha\cos(\alpha) + h\sin(\alpha) &= 0 \\ \tan(\alpha) &= -\frac{\alpha}{h} \end{aligned}$$

Therefore the eigenvalues are given by solution to

$$\tan(\alpha_n) = -\frac{\alpha_n}{h} \qquad n = 1, 2, 3, \cdots$$

And eigenfunctions are

$$X_n(x) = \sin(\alpha_n x)$$

The normalized eigenfunctions are

$$\phi_n(x) = \frac{X_n(x)}{\|X_n(x)\|}$$

But

$$\begin{aligned}
\|X_n(x)\|^2 &= \int_0^1 p(x)\, X_n^2(x)\, dx \\
&= \int_0^1 \sin^2(\alpha_n x)\, dx \\
&= \frac{1}{2}\int_0^1 1 - \cos(2\alpha_n x)\, dx \\
&= \frac{1}{2}\left(1 - \left[\frac{\sin(2\alpha_n x)}{2\alpha_n}\right]_0^1\right) \\
&= \frac{1}{2}\left(1 - \frac{1}{2\alpha_n}\left[\sin(2\alpha_n x)\right]_0^1\right) \\
&= \frac{1}{2}\left(1 - \frac{\sin(2\alpha_n)}{2\alpha_n}\right) \\
&= \frac{1}{2} - \frac{\sin(2\alpha_n)}{4\alpha_n}
\end{aligned}$$

But $\sin(2\alpha_n) = 2\sin\alpha_n \cos\alpha_n$ and $\alpha_n = -h\frac{\sin(\alpha_n)}{\cos(\alpha_n)}$, therefore the above becomes

$$\begin{aligned}
\|X_n(x)\|^2 &= \frac{1}{2} + \frac{2\sin\alpha_n \cos\alpha_n}{4h\frac{\sin(\alpha_n)}{\cos(\alpha_n)}} \\
&= \frac{1}{2} + \frac{\cos^2\alpha_n}{2h} \\
&= \frac{h + \cos^2\alpha_n}{2h}
\end{aligned}$$

Hence

$$\begin{aligned}
\phi_n(x) &= \frac{X_n(x)}{\sqrt{\frac{h+\cos^2\alpha_n}{2h}}} \\
&= \sqrt{\frac{2h}{h+\cos^2\alpha_n}}\sin(\alpha_n x)
\end{aligned}$$

Now we use generalized Fourier series to find the solution. Let

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t)\, \phi_n(x) \tag{1}$$

Substituting this back into the PDE gives

$$\sum_{n=1}^{\infty} B_n'(t)\, \phi_n(x) = k\sum_{n=1}^{\infty} B_n(t)\, \phi_n''(x)$$

But $\phi_n''(x) = -\lambda_n\phi_n(x) = -\alpha_n^2\phi_n(x)$. The above becomes

$$\sum_{n=1}^{\infty} B_n'(t)\,\phi_n(x) = -k\sum_{n=1}^{\infty} B_n(t)\,\alpha_n^2\phi_n(x)$$
$$B_n'(t) + k\alpha_n^2 B_n(t) = 0$$

The solution is

$$B_n(t) = B_n(0)\,e^{-k\alpha_n^2 t}$$

Hence (1) becomes

$$u(x,t) = \sum_{n=1}^{\infty} B_n(0)\,e^{-k\alpha_n^2 t}\phi_n(x)$$

At $t = 0$ the above becomes

$$f(x) = \sum_{n=1}^{\infty} B_n(0)\,\phi_n(x)$$

Therefore

$$\begin{aligned} B_n(0) &= \langle f(x), \phi_n(x)\rangle \\ &= \int_0^1 p(x)\,f(x)\,\phi_n(x)\,dx \\ &= \sqrt{\frac{2h}{h+\cos^2\alpha_n}}\int_0^1 f(x)\sin(\alpha_n x)\,dx \end{aligned}$$

Therefore

$$\begin{aligned} B_n(t) &= B_n(0)\,e^{-k\alpha_n^2 t} \\ &= \left(\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\int_0^1 f(x)\sin(\alpha_n x)\,dx\right)e^{-k\alpha_n^2 t} \end{aligned}$$

and solution (1) becomes

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\left(\int_0^1 f(x)\sin(\alpha_n x)\,dx\right)e^{-k\alpha_n^2 t}\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\sin(\alpha_n x) \\ &= \frac{2h}{h+\cos^2\alpha_n}\sum_{n=1}^{\infty}\left(\int_0^1 f(x)\sin(\alpha_n x)\,dx\right)e^{-k\alpha_n^2 t}\sin(\alpha_n x) \end{aligned}$$

Which is what required to show.

Part (b)

We need to show that the solution found in part (a) also satisfies the PDE when $-1 < x < 1$

$$u_t = ku_{xx} \qquad -1 < x < 1, t > 0$$

With boundary conditions (9)

$$u_x(-1,t) = hu(-1,t)$$
$$u_x(1,t) = -hu(1,t)$$

And initial conditions (10)

$$u(x,0) = f(x)$$

When $f(x)$ is odd.

The solution found in a already satisfies the above PDE with the second boundary conditions in (9). Since sine is odd then the solution in part(a) is also odd. Then its partial derivative is even in x, hence the first boundary conditions in (9) is also satisfied

$$u_x(-1,t) = hu(-1,t) = -u_x(1,t) = hu(1,t)$$

Finally we know that $u(x,0) = f(x)$ for $0 < x < 1$. Furthermore when $-1 < x < 0$ the fact that u and $f(x)$ are odd enables us to write

$$u(-x,0) = -u(x,0) = f(-x) = -f(x)$$

2.11.5 Section 77, Problem 2

where $\alpha_n = (2n-1)\pi/2$.

2. Heat transfer takes place at the surface $x = 0$ of a slab $0 \le x \le 1$ into a medium at temperature zero, according to the linear law of surface heat transfer, so that (Sec. 26)

$$u_x(0,t) = hu(0,t) \qquad (h > 0).$$

The other boundary conditions are as indicated in Fig. 61, and the unit of time is chosen so that $k = 1$ in the heat equation. By proceeding as in Sec. 77, derive the temperature formula

$$u(x,t) = \frac{hx+1}{h+1} - 2h\sum_{n=1}^{\infty} \frac{\sin \alpha_n(1-x)}{\alpha_n(h+\cos^2\alpha_n)} \exp(-\alpha_n^2 t),$$

where $\tan\alpha_n = -\alpha_n/h$ $(\alpha_n > 0)$.

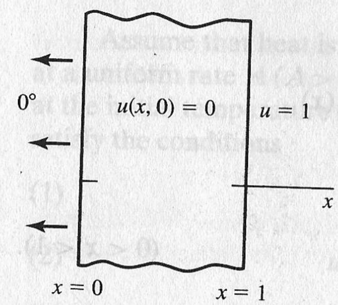

FIGURE 61

Suggestion: In simplifying the expression for the Fourier constants that arise, it is useful to note that

$$-\frac{h \sin \alpha_n}{\alpha_n^2} = \frac{\cos \alpha_n}{\alpha_n}.$$

Figure 2.101: Problem statement

<u>Solution</u>

Solve

$$u_t = u_{xx} \qquad 0 < x < 1, t > 0$$

With boundary conditions

$$u_x(0,t) - hu(0,t) = 0$$
$$u(1,t) = 1$$

With $h > 0$. And initial conditions $u(x,0) = f(x)$.

Because the second B.C. is not zero, we need to introduce a reference function $r(x)$ which satisfies the nonhomogeneous boundary conditions.

Let $r(x) = Ax + B$. When $x = 0$ then the first BC gives

$$A - hB = 0$$

And the second BC gives

$$A + B = 1$$

From the first equation $A = hB$. Substituting in the second equation give $hB + B = 1$ or

$B(1+h)=1$ or $B=\frac{1}{1+h}$. Hence $A=\frac{h}{1+h}$. Therefore

$$\begin{aligned} r(x) &= Ax+B \\ &= \frac{h}{1+h}x+\frac{1}{1+h} \\ &= \frac{hx+1}{1+h} \end{aligned} \tag{1}$$

To verify. $r_x=\frac{h}{1+h}$. When $x=0$ then $r(0)=\frac{1}{1+h}$. Hence $r_x(0)-hr(0)=\frac{h}{1+h}-h\frac{1}{1+h}=0$ as expected. And when $x=1$ then $r(1)=1$ as expected. Now that we found $r(x)$ then we write

$$u(x,t)=v(x,t)+r(x)$$

Where $v(x,t)$ is the solution to the homogenous PDE

$$v_t=v_{xx} \qquad 0<x<1, t>0$$

With boundary conditions

$$\begin{aligned} v_x(0,t)-hv(0,t) &= 0 \\ v(1,t) &= 0 \end{aligned}$$

We can now solve for $v(x,t)$ using separation of variables since boundary conditions are homogenous. Separation of variables gives

$$\begin{aligned} X''+\lambda X &= 0 \\ X'(0)-hX(0) &= 0 \\ X(1) &= 0 \end{aligned}$$

Using problem 5 section 72, the eigenfunctions and eigenvalues for the above are

$$\begin{aligned} \phi_n(x) &= \sqrt{\frac{2h}{h+\cos^2\alpha_n}}\sin(\alpha_n(1-x)) \qquad n=1,2,\cdots \\ \tan(\alpha_n) &= \frac{-\alpha_n}{h} \end{aligned}$$

With $\alpha_n>0$. Hence the solution $v(x,t)$ using generalized Fourier series is

$$v(x,t)=\sum_{n=1}^{\infty}B_n(t)\phi_n(x) \tag{2}$$

Substituting into the PDE $v_t=v_{xx}$ gives

$$\begin{aligned} \sum_{n=1}^{\infty}B_n'(t)\phi_n(x) &= \sum_{n=1}^{\infty}B_n(t)\phi_n''(x) \\ &= -\sum_{n=1}^{\infty}B_n(t)\alpha_n^2\phi_n(x) \end{aligned}$$

Therefore the ODE is

$$B_n'(t)+\alpha_n^2B_n(t)=0$$

The solution is

$$B_n(t) = B_n(0)\, e^{-\alpha_n^2 t}$$

Hence (2) becomes

$$v(x,t) = \sum_{n=1}^{\infty} B_n(0)\, e^{-\alpha_n^2 t} \phi_n(x)$$

And since $u(x,t) = v(x,t) + r(x)$ then

$$u(x,t) = \sum_{n=1}^{\infty} B_n(0)\, e^{-\alpha_n^2 t} \phi_n(x) + \frac{hx+1}{1+h}$$

Now we find $B_n(0)$ from initial conditions. At $t = 0$ the above becomes

$$\begin{aligned}
0 &= \sum_{n=1}^{\infty} B_n(0)\, \phi_n(x) + \frac{hx+1}{1+h} \\
-\frac{hx+1}{1+h} &= \sum_{n=1}^{\infty} B_n(0)\, \phi_n(x)
\end{aligned}$$

Hence

$$\begin{aligned}
B_n(0) &= \left\langle -\frac{hx+1}{1+h}, \phi_n(x) \right\rangle \\
&= -\int_0^1 p(x) \frac{hx+1}{1+h} \phi_n(x)\, dx \\
&= -\int_0^1 \frac{hx+1}{1+h} \sqrt{\frac{2h}{h+\cos^2\alpha_n}} \sin(\alpha_n(1-x))\, dx \\
&= -\frac{1}{1+h} \sqrt{\frac{2h}{h+\cos^2\alpha_n}} \int_0^1 (hx+1) \sin(\alpha_n(1-x))\, dx \qquad (3)
\end{aligned}$$

But

$$\begin{aligned}
\int_0^1 (hx+1)\sin(\alpha_n(1-x))\, dx &= \int_0^1 \sin(\alpha_n(1-x))\, dx + h\int_0^1 x\sin(\alpha_n(1-x))\, dx \\
&= \left[\frac{\cos(\alpha_n(1-x))}{\alpha_n}\right]_0^1 + h\left[\frac{\alpha_n x\cos(\alpha_n(1-x)) + \sin(\alpha_n(1-x))}{\alpha_n^2}\right]_0^1 \\
&= \frac{1-\cos(\alpha_n)}{\alpha_n} + \frac{h}{\alpha_n^2}\left[\alpha_n x\cos(\alpha_n(1-x)) + \sin(\alpha_n(1-x))\right]_0^1 \\
&= \frac{1-\cos(\alpha_n)}{\alpha_n} + \frac{h}{\alpha_n^2}\left[\alpha_n - \sin\alpha_n\right] \\
&= \frac{\alpha_n - \alpha_n\cos(\alpha_n) + h\alpha_n - h\sin\alpha_n}{\alpha_n^2}
\end{aligned}$$

But $\frac{\sin(\alpha_n)}{\cos(\alpha_n)} = -\frac{\alpha_n}{h}$ or $h\sin(\alpha_n) = -\alpha_n\cos(\alpha_n)$ or $-h\sin\alpha_n = \alpha_n\cos(\alpha_n)$, hence the above simplifies to

$$\begin{aligned}\int_0^1 (hx+1)\sin(\alpha_n(1-x))\,dx &= \frac{\alpha_n + h\alpha_n}{\alpha_n^2}\\ &= \frac{1+h}{\alpha_n}\end{aligned}$$

Therefore (3) becomes

$$\begin{aligned}B_n(0) &= \frac{-1}{1+h}\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\left(\frac{1+h}{\alpha_n}\right)\\ &= -\frac{1}{\alpha_n}\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\end{aligned}$$

Hence final solution becomes

$$\begin{aligned}u(x,t) &= \frac{hx+1}{1+h} + \sum_{n=1}^{\infty} B_n(0)\, e^{-\alpha_n^2 t}\phi_n(x)\\ &= \frac{hx+1}{1+h} + \sum_{n=1}^{\infty} B_n(0)\exp\left(-\alpha_n^2 t\right)\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\sin(\alpha_n(1-x))\\ &= \frac{hx+1}{1+h} + \sum_{n=1}^{\infty} -\frac{1}{\alpha_n}\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\exp\left(-\alpha_n^2 t\right)\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\sin(\alpha_n(1-x))\\ &= \frac{hx+1}{1+h} - 2h\sum_{n=1}^{\infty}\frac{\sin(\alpha_n(1-x))}{\alpha_n\left(h+\cos^2\alpha_n\right)}\exp\left(-\alpha_n^2 t\right)\end{aligned}$$

Which is what required to show.

Chapter 3

Study notes

Local contents

3.1 exam 1 notes 176

3.1 exam 1 notes

3.1.1 Chapter 1, sections 1-8 (Fourier series)

section 1

definition of left and right limits. definition of piecewise continuous function.

section 2

definition of Fourier cosine series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{2\pi}{T}x\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$ for $0 < x < \pi$.

section 3

Examples of Fourier cosine series

section 4

definition of Fourier sine series $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(n\frac{2\pi}{T}x\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin(nx)$ for $0 < x < \pi$.

section 5

Examples of Fourier sine series

section 6

Fourier series For period $T = 2\pi$

$$\begin{aligned} f(x) &\approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{2\pi}{T}x\right) + b_n \sin\left(n\frac{2\pi}{T}x\right) \qquad -\pi < x < \pi \\ &\approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \end{aligned}$$

Where

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad n = 0,1,2,\cdots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \qquad n = 1,2,\cdots \end{aligned}$$

If $f(x)$ is even then $b_n = 0$ and if $f(x)$ is odd, then $a_n = 0$.

section 7

Fourier series examples.

section 8 Adoption to different regions

Shows how F.S. on $-L < x < L$ can be obtained from know F.S. on $-\pi < x < \pi$. Not clear why example 2 on page 22 replaces $a = \frac{1}{\pi}$.

3.1.2 Chapter 2, sections 9-20 (Convergence of Fourier series)

section 9 (one sided derivatives)

$$f'_+(x_0) = \lim_{\substack{x\to x_0\\ x>x_0}} \frac{f(x) - f\left(x_0^+\right)}{x - x_0}$$
$$f'_-(x_0) = \lim_{\substack{x\to x_0\\ x>x_0}} \frac{f(x) - f\left(x_0^+\right)}{x - x_0}$$

Smooth function is one who is continuous and its derivative is also continuous. For example $f(x) = x^2$ is smooth, but $f(x) = |x|$ is not smooth.

Piecewise smooth function is one which $f(x)$ and $f'(x)$ are piecewise continuous.

section 10 (Properties of Fourier coefficients)

Bessel's inequalities

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \leq \frac{2}{\pi}\int_0^{\pi} \left[f(x)\right]^2 dx$$
$$\lim_{n\to\infty} a_n = 0$$
$$\sum_{n=1}^{\infty} b_n^2 \leq \frac{2}{\pi}\int_0^{\pi} \left[f(x)\right]^2 dx$$
$$\lim_{n\to\infty} b_n = 0$$

section 11 (Two Lemmas)

Lemma 1 If $f(x)$ is P.W.C. on $0 < x < \pi$ then

$$\lim_{N\to\infty} \int_0^{\pi} f(x) \sin\left(\left(N + \frac{1}{2}\right)x\right) dx = 0$$

Lemma 2 If $g(x)$ is P.W.C. on $0 < x < \pi$ and that $g'_+(0)$ exist, then

$$\lim_{N\to\infty}\int_0^{\pi} g(x)\frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{2\sin\frac{x}{2}}dx=\frac{\pi}{2}g\left(0^{+}\right)$$

Where $\frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{2\sin\frac{x}{2}}$ is called the Dirichlet kernel $D_N(x)$.

$$D_N(x)=\frac{1}{2}+\sum_{n=1}^{N}\cos(nx)$$
$$D_N(x)=\frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{2\sin\frac{x}{2}}$$
$$\int_0^{\pi} D_N(x)\,dx=\frac{\pi}{2}$$

Section 12 (Fourier theorem)

If $f(x)$ is P.W.C. on $-\pi<x<\pi$ and $f(x)$ is periodic on all of x with period 2π then at each x where $f'_{+}(x)$ and $f'_{-}(x)$ both exist, then $f(x)$ converges to the average of $f(x)$ at x which is $\frac{f\left(x^{+}\right)+f\left(x^{-}\right)}{2}$. Proof is long.

Section 13 (Related Fourier theorem)

Nothing new here. Seems same as last one. If $f(x)$ is PWC and $f'(x)$ is PWC, and $f(x)$ is periodic, then F.S. of $f(x)$ converges to mean of $f(x)$ at each point x.

Section 14 (Examples)

Examples on the Fourier theorem

Section 15 (Convergence on other intervals)

Nothing new here.

Section 16 (Lemma on absolute and uniform convergence)

If $f(x)$ is continuous on $-\pi<x<\pi$ (notice it has to be continuous, not PWC) and if $f(-\pi)=f(\pi)$ and $f'(x)$ is PWC on $-\pi<x<\pi$ then

$$\sum_{n=1}^{\infty}a_n^2+b_n^2$$

converges. Proof is given. And

$$\sum_{n=1}^{N} \alpha_n^2 + \beta_n^2 \leq \frac{1}{\pi} \int_{-\pi}^{\pi} \left[f'(x)\right]^2 dx \qquad N = 1, 2, 3, \cdots$$

Where

$$\begin{aligned} f'(x) &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos(nx) + \beta_n \sin(nx) \\ \alpha_0 &= 0 \\ \alpha_n &= nb_n \\ \beta_n &= na_n \end{aligned}$$

Section 17 (Absolute and uniform convergence of Fourier series)

M test is used to check if series is U.C. (uniform convergent). If we can find $\sum_{n=1}^{\infty} M_n$ which is convergent and M_n is positive constant, and where $\left|f_n(x)\right| \leq M_n$ for each n in $a < x < b$, then series $\sum_{n=1}^{\infty} f_n(x)$ is U.C.

Theorem If $f(x)$ is continuous on $-\pi \leq x \leq \pi$ and $f(-\pi) = f(\pi)$ and $f'(x)$ is PWC, then $f(x)$ both absolutely and uniformly convergent,

Section 18 (Gibbs phenomenon)

Not on exam.

Section 19 (Differentiation of Fourier series)

Same conditions as section 17 theorem. If $f(x)$ is continuous on $-\pi \leq x \leq \pi$ and $f(-\pi) = f(\pi)$ and $f'(x)$ is PWC, then F.S. of $f(x)$ can be differentiated term by term.

Section 20 (Integration of Fourier series)

As long as $f(x)$ is PWC, we can integrate F.S. term by term.

3.1.3 Chapter 3 (partial differential equations of physics)

Section 21 (Linear boundary value problem)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

And definitions.

Section 22 (1D heat PDE)

Flux is $\Phi = -K\frac{du}{dn}$ where K is thermal conductivity. Flux is amount of heat passing in normal direction per unit area in one second. Derivation of heat PDE

$$u_t = ku_{xx}$$

where k is thermal diffusivity $k = \frac{K}{\sigma\delta}$ where σ is specific heat and δ is density of material.

Section 23 (Related heat equations)

Nothing much here.

Section 24 (Laplace in cylindrical and spherical)

Just need to know the equations. Will be given in exam.

Section 25 (Derivations)

Not in exam

Section 26 (Boundary conditions)

Just need to know Neumann and Dirichlet.

Section 27 (Duhamel's principle)

Do not think this will be on exam.

Section 28 (Vibrating string)

Derivation of $y_{tt} = a^2 y_{xx}$ using physics. Will not be on exam.

Section 29 (Vibrations of bars and membranes)

Generalization of section 28.

Section 30 (General solution to wave equation)

To derive solution to $y_{tt} = a^2 y_{xx}$, use $u = x + at, v = x - at$ and the PDE becomes $y_{uv} = 0$ which has solution $y = \Phi(u) + \Psi(v)$ or

$$y(x,t) = \Phi(x + at) + \Psi(x - at)$$

Where initial conditions are $y(x,0) = f(x), y_t(x,0) = g(x)$ then the solution becomes

$$y(x,t) = \frac{1}{2}\Big(f(x+at) + f(x-at)\Big) + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds$$

Section 31 (Types of equations and boundary conditions)

1. Hyperbolic $B^2 - 4AC > 0$
2. Elliptic $B^2 - 4AC < 0$
3. parabolic $B^2 - 4AC = 0$

3.1.4 Chapter 4 (The Fourier method)

Section 32 (linear operators)

$$L(c_1u_1 + c_2u_2) = c_1Lu_1 + c_2Lu_2$$

Section 33 (Principle of superposition)

Suppose each function u_i satisfies a linear homogeneous differential equation or boundary value problem $Lu = 0$, then $\sum_{n=1}^{\infty} u_n$ also satisfies the same equation.

Section 34 (Examples of Principle of superposition)

Some examples. Go over.

Section 35 (Eigenvalues and eigenfunctions)

Show how to solve $X'' + \lambda X = 0$ for different boundary conditions.

Section 36 (A temperature problem)

Applying Eigenvalues and eigenfunctions to heat PDE on rod.

Section 37 (Vibrating string)

Applying Eigenvalues and eigenfunctions to wave PDE On string $u_{tt} = a^2u_{xx}$ with fixed on ends and have initial conditions.

Section 38 (Historical development)

Not on exam

Chapter 4

Exams

Local contents

4.1 exam 1 184
4.2 exam 2 186
4.3 Final exam 188

4.1 exam 1

Local contents

4.1.1 questions 184

4.1.1 questions

1. (30 points)

(1) Define the Fourier series over the interval $-c < x < c$ corresponding to piecewise continuous function $f(x)$.

(2) State the convergence theorem for such Fourier series.

(3) For what value a does the Fourier series over the interval $-1 < x < 1$ corresponding to the function

$$f(x) = e^x + a\,x$$

converge to $f(x)$ at $x = 1$.

2. (30 points)

Find eigenvalues and corresponding eigenfunctions.

$$X''(x) + \lambda\,X(x) = 0, \quad 0 < x < 1$$

subject to the boundary conditions $X'(0) = 0$ and $X(1) = 0$.

3. (40 points)

Solve the boundary value problem

$$y_{tt}(x,t) = y_{xx}(x,t) - y(x,t), \ 0 < x < \pi, \ t > 0;$$

$$y(0,t) = y(\pi,t) = 0; \ y(x,0) = 0, \ y_t(x,0) = 1.$$

4.2 exam 2

Local contents

4.2.1 questions 186

4.2.1 questions

1. (40 points)

With the aid of the expansion

$$\pi - x = 2\sum_{n=1}^{\infty} \frac{\sin nx}{n},\ \ 0 < x < \pi,$$

solve the following problem.

$$u_t(x,t) = u_{xx}(x,t) + t\,(\pi - x),\ \ 0 < x < \pi,\ \ t > 0;$$

source.

$$u(0,t) = 0,\ \ u(\pi,t) = 0;\ \ u(x,0) = 0.$$

2. (20 points) Verify that all of the conditions of the Fourier sine integral representation are satisfied by the function f defined by

$$f(x) := \begin{cases} x & \text{when } 0 \le x \le 1 \\ 2 - x & \text{when } 1 < x \le 2 \\ 0 & \text{when } x < 0 \text{ or } x > 2 \end{cases}$$

and show that for $0 < x < \infty$,

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{(2\,\sin\alpha - \sin 2\alpha)\,\sin\alpha\,x}{\alpha^2}\,d\alpha.$$

3. (40 points)

Find the bounded harmonic function $u(x, y)$ in the semi-infinite strip $0 < x < \infty$, $0 < y < 1$ that satisfies the conditions $u(x, 0) = 0$, $u(0, y) = 0$ and $u(x, 1) = f(x)$, where $f(x)$ is the function given in problem 2.

4.3 Final exam

Local contents

4.3.1 questions 188

4.3.1 questions

1. (30 points) Suppose both $f(x) = \sin(2\pi x)$ and $g(x) = \cos(2\pi x) + c$ are eigenfunctions corresponding to distinctive eigenvalues to the following Sturm-Liouville problem, where r, r' and q are all assumed to be continuous on $[0, 1]$. Find constant c.

$$[r(x)X'(x)]' + [q(x) + \lambda(x+1)]\, X(x) = 0, \quad 0 < x < 1;$$

$$X(0) = X(1), \quad X'(0) = X'(1).$$

2. (30 points) Solve for the eigenvalues and normalized eigenfunctions.

$$X'' + \lambda X = 0, \quad 0 < x < 1;$$

$$X(0) - X'(0) = 0, \quad X(1) + X'(1) = 0.$$

3. (40 points) Solve the boundary value problem

$$(1+t)\, u_t(x,t) = u_{xx}(x,t) \quad (0 < x < 1,\ t > 0).$$

$$u_x(0,t) = -1,\ u(1,t) = 0,\ u(x,0) = 0.$$