HW 8
MATH 4567 Applied Fourier Analysis
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1 Section 57, Problem 5

5. Find the bounded harmonic function $u(x, y)$ in the semi-infinite strip $0 < x < 1$, $y > 0$ that satisfies the conditions

$$u_y(x,0) = 0, \quad u(0,y) = 0, \quad u_x(1,y) = f(y).$$

Answer: $u(x,y) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\sinh \alpha x \cos \alpha y}{\alpha \cosh \alpha} \int_0^\infty f(s) \cos \alpha s \, ds \, d\alpha.$

Figure 1: Problem statement

<u>Solution</u>

$$\begin{aligned} \nabla^2 u\left(x,y\right) &= 0 \qquad \left(0<x<1, y>0\right) \\ u_y\left(x,0\right) &= 0 \\ u\left(0,y\right) &= 0 \\ u_x\left(1,y\right) &= f\left(y\right) \end{aligned}$$

As normal, we use separation of variables, ending in $\frac{X''}{X} + \frac{Y''}{Y} = -\lambda$. We will take the eigenvalue problem along the Y direction. This leads to

$$\begin{aligned} Y'' + \lambda Y &= 0 \\ Y'(0) &= 0 \end{aligned}$$

Where $\lambda = \alpha^2, \alpha > 0$. The steps that led to this were done before. Therefore the solution is

$$\begin{aligned} Y\left(y\right) &= c_1 \cos\left(\alpha y\right) + c_2 \sin\left(\alpha y\right) \\ Y'\left(y\right) &= -c_1 \alpha \sin\left(\alpha y\right) + c_2 \alpha \cos\left(\alpha y\right) \end{aligned}$$

At $y = 0$ the above gives

$$0 = c_2 \alpha$$

Which implies $c_2 = 0$. Hence the eigenfunctions are

$$Y_\alpha\left(y\right) = \cos\left(\alpha y\right)$$

With the eigenvalues being $\lambda = \alpha^2$ for all real positive values of α. The corresponding $X(x)$ ode is

$$\begin{aligned} X'' - \lambda X &= 0 \\ X(0) &= 0 \end{aligned}$$

The solution to this is $X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$, which at $x = 0$ gives

$$0 = c_1 + c_2$$

Which makes the solution as $X(x) = c_1 e^{\alpha x} - c_1 e^{-\alpha x} = c_1\left(e^{\alpha x} - e^{-\alpha x}\right) = 2c_1 \sinh(\alpha x) = c_3 \sinh(\alpha x)$. Therefore the general solution is given by the real form of the Fourier integral

$$u\left(x,y\right) = \int_0^\infty A(\alpha) \sinh(\alpha x) \cos\left(\alpha y\right) d\alpha \tag{1}$$

Taking derivative w.r.t. x gives

$$u_x\left(x,y\right) = \int_0^\infty A(\alpha)\, \alpha \cosh(\alpha x) \cos\left(\alpha y\right) d\alpha$$

At $x = 1$ the above becomes

$$f\left(y\right) = \int_0^\infty \left(A(\alpha)\, \alpha \cosh(\alpha)\right) \cos\left(\alpha y\right) d\alpha$$

Therefore

$$A(\alpha)\,\alpha\cosh(\alpha) = \frac{2}{\pi}\int_0^\infty f(y)\cos(\alpha y)\,d\alpha$$

$$A(\alpha) = \frac{2}{\pi\alpha\cosh(\alpha)}\int_0^\infty f(y)\cos(\alpha y)\,d\alpha$$

Substituting the above in (1) gives the solution

$$u(x,y) = \int_0^\infty\left(\frac{2}{\pi\alpha\cosh(\alpha)}\int_0^\infty f(s)\cos(\alpha s)\,ds\right)\sinh(\alpha x)\cos(\alpha y)\,d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty\frac{\sinh(\alpha x)\cos(\alpha y)}{\alpha\cosh(\alpha)}\left(\int_0^\infty f(s)\cos(\alpha s)\,ds\right)d\alpha$$

Which is the result required to show.

2 Section 58, Problem 5

5. (*a*) The face $x = 0$ of a semi-infinite solid $x \geq 0$ is insulated, and the initial temperature distribution is $f(x)$. Derive the temperature formula

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-x/(2\sqrt{kt})}^{\infty} f(x + 2\sigma\sqrt{kt})\, e^{-\sigma^2}\, d\sigma$$

$$+ \frac{1}{\sqrt{\pi}} \int_{x/(2\sqrt{kt})}^{\infty} f(-x + 2\sigma\sqrt{kt})\, e^{-\sigma^2}\, d\sigma.$$

(*b*) Show that if the function f in part (*a*) is defined by means of the equations

$$f(x) = \begin{cases} 1 & \text{when } 0 < x < c, \\ 0 & \text{when } \quad x > c, \end{cases}$$

then

$$u(x,t) = \frac{1}{2}\operatorname{erf}\left(\frac{c+x}{2\sqrt{kt}}\right) + \frac{1}{2}\operatorname{erf}\left(\frac{c-x}{2\sqrt{kt}}\right).$$

Figure 2: Problem statement

Solution

2.1 Part (a)

$$\begin{aligned} u_t(x,t) &= k u_{xx}(x,t) \qquad (0 < x < \infty, t > 0) \\ u(x,0) &= f(x) \\ u_x(0,t) &= 0 \end{aligned}$$

Applying separation of variables leads to

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda$$

Hence

$$\begin{aligned} X'' + \lambda X &= 0 \\ X'(0) &= 0 \\ |X(x)| &< M \end{aligned}$$

Since on semi-infinite domain, then only $\lambda > 0$ are possible eigenvalues. Let $\lambda = \alpha^2, \alpha > 0$, Where α takes on all positive real values. Then the solution to the eigenvalue ODE is

$$\begin{aligned} X_\alpha(x) &= c_1 \cos(\alpha x) + c_2 \sin(\alpha x) \\ X'_\alpha(x) &= -c_1 \alpha \sin(\alpha x) + c_2 \alpha \cos(\alpha x) \end{aligned}$$

At $x = 0$

$$0 = c_2 \alpha$$

Hence $c_2 = 0$ and the eigenfunctions are

$$X_\alpha(x) = \cos(\alpha x)$$

The time ODE is therefore $T' + \alpha^2 k T = 0$ which has solution $T = e^{-k\alpha^2 t}$. Hence the solution is given by the real Fourier integral

$$u(x,t) = \int_0^\infty A(\alpha)\, e^{-k\alpha^2 t} \cos(\alpha x)\, d\alpha \tag{1}$$

At $t = 0$, using initial conditions, then the above becomes

$$\begin{aligned} f(x) &= \int_0^\infty A(\alpha) \cos \alpha x d\alpha \\ A(\alpha) &= \frac{2}{\pi} \int_0^\infty f(s) \cos(\alpha s)\, ds \end{aligned} \tag{2}$$

Using (2) in (1) gives

$$u(x,t)=\int_0^\infty\left(\frac{2}{\pi}\int_0^\infty f(s)\cos(\alpha s)\,ds\right)e^{-k\alpha^2 t}\cos(\alpha x)\,d\alpha$$

Changing the order of integration

$$u(x,t)=\frac{1}{\pi}\int_0^\infty\int_0^\infty\left(e^{-k\alpha^2 t}\left[2\cos(\alpha x)\cos(\alpha s)\right]d\alpha\right)f(s)\,ds \tag{3}$$

Using trig identity $\cos(A)\cos(B)=\frac{\cos(A+B)+\cos(A-B)}{2}$, then

$$\begin{aligned}2\cos(\alpha x)\cos(\alpha s)&=\cos(\alpha x+\alpha s)+\cos(\alpha x-\alpha s)\\&=\cos(\alpha(x+s))+\cos(\alpha(x-s))\end{aligned}$$

Substituting the above in (3) gives

$$\begin{aligned}u(x,t)&=\frac{1}{\pi}\int_0^\infty\int_0^\infty\left(e^{-k\alpha^2 t}\left[\cos(\alpha(x+s))+\cos(\alpha(x-s))\right]d\alpha\right)f(s)\,ds\\&=\frac{1}{\pi}\int_0^\infty\left(\int_0^\infty e^{-k\alpha^2 t}\cos(\alpha(x+s))\,d\alpha+\int_0^\infty e^{-k\alpha^2 t}\cos(\alpha(x-s))\,d\alpha\right)f(s)\,ds\end{aligned}$$

Using the formula

$$\int_0^\infty e^{-\alpha^2 c}\cos(\alpha b)\,d\alpha=\frac{1}{2}\sqrt{\frac{\pi}{c}}\exp\left(-\frac{b^2}{4c}\right)$$

Where in our case $c=kt$ and $b=(x+s)$ for the first integral, and $b=(x-s)$ for the second integral. Using the above formula in (4) results in

$$u(x,t)=\frac{1}{\pi}\int_0^\infty\left(\frac{1}{2}\sqrt{\frac{\pi}{kt}}\exp\left(-\frac{(x+s)^2}{4kt}\right)+\frac{1}{2}\sqrt{\frac{\pi}{kt}}\exp\left(-\frac{(x-s)^2}{4kt}\right)\right)f(s)\,ds$$

For $t>0$. Hence the above becomes

$$u(x,t)=\frac{1}{2\sqrt{\pi kt}}\int_0^\infty f(s)\exp\left(-\frac{(x+s)^2}{4kt}\right)ds+\frac{1}{2\sqrt{\pi kt}}\int_0^\infty f(s)\exp\left(-\frac{(x-s)^2}{4kt}\right)ds$$

By writing $s=-x+2\sigma\sqrt{kt}$ for the first integral above, then $\frac{ds}{d\sigma}=2\sqrt{kt}$. When $s=0$ then $\sigma=\frac{x}{2\sqrt{kt}}$ and when $s=\infty$ then $\sigma=\infty$. And by writing $s=x+2\sigma\sqrt{kt}$ for the second integral above, then $\frac{ds}{d\sigma}=2\sqrt{kt}$. When $s=0$ then $\sigma=-\frac{x}{2\sqrt{kt}}$ Hence the above integral becomes

$$\begin{aligned}u(x,t)=&\frac{2\sqrt{kt}}{2\sqrt{\pi kt}}\int_{\frac{x}{2\sqrt{kt}}}^\infty f\left(-x+2\sigma\sqrt{kt}\right)\exp\left(-\frac{\left(-x+\left(x+2\sigma\sqrt{kt}\right)\right)^2}{4kt}\right)d\sigma\\&+\frac{2\sqrt{kt}}{2\sqrt{\pi kt}}\int_{-\frac{x}{2\sqrt{kt}}}^\infty f\left(x+2\sigma\sqrt{kt}\right)\exp\left(-\frac{\left(x-\left(x+2\sigma\sqrt{kt}\right)\right)^2}{4kt}\right)d\sigma\end{aligned}$$

Simplifying gives

$$\begin{aligned}u(x,t)&=\frac{1}{\sqrt{\pi}}\int_{-\frac{x}{2\sqrt{kt}}}^\infty f\left(x+2\sigma\sqrt{kt}\right)e^{-\frac{\left(-2\sigma\sqrt{kt}\right)^2}{4kt}}d\sigma+\frac{1}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{kt}}}^\infty f\left(-x+2\sigma\sqrt{kt}\right)e^{-\frac{\left(2\sigma\sqrt{kt}\right)^2}{4kt}}d\sigma\\&=\frac{1}{\sqrt{\pi}}\int_{-\frac{x}{2\sqrt{kt}}}^\infty f\left(x+2\sigma\sqrt{kt}\right)e^{-\sigma^2}d\sigma+\frac{1}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{kt}}}^\infty f\left(-x+2\sigma\sqrt{kt}\right)e^{-\sigma^2}d\sigma+\end{aligned}\tag{4}$$

Which is the result required to show.

2.2 Part b

$$f(x)=\begin{cases}1 & 0<x<c\\0 & x>0\end{cases}$$

Considering the first function in (4), where in the following $f(x) \equiv f\left(x + 2\sigma\sqrt{kt}\right)$ then (4) becomes

$$u(x,t) = \frac{1}{\sqrt{\pi}}\left(\int_0^{\frac{c+x}{2\sqrt{kt}}} e^{-\sigma^2} d\sigma + \int_0^{\frac{c-x}{2\sqrt{kt}}} e^{-\sigma^2} d\sigma\right)$$

But $\frac{2}{\sqrt{\pi}}\int_0^{\frac{c+x}{2\sqrt{kt}}} e^{-\sigma^2} d\sigma = \operatorname{erf}\left(\frac{c+x}{2\sqrt{kt}}\right)$ and $\frac{2}{\sqrt{\pi}}\int_0^{\frac{c-x}{2\sqrt{kt}}} e^{-\sigma^2} d\sigma = \operatorname{erf}\left(\frac{c-x}{2\sqrt{kt}}\right)$, hence the above becomes

$$u(x,t) = \frac{1}{2}\operatorname{erf}\left(\frac{c+x}{2\sqrt{kt}}\right) + \frac{1}{2}\operatorname{erf}\left(\frac{c-x}{2\sqrt{kt}}\right)$$

3 Section 58, Problem 7

7. Verify that for any constant C, the function

$$v(x,t) = Cxt^{-3/2}\exp\left(-\frac{x^2}{4kt}\right)$$

satisfies the heat equation $v_t = kv_{xx}$ when $x > 0$ and $t > 0$. Also, verify that for those values of x and t,

Thus show that $v(x,t)$ can be added to the solution (9) found in Sec. 58 to form other solutions of the problem there if the temperature function is not required to be bounded. Note that v is unbounded as x and t tend to zero (this can be seen by letting x vanish while $t = x^2$).

Figure 3: Problem statement

Solution

We need to substitute the solution $v(x,t) = Cxt^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}}$ into the PDE $v_t = kv_{xx}$ and see if it satisfies it.

$$\begin{aligned} v_t &= \frac{-3}{2}Cxt^{\frac{-5}{2}} e^{\frac{-x^2}{4kt}} + Cxt^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}}\left(\frac{x^2}{4kt^2}\right) \\ &= \frac{-3}{2}Cxt^{\frac{-5}{2}} e^{\frac{-x^2}{4kt}} + C\frac{x^3}{4kt^2}t^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}} \end{aligned}$$

And

$$\begin{aligned} v_x &= Ct^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}} - \frac{x^2}{2kt}Ct^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}} \\ v_{xx} &= \frac{-x}{2kt}Ct^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}} - \left(\frac{x}{kt}Ct^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}} - \frac{4x^3}{(4kt)^2}Ct^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}}\right) \\ &= \frac{-2x}{4kt}Ct^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}} - \left(\frac{x}{kt}Ct^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}} - \frac{x^3}{4k^2t^2}Ct^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}}\right) \\ &= \frac{-x}{2kt}Ct^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}} - \frac{x}{kt}Ct^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}} + \frac{4x^3}{(4kt)^2}Ct^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}} \\ &= -\frac{3}{2}\frac{x}{k}Ct^{\frac{-5}{2}} e^{\frac{-x^2}{4kt}} + C\frac{x^3}{4k^2t^2}t^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}} \end{aligned}$$

Hence $v_t = kv_{xx}$ becomes

$$\begin{aligned} \frac{-3}{2}Cxt^{\frac{-5}{2}} e^{\frac{-x^2}{4kt}} + C\frac{x^3}{4kt^2}t^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}} &= k\left(-\frac{3}{2}\frac{x}{k}Ct^{\frac{-5}{2}} e^{\frac{-x^2}{4kt}} + C\frac{x^3}{4k^2t^2}t^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}}\right) \\ \frac{-3}{2}Cxt^{\frac{-5}{2}} e^{\frac{-x^2}{4kt}} + C\frac{x^3}{4kt^2}t^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}} &= -\frac{3}{2}xCt^{\frac{-5}{2}} e^{\frac{-x^2}{4kt}} + C\frac{x^3}{4kt^2}t^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}} \\ 0 &= 0 \end{aligned}$$

Hence it is satisfied for any constant C.

Using $v(x,t) = Cxt^{\frac{-3}{2}} e^{\frac{-x^2}{4kt}}$, we see that $\lim_{x\to 0^+} v(x,t) = 0$. Also $\lim_{t\to 0^+} v(x,t) = 0$.

Since the solution to the heat PDE is now not required to be bounded and since $v(x,t)$ has zero initial conditions, then because the PDE is linear and homogeneous, then solution as $v(x,t)$ can be added to the solution in (9) using superposition.

4 Section 59, Problem 2

2. Derive this solution of the wave equation $y_{tt} = a^2 y_{xx}$ $(-\infty < x < \infty, t > 0)$, which satisfies the conditions $y(x, 0) = f(x)$ and $y_t(x, 0) = 0$ when $-\infty < x < \infty$:

$$y(x,t) = \frac{1}{\pi}\int_0^{\infty} \cos \alpha a t \int_{-\infty}^{\infty} f(s) \cos \alpha(s-x)\, ds\, d\alpha.$$

Also, reduce the solution to the form obtained in Example 1, Sec. 30:

$$y(x,t) = \frac{1}{2}\left[f(x+at) + f(x-at)\right].$$

Figure 4: Problem description

solution

Let $y(x,t) = X(x)T(t)$, then the PDE becomes

$$\begin{aligned} T''X &= a^2 X''T \\ \frac{1}{a^2}\frac{T''}{T} &= \frac{X''}{X} = -\lambda \end{aligned}$$

We take the $X(x)$ ode as the eigenvalue problem. Since the domain is infinite, then only positive eigenvalue are valid as was shown before. Let $\lambda = \alpha^2, \alpha > 0$. Hence the eigenfunctions are

$$X_\alpha(x) = A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)$$

The time ODE becomes

$$\begin{aligned} \frac{1}{a^2}\frac{T''}{T} &= -\alpha^2 \\ T'' + a^2\alpha^2 T &= 0 \end{aligned}$$

Which has the solution

$$T_\alpha(t) = C(\alpha)\cos(a\alpha t) + D(\alpha)\sin(a\alpha t)$$

Hence the solution is given by the Fourier real integral

$$\begin{aligned} y(x,t) &= \int_0^\infty T_\alpha(t) X_\alpha(x)\, d\alpha \qquad (1) \\ &= \int_0^\infty (C(\alpha)\cos(a\alpha t) + D(\alpha)\sin(a\alpha t))(A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x))\, d\alpha \\ &= \int_0^\infty C(\alpha)A(\alpha)\cos(a\alpha t)\cos(\alpha x)\, d\alpha + \int_0^\infty C(\alpha)B(\alpha)\cos(a\alpha t)\sin(\alpha x)\, d\alpha \\ &+ \int_0^\infty D(\alpha)A(\alpha)\sin(a\alpha t)\cos(\alpha x)\, d\alpha + \int_0^\infty D(\alpha)B(\alpha)\sin(a\alpha t)\sin(\alpha x)\, d\alpha \qquad (2) \end{aligned}$$

Taking time derivative

$$\begin{aligned} y_t(x,t) &= \int_0^\infty -a\alpha C(\alpha)A(\alpha)\sin(a\alpha t)\cos(\alpha x)\, d\alpha + \int_0^\infty a\alpha C(\alpha)B(\alpha)\sin(a\alpha t)\sin(\alpha x)\, d\alpha \\ &+ \int_0^\infty a\alpha D(\alpha)A(\alpha)\cos(a\alpha t)\cos(\alpha x)\, d\alpha + \int_0^\infty a\alpha D(\alpha)B(\alpha)\cos(a\alpha t)\sin(\alpha x)\, d\alpha \end{aligned}$$

At $t = 0$ the above becomes

$$0 = \int_0^\infty a\alpha D(\alpha)A(\alpha)\cos(\alpha x)\, d\alpha + \int_0^\infty a\alpha D(\alpha)B(\alpha)\sin(\alpha x)\, d\alpha$$

Which simplifies to

$$\begin{aligned} 0 &= \int_0^\infty D(\alpha)A(\alpha)\cos(\alpha x)\, d\alpha + \int_0^\infty D(\alpha)B(\alpha)\sin(\alpha x)\, d\alpha \\ &= \int_0^\infty D(\alpha)(A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x))\, d\alpha \end{aligned}$$

Therefore, since $A(\alpha), B(\alpha)$ can not be both zero, else eigenfunction is zero, then it must be that $D(\alpha) = 0$. Hence the solution in (2) becomes

$$y(x,t) = \int_0^\infty C(\alpha) A(\alpha) \cos(a\alpha t) \cos(\alpha x)\, d\alpha + \int_0^\infty C(\alpha) B(\alpha) \cos(a\alpha t) \sin(\alpha x)\, d\alpha \tag{3}$$

Let $C(\alpha) A(\alpha) = C_1(\alpha)$ and let $C(\alpha) B(\alpha) = C_2(\alpha)$ as two new constants, and the above becomes

$$y(x,t) = \int_0^\infty C_1(\alpha) \cos(a\alpha t) \cos(\alpha x)\, d\alpha + \int_0^\infty C_2(\alpha) \cos(a\alpha t) \sin(\alpha x)\, d\alpha$$

At $t = 0$ the above becomes

$$f(x) = \int_0^\infty C_1(\alpha) \cos(\alpha x)\, d\alpha + \int_0^\infty C_2(\alpha) \sin(\alpha x)\, d\alpha$$

Hence

$$\begin{aligned} C_1(\alpha) &= \frac{1}{\pi} \int_{-\infty}^\infty f(s) \cos(\alpha s)\, ds \\ C_2(\alpha) &= \frac{1}{\pi} \int_{-\infty}^\infty f(s) \sin(\alpha s)\, ds \end{aligned}$$

Therefore (3) becomes

$$\begin{aligned} y(x,t) &= \frac{1}{\pi} \int_0^\infty \left(\int_{-\infty}^\infty f(s) \cos(\alpha s)\, ds \right) \cos(a\alpha t) \cos(\alpha x)\, d\alpha \\ &+ \frac{1}{\pi} \int_0^\infty \left(\int_{-\infty}^\infty f(s) \sin(\alpha s)\, ds \right) \cos(a\alpha t) \sin(\alpha x)\, d\alpha \end{aligned}$$

Changing order of integrations in the above for both integrals results in

$$\begin{aligned} y(x,t) &= \frac{1}{\pi} \int_0^\infty \left(\int_{-\infty}^\infty \cos(a\alpha t) \cos(\alpha s) \cos(\alpha x)\, d\alpha \right) f(s)\, ds \\ &+ \frac{1}{\pi} \int_0^\infty \left(\int_{-\infty}^\infty \cos(a\alpha t) \sin(\alpha s) \sin(\alpha x)\, d\alpha \right) f(s)\, ds \end{aligned} \tag{4}$$

But

$$\begin{aligned} \cos(\alpha s) \cos(\alpha x) &= \frac{1}{2} \left(\cos(\alpha s + \alpha x) + \cos(\alpha s - \alpha x) \right) \\ &= \frac{1}{2} \left(\cos(\alpha(s + x)) + \cos(\alpha(s - x)) \right) \end{aligned}$$

and

$$\begin{aligned} \sin(\alpha s) \sin(\alpha x) &= \frac{1}{2} \left(\cos(\alpha s - \alpha x) - \cos(\alpha s + \alpha x) \right) \\ &= \frac{1}{2} \left(\cos(\alpha(s - x)) - \cos(\alpha(s + x)) \right) \end{aligned}$$

Substituting the above two relations back in (4) gives

$$\begin{aligned} y(x,t) &= \frac{1}{2\pi} \int_0^\infty \left(\int_{-\infty}^\infty \cos(a\alpha t) \left(\cos(\alpha(s + x)) + \cos(\alpha(s - x)) \right) d\alpha \right) f(s)\, ds \\ &+ \frac{1}{2\pi} \int_0^\infty \left(\int_{-\infty}^\infty \cos(a\alpha t) \left(\cos(\alpha(s - x)) - \cos(\alpha(s + x)) \right) d\alpha \right) f(s)\, ds \end{aligned}$$

Simplifying, terms cancel giving

$$\begin{aligned} y(x,t) &= \frac{1}{2\pi} \int_0^\infty \left(\int_{-\infty}^\infty \cos(a\alpha t) \left[\cos(\alpha(s - x)) + \cos(\alpha(s - x)) \right] d\alpha \right) f(s)\, ds \\ &= \frac{1}{\pi} \int_0^\infty \left(\int_{-\infty}^\infty \cos(a\alpha t) \cos(\alpha(s - x))\, d\alpha \right) f(s)\, ds \end{aligned}$$

Changing order of integration

$$y(x,t) = \frac{1}{\pi} \int_0^\infty \cos(a\alpha t) \int_{-\infty}^\infty f(s) \cos(\alpha(s - x))\, ds d\alpha$$

Which is the result required to show.

5 Section 59, Problem 3

3. Find the bounded harmonic function $u(x, y)$ in the strip $-\infty < x < \infty, 0 < y < b$ such that $u(x, 0) = 0$ and $u(x, b) = f(x)$ $(-\infty < x < \infty)$, where f is bounded and represented by its Fourier integral.

Answer: $u(x, y) = \frac{1}{\pi}\int_0^\infty \frac{\sinh \alpha y}{\sinh \alpha b}\int_{-\infty}^{\infty} f(s)\cos\alpha(s-x)\,ds\,d\alpha.$

Figure 5: Problem description

solution

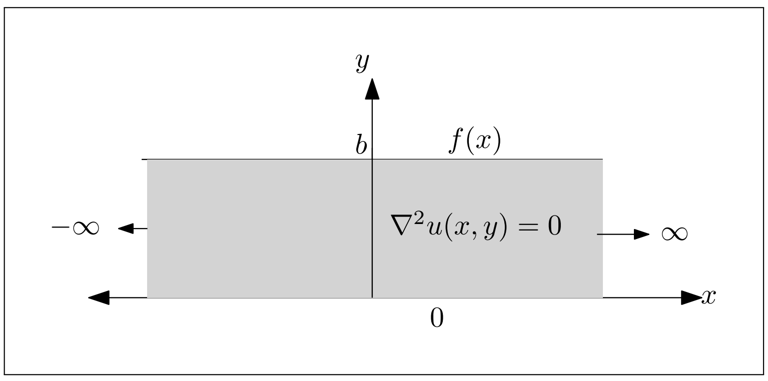

Figure 6: Solution domain for PDE

Let $u = X(x)Y(y)$, then $u_{xx} + y_{xx} = 0$ becomes

$$X''X + Y''X = 0$$
$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

Taking the eigenvalue ODE to be on the x axis, then

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

Hence

$$X'' + \lambda X = 0$$
$$|X(x)| < \infty$$

Hence λ can only be positive real. Let $\lambda = \alpha^2, \alpha > 0$. Therefore the eigenfunctions are

$$X_\alpha(x) = A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x \tag{1}$$

For the ODE $Y'' - Y\alpha^2 = 0$ the solution is

$$Y_\alpha(y) = C(\alpha)\cosh(\alpha y) + D(\alpha)\sinh(\alpha y) \tag{2}$$

Hence the solution is

$$\begin{aligned} u(x,y) &= \int_0^\infty X_\alpha(x)Y_\alpha(y)\,d\alpha \\ &= \int_0^\infty (A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x)\big(C(\alpha)\cosh(\alpha y) + D(\alpha)\sinh(\alpha y)\big)\,d\alpha \end{aligned} \tag{3}$$

When $y = 0$, the above becomes

$$0 = \int_0^\infty (A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x)\,C(\alpha)\,d\alpha$$

Which implies that $C(\alpha) = 0$. Therefore the solution (3) simplifies to

$$u(x,y) = \int_0^\infty (A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)) D(\alpha)\sinh(\alpha y)\, d\alpha$$
$$= \int_0^\infty A(\alpha) D(\alpha)\sinh(\alpha y)\cos\alpha x + B(\alpha) D(\alpha)\sinh(\alpha y)\sin(\alpha x)\, d\alpha$$

Let $A(\alpha) D(\alpha) = C_1(\alpha)$ and let $B(\alpha) D(\alpha) = C_2(\alpha)$, hence the above solution becomes

$$u(x,y) = \int_0^\infty C_1(\alpha)\sinh(\alpha y)\cos\alpha x + C_2(\alpha)\sinh(\alpha y)\sin(\alpha x)\, d\alpha \tag{4}$$

When $y = b$ the above becomes

$$f(x) = \int_0^\infty C_1(\alpha)\sinh(\alpha b)\cos\alpha x + C_2(\alpha)\sinh(\alpha b)\sin(\alpha x)\, d\alpha$$

Therefore

$$C_1(\alpha)\sinh(\alpha b) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(s)\cos(\alpha s)\, ds$$
$$C_1(\alpha) = \frac{1}{\pi\sinh(\alpha b)}\int_{-\infty}^{\infty} f(s)\cos(\alpha s)\, ds \tag{5}$$

And

$$C_2(\alpha)\sinh(\alpha b) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(s)\sin(\alpha s)\, ds$$
$$C_2(\alpha) = \frac{1}{\pi\sinh(\alpha b)}\int_{-\infty}^{\infty} f(s)\sin(\alpha s)\, ds \tag{6}$$

Using (5,6) in (4) gives

$$u(x,y) = \int_0^\infty \left(\frac{1}{\pi\sinh(\alpha b)}\int_{-\infty}^{\infty} f(s)\cos(\alpha s)\, ds\right)\sinh(\alpha y)\cos(\alpha x) + \left(\frac{1}{\pi\sinh(\alpha b)}\int_{-\infty}^{\infty} f(s)\sin(\alpha s)\, ds\right)\sinh$$
$$= \int_0^\infty \left(\frac{\sinh(\alpha y)}{\pi\sinh(\alpha b)}\int_{-\infty}^{\infty} f(s)\cos(\alpha s)\cos\alpha x ds\right) + \left(\frac{\sinh(\alpha y)}{\pi\sinh(\alpha b)}\int_{-\infty}^{\infty} f(s)\sin(\alpha s)\sin(\alpha x)\, ds\right) d\alpha$$
$$= \frac{1}{\pi}\int_0^\infty \frac{\sinh(\alpha y)}{\sinh(\alpha b)}\left(\int_{-\infty}^{\infty} f(s)\cos(\alpha s)\cos\alpha x + f(s)\sin(\alpha s)\sin(\alpha x)\, ds\right) d\alpha$$
$$= \frac{1}{\pi}\int_0^\infty \frac{\sinh(\alpha y)}{\sinh(\alpha b)}\left(\int_{-\infty}^{\infty} f(s)\left[\cos(\alpha s)\cos\alpha x + \sin(\alpha s)\sin(\alpha x)\right] ds\right) d\alpha$$
$$= \frac{1}{\pi}\int_0^\infty \frac{\sinh(\alpha y)}{\sinh(\alpha b)}\left(\int_{-\infty}^{\infty} f(s)\cos\alpha(s-x)\, ds\right) d\alpha$$

Which is the result required to show.