HW 5
MATH 4567 Applied Fourier Analysis
Spring 2019
University of Minnesota, Twin Cities

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November 2, 2019 Compiled on November 2, 2019 at 9:47pm [public]

Contents

1 Section 34, Problem 3

3. Verify that each of the functions

$$u_0 = y, \qquad u_n = \sinh ny \cos nx \qquad (n = 1, 2, \ldots)$$

satisfies Laplace's equation (Sec. 23)

$$u_{xx}(x, y) + u_{yy}(x, y) = 0 \qquad (0 < x < \pi,\ 0 < y < 2)$$

and the boundary conditions

$$u_x(0, y) = u_x(\pi, y) = 0, \qquad u(x, 0) = 0.$$

Then use the superposition principle in Sec. 33 to show formally, without considering questions of convergence, differentiability, or continuity, that the series

$$u(x, y) = A_0\, y + \sum_{n=1}^{\infty} A_n \sinh ny \cos nx$$

satisfies the same differential equation and boundary conditions.

Figure 1: Problem statement

<u>Solution</u>

The boundary conditions are

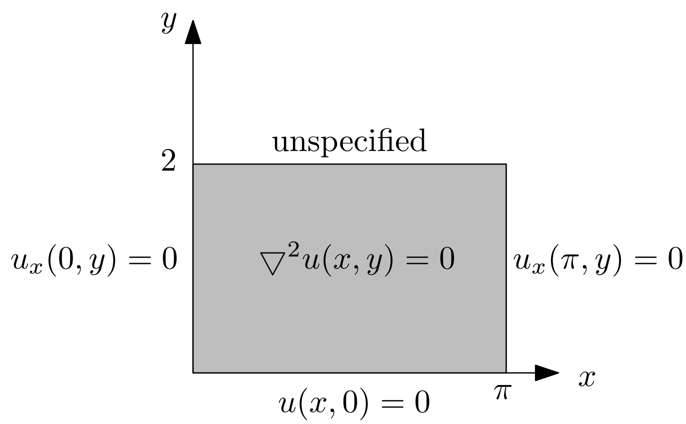


Figure 2: Boundary conditions

Let

$$u(x, y) = X(x)\, Y(y)$$

Substitution in the PDE $u_{xx} + y_{yy} = 0$ leads to

$$X''Y + Y''Y = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

Where λ is the separation constant. We obtain two ODE's

$$X'' + \lambda X = 0 \tag{1}$$

$$Y'' - \lambda Y = 0 \tag{2}$$

We use the $X(x)$ ODE (1) to determine the eigenvalues, since that ODE has both boundary conditions specified:

$$X'' + \lambda X = 0$$

$$X'(0) = 0$$

$$X'(\pi) = 0$$

Case $\lambda < 0$

Solution is

$$X(x) = A\cosh\left(\sqrt{-\lambda}x\right) + B\sinh\left(\sqrt{-\lambda}x\right)$$

$$X'(x) = A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}x\right) + B\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}x\right)$$

At $x = 0$ the above gives

$$0 = B\sqrt{-\lambda}\cosh(0)$$

$$= B\sqrt{-\lambda}$$

Hence $B = 0$ and the solution (3) reduces to

$$X(x) = A\cosh\left(\sqrt{-\lambda}x\right)$$

$$X'(x) = A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}x\right)$$

At $x = \pi$ the above becomes

$$0 = A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\pi\right)$$

For non-trivial solution we want $\sinh\left(\sqrt{-\lambda}\pi\right) = 0$, but $\sinh$ is only zero when its argument is zero, which is not possible here, since $\lambda \neq 0$. Therefore $\lambda < 0$ is not possible.

Case $\lambda = 0$

Solution becomes $X = Ax + B$. Hence $X' = A$. At $x = 0$ this leads to $A = 0$. Therefore the solution now becomes $X = B$. Hence $X' = 0$. Therefore the second boundary conditions at $x = \pi$ is automatically satisfied. Hence the solution is $X(x) = B$, a constant. We pick $X(x) = 1$. Therefore $\lambda = 0$ is eigenvalue with associated eigenfunction $X_0(x) = 1$.

Case $\lambda > 0$

The solution becomes

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$
$$X'(x) = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)$$

At $x = 0$ the above becomes

$$0 = B\sqrt{\lambda}$$

Hence $B = 0$ and the solution reduces to

$$X(x) = A\cos\left(\sqrt{\lambda}x\right)$$
$$X'(x) = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)$$

At $x = \pi$ the above gives

$$0 = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right)$$
$$\sin\left(\sqrt{\lambda}\pi\right) = 0$$

Therefore $\sqrt{\lambda}\pi = n\pi$ for $n = 1, 2, 3, \cdots$. Hence

$$\lambda_n = n^2 \qquad n = 1, 2, 3, \cdots$$

And the solution (corresponding eigenfunctions) is

$$X_n(x) = \cos\left(\sqrt{\lambda_n}x\right)$$
$$= \cos(nx)$$

In summary, the solution to the X ODE resulted in

$$X_0(x) = 1 \qquad n = 0 \tag{3}$$
$$X_n(x) = \cos(nx) \qquad n = 1, 2, 3, \cdots$$

Now we solve for the Y ODE

$$Y'' - \lambda Y = 0$$
$$Y(0) = 0$$

We are only given boundary conditions on bottom edge.

<u>case $\lambda = 0$</u>

$$Y = Ay + B$$

When $y = 0$ the above leads to $0 = B$. Hence the corresponding eigenfunction is $Y_0\left(y\right) = y$.

<u>case $\lambda > 0$</u>

The solution becomes

$$Y\left(y\right) = A\cosh\left(\sqrt{\lambda}y\right) + B\sinh\left(\sqrt{\lambda}y\right)$$

At $y=0$ the above gives

$$\begin{aligned}0 &= A\cosh(0)\\ &= A\end{aligned}$$

Hence the solution reduces to

$$Y(y) = B\sinh\left(\sqrt{\lambda}y\right)$$

Therefore the eigenfunctions for $n=1,2,3,\cdots$ are $Y_n(y)=\sinh(ny)$ since $\lambda_n=n^2$ for $n=1,2,3,\cdots$.

In summary, the solution to the Y ODE resulted in

$$\begin{aligned}Y_0(y) &= y \qquad n=0\\ Y_n(x) &= \sinh(ny) \qquad n=1,2,3,\cdots\end{aligned} \tag{4}$$

From (3,4) we see that

$$u_n(x,y) = X_n(x)\,Y_n(y)$$

For $n=0$ the above becomes

$$\begin{aligned}u_0(x,y) &= (1)(y)\\ &= y\end{aligned}$$

And for $n=1,2,3,\cdots$

$$\begin{aligned}u_n(x,y) &= \sinh(ny)\\ &= \cos(nx)\sinh(ny)\end{aligned}$$

Using superposition, then

$$\begin{aligned}u(x,y) &= X(x)\,Y(y)\\ &= A_0u_0 + \sum_{n=1}^{\infty}A_nu_n\\ &= A_0y + \sum_{n=1}^{\infty}A_n\cos(nx)\sinh(ny)\end{aligned}$$

QED.

2 Section 37, Problem 1

1. In Problem 3, Sec. 34, the functions

$$u_0 = y, \qquad u_n = \sinh ny \cos nx \qquad (n = 1, 2, \ldots)$$

were shown to satisfy Laplace's equation

$$u_{xx}(x, y) + u_{yy}(x, y) = 0 \qquad (0 < x < \pi,\ 0 < y < 2)$$

and the homogeneous boundary conditions

$$u_x(0, y) = u_x(\pi, y) = 0, \qquad u(x, 0) = 0.$$

After writing $u = X(x)Y(y)$ and separating variables, use the solutions of the Sturm-Liouville problem (1) in Sec. 35 to show how the functions u_0 and u_n $(n = 1, 2, \ldots)$ can be discovered. Then, by proceeding formally, derive the following solution of the boundary value problem that results when the nonhomogeneous condition $u(x, 2) = f(x)$ is included:

$$u(x, y) = A_0\, y + \sum_{n=1}^{\infty} A_n \sinh ny \cos nx,$$

where

$$A_0 = \frac{1}{2\pi}\int_0^{\pi} f(x)\, dx, \qquad A_n = \frac{2}{\pi \sinh 2n}\int_0^{\pi} f(x) \cos nx\, dx \quad (n = 1, 2, \ldots).$$

Figure 3: Problem statement

Solution

The boundary conditions now become as follows

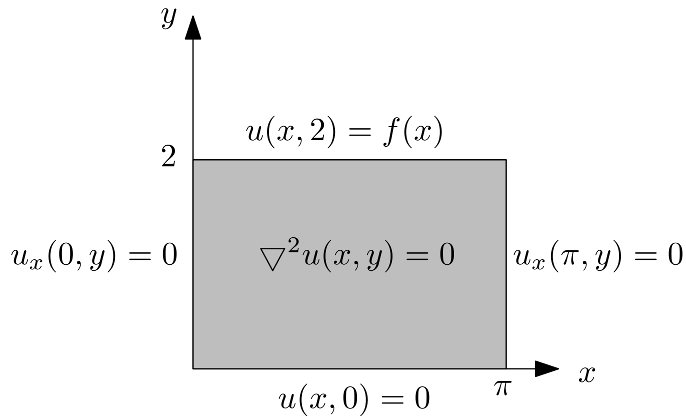


Figure 4: Boundary conditions

From the above problem we know the general solution is

$$u\left(x,y\right)=A_0 y+\sum_{n=1}^{\infty}A_n\cos\left(nx\right)\sinh\left(ny\right) \tag{1}$$

Now we impose the remaining boundary condition $u\left(x,2\right)=f\left(x\right)$. Therefore the above becomes

$$f\left(x\right)=2A_0+\sum_{n=1}^{\infty}A_n\cos\left(nx\right)\sinh\left(2n\right)$$

Multiplying both sides by $\cos\left(mx\right)$ integrating w.r.t. x from $x=0$ to $x=\pi$ results in

$$\int_0^\pi f\left(x\right)\cos\left(mx\right)dx=\int_0^\pi 2A_0\cos\left(mx\right)dx+\left[\int_0^\pi\sum_{n=1}^{\infty}A_n\cos\left(nx\right)\cos\left(mx\right)\sinh\left(2n\right)dx\right]$$

$$\int_0^\pi f\left(x\right)\cos\left(mx\right)dx=\int_0^\pi 2A_0\cos\left(mx\right)dx+\left[\sum_{n=1}^{\infty}A_n\sinh\left(2n\right)\left(\int_0^\pi\cos\left(nx\right)\cos\left(mx\right)dx\right)\right]$$

<u>case $m=0$</u>

$$\begin{aligned}\int_0^\pi f\left(x\right)dx&=\int_0^\pi 2A_0 dx\\&=2A_0\pi\\A_0&=\frac{1}{2\pi}\int_0^\pi f\left(x\right)dx\end{aligned} \tag{2}$$

<u>case $m=1,2,\cdots$</u>

$$\int_0^\pi f\left(x\right)\cos\left(mx\right)dx=\sum_{n=1}^{\infty}A_n\sinh\left(2n\right)\left(\int_0^\pi\cos\left(nx\right)\cos\left(mx\right)dx\right)$$

But $\int_0^\pi\cos\left(nx\right)\cos\left(mx\right)dx=0$ for all $m\neq n$ and $\frac{\pi}{2}$ when $m=n$. Hence the above simplifies to

$$\int_0^\pi f(x)\cos(mx)\,dx = \frac{\pi}{2}A_m \sinh(2m)$$

$$A_m = \frac{2}{\pi\sinh(2m)}\int_0^\pi f(x)\cos(mx)\,dx$$

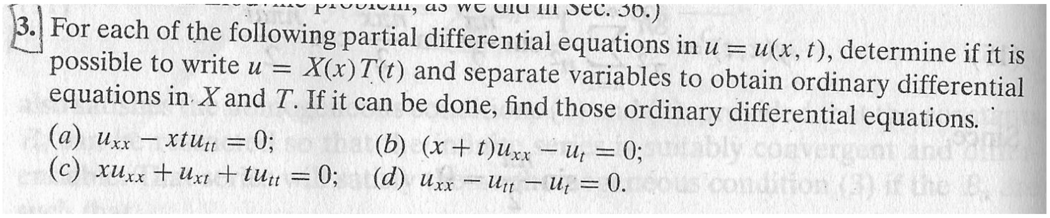
Since m is summation index, we can rename it to n and the above becomes

$$A_n = \frac{2}{\pi\sinh(2n)}\int_0^\pi f(x)\cos(nx)\,dx \tag{3}$$

Using (2,3) in (1) gives the final solution

$$u(x,y) = \left(\frac{1}{2\pi}\int_0^\pi f(x)\,dx\right)y + \sum_{n=1}^\infty \left(\frac{2}{\pi\sinh(2n)}\int_0^\pi f(x)\cos(nx)\,dx\right)\cos(nx)\sinh(ny)$$

3 Section 37, Problem 3



3. For each of the following partial differential equations in $u = u(x, t)$, determine if it is possible to write $u = X(x)T(t)$ and separate variables to obtain ordinary differential equations in X and T. If it can be done, find those ordinary differential equations.
(a) $u_{xx} - xu_{tt} = 0$; (b) $(x+t)u_{xx} - u_t = 0$;
(c) $xu_{xx} + u_{xt} + tu_{tt} = 0$; (d) $u_{xx} - u_{tt} - u_t = 0$.

Figure 5: Problem statement

3.1 Part (a)

$$u_{xx} - xtu_{tt} = 0$$

Let $u = X(x)T(t)$. Substituting this into the above PDE gives

$$X''T - xtT''X = 0$$

Dividing by $XT \neq 0$ gives

$$\frac{X''}{X} - xt\frac{T''}{T} = 0$$

Diving by x gives

$$\frac{1}{x}\frac{X''}{X} - t\frac{T''}{T} = 0$$

$$\frac{1}{x}\frac{X''}{X} = t\frac{T''}{T} = -\lambda$$

Hence it possible to separate them. The generated ODE's are

$$X'' + \lambda xX = 0$$

$$T'' + \lambda\frac{T}{t} = 0$$

3.2 Part (b)

$$(x+t)\,u_{xx} - u_t = 0$$

Let $u = X(x)T(t)$. Substituting this into the above PDE gives

$$(x+t)\,X''T - T'X = 0$$

Dividing by $XT \neq 0$ gives

$$x\frac{X''}{X} + t\frac{X''}{X} - \frac{T'}{T} = 0$$

It is not possible to separate them.

3.3 Part (c)

$$xu_{xx} + u_{xt} + tu_{tt} = 0$$

Let $u = X(x)T(t)$. Substituting this into the above PDE gives

$$\begin{aligned} xX''T - \frac{\partial}{\partial t}(X'T) + tT''X &= 0 \\ xX''T - X'T'X + tT''X &= 0 \end{aligned}$$

Dividing by $XT \neq 0$ gives

$$x\frac{X''}{X} - X'T' + t\frac{T''}{T} = 0$$

It is not possible to separate them.

3.4 Part (d)

$$u_{xx} - u_{tt} - u_t = 0$$

Let $u = X(x)T(t)$. Substituting this into the above PDE gives

$$X''T - T''X - T'X = 0$$

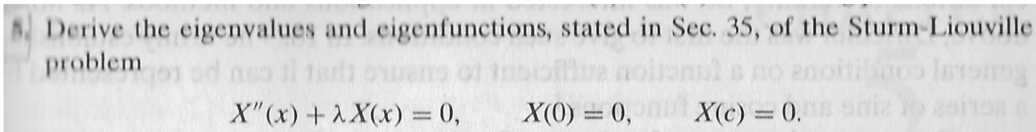
Dividing by $XT \neq 0$ gives

$$\begin{aligned} \frac{X''}{X} - \frac{T''}{T} - \frac{T'}{T} &= 0 \\ \frac{X''}{X} &= \frac{T''}{T} + \frac{T'}{T} = -\lambda \end{aligned}$$

It is possible to separate them. The ODE's are

$$\begin{aligned} X'' + \lambda X &= 0 \\ T'' + T' + \lambda T &= 0 \end{aligned}$$

4 Section 37, Problem 5



5. Derive the eigenvalues and eigenfunctions, stated in Sec. 35, of the Sturm-Liouville problem

$$X''(x) + \lambda X(x) = 0, \qquad X(0) = 0, \qquad X(c) = 0.$$

Figure 6: Problem statement

Case $\lambda < 0$

Solution is

$$X(x) = A\cosh\left(\sqrt{-\lambda}x\right) + B\sinh\left(\sqrt{-\lambda}x\right)$$

At $x = 0$ the above gives

$$0 = A$$

Hence the solution becomes

$$X(x) = B\sinh\left(\sqrt{-\lambda}x\right)$$

At $x = c$ the above becomes

$$0 = B\sinh\left(\sqrt{-\lambda}c\right)$$

For non-trivial solution we want $\sinh\left(\sqrt{-\lambda}c\right) = 0$. But $\sinh$ is zero only when its argument is zero. Which means $\sqrt{-\lambda}c = 0$ which is not possible. Hence $\lambda < 0$ is not possible.

Case $\lambda = 0$

Solution is

$$X(x) = Ax + B$$

At $x = 0$ the above gives

$$0 = B$$

Hence the solution becomes

$$X(x) = B$$

At $x = c$ the above becomes

$$0 = B$$

Which gives trivial solution. Hence $\lambda = 0$ is not possible.

Case $\lambda > 0$

Solution is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

At $x = 0$ the above gives

$$0 = A$$

Hence the solution becomes

$$X(x) = B\sin\left(\sqrt{\lambda}x\right)$$

At $x = c$ the above becomes

$$0 = B\sin\left(\sqrt{\lambda}c\right)$$

For non trivial solution we want $\sin\left(\sqrt{\lambda}c\right) = 0$ which implies

$$\sqrt{\lambda}c = n\pi \qquad n = 1, 2, 3, \cdots$$
$$\lambda_n = \left(\frac{n\pi}{c}\right)^2$$

Therefore the eigenvalues are $\lambda_n = \left(\frac{n\pi}{c}\right)^2$ for $n = 1, 2, 3, \cdots$ and the eigenfunctions are $X_n(x) = \sin\left(\frac{n\pi}{c}x\right)$ for $n = 1, 2, 3, \cdots$.

5 Section 39, Problem 2

center plane $x = \pi/2$.

2. Suppose that $f(x) = \sin x$ in Example 1, Sec. 39. Find $u(x, t)$ and verify the result fully.

Suggestion: Use the integration formula obtained in Problem 9, Sec. 5.

Answer: $u(x, t) = e^{-kt} \sin x$.

... found in Examples 1 and 2 in Sec. 39.

Figure 7: Problem statement

Solution

Example 1 is: Solve $u_t = ku_{xx}$ with $u(0,t) = 0$ and $u(\pi, t) = 0$. We now use initial conditions $u(x, 0) = \sin(x)$. The eigenvalues are $\lambda_n = n^2$ for $n = 1, 2, 3, \cdots$ and eigenfunctions are $\sin(nx)$. The general solution for this example is given in the book as

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-kn^2 t} \sin(nx)$$

At $t = 0$ the above becomes

$$\sin x = \sum_{n=1}^{\infty} B_n \sin(nx) \tag{1}$$

By comparing sides, we see that only $n = 1$ term exist. Hence $B_1 = 1$ and all other terms are zero. Hence the solution is, for $n = 1$

$$u(x,t) = e^{-kt} \sin(x)$$

To verify this, we start with (1) and multiply both sides by $\sin(mx)$ and integrate which gives

$$\begin{aligned}\int_0^{\pi} \sin x \sin(mx)\, dx &= \int_0^{\pi} \sum_{n=1}^{\infty} B_n \sin(nx) \sin(mx)\, dx \\ &= \sum_{n=1}^{\infty} B_n \left(\int_0^{\pi} \sin(nx) \sin(mx)\, dx \right)\end{aligned}$$

But $\int_0^{\pi} \sin(nx) \sin(mx)\, dx = 0$ for $m \neq n$ and $\frac{\pi}{2}$ for $n = m$. Hence the above gives

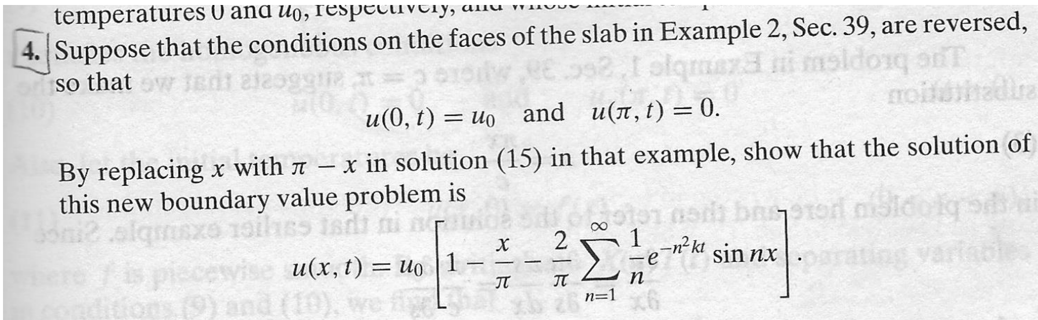
$$\int_0^{\pi} \sin x \sin(mx)\, dx = B_m \frac{\pi}{2}$$

Similarly, $\int_0^{\pi} \sin x \sin(mx)\, dx = 0$ for $m \neq 1$ and $\frac{\pi}{2}$ when $m = 1$, therefore the above becomes

$$\begin{aligned}\frac{\pi}{2} &= B_1 \frac{\pi}{2} \\ B_1 &= 1\end{aligned}$$

And all other $B_n = 0$. Which gives the same result obtain above, which is $u(x,t) = e^{-kt} \sin(x)$

6 Section 39, Problem 4


temperatures 0 and u_0, respectively, and whose initial ...

4. Suppose that the conditions on the faces of the slab in Example 2, Sec. 39, are reversed, so that

$$u(0,t)=u_0 \quad \text{and} \quad u(\pi,t)=0.$$

By replacing x with $\pi - x$ in solution (15) in that example, show that the solution of this new boundary value problem is

$$u(x,t)=u_0\left[1-\frac{x}{\pi}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-n^2kt}\sin nx\right].$$

Figure 8: Problem statement

<u>Solution</u>

We need to solve

$$u_t = ku_{xx} \qquad t>0, 0<x<\pi$$

With boundary conditions

$$u(0,t)=u_0$$
$$u(\pi,t)=0$$

And initial conditions

$$u(x,0)=0$$

Solution (15) is

$$u(x,t)=\frac{u_0}{\pi}\left[x+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2kt}\sin(nx)\right] \tag{15}$$

Replacing x by $\pi - x$ in (15) gives

$$\begin{aligned} u(x,t) &= \frac{u_0}{\pi}\left[(\pi-x)+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2kt}\sin(n(\pi-x))\right] \\ &= \frac{u_0}{\pi}(\pi-x)+2\frac{u_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2kt}\sin(n\pi-nx) \end{aligned} \tag{2}$$

Using $\sin(A-B)=\sin A\cos B+\cos A\sin B$, then

$$\sin(n\pi-nx)=\sin(n\pi)\cos(nx)+\cos(n\pi)\sin(nx)$$

But $\sin(n\pi)=0$ since n is integer and $\cos(n\pi)=(-1)^n$, then $\sin(n\pi-nx)=(-1)^n\sin(nx)$.

Substituting this in (2) gives

$$\begin{aligned} u(x,t) &= u_0 - u_0\frac{x}{\pi} + 2\frac{u_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2kt}(-1)^n\sin(nx) \\ &= u_0\left[1 - \frac{x}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n}e^{-n^2kt}\sin(nx)\right] \\ &= u_0\left[1 - \frac{x}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-n^2kt}\sin(nx)\right] \end{aligned}$$

Which is the result required.