HW 2
MATH 4567 Applied Fourier Analysis
Spring 2019
University of Minnesota, Twin Cities

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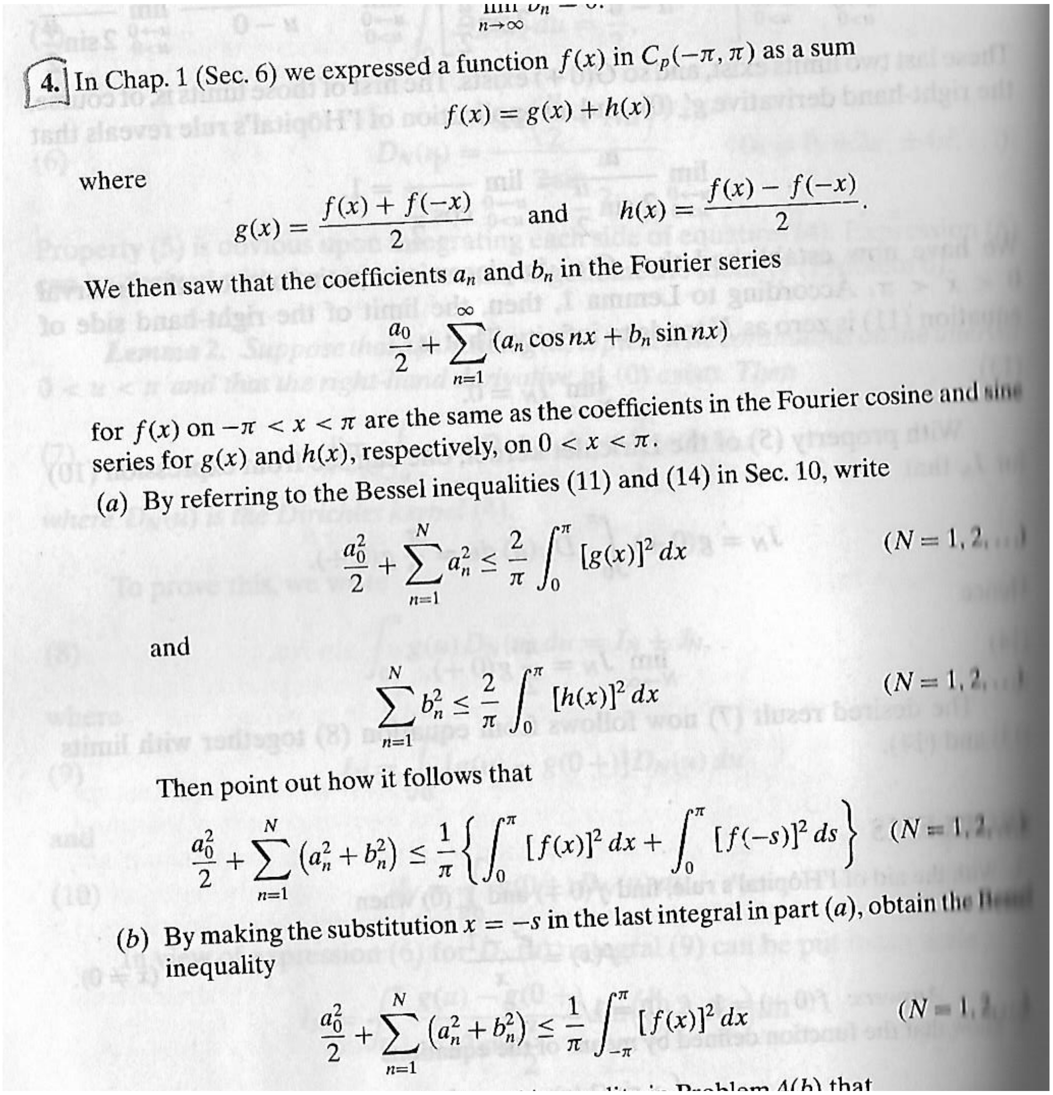
Contents

1 Section 11, Problem 4



$\lim_{n\to\infty} b_n = 0.$

4. In Chap. 1 (Sec. 6) we expressed a function $f(x)$ in $C_p(-\pi,\pi)$ as a sum

$$f(x) = g(x) + h(x)$$

where

$$g(x) = \frac{f(x)+f(-x)}{2} \quad \text{and} \quad h(x) = \frac{f(x)-f(-x)}{2}.$$

We then saw that the coefficients a_n and b_n in the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

for $f(x)$ on $-\pi < x < \pi$ are the same as the coefficients in the Fourier cosine and sine series for $g(x)$ and $h(x)$, respectively, on $0 < x < \pi$.

(*a*) By referring to the Bessel inequalities (11) and (14) in Sec. 10, write

$$\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 \le \frac{2}{\pi}\int_0^{\pi}[g(x)]^2\,dx \qquad (N = 1, 2, \ldots)$$

and

$$\sum_{n=1}^{N} b_n^2 \le \frac{2}{\pi}\int_0^{\pi}[h(x)]^2\,dx \qquad (N = 1, 2, \ldots)$$

Then point out how it follows that

$$\frac{a_0^2}{2} + \sum_{n=1}^{N}\left(a_n^2 + b_n^2\right) \le \frac{1}{\pi}\left\{\int_0^{\pi}[f(x)]^2\,dx + \int_0^{\pi}[f(-s)]^2\,ds\right\} \qquad (N = 1, 2, \ldots)$$

(*b*) By making the substitution $x = -s$ in the last integral in part (*a*), obtain the Bessel inequality

$$\frac{a_0^2}{2} + \sum_{n=1}^{N}\left(a_n^2 + b_n^2\right) \le \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx \qquad (N = 1, 2, \ldots)$$

Figure 1: Problem statement

1.1 Part (a)

Writing

$$\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 \le \frac{2}{\pi}\int_0^{\pi}\left[g(x)\right]^2 dx \tag{1}$$

$$\sum_{n=1}^{N} b_n^2 \le \frac{2}{\pi}\int_0^{\pi}\left[h(x)\right]^2 dx \tag{2}$$

Adding (1)+(2) gives

$$\begin{aligned}
\frac{a_0^2}{2}+\sum_{n=1}^{N}\left(a_n^2+b_n^2\right) &\leq \frac{2}{\pi}\int_0^{\pi}\left[g(x)\right]^2+\left[h(x)\right]^2 dx\\
&=\frac{2}{\pi}\int_0^{\pi}\left[\frac{f(x)+f(-x)}{2}\right]^2+\left[\frac{f(x)-f(-x)}{2}\right]^2 dx\\
&=\frac{2}{\pi}\int_0^{\pi}\frac{f^2(x)+f^2(-x)+2f(x)f(-x)}{4}+\frac{f^2(x)+f^2(-x)-2f(x)f(-x)}{4}dx\\
&=\frac{1}{2\pi}\int_0^{\pi}f^2(x)+f^2(-x)+f^2(x)+f^2(-x)\,dx\\
&=\frac{1}{2\pi}\int_0^{\pi}2f^2(x)+2f^2(-x)\,dx\\
&=\frac{1}{\pi}\left(\int_0^{\pi}f^2(x)+f^2(-x)\,dx\right)\\
&=\frac{1}{\pi}\left(\int_0^{\pi}\left[f(x)\right]^2dx+\int_0^{\pi}\left[f(-s)\right]^2ds\right) \qquad (3)
\end{aligned}$$

1.2 Part (b)

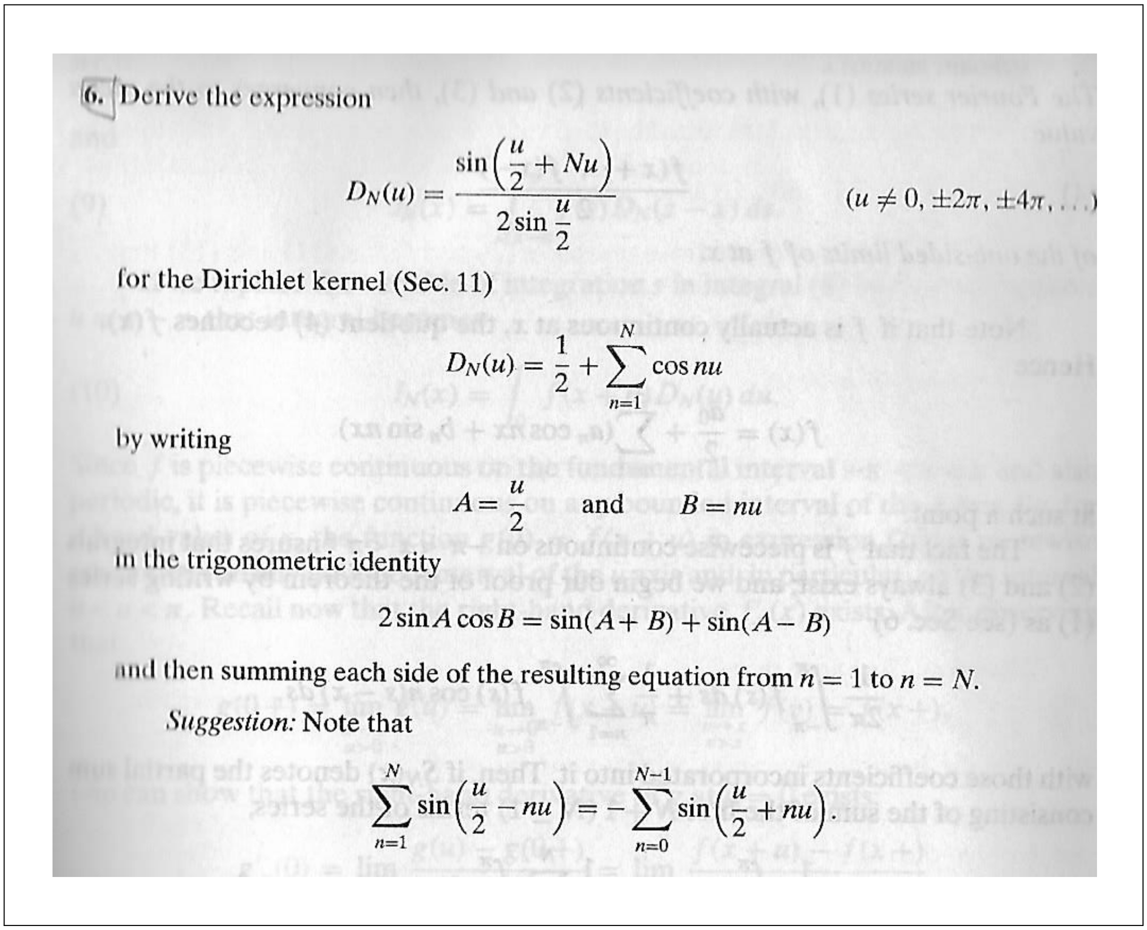
Let $x=-s$ in the last integral. Therefore $dx=-ds$. When $s=0$ then $x=0$ and when $s=\pi$ then $x=-\pi$, then (3) becomes

$$\begin{aligned}
\frac{a_0^2}{2}+\sum_{n=1}^{N}\left(a_n^2+b_n^2\right) &\leq \frac{1}{\pi}\left(\int_0^{\pi}\left[f(x)\right]^2dx+\int_0^{-\pi}\left[f(x)\right]^2(-dx)\right)\\
&=\frac{1}{\pi}\left(\int_0^{\pi}\left[f(x)\right]^2dx-\int_0^{-\pi}\left[f(x)\right]^2dx\right)
\end{aligned}$$

But $\int_0^{-\pi}=-\int_{-\pi}^{0}$ and the above becomes

$$\begin{aligned}
\frac{a_0^2}{2}+\sum_{n=1}^{N}\left(a_n^2+b_n^2\right) &\leq \frac{1}{\pi}\left(\int_0^{\pi}\left[f(x)\right]^2dx+\int_{-\pi}^{0}\left[f(x)\right]^2dx\right)\\
&=\frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(x)\right]^2dx
\end{aligned}$$

2 Section 11, Problem 6


6. Derive the expression

$$D_N(u) = \frac{\sin\left(\frac{u}{2} + Nu\right)}{2\sin\frac{u}{2}} \qquad (u \neq 0, \pm 2\pi, \pm 4\pi, \dots)$$

for the Dirichlet kernel (Sec. 11)

$$D_N(u) = \frac{1}{2} + \sum_{n=1}^{N} \cos nu$$

by writing

$$A = \frac{u}{2} \quad \text{and} \quad B = nu$$

in the trigonometric identity

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

and then summing each side of the resulting equation from $n = 1$ to $n = N$.

Suggestion: Note that

$$\sum_{n=1}^{N} \sin\left(\frac{u}{2} - nu\right) = -\sum_{n=0}^{N-1} \sin\left(\frac{u}{2} + nu\right).$$

Figure 2: Problem statement

We want to show the following (I've used x instead of u as it is more natural).

$$\frac{1}{2} + \sum_{n=1}^{N} \cos nx = \frac{\sin\left(\left(N + \frac{1}{2}\right)x\right)}{2\sin\frac{x}{2}} \tag{1}$$

Or, similarly, we want to show the following

$$\sin\frac{x}{2} + \sum_{n=1}^{N} 2\sin\frac{x}{2}\cos nx = \sin\left(\left(N + \frac{1}{2}\right)x\right) \tag{2}$$

We will now work on the left side of (2) only and see if we can simplify it to obtain the right side of (2). Writing the LHS of (2) as

$$\sin\frac{x}{2} + \sum_{n=1}^{N} 2\sin\frac{x}{2}\cos nx = \sin\frac{x}{2} + \sum_{n=1}^{N} 2\sin A\cos B \tag{3}$$

Where $A=\frac{x}{2}, B=nx$. But $\sin A\cos B=\frac{1}{2}\left(\sin\left(A+B\right)+\sin\left(A-B\right)\right)$. Hence (3) becomes

$$\begin{aligned}
\sin\frac{x}{2}+\sum_{n=1}^{N}2\sin\frac{x}{2}\cos nx &= \sin\frac{x}{2}+\sum_{n=1}^{N}\sin\left(A+B\right)+\sin\left(A-B\right)\\
&= \sin\frac{x}{2}+\sum_{n=1}^{N}\sin\left(\frac{x}{2}+nx\right)+\sin\left(\frac{x}{2}-nx\right)\\
&= \sin\frac{x}{2}+\sum_{n=1}^{N}\sin\left(\left(n+\frac{1}{2}\right)x\right)+\sin\left(\left(\frac{1}{2}-n\right)x\right)\\
&= \sin\frac{x}{2}+\sum_{n=1}^{N}\sin\left(\left(n+\frac{1}{2}\right)x\right)-\sin\left(\left(n-\frac{1}{2}\right)x\right)
\end{aligned}$$

Expanding few terms to see the pattern shows

$$\begin{aligned}
\sin\frac{x}{2}+\sum_{n=1}^{N}\sin\left(\left(n+\frac{1}{2}\right)x\right)-\sin\left(\left(n-\frac{1}{2}\right)x\right) &= \sin\frac{x}{2}+\left[\sin\left(\left(1+\frac{1}{2}\right)x\right)-\sin\left(\left(1-\frac{1}{2}\right)x\right)\right]\\
&\quad+\left[\sin\left(\left(2+\frac{1}{2}\right)x\right)-\sin\left(\left(2-\frac{1}{2}\right)x\right)\right]\\
&\quad+\left[\sin\left(\left(3+\frac{1}{2}\right)x\right)-\sin\left(\left(3-\frac{1}{2}\right)x\right)\right]+\cdots
\end{aligned}$$

Or

$$\begin{aligned}
\sum_{n=1}^{N}\sin\left(\left(n+\frac{1}{2}\right)x\right)-\sin\left(\left(n-\frac{1}{2}\right)x\right) &= \sin\frac{x}{2}+\left[\sin\left(\frac{3}{2}x\right)-\sin\left(\frac{1}{2}x\right)\right]\\
&\quad+\left[\sin\left(\frac{5}{2}x\right)-\sin\left(\frac{3}{2}x\right)\right]\\
&\quad+\left[\sin\left(\frac{7}{2}x\right)-\sin\left(\frac{5}{2}x\right)\right]+\cdots
\end{aligned}$$

We see that all terms cancel except for the term before the last term, which is $\sin\left(\left(N+\frac{1}{2}\right)x\right)$. (In the above limited expansion of terms, this will be the term $\sin\left(\frac{7}{2}x\right)$ which remains.) Hence as $n\to N$, the above simplifies to

$$\sin\frac{x}{2}+\sum_{n=1}^{N}\sin\left(\frac{x}{2}+nx\right)+\sin\left(\frac{x}{2}-nx\right)=\sin\left(\left(N+\frac{1}{2}\right)x\right)$$

Which is (2) which was obtained from (1). Hence (1) was verified to be valid.

3 Section 14, Problem 2

2. For each of the following functions, point out why its Fourier series on the interval $-\pi < x < \pi$ is convergent when $-\pi \le x \le \pi$, and state the sum of the series when $x = \pi$:

(a) the function

$$f(x) = \begin{cases} -\pi/2 & \text{when } -\pi < x < 0, \\ \pi/2 & \text{when } 0 < x < \pi, \end{cases}$$

whose series was found in Problem 1, Sec. 7;

(b) the function

$$f(x) = e^{ax} \quad (a \neq 0),$$

whose series was found in Problem 4, Sec. 7.

Answers: (a) sum $= 0$; (b) sum $= \cosh a\pi$.

Figure 3: Problem statement

3.1 Part (a)

The Fourier series for $f(x)$ is convergent since $f(x)$, after periodic extension, satisfies the 3 points of the Fourier theorem in the textbook at page 35

Theorem. *Suppose that*

(i) *f is piecewise continuous on the interval $-\pi < x < \pi$;*

(ii) *f is periodic, with period 2π, on the entire x axis;*

(iii) *$x(-\infty < x < \infty)$ is a point at which the one-sided derivatives $f'_+(x)$ and $f'_-(x)$ both exist.*

Figure 4: Fourier theorem

Point (i) is satisfied since $f(x)$ is piecewise continuous and also point (ii) when doing periodic extension. Also point (iii) is satisfied, since the left sided and right sides limit exist at each x.

```
In[ ]:= Clear[f];
        f[x_ /; -Pi < x < Pi] := Piecewise[{{-Pi / 2, -Pi < x < 0}, {Pi / 2, 0 < x < Pi}}]
        f[x_ /; x > Pi] := f[x - 2 Pi];
        f[x_ /; x < -Pi] := f[x + 2 Pi];
        Plot[f[x], {x, -4 Pi, 4 Pi}, PlotStyle → {Thick, Red}, Mesh → None,
         GridLines → Automatic, GridLinesStyle → LightGray,
         Ticks → {Range[-4 Pi, 4 Pi, Pi], Automatic},
         Exclusions → {x == -3 Pi, x == -Pi, x == 0, x == Pi, x == 3 Pi},
         Mesh → None, ExclusionsStyle → Dashed]
```

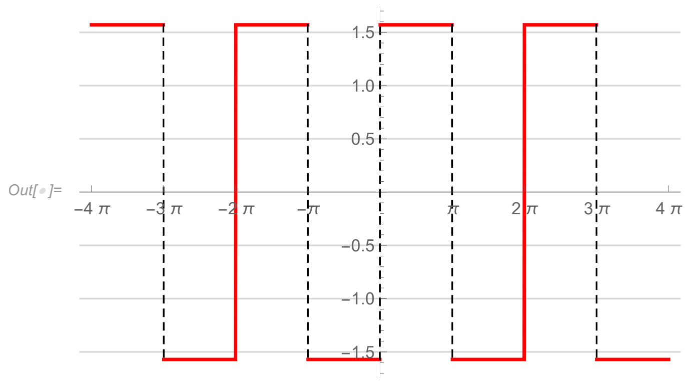


Figure 5: $f(x)$ after periodic extension

Therefore the Fourier series will converge to the average of the function $f(x)$ at $x = \pi$. This average is

$$\frac{f(\pi^-) + f(\pi^+)}{2} = \frac{\frac{\pi}{2} - \frac{\pi}{2}}{2} = 0$$

3.2 Part (b)

The Fourier series for $f(x) = e^{ax}$ is convergent since $f(x)$, after periodic extension, satisfies the 3 points of the Fourier theorem in the textbook at page 35. Point (i) is satisfied is piecewise continuous and also point (ii) when doing periodic extension. Also point (iii) is satisfied, since the left sided and right sides limit exist at each x. Here is a plot, using $a = \frac{1}{4}$ for illustration

In[∘]:=
```
Clear[f];
a = 1 / 4;
f[x_ /; -Pi < x < Pi] := Exp[a x];
f[x_ /; x > Pi] := f[x - 2 Pi];
f[x_ /; x < -Pi] := f[x + 2 Pi];
Plot[f[x], {x, -4 Pi, 4 Pi}, PlotStyle → {Thick, Red}, Mesh → None,
 GridLines → Automatic, GridLinesStyle → LightGray,
 Ticks → {Range[-4 Pi, 4 Pi, Pi], Automatic},
 Exclusions → {x == -3 Pi, x == -Pi, x == 0, x == Pi, x == 3 Pi},
 Mesh → None, ExclusionsStyle → Dashed,
 AxesOrigin → {0, 0}]
```

Out[∘]=

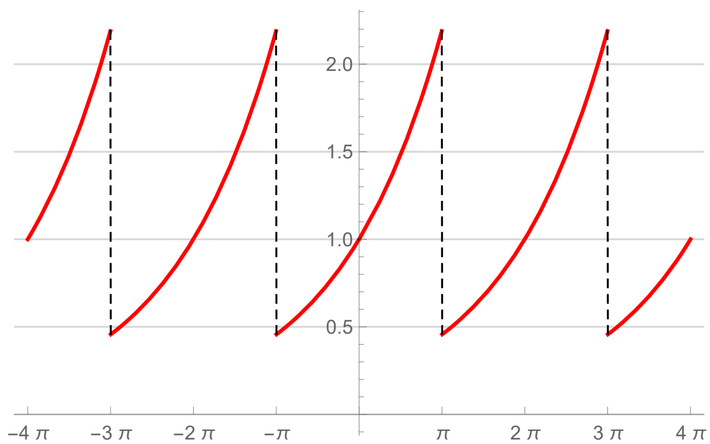
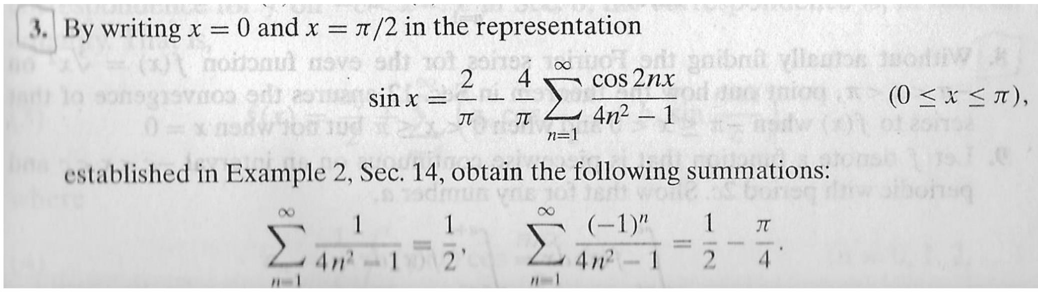


Figure 6: $f(x) = e^{ax}$ after periodic extension (Using $a = \frac{1}{4}$

Therefore the Fourier series will converge to the average of the function $f(x)$ at $x = \pi$. This average is

$$\frac{f(\pi^-) + f(\pi^+)}{2} = \frac{e^{a\pi} + e^{-a\pi}}{2} = \cosh(a\pi)$$

4 Section 14, Problem 3



3. By writing $x = 0$ and $x = \pi/2$ in the representation

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1} \qquad (0 \le x \le \pi),$$

established in Example 2, Sec. 14, obtain the following summations:

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = \frac{1}{2} - \frac{\pi}{4}.$$

Figure 7: Problem statement

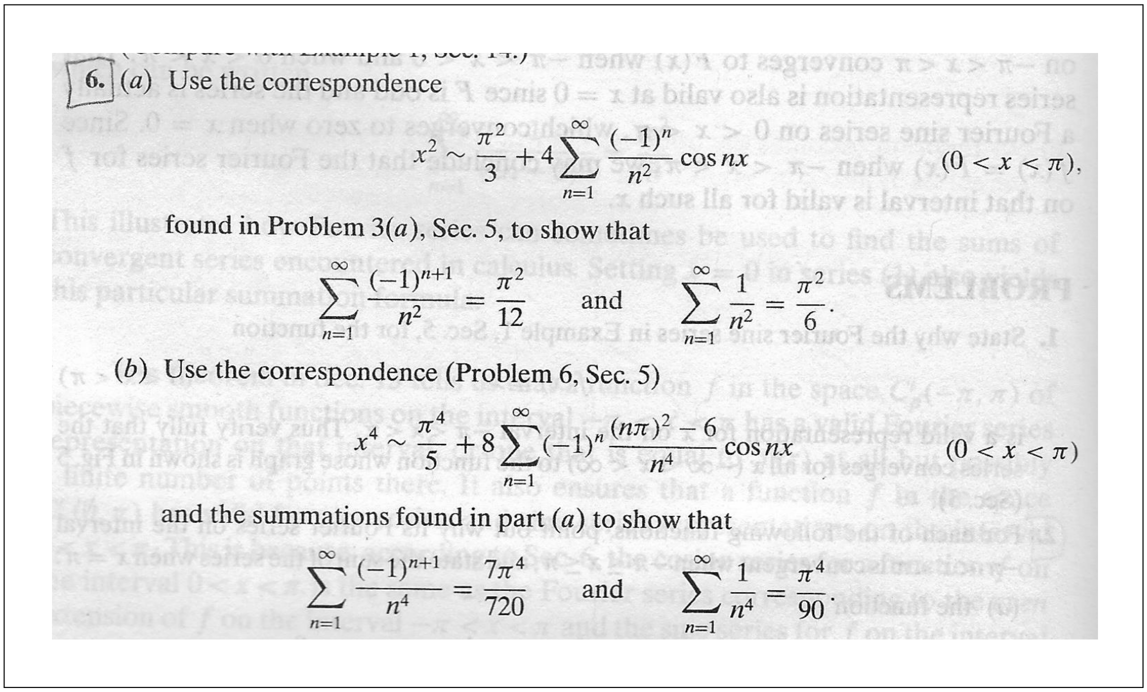
Substituting $x = 0$ in the given representation gives

$$\begin{aligned} 0 &= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \\ -2 &= -4\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \\ \frac{1}{2} &= \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \end{aligned}$$

And substituting $x = \frac{\pi}{2}$ in the given representation gives

$$\begin{aligned} 1 &= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{4n^2 - 1} \\ 1 - \frac{2}{\pi} &= -\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} \\ \pi - 2 &= -4\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} \\ \frac{1}{2} - \frac{\pi}{4} &= \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} \end{aligned}$$

5 Section 14, Problem 6



6. (*a*) Use the correspondence

$$x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \qquad (0 < x < \pi),$$

found in Problem 3(*a*), Sec. 5, to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(*b*) Use the correspondence (Problem 6, Sec. 5)

$$x^4 \sim \frac{\pi^4}{5} + 8\sum_{n=1}^{\infty} (-1)^n \frac{(n\pi)^2 - 6}{n^4} \cos nx \qquad (0 < x < \pi)$$

and the summations found in part (*a*) to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Figure 8: Problem statement

5.1 Part (a)

$$x^2 \sim \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \tag{1}$$

Letting $x = 0$ in (1) gives (After doing periodic extension, then $x = 0$ is now in the domain).

$$\begin{aligned} 0 &= \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \\ -\frac{1}{3}\pi^2 &= 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \\ -\frac{\pi^2}{12} &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \end{aligned}$$

Multiplying both sides by -1 gives the result needed

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

Now we need to obtain the second result $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Let $x = \pi$ in (1) (After doing periodic extension, then $x = \pi$ is now in the domain) gives

$$\pi^2 = \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n$$
$$\pi^2 - \frac{1}{3}\pi^2 = 4\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$
$$\frac{1}{6}\pi^2 = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

But $\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ since the power $2n$ is always even. This gives the result needed

$$\frac{1}{6}\pi^2 = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

5.2 Part (b)

$$x^4 \sim \frac{\pi^4}{5} + 8\sum_{n=1}^{\infty} (-1)^n \frac{(n\pi)^2 - 6}{n^4} \cos nx \tag{2}$$

Letting $x = 0$ in (2) gives

$$0 = \frac{\pi^4}{5} + 8\sum_{n=1}^{\infty} (-1)^n \frac{(n\pi)^2 - 6}{n^4}$$
$$-\frac{\pi^4}{5} = 8\left(\sum_{n=1}^{\infty} (-1)^n \frac{(n\pi)^2}{n^4} - 6\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}\right)$$
$$\frac{\pi^4}{5} = 8\left(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n\pi)^2}{n^4} + 6\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}\right)$$
$$\frac{\pi^4}{5} = 8\left(\pi^2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} + 6\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}\right)$$

But from part (a), we found that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$. Using this in the above results in

$$
\begin{aligned}
\frac{\pi^4}{5} &= 8\left(\pi^2\left(\frac{\pi^2}{12}\right)+6\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\right)\\
\frac{\pi^4}{5} &= \frac{8}{12}\pi^4+48\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\\
\frac{\pi^4}{5}-\frac{8\pi^4}{12} &= 48\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\\
-\frac{7}{15}\pi^4 &= 48\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\\
-\frac{7}{720}\pi^4 &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}
\end{aligned}
$$

Multiplying both sides by -1 gives the result needed

$$
\frac{7}{720}\pi^4=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}
$$

Now we need to obtain the second result $\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^2}{90}$. Let $x=\pi$ in (2) gives

$$
\begin{aligned}
\pi^4 &= \frac{\pi^4}{5}+8\sum_{n=1}^{\infty}(-1)^n\,\frac{(n\pi)^2-6}{n^4}\,(-1)^n\\
&= \frac{\pi^4}{5}+8\sum_{n=1}^{\infty}(-1)^{2n}\,\frac{(n\pi)^2-6}{n^4}
\end{aligned}
$$

But $(-1)^{2n}=1$ for all n. The above simplifies to

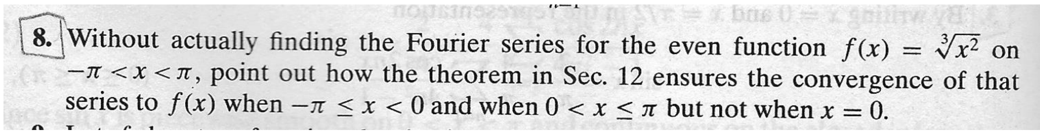
$$
\begin{aligned}
\pi^4 &= \frac{\pi^4}{5}+8\sum_{n=1}^{\infty}\frac{(n\pi)^2-6}{n^4}\\
\pi^4-\frac{\pi^4}{5} &= 8\left(\sum_{n=1}^{\infty}\frac{(n\pi)^2}{n^4}-6\sum_{n=1}^{\infty}\frac{1}{n^4}\right)\\
\frac{4\pi^4}{5} &= 8\left(\pi^2\sum_{n=1}^{\infty}\frac{1}{n^2}-6\sum_{n=1}^{\infty}\frac{1}{n^4}\right)
\end{aligned}
$$

But from part(a) we found that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ hence the above simplifies to

$$\begin{aligned}\frac{4\pi^4}{5} &= 8\left(\pi^2\left(\frac{\pi^2}{6}\right)-6\sum_{n=1}^{\infty}\frac{1}{n^4}\right)\\ \frac{4\pi^4}{40} &= \frac{\pi^4}{6}-6\sum_{n=1}^{\infty}\frac{1}{n^4}\\ -\frac{1}{15}\pi^4 &= -6\sum_{n=1}^{\infty}\frac{1}{n^4}\\ \frac{1}{90}\pi^4 &= \sum_{n=1}^{\infty}\frac{1}{n^4}\end{aligned}$$

Which is the result we are asked to show.

6 Section 14, Problem 8



8. Without actually finding the Fourier series for the even function $f(x) = \sqrt[3]{x^2}$ on $-\pi < x < \pi$, point out how the theorem in Sec. 12 ensures the convergence of that series to $f(x)$ when $-\pi \le x < 0$ and when $0 < x \le \pi$ but not when $x = 0$.

Figure 9: Problem statement

We first notice that the function $f(x)$ is not differentiable at $x = 0$

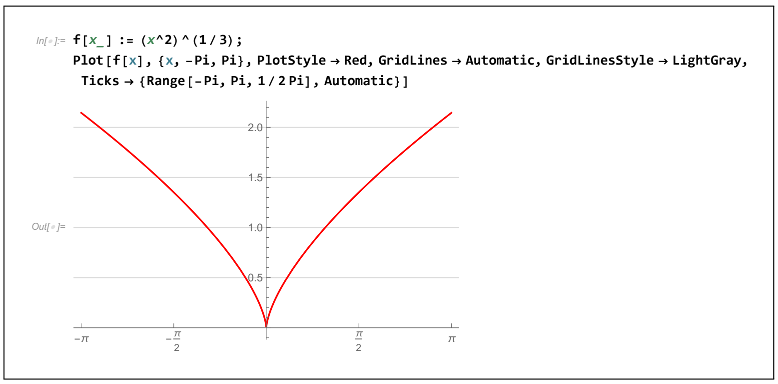


Figure 10: plot of $(x^2)^{1/3}$

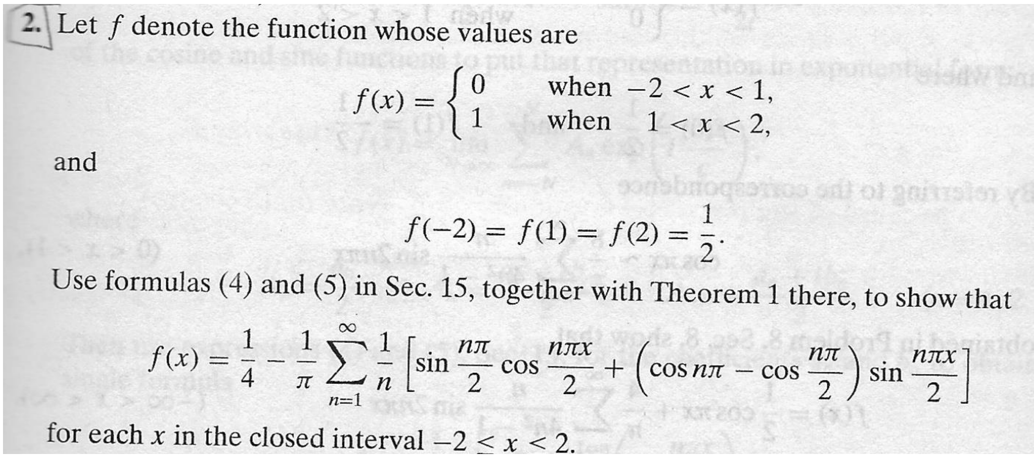
This is because, when $x_0 = 0$ the left sided derivative is equal to the right sided derivative

$$\lim_{\substack{x \to x_0 \\ x < x_0}} f(x) = f'_{-}(x_0) \neq \lim_{\substack{x \to x_0 \\ x > x_0}} f(x) = f'_{+}(x_0)$$

Since $f'_{-}(0) = -\infty$ while $f'_{+}(0) = +\infty$. The function is therefore piecewise continuous on each $-\pi < x < \pi$ but it is not differentiable at $x = 0$. But Fourier theorem, looking at point (iii) in the book, only says that if $f'_{-}(x_0)$ exist and if $f'_{+}(x_0)$ exist, then the Fourier series converges to the average of $f(x)$ at point x_0. In this example $f'_{-}(0) = -\infty$ and $f'_{+}(0) = +\infty$, which means these limits do not exist.

Hence we see that point (i) and (ii) in the Fourier theorem in the book are satisfied, but it is point (iii) which not satisfied at $x = 0$. Therefore Fourier series does not converge to $f(x)$ at $x = 0$ only while on other x in the domain it does.

7 Section 15, Problem 2



2. Let f denote the function whose values are

$$f(x) = \begin{cases} 0 & \text{when } -2 < x < 1, \\ 1 & \text{when } \ \ 1 < x < 2, \end{cases}$$

and

$$f(-2) = f(1) = f(2) = \frac{1}{2}.$$

Use formulas (4) and (5) in Sec. 15, together with Theorem 1 there, to show that

$$f(x) = \frac{1}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\sin\frac{n\pi}{2}\cos\frac{n\pi x}{2} + \left(\cos n\pi - \cos\frac{n\pi}{2}\right)\sin\frac{n\pi x}{2}\right]$$

for each x in the closed interval $-2 \leq x \leq 2$.

Figure 11: Problem statement

A plot of the function $f(x)$ and its periodic extension is given below

```
In[]:= f[x_] := Piecewise[{{0, -2 < x < 1}, {1, 1 < x < 2}}]
Plot[f[x], {x, -2, 2.01}, PlotStyle → Red, GridLines → Automatic, GridLinesStyle → LightGray,
 Ticks → {Range[-4, 4, 1], Automatic}, ExclusionsStyle → Dashed, Exclusions → {x == 1, x == 2, x == -2}]
Out[]=
```

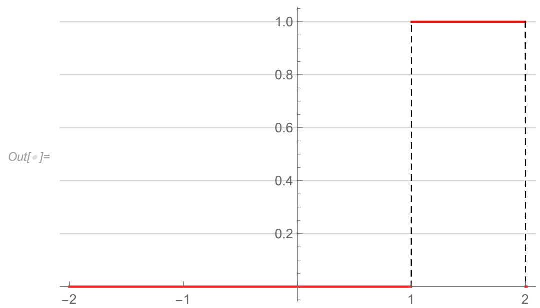


Figure 12: plot of $f(x)$ over one period

```
Clear[f];
f[x_ /; -2 < x < 2] := Piecewise[{{0, -2 < x < 1}, {1, 1 < x < 2}}]
f[x_ /; x > 2] := f[x - 4];
f[x_ /; x < -2] := f[x + 4];
Plot[f[x], {x, -8, 8}, PlotStyle → {Thick, Red}, Mesh → None,
 GridLines → Automatic, GridLinesStyle → LightGray,
 Ticks → {Range[-8, 8, 1], Automatic},
 Exclusions → {x == -7, x == -6, x == -3, x == -2, x == 1, x == 2, x == 5, x == 6},
 Mesh → None, ExclusionsStyle → Dashed]
```

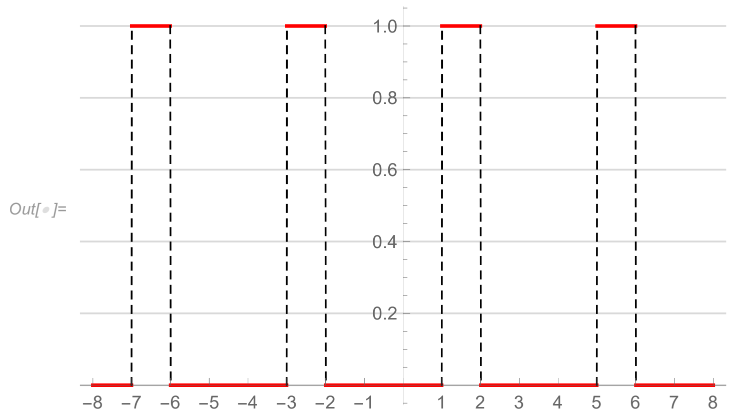


Figure 13: plot of $f(x)$ extended to become periodic. Showing 3 periods

The Fourier transform of $f(x)$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T} nx\right) + b_n \sin\left(\frac{2\pi}{T} nx\right)$$

Where T is the period of the function (after periodic extension) which is 4. Hence the above becomes

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{2} nx\right) + b_n \sin\left(\frac{\pi}{2} nx\right)$$

Since $f(x)$ meets the requirements of the Fourier theorem on page 35 of the text (at points of discontinues, the function is $\frac{1}{2}$ which is the average at those points), then $\sim$ can be replaced by $=$ above

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{2} nx\right) + b_n \sin\left(\frac{\pi}{2} nx\right) \tag{1}$$

Where

$$a_0 = \frac{1}{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx = \frac{1}{\frac{4}{2}} \int_{-\frac{4}{2}}^{\frac{4}{2}} f(x)\,dx = \frac{1}{2} \int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left(\int_{-2}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx\right)$$
$$= \frac{1}{2}\left(\int_{1}^{2} dx\right) = \frac{1}{2}(x)_1^2 = \frac{1}{2}$$

And

$$a_n = \frac{1}{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \cos\left(\frac{2\pi}{T} nx\right) dx = \frac{1}{2} \int_{-2}^{2} f(x) \cos\left(\frac{\pi}{2} nx\right) dx$$

$$= \frac{1}{2}\left(\int_{-2}^{1} f(x) \cos\left(\frac{\pi}{2} nx\right) dx + \int_{1}^{2} f(x) \cos\left(\frac{\pi}{2} nx\right) dx\right)$$

$$= \frac{1}{2} \int_{1}^{2} f(x) \cos\left(\frac{\pi}{2} nx\right) dx$$

$$= \frac{1}{2} \int_{1}^{2} \cos\left(\frac{\pi}{2} nx\right) dx$$

$$= \frac{1}{2}\left[\frac{\sin\left(\frac{\pi}{2} nx\right)}{\frac{\pi n}{2}}\right]_1^2$$

$$= \frac{1}{\pi n}\left(\sin(\pi n) - \sin\left(\frac{\pi n}{2}\right)\right)$$

$$= \frac{-1}{\pi n} \sin\left(\frac{\pi n}{2}\right)$$

And

$$b_n = \frac{1}{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \sin\left(\frac{2\pi}{T} nx\right) dx = \frac{1}{2} \int_{-2}^{2} f(x) \sin\left(\frac{\pi}{2} nx\right) dx$$

$$= \frac{1}{2}\left(\int_{-2}^{1} f(x) \sin\left(\frac{\pi}{2} nx\right) dx + \int_{1}^{2} f(x) \sin\left(\frac{\pi}{2} nx\right) dx\right)$$

$$= \frac{1}{2} \int_{1}^{2} f(x) \sin\left(\frac{\pi}{2} nx\right) dx$$

$$= \frac{1}{2} \int_{1}^{2} \sin\left(\frac{\pi}{2} nx\right) dx$$

$$= \frac{-1}{2}\left[\frac{\cos\left(\frac{\pi}{2} nx\right)}{\frac{\pi n}{2}}\right]_1^2$$

$$= \frac{-1}{\pi n}\left[\cos(\pi n) - \cos\left(\frac{\pi}{2} n\right)\right]$$

$$= \frac{-1}{\pi n}\left[\cos(\pi n) - \cos\left(\frac{\pi n}{2}\right)\right]$$

Using these results in (1) gives

$$
\begin{aligned}
f(x) &= \frac{1}{4} + \sum_{n=1}^{\infty} \left(\frac{-1}{\pi n} \sin\left(\frac{\pi n}{2}\right) \right) \cos\left(\frac{\pi}{2} n x\right) + \left(\frac{-1}{\pi n} \left[\cos(\pi n) - \cos\left(\frac{\pi n}{2}\right) \right] \right) \sin\left(\frac{\pi}{2} n x\right) \\
&= \frac{1}{4} - \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sin\left(\frac{\pi n}{2}\right) \right) \cos\left(\frac{\pi}{2} n x\right) + \frac{1}{n} \left(\cos(\pi n) - \cos\left(\frac{\pi n}{2}\right) \right) \sin\left(\frac{\pi}{2} n x\right) \\
&= \frac{1}{4} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[\sin\left(\frac{\pi n}{2}\right) \cos\left(\frac{\pi}{2} n x\right) + \left(\cos(\pi n) - \cos\left(\frac{\pi n}{2}\right) \right) \sin\left(\frac{\pi}{2} n x\right) \right]
\end{aligned}
$$

Which is the result we are asked to show. To verify this, the following shows the convergence to $f(x)$ when using more and more terms in the series.

```
fApprox[x_, nTerms_] := 1/4 - 1/π Sum[1/n (Sin[π n/2] Cos[π n x/2] + (Cos[π n] - Cos[π n/2]) Sin[π n x/2]), {n, 1, nTerms}];
Clear[f];
f[x_ /; -2 < x < 2] := Piecewise[{{0, -2 < x < 1}, {1, 1 < x < 2}}]
f[x_ /; x > 2] := f[x - 4];
f[x_ /; x < -2] := f[x + 4];
Grid[Partition[Table[Plot[{f[x], fApprox[x, n]}, {x, -Pi, Pi},
    PlotStyle → {Blue, Red},
    PlotLabel → Style[Row[{"Using ", n, " terms"}], Bold],
    ImageSize → 250],
   {n, 1, 10}], 2], Frame → All, FrameStyle → Gray]
```

Figure 14: Code used to draw the plot

Out[∘]=

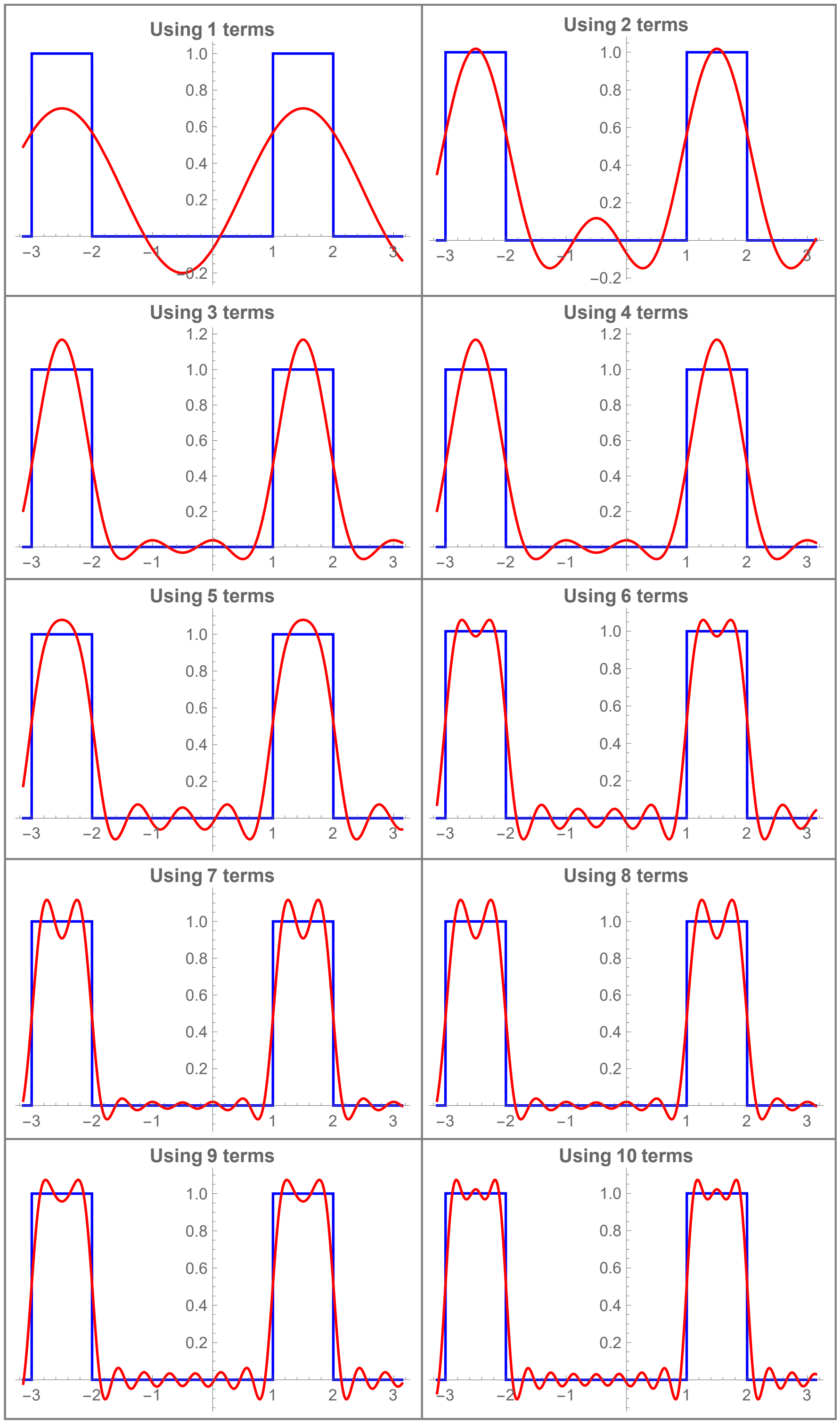


Figure 15: Fourier series approximation as more terms added

We notice that the Fourier series approximation converges to $\frac{1}{2}$ at the points of discontinuities.

But these are the actual values of $f(x)$ at those points.

8 Section 15, Problem 8

8. After writing the Fourier series representation (3), Sec. 15, as

$$f(x) = \frac{a_0}{2} + \lim_{N\to\infty} \sum_{n=1}^{N} \left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c} \right),$$

use the exponential forms†

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

of the cosine and sine functions to put that representation in exponential form:

$$f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} A_n \exp\left(i\frac{n\pi x}{c} \right),$$

where

$$A_0 = \frac{a_0}{2}, \qquad A_n = \frac{a_n - ib_n}{2}, \qquad A_{-n} = \frac{a_n + ib_n}{2} \qquad (n = 1, 2, \ldots).$$

Then use expressions (4) and (5), Sec. 15, for the coefficients a_n and b_n to obtain the single formula

$$A_n = \frac{1}{2c}\int_{-c}^{c} f(x)\exp\left(-i\frac{n\pi x}{c} \right) dx \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Figure 16: Problem statement

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + \lim_{N\to\infty} \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi}{c}x\right) + b_n \sin\left(\frac{n\pi}{c}x\right) \\
&= \frac{a_0}{2} + \lim_{N\to\infty} \sum_{n=1}^{N} a_n \left(\frac{e^{i\frac{n\pi}{c}x} + e^{-i\frac{n\pi}{c}x}}{2} \right) + b_n \left(\frac{e^{i\frac{n\pi}{c}x} - e^{-i\frac{n\pi}{c}x}}{2i} \right) \\
&= \frac{a_0}{2} + \lim_{N\to\infty} \sum_{n=1}^{N} a_n \left(\frac{e^{i\frac{n\pi}{c}x} + e^{-i\frac{n\pi}{c}x}}{2} \right) - ib_n \left(\frac{e^{i\frac{n\pi}{c}x} - e^{-i\frac{n\pi}{c}x}}{2} \right) \\
&= \frac{a_0}{2} + \lim_{N\to\infty} \sum_{n=1}^{N} e^{i\frac{n\pi}{c}x} \left(\frac{a_n - ib_n}{2} \right) + e^{-i\frac{n\pi}{c}x} \left(\frac{a_n + ib_n}{2} \right) \\
&= \frac{a_0}{2} + \lim_{N\to\infty} \sum_{n=1}^{N} e^{i\frac{n\pi}{c}x} \left(\frac{a_n - ib_n}{2} \right) + \sum_{n=-N}^{-1} e^{i\frac{n\pi}{c}x} \left(\frac{a_n + ib_n}{2} \right) \qquad (1)
\end{aligned}$$

Let

$$A_n = \begin{cases} \left(\frac{a_n - ib_n}{2}\right) & n > 0 \\ \frac{a_0}{2} & n = 0 \\ \left(\frac{a_n + ib_n}{2}\right) & n < 0 \end{cases}$$

Then (1) can be written as

$$f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} A_n e^{i\frac{n\pi}{c}x}$$

Since

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\left(\frac{n\pi}{c}x\right)dx \qquad n = 0, 1, 2, \cdots$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\left(\frac{n\pi}{c}x\right)dx \qquad n = 1, 2, \cdots$$

Then $a_n + ib_n$ gives

$$\begin{aligned} a_n - ib_n &= \frac{1}{c}\int_{-c}^{c} f(x)\cos\left(\frac{n\pi}{c}x\right)dx - i\frac{1}{c}\int_{-c}^{c} f(x)\sin\left(\frac{n\pi}{c}x\right)dx \\ &= \frac{1}{c}\left(\int_{-c}^{c} f(x)\cos\left(\frac{n\pi}{c}x\right)dx + \int_{-c}^{c} f(x)\left(-i\sin\left(\frac{n\pi}{c}x\right)\right)dx\right) \\ &= \frac{1}{c}\int_{-c}^{c} f(x)\left[\cos\left(\frac{n\pi}{c}x\right) - i\sin\left(\frac{n\pi}{c}x\right)\right]dx \\ &= \frac{1}{c}\int_{-c}^{c} f(x)e^{-i\frac{n\pi}{c}x}dx \end{aligned}$$

But $a_n - ib_n = 2A_n$ from first part of this problem. Hence the above becomes

$$A_n = \frac{1}{2c}\int_{-c}^{c} f(x)e^{-i\frac{n\pi}{c}x}dx \qquad n = 0, \pm 1, \pm 2, \cdots$$