University Course

Math 2243
Linear Algebra and Differential Equations

University of Minnesota, Twin Cities
Fall 2020

My Class Notes

Nasser M. Abbasi

Fall 2020

Contents

Chapter 1

Introduction

Local contents

1.1 Links

1. Instructor web page `http://www.math.umn.edu/~webst390/`
2. Canvas web page `https://canvas.umn.edu/courses/195168`

1.2 Schedule

MATH 2243 - 002 Linear Algebra and Differential Equations
Twin Cities/Rochester | Fall 2020 | Lecture

Class Details

Status	Closed	**Career**	Undergraduate
Class Number	21472	**Dates**	9/8/2020 - 12/16/2020
Session	001 Regular Academic Session	**Grading**	Student Option
Units	4 units	**Location**	Off Campus
Instruction Mode	Completely Online	**Campus**	Twin Cities
Class Components	Lecture Required		

Meeting Information

Days & Times	Room	Instructor	Meeting Dates
TuTh 6:00PM - 8:05PM	Twin Cities Remote	Lilly Webster	09/08/2020 - 12/16/2020

Class Availability

Class Capacity	27	**Wait List Capacity**	0
Enrollment Total	27	**Wait List Total**	0
Available Seats	0		

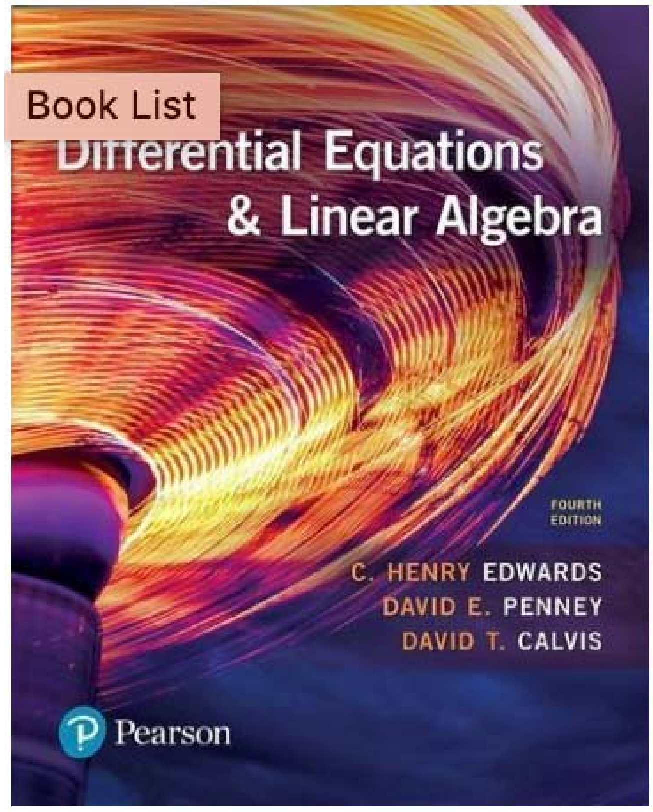
Description

Linear algebra: basis, dimension, matrices, eigenvalues/eigenvectors. Differential equations: first-order linear, separable; second-order linear with constant coefficients; linear systems with constant coefficients.

prereq: [1272 or 1282 or 1372 or 1572] w/grade of at least C-

Credit will not be granted if credit has been received for: MATH 2373, MATH 2471, MATH 2574H

1.3 Text book


Book List
Differential Equations
& Linear Algebra
FOURTH
EDITION
C. HENRY EDWARDS
DAVID E. PENNEY
DAVID T. CALVIS
Pearson

1.4 syllabus

MATH 2243 - SECTION 002 COURSE INFORMATION

Instructor: Lilly Webster
webst390@umn.edu
http://www.math.umn.edu/~webst390/

Canvas Page: https://canvas.umn.edu/courses/195168

Course Meetings: TTh 6:00 PM - 8:05 PM Central Time
See Canvas for Zoom link

Office Hours: TTh 5:00 PM - 6:00 PM Central Time
Also by appointment
See Canvas for Zoom link

Textbook: *Differential Equations and Linear Algebra* (4th edition) by Edwards and Penny

General Notes
I would prefer that you address me as Lilly with she/her/hers pronouns. If you prefer, you may call me Ms. Webster. You should not call me Professor Webster or Dr. Webster, as I am neither.
I strongly encourage you to come to office hours if you have questions. I also strongly encourage you to ask questions during class. In my experience, students who ask questions are much more successful in my courses. I am more than happy to talk at you for several hours a week, but our time will be much more useful if you can tell me what material is confusing to you. Keep in mind that, as with many math classes, the material in this course will build on itself throughout the semester. So, the earlier we address any issues the more likely you are to have success in this course.
The best way to get in touch with me is by email. I will respond to emails sent before 8pm on a weekday within a few hours. If you send an email after that time, I may not be able to respond until the next morning.
I understand that the circumstances of this semester are extremely unusual and that things are liable to change with little notice. I will do my best to be flexible with you as much as possible, and I hope that you will extend me the same courtesy. If your situation changes in a way that will affect your participation in the course, please let me know as soon as possible.

Virtual Learning Plan
This course will be conducted using a variety of online platforms. Canvas will be used for course communication, quizzes, recording grades, and posting all course content and assignments. I recommend checking your notification settings in Canvas so that you can be promptly informed about important course information. Gradescope will be used for

submitting and returning homework and exams. Zoom will be used for course meetings. The Gradescope and Zoom links within Canvas will take you directly to the relevant pages for our course, so I recommend using Canvas to access those platforms.
This course will include video and audio recordings. These will be used for educational purposes and will only be made available to students currently enrolled in this course. If you wish to share course recordings or other course content to anyone outside the course, you must get my permission first. I will inform you in advance of any class sessions that are being recorded. If your image or voice are on any class recording, I will obtain your permission before sharing that recording outside the course.

To give us the most flexibility possible, this course will be conducted as a "flipped" classroom. For each Zoom class session, there will be a series of videos that you are to watch before we meet. These videos will go over the new material from the textbook for each class. During class sessions, I will go over additional examples and answer any questions that you may have. To allow you sufficient time to watch the lecture videos, our virtual course meetings will last approximately 50 minutes (6:00 PM - 6:50 PM).

Office Hours and Additional Help
My office hours are 5:00 PM - 6:00 PM on Tuesdays and Thursdays. You can find the Zoom links for office hours on the Canvas page. If you want to meet with me outside of my usual office hours, send me an email with when you are available and I will do my best to find a time when we can meet. The other evening lecturer Eric Stucky has agreed that you may also go to his office hours for help. His office hours are 5:00 PM - 6:00 PM on Mondays and Wednesdays; see the Canvas page for Zoom links.
Between class sessions, I strongly encourage you to use the Discussions feature on Canvas. There are places there to ask questions about the course, about the material we are learning, and to find other students for a study group.
You may want to consider the free by-appointment tutoring available through the SMART Learning Commons. See `https://www.lib.umn.edu/smart` for more details.

Grading
Your final grade in this course will be calculated as quizzes 5%, homework 20%, two midterms 25% each, and the final exam 25%. The course grade lines will be adjusted based on the distribution of scores across all sections of the course, but grade lines for the total score will not be stricter than the following:

A: 90 -100%   B: 80 - 89%   C: 65 - 79%   D/F: 0-64%

I will also give grade lines for each midterm exam so you can get a sense of where you stand in the course.
If you have concerns about the grading of an assignment, it must be brought to my attention within 1 week of the assignment being returned. Send me an email or stop by my office hours and I'll be happy to look at it.

Quizzes
There will be a very short quiz at the start of each class session. The quiz will be available on Canvas from 5:30 PM - 6:15 PM on each class day and will have a 10-minute time limit. Quizzes will consist of two or three multiple choice questions on the material of the previous class and will not require significant computations. You may use your notes, the textbook, and any resources on Canvas for the quizzes.

Homework
There will be one homework assignment per week, due at the start of class on Thursdays. The assignment will be posted on Canvas at least one week prior to the due date. Homework solutions will be made available on Canvas after the assignment has been turned in. Homework should be submitted through the Gradescope portal. If you have not submitted assignments through Gradescope before, I recommend trying the practice assignment that is posted on Canvas.
You are allowed (and in fact encouraged) to work with other students on homework assignments. If you do that, please indicate on each problem who you worked with. If you use sources other than your notes, the textbook, and any resources on Canvas for your homework, you must indicate the source on each problem. You are not permitted to view, request, or look for solutions to any of the homework problems from solutions manuals, homework help websites, online forums, other students, or any other sources.
The problems that I assign for homework may not be sufficient for you to get comfortable with the material. The nearby problems in the textbook are a good opportunity to get more practice since the answers are in the back of the book. If you find you need more practice than the book provides, please let me know.

Exams
There will be two fifty-minute in-class midterm exams on October 15th and November 19th. This course has a common final exam, which will be given on December 17th from 12:00 PM - 3:00 PM. The material covered on each exam will be confirmed two weeks prior to the exam and review materials will be distributed one week prior to the exam. Exams will be distributed and submitted just like homework assignments, but will only be made available during the exam window. You will be required to sign an honor statement when you submit your exam and to be present on Zoom while working on the exam. You may not ask for or receive help from notes, textbooks, online resources, or other people on exams.
If you have an excused absence that will prevent you from taking an exam, let me know as soon as possible so we can find an acceptable solution.

Other Policies
You may use a calculator at all times; there are no restrictions on the type of calculators that are permitted. All work must still be written out completely, so that it can be understood by a person without a calculator. In general, assessments will be written so that they may be completed without the use of a calculator; exceptions to this will come with advance

warning. Unless stated otherwise, you should leave answers in an exact form like $\cos(2)$, e^2, or $\frac{3}{7}$ rather than giving decimal approximations. Whenever possible, you should simplify expressions such as $\ln e = 1$, $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, and $\frac{20}{5} = 4$.

Academic dishonesty of any kind will not be tolerated and is grounds for receiving an F or N for the entire course. Academic dishonesty includes (but is not limited to) plagarism, consulting unapproved resources on exams, obtaining exams without instructor permission, posting exam problems to online forums, and sharing the exam to other students.

You may drop this course without my approval and without a W on your transcript until September 21st. Between September 21st and November 16th, you may drop the course without my approval but you will get a W on your transcript. For more information, see `https://onestop.umn.edu/dates-and-deadlines/canceladd-deadlines`

The University of Minnesota is committed to providing equal access to learning opportunities for all students. The Disability Resource Center (DRC) is the campus office that collaborates with students who have disabilities to provide or arrange reasonable accomodations. Information is available on their website `https://disability.umn.edu/` or by calling 612-626-1333 or by sending an email to `ds@umn.edu`.

Inclusion Statement
The University of Minnesota provides equal access to and opportunity in its programs and facilities, without regard to race, color, creed, religion, national origin, gender, age, marital status, disability, public assistance status, veteran status, sexual orientation, gender identity, or gender expression. All students are valued in my classroom.
If you have a disability of any kind that requires accommodation for this course, please let me know so we can develop a plan to best meet your needs. If religious observances will conflict with class meetings or assignment due dates, please also let me know.

Chapter 2

HWs

Local contents

2.1 HW 1

Local contents

2.1.1 Problems listing

Homework 1 - Due September 17

Homework instructions: Complete the assigned problems on your own paper. Once you are finished, scan or photograph your work and upload it to Gradescope. When prompted, tell Gradescope where to find each problem.
You are allowed (and in fact encouraged) to work with other students on homework assignments. If you do that, please indicate on each problem who you worked with. If you use sources other than your notes, the textbook, and any resources on Canvas for your homework, you must indicate the source on each problem. You are not permitted to view, request, or look for solutions to any of the homework problems from solutions manuals, homework help websites, online forums, other students, or any other sources.

Textbook Problems:

- §3.1: 11, 15, 17
- §3.2: 7, 9, 15
- §3.3: 8, 11

Additional Problems:

1. Draw pictures to illustrate the three possibilities for the solution set of a linear system of two equations in two variables.
2. Give an example of a 3×3 matrix in echelon form with exactly 2 nonzero entries.
3. Give an example of a 3×3 matrix in reduced echelon form with exactly 4 nonzero entries.

1

2.1.2 Problem 11 section 3.1

Problem

use the method of elimination to determine whether the given linear system is consistent or inconsistent. For each consistent system, find the solution if it is unique; otherwise, describe the infinite solution set in terms of an arbitrary parameter t

$$\begin{aligned} 2x + 7y + 3z &= 11 \\ x + 3y + 2z &= 2 \\ 3x + 7y + 9z &= -12 \end{aligned}$$

Solution

The augmented matrix is

$$\begin{pmatrix} 2 & 7 & 3 & 11 \\ 1 & 3 & 2 & 2 \\ 3 & 7 & 9 & -12 \end{pmatrix}$$

Swapping R_2, R_1 gives

$$\begin{pmatrix} 1 & 3 & 2 & 2 \\ 2 & 7 & 3 & 11 \\ 3 & 7 & 9 & -12 \end{pmatrix}$$

$R_2 \to (-2)R_1 + R_2$ gives

$$\begin{pmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & -1 & 7 \\ 3 & 7 & 9 & -12 \end{pmatrix}$$

$R_3 \to (-3)R_1 + R_3$ gives

$$\begin{pmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & -1 & 7 \\ 0 & -2 & 3 & -18 \end{pmatrix}$$

$R_3 \to 2R_2 + R_3$ gives

$$\begin{pmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & -1 & 7 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

The leading variables are x, y, z. No free variables. Hence the system is consistent.

The equations after elimination are

$$\begin{aligned} x + 3y + 2z &= 2 \\ y - z &= 7 \\ z &= -4 \end{aligned}$$

Backsubstitution gives

$$z = -4$$

And

$$\begin{aligned} y - (-4) &= 7 \\ y &= 7 - 4 \\ &= 3 \end{aligned}$$

And

$$\begin{aligned} x + 3(3) + 2(-4) &= 2 \\ x &= 2 - 9 + 8 \\ &= 1 \end{aligned}$$

The solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$$

2.1.3 Problem 15 section 3.1

Problem

use the method of elimination to determine whether the given linear system is consistent or inconsistent. For each consistent system, find the solution if it is unique; otherwise, describe the infinite solution set in terms of an arbitrary parameter t

$$\begin{aligned} x+3y+2z&=5\\ x-y+3z&=3\\ 3x+y+8z&=10 \end{aligned}$$

Solution

The augmented matrix is

$$\begin{pmatrix} 1 & 3 & 2 & 5\\ 1 & -1 & 3 & 3\\ 3 & 1 & 8 & 10 \end{pmatrix}$$

$R_2 \to -R_1 + R_2$ gives

$$\begin{pmatrix} 1 & 3 & 2 & 5\\ 0 & -4 & 1 & -2\\ 3 & 1 & 8 & 10 \end{pmatrix}$$

$R_3 \to (-3)R_1 + R_3$ gives

$$\begin{pmatrix} 1 & 3 & 2 & 5\\ 0 & -4 & 1 & -2\\ 0 & -8 & 2 & -5 \end{pmatrix}$$

$R_3 \to (-2)R_2 + R_3$ gives

$$\begin{pmatrix} 1 & 3 & 2 & 5\\ 0 & -4 & 1 & -2\\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The equations after elimination are

$$\begin{aligned} x+3y+2z&=5\\ -4y+z&=-2\\ 0&=-1 \end{aligned}$$

Therefore the system is inconsistent due to the last row result above which is not valid. No solution exist.

2.1.4 Problem 17 section 3.1

Problem

use the method of elimination to determine whether the given linear system is consistent or inconsistent. For each consistent system, find the solution if it is unique; otherwise, describe the infinite solution set in terms of an arbitrary parameter t

$$\begin{aligned} 2x-y+4z&=7\\ 3x+2y-2z&=3\\ 5x+y+2z&=15 \end{aligned}$$

Solution

The augmented matrix is

$$\begin{pmatrix} 2 & -1 & 4 & 7\\ 3 & 2 & -2 & 3\\ 5 & 1 & 2 & 15 \end{pmatrix}$$

Scaling first row by 3 and second row by 2 gives

$$\begin{pmatrix} 6 & -3 & 12 & 21 \\ 6 & 4 & -4 & 6 \\ 5 & 1 & 2 & 15 \end{pmatrix}$$

$R_2 \rightarrow -R_1 + R_2$ gives

$$\begin{pmatrix} 6 & -3 & 12 & 21 \\ 0 & 7 & -16 & -15 \\ 5 & 1 & 2 & 15 \end{pmatrix}$$

Scaling first row by 5 and last row by 6 gives

$$\begin{pmatrix} 30 & -15 & 60 & 105 \\ 0 & 7 & -16 & -15 \\ 30 & 6 & 12 & 90 \end{pmatrix}$$

$R_3 \rightarrow -R_1 + R_3$ gives

$$\begin{pmatrix} 30 & -15 & 60 & 105 \\ 0 & 7 & -16 & -15 \\ 0 & 21 & -48 & -15 \end{pmatrix}$$

$R_3 \rightarrow -(3)R_2 + R_3$ gives

$$\begin{pmatrix} 30 & -15 & 60 & 105 \\ 0 & 7 & -16 & -15 \\ 0 & 0 & 0 & 30 \end{pmatrix}$$

The equations after elimination are

$$\begin{aligned} 30x - 15y + 60z &= 105 \\ 7y - 16z &= -15 \\ 0 &= 30 \end{aligned}$$

Therefore the system is <u>inconsistent</u> due to the last row result above which is not valid. No solution exist.

2.1.5 Problem 7 section 3.2

<u>Problem</u>

The linear systems in Problems 1–10 are in echelon form. Solve each by back substitution

$$\begin{aligned} x_1 + 2x_2 + 4x_3 - 5x_4 &= 17 \\ x_2 - 2x_3 + 7x_4 &= 7 \end{aligned}$$

<u>Solution</u>

The augmented matrix is

$$\begin{pmatrix} 1 & 2 & 4 & -5 & 17 \\ 0 & 1 & -2 & 7 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that the leading variables are x_1, x_2 and the free variables are x_3, x_4. Let $x_3 = s, x_4 = t$. From second row

$$\begin{aligned} x_2 - 2x_3 + 7x_4 &= 7 \\ x_2 - 2s + 7t &= 7 \\ x_2 &= 7 + 2s - 7t \end{aligned}$$

And from first row

$$\begin{aligned} x_1 + 2x_2 + 4x_3 - 5x_4 &= 17 \\ x_1 + 2(7 + 2s - 7t) + 4s - 5t &= 17 \\ x_1 &= 19t - 8s + 3 \end{aligned}$$

Hence the solution is

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} 19t - 8s + 3 \\ 7 + 2s - 7t \\ s \\ t \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 7 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -8 \\ 2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 19 \\ -7 \\ 0 \\ 1 \end{pmatrix} t \end{aligned}$$

There are infinite number of solutions.

2.1.6 Problem 9 section 3.2

Problem

The linear systems in Problems 1–10 are in echelon form. Solve each by back substitution

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 &= 6 \\ 3x_2 - x_3 - 2x_4 &= 2 \\ 3x_3 + 4x_4 &= 9 \\ x_4 &= 6 \end{aligned}$$

Solution

The augmented matrix is

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 6 \\ 0 & 3 & -1 & -2 & 2 \\ 0 & 0 & 3 & 4 & 9 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}$$

The leading variables are x_1, x_2, x_3, x_4. There are no free variables. Backsubstitution gives

$$x_4 = 6$$

And

$$\begin{aligned} 3x_3 + 4x_4 &= 9 \\ 3x_3 &= 9 - 4(6) \\ 3x_3 &= -15 \\ x_3 &= -5 \end{aligned}$$

And

$$\begin{aligned} 3x_2 - x_3 - 2x_4 &= 2 \\ 3x_2 + 5 - 12 &= 2 \\ x_2 &= 3 \end{aligned}$$

And

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 &= 6 \\ 2x_1 + 3 - 5 + 6 &= 6 \\ x_1 &= 1 \end{aligned}$$

Hence the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -5 \\ 6 \end{pmatrix}$$

2.1.7 Problem 15 section 3.2

Problem

In Problems 11–22, use elementary row operations to transform each augmented coefficient matrix to echelon form. Then solve the system by back substitution.

$$\begin{aligned} 3x_1 + x_2 - 3x_3 &= -4 \\ x_1 + x_2 + x_3 &= 1 \\ 5x_1 + 6x_2 + 8x_3 &= 8 \end{aligned}$$

Solution

The augmented matrix is

$$\begin{pmatrix} 3 & 1 & -3 & -4 \\ 1 & 1 & 1 & 1 \\ 5 & 6 & 8 & 8 \end{pmatrix}$$

Exchanging row 1 with row 2 gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -3 & -4 \\ 5 & 6 & 8 & 8 \end{pmatrix}$$

$R_2 \to (-3)R_1 + R_2$ gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -6 & -7 \\ 5 & 6 & 8 & 8 \end{pmatrix}$$

$R_3 \to (-5)R_1 + R_3$ gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -6 & -7 \\ 0 & 1 & 3 & 3 \end{pmatrix}$$

Exchanging row 2 with row 3 gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & -2 & -6 & -7 \end{pmatrix}$$

$R_3 \to (2)R_2 + R_3$ gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Hence the equations are

$$\begin{aligned} x_1 + x_2 + x_1 &= 1 \\ x_2 + 3x_3 &= 3 \\ 0 &= -1 \end{aligned}$$

Therefore the system is inconsistent due to the last row result above which is not valid. No solution exist.

2.1.8 Problem 8 section 3.3

Problem

Find the reduced echelon form

$$\begin{pmatrix} 1 & -4 & -5 \\ 3 & -9 & 3 \\ 1 & -2 & 3 \end{pmatrix}$$

Solution

First we convert the matrix to echelon form by elimination.

$R_2 \rightarrow (-3)R_1 + R_2$ gives

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 3 & 18 \\ 1 & -2 & 3 \end{pmatrix}$$

$R_3 \rightarrow (-1)R_1 + R_3$ gives

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 3 & 18 \\ 0 & 2 & 8 \end{pmatrix}$$

Scale second row by 3 and scale third row by 2 gives

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 6 & 36 \\ 0 & 6 & 24 \end{pmatrix}$$

$R_3 \rightarrow (-1)R_2 + R_3$ gives

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 6 & 36 \\ 0 & 0 & -12 \end{pmatrix}$$

The above is now in echelon form. We now convert it to reduced echelon form. Scaling the second row by $\frac{1}{6}$ gives

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 1 & 6 \\ 0 & 0 & -12 \end{pmatrix}$$

Scaling the third row by $\frac{-1}{12}$ gives

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

Starting from right to left, we now zero out all entries above the diagonal elements.

$R_2 \rightarrow (-6)R_3 + R_2$ gives

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_1 \rightarrow (5)R_3 + R_1$ gives

$$\begin{pmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_1 \rightarrow (4)R_2 + R_1$ gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The above is the reduced echelon form. It is the identity matrix.

2.1.9 Problem 11 section 3.3

Problem

Find the reduced echelon form

$$\begin{pmatrix}3 & 9 & 1\\ 2 & 6 & 7\\ 1 & 3 & -6\end{pmatrix}$$

Solution

First we convert the matrix to echelon form by elimination. Exchanging row 3 and first row gives

$$\begin{pmatrix}1 & 3 & -6\\ 2 & 6 & 7\\ 3 & 9 & 1\end{pmatrix}$$

$R_2 \to (-2)R_1 + R_2$ gives

$$\begin{pmatrix}1 & 3 & -6\\ 0 & 0 & 19\\ 3 & 9 & 1\end{pmatrix}$$

$R_2 \to (-3)R_1 + R_3$ gives

$$\begin{pmatrix}1 & 3 & -6\\ 0 & 0 & 19\\ 0 & 0 & 19\end{pmatrix}$$

$R_3 \to (-1)R_2 + R_3$ gives

$$\begin{pmatrix}1 & 3 & -6\\ 0 & 0 & 19\\ 0 & 0 & 0\end{pmatrix}$$

The above is now in echelon form. We now convert it to reduced echelon form. Scaling the second row by $\frac{1}{19}$ gives

$$\begin{pmatrix}1 & 3 & -6\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$$

Starting from right to left, we now zero out all entries above the leading elements starting from second row.

$R_1 \to (6)R_2 + R_1$ gives

$$\begin{pmatrix}1 & 3 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$$

The above is the reduced echelon form.

2.1.10 Problem 1 extra

Problem

Draw pictures to illustrate the three possibilities for the solution set of a linear system of two equations in two variables.

Solution

For homogeneous system

$$\begin{aligned}a_{11}x + a_{12}y &= 0\\ a_{21}x + a_{22}y &= 0\end{aligned}$$

There can be either one solution, which is the trivial solution $x = 0, y = 0$ where the two lines meet at the origin, or infinite number of solutions, which is when the two lines are on top of each others. The reason for this is that there is no intercept in the equation of the lines above. Only the slope of each line can change. Hence all lines must pass though the origin. This diagram illustrates this.

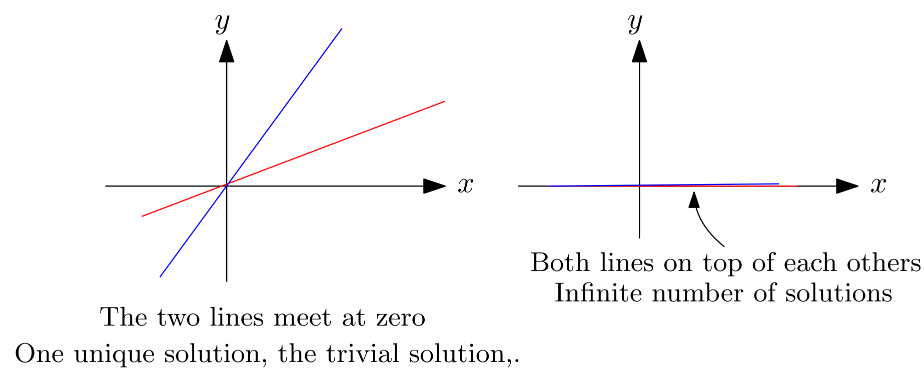


Figure 2.1: Possibilites for homogeneous system

For nonhomogeneous system

$$a_{11}x + a_{12}y = c_1$$
$$a_{21}x + a_{22}y = c_2$$

Now there can be three possibilities. Either one solution where the two lines meet or infinite number of solutions, which is when the two lines are on top of each others or no solutions, which is when the two lines are parallel but not on top of each others.

This diagram illustrates this.

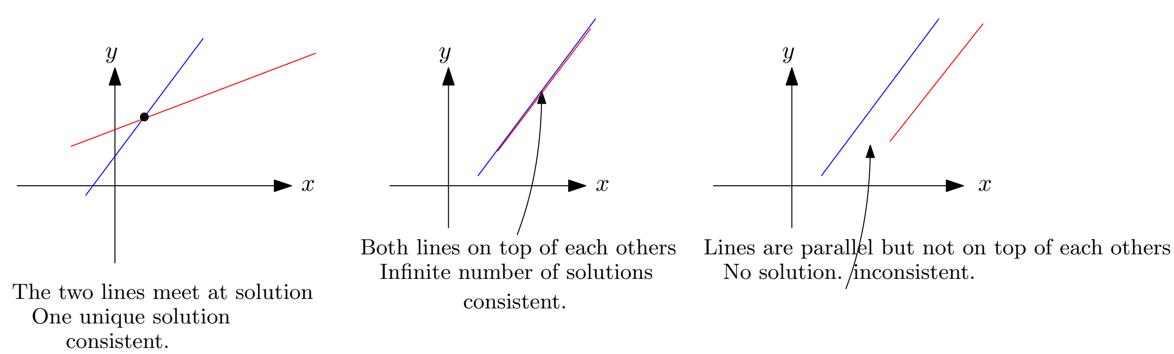


Figure 2.2: Possibilites for homogeneous system

2.1.11 Problem 2 extra

Problem

Give an example of a 3×3 matrix in echelon form with exactly 2 nonzero entries.

Solution

$$\begin{pmatrix} \bigstar & 0 & 0 \\ 0 & \bigstar & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Where $\bigstar$ is a nonzero entry.

2.1.12 Problem 3 extra

Problem

Give an example of a 3×3 matrix in reduced echelon form with exactly 4 nonzero entries.

Solution

$$\begin{pmatrix} 1 & 0 & \bigstar \\ 0 & 1 & \bigstar \\ 0 & 0 & 0 \end{pmatrix}$$

Where $\bigstar$ is a nonzero entry. In reduced echelon form only the leading entries (which must be 1) has to be zero.

2.1.13 key solution for HW 1

Homework 1 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

3.1.11 Our original system is

$$\begin{aligned} 2x + 7y + 3z &= 11 \\ x + 3y + 2z &= 2 \\ 3x + 7y + 9z &= -12 \end{aligned}$$

We add (-2) times the second equation to the first and (-3) times the second equation to the third.

$$\begin{aligned} y - z &= 7 \\ x + 3y + 2z &= 2 \\ -2y + 3z &= -18 \end{aligned}$$

Now, add 2 times the first equation to the third.

$$\begin{aligned} y - z &= 7 \\ x + 3y + 2z &= 2 \\ z &= -4 \end{aligned}$$

At this point, back substitution gives us the unique solution $x = 1, y = 3, z = -4$. The system is consistent.

3.1.15 Our original system is

$$\begin{aligned} x + 3y + 2z &= 5 \\ x - y + 3z &= 3 \\ 3x + y + 8z &= 10 \end{aligned}$$

Add (-1) times the first equation to the second and (-3) times the first equation to the third.

$$\begin{aligned} x + 3y + 2z &= 5 \\ -4y + z &= -2 \\ -8y + 2z &= -5 \end{aligned}$$

1

Add (-2) times the second equation to the third.

$$\begin{aligned} x + 3y + 2z &= 5 \\ -4y + z &= -2 \\ 0 &= -1 \end{aligned}$$

Our third equation is contradictory, so the system has no solutions and is inconsistent.

3.1.17 Our original system is

$$\begin{aligned} 2x - y + 4z &= 7 \\ 3x + 2y - 2z &= 3 \\ 5x + y + 2z &= 15 \end{aligned}$$

Add (-1) times the first equation to the second.

$$\begin{aligned} 2x - y + 4z &= 7 \\ x + 3y - 6z &= -4 \\ 5x + y + 2z &= 15 \end{aligned}$$

Add (-2) times the second equation to the first and (-5) times the second equation to the third.

$$\begin{aligned} -7y + 16z &= 15 \\ x + 3y - 6z &= -4 \\ -14y + 32z &= 35 \end{aligned}$$

Add (-2) times the first equation to the third.

$$\begin{aligned} -7y + 16z &= 15 \\ x + 3y - 6z &= -4 \\ 0 &= 5 \end{aligned}$$

Our third equation is contradictory, so the system has no solutions and is inconsistent.

3.2.7 Our system is

$$\begin{aligned} x_1 + 2x_2 + 4x_3 - 5x_4 &= 17 \\ x_2 - 2x_3 + 7x_4 &= 7 \end{aligned}$$

Our free variables are x_3 and x_4, so we set them equal to parameters: $x_3 = s, x_4 = t$. Solving for the other two variables gives $x_2 = 7 + 2s - 7t$ and $x_1 = 3 - 8s + 19t$.

3.2.9 Our system is

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 &= 6 \\ 3x_2 - x_3 - 2x_4 &= 2 \\ 3x_3 + 4x_4 &= 9 \\ x_4 &= 6 \end{aligned}$$

Working from the bottom up, we solve for each variable to get $x_4 = 6, x_3 = -5, x_2 = 3$, and $x_1 = 1$.

3.2.15 We do row operations to the augmented coefficient matrix. For brevity, we may do more than one row operation in a step.

$$\begin{bmatrix} 3 & 1 & -3 & -4 \\ 1 & 1 & 1 & 1 \\ 5 & 6 & 8 & 8 \end{bmatrix} \xrightarrow[(-5)R_2+R_3]{(-3)R_2+R_1} \begin{bmatrix} 0 & -2 & -6 & -7 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \end{bmatrix}$$
$$\xrightarrow{(2)R_3+R_1} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \end{bmatrix}$$
$$\xrightarrow{SWAP(R_1,R_2)} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 3 & 3 \end{bmatrix}$$
$$\xrightarrow{SWAP(R_2,R_3)} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

We continued with our reduction to reach echelon form, but notice that after the second step our first row corresponds to the equation $0 = 11$, a contradiction. So the system has no solutions.

3

3.3.8

$$\begin{bmatrix} 1 & -4 & -5 \\ 3 & -9 & 3 \\ 1 & -2 & 3 \end{bmatrix} \xrightarrow[(-3)R_1+R_2]{(-1)R_1+R_3} \begin{bmatrix} 1 & -4 & -5 \\ 0 & 3 & 18 \\ 0 & 2 & 8 \end{bmatrix}$$

$$\xrightarrow[(\frac{1}{3})R_2]{(\frac{1}{2})R_3} \begin{bmatrix} 1 & -4 & -5 \\ 0 & 1 & 6 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{(-1)R_2+R_3} \begin{bmatrix} 1 & -4 & -5 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\xrightarrow{(-\frac{1}{2})R_3} \begin{bmatrix} 1 & -4 & -5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[(5)R_3+R_1]{(-6)R_3+R_2} \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(4)R_2+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.3.11

$$\begin{bmatrix} 3 & 9 & 1 \\ 2 & 6 & 7 \\ 1 & 3 & -6 \end{bmatrix} \xrightarrow[(-2)R_3+R_2]{(-3)R_3+R_1} \begin{bmatrix} 0 & 0 & 19 \\ 0 & 0 & 19 \\ 1 & 3 & -6 \end{bmatrix}$$

$$\xrightarrow{(-1)R_2+R_1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 19 \\ 1 & 3 & -6 \end{bmatrix}$$

$$\xrightarrow[(\frac{1}{19})R_2]{SWAP(R_1,R_3)} \begin{bmatrix} 1 & 3 & -6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{(6)R_2+R_1} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Additional Problems:

1. Check the class notes for sketches of each case.

   Unique solution – two intersecting lines

   No solutions – two parallel lines

   Infinitely many solutions – two identical lines

4

2. There are many possibilities. One option: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

3. There are multiple possibilities here, but all correct answers will have their nonzero entries in the same places. One option: $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

2.2 HW 2

Local contents

2.2.1 Problems listing

Homework 2 - Due September 24

Homework instructions: Complete the assigned problems on your own paper. Once you are finished, scan or photograph your work and upload it to Gradescope. When prompted, tell Gradescope where to find each problem.
You are allowed (and in fact encouraged) to work with other students on homework assignments. If you do that, please indicate on each problem who you worked with. If you use sources other than your notes, the textbook, and any resources on Canvas for your homework, you must indicate the source on each problem. You are not permitted to view, request, or look for solutions to any of the homework problems from solutions manuals, homework help websites, online forums, other students, or any other sources.

Textbook Problems:

- §3.4: 3, 5, 8, 11
- §3.5: 3, 10, 16
- §3.6: 4, 9, 21

Additional Problems:

1. Give an example of matrices A and B where $AB = BA$.
2. Give an example of matrices C and D where $CD \neq DC$.
3. Let A, B, and C be invertible $n \times n$ matrices. Is the product ABC invertible? If it is invertible, what is $(ABC)^{-1}$?
4. Let $T = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}$ be a diagonal matrix. What is $\det T$?

   Optional: Consider an $n \times n$ diagonal matrix T. That is, T has entries $t_1, t_2, \ldots, t_n$ on the main diagonal and 0's everywhere else. What is $\det T$? The required part of this problem asks you to answer this question for the case where $n = 3$.

1

2.2.2 Problem 3 section 3.4

Problem

In Problems 1-4, two matrices A and B and two numbers c and d are given. Compute the matrix $cA + dB$

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 7 \\ 3 & -1 \end{bmatrix}, B = \begin{bmatrix} -4 & 5 \\ 3 & 2 \\ 7 & 4 \end{bmatrix}, c = -2, d = 4$$

Solution

$$\begin{aligned} cA + dB &= -2\begin{bmatrix} 5 & 0 \\ 0 & 7 \\ 3 & -1 \end{bmatrix} + 4\begin{bmatrix} -4 & 5 \\ 3 & 2 \\ 7 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -10 & 0 \\ 0 & -14 \\ -6 & 2 \end{bmatrix} + \begin{bmatrix} -16 & 20 \\ 12 & 8 \\ 28 & 16 \end{bmatrix} \\ &= \begin{bmatrix} -26 & 20 \\ 12 & -6 \\ 22 & 18 \end{bmatrix} \end{aligned}$$

2.2.3 Problem 5 section 3.4

Problem

In Problems 5-12, two matrices A and B are given. Calculate whichever of the matrices AB and BA is defined.

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} -4 & 2 \\ 1 & 3 \end{bmatrix}$$

Solution

A dimension is 2×2 and B dimension is 2×2. So inner dimensions agree. Both AB and BA are defined. Using definition of matrix multiplication we obtain

$$\begin{aligned} AB &= \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -4 & 2 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -9 & 1 \\ -10 & 12 \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} BA &= \begin{bmatrix} -4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 8 \\ 11 & 5 \end{bmatrix} \end{aligned}$$

2.2.4 Problem 8 section 3.4

Problem

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{bmatrix}$$

Solution

A dimension is 2×3 and B dimension is 3×2. Hence AB is $(2 \times 3)(3 \times 2) = 2 \times 2$ matrix. Therefore inner dimensions agree. And BA is define since $(3 \times 2)(2 \times 3) = 3 \times 3$. Therefore inner dimensions agree.

$$AB = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 21 & 15 \\ 35 & 0 \end{bmatrix}$$

And

$$BA = \begin{bmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 0 & 9 \\ 7 & -20 & 13 \\ 16 & -25 & 38 \end{bmatrix}$$

2.2.5 Problem 11 section 3.4

Problem

$$A = \begin{bmatrix} 3 & -5 \end{bmatrix}, B = \begin{bmatrix} 2 & 7 & 5 & 6 \\ -1 & 4 & 2 & 3 \end{bmatrix}$$

Solution

A dimension is 1×2 and B dimension is 2×4. Hence AB is $(1 \times 2)(2 \times 4) = 1 \times 4$ matrix. Therefore inner dimensions agree. And BA is not defined since $(2 \times 4)(1 \times 2)$. Therefore inner dimensions do not agree. So only AB is defined here.

$$AB = \begin{bmatrix} 3 & -5 \end{bmatrix} \begin{bmatrix} 2 & 7 & 5 & 6 \\ -1 & 4 & 2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 11 & 1 & 5 & 3 \end{bmatrix}$$

2.2.6 Problem 3 section 3.5

Problem

In Problems 1-8, first apply the formulas in (9) to find A^{-1}. Then use A^{-1} (as in Example 5) to solve the system $Ax = b$.

$$A = \begin{bmatrix} 6 & 7 \\ 5 & 6 \end{bmatrix}, b = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Solution

Formula (9) is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 6 & 7 \\ 5 & 6 \end{bmatrix}^{-1} = \frac{1}{36 - 35} \begin{bmatrix} 6 & -7 \\ -5 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & -7 \\ -5 & 6 \end{bmatrix}$$

Hence

$$\begin{aligned} x &= A^{-1}b \\ &= \begin{bmatrix} 6 & -7 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 33 \\ -28 \end{bmatrix} \end{aligned}$$

2.2.7 Problem 10 section 3.5

Problem

In Problems 9-22, use the method of Example 7 to find the inverse A^{-1} of each given matrix A.

$$A = \begin{bmatrix} 5 & 7 \\ 4 & 6 \end{bmatrix}$$

Solution

The augmented matrix is

$$\begin{bmatrix} 5 & 7 & 1 & 0 \\ 4 & 6 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$ gives

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 4 & 6 & 0 & 1 \end{bmatrix}$$

$R_2 \to -4R_1 + R_2$ gives

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & -4 & 5 \end{bmatrix}$$

$R_2 \to \frac{1}{2}R_2$ gives

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & \frac{5}{2} \end{bmatrix}$$

$R_1 \to R_1 - R_2$ gives

$$\begin{bmatrix} 1 & 0 & 3 & -\frac{7}{2} \\ 0 & 1 & -2 & \frac{5}{2} \end{bmatrix}$$

Since the left side of the augments matrix is now the identity matrix, then we read A^{-1} from the right side. Hence

$$\begin{aligned} A^{-1} &= \begin{bmatrix} 3 & -\frac{7}{2} \\ -2 & \frac{5}{2} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 6 & -7 \\ -4 & 5 \end{bmatrix} \end{aligned}$$

2.2.8 Problem 16 section 3.5

Problem

$$A = \begin{bmatrix} 1 & -3 & -3 \\ -1 & 1 & 2 \\ 2 & -3 & -3 \end{bmatrix}$$

Solution

The augmented matrix is

$$A = \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 2 & -3 & -3 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_1 + R_2$ gives

$$A = \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 1 & 0 \\ 2 & -3 & -3 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to -(2)R_1 + R_3$ gives

$$A = \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 1 & 0 \\ 0 & 3 & 3 & -2 & 0 & 1 \end{bmatrix}$$

$R_2 \to 3R_2$ and $R_3 \to 2R_3$ gives

$$A = \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & -6 & -3 & 3 & 3 & 0 \\ 0 & 6 & 6 & -4 & 0 & 2 \end{bmatrix}$$

$R_3 \to R_2 + R_3$ gives

$$A = \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & -6 & -3 & 3 & 3 & 0 \\ 0 & 0 & 3 & -1 & 3 & 2 \end{bmatrix}$$

$R_2 \to R_2 + R_3$ gives

$$A = \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & -6 & 0 & 2 & 6 & 2 \\ 0 & 0 & 3 & -1 & 3 & 2 \end{bmatrix}$$

$R_1 \to R_1 + R_3$ gives

$$A = \begin{bmatrix} 1 & -3 & 0 & 0 & 3 & 2 \\ 0 & -6 & 0 & 2 & 6 & 2 \\ 0 & 0 & 3 & -1 & 3 & 2 \end{bmatrix}$$

$R_1 \to R_1 - \frac{1}{2}R_2$ gives

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & -6 & 0 & 2 & 6 & 2 \\ 0 & 0 & 3 & -1 & 3 & 2 \end{bmatrix}$$

$R_2 \to \frac{-1}{6}R_2$ gives

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{3} & -1 & \frac{-1}{3} \\ 0 & 0 & 3 & -1 & 3 & 2 \end{bmatrix}$$

$R_3 \to \frac{1}{3}R_3$ gives

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{3} & -1 & \frac{-1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & 1 & \frac{2}{3} \end{bmatrix}$$

Since the left side of the augments matrix is now the identity matrix, then we read A^{-1} from the right side. Hence

$$\begin{aligned} A^{-1} &= \begin{bmatrix} -1 & 0 & 1 \\ -\frac{1}{3} & -1 & \frac{-1}{3} \\ -\frac{1}{3} & 1 & \frac{2}{3} \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} -3 & 0 & 3 \\ -1 & -3 & -1 \\ -1 & 3 & 2 \end{bmatrix} \end{aligned}$$

2.2.9 Problem 4 section 3.6

Problem

Use cofactor expansions to evaluate the determinants in Problems 1-6. Expand along the row or column that minimizes the

amount of computation that is required.

$$A=\begin{bmatrix}5 & 11 & 8 & 7\\ 3 & -2 & 6 & 23\\ 0 & 0 & 0 & -3\\ 0 & 4 & 0 & 17\end{bmatrix}$$

Solution

Row 4 has most zeros. Hence expansion is on row 4.

$$\begin{aligned}|A| &= (-)(-3)\begin{vmatrix}5 & 11 & 8\\ 3 & -2 & 6\\ 0 & 4 & 0\end{vmatrix}\\ &= 3\begin{vmatrix}5 & 11 & 8\\ 3 & -2 & 6\\ 0 & 4 & 0\end{vmatrix}\end{aligned}$$

For $\begin{vmatrix}5 & 11 & 8\\ 3 & -2 & 6\\ 0 & 4 & 0\end{vmatrix}$ we expand on 3rd row. The above becomes

$$\begin{aligned}|A| &= 3\left((-)4\begin{vmatrix}5 & 8\\ 3 & 6\end{vmatrix}\right)\\ &= -12\begin{vmatrix}5 & 8\\ 3 & 6\end{vmatrix}\\ &= -12(30-24)\end{aligned}$$

Therefore

$$|A| = -72$$

2.2.10 Problem 9 section 3.6

Problem

In Problems 7-12, evaluate each given determinant after first simplifying the computation (as in Example 6) by adding an appropriate multiple of some row or column to another.

$$A=\begin{bmatrix}3 & -2 & 5\\ 0 & 5 & 17\\ 6 & -4 & 12\end{bmatrix}$$

Solution

Adding multiple of some row to another row does not change the determinant of a matrix. Same for adding multiple of some column to another column. We can take advantage of this to add more zeros to the matrix before applying the cofactor method to reduce the computation needed.

Let $R_3 \to -2R_1 + R_3$ gives

$$A=\begin{bmatrix}3 & -2 & 5\\ 0 & 5 & 17\\ 0 & 0 & 2\end{bmatrix}$$

Expansion on third row now gives

$$\begin{aligned}|A| &= (+)2\begin{vmatrix}3 & -2\\0 & 5\end{vmatrix}\\ &= 2(15)\end{aligned}$$

Therefore

$$|A| = 30$$

2.2.11 Problem 21 section 3.6

Problem

Use Cramer's rule to solve the systems in Problems 21-32.

$$\begin{aligned}3x + 4y &= 2\\ 5x + 7y &= 1\end{aligned}$$

Solution

The system in matrix form is

$$\begin{bmatrix}3 & 4\\5 & 7\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

Hence using Cramer's rule

$$x = \frac{\begin{vmatrix}2 & 4\\1 & 7\end{vmatrix}}{\begin{vmatrix}3 & 4\\5 & 7\end{vmatrix}} = \frac{14-4}{21-20} = 10$$

And

$$y = \frac{\begin{vmatrix}3 & 2\\5 & 1\end{vmatrix}}{\begin{vmatrix}3 & 4\\5 & 7\end{vmatrix}} = \frac{3-10}{21-20} = -7$$

Hence the solution is

$$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}10\\-7\end{bmatrix}$$

2.2.12 Additional problem 1

Problem

Give an example of matrices A and B where $AB = BA$

Solution

Let $A = \begin{bmatrix}a & b\\c & d\end{bmatrix}, B = \begin{bmatrix}e & f\\g & h\end{bmatrix}$. Then

$$\begin{aligned}AB &= \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}e & f\\g & h\end{bmatrix}\\ &= \begin{bmatrix}ae+bg & af+bh\\ce+dg & cf+dh\end{bmatrix}\end{aligned} \tag{1}$$

And

$$\begin{aligned}BA &= \begin{bmatrix}e & f\\g & h\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix}\\ &= \begin{bmatrix}ae+cf & be+df\\ag+ch & bg+dh\end{bmatrix}\end{aligned} \tag{2}$$

For (1,2) to be equal implies that

$$
\begin{aligned}
ae+bg &= ae+cf\\
af+bh &= be+df\\
ce+dg &= ag+ch\\
cf+dh &= bg+dh
\end{aligned}
$$

Simplifying gives

$$
\begin{aligned}
bg &= cf\\
af+bh &= be+df\\
ce+dg &= ag+ch\\
cf &= bg
\end{aligned}
$$

First equation is the same as the fourth. Hence the above becomes

$$
\begin{aligned}
bg &= cf\\
af+bh &= be+df\\
ce+dg &= ag+ch
\end{aligned}
$$

Let $a=1, b=2, c=3, d=4, e=5, f=6$. The above becomes

$$
\begin{aligned}
2g &= 18\\
6+2h &= 10+24\\
15+4g &= g+3h
\end{aligned}
$$

or

$$
\begin{aligned}
g &= 9\\
h &= 14
\end{aligned}
$$

Hence and example is

$$
\begin{aligned}
A &= \begin{bmatrix} a & b\\ c & d\end{bmatrix} = \begin{bmatrix} 1 & 2\\ 3 & 4\end{bmatrix}\\
B &= \begin{bmatrix} e & f\\ g & h\end{bmatrix} = \begin{bmatrix} 5 & 6\\ 9 & 14\end{bmatrix}
\end{aligned}
$$

To verify

$$
\begin{aligned}
AB &= \begin{bmatrix} 1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix} 5 & 6\\ 9 & 14\end{bmatrix} = \begin{bmatrix} 23 & 34\\ 51 & 74\end{bmatrix}\\
BA &= \begin{bmatrix} 5 & 6\\ 9 & 14\end{bmatrix}\begin{bmatrix} 1 & 2\\ 3 & 4\end{bmatrix} = \begin{bmatrix} 23 & 34\\ 51 & 74\end{bmatrix}
\end{aligned}
$$

2.2.13 Additional problem 2

Problem

Give an example of matrices C and D where $CD \neq DC$.

Solution

From the last problem, we found a solution that makes $CD = DC$ to be

$$
\begin{aligned}
g &= 9\\
h &= 14
\end{aligned}
$$

So any other value will make $CD \neq DC$. Hence an example is

$$C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
$$D = \begin{bmatrix} e & f+1 \\ g & h \end{bmatrix} = \begin{bmatrix} 5 & 6+1 \\ 9 & 14 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 9 & 14 \end{bmatrix}$$

To verify

$$CD = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ 9 & 14 \end{bmatrix} = \begin{bmatrix} 23 & 35 \\ 51 & 77 \end{bmatrix}$$

But

$$DC = \begin{bmatrix} 5 & 7 \\ 9 & 14 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 26 & 38 \\ 51 & 74 \end{bmatrix}$$

Hence $CD \neq DC$

2.2.14 Additional problem 3

Problem

Let $A; B$, and C be invertible $n \times n$ matrices. Is the product ABC invertible? If it is invertible, what is $(ABC)^{-1}$?

Solution

Let $ABC = D$. Premultiplying both sides by A^{-1} gives

$$A^{-1}ABC = A^{-1}D$$
$$BC = A^{-1}D$$

Premultiplying both sides by B^{-1} gives

$$B^{-1}BC = B^{-1}A^{-1}D$$
$$B = B^{-1}A^{-1}D$$

Premultiplying both sides by C^{-1} gives

$$I = \left(C^{-1}B^{-1}A^{-1}\right)D \tag{1}$$

Starting with $ABC = D$ again, but now post multiplying both sides by C^{-1} gives

$$ABCC^{-1} = DC^{-1}$$
$$AB = DC^{-1}$$

Post multiplying both sides by B^{-1} gives

$$ABB^{-1} = DC^{-1}B^{-1}$$
$$A = DC^{-1}B^{-1}$$

Post multiplying both sides by A^{-1} gives

$$I = D\left(C^{-1}B^{-1}A^{-1}\right) \tag{2}$$

Comparing (1,2) we see that

$$\left(C^{-1}B^{-1}A^{-1}\right)D = D\left(C^{-1}B^{-1}A^{-1}\right) = I \tag{3}$$

This means $C^{-1}B^{-1}A^{-1}$ is the inverse of D by definition (page 177 of book) which says if $AB = BA = I$ then B is the inverse of A.

But D is the product of ABC. Hence the product is invertible. And from (3), its inverse is given by

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

2.2.15 Additional problem 4

Problem

Let $T = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}$ be diagonal matrix. What is det(T)?

Solution

The determinant of a diagonal matrix is the product of the elements on the diagonal. Hence

$$\det(T) = t_1 t_2 t_3$$

This comes from expansion over any row or column. For example, expansion along row 1 gives

$$\begin{aligned} \det(T) &= t_1 \begin{vmatrix} t_2 & 0 \\ 0 & t_3 \end{vmatrix} \\ &= t_1 t_2 \det(\begin{bmatrix} t_3 \end{bmatrix}) \\ &= t_1 t_2 t_3 \end{aligned}$$

Note that the sign of the elements are all positive for 3×3 since n is odd here.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

2.2.16 Additional problem. Optional

Problem

Optional: Consider an $n \times n$ diagonal matrix T. What is $det(T)$? The required part of this problem asks you to answer this question for the case where $n = 3$.

Solution

$$T = \begin{bmatrix} t_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & t_2 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & t_3 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \ddots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & t_n \end{bmatrix}$$

det(T) is the product of all elements on the diagonal. This comes from expansion over any row. For example, expansion on row 1 gives

$$\begin{aligned} \det(T) &= t_1 \begin{vmatrix} t_2 & 0 & \cdots & \cdots & 0 \\ 0 & t_3 & \cdots & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & 0 \\ \cdots & \cdots & \cdots & \cdots & t_n \end{vmatrix} \\ &= t_1 t_2 \begin{vmatrix} t_3 & \cdots & \cdots & 0 \\ \cdots & t_4 & \cdots & 0 \\ \cdots & \cdots & \ddots & 0 \\ \cdots & \cdots & \cdots & t_n \end{vmatrix} \\ &= t_1 t_2 t_3 \begin{vmatrix} t_4 & \cdots & 0 \\ \cdots & \ddots & 0 \\ \cdots & \cdots & t_n \end{vmatrix} \end{aligned}$$

And so on until the last entry

$$\begin{aligned}\det(T) &= t_1 t_2 t_3 \cdots t_n \\ &= \prod_{i=1}^{n} t_n\end{aligned}$$

Note on the sign. In expansion, we have to take account of sign changes. If n is odd, then the sign of the elements are all positive on the diagonal as in case $n = 3$ above. So we do not need to worry about this case.

For even n, the sign on diagoanl also remains positive, since the formula is $(-1)^{i+j}$ where i, j are the index of the diagonal elements, and this always adds to even number since $i = j$ on the diagonal. For an example for $n = 4$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

We see that product on the diagonal always has positive signs.

2.2.17 key solution for HW 2

Homework 2 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

3.4.3

$$(-2)\begin{bmatrix}5 & 0\\ 0 & 7\\ 3 & -1\end{bmatrix} + (4)\begin{bmatrix}-4 & 5\\ 3 & 2\\ 7 & 4\end{bmatrix} = \begin{bmatrix}-26 & 20\\ 12 & -6\\ 22 & 18\end{bmatrix}$$

3.4.5

$$\begin{aligned} AB &= \begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}\begin{bmatrix}-4 & 2\\ 1 & 3\end{bmatrix}\\ &= \begin{bmatrix}-9 & 1\\ -10 & 12\end{bmatrix}\\ BA &= \begin{bmatrix}-4 & 2\\ 1 & 3\end{bmatrix}\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}\\ &= \begin{bmatrix}-2 & 8\\ 11 & 5\end{bmatrix}\end{aligned}$$

3.4.8

$$\begin{aligned} AB &= \begin{bmatrix}1 & 0 & 3\\ 2 & -5 & 4\end{bmatrix}\begin{bmatrix}3 & 0\\ -1 & 4\\ 6 & 5\end{bmatrix}\\ &= \begin{bmatrix}21 & 15\\ 35 & 0\end{bmatrix}\\ BA &= \begin{bmatrix}3 & 0\\ -1 & 4\\ 6 & 5\end{bmatrix}\begin{bmatrix}1 & 0 & 3\\ 2 & -5 & 4\end{bmatrix}\\ &= \begin{bmatrix}3 & 0 & 9\\ 7 & -20 & 13\\ 16 & -25 & 38\end{bmatrix}\end{aligned}$$

3.4.11

$$\begin{aligned} AB &= \begin{bmatrix}3 & -5\end{bmatrix}\begin{bmatrix}2 & 7 & 5 & 6\\ -1 & 4 & 2 & 3\end{bmatrix}\\ &= \begin{bmatrix}11 & 1 & 5 & 3\end{bmatrix}\end{aligned}$$

The product BA is not defined because B is 2×4 and A is 1×2, so the dimensions do not match.

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3.5.3 For $A = \begin{bmatrix} 6 & 7 \\ 5 & 6 \end{bmatrix}$, our formula for 2×2 matrices tells us that $A^{-1} = \frac{1}{36-35} \begin{bmatrix} 6 & -7 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -7 \\ -5 & 6 \end{bmatrix}$. So if $A\vec{x} = \vec{b}$, we have that

$$\begin{aligned} \vec{x} &= A^{-1}\vec{b} \\ &= \begin{bmatrix} 6 & -7 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 33 \\ -28 \end{bmatrix} \end{aligned}$$

3.5.10 We adjoin an identity matrix and row reduce:

$$\begin{aligned} \begin{bmatrix} 5 & 7 & 1 & 0 \\ 4 & 6 & 0 & 1 \end{bmatrix} &\xrightarrow{-R_2+R_1} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 4 & 6 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{-4R_1+R_2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & -4 & 5 \end{bmatrix} \\ &\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & \frac{5}{2} \end{bmatrix} \\ &\xrightarrow{-R_2+R_1} \begin{bmatrix} 1 & 0 & 3 & -\frac{7}{2} \\ 0 & 1 & -2 & \frac{5}{2} \end{bmatrix} \end{aligned}$$

So the inverse matrix is $\begin{bmatrix} 3 & -\frac{7}{2} \\ -2 & \frac{5}{2} \end{bmatrix}$.

3.5.16 We adjoin an identity matrix and row reduce:

$$\begin{aligned} \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 2 & -3 & -3 & 0 & 0 & 1 \end{bmatrix} &\xrightarrow[-2R_1+R_3]{R_1+R_2} \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 1 & 0 \\ 0 & 3 & 3 & -2 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{R_3+R_2} \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 3 & 3 & -2 & 0 & 1 \end{bmatrix} \\ &\xrightarrow[3R_2+R_1]{-3R_2+R_3} \begin{bmatrix} 1 & 0 & 3 & -2 & 3 & 3 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & -3 & 1 & -3 & -2 \end{bmatrix} \\ &\xrightarrow{-\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 3 & -2 & 3 & 3 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & 1 & \frac{2}{3} \end{bmatrix} \\ &\xrightarrow[-3R_3+R_1]{-2R_3+R_2} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{3} & -1 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & 1 & \frac{2}{3} \end{bmatrix} \end{aligned}$$

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So the inverse is $\begin{bmatrix} -1 & 0 & 1 \\ -\frac{1}{3} & -1 & -\frac{1}{3} \\ -\frac{1}{3} & 1 & \frac{2}{3} \end{bmatrix}$

3.6.4 First, we expand along the third row:

$$\det \begin{bmatrix} 5 & 11 & 8 & 7 \\ 3 & -2 & 6 & 23 \\ 0 & 0 & 0 & -3 \\ 0 & 4 & 0 & 17 \end{bmatrix} = (-1)(-3)\det \begin{bmatrix} 5 & 11 & 8 \\ 3 & -2 & 6 \\ 0 & 4 & 0 \end{bmatrix}$$

Next, we expand along the third row:

$$(3)\det \begin{bmatrix} 5 & 11 & 8 \\ 3 & -2 & 6 \\ 0 & 4 & 0 \end{bmatrix} = (3)(-1)(4)\det \begin{bmatrix} 5 & 8 \\ 3 & 6 \end{bmatrix}$$

Finally, we evaluate using our formula for 2×2 determinants:

$$(-12)(5^*6 - 8^*3) = (-12)(6) = -72$$

3.6.9 Before proceeding with our calculation, we do the row operation $(-2)R_1 + R_3$ which does not change the determinant:

$$\det \begin{bmatrix} 3 & -2 & 5 \\ 0 & 5 & 17 \\ 6 & -4 & 12 \end{bmatrix} = \det \begin{bmatrix} 3 & -2 & 5 \\ 0 & 5 & 17 \\ 0 & 0 & 2 \end{bmatrix}$$

We expand this along the new and improved first column:

$$\begin{aligned} \det \begin{bmatrix} 3 & -2 & 5 \\ 0 & 5 & 17 \\ 0 & 0 & 2 \end{bmatrix} &= (+1)(3)\det \begin{bmatrix} 5 & 17 \\ 0 & 2 \end{bmatrix} \\ &= (3)(5^*2 - 17^*0) \\ &= 30 \end{aligned}$$

3.6.21 Our system is

$$\begin{aligned} 3x + 4y &= 2 \\ 5x + 7y &= 1 \end{aligned}$$

The coefficient matrix has determinant $\det \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} = 3^*7 - 4^*5 = 1$, so since this nonzero we can proceed with Cramer's Rule. Cramer's Rule tells us that the unique solution

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to this system is

$$x = \frac{\det\begin{bmatrix}2 & 4\\1 & 7\end{bmatrix}}{\det\begin{bmatrix}3 & 4\\5 & 7\end{bmatrix}} = \frac{2{*}7 - 4{*}1}{3{*}7 - 4{*}5} = 10$$

$$y = \frac{\det\begin{bmatrix}3 & 2\\5 & 1\end{bmatrix}}{\det\begin{bmatrix}3 & 4\\5 & 7\end{bmatrix}} = \frac{3{*}1 - 2{*}5}{3{*}7 - 4{*}5} = -7$$

Additional Problems:

1. Some possible easy choices here include taking $A = B$, taking $B = A^{-1}$, taking one of the matrices to be the identity, or choosing 1×1 matrices.

2. One choice that works here is $C = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$, $D = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$. We compute

$$CD = \begin{bmatrix}2 & 1\\4 & 3\end{bmatrix}$$
$$DC = \begin{bmatrix}3 & 4\\1 & 2\end{bmatrix}$$

   If you pick random entries for your matrices, odds are that they will work for this problem. Another easy choice would be picking C to be $n \times m$ and D to be $m \times n$ where $m \neq n$. Then CD and DC are different sizes, so are certainly not equal!

3. The product ABC is invertible, and the inverse is $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$. To see why this is the inverse, we compute:

$$\begin{aligned}(ABC)(C^{-1}B^{-1}A^{-1}) &= AB(CC^{-1})B^{-1}A^{-1}\\ &= ABIB^{-1}A^{-1}\\ &= A(BB^{-1})A^{-1}\\ &= AIA^{-1}\\ &= AA^{-1}\\ &= I\end{aligned}$$

   You can calulate similarly that $(C^{-1}B^{-1}A^{-1})(ABC) = I$.

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4. The determinant of $T = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}$ is $t_1 t_2 t_3$. The cofactor expansion is straightforward here, no matter which row or column you choose to use.

   For an $n \times n$ diagonal matrix T, the determinant is the product $t_1 t_2 \cdots t_n$. In mathematics, we usually write a product like this as $\prod_{i=1}^{n} t_i$. This kind of notation is called "product notation" and works very similarly to summation notation that you may be already familiar with.

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2.3 HW 3

Local contents

2.3.1 Problems listing

HOMEWORK 3 - DUE OCTOBER 1

Homework instructions: Complete the assigned problems on your own paper. Once you are finished, scan or photograph your work and upload it to Gradescope. When prompted, tell Gradescope where to find each problem.
You are allowed (and in fact encouraged) to work with other students on homework assignments. If you do that, please indicate on each problem who you worked with. If you use sources other than your notes, the textbook, and any resources on Canvas for your homework, you must indicate the source on each problem. You are not permitted to view, request, or look for solutions to any of the homework problems from solutions manuals, homework help websites, online forums, other students, or any other sources.

Textbook Problems:

- §3.7: 3
- §4.1: 1, 19, 23, 27
- §4.2: 2, 4, 17, 21

Additional Problems:

1. My fictional company Linear Algebra Inc had a stock price of $10 on day 1, $15 on day 2, and $10 on day 3. Interpolate this data with a quadratic polynomial $f(t) = a + bt + ct^2$, where t is the day and $f(t)$ is the price on day t.

   Is it a good idea to use $f(t)$ to predict the stock price of Linear Algebra Inc on day 4?
2. Geometrically, what do subspaces of $\mathbb{R}^2$ look like?
3. Let A be an $n \times n$ matrix and consider the linear system $A\vec{x} = \vec{b}$. If I know that the solution set to this linear system is a subspace of $\mathbb{R}^n$, what can you say about $\vec{b}$?

1

2.3.2 Problem 3 section 3.7

In each of Problems 1–10, $n+1$ data points are given. Find the n^{th} degree polynomial $y=f(x)$ that fits these points.

$$\left(x,y\right)=\{(0,3),(1,1),(2,-5)\}$$

Solution

Since $n+1=3$, then $n=2$. Therefore we need degree 2 polynomial

$$f(x)=A+Bx+Cx^2$$

From the data given, we obtain the following three equations

$$\begin{aligned}3&=A\\1&=A+B+C\\-5&=A+2B+4C\end{aligned}$$

This gives the system

$$\begin{bmatrix}1&0&0\\1&1&1\\1&2&4\end{bmatrix}\begin{bmatrix}A\\B\\C\end{bmatrix}=\begin{bmatrix}3\\1\\-5\end{bmatrix}$$

Augmented matrix is

$$\begin{bmatrix}1&0&0&3\\1&1&1&1\\1&2&4&-5\end{bmatrix}$$

$R_2\rightarrow -R_1+R_2$ gives

$$\begin{bmatrix}1&0&0&3\\0&1&1&-2\\1&2&4&-5\end{bmatrix}$$

$R_3\rightarrow -R_1+R_3$ gives

$$\begin{bmatrix}1&0&0&3\\0&1&1&-2\\0&2&4&-8\end{bmatrix}$$

$R_3\rightarrow -2R_2+R_3$ gives

$$\begin{bmatrix}1&0&0&3\\0&1&1&-2\\0&0&2&-4\end{bmatrix}$$

Hence we obtain the system in Echelon form as

$$\begin{bmatrix}1&0&0\\0&1&1\\0&0&2\end{bmatrix}\begin{bmatrix}A\\B\\C\end{bmatrix}=\begin{bmatrix}3\\-2\\-4\end{bmatrix}$$

Back substitution: Last row gives $2C=-4$ or $C=-2$. Second row gives $B-C=-2$ or $B=0$. First row gives $A=3$. Therefore the solution is

$$\begin{bmatrix}A\\B\\C\end{bmatrix}=\begin{bmatrix}3\\0\\-2\end{bmatrix}$$

The polynomial is

$$\begin{aligned}f(x)&=A+Bx+Cx^2\\&=3-2x^2\end{aligned}$$

Here is plot of the solution fitted on the points

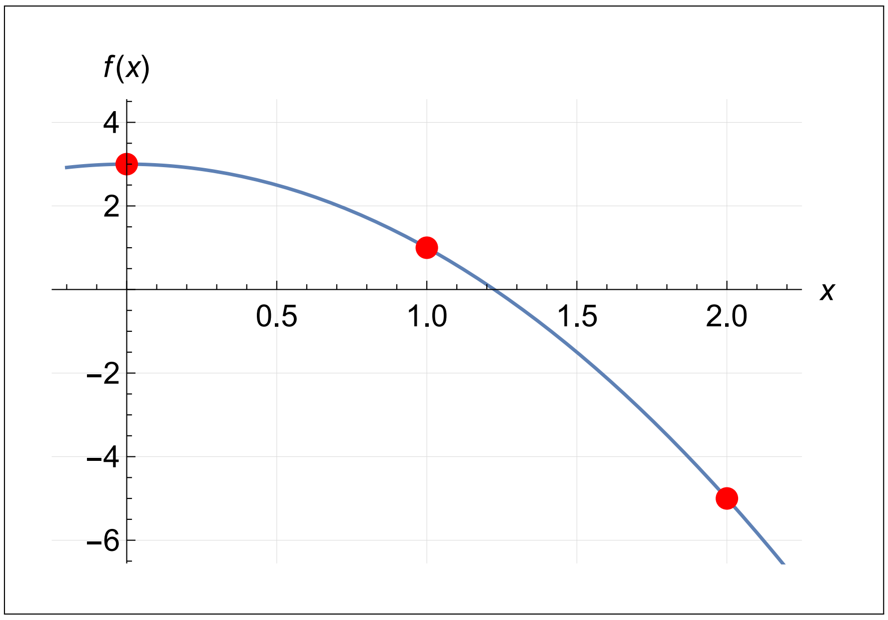

Figure 2.3: Fitted polynomial plot

```
p1 = ListPlot[{{0, 3}, {1, 1}, {2, -5}}, PlotStyle → {PointSize[.03], Red}];
p2 = Plot[3 + 0 x - 2 x^2, {x, -.2, 2.2}, AxesLabel → {x, f[x]},
   BaseStyle → 14, GridLines → Automatic, GridLinesStyle → LightGray];
p = Show[p2, p1, PlotRange → {Automatic, {-6, 4}}];
```

Figure 2.4: Code used for the above plot

2.3.3 Problem 1 section 4.1

In Problems 1–4, find $\left|\vec{a}-\vec{b}\right|, 2\vec{a}+\vec{b}, 3\vec{a}-4\vec{b}$

$$\vec{a} = (2,5,-4), \vec{b} = (1,-2,-3)$$

<u>Solution</u>

$$\begin{aligned}\left|\vec{a}-\vec{b}\right| &= |(2,5,-4)-(1,-2,-3)| \\ &= |(2-1,5+2,-4+3)| \\ &= |(1,7,-1)| \\ &= \sqrt{1+49+1} \\ &= \sqrt{51}\end{aligned}$$

And

$$\begin{aligned}2\vec{a}+\vec{b} &= 2(2,5,-4)+(1,-2,-3) \\ &= (4,10,-8)+(1,-2,-3) \\ &= (4+1,10-2,-8-3) \\ &= (5,8,-11)\end{aligned}$$

And

$$\begin{aligned}3\vec{a}-4\vec{b} &= 3(2,5,-4)-4(1,-2,-3) \\ &= (6,15,-12)-(4,-8,-12) \\ &= (6-4,15+8,-12+12) \\ &= (2,23,0)\end{aligned}$$

2.3.4 Problem 19 section 4.1

In Problems 19–24, use the method of Example 3 to determine whether the given vectors $\vec{u}, \vec{v}$, and $\vec{w}$ are linearly independent or dependent. If they are linearly dependent, find scalars a, b, and c not all zero such that $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$

$$\begin{aligned}\vec{u} &= (2,0,1)\\ \vec{v} &= (-3,1,-1)\\ \vec{w} &= (0,-2,-1)\end{aligned}$$

Solution

We set up $Ax = 0$ and solve for x where x here is (a, b, c) vector. If x is the trivial solution, then the vectors are linearly independent. If we find non-trivial solution, then the vectors are linearly dependent.

$$a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$$

$$a\begin{bmatrix}2\\0\\1\end{bmatrix} + b\begin{bmatrix}-3\\1\\-1\end{bmatrix} + c\begin{bmatrix}0\\-2\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}2 & -3 & 0\\0 & 1 & -2\\1 & -1 & -1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix}2 & -3 & 0\\0 & 1 & -2\\1 & -1 & -1\end{bmatrix}$$

$R_3 \to -\frac{1}{2}R_1 + R_3$ gives

$$\begin{bmatrix}2 & -3 & 0\\0 & 1 & -2\\0 & \frac{1}{2} & -1\end{bmatrix}$$

$R_3 \to \frac{-1}{2}R_2 + R_3$ gives

$$\begin{bmatrix}2 & -3 & 0\\0 & 1 & -2\\0 & 0 & 0\end{bmatrix}$$

Hence the system becomes

$$\begin{bmatrix}2 & -3 & 0\\0 & 1 & -2\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

a, b are leading variables and c is free variable. Let $c = t$ which can be any value. Then $b = 2t$ and $2a - 3b = 0$ or $a = 3t$. Hence solution is

$$\begin{bmatrix}a\\b\\c\end{bmatrix} = t\begin{bmatrix}3\\2\\1\end{bmatrix}$$

There are infinite solutions. We need only one non-zero solution to show that the vectors are linearly dependent. Let $t = 1$

$$\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}3\\2\\1\end{bmatrix}$$

Hence vectors are linearly dependent

$$3\vec{u} + 2\vec{v} + \vec{w} = \vec{0}$$

2.3.5 Problem 23 section 4.1

In Problems 19–24, use the method of Example 3 to determine whether the given vectors $\vec{u}, \vec{v}$, and $\vec{w}$ are linearly independent or dependent. If they are linearly dependent, find scalars a, b, and c not all zero such that $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$

$$
\begin{aligned}
\vec{u} &= (2,0,3) \\
\vec{v} &= (5,4,-2) \\
\vec{w} &= (2,-1,1)
\end{aligned}
$$

Solution

We set up $Ax = 0$ and solve for x where x here is (a, b, c) vector. If x is the trivial solution, then the vectors are linearly independent. If we find non-trivial solution, then the vectors are linearly dependent.

$$
a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}
$$

$$
a\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + b\begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} + c\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 2 & 5 & 2 \\ 0 & 4 & -1 \\ 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

Augmented matrix

$$
\begin{bmatrix} 2 & 5 & 2 \\ 0 & 4 & -1 \\ 3 & -2 & 1 \end{bmatrix}
$$

$R_3 \to -3R_1 + 2R_3$ gives

$$
\begin{bmatrix} 2 & 5 & 2 \\ 0 & 4 & -1 \\ 0 & -19 & -4 \end{bmatrix}
$$

$R_3 \to -19R_2 + 4R_3$ gives

$$
\begin{bmatrix} 2 & 5 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 3 \end{bmatrix}
$$

Hence the system in Echelon form becomes

$$
\begin{bmatrix} 2 & 5 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

Last row gives $c = 0$. Second row gives $4b = 0$ or $b = 0$. First row gives $2a = 0$ or $a = 0$.

Therefore the vectors are linearly independent because only the trivial solution exist.

2.3.6 Problem 27 section 4.1

In Problems 25–28, express the vector $\vec{t}$ as a linear combination of the vectors $\vec{u}, \vec{v}$, and $\vec{w}$.

$$
\vec{t} = (0,0,19), \vec{u} = (1,4,3), \vec{v} = (-1,-2,2), \vec{w} = (4,4,1)
$$

Solution

In system form we are looking for

$$a\vec{u} + b\vec{v} + c\vec{w} = \vec{t}$$

$$a\begin{bmatrix}1\\4\\3\end{bmatrix} + b\begin{bmatrix}-1\\-2\\2\end{bmatrix} + c\begin{bmatrix}4\\4\\1\end{bmatrix} = \begin{bmatrix}0\\0\\19\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & 4\\4 & -2 & 4\\3 & 2 & 1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\19\end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix}1 & -1 & 4 & 0\\4 & -2 & 4 & 0\\3 & 2 & 1 & 19\end{bmatrix}$$

$R_2 \to -4R_1 + R_2$ gives

$$\begin{bmatrix}1 & -1 & 4 & 0\\0 & 2 & -12 & 0\\3 & 2 & 1 & 19\end{bmatrix}$$

$R_3 \to -3R_1 + R_3$ gives

$$\begin{bmatrix}1 & -1 & 4 & 0\\0 & 2 & -12 & 0\\0 & 5 & -11 & 19\end{bmatrix}$$

$R_3 \to -\frac{5}{2}R_2 + R_3$ gives

$$\begin{bmatrix}1 & -1 & 4 & 0\\0 & 2 & -12 & 0\\0 & 0 & 19 & 19\end{bmatrix}$$

The above is Echelon form. Hence the system is

$$\begin{bmatrix}1 & -1 & 4\\0 & 2 & -12\\0 & 0 & 19\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\19\end{bmatrix}$$

Last row gives $19c = 19$ or $c = 1$. Second row gives $2b - 12c = 0$ or $b = 6$. First row gives $a - b + 4c = 0$ or $a = b - 4c$ or $a = 6 - 4 = 2$. Hence

$$a\vec{u} + b\vec{v} + c\vec{w} = \vec{t}$$

$$2\vec{u} + 6\vec{v} + \vec{w} = \vec{t}$$

2.3.7 Problem 2 section 4.2

Apply Theorem 1 to determine whether or not W is a subspace of $\mathbb{R}^n$.

W is the set of all vectors in $\mathbb{R}^3$ such that $x_1 = 5x_2$

Solution

Theorem 1 at page 225 gives conditions for subspace:

The non empty subset W of the vector space V is a subspace of V if and only if it satisfies the following two conditions:

1. If $\vec{u}$ and $\vec{v}$ are vectors in W , then $\vec{u} + \vec{v}$ is also in W.
2. If $\vec{u}$ is in W and c is a scalar, then the vector $c\vec{u}$ is also in W.

Let $\vec{u} = (x_1, x_2, x_3)$ and $\vec{v} = \left(y_1, y_2, y_3\right)$ where $x_1 = 5x_2$ and $y_1 = 5y_2$. then (1) becomes

$$\begin{aligned}\vec{u} + \vec{v} &= (x_1, x_2, x_3) + \left(y_1, y_2, y_3\right)\\ &= \left(x_1 + y_1, x_2 + y_2, x_3 + y_3\right)\end{aligned}$$

Then

$$\begin{aligned}x_1 + y_1 &= 5x_2 + 5y_2\\ &= 5\left(x_2 + y_2\right)\end{aligned}$$

Hence closed under addition. Condition (2) says

$$\begin{aligned}c\vec{u} &= c(x_1, x_2, x_3)\\ &= (cx_1, cx_2, cx_3)\end{aligned}$$

Hence $cx_1 = c(5x_2) = 5(cx_2)$. Therefore closed under scalar multiplication as well. Therefore this is a subspace.

2.3.8 Problem 4 section 4.2

Apply Theorem 1 to determine whether or not W is a subspace of $\mathbb{R}^n$.

W is the set of all vectors in $\mathbb{R}^3$ such that $x_1 + x_2 + x_3 = 1$

Solution

Theorem 1 at page 225 gives conditions for subspace:

The non empty subset W of the vector space V is a subspace of V if and only if it satisfies the following two conditions:

1. If $\vec{u}$ and $\vec{v}$ are vectors in W , then $\vec{u} + \vec{v}$ is also in W.
2. If $\vec{u}$ is in W and c is a scalar, then the vector $c\vec{u}$ is also in W.

Let $\vec{u} = (x_1, x_2, x_3)$ and $\vec{v} = \left(y_1, y_2, y_3\right)$ where $x_1 + x_2 + x_3 = 1$ and $y_1 + y_2 + y_3 = 1$. then (1) becomes

$$\begin{aligned}\vec{u} + \vec{v} &= (x_1, x_2, x_3) + \left(y_1, y_2, y_3\right)\\ &= \left(x_1 + y_1, x_2 + y_2, x_3 + y_3\right)\end{aligned}$$

Then

$$\begin{aligned}x_1 + y_1 + x_2 + y_2 + x_3 + y_3 &= (x_1 + x_2 + x_3) + \left(y_1 + y_2 + y_3\right)\\ &= 1 + 1\\ &= 2\end{aligned}$$

Therefore this is not closed under addition since $\vec{u}+\vec{v}$ does not satisfy (1). Hence not a subspace.

2.3.9 Problem 17 section 4.2

In Problems 15–18, apply the method of Example 5 to find two solution vectors $\vec{u}$ and $\vec{v}$ such that the solution space is the set

of all linear combinations of the form $s\vec{u} + t\vec{v}$

$$\begin{aligned}x_1 + 3x_2 + 8x_3 - x_4 &= 0\\ x_1 - 3x_2 - 10x_3 + 5x_4 &= 0\\ x_1 + 4x_2 + 11x_3 - 2x_4 &= 0\end{aligned}$$

(notice: typo in book. Last term in second equation is $5x_5$ in book, but it should be $5x_4$).

Solution

System is

$$\begin{bmatrix} 1 & 3 & 8 & -1 \\ 1 & -3 & -10 & 5 \\ 1 & 4 & 11 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix} 1 & 3 & 8 & -1 \\ 1 & -3 & -10 & 5 \\ 1 & 4 & 11 & -2 \end{bmatrix}$$

$R_2 \rightarrow -R_1 + R_2$ gives

$$\begin{bmatrix} 1 & 3 & 8 & -1 \\ 0 & -6 & -18 & 6 \\ 1 & 4 & 11 & -2 \end{bmatrix}$$

$R_3 \rightarrow -R_1 + R_3$ gives

$$\begin{bmatrix} 1 & 3 & 8 & -1 \\ 0 & -6 & -18 & 6 \\ 0 & 1 & 3 & -1 \end{bmatrix}$$

$R_3 \rightarrow \frac{1}{6}R_2 + R_3$ gives

$$\begin{bmatrix} 1 & 3 & 8 & -1 \\ 0 & -6 & -18 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading variables are x_1, x_2. Free variables are x_3, x_4. Let $x_4 = t, x_3 = s$. The system becomes

$$\begin{bmatrix} 1 & 3 & 8 & -1 \\ 0 & -6 & -18 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From second row, $-6x_2 - 18s + 6t = 0$ or $x_2 = -\frac{18s-6t}{6} = -3s + t$.

From first row, $x_1 + 3x_2 + 8s - t = 0$. Hence $x_1 = -3x_2 - 8s + t$ or $x_1 = -3(-3s + t) - 8s + t$ or $x_1 = s - 2t$. Therefore the solution is

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} s - 2t \\ -3s + t \\ s \\ t \end{bmatrix} \\ &= s \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \\ &= s\vec{u} + t\vec{v} \end{aligned}$$

Therefore he solution space is the set of all linear combinations of the form $s\vec{u} + t\vec{v}$

2.3.10 Problem 21 section 4.2

In Problems 19–22, reduce the given system to echelon form to find a single solution vector $\vec{u}$ such that the solution space is

the set of all scalar multiples of $\vec{u}$.

$$\begin{aligned} x_1 + 7x_2 + 2x_3 - 3x_4 &= 0 \\ 2x_1 + 7x_2 + x_3 - 4x_4 &= 0 \\ 3x_1 + 5x_2 - x_3 - 5x_4 &= 0 \end{aligned}$$

Solution

System is

$$\begin{bmatrix} 1 & 7 & 2 & -3 \\ 2 & 7 & 1 & -4 \\ 3 & 5 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix} 1 & 7 & 2 & -3 \\ 2 & 7 & 1 & -4 \\ 3 & 5 & -1 & -5 \end{bmatrix}$$

$R_2 \rightarrow -2R_1 + R_2$ gives

$$\begin{bmatrix} 1 & 7 & 2 & -3 \\ 0 & -7 & -3 & 2 \\ 3 & 5 & -1 & -5 \end{bmatrix}$$

$R_3 \rightarrow -3R_1 + R_3$ gives

$$\begin{bmatrix} 1 & 7 & 2 & -3 \\ 0 & -7 & -3 & 2 \\ 0 & -16 & -7 & 4 \end{bmatrix}$$

$R_3 \rightarrow \frac{-16}{7} R_2 + R_3$ gives

$$\begin{bmatrix} 1 & 7 & 2 & -3 \\ 0 & -7 & -3 & 2 \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} \end{bmatrix}$$

Hence the system in Echelon form is

$$\begin{bmatrix} 1 & 7 & 2 & -3 \\ 0 & -7 & -3 & 2 \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Leading variables are x_1, x_2, x_3. Free variable is $x_4 = t$. Last row gives $-\frac{1}{7}x_3 - \frac{4}{7}t = 0$. Hence $x_3 = -4t$. Second row gives $-7x_2 - 3x_3 + 2x_4 = 0$ or $-7x_2 = 3x_3 - 2x_4$ or $-7x_2 = 3(-4t) - 2(t)$. Hence $-7x_2 = -14t$ or $x_2 = 2t$.

First row gives $x_1 + 7x_2 + 2x_3 - 3x_4 = 0$ or $x_1 = -7(2t) - 2(-4t) + 3(t)$. Hence $x_1 = -3t$. Therefore the solution is

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} -3t \\ 2t \\ -4t \\ t \end{bmatrix} \\ &= t \begin{bmatrix} -3 \\ 2 \\ -4 \\ 1 \end{bmatrix} \\ &= t\vec{u} \end{aligned}$$

The solution space is the set of all scalar multiples of $\vec{u}$.

2.3.11 Additional problem 1

My fictional company Linear Algebra Inc had a stock price of \$10 on day 1, \$15 on day 2, and \$10 on day 3. Interpolate this data with a quadratic polynomial $f(t) = a + bt + ct^2$,where

t is the day and $f(t)$ is the price on day t. Is it a good idea to use $f(t)$ to predict the stock price of Linear Algebra Inc on day 4?

Solution

Data is $(1,10),(2,15),(3,10)$. Therefore we obtain 3 equations using $f(t) = a + bt + ct^2$ as

$$\begin{aligned}10 &= a + b + c\\ 15 &= a + 2b + 4c\\ 10 &= a + 3b + 9c\end{aligned}$$

Which gives the system

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 4\\ 1 & 3 & 9\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}10\\ 15\\ 10\end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix}1 & 1 & 1 & 10\\ 1 & 2 & 4 & 15\\ 1 & 3 & 9 & 10\end{bmatrix}$$

$R_2 \to -R_1 + R_2$ gives

$$\begin{bmatrix}1 & 1 & 1 & 10\\ 0 & 1 & 3 & 5\\ 1 & 3 & 9 & 10\end{bmatrix}$$

$R_3 \to -R_1 + R_3$ gives

$$\begin{bmatrix}1 & 1 & 1 & 10\\ 0 & 1 & 3 & 5\\ 0 & 2 & 8 & 0\end{bmatrix}$$

$R_3 \to -2R_2 + R_3$ gives

$$\begin{bmatrix}1 & 1 & 1 & 10\\ 0 & 1 & 3 & 5\\ 0 & 0 & 2 & -10\end{bmatrix}$$

Hence the system in Echelon form is

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 3\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}10\\ 5\\ -10\end{bmatrix}$$

Leading variables are a, b, c. There are no free variables. From last row $2c = -10$ hence $c = -5$. From second row $b + 3c = 5$ or $b = 5 - 3c$ or $b = 5 - 3(-5)$ or $b = 20$. From first row $a + b + c = 10$. Hence $a = 10 - b - c$ or $a = 10 - 20 + 5$ or $a = -5$. The solution is

$$\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}-5\\ 20\\ -5\end{bmatrix}$$

Therefore, the interpolation polynomial is

$$f(t) = a + bt + ct^2$$

Or

$$\boxed{f(t) = -5 + 20t - 5t^2}$$

It is not good idea to use $f(t)$ to predict the price outside the range of interpolation, which is $t = 1 \cdots 3$. Doing so is extrapolation and can produce wrong prediction. For example, using $t = 4$ gives $f(4) = -5$ dollars as stock price, which is not possible. The lowest value a stock can have is zero dollars, which is when the company go bankrupt.

Here is plot of the solution fitted on the points

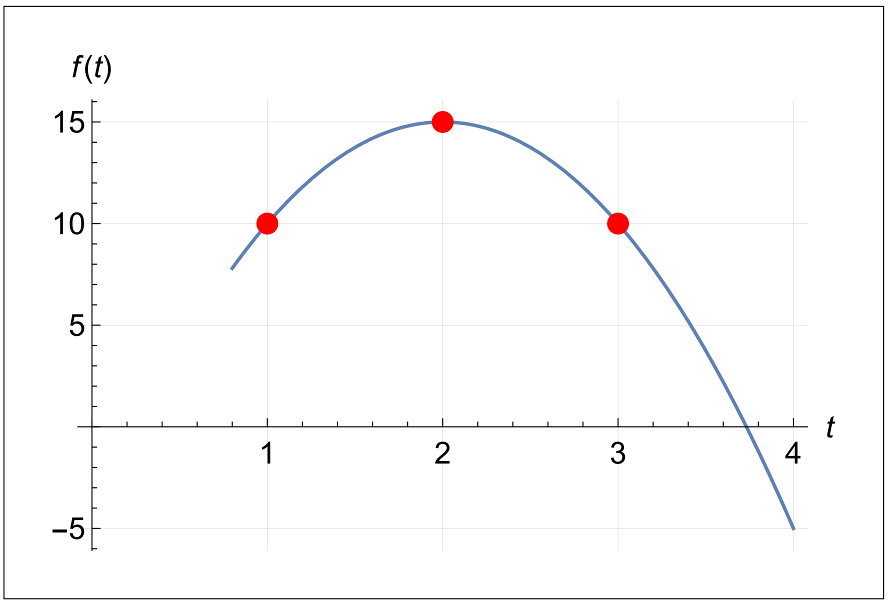

Figure 2.5: Fitted polynomial plot

```
p1 = ListPlot[{{1, 10}, {2, 15}, {3, 10}}, PlotStyle → {PointSize[.03], Red}];
p2 = Plot[-5 + 20 t - 5 t^2, {t, 0.8, 4}, AxesLabel → {t, f[t]},
   BaseStyle → 14, GridLines → Automatic, GridLinesStyle → LightGray];
p = Show[p2, p1, AxesOrigin → {0, 0}, PlotRange → All];
```

Figure 2.6: Code used for the above plot

2.3.12 Additional problem 2

Geometrically, what do subspaces of $\mathbb{R}^2$ look like?

Solution

A Subspace of $\mathbb{R}^2$ is all straight lines that pass through the origin. So each straight lines that pass through the origin is a subspace. This shows there are infinite number of subspaces.

Another subspace of $\mathbb{R}^2$ is just the origin $\vec{0}$. And $\mathbb{R}^2$ itself is subspace of itself.

2.3.13 Additional problem 3

Let A be an $n \times n$ matrix and consider the linear system $A\vec{x} = \vec{b}$. If I know that the solution set to this linear system is a subspace of $\mathbb{R}^n$, what can you say about $\vec{b}$?

Solution

The vector $\vec{b}$ is the zero vector. This is by theorem 2, page 226 in the textbook.

2.3.14 key solution for HW 3

Homework 3 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

3.7.3 We need to interpolate $(0,3),(1,1),(2,-5)$. Since we have 3 points, we use a degree 2 polynomial $f(x)=a_0+a_1x+a_2x^2$. This gives the linear system

$$\begin{aligned} a_0 &= 3 \\ a_0+a_1+a_2 &= 1 \\ a_0+2a_1+4a_2 &= -5 \end{aligned}$$

The first equation says $a_0=3$, so substituting we get the 2×2 system

$$\begin{aligned} a_1+a_2 &= -2 \\ 2a_1+4a_2 &= -8 \end{aligned}$$

We solve by row reduction:

$$\begin{bmatrix} 1 & 1 & -2 \\ 2 & 4 & -8 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & -4 \end{bmatrix}$$

So $a_2=-2$ and thus $a_1=0$. So our polynomial is $f(x)=3-2x^2$.

4.1.1 We are given $\vec{a}=(2,5,-4)$ and $\vec{b}=(1,-2,-3)$. We calculate

$$\begin{aligned} \left|\vec{a}-\vec{b}\right| &= |(1,7,-1)| \\ &= \sqrt{1+49+1} \\ &= \sqrt{51} \\ 2\vec{a}+\vec{b} &= (4,10,-8)+(1,-2,-3) \\ &= (5,8,-11) \\ 3\vec{a}-4\vec{b} &= (6,15-12)-(4,-8,-12) \\ &= (2,23,0) \end{aligned}$$

4.1.19 We are given $\vec{u}=(2,0,1),\vec{v}=(-3,1,-1),\vec{w}=(0,-2,-1)$. We are asked to use row

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reduction in this case.

$$\begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{bmatrix} \xrightarrow{-2R_3+R_1} \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{bmatrix}$$
$$\xrightarrow{R_2+R_1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{bmatrix}$$
$$\xrightarrow{R_2+R_1} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

We have a free variable, so these vectors are linearly dependent. To find a particular linear combination, we choose a value for c, say $c = 2$. Then we solve to get $b = 4, a = 6$. So we have $6\vec{u} + 4\vec{v} + 2\vec{w} = \vec{0}$.

4.1.23 We are given $\vec{u} = (2, 0, 3), \vec{v} = (5, 4, -2), \vec{w} = (2, -1, 1)$. We do the row reduction:

$$\begin{bmatrix} 2 & 5 & 2 \\ 0 & 4 & -1 \\ 3 & -2 & 1 \end{bmatrix} \xrightarrow[2R_3]{3R_1} \begin{bmatrix} 6 & 15 & 6 \\ 0 & 4 & -1 \\ 6 & -4 & 2 \end{bmatrix}$$
$$\xrightarrow{-R_1+R_3} \begin{bmatrix} 6 & 15 & 6 \\ 0 & 4 & -1 \\ 0 & -19 & -4 \end{bmatrix}$$
$$\xrightarrow{4R_2+R_3} \begin{bmatrix} 6 & 15 & 6 \\ 0 & 4 & -1 \\ 0 & -3 & 0 \end{bmatrix}$$
$$\xrightarrow{R_3+R_2} \begin{bmatrix} 6 & 15 & 6 \\ 0 & 1 & -1 \\ 0 & -3 & 0 \end{bmatrix}$$
$$\xrightarrow{3R_2+R_3} \begin{bmatrix} 6 & 15 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{bmatrix}$$

We have an echelon form matrix with a leading entry in every row, so the homogeneous system has only the trivial solution. Hence the vectors are linearly independent.

4.1.27 We want to write $\vec{t} = a\vec{u} + b\vec{v} + c\vec{w}$, so we set up and reduce an augmented matrix:

$$\begin{bmatrix} 1 & -1 & 4 & 0 \\ 4 & -2 & 4 & 0 \\ 3 & 2 & 1 & 19 \end{bmatrix} \xrightarrow[-3R_1+R_3]{-4R_1+R_2} \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 2 & -12 & 0 \\ 0 & 5 & -11 & 19 \end{bmatrix}$$
$$\xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 2 & -12 & 0 \\ 0 & 1 & 13 & 19 \end{bmatrix}$$
$$\xrightarrow{-2R_3+R_2} \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 0 & -38 & -38 \\ 0 & 1 & 13 & 19 \end{bmatrix}$$
$$\xrightarrow[\frac{-1}{38}R_3]{(R_2,R_3)} \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & 13 & 19 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Now back substitution gives us $c = 1$, $b = 6$, and $a = 2$. So $\vec{t} = 2\vec{u} + 6\vec{v} + \vec{w}$.

4.2.2 W is a subspace. To see why, suppose that both $\vec{x}$ and $\vec{y}$ are in W. Then for any scalar c, $c\vec{x} = (cx_1, cx_2, cx_3)$. Since we know $x_1 = 5x_2$, we have $cx_1 = 5(cx_2)$. So $c\vec{x}$ is in W. Also, $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$. We know $x_1 = 5x_2$ and $y_1 = 5y_2$, so $x_1 + y_1 = 5(x_2 + y_2)$. Hence $\vec{x} + \vec{y}$ is in W. This shows closure under scalar multiplication and under addition.

4.2.4 W is not a subspace. It fails everything pretty badly, but an easy way to see it is not a subspace is that it does not contain $\vec{0}$ since $0 + 0 + 0 \neq 1$.

4.2.17 *There is a typo in this problem in the book. The second equation is meant to have* x_4 *in place of* x_5 *and I have solved that version. If you solved it correctly as written, you received full points as well.*

We do row reduction to our system:

$$\begin{bmatrix} 1 & 3 & 8 & -1 \\ 1 & -3 & -10 & 5 \\ 1 & 4 & 11 & -2 \end{bmatrix} \xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 3 & 8 & -1 \\ 0 & -6 & -18 & 6 \\ 0 & 1 & 3 & -1 \end{bmatrix}$$
$$\xrightarrow[R_2+R_3]{\frac{1}{6}R_2} \begin{bmatrix} 1 & 3 & 8 & -1 \\ 0 & -1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\xrightarrow[-R_2]{3R_2+R_1} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have two free variables. We set $x_3 = s$ and $x_4 = t$ and then use back substitution

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to find $x_2 = -3s + t$ and $x_1 = s - 2t$. So our solution vectors look like

$$\vec{x} = \begin{bmatrix} s-2t \\ -3s+t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

4.2.21 We do row reduction to our system:

$$\begin{bmatrix} 1 & 7 & 2 & -3 \\ 2 & 7 & 1 & -4 \\ 3 & 5 & -1 & -5 \end{bmatrix} \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 7 & 2 & -3 \\ 0 & -7 & -3 & 2 \\ 0 & -16 & -7 & 4 \end{bmatrix}$$
$$\xrightarrow[-2R_2+R_3]{R_2+R_1} \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & -7 & -3 & 2 \\ 0 & -2 & -1 & 0 \end{bmatrix}$$
$$\xrightarrow{-4R_3+R_2} \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -1 & 0 \end{bmatrix}$$
$$\xrightarrow{2R_2+R_3} \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$
$$\xrightarrow[-R_3+R_2]{R_3+R_1} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

We have one free variable, so we set $x_4 = t$. Back substitution gives us $x_3 = -4t, x_2 = 2t, x_1 = -3t$. So a typical solution looks like

$$\vec{x} = \begin{bmatrix} -3t \\ 2t \\ -4t \\ t \end{bmatrix} = t\begin{bmatrix} -3 \\ 2 \\ -4 \\ 1 \end{bmatrix}$$

Additional Problems:

1. We need to interpolate the points $(1, 10), (2, 15), (3, 10)$ with a quadratic $f(t) = a+bt+ct^2$. This sets up the linear system

$$\begin{aligned} a + b + c &= 10 \\ a + 2b + 4c &= 15 \\ a + 3b + 9c &= 10 \end{aligned}$$

The polynomial we get after solving is $f(t) = -5 + 20t - 5t^2$.

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It is a *very bad* idea to use this polynomial to predict the price on day 4. For one thing, $f(4) = -5$ and a price of -\$5 is absurd. It also doesn't make very much sense that stock prices would behave like a parabola, where the value either always increases after a certain time or always decreases after a certain time. I'm not an enconomist, but I imagine that most stock prices would go up for a while, then down for a while, then up for a while, then down for a while, and so on. Using this polynomial to determine your investing strategy would be a great way to lose all your money.

2. Proper subspaces of $\mathbb{R}^2$ look like lines through $(0,0)$. There is also the subspace that is all of $\mathbb{R}^2$ and the subspace that is just $\vec{0}$.

3. The solution set to $A\vec{x} = \vec{b}$ is a subspace of $\mathbb{R}^n$ if and only if $\vec{b} = \vec{0}$. On the one hand, we know that the solution set for a homogeneous linear system is always a subspace. On the other hand, if the solutions to $A\vec{x} = \vec{b}$ forms a subspace, then for any solution $\vec{x}_0$ we know by closure under scalar multiplication that $2\vec{x}_0$ is also a solution. So $\vec{b} = A(2\vec{x}_0) = 2A\vec{x}_0 = 2\vec{b}$, which only works when $\vec{b} = \vec{0}$.

2.4 HW 4

Local contents

2.4.1 Problems listing

Homework 4 - Due October 8

Homework instructions: Complete the assigned problems on your own paper. Once you are finished, scan or photograph your work and upload it to Gradescope. When prompted, tell Gradescope where to find each problem.
You are allowed (and in fact encouraged) to work with other students on homework assignments. If you do that, please indicate on each problem who you worked with. If you use sources other than your notes, the textbook, and any resources on Canvas for your homework, you must indicate the source on each problem. You are not permitted to view, request, or look for solutions to any of the homework problems from solutions manuals, homework help websites, online forums, other students, or any other sources.

Textbook Problems:

- §4.3: 9, 17, 18
- §4.4: 6, 16, 20
- §4.5: 5, 7, 15

Additional Problems:

1. Let $\vec{v}_1$ and $\vec{v}_2$ be any linearly independent vectors. Show that $\vec{u}_1 = 2\vec{v}_1$ and $\vec{u}_2 = \vec{v}_1 + \vec{v}_2$ are also linearly independent.
2. In section 4.2, we looked at the set W consisting of all vectors in $\mathbb{R}^3$ where $x_1 = 5x_2$ and determined it was a subspace of $\mathbb{R}^3$. Find a basis for W. What is the dimension of W?
3. Let $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a set of linearly independent vectors and suppose that $\vec{v}$ is not an element of $\operatorname{span} S$. Show that $S' = \{\vec{v}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

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2.4.2 Problem 9 section 4.3

In Problems 9–16, express the indicated vector $\vec{w}$ as a linear combination of the given vectors $\vec{v}_1; \vec{v}_2 \cdots \vec{v}_k$ if this is possible. If not, show that it is impossible

$$\vec{w} = \begin{bmatrix} 1 \\ 0 \\ -7 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

solution

Let $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$. In matrix form this becomes

$$c_1 \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -7 \end{bmatrix}$$

Augmented matrix is

$$\begin{bmatrix} 5 & 3 & 1 \\ 3 & 2 & 0 \\ 4 & 5 & -7 \end{bmatrix}$$

$R_1 \to 3R_1$ and $R_2 \to 5R_2$ gives

$$\begin{bmatrix} 15 & 9 & 3 \\ 15 & 10 & 0 \\ 4 & 5 & -7 \end{bmatrix}$$

$R_2 \to -R_1 + R_2$ gives

$$\begin{bmatrix} 15 & 9 & 3 \\ 0 & 1 & -3 \\ 4 & 5 & -7 \end{bmatrix}$$

$R_1 \to 4R_1$ and $R_3 \to 15R_3$ gives

$$\begin{bmatrix} 60 & 36 & 12 \\ 0 & 1 & -3 \\ 60 & 75 & -105 \end{bmatrix}$$

$R_3 \to -R_1 + R_3$ gives

$$\begin{bmatrix} 60 & 36 & 12 \\ 0 & 1 & -3 \\ 0 & 39 & -117 \end{bmatrix}$$

$R_3 \to -39R_2 + R_3$ gives

$$\begin{bmatrix} 60 & 36 & 12 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence the system becomes

$$\begin{bmatrix} 60 & 36 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -3 \\ 0 \end{bmatrix}$$

From second row $c_2 = -3$ and from first row $60c_1 + 36(c_2) = 12$ or $c_1 = \frac{12-36(-3)}{60} = 2$. Hence

$$\vec{w} = 2\vec{v}_1 - 3\vec{v}_2$$

$\vec{w}$ is linear combination.

2.4.3 Problem 17 section 4.3

In Problems 17–22, three vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are given. If they are linearly independent, show this; otherwise find a nontrivial linear combination of them that is equal to the zero vector.

$$\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}2\\-3\\4\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}3\\5\\2\end{bmatrix}$$

solution

The vectors are Linearly independent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

only when $c_1 = c_2 = c_3 = 0$. If we can find at least one c_i where the above is true, then the vectors are Linearly dependent.

Writing the above as $A\vec{c} = \vec{0}$ gives

$$\begin{bmatrix}1 & 2 & 3\\0 & -3 & 5\\1 & 4 & 2\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix}1 & 2 & 3 & 0\\0 & -3 & 5 & 0\\1 & 4 & 2 & 0\end{bmatrix}$$

$R_3 \to -R_1 + R_3$ gives

$$\begin{bmatrix}1 & 2 & 3 & 0\\0 & -3 & 5 & 0\\0 & 2 & -1 & 0\end{bmatrix}$$

$R_3 \to R_3, R_2 \to 2R_2$ gives

$$\begin{bmatrix}1 & 2 & 3 & 0\\0 & -6 & 10 & 0\\0 & 6 & -3 & 0\end{bmatrix}$$

$R_3 \to R_2 + R_3$ gives

$$\begin{bmatrix}1 & 2 & 3 & 0\\0 & -6 & 10 & 0\\0 & 0 & 7 & 0\end{bmatrix}$$

Hence the original system (1) in Echelon form becomes

$$\begin{bmatrix}1 & 2 & 3\\0 & -6 & 10\\0 & 0 & 7\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Leading variables are c_1, c_2, c_3. Since there are no free variables, then only the trivial solution exist. We see this by backsubstitution. Last row gives $c_3 = 0$. Second row gives $c_2 = 0$ and first row gives $c_1 = 0$.

Since all $c_i = 0$, then the vectors are Linearly independent.

2.4.4 Problem 18 section 4.3

In Problems 17–22, three vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are given. If they are linearly independent, show this; otherwise find a nontrivial linear combination of them that is equal to the zero vector.

$$\vec{v}_1 = \begin{bmatrix}2\\0\\-3\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}4\\-5\\-6\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}-2\\1\\3\end{bmatrix}$$

solution

The vectors are Linearly independent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

only when $c_1 = c_2 = c_3 = 0$. If we can find at least one c_i where the above is true, then the vectors are Linearly dependent.

Writing the above as $A\vec{c} = \vec{0}$ gives

$$\begin{bmatrix} 2 & 4 & -2 \\ 0 & -5 & 1 \\ -3 & -6 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 0 & -5 & 1 & 0 \\ -3 & -6 & 3 & 0 \end{bmatrix}$$

$R_1 \to 3R_1, R_3 \to 2R_3$ gives

$$\begin{bmatrix} 6 & 12 & -6 & 0 \\ 0 & -5 & 1 & 0 \\ -6 & -12 & 6 & 0 \end{bmatrix}$$

$R_3 \to R_1 + R_3$ gives

$$\begin{bmatrix} 6 & 12 & -6 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the system (1) becomes

$$\begin{bmatrix} 6 & 12 & -6 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The leading variables are c_1, c_2 and free variable is c_3. Since there is a free variable, then the vectors are Linearly dependent. To see this, let $c_3 = t$. From second row $-5c_2 + t = 0$ or $c_2 = \frac{1}{5}t$. From first row $6c_1 + 12c_2 - 6t = 0$. Or $c_1 = \frac{6t - 12\left(\frac{1}{5}t\right)}{6} = \frac{3}{5}t$. Hence

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{5}t \\ \frac{1}{5}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix} = \frac{1}{5}t \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

Taking $\tilde{t} = 5$ the above becomes

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

Therefore we found one solution where

$$\begin{aligned} c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 &= \vec{0} \\ 3\vec{v}_1 + \vec{v}_2 + 5\vec{v}_3 &= \vec{0} \end{aligned}$$

not all c_i zero. Hence linearly dependent vectors.

2.4.5 Problem 6 section 4.4

In Problems 1–8, determine whether or not the given vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$

$$\vec{v}_1 = \begin{bmatrix}0\\0\\1\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}0\\1\\2\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

solution

If the vectors are Linearly independent, then they form basis. To check, we solve $A\vec{c} = \vec{0}$ and see if the solution is the trivial solution or not. If the solution is the trivial solution, then the vectors are linearly independent and hence form basis.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

Writing the above as $A\vec{c} = \vec{0}$ gives

$$\begin{bmatrix}0&0&1\\0&1&2\\1&2&3\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix}0&0&1&0\\0&1&2&0\\1&2&3&0\end{bmatrix}$$

Since the pivot (1,1) is pivot, we replace R_1 with R_3 first.

$$\begin{bmatrix}1&2&3&0\\0&1&2&0\\0&0&1&0\end{bmatrix}$$

This is in Echelon form. No free variables. Therefore, the solution is the trivial solution. Eq (1) becomes

$$\begin{bmatrix}1&2&3\\0&1&2\\0&0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Which shows that $c_1 = 0, c_2 = 0, c_3 = 0$. Hence the vectors form a basis for $\mathbb{R}^3$

2.4.6 Problem 16 section 4.4

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$\begin{aligned}x_1 + 3x_2 + 4x_3 &= 0\\3x_1 + 8x_2 + 7x_3 &= 0\end{aligned}$$

solution

$A\vec{x} = \vec{0}$ gives

$$\begin{bmatrix}1&3&4\\3&8&7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix}1&3&4&0\\3&8&7&0\end{bmatrix}$$

$R_2 \to -3R_1 + R_2$ gives

$$\begin{bmatrix}1&3&4&0\\0&-1&-5&0\end{bmatrix}$$

Hence the leading variables are x_1, x_2 and the free variable is $x_3 = t$. The system becomes

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Last row gives $-x_2 - 5x_3 = 0$ or $-x_2 = 5t$. Hence $x_2 = -5t$. From first row, $x_1 + 3x_2 + 4x_3 = 0$, or $x_1 = -3x_2 - 4x_3$ or $x_1 = -3(-5t) - 4t = 11t$. Therefore the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11t \\ -5t \\ t \end{bmatrix} = t \begin{bmatrix} 11 \\ -5 \\ 1 \end{bmatrix}$$

Let $t = 1$. The basis is

$$\begin{bmatrix} 11 \\ -5 \\ 1 \end{bmatrix}$$

A one dimensional subspace.

2.4.7 Problem 20 section 4.4

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$\begin{aligned} x_1 - 3x_2 - 10x_3 + 5x_4 &= 0 \\ x_1 + 4x_2 + 11x_3 - 2x_4 &= 0 \\ x_1 + 3x_2 + 8x_3 - x_4 &= 0 \end{aligned}$$

solution

$A\vec{x} = \vec{0}$ gives

$$\begin{bmatrix} 1 & -3 & -10 & 5 \\ 1 & 4 & 11 & -2 \\ 1 & 3 & 8 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -3 & -10 & 5 & 0 \\ 1 & 4 & 11 & -2 & 0 \\ 1 & 3 & 8 & -1 & 0 \end{bmatrix}$$

$R_2 \to -R_1 + R_2$ gives

$$\begin{bmatrix} 1 & -3 & -10 & 5 & 0 \\ 0 & 7 & 21 & -7 & 0 \\ 1 & 3 & 8 & -1 & 0 \end{bmatrix}$$

$R_3 \to -R_1 + R_3$ gives

$$\begin{bmatrix} 1 & -3 & -10 & 5 & 0 \\ 0 & 7 & 21 & -7 & 0 \\ 0 & 6 & 18 & -6 & 0 \end{bmatrix}$$

$R_3 \to 7R_3$ and $R_2 \to 6R_2$ gives

$$\begin{bmatrix} 1 & -3 & -10 & 5 & 0 \\ 0 & 42 & 126 & -42 & 0 \\ 0 & 42 & 126 & -42 & 0 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$ gives

$$\begin{bmatrix} 1 & -3 & -10 & 5 & 0 \\ 0 & 42 & 126 & -42 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading variables are x_1, x_2 Free variables are $x_3 = t, x_4 = s$. The system becomes

$$\begin{bmatrix}1 & -3 & -10 & 5\\0 & 42 & 126 & -42\\0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

second row gives $42x_2 + 126x_3 - 42x_4 = 0$ or $42x_2 = -126t + 42s$ or $x_2 = -\frac{126}{42}t + \frac{42}{42}s = -3t + s$.
First row gives $x_1 - 3x_2 - 10x_3 + 5x_4 = 0$ or $x_1 = 3x_2 + 10x_3 - 5x_4$ or $x_1 = 3(-3t+s) + 10t - 5s = t - 2s$.
Hence the solution is

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}t-2s\\-3t+s\\t\\s\end{bmatrix} = t\begin{bmatrix}1\\-3\\1\\0\end{bmatrix} + s\begin{bmatrix}-2\\1\\0\\1\end{bmatrix}$$

Let $t = 1, s = 1$. The basis are

$$\begin{bmatrix}1\\-3\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\1\end{bmatrix}$$

A two dimensional subspace.

2.4.8 Problem 5 section 4.5

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix}1 & 1 & 1 & 1\\3 & 1 & -3 & 4\\2 & 5 & 11 & 12\end{bmatrix}$$

solution

We start by converting the matrix to reduced Echelon form.

$R_2 \to -3R_1 + R_2$ gives

$$\begin{bmatrix}1 & 1 & 1 & 1\\0 & -2 & -6 & 3\\2 & 5 & 11 & 12\end{bmatrix}$$

$R_3 \to -2R_1 + R_3$ gives

$$\begin{bmatrix}1 & 1 & 1 & 1\\0 & -2 & -6 & 3\\0 & 3 & 9 & 10\end{bmatrix}$$

$R_2 \to 3R_2$ and $R_3 \to 2R_3$ gives

$$\begin{bmatrix}1 & 1 & 1 & 1\\0 & -6 & -18 & 9\\0 & 6 & 18 & 20\end{bmatrix}$$

$R_3 \to R_2 + R_3$ gives

$$\begin{bmatrix}1 & 1 & 1 & 1\\0 & -6 & -18 & 9\\0 & 0 & 0 & 29\end{bmatrix}$$

Now to start the reduce Echelon form phase. The pivots all needs to be 1.

$R_2 \to \frac{-1}{6}R_2$ and $R_3 \to \frac{1}{29}R_3$ gives

$$\begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 3 & \frac{3}{2}\\0 & 0 & 0 & 1\end{bmatrix}$$

Now we need to zero all elements above each pivot.

$R_2 \to R_2 - \frac{3}{2}R_2$ gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_3$ gives

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$ gives

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The above is now in reduced Echelon form. Now we can answer the question. The basis for the row space are all the rows which are not zero. Hence row space basis are (I prefer to show all basis as column vectors, instead of row vectors. This just makes it easier to read them).

$$\begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The dimension is 3. The column space correspond to pivot columns in original A. These are column $1, 2, 4$. Hence basis for column space are

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 12 \end{bmatrix}$$

The dimension is 3. We notice that the dimension of the row space and the column space is equal as expected. (This is called the rank of A. Hence $\text{rank}(A) = 3$.)

The Null space of A has dimension 1, since there is only one free variable (x_3). We see that the number of columns of A (which is 4) is therefore the sum of column space dimension (or the rank) and the null space dimension as expected.

2.4.9 Problem 7 section 4.5

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 1 & 4 & 5 & 16 \\ 1 & 3 & 3 & 13 \\ 2 & 5 & 4 & 23 \end{bmatrix}$$

solution

We start by converting the matrix to reduced Echelon form.

$R_2 \to -R_1 + R_2$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 3 & 6 & 9 \\ 1 & 3 & 3 & 13 \\ 2 & 5 & 4 & 23 \end{bmatrix}$$

$R_3 \to -R_1 + R_3$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 3 & 6 & 9 \\ 0 & 2 & 4 & 6 \\ 2 & 5 & 4 & 23 \end{bmatrix}$$

$R_4 \to -2R_1 + R_4$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 3 & 6 & 9 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix}$$

$R_2 \to 2R_2$ and $R_3 \to 3R_3$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 6 & 12 & 18 \\ 0 & 6 & 12 & 18 \\ 0 & 3 & 6 & 9 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 6 & 12 & 18 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 9 \end{bmatrix}$$

$R_4 \to -\frac{1}{2}R_2 + R_4$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 6 & 12 & 18 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot (leading) columns are $1,2$ and free variables go with $3,4$ columns. The Null space of A is therefore have dimension 2. We now convert it to reduced Echelon form.

$R_2 \to \frac{1}{6}R_2$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - R_2$ gives

$$\begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is reduced Echelon form. The basis for the row space are all the rows which are not zero. Hence row space basis are (dimension 2)

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

The column space correspond to pivot columns in original A. These are columns $1,2$. Hence basis for column space are (dimension 2)

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \\ 5 \end{bmatrix} \right\}$$

We notice that the dimension of the row space and the column space is equal as expected.

The Null space of A has dimension 2, since there is two free variables. We see that the number of columns of A (which is 4) is therefore the sum of column space dimension and the null space dimension as expected.

2.4.10 Problem 15 section 4.5

In Problems 13–16, a set S of vectors in $\mathbb{R}^4$ is given. Find a subset of S that forms a basis for the subspace of $\mathbb{R}^4$ spanned by S

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

solution

We set up a matrix made of the above vectors, then find the dimensions of the column space.

$$\begin{bmatrix} 3 & 2 & 4 & 1 \\ 2 & 1 & 3 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 1 & 3 & 4 \end{bmatrix}$$

$R_1 \to 2R_1$ and $R_2 \to 3R_2$ and $R_3 \to 2R_3$ and $R_4 \to 3R_4$. This gives

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 6 & 3 & 9 & 6 \\ 6 & 6 & 6 & 9 \\ 6 & 3 & 9 & 12 \end{bmatrix}$$

$R_2 \to -R_1 + R_2$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 6 & 6 & 6 & 9 \\ 6 & 3 & 9 & 12 \end{bmatrix}$$

$R_3 \to -R_1 + R_3$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 2 & -2 & 7 \\ 6 & 3 & 9 & 12 \end{bmatrix}$$

$R_4 \to -R_1 + R_4$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 2 & -2 & 7 \\ 0 & -1 & 1 & 10 \end{bmatrix}$$

$R_3 \to 2R_2 + R_3$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 15 \\ 0 & -1 & 1 & 10 \end{bmatrix}$$

$R_4 \to -R_2 + R_4$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$R_4 \to 15R_4$ and $R_3 \to 6R_3$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 90 \\ 0 & 0 & 0 & 90 \end{bmatrix}$$

$R_4 \to R_3 + R_4$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 90 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the pivot columns are $1, 2, 4$. Therefore the column space basis are $\vec{v}_1, \vec{v}_2, \vec{v}_4$ given by

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

The above is the subset required.

2.4.11 Additional problem 1

Let $\vec{v}_1$ and $\vec{v}_2$ be any linearly independent vectors. Show that $\vec{u}_1 = 2\vec{v}_1$ and $\vec{u}_2 = \vec{v}_1 + \vec{v}_2$ are also linearly independent.

solution

We want to solve for c_1, c_2 from

$$c_1\vec{u}_1 + c_2\vec{u}_2 = \vec{0} \tag{1}$$

And see if the solution is only the trivial solution or not. The above becomes

$$\begin{aligned} c_1\left(2\vec{v}_1\right) + c_2\left(\vec{v}_1 + \vec{v}_2\right) &= \vec{0} \\ 2c_1\vec{v}_1 + c_2\vec{v}_1 + c_2\vec{v}_2 &= \vec{0} \\ (2c_1 + c_2)\vec{v}_1 + c_2\vec{v}_2 &= \vec{0} \end{aligned}$$

Let $2c_1 + c_2 = c_3$ a new constant. The above becomes

$$c_3\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$$

But we are told that $\vec{v}_1$ and $\vec{v}_2$ be any linearly independent. Therefore only choice for the above is that $c_2 = 0, c_3 = 0$. But $c_3 = 2c_1 + c_2$ which means that $c_1 = 0$. Therefore we just showed that $c_1 = c_2 = 0$ is only solution to (1). This implies that $\vec{u}_1, \vec{u}_2$ are linearly independent vectors.

2.4.12 Additional problem 2

In section 4.2, we looked at the set W consisting of all vectors in $\mathbb{R}^3$ where $x_1 = 5x_2$ and determined it was a subspace of $\mathbb{R}^3$. Find a basis for W. What is the dimension of W?

solution

Let $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Let $x_2 = t, x_3 = s$. Therefore

$$\begin{aligned} \vec{v} &= \begin{bmatrix} 5t \\ t \\ s \end{bmatrix} \\ &= t\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Hence basis for W are

$$\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

And the dimension of W is 2.

2.4.13 Additional problem 3

Let $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a set of linearly independent vectors and suppose that $\vec{v}$ is not an element of span S. Show that $S' = \{\vec{v}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

solution

Proof by contradiction. Assuming the vectors $\vec{v}, \vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent. Therefore we can find constants c_1, c_2, c_3, c_4 not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v} = \vec{0}$$

Or

$$-\frac{c_1}{c_4}\vec{v}_1 - \frac{c_1}{c_4}\vec{v}_2 - \frac{c_1}{c_4}\vec{v}_3 = \vec{v}$$

Renaming the constants gives

$$C_1\vec{v}_1 + C_2\vec{v}_2 + C_3\vec{v}_3 = \vec{v} \tag{1}$$

The above says, we can represent $\vec{v}$ as linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$. But $\vec{v}$ is not in the span of S, which means we can not reach $\vec{v}$ using any linear combination of the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Hence (1) is not possible.

Therefore our assumption that the vectors are linearly dependent is invalid. Hence they must be linearly independent.

2.4.14 key solution for HW 4

Homework 4 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

4.3.9 We need to write $(1, 0, -7)$ as a linear combination of $(5, 3, 4)$ and $(3, 2, 5)$. We set up the augmented matrix and row reduce:

$$\begin{bmatrix} 5 & 3 & 1 \\ 3 & 2 & 0 \\ 4 & 5 & -7 \end{bmatrix} \xrightarrow{-R_3+R_1} \begin{bmatrix} 1 & -2 & 8 \\ 3 & 2 & 0 \\ 4 & 5 & -7 \end{bmatrix}$$
$$\xrightarrow[-4R_1+R_3]{-3R_1+R_2} \begin{bmatrix} 1 & -2 & 8 \\ 0 & 8 & -24 \\ 0 & 13 & -39 \end{bmatrix}$$
$$\xrightarrow[\frac{1}{13}R_3]{\frac{1}{8}R_2} \begin{bmatrix} 1 & -2 & 8 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix}$$
$$\xrightarrow[2R_2+R_1]{-R_2+R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

This system has the unique solution $c_2 = -3$ and $c_1 = 2$, so

$$(1, 0, -7) = 2(5, 3, 4) - 3(3, 2, 5)$$

4.3.17 We determine linear independence by row reduction:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & 5 \\ 1 & 4 & 2 \end{bmatrix} \xrightarrow{-R_1+R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & 5 \\ 0 & 2 & -1 \end{bmatrix}$$
$$\xrightarrow{2R_3+R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 2 & -1 \end{bmatrix}$$
$$\xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & -7 \end{bmatrix}$$

Since we have leading entries in all three columns, the homogeneous system has a unique solution and thus the vectors are linearly independent.

1

4.3.18 We determine linear independence by row reduction:

$$\begin{bmatrix} 2 & 4 & -2 \\ 0 & -5 & 1 \\ -3 & -6 & 3 \end{bmatrix} \xrightarrow{\frac{3}{2}R_1+R_3} \begin{bmatrix} 2 & 4 & -2 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have c_3 as a free variable. With foresight to prevent fractions, we set $c_3 = 5$. Back substitution gives $c_2 = 1$ and $c_1 = 3$. These vectors are linearly dependent, and we have a nontrivial linear combination equalling zero:

$$3(2, 0, -3) + (4, -5, -6) + 5(-2, 1, 3) = (0, 0, 0)$$

4.4.6 We have 3 vectors in $\mathbb{R}^3$, so it suffices to compute a determinant:

$$\begin{aligned} \det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} &= \det \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \\ &= -1 \end{aligned}$$

This determinant is nonzero, so the vectors form a basis.

4.4.16 We do row reduction to our system:

$$\begin{aligned} \begin{bmatrix} 1 & 3 & 4 \\ 3 & 8 & 7 \end{bmatrix} &\xrightarrow{-3R_1+R_2} \begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & -5 \end{bmatrix} \\ &\xrightarrow[-R_2]{3R_2+R_1} \begin{bmatrix} 1 & 0 & -11 \\ 0 & 1 & 5 \end{bmatrix} \end{aligned}$$

We have a free variable $x_3 = t$ and we solve to get $x_2 = -5t$, $x_1 = 11t$. So a typical solution looks like $\vec{x} = t(11, -5, 1)$ and thus a basis for the solution space is $\{(11, -5, 1)\}$.

4.4.20 We do row reduction to our system:

$$\begin{aligned} \begin{bmatrix} 1 & -3 & -10 & 5 \\ 1 & 4 & 11 & -2 \\ 1 & 3 & 8 & -1 \end{bmatrix} &\xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & -3 & -10 & 5 \\ 0 & 7 & 21 & -7 \\ 0 & 6 & 18 & -6 \end{bmatrix} \\ &\xrightarrow[\frac{1}{6}R_3]{\frac{1}{7}R_2} \begin{bmatrix} 1 & -3 & -10 & 5 \\ 0 & 1 & 3 & -1 \\ 0 & 1 & 3 & -1 \end{bmatrix} \\ &\xrightarrow[3R_2+R_1]{-R_2+R_3} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

We have free variables $x_3 = s$, $x_4 = t$ and we solve to get $x_2 = -3s + t$, $x_1 = s - 2t$. A typical solution looks like $\vec{x} = s(1, -3, 1, 0) + t(-2, 1, 0, 1)$ so our basis for the solution space is $\{(1, -3, 1, 0), (-2, 1, 0, 1)\}$.

4.5.5 We reduce the given matrix into echelon form (steps omitted):

$$\begin{bmatrix}1 & 1 & 1 & 1\\ 3 & 1 & -3 & 4\\ 2 & 5 & 11 & 12\end{bmatrix} \to \begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 3 & 11\\ 0 & 0 & 0 & 1\end{bmatrix}$$

A basis for the row space is the nonzero rows of the echelon matrix: $\{(1,1,1,1),(0,1,3,11),(0,0,0,1)\}$.

The pivot columns of the echelon matrix are 1, 2, and 4. So a basis for the column space is the corresponding columns of our original matrix: $\{(1,3,2),(1,1,5),(1,4,12)\}$.

4.5.7 We reduce the given matrix into echelon form (steps omitted):

$$\begin{bmatrix}1 & 1 & -1 & 7\\ 1 & 4 & 5 & 16\\ 1 & 3 & 3 & 13\\ 2 & 5 & 4 & 23\end{bmatrix} \to \begin{bmatrix}1 & 1 & -1 & 7\\ 0 & 1 & 2 & 3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

A basis for the row space is the nonzero rows of the echelon matrix: $\{(1,1,-1,7),(0,1,2,3)\}$.

The pivot columns of the echelon matrix are 1 and 2. So a basis for the column space is the corresponding columns of our original matrix: $\{(1,1,1,2),(1,4,3,5)\}$.

4.5.15 To find a subset of S that is a basis for $\operatorname{span} S$, we put the vectors in the columns of a matrix and find a basis for the column space. First, we do row reduction (steps omitted):

$$\begin{bmatrix}3 & 2 & 4 & 1\\ 2 & 1 & 3 & 2\\ 2 & 2 & 2 & 3\\ 2 & 1 & 3 & 4\end{bmatrix} \to \begin{bmatrix}1 & 1 & 1 & -1\\ 0 & -1 & 1 & 4\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

The first, second, and fourth columns are pivot columns, so vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_4$ make up a basis for $\operatorname{span} S$.

Additional Problems:

1. Suppose that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent. To show $\vec{u}_1$ and $\vec{u}_2$ are independent, we set up a homogeneous system:

$$\begin{aligned} c_1\vec{u}_1 + c_2\vec{u}_2 &= \vec{0}\\ c_1(2\vec{v}_1) + c_2(\vec{v}_1 + \vec{v}_2) &= \vec{0}\\ (2c_1 + c_2)\vec{v}_1 + c_2\vec{v}_2 &= \vec{0} \end{aligned}$$

This is a linear combination of the $\vec{v}_i$ equal to the zero vector, so since $\vec{v}_1$ and $\vec{v}_2$ are linearly independent we have that $2c_1 + c_2 = 0$ and $c_2 = 0$. From the first equation, we can conclude $c_1 = 0$ as well so the $\vec{u}_i$ must be linearly independent as well.

3

2. W is the set of solutions to the homogeneous linear equation $x_1 - 5x_2 = 0$. So x_3 and x_2 are free variables and we set $x_2 = s, x_3 = t$. Solving, we get $x_1 = 5s$. So, we have

$$\vec{x} = \begin{bmatrix} 5s \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The basis vectors for W are thus $(5, 1, 0)$ and $(0, 0, 1)$.

3. To show S' is linearly independent, we set up the homogeneous linear system

$$c\vec{v} + c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

If $c \neq 0$, then we can write

$$\vec{v} = -\frac{c_1}{c}\vec{v}_1 - \frac{c_2}{c}\vec{v}_2 - \frac{c_3}{c}\vec{v}_3$$

which would mean that $\vec{v}$ is in the span of S (something we assumed was false). So we must have $c = 0$. Then our system is

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

Since we know S is linearly independent, we can conclude that $c_1 = c_2 = c_3 = 0$. So all constants must be 0 and thus S' is linearly independent.

2.5 HW 5

Local contents

2.5.1 Problems listing

Homework 5 - Due October 15

Homework instructions: Complete the assigned problems on your own paper. Once you are finished, scan or photograph your work and upload it to Gradescope. When prompted, tell Gradescope where to find each problem.
You are allowed (and in fact encouraged) to work with other students on homework assignments. If you do that, please indicate on each problem who you worked with. If you use sources other than your notes, the textbook, and any resources on Canvas for your homework, you must indicate the source on each problem. You are not permitted to view, request, or look for solutions to any of the homework problems from solutions manuals, homework help websites, online forums, other students, or any other sources.

Textbook Problems:

- §4.7: 7, 10
- §1.1: 5, 17
- §5.1: 3, 5, 7, 33, 35, 39

Additional Problems:

1. Let $\mathcal{P}_2$ be the subspace of polynomials of degree at most 2. So elements of $\mathcal{P}_2$ look like $a_0 + a_1x + a_2x^2$. Show that $\{3 + x, 1 + x + x^2, x - 2x^2\}$ is a basis for $\mathcal{P}_2$.
2. Find the general solution to the differential equation $y'' - 25y = 0$. What is the particular solution if I give you initial conditions $y(0) = a$ and $y'(0) = b$?

1

2.5.2 Problem 7 section 4.7

In Problems 5–8, determine whether or not each indicated set of functions is a subspace of the space F of all real-valued functions on $\mathbb{R}$.

The set of all f such that $f(0) = 0$ and $f(1) = 1$

Solution

Let f, g be two functions such that $f(0) = 0, g(0) = 0$ and $f(1) = 1, g(1) = 1$ in F. Let us check if it is closed under addition

$$f(0) + g(0) = 0 + 0 = 0$$

OK.

$$f(1) + g(1) = 1 + 1 = 2 \neq 1$$

Hence not closed under addition. Therefore not a subspace.

2.5.3 Problem 10 section 4.7

In Problems 9–12, a condition on the coefficients of a polynomial $a_0 + a_1x + a_2x^2 + a_3x^3$ is given. Determine whether or not the set of all such polynomials satisfying this condition is a subspace of the space P of all polynomials

$$a_0 = a_1 = 0$$

Solution

Let

$$\begin{aligned} p_1(x) &= a_2x^2 + a_3x^3 \\ p_2(x) &= b_2x^2 + b_3x^3 \end{aligned}$$

Checking if closed under scalar multiplication. Let c be some scalar. Hence

$$\begin{aligned} cp_1(x) &= c\left(a_2x^2 + a_3x^3\right) \\ &= (ca_2)x^2 + (ca_3)x^3 \\ &= A_2x^2 + A_3x^3 \end{aligned}$$

Therefore closed. Now checking if closed under addition.

$$\begin{aligned} p_1(x) + p_2(x) &= a_2x^2 + a_3x^3 + b_2x^2 + b_3x^3 \\ &= (a_2 + b_2)x^2 + (a_3 + b_3)x^3 \\ &= A_2x^2 + A_3x^3 \end{aligned}$$

Therefore Closed under addition. Also the zero polynomial in included when $a_2 = a_3 = 0$.

Therefore this is a subspace.

2.5.4 Problem 5 section 1.1

In Problems 1 through 12, verify by substitution that each given function is a solution of the given differential equation. Throughout these problems, primes denote derivatives with respect to x.

$$\begin{aligned} y' &= y + 2e^{-x} \\ y &= e^x - e^{-x} \end{aligned} \tag{A}$$

Solution

Using the solution given, we see that

$$\begin{aligned} y' &= e^x - (-e^{-x}) \\ &= e^x + e^{-x} \end{aligned} \tag{1}$$

Substituting (1) into EQ. (A) gives

$$\begin{aligned} e^x + e^{-x} &= (e^x - e^{-x}) + 2e^{-x} \\ e^x + e^{-x} &= e^x + e^{-x} \\ 0 &= 0 \end{aligned}$$

Hence the solution given satisfies the ODE.

2.5.5 Problem 17 section 1.1

In Problems 17 through 26, first verify that $y(x)$ satisfies the given differential equation. Then determine a value of the constant C so that $y(x)$ satisfies the given initial condition. Use a computer or graphing calculator (if desired) to sketch several typical solutions of the given differential equation, and highlight the one that satisfies the given initial condition.

$$\begin{aligned} y' + y &= 0 \qquad \text{(A)} \\ y(x) &= Ce^{-x} \\ y(0) &= 2 \end{aligned}$$

Solution

Using the solution given, we see that

$$y' = -Ce^{-x} \qquad (1)$$

Substituting (1) into EQ. (A) gives

$$\begin{aligned} -Ce^{-x} + Ce^{-x} &= 0 \\ 0 &= 0 \end{aligned}$$

Hence the solution gives satisfies the ODE.

When $x = 0$ the solution becomes

$$\begin{aligned} 2 &= Ce^{-(0)} \\ &= C \end{aligned}$$

Hence $C = 2$ and the particular solution becomes

$$y(x) = 2e^{-x}$$

The following are some solutions plots for different C

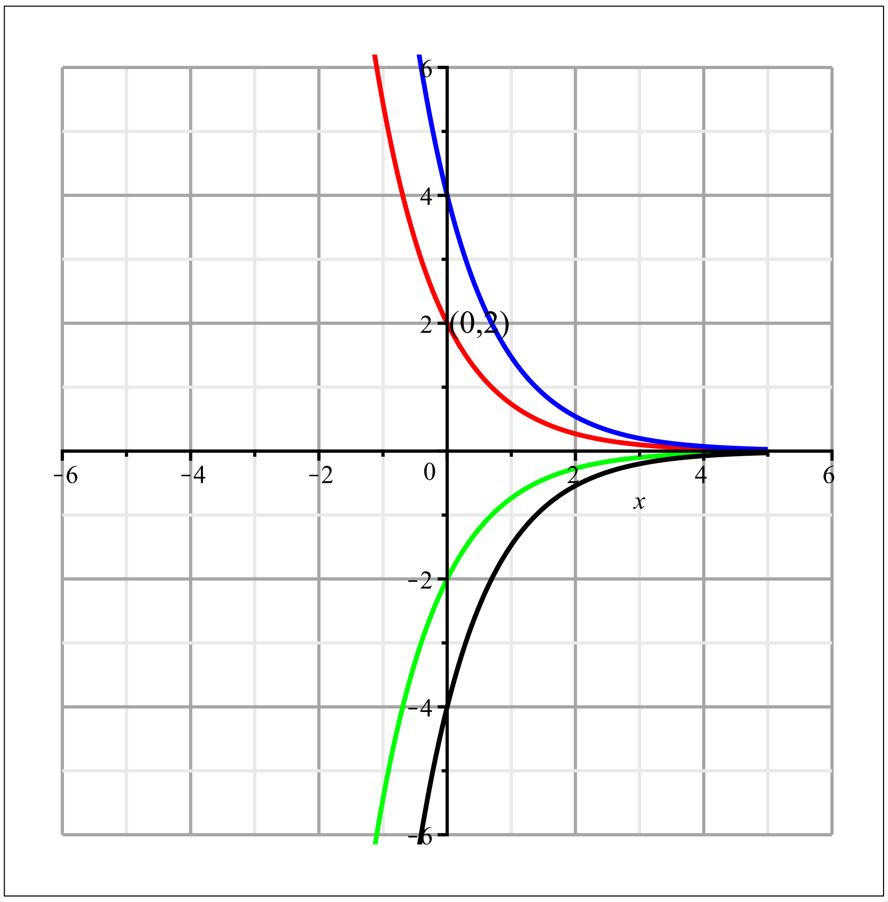

Figure 2.7: Plot of serveral solution with different c. Red solution is one given in problem.

```
restart;
f:=(x,c)->c*exp(-x)
p1:=plot(f(x,2),x=-5..5,gridlines=true,view=[-6..6, -6..6],color=red):
p2:=plot(f(x,4),x=-5..5,gridlines=true,view=[-6..6, -6..6],color=blue):
p3:=plot(f(x,-2),x=-5..5,gridlines=true,view=[-6..6, -6..6],color=green):
p4:=plot(f(x,-4),x=-5..5,gridlines=true,view=[-6..6, -6..6],color=black):
T:=plots:-textplot([[.5,2,"(0,2)"]], font=[times,16],tickmarks=NULL):
plots:-display([p1,p2,p3,p4,T]);
```

2.5.6 Problem 3 section 5.1

A homogeneous second-order linear differential equation, two functions y_1 and y_2, and a pair of initial conditions are given. First verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1y_1 + c_2y_2$ that satisfies the given initial conditions. Primes denote derivatives with respect to x.

$$
\begin{aligned}
y'' + 4y &= 0 \qquad (1)\\
y_1 &= \cos 2x\\
y_2 &= \sin 2x\\
y(0) &= 3\\
y'(0) &= 8
\end{aligned}
$$

Solution

Checking if $y_1(x)$ is a solution. Since

$$
\begin{aligned}
y_1' &= -2\sin 2x \qquad (2)\\
y_1'' &= -4\cos 2x \qquad (3)
\end{aligned}
$$

Substituting the above equations back into (1) gives

$$
\begin{aligned}
(-4\cos 2x) + 4\cos 2x &= 0\\
0 &= 0
\end{aligned}
$$

Hence y_1 is a solution. We do the same for y_2

$$y_2' = 2\cos 2x \tag{4}$$
$$y_2'' = -4\sin 2x \tag{5}$$

Substituting (4,5) back into (1) gives

$$\begin{aligned}(-4\sin 2x) + 4(\sin 2x) &= 0\\ 0 &= 0\end{aligned}$$

Hence y_2 is a solution. Let general solution be

$$\begin{aligned}y(x) &= c_1 y_1(x) + c_2 y_2(x)\\ &= c_1\cos 2x + c_2\sin 2x \end{aligned}\tag{6}$$

Applying the first initial conditions $y(0) = 3$ in (6) gives

$$3 = c_1$$

Hence (6) now becomes

$$y(x) = 3\cos 2x + c_2\sin 2x \tag{7}$$

Taking derivative of the above gives

$$y'(x) = -6\sin 2x + 2c_2\cos 2x$$

Applying the second initial conditions $y'(0) = 8$ in the above gives

$$\begin{aligned}8 &= 2c_2\\ c_2 &= 4\end{aligned}$$

Therefore the general solution (6) becomes

$$\boxed{y(x) = 3\cos 2x + 4\sin 2x} \tag{8}$$

2.5.7 Problem 5 section 5.1

A homogeneous second-order linear differential equation, two functions y_1 and y_2, and a pair of initial conditions are given. First verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1y_1 + c_2y_2$ that satisfies the given initial conditions. Primes denote derivatives with respect to x.

$$y'' - 3y' + 2y = 0 \tag{1}$$
$$\begin{aligned}y_1 &= e^x\\ y_2 &= e^{2x}\\ y(0) &= 1\\ y'(0) &= 0\end{aligned}$$

Solution

Checking if $y_1(x)$ is a solution. Since

$$y_1' = e^x \tag{2}$$
$$y_1'' = e^x \tag{3}$$

Substituting the above equations back into (1) gives

$$\begin{aligned}e^x - 3e^x + 2e^x &= 0\\ 0 &= 0\end{aligned}$$

Hence y_1 is a solution. We do the same for y_2

$$y_2' = 2e^{2x} \tag{4}$$
$$y_2'' = 4e^{2x} \tag{5}$$

Substituting (4,5) back into (1) gives

$$\begin{aligned}\left(4e^{2x}\right) - 3\left(2e^{2x}\right) + 2\left(e^{2x}\right) &= 0\\ 4e^{2x} - 6e^{2x} + 2e^{2x} &= 0\\ 0 &= 0\end{aligned}$$

Hence y_2 is a solution. Let general solution be

$$\begin{aligned}y(x) &= c_1 y_1 + c_2 y_2\\ &= c_1 e^x + c_2 e^{2x} \end{aligned}\tag{6}$$

Applying the first initial conditions $y(0) = 1$ in (6) gives

$$1 = c_1 + c_2 \tag{7}$$

Taking derivative of Eq. (6) gives

$$y'(x) = c_1 e^x + 2c_2 e^{2x}$$

Applying the second initial conditions $y'(0) = 0$ in the above gives

$$0 = c_1 + 2c_2 \tag{8}$$

We have two equations (7,8) to solve for the 2 unknowns c_1, c_2. (7)-(8) gives

$$c_2 = -1$$

Hence from (7) $c_1 = 1 - c_2 = 1 + 1 = 2$. Therefore the solution (6) now becomes

$$y(x) = 2e^x - e^{2x}$$

2.5.8 Problem 7 section 5.1

A homogeneous second-order linear differential equation, two functions y_1 and y_2, and a pair of initial conditions are given. First verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1 y_1 + c_2 y_2$ that satisfies the given initial conditions. Primes denote derivatives with respect to x.

$$\begin{aligned}y'' + y' &= 0 \quad (1)\\ y_1 &= 1\\ y_2 &= e^{-x}\\ y(0) &= -2\\ y'(0) &= 8\end{aligned}$$

Solution

Checking if $y_1(x)$ is a solution. Since

$$y_1' = 0 \tag{2}$$
$$y_1'' = 0 \tag{3}$$

Substituting the above equations back into (1) gives

$$\begin{aligned}0 + 0 &= 0\\ 0 &= 0\end{aligned}$$

Hence y_1 is a solution. We do the same for y_2

$$y_2' = -e^{-x} \tag{4}$$
$$y_2'' = e^{-x} \tag{5}$$

Substituting (4,5) back into (1) gives

$$\begin{aligned}(e^{-x}) - e^{-x} &= 0\\ 0 &= 0\end{aligned}$$

Hence y_2 is a solution. Let general solution be

$$\begin{aligned}y(x) &= c_1 y_1 + c_2 y_2\\ &= c_1 + c_2 e^{-x} \end{aligned} \tag{6}$$

Applying the first initial conditions $y(0) = -2$ in (6) gives

$$-2 = c_1 + c_2 \tag{7}$$

Taking derivative of Eq. (6) gives

$$y'(x) = -c_2 e^{-x}$$

Applying the second initial conditions $y'(0) = 8$ in the above gives

$$\begin{aligned}8 &= -c_2\\ c_2 &= -8 \end{aligned} \tag{8}$$

Hence from (7)

$$\begin{aligned}-2 &= c_1 + c_2\\ &= c_1 - 8\\ c_1 &= 6\end{aligned}$$

Therefore the solution (6) now becomes

$$\begin{aligned}y(x) &= c_1 + c_2 e^{-x}\\ &= 6 - 8e^{-x}\end{aligned}$$

2.5.9 Problem 33 section 5.1

Apply Theorems 5 and 6 to find general solutions of the differential equations given in Problems 33 through 42. Primes denote derivatives with respect to x.

$$y'' - 3y' + 2y = 0$$

Solution

The characteristic equation is

$$\begin{aligned}r^2 - 3r + 2 &= 0\\ (r-1)(r-2) &= 0\end{aligned}$$

Hence the roots are $r_1 = 1, r_2 = 2$. Therefore the general solution is

$$\begin{aligned}y(x) &= Ae^{r_1 x} + Be^{r_2 x}\\ &= Ae^{x} + Be^{2x}\end{aligned}$$

Where A, B are the constants of integrations which are found from initial conditions.

2.5.10 Problem 35 section 5.1

Apply Theorems 5 and 6 to find general solutions of the differential equations given in Problems 33 through 42. Primes denote derivatives with respect to x.

$$y'' + 5y' = 0$$

Solution

The characteristic equation is

$$\begin{aligned} r^2 + 5r &= 0 \\ (r+5)r &= 0 \end{aligned}$$

Hence the roots are $r_1 = 0, r_2 = -5$. Therefore the general solution is

$$\begin{aligned} y(x) &= Ae^{r_1 x} + Be^{r_2 x} \\ &= A + Be^{-5x} \end{aligned}$$

Where A, B are the constants of integrations which are found from initial conditions.

2.5.11 Problem 39 section 5.1

Apply Theorems 5 and 6 to find general solutions of the differential equations given in Problems 33 through 42. Primes denote derivatives with respect to x.

$$4y'' + 4y' + y = 0$$

Solution

The characteristic equation is

$$\begin{aligned} 4r^2 + 4r + 1 &= 0 \\ r^2 + r + \frac{1}{4} &= 0 \\ \left(r + \frac{1}{2}\right)^2 &= 0 \end{aligned}$$

Hence the root is $r = -\frac{1}{2}$. A double root. Therefore the general solution is

$$\begin{aligned} y(x) &= Ae^{rx} + Bxe^{rx} \\ &= Ae^{-\frac{1}{2}x} + Bxe^{-\frac{1}{2}x} \end{aligned}$$

Where A, B are the constants of integrations which are found from initial conditions.

2.5.12 Additional problem 1

Let P_2 be subspace of polynomials of degree at most 2. So elements of P_2 look like $a_0 + a_1 x + a_2 x^2$. Show that $\left\{3 + x, 1 + x + x^2, x - 2x^2\right\}$ is basis for P_2

Solution

Assuming these are basis, then we can write

$$a_0 + a_1 x + a_2 x^2 = c_1(3 + x) + c_2\left(1 + x + x^2\right) + c_3\left(x - 2x^2\right)$$

For constants c_1, c_2, c_3. If we can find unique solution for the c_i then these are basis. The above becomes

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 &= 3c_1 + c_2 + xc_1 + xc_2 + xc_3 + x^2 c_2 - 2x^2 c_3 \\ &= (3c_1 + c_2) + x(c_1 + c_2 + c_3) + x^2(c_2 - 2c_3) \end{aligned}$$

Comparing coefficients gives the equations

$$\begin{aligned} a_0 &= 3c_1 + c_2 \\ a_1 &= c_1 + c_2 + c_3 \\ a_2 &= c_2 - 2c_3 \end{aligned}$$

In Matrix form the above becomes

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Augmented matrix is

$$\begin{bmatrix} 3 & 1 & 0 & a_0 \\ 1 & 1 & 1 & a_1 \\ 0 & 1 & -2 & a_2 \end{bmatrix}$$

Replacing row 2 with row 1 gives

$$\begin{bmatrix} 1 & 1 & 1 & a_1 \\ 3 & 1 & 0 & a_0 \\ 0 & 1 & -2 & a_2 \end{bmatrix}$$

$R_2 \to -3R_1 + R_2$ gives

$$\begin{bmatrix} 1 & 1 & 1 & a_1 \\ 0 & -2 & -3 & a_0 - 3a_1 \\ 0 & 1 & -2 & a_2 \end{bmatrix}$$

$R_3 \to R_2 + 2R_3$ gives

$$\begin{bmatrix} 1 & 1 & 1 & a_1 \\ 0 & -2 & -3 & a_0 - 3a_1 \\ 0 & 0 & -7 & a_0 - 3a_1 + 2a_2 \end{bmatrix}$$

The matrix is now in Echelon form. We see that there are no free variables. Only leading variables c_1, c_2, c_3. This implies we have unique solution. Which means we can solve for c_1, c_2, c_3 in terms of a_1, a_2, a_3. We are not asked to complete the solution, only to say if these are basis. So we can stop here.

This shows that $\left\{3 + x, 1 + x + x^2, x - 2x^2\right\}$ are basis for P_2.

2.5.13 Additional problem 2

Find the general solution for $y'' - 25y = 0$. What is the particular solution for $y(0) = a, y'(0) = b$?

Solution

The characteristic equation is

$$\begin{aligned} r^2 - 25 &= 0 \\ r &= \pm 5 \end{aligned}$$

Two distinct real roots $r_1 = 5, r_2 = -5$. Therefore the general solution is

$$\begin{aligned} y(x) &= c_1 e^{r_1 x} + c_2 e^{r_2 x} \\ &= c_1 e^{5x} + c_2 e^{-5x} \end{aligned} \tag{1}$$

Now we apply the initial conditions. The first one $y(0) = a$ applied to the above gives

$$a = c_1 + c_2 \tag{2}$$

Taking derivative of (1) gives

$$y' = 5c_1 e^{5x} - 5c_2 e^{-5x}$$

Applying second initial conditions $y'(0) = b$ to the above gives

$$b = 5c_1 - 5c_2 \tag{3}$$

Multiplying (2) by 5 and adding the result to Eq (3) gives

$$\begin{aligned}5a + b &= (5c_1 + 5c_2) + (5c_1 - 5c_2)\\5a + b &= 10c_1\end{aligned}$$

Hence

$$c_1 = \frac{5a + b}{10}$$

From (2) we now solve for c_2

$$\begin{aligned}a &= \frac{5a + b}{10} + c_2\\c_2 &= a - \frac{5a + b}{10}\\&= \frac{a}{2} - \frac{b}{10}\end{aligned}$$

Now that we found both constants, the particular solution becomes

$$\begin{aligned}y(x) &= c_1 e^{5x} + c_2 e^{-5x}\\&= \left(\frac{a}{2} + \frac{b}{10}\right)e^{5x} + \left(\frac{a}{2} - \frac{b}{10}\right)e^{-5x}\end{aligned}$$

2.5.14 key solution for HW 5

Homework 5 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

4.7.7 This is not a subspace. It fails everything fairly badly, but I'll go through why it fails to be closed under addition. Suppose that f and g are functions where $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Then $(f + g)(1) = f(1) + g(1) = 1 + 1 = 2$ so $f + g$ is not in the set.

4.7.10 This is a subspace. Polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ where $a_0 = a_1 = 0$ are of the form $bx^2 + cx^3$. If we add two such polynomials, we have

$$(b_1x^2 + c_1x^3) + (b_2x^2 + c_2x^3) = (b_1 + b_2)x^2 + (c_1 + c_2)x^3$$

The result here is another polynomial of this form. Similarly, when we scale we get

$$k(bx^2 + cx^3) = (kb)x^2 + (kc)x^3$$

Again, the result is in the set. So we have a subspace.

1.1.5 We have the differential equation $y' = y + 2e^{-x}$. We need to check that $y = e^x - e^{-x}$ is a solution. We compute:

$$\begin{aligned} \frac{d}{dx}(e^x - e^{-x}) &= e^x - (-e^{-x}) \\ &= e^x + e^{-x} \\ &= e^x - e^{-x} + 2e^{-x} \\ &= y + 2e^{-x} \end{aligned}$$

So this is indeed a solution.

1.1.17 We have the differential equation $y' + y = 0$. First, we check that $y(x) = Ce^{-x}$ is a solution:

$$\begin{aligned} \frac{d}{dx}(Ce^{-x}) + (Ce^{-x}) &= -Ce^{-x} + Ce^{-x} \\ &= 0 \end{aligned}$$

We need to find the value of C so that $y(0) = 2$. We have $y(0) = Ce^0 = C$, so $C = 2$ is the necessary value.

1

5.1.3 We omit the verification that y_1 and y_2 are solutions. Our general solution is $y(x) = c_1 \cos 2x + c_2 \sin 2x$. This has derivative $y'(x) = -2c_1 \sin 2x + 2c_2 \cos 2x$. So, our initial conditions tell us that

$$\begin{aligned} y(0) &= c_1 \cos 0 + c_2 \sin 0 \\ 3 &= c_1 \\ y'(0) &= -2c_1 \sin 0 + 2c_2 \cos 0 \\ 8 &= 2c_2 \end{aligned}$$

So, our particular solution is $y(x) = 3\cos 2x + 4 \sin 2x$.

5.1.5 We omit the verification that y_1 and y_2 are solutions. Our general solution is $y(x) = c_1 e^x + c_2 e^{2x}$. This has derivative $y'(x) = c_1 e^x + 2c_2 e^{2x}$. So, our initial conditions tell us that

$$\begin{aligned} y(0) &= c_1 e^0 + c_2 e^0 \\ 1 &= c_1 + c_2 \\ y'(0) &= c_1 e^0 + 2c_2 e^0 \\ 0 &= c_1 + 2c_2 \end{aligned}$$

Subtracting our equations gives $c_2 = -1$, and we substitute to get that $c_1 = 2$. So, our particular solution is $y(x) = 2e^x - e^{2x}$.

5.1.7 We omit the verification that y_1 and y_2 are solutions. Our general solution is $y(x) = c_1 + c_2 e^{-x}$. This has derivative $y'(x) = -c_2 e^{-x}$. So, our initial conditions tell us that

$$\begin{aligned} y(0) &= c_1 + c_2 e^0 \\ -2 &= c_1 + c_2 \\ y'(0) &= -c_2 e^0 \\ 8 &= -c_2 \end{aligned}$$

So, $c_2 = -8$ and thus $c_1 = 6$. So our particular solution is $y(x) = 6 - 8e^{-x}$.

5.1.33 We have characteristic equation $r^2 - 3r + 2 = (r-1)(r-2)$ so we have roots $r = 1, 2$. This gives us general solution $y(x) = c_1 e^x + c_2 e^{2x}$.

5.1.35 We have characteristic equation $r^2 + 5r = r(r+5)$ so we have roots $r = 0, -5$. This gives us general solution $y(x) = c_1 e^{0x} + c_2 e^{-5x} = c_1 + c_2 e^{-5x}$.

5.1.39 We have characteristic equation $4r^2 + 4r + 1 = (2r+1)(2r+1)$ so we have repeated root $r = -\frac{1}{2}$. This gives us general solution $y(x) = c_1 e^{-\frac{x}{2}} + c_2 x e^{-\frac{x}{2}}$.

Additional Problems:

2

1. We wish to show that $\{3+x, 1+x+x^2, x-2x^2\}$ is a basis for $\mathcal{P}_2$. So, we need to show that the equation

$$c_1(3+x)+c_2(1+x+x^2)+c_3(x-2x^2)=a_0+a_1x+a_2x^2$$

has a unique solution for each value of a_0, a_1, a_2. Rearranging terms, we have

$$(3c_1+c_2)+(c_1+c_2+c_3)x+(c_2-2c_3)x^2=a_0+a_1x+a_2x^2$$

Equating the coefficients of each power of x, we get the linear system

$$\begin{bmatrix}3 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & -2\end{bmatrix}\begin{bmatrix}c_1\\ c_2\\ c_3\end{bmatrix}=\begin{bmatrix}a_0\\ a_1\\ a_2\end{bmatrix}$$

If we can show this matrix is invertible, we will be done. This matrix looks annoying to row reduce, so I'll compute the determinant by expanding along the first row:

$$\begin{aligned}\det\begin{bmatrix}3 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & -2\end{bmatrix}&=(+3)\det\begin{bmatrix}1 & 1\\ 1 & -2\end{bmatrix}+(-1)\det\begin{bmatrix}1 & 1\\ 0 & -2\end{bmatrix}\\ &=3(-2-1)-(-2-0)\\ &=-7\end{aligned}$$

Since this determinant is nonzero, the matrix is invertible and the system we are considering always has a unique solution.

2. We have the initial value problem $y''-25y=0$, $y(0)=a$, $y'(0)=b$.

The characteristic equaiton is r^2-25 which has roots $r=\pm 5$. So our general solution is $y(x)=c_1e^{5x}+c_2e^{-5x}$. We compute $y'(x)=5c_1e^{5x}-5c_2e^{-5x}$, so our initial conditions give us the system

$$\begin{aligned}c_1+c_2&=a\\ 5c_1-5c_2&=b\end{aligned}$$

We can write this system in matrix form as

$$\begin{bmatrix}1 & 1\\ 5 & -5\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}a\\ b\end{bmatrix}$$

We have the matrix inverse $\begin{bmatrix}1 & 1\\ 5 & -5\end{bmatrix}^{-1}=\frac{1}{-5-5}\begin{bmatrix}-5 & -1\\ -5 & 1\end{bmatrix}$ so

$$\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & \frac{1}{10}\\ \frac{1}{2} & -\frac{1}{10}\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}\frac{a}{2}+\frac{b}{10}\\ \frac{a}{2}-\frac{b}{10}\end{bmatrix}$$

This solves for the constants c_1 and c_2 in terms of the given initial values a and b.

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2.6 HW 6

Local contents

2.6.1 Problems listing

Homework 6 - Due October 22

Homework instructions: Complete the assigned problems on your own paper. Once you are finished, scan or photograph your work and upload it to Gradescope. When prompted, tell Gradescope where to find each problem.
You are allowed (and in fact encouraged) to work with other students on homework assignments. If you do that, please indicate on each problem who you worked with. If you use sources other than your notes, the textbook, and any resources on Canvas for your homework, you must indicate the source on each problem. You are not permitted to view, request, or look for solutions to any of the homework problems from solutions manuals, homework help websites, online forums, other students, or any other sources.

Textbook Problems:

- §5.2: 9, 16, 24
- §5.3: 8, 11, 14, 18

Additional Problems:

1. This problem will walk you through finding the general solution of the differential equation

$$y^{(7)} - 2y^{(6)} + 9y^{(5)} - 16y^{(4)} + 24y^{(3)} - 32y'' + 16y' = 0$$

   (a) Write the characteristic equation for this differential equation. Factor out the common factor of r.
   (b) Check that 1 is a root of the remaining polynomial. This means that $(r-1)$ is a factor, so use polynomial long division to factor it out. Repeat until 1 is no longer a factor of the remaining polynomial.
   (c) The remaining polynomial should be of the form ar^4+br^2+c. Make the substitution $x=r^2$ and factor the quadratic ax^2+bx+c.
   (d) Substitute back $x=r^2$ and find the roots of whatever remains.
   (e) List all the roots of the characteristic polynomial and their multiplicities. Use this list to write down the general solution. Since this differential equation is of order 7, your general solution should have 7 terms.

1

2.6.2 Problem 9 section 5.2

In Problems 7 through 12, use the Wronskian to prove that the given functions are linearly independent on the indicated interval.

$$f(x) = e^x, g(x) = \cos x, h(x) = \sin x$$

On the real line.

Solution

$$W(x) = \begin{bmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{bmatrix}$$

Hence

$$W(x) = \begin{bmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{bmatrix}$$

The determinant is, expanding along first row is

$$\begin{aligned} |W(x)| &= e^x \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} - \cos x \begin{vmatrix} e^x & \cos x \\ e^x & -\sin x \end{vmatrix} + \sin x \begin{vmatrix} e^x & -\sin x \\ e^x & -\cos x \end{vmatrix} \\ &= e^x\left(\sin^2 x + \cos^2 x\right) - \cos x(-e^x \sin x - e^x \cos x) + \sin x(-e^x \cos x + e^x \sin x) \end{aligned}$$

But $\sin^2 x + \cos^2 x = 1$ and the above simplifies to

$$\begin{aligned} |W(x)| &= e^x - \left(-e^x \sin x \cos x - e^x \cos^2 x\right) + \left(-e^x \cos x \sin x + e^x \sin^2 x\right) \\ &= e^x + e^x \sin x \cos x + e^x \cos^2 x - e^x \cos x \sin x + e^x \sin^2 x \\ &= e^x + e^x \cos^2 x + e^x \sin^2 x \\ &= e^x + e^x\left(\sin^2 x + \cos^2 x\right) \\ &= 2e^x \end{aligned}$$

And since e^x is never zero on the real line, then $|W(x)| \neq 0$ Hence functions are linearly independent.

2.6.3 Problem 16 section 5.2

In Problems 13 through 20, a third-order homogeneous linear equation and three linearly independent solutions are given. Find a particular solution satisfying the given initial conditions.

$$\begin{aligned} y''' - 5y'' + 8y' - 4y &= 0 \\ y_1 &= e^x \\ y_2 &= e^{2x} \\ y_3 &= xe^{2x} \end{aligned}$$

I.C. are

$$y(0) = 1, y'(0) = 4, y''(0) = 0$$

Solution

The general solution is

$$\begin{aligned} y(x) &= c_1 y_2 + c_2 y_2 + c_3 y_3 \\ &= c_1 e^x + c_2 e^{2x} + c_3 xe^{2x} \end{aligned} \tag{1}$$

At $y(0) = 0$ the above becomes

$$1 = c_1 + c_2 \tag{2}$$

Taking derivative of (1) gives

$$y'(x) = c_1 e^x + 2c_2 e^{2x} + c_3\left(e^{2x} + 2xe^{2x}\right)$$

At $y'(0) = 4$ the above becomes

$$4 = c_1 + 2c_2 + c_3 \tag{3}$$

Taking derivative of $y''(x)$ gives

$$\begin{aligned} y''(x) &= c_1 e^x + 4c_2 e^{2x} + c_3\left(2e^{2x} + 2\left(e^{2x} + 2xe^{2x}\right)\right) \\ &= c_1 e^x + 4c_2 e^{2x} + c_3\left(2e^{2x} + 2e^{2x} + 4xe^{2x}\right) \end{aligned}$$

At $y''(0) = 0$ the above becomes

$$0 = c_1 + 4c_2 + 4c_3 \tag{4}$$

Equations (2,3,4) are now solved for c_1, c_2, c_3

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 4 & 4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 4 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 4 & -1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -10 \end{bmatrix}$$

The above is Echelon form. Hence the system becomes

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -10 \end{bmatrix}$$

From last row, $c_3 = -10$. From second row $c_2 + c_3 = 3$ or $c_2 = 13$. From first row $c_1 + c_2 = 1$. Hence $c_1 = -12$. Therefore

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -12 \\ 13 \\ -10 \end{bmatrix}$$

Substituting these values back in general solution (1) gives the solution that satisfies these initial conditions as

$$\begin{aligned} y(x) &= c_1 e^x + c_2 e^{2x} + c_3 xe^{2x} \\ &= -12e^x + 13e^{2x} - 10xe^{2x} \end{aligned}$$

2.6.4 Problem 19 section 5.2

In Problems 13 through 20, a third-order homogeneous linear equation and three linearly independent solutions are given. Find a particular solution satisfying the given initial conditions.

$$\begin{aligned} x^3y''' - 3x^2y'' + 6xy' - 6y &= 0 \\ y_1 &= x \\ y_2 &= x^2 \\ y_3 &= x^3 \end{aligned}$$

I.C. are

$$y(1) = 6, y'(1) = 14, y''(1) = 22$$

<u>Solution</u>

The general solution is

$$\begin{aligned} y(x) &= c_1y_2 + c_2y_2 + c_3y_3 \\ &= c_1x + c_2x^2 + c_3x^3 \end{aligned} \tag{1}$$

At $y(1) = 0$ the above becomes

$$6 = c_1 + c_2 + c_3 \tag{2}$$

Taking derivative of (1) gives

$$y'(x) = c_1 + 2c_2x + 3c_3x^2$$

At $y'(1) = 14$ the above becomes

$$14 = c_1 + 2c_2 + 3c_3 \tag{3}$$

Taking derivative of $y'(x)$ gives

$$y''(x) = 2c_2 + 6c_3x$$

At $y''(1) = 22$ the above becomes

$$22 = 2c_2 + 6c_3 \tag{4}$$

Equations (2,3,4) are now solved for c_1, c_2, c_3

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 22 \end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 0 & 2 & 6 & 22 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 6 & 22 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

The above is Echelon form. Hence the system becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 6 \end{bmatrix}$$

From last row, $2c_3 = 6$ or $c_3 = 3$. From second row $c_2 + 2c_3 = 8$ or $c_2 = 8 - 2(3) = 2$. From first row $c_1 + c_2 + c_3 = 6$. Hence $c_1 = 6 - 2 - 3 = 1$. Therefore

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Substituting these values back in general solution (1) gives the solution that satisfies these initial conditions as

$$\begin{aligned} y(x) &= c_1 x + c_2 x^2 + c_3 x^3 \\ &= x + 2x^2 + 3x^3 \end{aligned}$$

2.6.5 Problem 24 section 5.2

In Problems 21 through 24, a nonhomogeneous differential equation, a complementary solution y_c, and a particular solution

y_p are given. Find a solution satisfying the given initial conditions.

$$\begin{aligned} y'' - 2y' + 2y &= 2x \\ y_c &= c_1 e^x \cos x + c_2 e^x \sin x \\ y_p &= x + 1 \end{aligned}$$

I.C. are

$$y(0) = 4, y'(0) = 8$$

Solution

The general solution is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= c_1 e^x \cos x + c_2 e^x \sin x + x + 1 \qquad (1) \end{aligned}$$

At $y(0) = 4$ the above becomes (using $e^0 = 1, \cos 0 = 1, \sin 0 = 0$)

$$4 = c_1 + 1 \qquad (2)$$

Taking derivative of (1) gives

$$y'(x) = c_1(e^x \cos x - e^x \sin x) + c_2 e^x \cos x + 1$$

At $y'(0) = 8$ the above becomes

$$\begin{aligned} 8 &= c_1(1 - 0) + c_2 + 1 \\ 8 &= c_1 + c_2 + 1 \qquad (3) \end{aligned}$$

We have two equations (2,3) to solve for c_1, c_2. From (3) we see that $c_1 = 3$. Hence from (3) $8 = 3 + c_2 + 1$ or $c_2 = 4$. Therefore the solution in (1) becomes

$$\begin{aligned} y(x) &= c_1 e^x \cos x + c_2 e^x \sin x + x + 1 \\ &= 3e^x \cos x + 4e^x \sin x + x + 1 \\ &= e^x(3\cos x + 4\sin x) + x + 1 \end{aligned}$$

2.6.6 Problem 8 section 5.3

Find the general solutions of the differential equations in Problems 1 through 20.

$$y'' - 6y' + 13y = 0$$

Solution This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where here we see that $A=1, B=-6, C=13$.

Let the solution be $y(x)=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x}-6\lambda e^{\lambda x}+13 e^{\lambda x}=0 \tag{1}$$

Since $e^{\lambda x}\neq 0$, then dividing Eq. (1) throughout by$e^{\lambda x}$ results in

$$\lambda^2-6\lambda+13=0 \tag{2}$$

Eq. (2) is the characteristic equation of the ODE. We need to determine its roots to find the general solution. Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=-6, C=13$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{6}{(2)(1)}\pm\frac{1}{(2)(1)}\sqrt{-6^2-(4)(1)(13)}\\&=3\pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=3+2i\\\lambda_2&=3-2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2}=\alpha\pm i\beta$$

Where $\alpha=3$ and $\beta=2$. Therefore the final solution, when using Euler relation, can be written as

$$y(x)=e^{\alpha x}\big(c_1\cos(\beta x)+c_2\sin(\beta x)\big)$$

Which becomes

$$y(x)=e^{3x}(c_1\cos(2x)+c_2\sin(2x))$$

2.6.7 Problem 11 section 5.3

Find the general solutions of the differential equations in Problems 1 through 20.

$$y^{(4)}(x)-8y^{(3)}+16y''=0$$

Solution

We start by writing the characteristic equation of the ODE

$$\lambda^4-8\lambda^3+16\lambda^2=0$$

We now solve for the roots of the above equation. Writing the above as

$$\lambda^2\big(\lambda^2-8\lambda+16\big)=0$$

We see that $\lambda^2=0$ gives $\lambda=0$ with multiplicity 2. The equation $\lambda^2-8\lambda+16=0$ can be factored to $(\lambda-4)(\lambda-4)=0$. Therefor $\lambda=4$ with multiplicity 2.

Hence the roots are

$$\begin{aligned}\lambda_1&=0\\\lambda_2&=0\\\lambda_3&=4\\\lambda_4&=4\end{aligned}$$

This table summarizes the result

root	multiplicity	type of root
0	2	real root
4	2	real root

For a real root λ with multiplicity one, we obtain a basis solution of the form $e^{\lambda x}$ and real root λ with multiplicity two we obtain basis solutions $\left\{e^{\lambda x}, xe^{\lambda x}\right\}$. Therefore the solution is

$$\begin{aligned} y(x) &= c_2 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x} + c_2 e^{\lambda_3 x} + c_2 x e^{\lambda_3 x} \\ &= c_2 + c_2 x + c_2 e^{4x} + c_2 x e^{4x} \end{aligned}$$

2.6.8 Problem 14 section 5.3

Find the general solutions of the differential equations in Problems 1 through 20.

$$y^{(4)}(x) + 3y'' - 4y = 0$$

Solution

We start by writing the characteristic equation

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

Let

$$z = \lambda^2$$

The characteristic becomes

$$z^2 + 3z - 4 = 0$$

Factoring the above gives

$$(z + 4)(z - 1) = 0$$

Hence $z = -4, z = 1$. When $z = -4$, then $\lambda = \pm\sqrt{-4} = \pm 2i$. And when $z = 1$, then $\lambda = \pm\sqrt{1} = \pm 1$. Therefore the roots are

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -1 \\ \lambda_3 &= 2i \\ \lambda_4 &= -2i \end{aligned}$$

This table summarizes the result

root	multiplicity	type of root
-1	1	real root
1	1	real root
$\pm 2i$	1	complex conjugate root

For a real root λ with multiplicity one, we obtain a basis of the form $c_1 e^{\lambda x}$ and for a complex conjugate root of the form $a \pm ib$ we obtain basis solution of the form $e^{ax}(c_1 \cos(bx) + c_2 \sin(bx))$. Therefore the final solution, using $a = 0, b = 2$ is

$$y(x) = c_1 e^{-x} + c_2 e^{x} + c_3 \cos(2x) + c_4 \sin(2x)$$

2.6.9 Problem 18 section 5.3

Find the general solutions of the differential equations in Problems 1 through 20.

$$y^{(4)}(x) = 16y$$

Solution

We start by writing the characteristic equation

$$\lambda^4 = 16$$

Let

$$z = \lambda^2$$

The characteristic becomes

$$z^2 = 16$$

Hence $z = \pm 4$. When $z = 4$ then $\lambda = \pm\sqrt{4} = \pm 2$. And when $z = -4$ then $\lambda = \pm\sqrt{-4} = \pm 2i$. Hence the roots are

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\\ \lambda_3 &= 2i\\ \lambda_4 &= -2i\end{aligned}$$

This table summarizes the result

root	multiplicity	type of root
-2	1	real root
2	1	real root
$\pm 2i$	1	complex conjugate root

As in the earlier problem, we now can write the general solution as

$$y(x) = e^{-2x}c_1 + c_2e^{2x} + c_3\cos(2x) + c_4\sin(2x)$$

2.6.10 Additional problem 1

Find the general solutions of the differential equations in Problems 1 through 20.

$$y^{(7)}(x) - 2y^{(6)} + 9y^{(5)} - 16y^{(4)} + 24y^{(3)} - 32y'' + 16y' = 0$$

Solution

2.6.10.1 Part a

The characteristic equation is

$$\begin{aligned}r^7 - 2r^6 + 9r^5 - 16r^4 + 24r^3 - 32r^2 + 16r &= 0\\ r\left(r^6 - 2r^5 + 9r^4 - 16r^3 + 24r^2 - 32r + 16\right) &= 0\end{aligned}$$

Hence one root is $r = 0$. And now we need to solve

$$r^6 - 2r^5 + 9r^4 - 16r^3 + 24r^2 - 32r + 16 = 0$$

2.6.10.2 Part b

Substituting $r = 1$ in the above gives

$$\begin{aligned}1 - 2 + 9 - 16 + 24 - 32 + 16 &= 0\\ 0 &= 0\end{aligned}$$

Therefore $(r-1)$ is a factor. Doing long division (do not know how type polynomial division in Latex, please see scanned hand solution in appendix of this problem).

$$\frac{r^6 - 2r^5 + 9r^4 - 16r^3 + 24r^2 - 32r + 16}{(r-1)} = r^5 - r^4 + 8r^3 - 8r^2 + 16r - 16$$

Hence

$$r^6 - 2r^5 + 9r^4 - 16r^3 + 24r^2 - 32r + 16 = (r-1)\left(r^5 - r^4 + 8r^3 - 8r^2 + 16r - 16\right)$$

Substituting $r = 1$ in $\left(r^5 - r^4 + 8r^3 - 8r^2 + 16r - 18\right)$ gives

$$r^5 - r^4 + 8r^3 - 8r^2 + 16r - 18 \rightarrow 1 - 1 + 8 - 8 + 16 - 16 = 0$$

Hence $(r-1)$ is a factor of $\left(r^5 - r^4 + 8r^3 - 8r^2 + 16r - 16\right)$. Therefore we now need to do long division

$$\frac{r^5 - r^4 + 8r^3 - 8r^2 + 16r - 16}{(r-1)} = r^4 + 8r^2 + 16$$

Hence now we have

$$r^6 - 2r^5 + 9r^4 - 16r^3 + 24r^2 - 32r + 16 = (r-1)(r-1)\left(r^4 + 8r^2 + 16\right)$$

2.6.10.3 Part c

Looking at $r^4 + 8r^2 + 16 = 0$, let $z = r^2$. Therefore $r^4 + 8r^2 + 16$ becomes $z^2 + 8z + 16 = 0$, This can be factored to $(z+4)(z+4) = 0$. Hence roots are $z = -4$ which is double root.

2.6.10.4 Part d

Therefore when $z = -4$ then $r = \pm\sqrt{-4} = \pm 2i$ with multiplicity 2 since $z = -4$ is double root. Therefore the final factorization is

$$r^6 - 2r^5 + 9r^4 - 16r^3 + 24r^2 - 32r + 16 = (r-1)(r-1)(r-2i)(r+2i)(r-2i)(r+2i)$$

2.6.10.5 Part e

This table summarizes the result

root	multiplicity	type of root
0	1	real root
1	2	real root
$\pm 2i$	2	complex conjugate

Now we are above to write down the general solution.

$$\begin{aligned}
y(x) &= c_1 e^{0x} + (c_2 e^x + c_3 x e^x) + \left(c_4 e^{2ix} + c_5 x e^{2ix}\right) + \left(c_6 e^{-2ix} + c_7 x e^{-2ix}\right) \\
&= c_1 + (c_2 e^x + c_3 x e^x) + \left(c_4 e^{2ix} + c_5 x e^{2ix}\right) + \left(c_6 e^{-2ix} + c_7 x e^{-2ix}\right) \\
&= c_1 + (c_2 e^x + c_3 x e^x) + \left(c_4 e^{2ix} + c_6 e^{-2ix}\right) + x\left(c_5 e^{2ix} + c_7 e^{-2ix}\right)
\end{aligned}$$

We see the above has 7 terms. But using Euler relation, we can write $\left(e^{2ix} + e^{-2ix}\right)$ using trig functions. The above becomes

$$y(x) = c_1 + (c_2 e^x + c_3 x e^x) + (c_4 \cos 2x + c_5 \sin 2x) + x(c_6 \cos 2x + c_7 \sin 2x)$$

(constants of integrations kept the same as originally for simplicity, since it does not matter as these are found from initial conditions if given).

2.6.10.6 Appendix

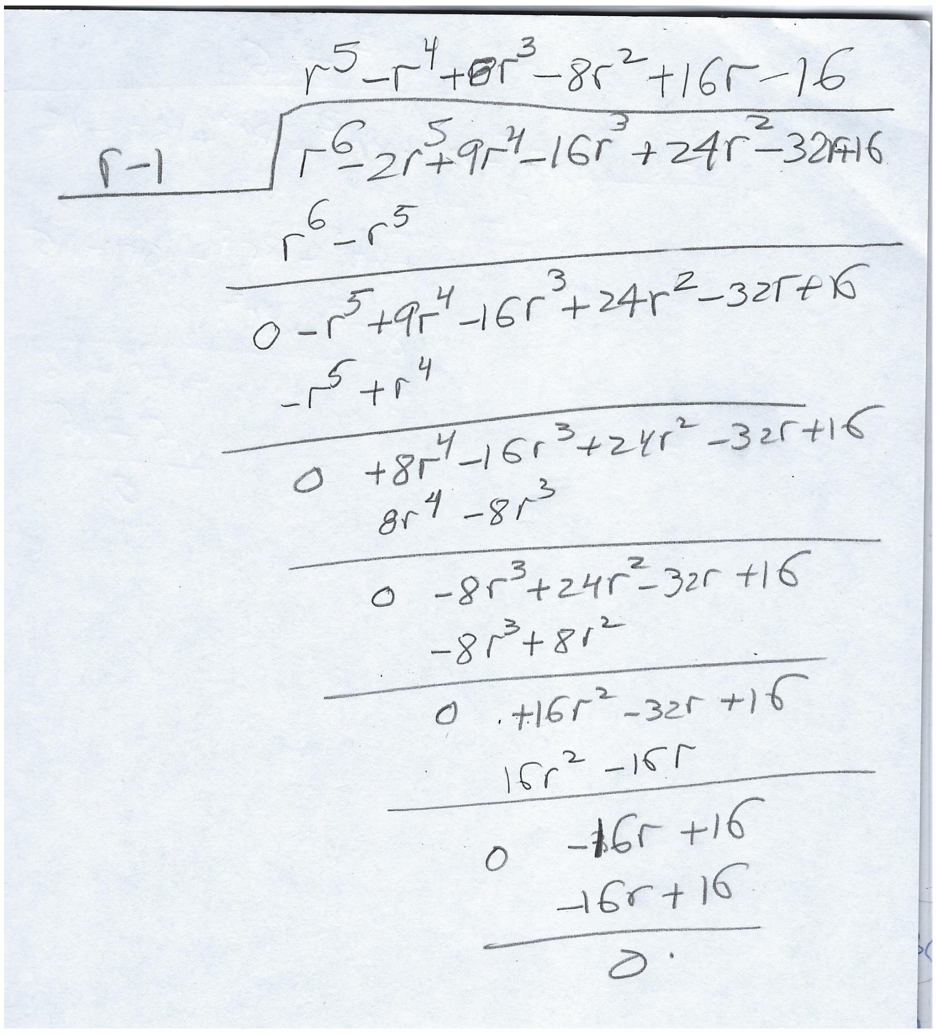

Figure 2.8: First long division

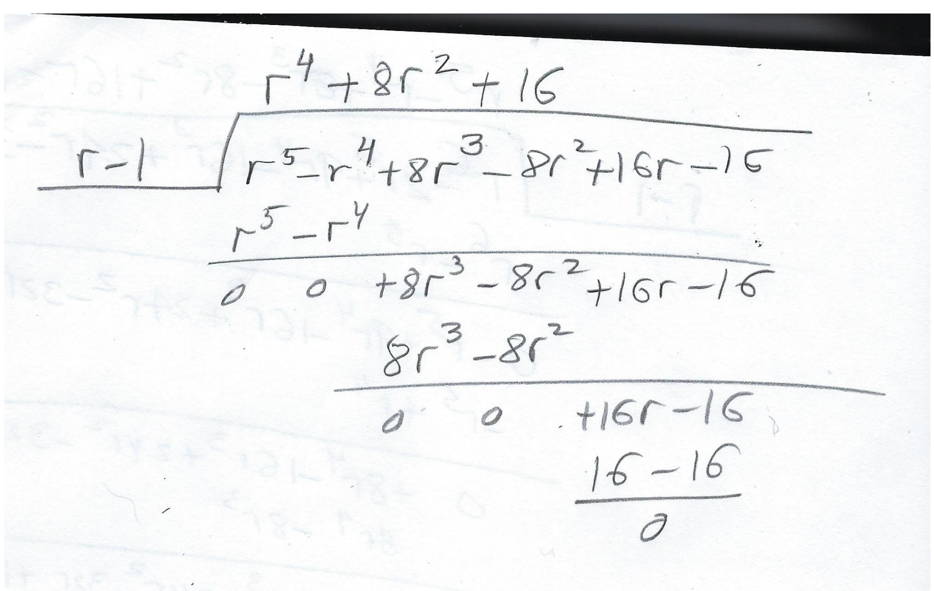

Figure 2.9: Second long division

2.6.11 key solution for HW 6

HOMEWORK 6 – SOLUTIONS

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

5.2.9 We compute

$$\begin{aligned}
W(x) &= \det \begin{bmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{bmatrix} \\
&= e^x \det \begin{bmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{bmatrix} - \cos x \det \begin{bmatrix} e^x & \cos x \\ e^x & -\sin x \end{bmatrix} + \sin x \det \begin{bmatrix} e^x & -\sin x \\ e^x & -\cos x \end{bmatrix} \\
&= e^x(\sin^2 x + \cos^2 x) - \cos x(-e^x \sin x - e^x \cos x) + \sin x(-e^x \cos x + e^x \sin x) \\
&= e^x + e^x(\cos x \sin x + \cos^2 x - \sin x \cos x + \sin^2 x) \\
&= 2e^x
\end{aligned}$$

Now $W(x) = 2e^x$ is not identically 0, so the functions are linearly independent.

5.2.16 Our general solution is $y = c_1 e^x + c_2 e^{2x} + c_3 x e^{2x}$. The given initial conditions give us the following equations:

$$\begin{aligned}
y &= c_1 e^x + c_2 e^{2x} + c_3 x e^{2x} \\
y(0) &= c_1 + c_2 = 1 \\
y' &= c_1 e^x + 2c_2 e^{2x} + c_3 e^{2x} + 2c_3 x e^{2x} \\
y'(0) &= c_1 + 2c_2 + c_3 = 4 \\
y'' &= c_1 e^x + 4c_2 e^{2x} + 4c_3 e^{2x} + 4c_3 x e^{2x} \\
y''(0) &= c_1 + 4c_2 + 4c_3 = 0
\end{aligned}$$

We have a system of three equations in three variables which we solve by row reducing the augmented matrix.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 4 \\ 1 & 4 & 4 & 0 \end{bmatrix} \xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 4 & -1 \end{bmatrix}$$

$$\xrightarrow{-3R_2+R_3} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -10 \end{bmatrix}$$

Now we back substitute to get $c_3 = -10, c_2 = 13, c_1 - 12$. So, the particular solution here is $y = -12e^x + 13e^{2x} - 10xe^{2x}$.

1

5.2.24 The general form of a solution is $y = c_1e^x \cos x + c_2e^x \sin x + x + 1$. The initial conditions give us the following:

$$\begin{aligned} y &= c_1e^x \cos x + c_2e^x \sin x + x + 1 \\ y(0) &= c_1 + 1 = 4 \\ y' &= c_1e^x \cos x - c_1e^x \sin x + c_2e^x \sin x + c_2e^x \cos x + 1 \\ y'(0) &= c_1 + c_2 + 1 = 8 \end{aligned}$$

We can solve immediately to get $c_1 = 3, c_2 = 4$. So the solution is $y = 3e^x \cos x + 4e^x \sin x + x + 1$.

5.3.8 We have $y'' - 6y' + 13y = 0$ with characteristic equation $r^2 - 6r + 13$. This doesn't factor obviously, so we use the quadratic equation:

$$\begin{aligned} r &= \frac{6 \pm \sqrt{36 - 52}}{2} \\ &= 3 \pm 2i \end{aligned}$$

In this case, we have complex conjugate roots $a \pm bi$ where $a = 3$ and $b = 2$. So our general solution is $y = c_1e^{3x} \cos(2x) + c_2e^{3x} \sin(2x)$.

5.3.11 We have $y^{(4)} - 8y^{(3)} + 16y'' = 0$ with characteristic equation $r^4 - 8r^3 + 16r^2 = r^2(r^2 - 8r + 16)$. We can factor the remaining quadratic easily as $(r - 4)^2$. So we have roots $r = 0, 4$ each of multiplicity 2.

Our general solution is $y = c_1 + c_2x + c_3e^{4x} + c_4xe^{4x}$.

5.3.14 We have characteristic equation $r^4 + 3r^2 - 4$. Mentally making the substitution $x = r^2$, we can see that there is a factorization $(r^2 + 4)(r^2 - 1) = (r^2 + 4)(r + 1)(r - 1)$. The roots are thus $\pm 2i, \pm 1$.

Our general solution is $y = c_1e^x + c_2e^{-x} + c_3 \cos(2x) + c_4 \sin(2x)$.

5.3.18 We rewrite the differential equation as $y^{(4)} - 16y = 0$ so the characteristic equation is $r^4 - 16$. This is a difference of squares, so factors as $(r^2 - 4)(r^2 + 4)$. We can factor further using difference of squares to get $(r-2)(r+2)(r^2+4)$. So the roots are $\pm 2, \pm 2i$.

The general solution is $y = c_1e^{2x} + c_2e^{-2x} + c_3 \cos(2x) + c_3 \sin(2x)$.

Additional Problems:

1. We have the differential equation

$$y^{(7)} - 2y^{(6)} + 9y^{(5)} - 16y^{(4)} + 24y^{(3)} - 32y'' + 16y' = 0$$

(a) The characteristic equation is $r^7 - 2r^6 + 9r^5 - 16r^4 + 24r^3 - 32r^2 + 16r$ which factors as $r(r^6 - 2r^5 + 9r^4 - 16r^3 + 24r^2 - 32r + 16)$

2

(b) 1 is a root as $1 - 2 + 9 - 16 + 24 - 32 + 16 = 0$. Polynomial long division (steps omitted) gives us the factorization $r(r-1)(r^5 - r^4 + 8r^3 - 8r^2 + 16r - 16)$.

1 is still a root of the degree 5 factor as $1 - 1 + 8 - 8 + 16 - 16 = 0$, so we do polynomial long division (steps omitted again) again to get the factorization $r(r-1)^2(r^4 + 8r^2 + 16)$.

1 is no longer a root, since $1 + 8 + 16 = 25$.

(c) We have $x^2 + 8x + 16 = (x+4)(x+4)$.

(d) Our characteristic polynomial factors as $r(r-1)^2(r^2+4)^2$. The roots of the $r^2 + 4$ factor are $\pm 2i$.

(e) Our roots are 0 (mult. 1), 1 (mult. 2), and $\pm 2i$ (each of mult. 2). This gives us the following general solution

$$y(x) = c_1 + c_2 e^x + c_3 x e^x + c_4 \cos(2x) + c_5 \sin(2x) + c_6 x \cos(2x) + c_7 x \sin(2x)$$

As expected, we have seven terms in the general solution.

2.7 HW 7

Local contents

2.7.1 Problems listing

Homework 7 - Due October 29

Homework instructions: Complete the assigned problems on your own paper. Once you are finished, scan or photograph your work and upload it to Gradescope. When prompted, tell Gradescope where to find each problem.
You are allowed (and in fact encouraged) to work with other students on homework assignments. If you do that, please indicate on each problem who you worked with. If you use sources other than your notes, the textbook, and any resources on Canvas for your homework, you must indicate the source on each problem. You are not permitted to view, request, or look for solutions to any of the homework problems from solutions manuals, homework help websites, online forums, other students, or any other sources.

Textbook Problems:

- §5.5: 3, 9, 11, 23, 32
- §6.1: 7, 17, 21, 25, 29

Additional Problems:

1. On the last homework, you found the general solution for the differential equation

$$y^{(7)} - 2y^{(6)} + 9y^{(5)} - 16y^{(4)} + 24y^{(3)} - 32y'' + 16y' = 0$$

Using your solution to that problem, find the appropriate form for of a particular solution y_p to the differential equation below. Do *not* find the values of the coefficients!

$$y^{(7)} - 2y^{(6)} + 9y^{(5)} - 16y^{(4)} + 24y^{(3)} - 32y'' + 16y' = e^{2x} + x\sin x + x^2$$

2. Let $A = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}$ where t_1, t_2, t_3 are distinct real numbers. Find the eigenvalues of A and the corresponding eigenvectors.

3. *This problem is optional.* Extend the result in problem 2 to the case of $n \times n$ matrices. That is, let A be a matrix with entries $t_1, t_2, \ldots, t_n$ on the main diagonal and 0s everywhere else, where the t_i are distinct real numbers. Find the eigenvalues and corresponding eigenvectors.

1

2.7.2 Problem 3 section 5.5

In Problems 1 through 20, find a particular solution y_p of the given equation.

$$y'' - y' - 6y = 2\sin 3x \tag{A}$$

<u>Solution</u>

The first step is to find the homogeneous solution y_h in order to determine the basis solutions to check for any duplication with basis solutions for the particular solution.

$$y'' - y' - 6y = 0$$

The characteristic equation is

$$\begin{aligned} r^2 - r - 6 &= 0 \\ (r-3)(r+2) &= 0 \end{aligned}$$

Hence the roots are $r_1 = 3, r_2 = -2$. Therefore the basis solutions are

$$\left\{e^{3x}, e^{-2x}\right\} \tag{1}$$

Which implies

$$y_h = c_1 e^{3x} + c_2 e^{-2x}$$

Now that we found the basis solution, we turn our attention to finding y_p. The RHS is $\sin 3x$. Looking at this function and all possible derivatives gives

$$\{\sin 3x, \cos 3x\} \tag{2}$$

Notice that we ignore any leading coefficients when doing this. Now we compare the above to the basis of the homogeneous solution found in (1) to check if there are duplication in basis or not. <u>There is no duplication</u>. Therefore we assume that particular solution y_p is a linear combination of the functions in (2). This implies that

$$\begin{aligned} y_p &= A\sin 3x + B\cos 3x \\ y_p' &= 3A\cos 3x - 3B\sin 3x \\ y_p'' &= -9A\sin 3x - 9B\cos 3x \end{aligned}$$

Substituting the above back in original ODE (A) gives

$$\begin{aligned} y_p'' - y_p' - 6y_p &= 2\sin 3x \\ (-9A\sin 3x - 9B\cos 3x) - (3A\cos 3x - 3B\sin 3x) - 6(A\sin 3x + B\cos 3x) &= 2\sin 3x \\ \sin(3x)(-9A + 3B - 6A) + \cos(3x)(-9B - 3A - 6B) &= 2\sin 3x \\ \sin(3x)(-15A + 3B) + \cos(3x)(-15B - 3A) &= 2\sin 3x \end{aligned}$$

Comparing coefficients gives

$$-15A + 3B = 2 \tag{3}$$
$$-15B - 3A = 0 \tag{4}$$

Multiplying first equation by 5 and adding result to second equation gives

$$\begin{aligned} (-75A + 15B) + (-15B - 3A) &= 10 \\ -78A &= 10 \\ A &= -\frac{10}{78} \\ &= -\frac{5}{39} \end{aligned}$$

From (3)

$$\begin{aligned}
-15\left(-\frac{5}{39}\right)+3B &= 2\\
\frac{25}{13}+3B &= 2\\
B &= \frac{2-\frac{25}{13}}{3}\\
&= \frac{1}{39}
\end{aligned}$$

Hence the particular solution is

$$\begin{aligned}
y_p &= A\sin 3x + B\cos 3x\\
&= -\frac{5}{39}\sin 3x + \frac{1}{39}\cos 3x\\
&= \frac{1}{39}(\cos 3x - 5\sin 3x)
\end{aligned}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p\\
&= c_1 e^{3x} + c_2 e^{-2x} + \frac{1}{39}(\cos 3x - 5\sin 3x)
\end{aligned}$$

2.7.3 Problem 9 section 5.5

In Problems 1 through 20, find a particular solution y_p of the given equation.

$$y'' + 2y' - 3y = 1 + xe^x \tag{A}$$

Solution

The first step is to find the homogeneous solution y_h in order to determine the basis solutions to check for any duplication with basis solutions for the particular solution.

$$y'' + 2y' - 3y = 0$$

The characteristic equation is

$$\begin{aligned}
r^2 + 2r - 3 &= 0\\
(r+3)(r-1) &= 0
\end{aligned}$$

Hence the roots are $r_1 = -3, r_2 = 1$. Therefore the basis solutions are

$$\left\{e^{-3x}, e^x\right\} \tag{1}$$

Which implies

$$y_h = c_1 e^{-3x} + c_2 e^x$$

Now that we found the basis solution, we turn our attention to finding y_p. The RHS is $1 + xe^x$. Hence it basis functions are

$$\{1, xe^x\}$$

taking derivatives of each basis gives

$$\{1, (xe^x, e^x)\} \tag{2}$$

Where we used () to group all basis generated from same one.

Now we compare the above to the basis of the homogeneous solution found in (1) to check if there are duplication in basis or not. We see duplication since e^x is basis in both (1) and (2). Therefore we multiply the group which generated e^x by x. The the above now becomes

$$\left\{1, \left(x^2e^x, xe^x\right)\right\} \tag{2A}$$

We compare again (1) against (2A) and now we see no duplication. Therefore we assume that particular solution y_p is a linear combination of the functions in (2A). This implies that

$$\begin{aligned}
y_p &= A + Bx^2e^x + Cxe^x \\
y_p' &= 2Bxe^x + Bx^2e^x + Ce^x + Cxe^x \\
y_p'' &= 2Be^x + 2Bxe^x + 2Bxe^x + Bx^2e^x + Ce^x + Ce^x + Cxe^x \\
&= xe^x(2B + 2B + C) + x^2e^x(B) + e^x(2B + 2C)
\end{aligned}$$

Substituting the above back in original ODE (A) gives

$$\begin{aligned}
y_p'' + 2y_p' - 3y_p &= 1 + xe^x \\
xe^x(2B + 2B + C) + x^2e^x(B) + e^x(2B + 2C) + 2\left(2Bxe^x + Bx^2e^x + Ce^x + Cxe^x\right) - 3\left(A + Bx^2e^x + Cxe^x\right) &= 1 + xe^x \\
xe^x(2B + 2B + C + 4B + 2C - 3C) + e^x(2B + 2C + 2C) + x^2e^x(B + 2B - 3B) - 3A &= 1 + xe^x \\
xe^x(8B) + e^x(2B + 4C) - 3A &= 1 + xe^x
\end{aligned}$$

Comparing coefficients

$$\begin{aligned}
-3A &= 1 \\
2B + 4C &= 0 \\
8B &= 1
\end{aligned}$$

Hence $B = \frac{1}{8}$ and from second equation $4C = -\frac{2}{8}$, or $C = -\frac{1}{16}$ and $A = -\frac{1}{3}$. Therefore the particular solution is

$$\begin{aligned}
y_p &= A + Bx^2e^x + Cxe^x \\
&= \frac{-1}{3} + \frac{1}{8}x^2e^x - \frac{1}{16}xe^x \\
&= -\frac{1}{3} + \frac{1}{16}\left(2x^2 - x\right)e^x
\end{aligned}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= c_1e^{-3x} + c_2e^x - \frac{1}{3} + \frac{1}{16}\left(2x^2 - x\right)e^x
\end{aligned}$$

2.7.4 Problem 11 section 5.5

In Problems 1 through 20, find a particular solution y_p of the given equation.

$$y^{(3)} + 4y' = 3x - 1 \tag{A}$$

Solution

The first step is to find the homogeneous solution y_h in order to determine the basis solutions to check for any duplication with basis solutions for the particular solution.

$$y^{(3)} + 4y' = 0$$

The characteristic equation is

$$\begin{aligned}
r^3 + 4r &= 0 \\
r\left(r^2 + 4\right) &= 0
\end{aligned}$$

Hence the roots are $r_1 = 0, r_2 = \pm 2i$. Therefore the basis solutions are

$$\{1, \cos(2x), \sin(2x)\} \tag{1}$$

Which implies

$$y_h = c_1 + c_2\cos(2x) + c_3\sin(2x)$$

Now that we found the basis solution, we turn our attention to finding y_p. The RHS is $3x-1$. Hence it basis functions are

$$\{1,x\} \tag{2}$$

Taking derivatives does not add any new basis. Now we compare the above to the basis of the homogeneous solution found in (1) to check if there are duplication in basis or not. We see duplication the constant is in both (1) and (2). Therefore we multiply the group by x. We took the whole basis as one group, since the constant 1 above is generated by taking derivative of x, so it is really in the same group. The above now becomes, after multiplying everything by x

$$\left\{x,x^2\right\} \tag{2A}$$

We compare again (1) against (2A) and now we see no duplication. Therefore we assume that particular solution y_p is a linear combination of the functions in (2A). This implies that

$$\begin{aligned}
y_p &= Ax+Bx^2\\
y_p' &= A+2Bx\\
y_p'' &= 2B\\
y_p^{(3)} &= 0
\end{aligned}$$

Substituting the above back in original ODE (A) gives

$$\begin{aligned}
y_p^{(3)}+y_p' &= 3x-1\\
0+4(A+2Bx) &= 3x-1\\
4A+8Bx &= 3x-1
\end{aligned}$$

Comparing coefficients

$$\begin{aligned}
4A &= -1\\
8B &= 3
\end{aligned}$$

Hence $A=-\frac{1}{4},B=\frac{3}{8}$. Therefore the particular solution is

$$\begin{aligned}
y_p &= Ax+Bx^2\\
&= -\frac{1}{4}x+\frac{3}{8}x^2\\
&= \frac{1}{8}\left(3x^2-2x\right)
\end{aligned}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h+y_p\\
&= c_1+c_2\cos(2x)+c_3\sin(2x)-\frac{1}{4}x+\frac{3}{8}x^2
\end{aligned}$$

2.7.5 Problem 23 section 5.5

In Problems 21 through 30, set up the appropriate form of a particular solution y_p, but do not determine the values of the coefficients.

$$y''+4y=3x\cos(2x) \tag{A}$$

Solution

The first step is to find the homogeneous solution y_h in order to determine the basis solutions to check for any duplication with basis solutions for the particular solution.

$$y''+4y=0$$

The characteristic equation is

$$r^2 + 4 = 0$$

Hence the roots are $r = \pm 2i$. Therefore the basis solutions are

$$\{\cos(2x), \sin(2x)\} \tag{1}$$

Which implies

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

Now that we found the basis solution, we turn our attention to finding y_p. The RHS is $3x\cos(2x)$. Hence it basis functions are

$$\{x\cos(2x)\} \tag{2}$$

Taking all possible derivatives of the above gives

$$\{x\cos(2x), \cos(2x), x\sin(2x), \sin(2x)\} \tag{2A}$$

Where in the above all signs and coefficients were ignored.

Now we compare the above to the basis of the homogeneous solution found in (1) to check if there are duplication in basis or not. We see duplication as $\cos(2x), \sin(2x)$ are in both. Therefore we multiply the group by x. We took the whole basis as one group since everything above was generated from (2). The above now becomes, after multiplying each term by x

$$\left\{x^2\cos(2x), x\cos(2x), x^2\sin(2x), x\sin(2x)\right\} \tag{2B}$$

Now we compare (2B) again with (1) and see no duplication. Hence

$$y_p = Ax^2\cos(2x) + Bx\cos(2x) + Cx^2\sin(2x) + Dx\sin(2x)$$

2.7.6 Problem 32 section 5.5

Solve the initial value problems in Problems 31 through 40.

$$\begin{aligned} y'' + 3y' + 2y &= e^x \qquad \text{(A)} \\ y(0) &= 0 \\ y'(0) &= 3 \end{aligned}$$

Solution

The first step is to find the homogeneous solution y_h in order to determine the basis solutions to check for any duplication with basis solutions for the particular solution.

$$y'' + 3y' + 2 = 0$$

The characteristic equation is

$$\begin{aligned} r^2 + 3r + 2 &= 0 \\ (r+2)(r+1) &= 0 \end{aligned}$$

Hence the roots are $r_1 = -2, r_2 = -1$. Therefore the basis solutions are

$$\left\{e^{-2x}, e^{-x}\right\} \tag{1}$$

Which implies

$$y_h = c_1 e^{-2x} + c_2 e^{-x}$$

Now that we found the basis solution, we turn our attention to finding y_p. The RHS is e^x. Hence it basis functions are

$$\{e^x\} \tag{2}$$

Taking all derivatives does not any terms. We also see no duplication between (2) and (1). Hence let

$$
\begin{aligned}
y_p &= Ae^x \\
y_p' &= Ae^x \\
y_p'' &= Ae^x
\end{aligned}
$$

Substituting these into (A) gives

$$
\begin{aligned}
y_p'' + 3y_p' + 2y_p &= e^x \\
Ae^x + 3Ae^x + 2Ae^x &= e^x \\
e^x(A + 3A + 2A) &= e^x
\end{aligned}
$$

Hence

$$
\begin{aligned}
6A &= 1 \\
A &= \frac{1}{6}
\end{aligned}
$$

Therefore

$$
y_p = \frac{1}{6}e^x
$$

Therefore the complete solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= c_1e^{-2x} + c_2e^{-x} + \frac{1}{6}e^x \qquad (3)
\end{aligned}
$$

We are now ready to apply the initial conditions. $y(0) = 0, y'(0) = 3$. Applying first IC to (3) gives

$$
0 = c_1 + c_2 + \frac{1}{6} \qquad (4)
$$

Taking derivative of (3) gives

$$
y' = -2c_1e^{-2x} - c_2e^{-x} + \frac{1}{6}e^x
$$

Applying second IC to the above gives

$$
3 = -2c_1 - c_2 + \frac{1}{6} \qquad (5)
$$

We now need to solve (4,5) for c_1, c_2. Adding (4,5) gives

$$
\begin{aligned}
3 &= \left(c_1 + c_2 + \frac{1}{6}\right) + \left(-2c_1 - c_2 + \frac{1}{6}\right) \\
3 &= \frac{1}{3} - c_1 \\
c_1 &= \frac{1}{3} - 3 \\
&= -\frac{8}{3}
\end{aligned}
$$

From (4)

$$
\begin{aligned}
0 &= c_1 + c_2 + \frac{1}{6} \\
0 &= -\frac{8}{3} + c_2 + \frac{1}{6} \\
c_2 &= \frac{5}{2}
\end{aligned}
$$

Therefore the complete solution (3) becomes

$$
\begin{aligned}
y(x) &= c_1e^{-2x} + c_2e^{-x} + \frac{1}{6}e^x \\
&= -\frac{8}{3}e^{-2x} + \frac{5}{2}e^{-x} + \frac{1}{6}e^x \\
&= \frac{1}{6}\left(-16e^{-2x} + 15e^{-x} + e^x\right)
\end{aligned}
$$

2.7.7 Problem 7 section 6.1

In Problems 1 through 26, find the (real) eigenvalues and associated eigenvectors of the given matrix A. Find a basis for each eigenspace of dimension 2 or larger.

$$A = \begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix}$$

<u>Solution</u>

We first need to find the eigenvalues. These are found by solving $|A - \lambda I| = 0$. Hence

$$\begin{aligned} \begin{vmatrix} 10-\lambda & -8 \\ 6 & -4-\lambda \end{vmatrix} &= 0 \\ (10-\lambda)(-4-\lambda) + 48 &= 0 \\ \lambda^2 - 6\lambda + 8 &= 0 \\ (\lambda-4)(\lambda-2) &= 0 \end{aligned}$$

Hence $\lambda_1 = 4, \lambda_2 = 2$. For each eigenvalue we find its associated eigenvectors.

<u>$\lambda_1 = 4$</u>

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{aligned} \begin{bmatrix} 10-4 & -8 \\ 6 & -4-4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 6 & -8 \\ 6 & -8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \tag{1}$$

Augmented matrix

$$\begin{bmatrix} 6 & -8 & 0 \\ 6 & -8 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 6 & -8 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore (1) becomes

$$\begin{bmatrix} 6 & -8 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1A}$$

v_2 is free variable. Let $v_2 = 1$. Then from first row $6v_1 - 8 = 0$ or $v_1 = \frac{8}{6}$. Hence

$$\vec{v}_1 = \begin{bmatrix} \frac{8}{6} \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

<u>$\lambda_1 = 2$</u>

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{aligned} \begin{bmatrix} 10-2 & -8 \\ 6 & -4-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 8 & -8 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \tag{1}$$

Augmented matrix

$$\begin{bmatrix} 8 & -8 & 0 \\ 6 & -6 & 0 \end{bmatrix}$$

$R_2 \to R_2 - \frac{6}{8}R_1$

$$\begin{bmatrix} 8 & -8 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore (1) becomes

$$\begin{bmatrix} 8 & -8 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1A}$$

v_2 is free variable. Let $v_2 = 1$. Then from first row $8v_1 - 8 = 0$ or $v_1 = 1$. Hence

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This table gives summary of the result

eigenvalue λ	associated eigenvector $\vec{v}$
4	$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$
2	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2.7.8 Problem 17 section 6.1

In Problems 1 through 26, find the (real) eigenvalues and associated eigenvectors of the given matrix A. Find a basis for each eigenspace of dimension 2 or larger.

$$A = \begin{bmatrix} 3 & 5 & -2 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Solution

We first need to find the eigenvalues. These are found by solving $|A - \lambda I| = 0$. Hence

$$\begin{vmatrix} 3-\lambda & 5 & -2 \\ 0 & 2-\lambda & 0 \\ 0 & 2 & 1-\lambda \end{vmatrix} = 0$$

Expanding along the first columns.

$$(3-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix} = 0$$
$$(3-\lambda)(2-\lambda)(1-\lambda) = 0$$

Hence roots (eigenvalues) are $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1$. For each eigenvalue we find its associated eigenvectors.

$\lambda_1 = 3$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} 3-3 & 5 & -2 \\ 0 & 2-3 & 0 \\ 0 & 2 & 1-3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 5 & -2 \\ 0 & -1 & 0 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

$R_3 \to R_2 + 2R_2$

$$\begin{bmatrix} 0 & 5 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Free variable is v_1. Let $v_1 = 1$. Last row gives $v_3 = 0$. Second row gives $v_2 = 0$. Hence the eigenvector is

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\underline{\lambda_1 = 2}$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} 3-2 & 5 & -2 \\ 0 & 2-2 & 0 \\ 0 & 2 & 1-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & -2 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

Swap R_2, R_3 (for clarify only)

$$\begin{bmatrix} 1 & 5 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Free variable is v_3. Let $v_3 = 1$. From second row $2v_2 - v_3 = 0$. Hence $v_2 = \frac{1}{2}$. First row gives $v_1 + 5v_2 - 2v_3 = 0$. Hence $v_1 = -5\left(\frac{1}{2}\right) + 2 = -\frac{1}{2}$. Hence the eigenvector is

$$\vec{v}_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$\underline{\lambda_1 = 1}$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} 3-1 & 5 & -2 \\ 0 & 2-1 & 0 \\ 0 & 2 & 1-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & -2 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 2 & 5 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Free variable is v_3. Let $v_3 = 1$. From second row $v_2 = 0$. First row gives $2v_1 = 2v_3$. Hence $v_1 = 1$. Hence the eigenvector is

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

This table gives summary of the result

eigenvalue λ	associated eigenvector $\vec{v}$
3	$\begin{bmatrix}1\\0\\0\end{bmatrix}$
2	$\begin{bmatrix}-1\\1\\2\end{bmatrix}$
1	$\begin{bmatrix}1\\0\\1\end{bmatrix}$

2.7.9 Problem 21 section 6.1

In Problems 1 through 26, find the (real) eigenvalues and associated eigenvectors of the given matrix A. Find a basis for each eigenspace of dimension 2 or larger.

$$A=\begin{bmatrix}4 & -3 & 1\\2 & -1 & 1\\0 & 0 & 2\end{bmatrix}$$

Solution

We first need to find the eigenvalues. These are found by solving $|A-\lambda I|=0$. Hence

$$\begin{vmatrix}4-\lambda & -3 & 1\\2 & -1-\lambda & 1\\0 & 0 & 2-\lambda\end{vmatrix}=0$$

Expanding along the last row

$$\begin{aligned}(-1)^{3+3}(2-\lambda)\begin{vmatrix}4-\lambda & -3\\2 & -1-\lambda\end{vmatrix}&=0\\(2-\lambda)((4-\lambda)(-1-\lambda)+6)&=0\\(2-\lambda)\left(\lambda^2-3\lambda+2\right)&=0\\(2-\lambda)(\lambda-2)(\lambda-1)&=0\end{aligned}$$

Hence the eigenvalues are $\lambda_1=2$ of algebraic multiplicity 2 and $\lambda_2=1$. For each eigenvalue we find its associated eigenvectors.

$\lambda_1=2$

We nee to solve $A\vec{v}=\lambda\vec{v}$. This becomes $(A-\lambda I)\vec{v}=\vec{0}$. Therefore

$$\begin{bmatrix}4-2 & -3 & 1\\2 & -1-2 & 1\\0 & 0 & 2-2\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}2 & -3 & 1\\2 & -3 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\tag{1}$$

$R_2\to R_2-R_1$

$$\begin{bmatrix}2 & -3 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Free variables are $v_3.v_2$. This means this is a complete eigenvalue. Since it has algebraic multiplicity of 2 and have a geometric multiplicity of 2 as well. This means we can find two linearly independent eigenvectors from it. Let $v_2 = s, v_3 = t$. First row gives $2v_1 - 3s + t = 0$ or $v_1 = \frac{3}{2}s - \frac{1}{2}t$. Hence the solution is

$$\begin{aligned}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} \frac{3}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix} \\ &= s\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}\end{aligned}$$

Therefore the basis (eigenvectors) are

$$\left\{ \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\} \rightarrow \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

Now that we found the eigenvectors associated with $\lambda_1 = 2$, we will do the same for second eigenvalue.

$\underline{\lambda_2 = 1}$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{aligned}\begin{bmatrix} 4-1 & -3 & 1 \\ 2 & -1-1 & 1 \\ 0 & 0 & 2-1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (1)\end{aligned}$$

$R_2 \rightarrow R_2 - \frac{2}{3}R_1$

$$\begin{bmatrix} 3 & -3 & 1 \\ 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_3 + \frac{3}{2}R_2$

$$\begin{bmatrix} 3 & -3 & 1 \\ 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Free variable is v_2, leading variables are v_1, v_3. Let $v_2 = 1$. From second row, $v_3 = 0$.First row gives $3v_1 = 3$. Hence $v_1 = 1$. Hence the eigenvector is

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

This table gives summary of the result

eigenvalue λ	associated eigenvector $\vec{v}$
2 (multiplicity 2)	$\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$
1	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

2.7.10 Problem 25 section 6.1

In Problems 1 through 26, find the (real) eigenvalues and associated eigenvectors of the given matrix A. Find a basis for each eigenspace of dimension 2 or larger.

$$A=\begin{bmatrix}1&0&1&0\\0&1&1&0\\0&0&2&0\\0&0&0&2\end{bmatrix}$$

Solution

We first need to find the eigenvalues. These are found by solving $|A-\lambda I|=0$. Hence

$$\begin{vmatrix}1-\lambda&0&1&0\\0&1-\lambda&1&0\\0&0&2-\lambda&0\\0&0&0&2-\lambda\end{vmatrix}=0$$

Since this is an upper triangle matrix, then the determinant is the product of the diagonal. Hence the above reduces to

$$(1-\lambda)^2(2-\lambda)^2=0$$

Therefore the eigenvalues are $\lambda_1=1$ of algebraic multiplicity 2 and $\lambda_2=2$ also of algebraic multiplicity 2. For each eigenvalue we find its associated eigenvectors.

$\lambda_1=1$

We nee to solve $A\vec{v}=\lambda\vec{v}$. This becomes $(A-\lambda I)\vec{v}=\vec{0}$. Therefore

$$\begin{bmatrix}1-1&0&1&0\\0&1-1&1&0\\0&0&2-1&0\\0&0&0&2-1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}0&0&1&0\\0&0&1&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$R_2\rightarrow R_2-R_1$

$$\begin{bmatrix}0&0&1&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Swapping R_3,R_2 (for clarify)

$$\begin{bmatrix}0&0&1&0\\0&0&1&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$R_2\rightarrow R_2-R_1$

$$\begin{bmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Swapping R_4,R_2 (for clarify)

$$\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Hence leading variables are v_3, v_4 free variables are v_1, v_2. Let $v_1 = s, v_2 = t$. Second row gives $v_4 = 0$. First row gives $v_3 = 0$. Therefore the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} s \\ t \\ 0 \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Therefore the two eigenvectors associated with this eigenvalues are

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$\underline{\lambda_2 = 2}$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} 1-2 & 0 & 1 & 0 \\ 0 & 1-2 & 1 & 0 \\ 0 & 0 & 2-2 & 0 \\ 0 & 0 & 0 & 2-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence leading variables are v_1, v_2 free variables are v_3, v_4 Let $v_3 = s, v_4 = t$. Second row gives $-v_2 + v_3 = 0$ or $v_2 = s$. First row gives $-v_1 + s = 0$ or $v_1 = s$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} s \\ s \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore the two eigenvectors associated with this eigenvalues are

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

This table gives summary of the result

eigenvalue λ	associated eigenvector $\vec{v}$
1 (multiplicity 2)	$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$
2 (multiplicity 2)	$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

2.7.11 Problem 29 section 6.1

Find the complex conjugate eigenvalues and corresponding eigenvectors of the matrices given in Problems 27 through 32

$$A = \begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix}$$

Solution

We first need to find the eigenvalues. These are found by solving $|A - \lambda I| = 0$. Hence

$$\begin{vmatrix} -\lambda & -3 \\ 12 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 36 = 0$$

Hence $\lambda = \pm 6i$. For each eigenvalue we find its associated eigenvectors.

$\lambda_1 = 6i$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} -6i & -3 \\ 12 & -6i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + \frac{12}{6i} R_1$

$$\begin{bmatrix} -6i & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Leading variable is v_1, free variable is v_2. Let $v_2 = 1$. From first row $-6iv_1 - 3v_2 = 0$ or $v_1 = -\frac{3}{6i} = -\frac{1}{2i} = \frac{1}{2}i$. Hence the eigenvector is

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{2}i \\ 1 \end{bmatrix} = \begin{bmatrix} i \\ 2 \end{bmatrix}$$

$\lambda_1 = -6i$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} +6i & -3 \\ 12 & +6i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - \frac{12}{6i} R_1$

$$\begin{bmatrix} 6i & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Leading variable is v_1, free variable is v_2. Let $v_2 = 1$. From first row $6iv_1 - 3v_2 = 0$ or $v_1 = \frac{3}{6i} = \frac{1}{2i} = -\frac{1}{2}i$. Hence the eigenvector is

$$\vec{v}_1 = \begin{bmatrix} -\frac{1}{2}i \\ 1 \end{bmatrix} = \begin{bmatrix} -i \\ 2 \end{bmatrix}$$

This table gives summary of the result

eigenvalue λ	associated eigenvector $\vec{v}$
$6i$	$\begin{bmatrix} i \\ 2 \end{bmatrix}$
$-6i$	$\begin{bmatrix} -i \\ 2 \end{bmatrix}$

2.7.12 Additional problem 1

Find particular solution to

$$y^{(7)}(x) - 2y^{(6)} + 9y^{(5)} - 16y^{(4)} + 24y^{(3)} - 32y'' + 16y' = e^{2x} + x\sin x + x^2 \tag{1}$$

Solution

From HW6, we found y_h as

$$y(x) = c_1 + (c_2e^x + c_3xe^x) + (c_4\cos 2x + c_5\sin 2x) + x(c_6\cos 2x + c_7\sin 2x)$$

Therefore, we see the basis functions for y_h are

$$\{1, e^x, xe^x, \cos 2x, \sin 2x, x\cos 2x, x\sin 2x\} \tag{2}$$

Looking at RHS of (1), we see the basis functions for y_p are

$$\left\{e^{2x}, x^2, x\sin x\right\}$$

Taking derivative e^{2x} does not generate new basis. Taking derivative of x^2 generates $x,1$. And taking derivative of $x\sin x$ generates $\sin x, x\cos x, \cos x$. Hence the above becomes

$$\left\{e^{2x}, \left(x^2, x, 1\right), (x\sin x, \sin x, x\cos x, \cos x)\right\} \tag{3}$$

There are 3 groups. Comparing (2,3) we see there is one duplication, which is the constant term. Hence we need to multiply that one group by x. The above becomes

$$\left\{e^{2x}, x\left(x^2, x, 1\right), (x\sin x, \sin x, x\cos x)\right\} = \left\{e^{2x}, \left(x^3, x^2, x\right), (x\sin x, \sin x, x\cos x, \cos x)\right\} \tag{3A}$$

Now we again compare (3A) and (2). Now there is no duplication. Therefore the particular solution is

$$y_p = A_1e^{2x} + A_2\left(x^3\right) + A_3\left(x^2\right) + A_4(x) + A_5(x\sin x) + A_6(\sin x) + A_7(x\cos x) + A_8\cos x$$

2.7.13 Additional problem 2

Let $A = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}$ where t_1, t_2, t_3 are distinct real numbers. Find the eigenvalues of A and the corresponding eigenvectors.

Solution

We first need to find the eigenvalues. These are found by solving $|A - \lambda I| = 0$. Hence

$$\begin{vmatrix} t_1 - \lambda & 0 & 0 \\ 0 & t_2 - \lambda & 0 \\ 0 & 0 & t_3 - \lambda \end{vmatrix} = 0$$

Since this is a diagonal matrix, then the determinant is the product of the diagonal. Hence the above reduces to

$$(t_1 - \lambda)(t_2 - \lambda)(t_3 - \lambda) = 0$$

Hence the eigenvalues are $\lambda_1 = t_1, \lambda_2 = t_2, \lambda_3 = t_3$. For each eigenvalue we find its associated eigenvectors.

$\lambda_1 = t_1$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} t_1 - t_1 & 0 & 0 \\ 0 & t_2 - t_1 & 0 \\ 0 & 0 & t_3 - t_1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2 - t_1 & 0 \\ 0 & 0 & t_3 - t_1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Leading variables are v_2, v_3 and free variables is v_1. Let $v_1 = s$. Second and third rows give $v_2 = 0, v_3 = 0$, this is because t_1, t_2, t_3 are distinct real numbers therefore $t_2 - t_1 \neq 0, t_3 - t_1 \neq 0$. Therefore the solution is

$$\vec{v}_1 = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $s = 1$ this gives the eigenvector

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\underline{\lambda_2 = t_2}$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} t_1 - t_2 & 0 & 0 \\ 0 & t_2 - t_2 & 0 \\ 0 & 0 & t_3 - t_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} t_1 - t_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t_3 - t_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Leading variables are v_1, v_3 and free variable is v_2. Let $v_2 = s$. First and third rows give $v_1 = 0, v_3 = 0$, this is because t_1, t_2, t_3 are distinct real numbers therefore $t_1 - t_2 \neq 0, t_3 - t_2 \neq 0$. Therefore the solution is

$$\vec{v}_2 = \begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $s = 1$ this gives the eigenvector

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\underline{\lambda_3 = t_3}$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} t_1 - t_3 & 0 & 0 \\ 0 & t_2 - t_3 & 0 \\ 0 & 0 & t_3 - t_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} t_1 - t_3 & 0 & 0 \\ 0 & t_2 - t_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Leading variables are v_1, v_2 and free variable is v_3. Let $v_3 = s$. First and third rows give $v_1 = 0, v_2 = 0$, this is because t_1, t_2, t_3 are distinct real numbers therefore $t_1 - t_3 \neq 0, t_2 - t_3 \neq 0$. Therefore the solution is

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} = s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

For $s = 1$ this gives the eigenvector

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This table gives summary of the result

eigenvalue λ	associated eigenvector $\vec{v}$
t_1	$\begin{bmatrix}1\\0\\0\end{bmatrix}$
t_2	$\begin{bmatrix}0\\1\\0\end{bmatrix}$
t_2	$\begin{bmatrix}0\\0\\1\end{bmatrix}$

2.7.14 Additional optional problem 3

Extend the result in problem 2 to the case of $n \times n$ matrices. That is, let A be a matrix with entries $t_1, t_2, \cdots, t_n$ on the main diagonal and 0s everywhere else, where the t_i are distinct real numbers. Find the eigenvalues and corresponding eigenvectors.

Solution

This follows immediately from the last problem. Therefore each eigenvalue will be $\lambda_1 = t_1, \lambda_2 = t_2, \cdots, \lambda_n = t_n$. And corresponding eigenvectors are (each eigenvector is $n \times 1$.

$$v_1 = \begin{bmatrix}1\\0\\0\\\vdots\\0\\0\end{bmatrix}, v_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\\0\end{bmatrix}, \cdots, v_n = \begin{bmatrix}0\\0\\0\\\vdots\\0\\1\end{bmatrix}$$

These eigenvectors are the standard basis for $\mathbb{R}^n$.

2.7.15 key solution for HW 7

Homework 7 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

5.5.3 We have characteristic equation $r^2 - r - 6 = (r-3)(r+2)$. So the complementary solution is $y_c = c_1 e^{3x} + c_2 e^{-2x}$.

We have term $\sin 3x$ with derivative $\cos 3x$ and no repetition. So we set $y_p = A\sin 3x + B\cos 3x$. We compute

$$\begin{aligned} y_p'' - y' - 6y &= (-9A\sin 3x - 9B\cos 3x) - (3A\cos 3x - 3B\sin 3x) - 6(A\sin 3x + B\cos 3x) \\ &= (-3A - 15B)\cos 3x + (-15A + 3B)\sin 3x \end{aligned}$$

We now have the equations $-3A - 15B = 0$ and $-15A + 3B = 2$ which solve to $A = -5/39, B = 1/39$. So the particular solution is $y_p = \frac{-5}{39}\sin 3x + \frac{1}{39}\cos 3x$.

5.5.9 We have characteristic equation $r^2 + 2r - 3 = (r-1)(r+3)$. So the complementary solution is $y_c = c_1 e^x + c_2 e^{-3x}$.

We have term 1 with no derivatives and term xe^x with derivative e^x. We have repetition so we bump to x^2e^x, xe^x So we set $y_p = A + Bxe^x + Cx^2e^x$. We compute

$$\begin{aligned} y_p'' + 2y_p' - 3y_p &= ((2B+2C)e^x + (B+4C)xe^x + Cx^2e^x) \\ &\quad + 2(Be^x + (B+2C)xe^x + Cx^2e^x) \\ &\quad - 3(A + Bxe^x + Cx^2e^x) \\ &= -3A + (4B+2C)e^x + (8C)xe^x + (0)x^2e^x \end{aligned}$$

We now have the equations $-3A = 1$, $4B + 2C = 0$, and $8C = 1$ which solve to $A = -1/3, B = -1/16, C = 1/8$. So our particular solution is $y_p = \frac{-1}{3} - \frac{1}{16}xe^x + \frac{1}{8}x^2e^x$.

5.5.11 We have characteristic equation $r^3 + 4r = r(r^2+4)$ with roots $0, \pm 2i$. So our complementary solution is $y_c = c_1 + c_2\cos 2x + c_3 \sin 2x$.

We have terms $x, 1$ with duplication, so we bump to x^2, x. So we set $y_p = Ax + Bx^2$. We compute

$$\begin{aligned} y_p^{(3)} + 4y_p' &= (0) + 4(A + 2Bx) \\ &= 4A + 8Bx \end{aligned}$$

We have equations $4A = -1$ and $8B = 3$ which solve to $A = -1/4, B = 3/8$. So the particular solution is $y_p = \frac{-1}{4}x + \frac{3}{8}x^2$.

1

5.5.23 We have characteristic equation $r^2 + 4$ with roots $\pm 2i$, so the complementary solution is $y_c = c_1 \cos 2x + c_2 \sin 2x$.

We have term $x \cos 2x$ with derivatives $x \sin 2x, \sin 2x, \cos 2x$. We have duplication, so we bump up to $x^2 \sin 2x, x^2 \cos 2x, x \sin 2x, x \cos 2x$. This gives us a particular solution of the form

$$y_p = Ax \cos 2x + Bx \sin 2x + Cx^2 \cos 2x + Dx^2 \sin 2x$$

5.5.32 We have characteristic equation $r^2 + 3r + 2 = (r+2)(r+1)$, so we have complementary solution $y_c = c_1 e^{-2x} + c_2 e^{-x}$.

We have term e^x and no repetition, so we set $y_p = Ae^x$. We compute

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= Ae^x + 3Ae^x + 2Ae^x \\ &= 6Ae^x \end{aligned}$$

So we have the equation $6A = 1$ and thus our particular solution is $y_p = \frac{1}{6}e^x$.

The form of a general solution is now $y = y_c + y_p = c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{6}e^x$. The initial conditions give us the following:

$$\begin{aligned} y(0) &= c_1 e^0 + c_2 e^0 + \frac{1}{6}e^0 \\ 0 &= c_1 + c_2 + \frac{1}{6} \\ y'(x) &= -2c_1 e^{-2x} - c_2 e^{-x} + \frac{1}{6}e^x \\ y'(0) &= -2c_1 e^0 - c_2 e^0 + \frac{1}{6}e^0 \\ 3 &= -2c_1 - c_2 + \frac{1}{6} \end{aligned}$$

So we have equations $c_1 + c_2 = -\frac{1}{6}$ and $-2c_1 - c_2 = \frac{17}{6}$. This system solves to $c_1 = -\frac{8}{3}$ and $c_2 = \frac{5}{2}$. So, our solution in this case is $y(x) = -\frac{8}{3}e^{-2x} + \frac{5}{2}e^{-x} + \frac{1}{6}e^x$.

6.1.7 First, we compute the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 10-\lambda & -8 \\ 6 & -4-\lambda \end{bmatrix} \\ &= (10-\lambda)(-4-\lambda) + 48 \\ &= \lambda^2 - 6\lambda + 8 \\ &= (\lambda - 4)(\lambda - 2) \end{aligned}$$

The eigenvalues are $\lambda_1 = 4, \lambda_2 = 2$.

For $\lambda_1 = 4$, we have the matrix $\begin{bmatrix} 6 & -8 \\ 6 & -8 \end{bmatrix}$ which reduces to $\begin{bmatrix} 3 & -4 \\ 0 & 0 \end{bmatrix}$. This is a dimension 1 eigenspace with eigenvector $\vec{v}_1 = (4, 3)$.

For $\lambda_2 = 2$, we have the matrix $\begin{bmatrix} 8 & -8 \\ 6 & -6 \end{bmatrix}$ which reduces to $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. This is a dimension 1 eigenspace with eigenvector $\vec{v}_2 = (1, 1)$.

6.1.17 First, we compute the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 5 & -2 \\ 0 & 2-\lambda & 0 \\ 0 & 2 & 1-\lambda \end{bmatrix} \\ &= (3-\lambda) \det \begin{bmatrix} 2-\lambda & 0 \\ 2 & 1-\lambda \end{bmatrix} \\ &= (3-\lambda)(2-\lambda)(1-\lambda) \end{aligned}$$

The eigenvalues are $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1$.

For $\lambda_1 = 3$, we have the matrix $\begin{bmatrix} 0 & 5 & -2 \\ 0 & -1 & 0 \\ 0 & 2 & -2 \end{bmatrix}$ which reduces to $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. This is a dimension 1 eigenspace with eigenvector $\vec{v}_1 = (1, 0, 0)$.

For $\lambda_2 = 2$, we have the matrix $\begin{bmatrix} 1 & 5 & -2 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix}$ which reduces to $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. This is a dimension 1 eigenspace with eigenvector $\vec{v}_2 = (-1, 1, 2)$.

For $\lambda_3 = 1$, we have the matrix $\begin{bmatrix} 2 & 5 & -2 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ which reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This is a dimension 1 eigenspace with eigenvector $\vec{v}_3 = (1, 0, 1)$.

6.1.21 First, we compute the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 4-\lambda & -3 & 1 \\ 2 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} \\ &= (2-\lambda) \det \begin{bmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{bmatrix} \\ &= (2-\lambda)\left[(4-\lambda)(-1-\lambda) + 6\right] \\ &= (2-\lambda)\left[\lambda^2 - 3\lambda + 2\right] \\ &= (2-\lambda)(\lambda-2)(\lambda-1) \end{aligned}$$

The eigenvalues are $\lambda_1 = 1, \lambda_2 = 2$ with λ_2 occuring with multiplicity 2.

For $\lambda_1 = 1$, we have the matrix $\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ which reduces to $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. This is a 1-dimensional eigenspace with eigenvector $\vec{v}_1 = (1, 1, 0)$.

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For $\lambda_2 = 2$, we have the matrix $\begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ which reduces to $\begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This is a 2-dimensional eigenspace with basis vectors $\vec{v}_2 = (1, 0, -2)$ and $\vec{v}_3 = (3, 2, 0)$.

6.1.25 We compute the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix} \\ &= (1-\lambda)^2(2-\lambda)^2 \end{aligned}$$

Here we are using that the determinant of an upper triangular matrix is the product of the diagonal entries, but you can also see this by successively expanding along the first column. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$, each of multiplicity 2.

For $\lambda_1 = 1$, we have the matrix $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ which reduces to $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ We have a dimension 2 eigenspace with basis vectors $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$.

For $\lambda_2 = 2$, we have the matrix $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which is already sufficiently reduced. This gives a dimension 2 eigenspace with basis vectors $(0, 0, 0, 1)$ and $(1, 1, 1, 0)$.

6.1.29 We compute the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} -\lambda & -3 \\ 12 & -\lambda \end{bmatrix} \\ &= \lambda^2 + 36 \end{aligned}$$

The eigenvalues are $\pm 6i$.

For $\lambda_1 = 6i$, we have matrix $\begin{bmatrix} -6i & -3 \\ 12 & -6i \end{bmatrix}$. We row reduce this matrix:

$$\begin{aligned} \begin{bmatrix} -6i & -3 \\ 12 & -6i \end{bmatrix} &\xrightarrow{-2iR_1 + R_2} \begin{bmatrix} -6i & -3 \\ 0 & 0 \end{bmatrix} \\ &\xrightarrow{-\frac{1}{3}R_1} \begin{bmatrix} 2i & 1 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

We have a dimension 1 eigenspace containing the eigenvector $\vec{v}_1 = (-1, 2i)$.

For the conjugate eigenvalue $\lambda_2 = -6i$, we have the conjugate eigenvector $\vec{v}_2 = (-1, -2i)$.

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Additional Problems:

1. On homework 6, we found the general solution to the homogeneous equation, so we have the complementary solution in this case:

$$y_c(x) = c_1 + c_2e^x + c_3xe^x + c_4\cos(2x) + c_5\sin(2x) + c_6x\cos(2x) + c_7x\sin(2x)$$

The term e^{2x} is not duplicated. The term $x\sin x$ has derivatives $x\cos x, \sin x, \cos x$ and there is no duplication. The term x^2 has derivatives $x, 1$ and there is duplication. It is enough to bump up by a factor of x, so we get new terms x^3, x^2, x. So, the form of our particular solution is

$$y_p = Ae^{2x} + Bx\sin x + Cx\cos x + D\sin x + E\cos x + Fx^3 + Gx^2 + Hx$$

2. We compute the characteristic polynomial:

$$\begin{aligned}\det(A - \lambda I) &= \det\begin{bmatrix} t_1-\lambda & 0 & 0 \\ 0 & t_2-\lambda & 0 \\ 0 & 0 & t_3-\lambda\end{bmatrix} \\ &= (t_1-\lambda)(t_2-\lambda)(t_3-\lambda)\end{aligned}$$

The computation of determinants like this was done in Additional Problem 4 of Homework 2. This has three distinct roots, t_1, t_2, t_3.

For $\lambda = t_1$, $A - \lambda I$ is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2-t_1 & 0 \\ 0 & 0 & t_3-t_1\end{bmatrix}$. Notice that since t_1, t_2, t_3 are all distinct, $t_2 - t_1 \neq 0$ and $t_3 - t_1 \neq 0$ so we only have a free variable in the first column. A solution to this system is $\vec{v_1} = (1, 0, 0)$, so this is an eigenvector corresponding to t_1.

For $\lambda = t_2$, $A - \lambda I$ is $\begin{bmatrix} t_1-t_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t_3-t_2\end{bmatrix}$. A solution to this system is $\vec{v_2} = (0, 1, 0)$, so this is an eigenvector corresponding to t_2.

For $\lambda = t_3$, $A - \lambda I$ is $\begin{bmatrix} t_1-t_3 & 0 & 0 \\ 0 & t_2-t_3 & 0 \\ 0 & 0 & 0\end{bmatrix}$. A solution to this system is $\vec{v_3} = (0, 0, 1)$, so this is an eigenvector corresponding to t_3.

3. Perhaps not surprisingly, the characteristic polynomial is

$$(t_1-\lambda)(t_2-\lambda)\cdots(t_n-\lambda) = \prod_{i=1}^{n}(t_i-\lambda)$$

So each t_i is an eigenvalue.

If we set $\lambda = t_i$, $A - \lambda I$ has a 0 in the i-th diagonal element. A solution vector is $\vec{e_i}$, the standard basis vector with 1 in the i-th position and 0 elsewhere.

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2.8 HW 8

Local contents

2.8.1 Problems listing

Homework 8 - Due November 5

Homework instructions: Complete the assigned problems on your own paper. Once you are finished, scan or photograph your work and upload it to Gradescope. When prompted, tell Gradescope where to find each problem.
You are allowed (and in fact encouraged) to work with other students on homework assignments. If you do that, please indicate on each problem who you worked with. If you use sources other than your notes, the textbook, and any resources on Canvas for your homework, you must indicate the source on each problem. You are not permitted to view, request, or look for solutions to any of the homework problems from solutions manuals, homework help websites, online forums, other students, or any other sources.

Textbook Problems:

- §6.2: 7, 15, 19
- §6.3: 7, 13, 25

Additional Problems:

1. For $n \times n$ matrices A and B, we say that **A is similar to B** if there is an invertible matrix P so that $A = PBP^{-1}$. So, in order to show that A is similar to B, you need to (1) say what the matrix P is in that case (2) check that your choice of P is invertible and (3) explain why the equation $A = PBP^{-1}$ is true.
   (a) Let A be any $n \times n$ matrix. Show that A is similar to itself.
   (b) Let A, B be $n \times n$ matrices. Suppose that A is similar to B. Show that B is similar to A.
   (c) Let A, B, C be $n \times n$ matrices. Suppose that A is similar to B and that B is similar to C. Show that A is similar to C.

   Cultural Aside: A matrix is **diagonalizable** if it is similar to a diagonal matrix. For a matrix that isn't diagonalizable, it may still be useful to find a nice matrix that it is similar to (even if we can't get to something quite as nice as a diagonal matrix). If you are interested in these not-quite-diagonal nice matrices, you should look up "Jordan normal form."

   Further Cultural Aside: These three properties are called (a) reflexivity, (b) symmetry, and (c) transitivity. A relation like "is similar to" that satisfies all three properties is called an *equivalence relation*. Equivalence relations behave a lot like "is equal to" and are nearly as useful in mathematics. In fact, we have already seen another example of an equivalence relation in this class: "is row equivalent to".

2. The **Fibonacci sequence** begins as

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 \ldots$$

1

It can be defined recursively by $f_0 = 0$ and $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$. That is, we begin the sequence with $1, 1$ and then each successive term is the sum of the two previous terms. (This sequence is usually interpreted as the number of pairs of rabbits in a population.)

Notice that we can write

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \begin{bmatrix} f_n + f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix}$$

Giving some labels to things, let $\vec{x}_n = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}$ and let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then we have $\vec{x}_n = A\vec{x}_{n-1}$. So, $\vec{x}_n = A^n\vec{x}_0$ where $\vec{x}_0 = \begin{bmatrix} f_1 \\ f_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(a) Find the eigenvalues and corresponding eigenvectors of A. (I recommend using the quadratic formula to find the roots of the characteristic polynomial.)

(b) Find a diagonalization $A = PDP^{-1}$ of A. Use this write down a formula for A^n.

(c) Use the fact that $\vec{x}_n = A^n\vec{x}_0$ to write down a formula for f_n.

Hint: It may be useful to denote the number $\frac{1+\sqrt{5}}{2} \approx 1.61803$ by φ. This number is called the **golden ratio** and satisfies

$$\frac{1-\sqrt{5}}{2} = -\frac{1}{\varphi} = 1 - \varphi \approx -0.61803$$

2.8.2 Problem 7, section 6.2

In Problems 1 through 28, determine whether or not the given matrix A is diagonalizable. If it is, find a diagonalizing matrix P and a diagonal matrix D such that $P^{-1}AP = D$

$$\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix}$$

solution The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\det\left(\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &= 0 \\
\det\begin{bmatrix} 6-\lambda & -10 \\ 2 & -3-\lambda \end{bmatrix} &= 0 \\
(6-\lambda)(-3-\lambda) + 20 &= 0 \\
\lambda^2 - 3\lambda + 2 &= 0 \\
(\lambda - 2)(\lambda - 1) &= 0
\end{aligned}$$

The eigenvalues are the roots of the above characteristic polynomial. From the above, these are

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= 1
\end{aligned}$$

This table summarizes the result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector.

$\lambda = 1$

We need now to determine the eigenvector $\vec{v}$ where

$$\begin{aligned}
A\vec{v} &= \lambda\vec{v} \\
A\vec{v} - \lambda\vec{v} &= \vec{0} \\
(A - \lambda I)\vec{v} &= \vec{0} \\
\left(\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 5 & -10 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 5 & -10 & 0 \\ 2 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{5} \implies \left[\begin{array}{cc|c} 5 & -10 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & -10 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables. First row gives $5v_1 = 10t$ or $v_1 = 2t$. Hence the eigenvector for this eigenvalue is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\underline{\lambda = 2}$

We need now to determine the eigenvector $\vec{v}$ where

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ A\vec{v} - \lambda\vec{v} &= \vec{0} \\ (A - \lambda I)\vec{v} &= \vec{0} \\ \left(\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 4 & -10 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -10 & 0 \\ 2 & -5 & 0 \end{array} \right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 4 & -10 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -10 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables. First row gives $4v_1 = 10v_2$ or $v_1 = \frac{5t}{2}$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

The following table summarizes the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$$

Therefore

$$A = PDP^{-1}$$

$$\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}^{-1}$$

2.8.3 Problem 15 section 6.2

In Problems 1 through 28, determine whether or not the given matrix A is diagonalizable. If it is, find a diagonalizing matrix P and a diagonal matrix D such that $P^{-1}AP = D$

$$\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det \left(\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$$

$$\det \begin{bmatrix} 3-\lambda & -3 & 1 \\ 2 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$$

Expanding along last row gives

$$(-1)^{3+3}(1-\lambda) \begin{vmatrix} 3-\lambda & -3 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)((3-\lambda)(-2-\lambda)+6) = 0$$

$$(1-\lambda)\left(\lambda^2 - \lambda\right) = 0$$

$$(1-\lambda)\lambda(\lambda-1) = 0$$

The eigenvalues are the roots of the above characteristic polynomial. These are seen to be

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

This table summarizes the result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
1	2	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector.

$\underline{\lambda = 0}$

We need now to determine the eigenvector $\vec{v}$ where

$$A\vec{v} = \lambda\vec{v}$$
$$A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$(A - \lambda I)\vec{v} = \vec{0}$$
$$\left(\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\left(\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & -3 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 3 & -3 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} 3 & -3 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -3 & 1 \\ 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables. Second row gives $v_3 = 0$. First row gives $3v_1 - 3v_2 = 0$ or $v_1 = t$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\underline{\lambda = 1}$

We need now to determine the eigenvector $\vec{v}$ where

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ A\vec{v} - \lambda\vec{v} &= \vec{0} \\ (A - \lambda I)\vec{v} &= \vec{0} \\ \left(\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables. First row gives $2v_1 - 3v_2 + v_3 = 0$ or $v_1 = \frac{3t}{2} - \frac{s}{2}$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} - \frac{s}{2} \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{s}{2} \\ 0 \\ s \end{bmatrix}$$
$$= t\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}\right)$$

Which can be normalized to

$$\left(\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}\right)$$

The following table summarizes the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
0	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
1	2	2	No	$\begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvalues found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$P = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{-1}$$

2.8.4 Problem 19 section 6.2

In Problems 1 through 28, determine whether or not the given matrix A is diagonalizable. If it is, find a diagonalizing matrix P and a diagonal matrix D such that $P^{-1}AP = D$

$$\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix}$$

Solution

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\det\left(\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) &= 0 \\
\det\begin{bmatrix} 1-\lambda & 1 & -1 \\ -2 & 4-\lambda & -1 \\ -4 & 4 & 1-\lambda \end{bmatrix} &= 0 \\
-\lambda^3 + 6\lambda^2 - 11\lambda + 6 &= 0
\end{aligned}$$

Expanding along first row gives

$$\begin{aligned}
(1-\lambda)\begin{vmatrix} 4-\lambda & -1 \\ 4 & 1-\lambda \end{vmatrix} - \begin{vmatrix} -2 & -1 \\ -4 & 1-\lambda \end{vmatrix} - \begin{vmatrix} -2 & 4-\lambda \\ -4 & 4 \end{vmatrix} &= 0 \\
(1-\lambda)((4-\lambda)(1-\lambda) + 4) - (-2(1-\lambda) - 4) - (-8 + 4(4-\lambda)) &= 0 \\
-\lambda^3 + 6\lambda^2 - 13\lambda + 8 - (2\lambda - 6) - (8 - 4\lambda) &= 0 \\
-\lambda^3 + 6\lambda^2 - 11\lambda + 6 &= 0 \\
\lambda^3 - 6\lambda^2 + 11\lambda - 6 &= 0
\end{aligned}$$

Trying $\lambda = 1$

$$\begin{aligned}
1^3 - 6 + 11 - 6 &= 0 \\
0 &= 0
\end{aligned}$$

Hence $(\lambda - 1)$ is a factor. Doing long division $\frac{\lambda^3-6\lambda^2+11\lambda-6}{(\lambda-1)} = \lambda^2 - 5\lambda + 6$. This can be factored as $(\lambda - 2)(\lambda - 3)$. Therefore

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3)$$

Hence the eigenvalues are

$$\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= 2 \\
\lambda_3 &= 3
\end{aligned}$$

This table summarizes the result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector.

$\lambda = 1$

We need now to determine the eigenvector $\vec{v}$ where

$$A\vec{v} = \lambda\vec{v}$$
$$A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$(A - \lambda I)\vec{v} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\left(\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 & -1 \\ -2 & 3 & -1 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ -2 & 3 & -1 & 0 \\ -4 & 4 & 0 & 0 \end{array}\right]$$

current pivot $A(1,1)$ is zero. Hence we need to replace current pivot row with one non-zero. Replacing row 1 with row 2 gives

$$\left[\begin{array}{ccc|c} -2 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -4 & 4 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables. Second row gives $v_2 = v_3 = t$. First row gives $-2v_1 + 3v_2 - v_3 = 0$ or $-2v_1 = -3t + t = -2t$. Hence $v_1 = t$. Therefore the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda = 2}$

We need now to determine the eigenvector $\vec{v}$ where

$$\begin{aligned}
A\vec{v} &= \lambda\vec{v} \\
A\vec{v} - \lambda\vec{v} &= \vec{0} \\
(A - \lambda I)\vec{v} &= \vec{0} \\
\left(\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -1 & 1 & -1 \\ -2 & 2 & -1 \\ -4 & 4 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ -2 & 2 & -1 & 0 \\ -4 & 4 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -4 & 4 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - 4R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Third row gives $v_3 = 0$. First row gives $-v_1 + v_2 = 0$ or $v_1 = t$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t=1$ then the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\underline{\lambda = 3}$

We need now to determine the eigenvector $\vec{v}$ where

$$\begin{aligned}
A\vec{v} &= \lambda\vec{v} \\
A\vec{v} - \lambda\vec{v} &= \vec{0} \\
(A - \lambda I)\vec{v} &= \vec{0} \\
\left(\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -2 & 1 & -1 \\ -2 & 1 & -1 \\ -4 & 4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ -2 & 1 & -1 & 0 \\ -4 & 4 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 4 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]$$

current pivot $A(2,2)$ is zero. Hence we need to replace current pivot row with one non-zero. Replacing row 2 with row 3 gives

$$\left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables. Second row gives $v_2 = 0$. First row gives $-2v_1 = v_3 = t$. Hence $v_1 = -\frac{t}{2}$. Therefore

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

The following table summarizes the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -1 \\ -4 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}^{-1}$$

2.8.5 Problem 7 section 6.3

In Problems 1 through 10, a matrix A is given. Use the method of Example 1 to compute A^5.

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution

If A is diagonalizable, then by first writing $A = PDP^{-1}$ then $A^5 = PD^5P^{-1}$. And since D is diagonal matrix, it is easy to raise it to power. So the first step is to diagonalize A as we did in the above problems.

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 1-\lambda & 3 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} = 0$$

Expansion along the first column gives

$$(1-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$$
$$(1-\lambda)(2-\lambda)(2-\lambda) = 0$$

Therefore the eigenvalues are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

This table summarizes the result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	2	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector.

$\lambda = 1$

We need now to determine the eigenvector $\vec{v}$ where

$$A\vec{v} = \lambda\vec{v}$$
$$A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$(A - \lambda I)\vec{v} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\left(\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

current pivot $A(2,3)$ is zero. Hence we need to replace current pivot row with one non-zero. Replacing row 2 with row 3 gives

$$\left[\begin{array}{ccc|c} 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Second row gives $v_3 = 0$. First row also gives $v_2 = 0$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\underline{\lambda = 2}$

We need now to determine the eigenvector $\vec{v}$ where

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ A\vec{v} - \lambda\vec{v} &= \vec{0} \\ (A - \lambda I)\vec{v} &= \vec{0} \end{aligned}$$

$$\left(\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. First row gives $-v_1 = -3v_2 = -3t$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 3t \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 3t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} \\ &= t\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

The following table summarizes the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	1	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
2	2	2	No	$\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Now that we have diagonalized A, we can finally answer the question.

$$\begin{aligned} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}^5 &= \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}^5 \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^5 & 0 \\ 0 & 0 & 2^5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \end{aligned}$$

But

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} = \begin{bmatrix} 1 & 96 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}^5 = \begin{bmatrix} 1 & 96 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \tag{1}$$

We know need to find $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$. The augmented matrix is

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 3R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -3 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Since left half is now I then the right half is the inverse. Therefore $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Hence (1) becomes

$$\begin{aligned} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}^5 &= \begin{bmatrix} 1 & 96 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 93 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} \end{aligned}$$

2.8.6 Problem 13 section 6.3

Find A^{10}.

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & -4 & 1 \end{bmatrix}$$

<u>Solution</u>

If A is diagonalizable, then by first writing $A = PDP^{-1}$ then $A^{10} = PD^{10}P^{-1}$. And since D is diagonal matrix, it is easy to raise it to power. So the first step is to diagonalize A as we did in the above problems.

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & -4 & 1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det\left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & -4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) &= 0 \\ \det\begin{bmatrix} 1-\lambda & -1 & 1 \\ 2 & -2-\lambda & 1 \\ 4 & -4 & 1-\lambda \end{bmatrix} &= 0 \\ (1-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ -4 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 4 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 2 & -2-\lambda \\ 4 & -4 \end{vmatrix} &= 0 \\ (1-\lambda)((-2-\lambda)(1-\lambda)+4) + 2(1-\lambda) - 4 + (-8) - 4(-2-\lambda) &= 0 \\ -\lambda^3 - \lambda + 2 - 2\lambda - 2 + 4\lambda &= \\ \lambda - \lambda^3 &= 0 \\ \lambda\left(1-\lambda^2\right) &= 0 \end{aligned}$$

Therefore the eigenvalues are

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 1 \\ \lambda_3 &= -1 \end{aligned}$$

This table summarizes the result

eigenvalue	algebraic multiplicity	type of eigenvalue
−1	1	real eigenvalue
0	1	real eigenvalue
1	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector.

<u>$\lambda = -1$</u>

We need now to determine the eigenvector $\vec{v}$ where

$$
\begin{aligned}
A\vec{v} &= \lambda\vec{v} \\
A\vec{v} - \lambda\vec{v} &= \vec{0} \\
(A - \lambda I)\vec{v} &= \vec{0} \\
\left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & -4 & 1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & -4 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ 4 & -4 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 2 & -1 & 1 & 0 \\ 4 & -4 & 2 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & -4 & 2 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right]
$$

current pivot $A(2,2)$ is zero. Hence we need to replace current pivot row with one non-zero. Replacing row 2 with row 3 gives

$$
\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 2 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Second row gives $v_2 = 0$. First row gives $2v_1 + t = 0$ or $v_1 = -\frac{t}{2}$. Hence the solution is

$$
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}
$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$\underline{\lambda = 0}$

We need now to determine the eigenvector $\vec{v}$ where

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ A\vec{v} - \lambda\vec{v} &= \vec{0} \\ (A - \lambda I)\vec{v} &= \vec{0} \\ \left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & -4 & 1 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & -4 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & -4 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 4 & -4 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \implies \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 4 & -4 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - 4R_1 \implies \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_2 \implies \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Second row gives $v_3 = 0$. First row give $v_1 - t = 0$ or $v_1 = t$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\underline{\lambda = 1}$

We need now to determine the eigenvector $\vec{v}$ where

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ A\vec{v} - \lambda\vec{v} &= \vec{0} \\ (A - \lambda I)\vec{v} &= \vec{0} \\ \left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & -4 & 1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & -4 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 4 & -4 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 2 & -3 & 1 & 0 \\ 4 & -4 & 0 & 0 \end{array}\right]$$

current pivot $A(1,1)$ is zero. Hence we need to replace current pivot row with one non-zero. Replacing row 1 with row 2 gives

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & -4 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -3 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. From second row $-v_2 + t = 0$ or $v_2 = t$. First row gives $2v_1 - 3v_2 + t = 0$ or $2v_1 = 3v_2 - t$ or $v_1 = \frac{3t-t}{2} = t$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table summarizes the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-1	1	1	No	$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}^{-1}$$

Now that we have diagonalized A, we can finally answer the question.

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & -4 & 1 \end{bmatrix}^{10} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{10} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}^{-1}$$

But $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$. The above becomes

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & -4 & 1 \end{bmatrix}^{10} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}^{-1} \tag{1}$$

We now just need to find P^{-1}. Augmented matrix is

$$\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 3 & -1 & -1 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & -3 & 1 \end{bmatrix}$$

Now we start the reduced Echelon phase.

$R_3 \to -R_3$

$$\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 3 & -1 \end{bmatrix}$$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & -1 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 3 & -1 \end{bmatrix}$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 3 & -1 \end{bmatrix}$$

Since left half is now I then the inverse is the right half of the above augmented matrix. Hence

$$P^{-1} = \begin{bmatrix} 2 & -2 & 1 \\ -1 & 1 & 0 \\ -2 & 3 & -1 \end{bmatrix}$$

Substituting the above in (1) gives

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & -4 & 1 \end{bmatrix}^{10} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ -1 & 1 & 0 \\ -2 & 3 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

2.8.7 Problem 25 section 6.3

In Problems 25 through 30, a city-suburban population transition matrix A (as in Example 2) is given. Find the resulting long-term distribution of a constant total population between the city and its suburbs.

$$A = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$$

Solution

The first step is diagonalize $A = PDP^{-1}$ and then evaluate A^k in the limit as $k \to \infty$. Writing A as

$$A = \begin{bmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\det\left(\begin{bmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) &= 0 \\
\det \begin{bmatrix} \frac{9}{10} - \lambda & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} - \lambda \end{bmatrix} &= 0 \\
\left(\frac{9}{10} - \lambda\right)\left(\frac{9}{10} - \lambda\right) - \frac{1}{100} &= 0 \\
\frac{1}{100}(10\lambda - 9)^2 - \frac{1}{100} &= 0 \\
\frac{1}{100}\left((10\lambda - 9)^2 - 1\right) &= 0 \\
(10\lambda - 9)^2 - 1 &= 0 \\
100\lambda^2 - 180\lambda + 80 &= 0 \\
\lambda^2 - \frac{18}{10}\lambda + \frac{8}{10} &= 0 \\
(\lambda - 1)\left(\lambda - \frac{8}{10}\right) &= 0
\end{aligned}$$

Hence the eigenvalues are

$$\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= \frac{4}{5}
\end{aligned}$$

This table summarizes the result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
$\frac{4}{5}$	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector.

$\lambda = 1$

We need now to determine the eigenvector $\vec{v}$ where

$$
\begin{aligned}
A\vec{v} &= \lambda\vec{v}\\
A\vec{v} - \lambda\vec{v} &= \vec{0}\\
(A - \lambda I)\vec{v} &= \vec{0}\\
\left(\begin{bmatrix} \frac{9}{10} & \frac{1}{10}\\ \frac{1}{10} & \frac{9}{10}\end{bmatrix} - (1)\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix} v_1\\ v_2\end{bmatrix} &= \begin{bmatrix} 0\\ 0\end{bmatrix}\\
\left(\begin{bmatrix} \frac{9}{10} & \frac{1}{10}\\ \frac{1}{10} & \frac{9}{10}\end{bmatrix} - \begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix} v_1\\ v_2\end{bmatrix} &= \begin{bmatrix} 0\\ 0\end{bmatrix}\\
\begin{bmatrix} -\frac{1}{10} & \frac{1}{10}\\ \frac{1}{10} & -\frac{1}{10}\end{bmatrix}\begin{bmatrix} v_1\\ v_2\end{bmatrix} &= \begin{bmatrix} 0\\ 0\end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} -\frac{1}{10} & \frac{1}{10} & 0\\ \frac{1}{10} & -\frac{1}{10} & 0\end{array}\right]
$$

$$
R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -\frac{1}{10} & \frac{1}{10} & 0\\ 0 & 0 & 0\end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} -\frac{1}{10} & \frac{1}{10}\\ 0 & 0\end{bmatrix}\begin{bmatrix} v_1\\ v_2\end{bmatrix} = \begin{bmatrix} 0\\ 0\end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. First row gives $v_1 = t$. Hence the solution is

$$
\begin{bmatrix} v_1\\ v_2\end{bmatrix} = \begin{bmatrix} t\\ t\end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} v_1\\ v_2\end{bmatrix} = t\begin{bmatrix} 1\\ 1\end{bmatrix}
$$

Or, by letting $t = 1$ then the eigenvector is

$$
\begin{bmatrix} v_1\\ v_2\end{bmatrix} = \begin{bmatrix} 1\\ 1\end{bmatrix}
$$

$\lambda = \frac{4}{5}$

We need now to determine the eigenvector $\vec{v}$ where

$$
\begin{aligned}
A\vec{v} &= \lambda\vec{v}\\
A\vec{v} - \lambda\vec{v} &= \vec{0}\\
(A - \lambda I)\vec{v} &= \vec{0}\\
\left(\begin{bmatrix} \frac{9}{10} & \frac{1}{10}\\ \frac{1}{10} & \frac{9}{10}\end{bmatrix} - \left(\frac{4}{5}\right)\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix} v_1\\ v_2\end{bmatrix} &= \begin{bmatrix} 0\\ 0\end{bmatrix}\\
\left(\begin{bmatrix} \frac{9}{10} & \frac{1}{10}\\ \frac{1}{10} & \frac{9}{10}\end{bmatrix} - \begin{bmatrix} \frac{4}{5} & 0\\ 0 & \frac{4}{5}\end{bmatrix}\right)\begin{bmatrix} v_1\\ v_2\end{bmatrix} &= \begin{bmatrix} 0\\ 0\end{bmatrix}\\
\begin{bmatrix} \frac{1}{10} & \frac{1}{10}\\ \frac{1}{10} & \frac{1}{10}\end{bmatrix}\begin{bmatrix} v_1\\ v_2\end{bmatrix} &= \begin{bmatrix} 0\\ 0\end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{10} & \frac{1}{10} & 0 \\ \frac{1}{10} & \frac{1}{10} & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{10} & \frac{1}{10} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{10} & \frac{1}{10} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. First row gives $v_1 = -t$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table summarizes the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\frac{4}{5}$	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{4}{5} \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

And

$$\begin{aligned} \begin{bmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} \end{bmatrix}^k &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{4}{5} \end{bmatrix}^k \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{4}{5}\right)^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \end{aligned}$$

As $k \to \infty$ the term $\left(\frac{4}{5}\right)^k \to 0$. Hence in the limit the above becomes

$$\begin{aligned}\lim_{k\to\infty}\begin{bmatrix}\frac{9}{10} & \frac{1}{10}\\ \frac{1}{10} & \frac{9}{10}\end{bmatrix}^k &= \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}^{-1}\\ &= \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}^{-1}\end{aligned}$$

But $\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}^{-1} = \frac{1}{\det\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}^T = \frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}$. The above becomes

$$\begin{aligned}\lim_{k\to\infty}\begin{bmatrix}\frac{9}{10} & \frac{1}{10}\\ \frac{1}{10} & \frac{9}{10}\end{bmatrix}^k &= \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}\\ &= \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\end{aligned}$$

Therefore

$$\begin{aligned}\vec{x}_k &= A^k\vec{x}_0\\ \lim_{k\to\infty}\vec{x}_k &= \lim_{k\to\infty}A^k\vec{x}_0\\ &= \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}C_0\\ S_0\end{bmatrix}\\ &= \begin{bmatrix}\frac{1}{2}C_0 + \frac{1}{2}S_0\\ \frac{1}{2}C_0 + \frac{1}{2}S_0\end{bmatrix}\\ &= (C_0 + S_0)\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\end{aligned}$$

This means in long term each city will have half of the initial total population.

2.8.8 Additional problem 1

Solution

2.8.8.1 Part (a)

To show A is similar to itself, we need to show there exist P, such that $A = PAP^{-1}$, where P is matrix whose columns are linearly independent and hence invertible. Let $P = I$ (the identity matrix of same size as A). Hence $A = IAI^{-1}$. Since (a) I has linearly independent columns (basis vectors) and (b) I is clearly invertible and (c) $A = IAI^{-1}$ is true: Post multiplying both sides by I gives $AI^{-1} = IA$. But $AI^{-1} = AI$ and $IA = AI$ which means $AI = AI$ or $A = A$ which is true.

2.8.8.2 Part (b)

We are given that

$$A = PBP^{-1} \tag{1}$$

We need to show that $B = PAP^{-1}$. Starting with (1) given relation, and post multiplying both sides by P gives

$$\begin{aligned}AP &= PBP^{-1}P\\ AP &= PB\end{aligned}$$

Since $P^{-1}P = I$. pre multiplying both sides by P^{-1} gives

$$P^{-1}AP = B$$
$$P^{-1}AP = B$$

Let $P^{-1} = Q$. Then the above can also be written as

$$B = QAQ^{-1}$$

Hence B is similar to A.

2.8.8.3 Part (c)

We are given that

$$A = PBP^{-1} \tag{1}$$

And that

$$B = QCQ^{-1} \tag{2}$$

We need to show that $A = VCV^{-1}$ for some invertible matrix V. Substituting (2) into (1) gives

$$\begin{aligned} A &= P\left(QCQ^{-1}\right)P^{-1} \\ &= (PQ)C\left(Q^{-1}P^{-1}\right) \end{aligned}$$

But $Q^{-1}P^{-1} = (PQ)^{-1}$. The above becomes

$$A = (PQ)C(QP)^{-1}$$

Let $PQ = V$. The above becomes

$$A = VCV^{-1}$$

Hence A is similar to C.

2.8.9 Additional problem 2

<u>Solution</u>

$$\begin{aligned} \vec{x}_n &= A^n \vec{x}_0 \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

2.8.9.1 Part (a)

To find eigenvalues and eigenvectors of A.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &= 0 \\ \det\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} &= 0 \\ \lambda^2 - \lambda - 1 &= 0 \end{aligned}$$

The eigenvalues are the roots of the above characteristic polynomial. Using the quadratic formula $\lambda = \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2-4ac} = \frac{1}{2} \pm \frac{1}{2}\sqrt{(-1)^2-4(-1)} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4} = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$. Hence

$$\lambda_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{\sqrt{5}}{2}$$

This table summarizes the result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{1}{2} + \frac{\sqrt{5}}{2}$	1	real eigenvalue
$\frac{1}{2} - \frac{\sqrt{5}}{2}$	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector.

$\lambda = \frac{1}{2} + \frac{\sqrt{5}}{2}$

We need now to determine the eigenvector $\vec{v}$ where

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ A\vec{v} - \lambda\vec{v} &= \vec{0} \\ (A-\lambda I)\vec{v} &= \vec{0} \\ \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} + \frac{\sqrt{5}}{2} & 0 \\ 0 & \frac{1}{2} + \frac{\sqrt{5}}{2} \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{5}}{2} & 1 \\ 1 & -\frac{1}{2} - \frac{\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2} - \frac{\sqrt{5}}{2} & 1 & 0 \\ 1 & -\frac{1}{2} - \frac{\sqrt{5}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{\frac{1}{2} - \frac{\sqrt{5}}{2}} \implies \left[\begin{array}{cc|c} \frac{1}{2} - \frac{\sqrt{5}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2} - \frac{\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. First row gives $\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)v_1 = -t$. Hence $\frac{1-\sqrt{5}}{2}v_1 = -t$. or $v_1 = \frac{-2}{1-\sqrt{5}}t$ or $v_1 = \frac{2}{\sqrt{5}-1}t$. Or $v_1 = \frac{2(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)}t = \frac{2(\sqrt{5}+1)}{4}t = \frac{\sqrt{5}+1}{2}t$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}+1}{2}t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t\begin{bmatrix} \frac{\sqrt{5}+1}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}+1}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \sqrt{5}+1 \\ 1 \end{bmatrix}$$

$\underline{\lambda = \frac{1}{2} - \frac{\sqrt{5}}{2}}$

We need now to determine the eigenvector $\vec{v}$ where

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ A\vec{v} - \lambda\vec{v} &= \vec{0} \\ (A - \lambda I)\vec{v} &= \vec{0} \\ \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{5}}{2} & 0 \\ 0 & \frac{1}{2} - \frac{\sqrt{5}}{2} \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \frac{1}{2} + \frac{\sqrt{5}}{2} & 1 \\ 1 & \frac{\sqrt{5}}{2} - \frac{1}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2} + \frac{\sqrt{5}}{2} & 1 & 0 \\ 1 & \frac{\sqrt{5}}{2} - \frac{1}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{\frac{1}{2} + \frac{\sqrt{5}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2} + \frac{\sqrt{5}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2} + \frac{\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. First row gives $\left(\frac{1+\sqrt{5}}{2}\right) v_1 = -t$ or $v_1 = \frac{-2}{1+\sqrt{5}} t = \frac{-2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} t$ which simplifies to $v_1 = \frac{-2(1-\sqrt{5})t}{-4} = \frac{(1-\sqrt{5})t}{2}$ Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{(1-\sqrt{5})t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} \frac{(1-\sqrt{5})}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{(1-\sqrt{5})}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1-\sqrt{5} \\ 2 \end{bmatrix}$$

The following table summarizes the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
$\frac{1+\sqrt{5}}{2}$	1	1	No	$\begin{bmatrix} 1+\sqrt{5} \\ 2 \end{bmatrix}$
$\frac{1-\sqrt{5}}{2}$	1	1	No	$\begin{bmatrix} 1-\sqrt{5} \\ 2 \end{bmatrix}$

2.8.9.2 Part (b)

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

$$P = \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix}^{-1}$$

And now we can write

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n &= \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}^n \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix}^{-1} \end{aligned}$$

Using hint, let $\frac{1+\sqrt{5}}{2} = \varphi \approx 1.61803$ and $\frac{1-\sqrt{5}}{2} = 1 - \varphi \approx -0.61803$. The above becomes

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} 2\varphi & 2(1-\varphi) \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \varphi^n & 0 \\ 0 & (1-\varphi)^n \end{bmatrix} \begin{bmatrix} 2\varphi & 2(1-\varphi) \\ 2 & 2 \end{bmatrix}^{-1}$$

2.8.9.3 Part (c)

Since

$$\begin{aligned} \vec{x}_n &= A^n \vec{x}_0 \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

Then using result from part b, we can now write

$$\begin{aligned} \vec{x}_n &= A^n \vec{x}_0 \\ &= \overbrace{\begin{bmatrix} 2\varphi & 2(1-\varphi) \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \varphi^n & 0 \\ 0 & (1-\varphi)^n \end{bmatrix} \begin{bmatrix} 2\varphi & 2(1-\varphi) \\ 2 & 2 \end{bmatrix}^{-1}}^{A^n} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2\varphi\varphi^n & 2(1-\varphi)(1-\varphi)^n \\ 2\varphi^n & 2(1-\varphi)^n \end{bmatrix} \begin{bmatrix} 2\varphi & 2(1-\varphi) \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2\varphi^{n+1} & 2(1-\varphi)^{n+1} \\ 2\varphi^n & 2(1-\varphi)^n \end{bmatrix} \begin{bmatrix} 2\varphi & 2(1-\varphi) \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned} \tag{1}$$

But

$$
\begin{aligned}
\begin{bmatrix} 2\varphi & 2(1-\varphi) \\ 2 & 2 \end{bmatrix}^{-1} &= \frac{1}{\det\begin{bmatrix} 2\varphi & 2(1-\varphi) \\ 2 & 2 \end{bmatrix}}\begin{bmatrix} 2 & -2(1-\varphi) \\ -2 & 2\varphi \end{bmatrix} \\
&= \frac{1}{4\varphi - 4(1-\varphi)}\begin{bmatrix} 2 & -2(1-\varphi) \\ -2 & 2\varphi \end{bmatrix} \\
&= \frac{1}{8\varphi - 4}\begin{bmatrix} 2 & -2(1-\varphi) \\ -2 & 2\varphi \end{bmatrix} \\
&= \frac{1}{4\varphi - 2}\begin{bmatrix} 1 & \varphi - 1 \\ -1 & \varphi \end{bmatrix}
\end{aligned}
$$

Hence (1) becomes

$$
\begin{aligned}
\vec{x}_n &= \frac{1}{4\varphi-2}\begin{bmatrix} 2\varphi^{n+1} & 2(1-\varphi)^{n+1} \\ 2\varphi^n & 2(1-\varphi)^n \end{bmatrix}\begin{bmatrix} 1 & \varphi-1 \\ -1 & \varphi \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{4\varphi-2}\begin{bmatrix} 2\varphi^{n+1} - 2(1-\varphi)^{n+1} & 2\varphi(1-\varphi)^{n+1} - 2\varphi^{n+1} + 2\varphi^{n+2} \\ 2\varphi^n - 2(1-\varphi)^n & 2\varphi(1-\varphi)^n - 2\varphi^n + 2\varphi\varphi^n \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{4\varphi-2}\begin{bmatrix} 2\varphi^{n+1} - 2(1-\varphi)^{n+1} \\ 2\varphi^n - 2(1-\varphi)^n \end{bmatrix} \\
&= \frac{1}{2\varphi-1}\begin{bmatrix} \varphi^{n+1} - (1-\varphi)^{n+1} \\ \varphi^n - (1-\varphi)^n \end{bmatrix}
\end{aligned}
$$

But $\vec{x}_n = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}$, hence

$$
\begin{aligned}
f_n &= \frac{\varphi^n - (1-\varphi)^n}{2\varphi - 1} \\
&\approx \frac{1.61803^n - (-0.61803)^n}{2(1.61803) - 1} \\
&\approx \frac{1.61803^n - (-0.61803)^n}{2.2361}
\end{aligned}
$$

Check: we see from problem statement that $f_0 = 0, f_1 = 1, \cdots f_{12} = 144$. Let us check the formula above for f_{12}

$$
\begin{aligned}
f_{12} &= \frac{\varphi^{12} - (1-\varphi)^{12}}{2\varphi - 1} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{12} - \left(1 - \frac{1+\sqrt{5}}{2}\right)^{12}}{2\left(\frac{1+\sqrt{5}}{2}\right) - 1} \\
&= \frac{144\sqrt{5}}{\sqrt{5}} \\
&= 144
\end{aligned}
$$

<u>Verified</u> OK.

2.8.10 key solution for HW8

HOMEWORK 8 – SOLUTIONS

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

6.2.7 We have $A = \begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix}$. First, we compute eigenvalues.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 6-\lambda & -10 \\ 2 & -3-\lambda \end{bmatrix} \\ &= (6-\lambda)(-3-\lambda) + 20 \\ &= \lambda^2 - 3\lambda + 2 \\ &= (\lambda - 1)(\lambda - 2) \end{aligned}$$

For $\lambda_1 = 1$, we have

$$A - I = \begin{bmatrix} 5 & -10 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_1 = (2, 1)$.

For $\lambda_2 = 2$, we have

$$A - 2I = \begin{bmatrix} 4 & -10 \\ 2 & -5 \end{bmatrix} \to \begin{bmatrix} 2 & -5 \\ 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_2 = (5, 2)$.

We have 2 distinct eigenvalues, so A is diagonalizable. The diagonalization is

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}^{-1}$$

6.2.15 First, we compute eigenvalues.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & -3 & 1 \\ 2 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda) \det \begin{bmatrix} 3-\lambda & -3 \\ 2 & -2-\lambda \end{bmatrix} \\ &= (1-\lambda)\left[(3-\lambda)(-2-\lambda) + 6\right] \\ &= (1-\lambda)(\lambda^2 - \lambda) = -\lambda(\lambda - 1)^2 \end{aligned}$$

1

For $\lambda_1 = 0$, we have

$$A - 0I = \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_1 = (1, 1, 0)$.

For $\lambda_2 = 1$, we have

$$A - I = \begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have basis for the eigenspace $\{(3, 2, 0), (1, 0, -2)\}$.

We have one dimension 1 eigenspace and one dimension 2 eigenspace, so A is diagonalizable. The diagonalization is

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}^{-1}$$

6.2.19 First, we compute eigenvalues.

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 1 & -1 \\ -2 & 4-\lambda & -1 \\ -4 & 4 & 1-\lambda \end{bmatrix} \\
&= (1-\lambda)\det \begin{bmatrix} 4-\lambda & -1 \\ 4 & 1-\lambda \end{bmatrix} - \det \begin{bmatrix} -2 & -1 \\ -4 & 1-\lambda \end{bmatrix} - \det \begin{bmatrix} -2 & 4-\lambda \\ -4 & 4 \end{bmatrix} \\
&= (1-\lambda)\left[(4-\lambda)(1-\lambda) + 4\right] - (-2 + 2\lambda - 4) - (-8 + 16 - 4\lambda) \\
&= (1-\lambda)(\lambda^2 - 5\lambda + 8) + 2\lambda - 2 \\
&= (1-\lambda)(\lambda^2 - 5\lambda + 8 - 2) \\
&= (1-\lambda)(\lambda^2 - 5\lambda + 6) \\
&= (1-\lambda)(\lambda - 2)(\lambda - 3)
\end{aligned}$$

For $\lambda_1 = 1$, we have

$$A - I = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 3 & -1 \\ -4 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} -2 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_1 = (1, 1, 1)$.

2

For $\lambda_2 = 2$, we have

$$A - 2I = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 2 & -1 \\ -4 & 4 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_2 = (1, 1, 0)$

For $\lambda_3 = 3$, we have

$$A - 3I = \begin{bmatrix} -2 & 1 & -1 \\ -2 & 1 & -1 \\ -4 & 4 & -2 \end{bmatrix} \to \begin{bmatrix} -2 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_3 = (1, 0, -2)$.

We have 3 distinct eigenvalues, so A is diagonalizable. The diagonalization is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}^{-1}$$

6.3.7 We need to diagonalize first. So, we compute eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 3 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} \\ &= (1-\lambda)(2-\lambda)(2-\lambda) \end{aligned}$$

For $\lambda_1 = 1$, we have

$$A - I = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_1 = (1, 0, 0)$.

For $\lambda_2 = 2$, we have

$$A - 2I = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace is $\{(3, 1, 0), (0, 0, 1)\}$.

So we can diagonalize A as

$$A = PDP^{-1} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

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We compute the inverse of P:

$$\left[\begin{array}{cccccc}1&3&0&1&0&0\\0&1&0&0&1&0\\0&0&1&0&0&1\end{array}\right]\xrightarrow{-3R_2+R_1}\left[\begin{array}{cccccc}1&0&0&1&-3&0\\0&1&0&0&1&0\\0&0&1&0&0&1\end{array}\right]$$

Now $A^5 = PD^5P^{-1}$, so

$$\begin{aligned}A^5 &= \begin{bmatrix}1&3&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&0\\0&0&2\end{bmatrix}^5\begin{bmatrix}1&3&0\\0&1&0\\0&0&1\end{bmatrix}^{-1}\\ &= \begin{bmatrix}1&3&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&32&0\\0&0&32\end{bmatrix}\begin{bmatrix}1&-3&0\\0&1&0\\0&0&1\end{bmatrix}\\ &= \begin{bmatrix}1&96&0\\0&32&0\\0&0&32\end{bmatrix}\begin{bmatrix}1&-3&0\\0&1&0\\0&0&1\end{bmatrix}\\ &= \begin{bmatrix}1&93&0\\0&32&0\\0&0&32\end{bmatrix}\end{aligned}$$

6.3.13 To diagonalize, first we compute eigenvalues.

$$\begin{aligned}\det(A-\lambda I) &= \det\begin{bmatrix}1-\lambda&-1&1\\2&-2-\lambda&1\\4&-4&1-\lambda\end{bmatrix}\\ &= (1-\lambda)\det\begin{bmatrix}-2-\lambda&1\\-4&1-\lambda\end{bmatrix}+\det\begin{bmatrix}2&1\\4&1-\lambda\end{bmatrix}+\det\begin{bmatrix}2&-2-\lambda\\4&-4\end{bmatrix}\\ &= (1-\lambda)\left[(-2-\lambda)(1-\lambda)+4\right]+2-2\lambda-4+-8+8+4\lambda\\ &= (1-\lambda)(\lambda^2+\lambda+2)+2\lambda-2\\ &= (1-\lambda)(\lambda^2+\lambda+2-2)\\ &= \lambda(1-\lambda)(\lambda+1)\end{aligned}$$

For $\lambda_1 = 0$, we have

$$A-0I = \begin{bmatrix}1&-1&1\\2&-2&1\\4&-4&1\end{bmatrix}\to\begin{bmatrix}1&-1&1\\0&0&1\\0&0&0\end{bmatrix}$$

So we have eigenvector $\vec{v}_1 = (1,1,0)$.

For $\lambda_2 = -1$, we have

$$A+I = \begin{bmatrix}2&-1&1\\2&-1&1\\4&-4&2\end{bmatrix}\to\begin{bmatrix}2&-1&1\\0&1&0\\0&0&0\end{bmatrix}$$

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So we have eigenvector $\vec{v}_2 = (1, 0, -2)$.

For $\lambda_3 = 1$, we have

$$A - I = \begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 4 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -3 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_3 = (1, 1, 1)$.

So we can diagonalize A as

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix}^{-1}$$

We compute the inverse of P:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2+R_1]{-2R_2+R_3} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow[(-1)R_2]{-R_3+R_1} \begin{bmatrix} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix}$$

Now $A^{10} = PD^{10}P^{-1}$, so

$$\begin{aligned} A^{10} &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{10} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

6.3.25 We need to diagonalize $A = \begin{bmatrix} .9 & .1 \\ .1 & .9 \end{bmatrix}$. First, we find eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} .9 - \lambda & .1 \\ .1 & .9 - \lambda \end{bmatrix} \\ &= (.9 - \lambda)(.9 - \lambda) - .01 \\ &= \lambda^2 - 1.8\lambda + .8 \\ &= (\lambda - 1)(\lambda - .8) \end{aligned}$$

For $\lambda_1 = 1$, we have

$$A - I = \begin{bmatrix} -.1 & .1 \\ .1 & -.1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_1 = (1, 1)$.

For $\lambda_2 = .8$, we have

$$A - .8I = \begin{bmatrix} .1 & .1 \\ .1 & .1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_2 = (-1, 1)$.

Our diagonalization is thus

$$\begin{aligned} A &= PDP^{-1} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4/5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \end{aligned}$$

Using our 2×2 inverse formula, we get $P^{-1} = \frac{1}{1+1} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, so we have

$$\begin{aligned} A^k &= PD^kP^{-1} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (4/5)^k \end{bmatrix} \left(\frac{1}{2}\right) \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & -(4/5)^k \\ 1 & (4/5)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 + (4/5)^k & 1 - (4/5)^k \\ 1 - (4/5)^k & 1 + (4/5)^k \end{bmatrix} \end{aligned}$$

Now, we can use this to tell us what the population in the city and the suburbs is at any given time from a starting population of C_0 in the city and S_0 in the suburbs.

$$\begin{aligned} \begin{bmatrix} C_k \\ S_k \end{bmatrix} &= A^k \begin{bmatrix} C_0 \\ S_0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 + (4/5)^k & 1 - (4/5)^k \\ 1 - (4/5)^k & 1 + (4/5)^k \end{bmatrix} \begin{bmatrix} C_0 \\ S_0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} C_0 + C_0(4/5)^k + S_0 - S_0(4/5)^k \\ C_0 - C_0(4/5)^k + S_0 + S_0(4/5)^k \end{bmatrix} \end{aligned}$$

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As we let $k \to \infty$, we have $(4/5)^k \to 0$. So in the long run,

$$\begin{bmatrix} C_k \\ S_k \end{bmatrix} \approx \frac{1}{2} \begin{bmatrix} C_0 + S_0 \\ C_0 + S_0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (C_0 + S_0)$$

So in the long run, 50% of the total population will live in the city and 50% will live in the suburbs.

Additional Problems:

1. (a) Let A be an $n \times n$ matrix. For our matrix P, we take the $n \times n$ identity matrix I. Note that I is invertible and $I^{-1} = I$. We have $IAI^{-1} = IAI = A$, so A is similar to A.

   (b) Suppose that A is similar to B. So there is an invertible matrix P with $A = PBP^{-1}$. Multiplying this equation on the left by P^{-1} and on the right by P, we have $P^{-1}AP = B$. Since P is invertible, P^{-1} is invertible as well with inverse P. So, $B = (P^{-1})A(P^{-1})^{-1}$. Thus B is similar to A.

   (c) Suppose that A is similar to B and B is similar to C. So there are invertible matrices P and Q with $A = PBP^{-1}$ and $B = QCQ^{-1}$.

   Substituting this expression for B into the first equation, we get $A = PQCQ^{-1}P^{-1}$. Since both P and Q are invertible, PQ is invertible with inverse $Q^{-1}P^{-1}$. So, we have $A = (PQ)C(PQ)^{-1}$ and thus A is similar to C.

2. (a) We compute the characteristic polynomial

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \\ &= (1-\lambda)(-\lambda) - 1 \\ &= -\lambda + \lambda^2 - 1 \\ &= \lambda^2 - \lambda - 1 \end{aligned}$$

   Applying the quadratic formula, we have

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

   To avoid getting lost in square roots, we will write $\varphi = \frac{1+\sqrt{5}}{2}$. Notice then that $1 - \varphi = -\frac{1}{\varphi} = \frac{1-\sqrt{5}}{2}$ which will make a lot of our calculations easier. For $\lambda_1 = \varphi$, we have the matrix

$$A - \varphi I = \begin{bmatrix} 1-\varphi & 1 \\ 1 & -\varphi \end{bmatrix} \to \begin{bmatrix} 1-\varphi & 1 \\ 0 & 0 \end{bmatrix}$$

   So we have the eigenvector $\vec{v}_1 = (1, \varphi - 1)$. For $\lambda_2 = 1 - \varphi$, we have the matrix

$$A - (1-\varphi)I = \begin{bmatrix} \varphi & 1 \\ 1 & \varphi - 1 \end{bmatrix} \to \begin{bmatrix} \varphi & 1 \\ 0 & 0 \end{bmatrix}$$

   So we have the eigenvector $\vec{v}_2 = (1, -\varphi)$.

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(b) Using what we found in (a), we can write the diagonalization of A:

$$\begin{aligned}A &= PDP^{-1}\\ &= \begin{bmatrix}1 & 1\\ \varphi-1 & -\varphi\end{bmatrix}\begin{bmatrix}\varphi & 0\\ 0 & 1-\varphi\end{bmatrix}\begin{bmatrix}1 & 1\\ \varphi-1 & -\varphi\end{bmatrix}^{-1}\end{aligned}$$

Using our 2×2 matrix inverse formula, we can write

$$\begin{aligned}\begin{bmatrix}1 & 1\\ \varphi-1 & -\varphi\end{bmatrix}^{-1} &= \frac{1}{-\varphi-(-1+\varphi)}\begin{bmatrix}-\varphi & -1\\ 1-\varphi & 1\end{bmatrix}\\ &= \frac{1}{1-2\varphi}\begin{bmatrix}-\varphi & -1\\ 1-\varphi & 1\end{bmatrix}\end{aligned}$$

At this point, it is worthwhile to remember that we have set $\varphi = \frac{1+\sqrt{5}}{2}$. Using this, we can simplify $\frac{1}{1-2\varphi} = -\frac{1}{\sqrt{5}}$. So the inverse matrix is more simply

$$\begin{bmatrix}1 & 1\\ \varphi-1 & -\varphi\end{bmatrix}^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix}\varphi & 1\\ \varphi-1 & -1\end{bmatrix}$$

Since $A^n = PD^nP^{-1}$, we compute

$$\begin{aligned}A^n &= \begin{bmatrix}1 & 1\\ \varphi-1 & -\varphi\end{bmatrix}\begin{bmatrix}\varphi & 0\\ 0 & 1-\varphi\end{bmatrix}^n\begin{bmatrix}1 & 1\\ \varphi-1 & -\varphi\end{bmatrix}^{-1}\\ &= \frac{1}{\sqrt{5}}\begin{bmatrix}1 & 1\\ \varphi-1 & -\varphi\end{bmatrix}\begin{bmatrix}\varphi^n & 0\\ 0 & (1-\varphi)^n\end{bmatrix}\begin{bmatrix}\varphi & 1\\ \varphi-1 & -1\end{bmatrix}\\ &= \frac{1}{\sqrt{5}}\begin{bmatrix}\varphi^n & (1-\varphi)^n\\ \varphi^n(\varphi-1) & (-\varphi)(1-\varphi)^n\end{bmatrix}\begin{bmatrix}\varphi & 1\\ \varphi-1 & -1\end{bmatrix}\\ &= \frac{1}{\sqrt{5}}\begin{bmatrix}\varphi^{n+1}+(1-\varphi)^n(\varphi-1) & \varphi^n-(1-\varphi)^n\\ \varphi^{n+1}(\varphi-1)-\varphi(1-\varphi)^n(\varphi-1) & \varphi^n(\varphi-1)+\varphi(1-\varphi)^n\end{bmatrix}\\ &= \frac{1}{\sqrt{5}}\begin{bmatrix}\varphi^{n+1}-(1-\varphi)^{n+1} & \varphi^n-(1-\varphi)^n\\ (\varphi-1)(\varphi^{n+1}-\varphi(1-\varphi)^n) & \varphi^n(\varphi-1)+\varphi(1-\varphi)^n\end{bmatrix}\end{aligned}$$

At this point, it is useful to remember that $1-\varphi = -\frac{1}{\varphi}$ and $\varphi - 1 = \frac{1}{\varphi}$, so we can simplify a bit further to get

$$A^n = \frac{1}{\sqrt{5}}\begin{bmatrix}\varphi^{n+1}-(1-\varphi)^{n+1} & \varphi^n-(1-\varphi)^n\\ \varphi^n-(1-\varphi)^n & \varphi^{n-1}-(1-\varphi)^{n-1}\end{bmatrix}$$

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(c) We know that $\vec{x}_n = A^n \vec{x}_0 = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so

$$\begin{aligned} \vec{x}_n &= A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^{n+1} - (1-\varphi)^{n+1} & \varphi^n - (1-\varphi)^n \\ \varphi^n - (1-\varphi)^n & \varphi^{n-1} - (1-\varphi)^{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^{n+1} - (1-\varphi)^{n+1} \\ \varphi^n - (1-\varphi)^n \end{bmatrix} \end{aligned}$$

Now, we know that $\vec{x}_n = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}$, so this computation tells us that

$$\begin{aligned} f_n &= \frac{1}{\sqrt{5}} (\varphi^n - (1-\varphi)^n) \\ &= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] \end{aligned}$$

Aside: This is a remarkable formula. For one thing, it should be surprising that this expression *ever* yields an integer, let alone that it is an integer for every nonnegative integer n. The golden ratio φ is an irrational number, meaning that it cannot be expressed as a fraction of two integers. It is quite unusual for expressions involving irrational numbers to produce rational numbers. It is even more unusual for them to produce integers.

We should also be surprised that we can compute the 100th Fibonacci number without computing the 99 before it. Ostensibly, I only know that $f_{100} = f_{99} + f_{98}$ and I have to use the rule $f_{n+1} = f_n + f_{n-1}$ many many times before I can apply the initial values $f_0 = 0$ and $f_1 = 1$. But with this formula, I can use any calculator to immediately compute $f_{100} = 354,224,848,179,261,915,075$.

2.9 HW 9

Local contents

2.9.1 Problems listing

Homework 9 - Due November 12

Homework instructions: Complete the assigned problems on your own paper. Once you are finished, scan or photograph your work and upload it to Gradescope. When prompted, tell Gradescope where to find each problem.
You are allowed (and in fact encouraged) to work with other students on homework assignments. If you do that, please indicate on each problem who you worked with. If you use sources other than your notes, the textbook, and any resources on Canvas for your homework, you must indicate the source on each problem. You are not permitted to view, request, or look for solutions to any of the homework problems from solutions manuals, homework help websites, online forums, other students, or any other sources.

Textbook Problems:

- §7.1: 8
- §7.2: 1, 5, 9, 15, 19, 24, 28

Additional Problems:

1. There is a system of three brine tanks. Tanks 1 and 3 begin with 200 L of fresh water each and tank 2 begins with 100 L of water and 10 kg of salt.

   Water containing 2 kg of salt per liter is pumped into tank 1 at a rate of 15 L/min. The well-mixed solution is pumped from tank 1 to tank 2 at a rate of 20 L/min, from tank 2 to tank 3 at a rate of 20 L/min, and from tank 3 to tank 1 at a rate of 5 L/min. The well-mixed solution is pumped out of tank 3 at a rate of 15 L/min.

   (a) Draw and label a picture that illustrates this situation

   (b) Let $x_1(t), x_2(t)$, and $x_3(t)$ denote the amount of salt (in kilograms) in tanks 1, 2, and 3 respectively after t minutes. Write down differential equations for x_1', x_2', and x_3'.

   (c) Write the system of differential equations in (b) as a matrix equation

   $$\vec{x}' = P(t)\vec{x} + \vec{f}(t)$$

   What are the initial conditions $\vec{x}(0)$?

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2.9.2 Problem 8, section 7.1

Transform the given differential equations into an equivalent system of first-order differential equations.

$$\begin{aligned} x'' + 3x' + 4x - 2y &= 0 \\ y'' + 2y' - 3x + y &= \cos t \end{aligned}$$

<u>Solution</u>

There are two second order ODE's, hence we needs 4 states variables x_1, x_2, x_3, x_4 where (it is better and more standard to use x_i notation for all state variables. The book uses x_i, y_i which is not optimal. x_i will be used here for all state variables)

$$\begin{aligned} x_1 &= x \\ x_2 &= x' \\ x_3 &= y \\ x_4 &= y' \end{aligned} \tag{1}$$

Taking derivatives w.r.t time t gives

$$\begin{aligned} x_1' &= x' \\ &= x_2 \\ x_2' &= x'' \\ &= -\left(3x' + 4x - 2y\right) \\ &= -4x_1 - 3x_2 + 2x_3 \\ x_3' &= y' \\ &= x_4 \\ x_4' &= y'' \\ &= -\left(2y' - 3x + y\right) + \cos t \\ &= 3x_1 - x_3 - 2x_4 + \cos t \end{aligned}$$

Or in Matrix form (if needed)

$$\begin{aligned} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & -3 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos t \end{bmatrix} \\ \vec{x}' &= A\vec{x} + \vec{f}(t) \end{aligned}$$

2.9.3 Problem 1 section 7.2

Verify the product law for differentiation $(AB)' = A'B + AB'$

$$A(t) = \begin{bmatrix} t & 2t-1 \\ t^3 & \frac{1}{t} \end{bmatrix}, B(t) = \begin{bmatrix} 1-t & 1+t \\ 3t^2 & 4t^3 \end{bmatrix}$$

<u>Solution</u>

$$\begin{aligned} AB &= \begin{bmatrix} t & 2t-1 \\ t^3 & t^{-1} \end{bmatrix} \begin{bmatrix} 1-t & 1+t \\ 3t^2 & 4t^3 \end{bmatrix} \\ &= \begin{bmatrix} t\left(6t^2 - 4t + 1\right) & t\left(8t^3 - 4t^2 + t + 1\right) \\ t\left(-t^3 + t^2 + 3\right) & t^2\left(t^2 + t + 4\right) \end{bmatrix} \\ &= \begin{bmatrix} 6t^3 - 4t^2 + t & 8t^4 - 4t^3 + t^2 + t \\ -t^4 + t^3 + 3t & t^4 + t^3 + 4t^2 \end{bmatrix} \end{aligned}$$

Taking derivative of each entry w.r.t t gives

$$(AB)' = \begin{bmatrix} 18t^2 - 8t + 1 & 32t^3 - 12t^2 + 2t + 1 \\ -4t^3 + 3t^2 + 3 & 4t^3 + 3t^2 + 8t \end{bmatrix} \tag{1}$$

Now

$$\begin{aligned} A'(t) &= \frac{d}{dt}\begin{bmatrix} t & 2t-1 \\ t^3 & \frac{1}{t} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3t^2 & -t^{-2} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} A'(t)B(t) &= \begin{bmatrix} 1 & 2 \\ 3t^2 & -t^{-2} \end{bmatrix}\begin{bmatrix} 1-t & 1+t \\ 3t^2 & 4t^3 \end{bmatrix} \\ &= \begin{bmatrix} 6t^2 - t + 1 & 8t^3 + t + 1 \\ -3t^3 + 3t^2 - 3 & t\left(3t^2 + 3t - 4\right) \end{bmatrix} \\ &= \begin{bmatrix} 6t^2 - t + 1 & 8t^3 + t + 1 \\ -3t^3 + 3t^2 - 3 & 3t^3 + 3t^2 - 4t \end{bmatrix} \end{aligned} \tag{2}$$

And

$$\begin{aligned} B'(t) &= \frac{d}{dt}\begin{bmatrix} 1-t & 1+t \\ 3t^2 & 4t^3 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 6t & 12t^2 \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} A(t)B'(t) &= \begin{bmatrix} t & 2t-1 \\ t^3 & \frac{1}{t} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 6t & 12t^2 \end{bmatrix} \\ &= \begin{bmatrix} t(12t-7) & t\left(24t^2 - 12t + 1\right) \\ 6 - t^3 & t^3 + 12t \end{bmatrix} \\ &= \begin{bmatrix} 12t^2 - 7t & 24t^3 - 12t^2 + t \\ 6 - t^3 & t^3 + 12t \end{bmatrix} \end{aligned} \tag{3}$$

Therefore, from (2,3)

$$\begin{aligned} A'B + AB' &= \begin{bmatrix} 6t^2 - t + 1 & 8t^3 + t + 1 \\ -3t^3 + 3t^2 - 3 & 3t^3 + 3t^2 - 4t \end{bmatrix} + \begin{bmatrix} 12t^2 - 7t & 24t^3 - 12t^2 + t \\ 6 - t^3 & t^3 + 12t \end{bmatrix} \\ &= \begin{bmatrix} \left(6t^2 - t + 1\right) + \left(12t^2 - 7t\right) & \left(8t^3 + t + 1\right) + \left(24t^3 - 12t^2 + t\right) \\ \left(-3t^3 + 3t^2 - 3\right) + \left(6 - t^3\right) & \left(3t^3 + 3t^2 - 4t\right) + \left(t^3 + 12t\right) \end{bmatrix} \\ &= \begin{bmatrix} 18t^2 - 8t + 1 & 32t^3 - 12t^2 + 2t + 1 \\ -4t^3 + 3t^2 + 3 & 4t^3 + 3t^2 + 8t \end{bmatrix} \end{aligned} \tag{4}$$

Comparing (1) and (4) shows they are the same. Therefore $(AB)' = A'B + AB'$ has been verified.

2.9.4 Problem 5 section 7.2

Write the given system in the form $\vec{x}' = P(t)\vec{x} + \vec{f}(t)$

$$\begin{aligned} x' &= 2x + 4y + 3e^t \\ y' &= 5x - y - t^2 \end{aligned}$$

Solution

There are two first order ODE's, hence we needs 2 states variables x_1, x_2. Let

$$\begin{aligned} x_1 &= x \\ x_2 &= y \end{aligned} \tag{1}$$

Taking derivatives w.r.t time t gives

$$\begin{aligned} x_1' &= x' \\ &= 2x + 4y + 3e^t \\ &= 2x_1 + 4x_2 + 3e^t \\ x_2' &= y' \\ &= 5x - y - t^2 \\ &= 5x_1 - x_2 - t^2 \end{aligned}$$

Or in Matrix form

$$\begin{aligned} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} &= \overbrace{\begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix}}^{P(t)} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} 3e^t \\ -t^2 \end{bmatrix}}^{\vec{f}(t)} \\ \vec{x}' &= P\vec{x} + \vec{f}(t) \end{aligned}$$

Or using book notation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3e^t \\ -t^2 \end{bmatrix}$$

2.9.5 Problem 9 section 7.2

Write the given system in the form $\vec{x}' = P(t)\vec{x} + \vec{f}(t)$

$$\begin{aligned} x' &= 3x - 4y + z + t \\ y' &= x - 3z + t^2 \\ z' &= 6y - 7z + t^3 \end{aligned}$$

Solution

There are three first order ODE's, hence we needs 3 states variables x_1, x_2, x_3. Let

$$\begin{aligned} x_1 &= x \\ x_2 &= y \\ x_3 &= z \end{aligned} \tag{1}$$

Taking derivatives w.r.t time t gives

$$\begin{aligned} x_1' &= x' \\ &= 3x - 4y + z + t \\ &= 3x_1 - 4x_2 + x_3 + t \\ x_2' &= y' \\ &= x - 3z + t^2 \\ &= x_1 - 3x_3 + t^2 \\ x_3' &= 6y - 7z + t^3 \\ &= 6x_2 - 7x_3 + t^3 \end{aligned}$$

Or in Matrix form

$$\begin{aligned} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} &= \overbrace{\begin{bmatrix} 3 & -4 & 1 \\ 1 & 0 & -3 \\ 0 & 6 & -7 \end{bmatrix}}^{P(t)} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \overbrace{\begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}}^{\vec{f}(t)} \\ \vec{x}' &= P\vec{x} + \vec{f}(t) \end{aligned}$$

Or using book notation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 3 & -4 & 1 \\ 1 & 0 & -3 \\ 0 & 6 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$$

2.9.6 Problem 15 section 7.2

First verify that the given vectors are solutions of the given system. Then use the Wronskian to show that they are linearly independent. Finally, write the general solution of the system

$$\begin{aligned} \vec{x}' &= \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \vec{x} \\ \vec{x}_1 &= e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \vec{x}_2 &= e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix} \end{aligned}$$

Solution

The system is

$$\vec{x}'(t) = A\vec{x} \tag{1}$$

To verify each vector solution, we will check if the LHS is the same as the RHS. The LHS of (1) is

$$\begin{aligned} \frac{d}{dt}\vec{x}_1(t) &= \frac{d}{dt} e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= 2e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= 2\vec{x}_1(t) \end{aligned} \tag{2}$$

The RHS of (1) is

$$\begin{aligned} A\vec{x}_1 &= \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \vec{x}_1 \\ &= \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= 2e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= 2\vec{x}_1(t) \end{aligned} \tag{3}$$

Comparing (1,2) shows they are the same. Now we do the same for the second vector solution. The LHS of (1) is

$$\begin{aligned} \frac{d}{dt}\vec{x}_2(t) &= \frac{d}{dt} e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix} \\ &= -2e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix} \\ &= -2\vec{x}_2(t) \end{aligned} \tag{4}$$

The RHS of (1) is

$$
\begin{aligned}
A\vec{x}_2 &= \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\vec{x}_2 \\
&= \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix} \\
&= e^{-2t}\begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} \\
&= e^{-2t}\begin{bmatrix} -2 \\ -10 \end{bmatrix} \\
&= -2e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix} \\
&= -2\vec{x}_2(t) \qquad (5)
\end{aligned}
$$

Comparing (4,5) shows they are the same. Both solution vectors verified. The Wronskian is the determinant of the matrix whose columns are the vectors $\vec{x}_1(t), \vec{x}_2(t)$. Hence

$$
\begin{aligned}
\begin{vmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{vmatrix} &= 5\left(e^{2t}e^{-2t}\right) - e^{2t}e^{-2t} \\
&= 5 - 1 \\
&= 4
\end{aligned}
$$

Since the determinant is not zero (anywhere), then $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent.

The general solution is linear combination of the basis vector solutions. Therefore

$$
\begin{aligned}
\vec{x}(t) &= c_1\vec{x}_1(t) + c_2\vec{x}_2(t) \\
&= c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix} \\
&= \begin{bmatrix} c_1 e^{2t} + c_2 e^{-2t} \\ c_1 e^{2t} + 5c_2 e^{-2t} \end{bmatrix}
\end{aligned}
$$

2.9.7 Problem 19 section 7.2

First verify that the given vectors are solutions of the given system. Then use the Wronskian to show that they are linearly independent. Finally, write the general solution of the system

$$
\begin{aligned}
\vec{x}' &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\vec{x} \\
\vec{x}_1 &= e^{2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
\vec{x}_2 &= e^{-t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\
\vec{x}_2 &= e^{-t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}
\end{aligned}
$$

<u>Solution</u>

The system is

$$
\vec{x}'(t) = A\vec{x} \qquad (1)
$$

To verify each vector solution, we will check if the LHS is the same as the RHS. The LHS of (1) is

$$\begin{aligned}
\frac{d}{dt}\vec{x}_1(t) &= \frac{d}{dt}e^{2t}\begin{bmatrix}1\\1\\1\end{bmatrix}\\
&= 2e^{2t}\begin{bmatrix}1\\1\\1\end{bmatrix}\\
&= 2\vec{x}_1(t) \qquad (2)
\end{aligned}$$

The RHS of (1) is

$$\begin{aligned}
A\vec{x}_1 &= \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\vec{x}_1\\
&= e^{2t}\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}\\
&= e^{2t}\begin{bmatrix}2\\2\\2\end{bmatrix}\\
&= 2e^{2t}\begin{bmatrix}1\\1\\1\end{bmatrix}\\
&= 2\vec{x}_1(t) \qquad (3)
\end{aligned}$$

Comparing (1,2) shows they are the same. Now we do the same for $\vec{x}_2(t)$. The LHS of (1) is

$$\begin{aligned}
\frac{d}{dt}\vec{x}_2(t) &= \frac{d}{dt}e^{-t}\begin{bmatrix}1\\0\\-1\end{bmatrix}\\
&= -e^{-t}\begin{bmatrix}1\\0\\-1\end{bmatrix}\\
&= -\vec{x}_2(t) \qquad (4)
\end{aligned}$$

The RHS of (1) is

$$\begin{aligned}
A\vec{x}_2 &= \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\vec{x}_2\\
&= e^{-t}\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}\\
&= e^{-t}\begin{bmatrix}-1\\0\\1\end{bmatrix}\\
&= -e^{-t}\begin{bmatrix}1\\0\\-1\end{bmatrix}\\
&= -\vec{x}_2(t) \qquad (5)
\end{aligned}$$

Comparing (4,5) shows they are the same. Now we do the same for $\vec{x}_3(t)$. The LHS of (1) is

$$\begin{aligned}\frac{d}{dt}\vec{x}_3(t) &= \frac{d}{dt}e^{-t}\begin{bmatrix}0\\1\\-1\end{bmatrix}\\ &= -e^{-t}\begin{bmatrix}0\\1\\-1\end{bmatrix}\\ &= -\vec{x}_3(t) \qquad (6)\end{aligned}$$

The RHS of (1) is

$$\begin{aligned}A\vec{x}_3 &= \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\vec{x}_3\\ &= e^{-t}\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\begin{bmatrix}0\\1\\-1\end{bmatrix}\\ &= e^{-t}\begin{bmatrix}0\\-1\\1\end{bmatrix}\\ &= -e^{-t}\begin{bmatrix}0\\1\\-1\end{bmatrix}\\ &= -\vec{x}_3(t) \qquad (7)\end{aligned}$$

Comparing (6,7) shows they are the same. All three vectors solutions verified. The Wronskian is the determinant of the matrix whose columns are the vectors $\vec{x}_1(t), \vec{x}_2(t), \vec{x}_3(t)$. Hence

$$\begin{aligned}\begin{vmatrix}e^{2t}&e^{-t}&0\\e^{2t}&0&e^{-t}\\e^{2t}&-e^{-t}&-e^{-t}\end{vmatrix} &= e^{2t}\begin{vmatrix}0&e^{-t}\\-e^{-t}&-e^{-t}\end{vmatrix} - e^{-t}\begin{vmatrix}e^{2t}&e^{-t}\\e^{2t}&-e^{-t}\end{vmatrix}\\ &= e^{2t}\left(e^{-2t}\right) - e^{-t}\left(e^{t}-e^{t}\right)\\ &= 1 - e^{-t}(0)\\ &= 1\end{aligned}$$

Since the determinant is not zero (anywhere), then $\vec{x}_1(t), \vec{x}_2(t), \vec{x}_3(t)$ are linearly independent. The general solution is linear combination of the basis vector solutions. Therefore

$$\begin{aligned}\vec{x}(t) &= c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)\\ &= c_1e^{2t}\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2e^{-t}\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_3e^{-t}\begin{bmatrix}0\\1\\-1\end{bmatrix}\\ &= \begin{bmatrix}c_1e^{2t}+c_2e^{-t}\\c_1e^{2t}+c_3e^{-t}\\c_1e^{2t}+c_2e^{-t}+c_3e^{-t}\end{bmatrix}\end{aligned}$$

2.9.8 Problem 24 section 7.2

Find the particular solution of the indicated linear system that satisfies the given initial conditions. The system of problem 15.

$$\vec{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \vec{x}$$

$$\vec{x}_1 = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{x}_2 = e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$x_1(0) = 0, x_2(0) = 5$$

Solution

The general solution is

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix} \tag{1}$$

At $t = 0$, the above becomes

$$\begin{bmatrix} 0 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + 5c_2 \end{bmatrix}$$

Two equations with two unknown. From first equation $c_1 = -c_2$. Substituting in the second equation gives $5 = -c_2 + 5c_2$ or $4c_2 = 5$. Hence $c_2 = \frac{5}{4}$. Therefore $c_1 = -\frac{5}{4}$.Therefore the solution that satisfies the initial conditions is, from (1)

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = -\frac{5}{4} e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{5}{4} e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{4} e^{2t} + \frac{5}{4} e^{-2t} \\ -\frac{5}{4} e^{2t} + \frac{25}{4} e^{-2t} \end{bmatrix}$$

Or

$$\vec{x}(t) = \frac{5}{4} \left(-\vec{x}_1(t) + \vec{x}_2(t) \right)$$

2.9.9 Problem 28 section 7.2

Find the particular solution of the indicated linear system that satisfies the given initial conditions. The system of problem 19.

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}$$

$$\vec{x}_1 = e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{x}_2 = e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{x}_2 = e^{-t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$x_1(0) = 10, x_2(0) = 12, x_3(0) = -1$$

Solution

The general solution is

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 e^{-t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \tag{1}$$

At $t = 0$, the above becomes

$$\begin{bmatrix} 10 \\ 12 \\ -1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 10 \\ 12 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + c_3 \\ c_1 - c_2 - c_3 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ -1 \end{bmatrix} \tag{2}$$

The augmented system is

$$\begin{bmatrix} 1 & 1 & 0 & 10 \\ 1 & 0 & 1 & 12 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 10 \\ 0 & -1 & 1 & 2 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 10 \\ 0 & -1 & 1 & 2 \\ 0 & -2 & -1 & -11 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 0 & 10 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & -3 & -15 \end{bmatrix}$$

Therefore the system (2) now is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ -15 \end{bmatrix} \tag{3}$$

Last row gives $c_3 = 5$. Second row gives $-c_2 + c_3 = 2$. Hence $-c_2 = 2 - 5 = -3$ or $c_2 = 3$. First row gives $c_1 + c_2 = 10$. Hence $c_1 = 10 - 3 = 7$. Therefore

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 5 \end{bmatrix}$$

Substituting these in (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = 7e^{2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3e^{-t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 5e^{-t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

or

$$\vec{x}(t) = 7\vec{x}_1(t) + 3\vec{x}_2(t) + 5\vec{x}_3(t)$$

2.9.10 Additional problem 1

There is a system of three brine tanks. Tanks 1 and 3 begin with 200 L of fresh water each and tank 2 begins with 100 L of water and 10 kg of salt.

Water containing 2 kg of salt per liter is pumped into tank 1 at a rate of 15 L/min. The well-mixed solution is pumped from tank 1 to tank 2 at a rate of 20 L/min, from tank 2 to tank 3 at a rate of 20 L/min, and from tank 3 to tank 1 at a rate of 5 L/min. The well-mixed solution is pumped out of tank 3 at a rate of 15 L/min.

(a) Draw and label a picture that illustrates this situation. (b) Let $x_1(t), x_2(t), x_3(t)$ denote the amount of salt (in kilograms) in tanks 1, 2, and 3 respectively after t minutes. Write down differential equations for $x_1'(t), x_2'(t), x_3'(t)$.(c) Write the system of differential equations in (b) as a matrix equation $\vec{x}' = P\vec{x} + \vec{f}(t)$. What are the initial conditions $\vec{x}(0)$?

Solution

2.9.10.1 Part (a)

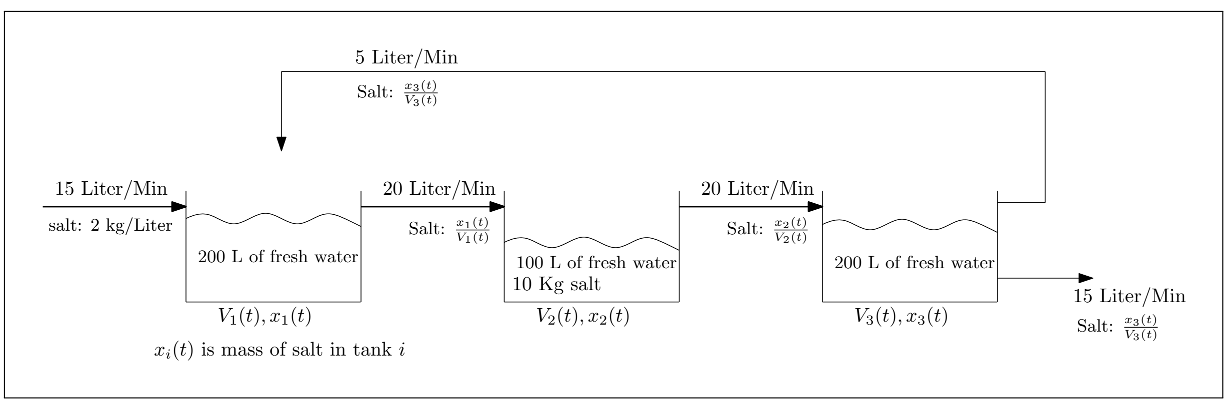


Figure 2.10: Diagram description of the problem

2.9.10.2 Part (b)

$$\begin{aligned} x_1'(t) &= \text{rate of flow in} - \text{ rate of flow out} \\ &= \left(15\left(\frac{\text{L}}{\text{min}}\right)2\left(\frac{\text{kg}}{\text{L}}\right)\right) + \left(5\left(\frac{\text{L}}{\text{min}}\right)\frac{x_3(t)}{V_3(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) - \left(20\left(\frac{\text{L}}{\text{min}}\right)\frac{x_1(t)}{V_1(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) \end{aligned} \tag{1}$$

And

$$\begin{aligned} x_2'(t) &= \text{rate of flow in} - \text{ rate of flow out} \\ &= \left(20\left(\frac{\text{L}}{\text{min}}\right)\frac{x_1(t)}{V_1(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) - \left(20\left(\frac{\text{L}}{\text{min}}\right)\frac{x_2(t)}{V_2(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) \end{aligned} \tag{2}$$

And

$$\begin{aligned} x_3'(t) &= \text{rate of flow in} - \text{ rate of flow out} \\ &= \left(20\left(\frac{\text{L}}{\text{min}}\right)\frac{x_2(t)}{V_2(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) - \left(15\left(\frac{\text{L}}{\text{min}}\right)\frac{x_3(t)}{V_3(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) - \left(5\left(\frac{\text{L}}{\text{min}}\right)\frac{x_3(t)}{V_3(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) \end{aligned} \tag{3}$$

The volume of water at time t is found as follows. $V(t) = V(0) + (\text{rate in-rate out})t$. Therefore

$$\begin{aligned} V_1(t) &= V_1(0)(\text{L}) + (15+5-20)\left(\frac{\text{L}}{\text{min}}\right)t \\ &= 200 + 0t \\ &= 200 \end{aligned} \tag{4}$$

And

$$\begin{aligned} V_2(t) &= V_2(0)(\text{L}) + (20-20)\left(\frac{\text{L}}{\text{min}}\right)t \\ &= 100 + 0t \\ &= 100 \end{aligned} \tag{5}$$

And

$$\begin{aligned} V_3(t) &= V_3(0)(\text{L}) + (20-5-15)\left(\frac{\text{L}}{\text{min}}\right)t \\ &= 200 + 0t \\ &= 200 \end{aligned} \tag{6}$$

We see from the above, that the volume of water in each tank is constant over time. Now, substituting (4,5,6) into (1,2,3) gives the equations needed.

$$\begin{aligned} x_1'(t) &= 30 + \frac{5}{200}x_3 - \frac{20}{200}x_1 \\ &= 30 + \frac{1}{40}x_3 - \frac{1}{10}x_1 \end{aligned} \tag{7}$$

And

$$\begin{aligned} x_2'(t) &= \frac{20}{200}x_1 - \frac{20}{100}x_2 \\ &= \frac{1}{10}x_1 - \frac{2}{10}x_2 \end{aligned} \tag{8}$$

And

$$\begin{aligned} x_3'(t) &= \frac{20}{100}x_2 - \frac{15}{200}x_3 - \frac{5}{200}x_3 \\ &= \frac{2}{10}x_2 - \frac{1}{10}x_3 \end{aligned} \tag{9}$$

In summary, the differential equations are

$$\begin{aligned} x_1'(t) &= 30 + \frac{1}{40}x_3(t) - \frac{1}{10}x_1(t) \\ x_2'(t) &= \frac{1}{10}x_1(t) - \frac{2}{10}x_2(t) \\ x_3'(t) &= \frac{2}{10}x_2(t) - \frac{1}{10}x_3(t) \end{aligned}$$

2.9.10.3 Part (c)

In Matrix form, the solution found in part b is

$$\begin{aligned} \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} &= \begin{bmatrix} -\frac{1}{10} & 0 & \frac{1}{40} \\ \frac{1}{10} & -\frac{2}{10} & 0 \\ 0 & \frac{2}{10} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 30 \\ 0 \\ 0 \end{bmatrix} \\ &= \frac{1}{10}\begin{bmatrix} -1 & 0 & \frac{1}{4} \\ 1 & -2 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 30 \\ 0 \\ 0 \end{bmatrix} \\ \vec{x}' &= P\vec{x} + \vec{f}(t) \end{aligned}$$

The initial conditions are

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} (\text{kg})$$

2.9.11 key solution for HW9

Homework 9 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

7.1.8 Our original differential equations are

$$\begin{aligned} x'' + 3x' + 4x - 2y &= 0 \\ y'' + 2y' - 3x + y &= \cos t \end{aligned}$$

We define new functions $x_1 = x, x_2 = x', x_3 = y, x_4 = y'$. Since x'' and y'' are the highest derivatives we have, we can stop defining new functions at the first derivatives. We make the appropriate substitutions, and add additional equations to explicate the relationships between our newly defined variables.

$$\begin{aligned} x_2' + 3x_2 + 4x_1 - 2x_3 &= 0 \\ x_4' + 2x_4 - 3x_1 + x_3 &= \cos t \\ x_3' &= x_4 \\ x_1' &= x_2 \end{aligned}$$

Note that the first derivative of each of our functions appears exactly once, which indicates we have enough equations.

7.2.1 We have $A(t) = \begin{bmatrix} t & 2t-1 \\ t^3 & \frac{1}{t} \end{bmatrix}$ and $B(t) = \begin{bmatrix} 1-t & 1+t \\ 3t^2 & 4t^3 \end{bmatrix}$. We compute

$$\begin{aligned} \frac{d}{dt}(AB) &= \frac{d}{dt}\begin{bmatrix} t(1-t)+(2t-1)3t^2 & t(1+t)+(2t-1)4t^3 \\ t^3(1-t)+3t & t^3(1+t)+4t^2 \end{bmatrix} \\ &= \frac{d}{dt}\begin{bmatrix} t-4t^2+6t^3 & t+t^2+8t^4-4t^3 \\ t^3-t^4+3t & t^3+t^4+4t^2 \end{bmatrix} \\ &= \begin{bmatrix} 1-8t+18t^2 & 1+2t+32t^3-12t^2 \\ 3t^2-4t^3+3 & 3t^2+4t^3+8t \end{bmatrix} \end{aligned}$$

On the other hand, we have

$$\begin{aligned} A'B + AB' &= \begin{bmatrix} 1 & 2 \\ 3t^2 & -t^{-2} \end{bmatrix}\begin{bmatrix} 1-t & 1+t \\ 3t^2 & 4t^3 \end{bmatrix} + \begin{bmatrix} t & 2t-1 \\ t^3 & \frac{1}{t} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 6t & 12t^2 \end{bmatrix} \\ &= \begin{bmatrix} 1-t+6t^2 & 1+t+8t^3 \\ 3t^2-3t^3-3 & 3t^2+3t^3-4t \end{bmatrix} + \begin{bmatrix} -7t+12t^2 & t+24t^3-12t^2 \\ -t^3+6 & t^3+12t \end{bmatrix} \\ &= \begin{bmatrix} 1-8t+18t^2 & 1+2t-12t^2+32t^3 \\ 3+3t^2-4t^3 & 8t+3t^2+4t^3 \end{bmatrix} \end{aligned}$$

A few of the entries need rearranging, but the resulting matrices are indeed equal.

1

7.2.5 We can rewrite the system as

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3e^t \\ -t^2 \end{bmatrix}$$

7.2.9 We can rewrite the system as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} 3 & -4 & 1 \\ 1 & 0 & -3 \\ 0 & 6 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$$

7.2.15 $\vec{x}_1$ is a solution, since

$$2e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{2t} \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{2t} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$\vec{x}_2$ is a solution, since

$$-2e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = e^{-2t} \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = e^{-2t} \begin{bmatrix} -2 \\ -10 \end{bmatrix}$$

These solutions are linearly independent, since the Wronskian is

$$e^{2t}e^{-2t} \det \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} = 4$$

The general solution is

$$\vec{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

7.2.19 $\vec{x}_1$ is a solution, since

$$2e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = e^{2t} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = e^{2t} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$\vec{x}_2$ is a solution, since

$$-e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = e^{2t} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\vec{x}_3$ is a solution, since

$$-e^{-t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = e^{2t} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = e^{-t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

2

These solutions are linearly independent, since the Wronskian is

$$e^{2t}e^{-t}e^{-t}\det\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 1 & -1 & -1\end{bmatrix} = \det\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix} - \det\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$
$$= 1-(-2)$$
$$= 3$$

The general solution is

$$\vec{x}(t) = c_1e^{2t}\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2e^{-t}\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_3e^{-t}\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

7.2.24 Using the general solution we wrote down in problem 15,

$$\vec{x}(0) = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\5\end{bmatrix} = \begin{bmatrix}c_1+c_2\\ c_1+5c_2\end{bmatrix}$$

Given that $\vec{x}(0) = \begin{bmatrix}0\\5\end{bmatrix}$, we have the linear system

$$\begin{bmatrix}1 & 1\\ 1 & 5\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}0\\5\end{bmatrix}$$

We row reduce

$$\begin{bmatrix}1 & 1 & 0\\ 1 & 5 & 5\end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix}1 & 1 & 0\\ 0 & 4 & 5\end{bmatrix}$$

Now we can solve to get $c_2 = 5/4$ and $c_1 = -5/4$. The particular solution is thus $\vec{x} = -\frac{5}{4}\vec{x}_1 + \frac{5}{4}\vec{x}_2$

7.2.28

Using the general solution we wrote down in problem 19,

$$\vec{x}(0) = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_3\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

Given the initial conditions, we have the linear system

$$\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 1 & -1 & -1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\\ c_3\end{bmatrix} = \begin{bmatrix}10\\12\\-1\end{bmatrix}$$

3

We row reduce

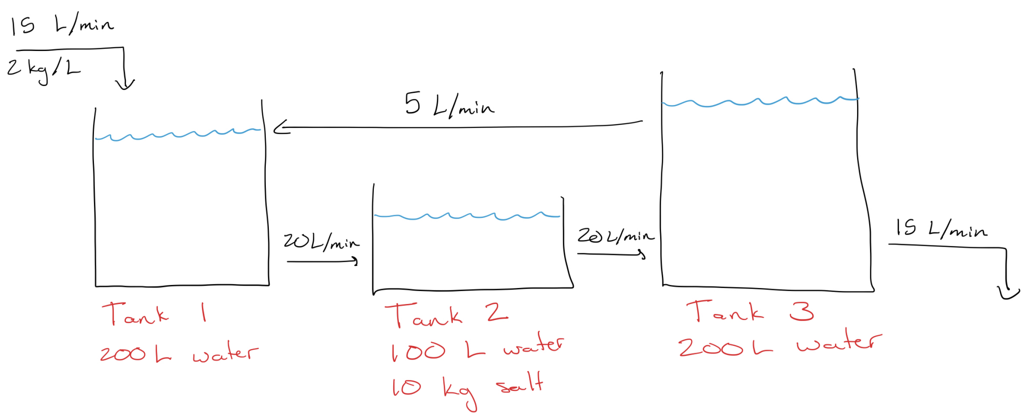
$$\begin{bmatrix} 1 & 1 & 0 & 10 \\ 1 & 0 & 1 & 12 \\ 1 & -1 & -1 & -1 \end{bmatrix} \xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 1 & 0 & 10 \\ 0 & -1 & 1 & 2 \\ 0 & -2 & -1 & -11 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 1 & 0 & 10 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & -3 & -15 \end{bmatrix}$$

We now solve to get $c_3 = 5$, $c_2 = 3$, and $c_1 = 7$. The particular solution is thus $\vec{x} = 7\vec{x}_1 + 3\vec{x}_2 + 5\vec{x}_3$

Additional Problems:

1. The picture below illustrates this system:



Note that each tank has 20 L of water flowing in and 20 L of water flowing out, so all volumes are constant. We calculate the change in salt as (rate in) ∗ (concentration in) − (rate out) ∗ (concentration out). This gives us the system of differential equations

$$x_1' = 30 + 5\frac{x_3}{200} - 20\frac{x_1}{200} = -\frac{x_1}{10} + \frac{x_3}{40} + 30$$

$$x_2' = 20\frac{x_1}{200} - 20\frac{x_2}{100} = \frac{x_1}{10} - \frac{x_2}{5}$$

$$x_3' = 20\frac{x_2}{100} - 5\frac{x_3}{200} - 15\frac{x_3}{200} = \frac{x_2}{5} - \frac{x_3}{10}$$

The matrix version of this system is

$$\vec{x}' = \begin{bmatrix} -1/10 & 0 & 1/40 \\ 1/10 & -1/5 & 0 \\ 0 & 1/5 & -1/10 \end{bmatrix} \vec{x} + \begin{bmatrix} 30 \\ 0 \\ 0 \end{bmatrix}$$

4

Our initial condition is

$$\vec{x}(0)=\begin{bmatrix}0\\10\\0\end{bmatrix}$$

5

2.10 HW 10

Local contents

2.10.1 Problems listing

HOMEWORK 10 - DUE NOVEMBER 19

Homework instructions: Complete the assigned problems on your own paper. Once you are finished, scan or photograph your work and upload it to Gradescope. When prompted, tell Gradescope where to find each problem.
You are allowed (and in fact encouraged) to work with other students on homework assignments. If you do that, please indicate on each problem who you worked with. If you use sources other than your notes, the textbook, and any resources on Canvas for your homework, you must indicate the source on each problem. You are not permitted to view, request, or look for solutions to any of the homework problems from solutions manuals, homework help websites, online forums, other students, or any other sources.

Textbook Problems:

- §7.3: 4, 6, 8, 18, 38 (**do not** sketch direction fields or solution curves, **do** find particular solutions when asked for)

Additional Problems:

1. Consider the differential equation

$$x^{(3)} + x'' - 2x' = 0$$

   (a) Transform this into an equivalent system of first-order differential equations.
   (b) Write the system from (a) as $\mathbf{x}' = A\mathbf{x}$ (the matrix A should be 3×3).
   (c) Use the eigenvalue method to solve the system in (b)
   (d) Using your solution to (c), what is the general solution $x(t)$ to the given differential equation?

2. Consider the following system of brine tanks:

   There are three tanks. Tank 1 contains 20 L of water, tank 2 contains 30 L of water, and tank 3 contains 60L of water. Fresh water is pumped into tank 1 at a rate of 120 L/min. The well-mixed solution is pumped from tank 1 to tank 2, from tank 2 to tank 3, and out of tank 3 all at a rate of 120 L/min.

   (a) Draw and label a diagram describing the system
   (b) Let $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ be the vector function of the amount of salt in each tank at time t. Write a differential equation $\mathbf{x}' = A\mathbf{x}$ describing the system.
   (c) Find the general solution to the differential equation you wrote in (b) using the eigenvalue method
   (d) Initially, there is 100 kg of salt in tank 1 and 20 kg of salt in tank 2. Find the particular solution corresponding to these initial conditions.

1

2.10.2 Problem 4, section 7.3

In Problems 1 through 16, apply the eigenvalue method of this section to find a general solution of the given system. If initial values are given, find also the corresponding particular solution.

$$
\begin{aligned}
x_1'(t) &= 4x_1(t) + x_2(t)\\
x_2'(t) &= 6x_1(t) - x_2(t)
\end{aligned}
$$

Solution

This is a system of linear ODE's which can be written as

$$\vec{x}'(t) = A\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t)\\ x_2'(t)\end{bmatrix} = \begin{bmatrix} 4 & 1\\ 6 & -1\end{bmatrix}\begin{bmatrix} x_1(t)\\ x_2(t)\end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & 1\\ 6 & -1\end{bmatrix} - \lambda\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & 1\\ 6 & -1-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$
\begin{aligned}
\lambda^2 - 3\lambda - 10 &= 0\\
(\lambda - 5)(\lambda + 2) &= 0
\end{aligned}
$$

The roots are therefore

$$
\begin{aligned}
\lambda_1 &= 5\\
\lambda_2 &= -2
\end{aligned}
$$

Next, the eigenvectors for each eigenvalue are found.

eigenvalue -2

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$
\begin{aligned}
\left(\begin{bmatrix} 4 & 1\\ 6 & -1\end{bmatrix} - (-2)\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix} v_1\\ v_2\end{bmatrix} &= \begin{bmatrix} 0\\ 0\end{bmatrix}\\
\begin{bmatrix} 6 & 1\\ 6 & 1\end{bmatrix}\begin{bmatrix} v_1\\ v_2\end{bmatrix} &= \begin{bmatrix} 0\\ 0\end{bmatrix}
\end{aligned}
$$

Using first row, and letting $v_2 = 1$ gives $6v_1 = -1$ or $v_1 = \frac{-1}{6}$. Hence the eigenvector is

$$\vec{v}_1 = \begin{bmatrix} \frac{-1}{6}\\ 1\end{bmatrix}$$

normalizing the eigenvector gives

$$\vec{v}_1 = \begin{bmatrix} -1\\ 6\end{bmatrix}$$

eigenvalue 5

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using first row, and letting $v_2 = 1$ gives $-v_1 = -1$ or $v = 1$. Hence the eigenvector is

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue.

If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
5	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
−2	1	1	No	$\begin{bmatrix} -1 \\ 6 \end{bmatrix}$

Now that the we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective.

Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= e^{5t}\,\vec{v}_1 \\ &= e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Since eigenvalue −2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= e^{-2t}\,\vec{v}_2 \\ &= e^{-2t}\begin{bmatrix} -1 \\ 6 \end{bmatrix} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{5t} - c_2 e^{-2t} \\ c_1 e^{5t} + 6c_2 e^{-2t} \end{bmatrix}$$

2.10.3 Problem 6, section 7.3

In Problems 1 through 16, apply the eigenvalue method of this section to find a general solution of the given system. If initial values are given, find also the corresponding particular solution.

$$
\begin{aligned}
x_1'(t) &= 9x_1(t) + 5x_2(t) \\
x_2'(t) &= -6x_1(t) - 2x_2(t)
\end{aligned}
$$

With initial conditions

$$x_1(0) = 1, x_2(0) = 0$$

Solution

This is a system of linear ODE's which can be written as

$$\vec{x}'(t) = A\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{bmatrix} \right) = 0$$

Which gives the characteristic equation

$$
\begin{aligned}
\lambda^2 - 7\lambda + 12 &= 0 \\
(\lambda - 3)(\lambda - 4) &= 0
\end{aligned}
$$

The roots of the above are therefore

$$
\begin{aligned}
\lambda_1 &= 3 \\
\lambda_2 &= 4
\end{aligned}
$$

Next, the eigenvectors for each eigenvalue are found.

eigenvalue 3

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$
\begin{aligned}
\left(\begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 6 & 5 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

Using first row, and letting $v_2 = 1$ gives $6v_1 = -5$ or $v_1 = \frac{-5}{6}$. Hence the eigenvector is

$$\vec{v}_1 = \begin{bmatrix} \frac{-5}{6} \\ 1 \end{bmatrix}$$

Normalizing gives

$$\vec{v}_1 = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

eigenvalue 4

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 5 & 5 \\ -6 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using first row, and letting $v_2 = 1$ gives $5v_1 = -5$ or $v_1 = -1$. Hence the eigenvector is

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue.

If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
4	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} -5 \\ 6 \end{bmatrix}$

Now that the we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective.

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{4t} \\ &= e^{4t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{3t} \\ &= e^{3t}\begin{bmatrix} -5 \\ 6 \end{bmatrix} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{4t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{4t} - 5c_2 e^{3t} \\ c_1 e^{4t} + 6c_2 e^{3t} \end{bmatrix} \tag{1}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 0 \end{bmatrix}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_1 - 5c_2 \\ c_1 + 6c_2 \end{bmatrix}$$

Adding first equation to second gives $1 = c_2$. From second equation this gives $0 = c_1 + 6$ or $c_1 = -6$

$$c_1 = -6$$
$$c_2 = 1$$

Substituting these constants back in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 6e^{4t} - 5e^{3t} \\ -6e^{4t} + 6e^{3t} \end{bmatrix}$$

2.10.4 Problem 8, section 7.3

In Problems 1 through 16, apply the eigenvalue method of this section to find a general solution of the given system. If initial values are given, find also the corresponding particular solution.

$$x_1'(t) = x_1(t) - 5x_2(t)$$
$$x_2'(t) = x_1(t) - x_2(t)$$

Solution

This is a system of linear ODE's which can be written as

$$\vec{x}'(t) = A\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4 = 0$$
$$\lambda^2 = -4$$

Therefore the roots are

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

Next, the eigenvectors for each eigenvalue are found.

eigenvalue $2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} - (2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From first row, letting $v_2 = 1$ then $(1-2i)v_1 = 5$ or $v_1 = \frac{5}{1-2i} = \frac{(1+2i)5}{(1-2i)(1+2i)} = \frac{(1+2i)5}{1-4i^2} = \frac{(1+2i)5}{5} = 1+2i$. Hence

$$\vec{v}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$$

eigenvalue $-2i$

The second eigenvector is complex conjugate of the first. Therefore

$$\vec{v}_2 = \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue.

If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2i$	1	1	No	$\begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$
$-2i$	1	1	No	$\begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$

Now that the we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective.

Since eigenvalue $2i$ is complex, then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{2it} \\ &= e^{2it}\begin{bmatrix} 1+2i \\ 1 \end{bmatrix} \end{aligned}$$

Since eigenvalue $-2i$ is complex, then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{-2it} \\ &= e^{-2it}\begin{bmatrix} 1-2i \\ 1 \end{bmatrix} \end{aligned}$$

The complex eigenvectors found above, which are complex conjugate of each others, are now converted to real basis as follows (Using Euler relation that $e^{i\theta} = \cos\theta + i\sin\theta$).

First we break $\vec{x}_1(t)$ into real part and imaginary part (we could also have done this using $\vec{x}_2(t)$, either one will work.

$$
\begin{aligned}
\vec{x}_1(t) &= e^{2it}\begin{bmatrix} 1+2i \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{2it}(1+2i) \\ e^{2it} \end{bmatrix} \\
&= \begin{bmatrix} (\cos 2t + i\sin 2t)(1+2i) \\ \cos 2t + i\sin 2t \end{bmatrix} \\
&= \begin{bmatrix} (\cos 2t + i\sin 2t) + 2i(\cos 2t + i\sin 2t) \\ \cos 2t + i\sin 2t \end{bmatrix} \\
&= \begin{bmatrix} \cos 2t + i\sin 2t + 2i\cos 2t - 2\sin 2t \\ \cos 2t + i\sin 2t \end{bmatrix} \\
&= \begin{bmatrix} (\cos 2t - 2\sin 2t) + i(2\cos 2t + \sin 2t) \\ \cos 2t + i(\sin 2t) \end{bmatrix} \qquad (1)
\end{aligned}
$$

Therefore, using

$$
\begin{aligned}
\vec{x}_1(t) &= \operatorname{Re}\left(\vec{x}_1(t)\right) \\
\vec{x}_2(t) &= \operatorname{Im}\left(\vec{x}_1(t)\right)
\end{aligned}
$$

From (1) this gives

$$
\begin{aligned}
\vec{x}_1(t) &= \begin{bmatrix} \cos 2t - 2\sin 2t \\ \cos 2t \end{bmatrix} \\
\vec{x}_2(t) &= \begin{bmatrix} 2\cos 2t + \sin 2t \\ \sin 2t \end{bmatrix}
\end{aligned}
$$

The final solution becomes

$$
\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)
$$

Or

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1\begin{bmatrix} \cos(2t) - 2\sin(2t) \\ \cos(2t) \end{bmatrix} + c_2\begin{bmatrix} 2\cos(2t) + \sin(2t) \\ \sin(2t) \end{bmatrix}
$$

Hence

$$
\begin{aligned}
x_1(t) &= c_1\cos(2t) - 2c_1\sin(2t) + 2c_2\cos(2t) + c_2\sin(2t) \\
x_2(t) &= c_1\cos(2t) + c_2\sin(2t)
\end{aligned}
$$

Or

$$
\begin{aligned}
x_1(t) &= (c_1 + 2c_2)\cos(2t) + (c_2 - 2c_1)\sin(2t) \\
x_2(t) &= c_1\cos(2t) + c_2\sin(2t)
\end{aligned}
$$

2.10.5 Problem 18, section 7.3

In Problems 17 through 25, the eigenvalues of the coefficient matrix can be found by inspection and factoring. Apply the eigenvalue method to find a general solution of each system

$$
\begin{aligned}
x_1'(t) &= x_1 + 2x_2 + 2x_3 \\
x_2'(t) &= 2x_1 + 7x_2 + x_3 \\
x_3'(t) &= 2x_1 + x_2 + 7x_3
\end{aligned}
$$

Solution

This is a system of linear ODE's which can be written as

$$\vec{x}'(t) = A\vec{x}(t)$$

Or in matrix form

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$$

Or

$$\det\left(\begin{bmatrix} 1-\lambda & 2 & 2 \\ 2 & 7-\lambda & 1 \\ 2 & 1 & 7-\lambda \end{bmatrix} \right) = 0$$

Expanding along first row gives

$$\begin{aligned}
(1-\lambda)\begin{vmatrix} 7-\lambda & 1 \\ 1 & 7-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 2 & 7-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & 7-\lambda \\ 2 & 1 \end{vmatrix} &= 0 \\
(1-\lambda)\left((7-\lambda)^2 - 1\right) - 2(2(7-\lambda) - 2) + 2(2 - 2(7-\lambda)) &= 0 \\
(1-\lambda)(7-\lambda)^2 - (1-\lambda) - 4(7-\lambda) + 4 + 4 - 4(7-\lambda) &= 0 \\
(1-\lambda)(7-\lambda)^2 - 1 + \lambda - 28 + 4\lambda + 8 - 28 + 4\lambda &= 0 \\
(1-\lambda)(7-\lambda)^2 + 9\lambda - 49 &= 0 \\
(1-\lambda)\left(\lambda^2 - 14\lambda + 49\right) + 9\lambda - 49 &= 0 \\
-\lambda^3 + 15\lambda^2 - 63\lambda + 49 + 9\lambda - 49 &= 0 \\
-\lambda^3 + 15\lambda^2 - 63\lambda + 9\lambda &= 0 \\
-\lambda^3 + 15\lambda^2 - 54\lambda &= 0 \\
\lambda^3 - 15\lambda^2 + 54\lambda &= 0 \\
\lambda\left(\lambda^2 - 15\lambda + 54\right) &= 0 \\
\lambda(\lambda - 6)(\lambda - 9) &= 0
\end{aligned}$$

Therefore the roots are

$$\begin{aligned}
\lambda_1 &= 9 \\
\lambda_2 &= 6 \\
\lambda_3 &= 0
\end{aligned}$$

Next, the eigenvectors for each eigenvalue are found.

eigenvalue 0

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\begin{aligned}
\left(\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 2 & 7 & 1 & 0 \\ 2 & 1 & 7 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 3 & -3 & 0 \\ 2 & 1 & 7 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system in Echelon form is

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_3 and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of free variables gives $v_1 = -4t, v_2 = t$. Hence the solution is

$$\vec{v}_1 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -4t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\vec{v}_1 = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

<u>eigenvalue 6</u>

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} - (6)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -5 & 2 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 2 & 0 \\ 0 & \frac{9}{5} & \frac{9}{5} & 0 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 2 & 0 \\ 0 & \frac{9}{5} & \frac{9}{5} & 0 \\ 0 & \frac{9}{5} & \frac{9}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 2 & 0 \\ 0 & \frac{9}{5} & \frac{9}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system in Echelon form is

$$\begin{bmatrix} -5 & 2 & 2 \\ 0 & \frac{9}{5} & \frac{9}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_3 and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of the free variable gives equations $v_1 = 0, v_2 = -t$. Hence the eigenvector is

$$\vec{v}_2 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

eigenvalue 9

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} - (9) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -8 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -8 & 2 & 2 & 0 \\ 2 & -2 & 1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -8 & 2 & 2 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 2 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -8 & 2 & 2 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -8 & 2 & 2 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -8 & 2 & 2 \\ 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_3 and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of the free variable gives equations $v_1 = \frac{t}{2}, v_2 = t$. Hence the eigenvector is

$$\vec{v}_3 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

Letting $t = 1$ the normalized eigenvector is

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue.

If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	1	1	No	$\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$
6	1	1	No	$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$
9	1	1	No	$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Now that the we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^0 \\ &= \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{6t} \\ &= e^{6t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

Since eigenvalue 9 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{9t} \\ &= e^{9t}\begin{bmatrix}1\\2\\2\end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix} = c_1\begin{bmatrix}-4\\1\\1\end{bmatrix} + c_2 e^{6t}\begin{bmatrix}0\\-1\\1\end{bmatrix} + c_3 e^{9t}\begin{bmatrix}1\\2\\2\end{bmatrix}$$

Or

$$\begin{aligned}x_1(t) &= -4c_1 + c_3 e^{9t} \\ x_2(t) &= c_1 - c_2 e^{6t} + 2c_3 e^{9t} \\ x_3(t) &= c_1 + c_2 e^{6t} + 2c_3 e^{9t}\end{aligned}$$

2.10.6 Problem 38, section 7.3

For each matrix A given in Problems 38 through 40, the zeros in the matrix make its characteristic polynomial easy to calculate. Find the general solution of $\vec{x}'(t) = A\vec{x}(t)$

$$A = \begin{bmatrix}1&0&0&0\\2&2&0&0\\0&3&3&0\\0&0&4&4\end{bmatrix}$$

Solution

This is a system of linear ODE's which can be written as

$$\vec{x}'(t) = A\vec{x}(t)$$

Or

$$\begin{bmatrix}x_1'(t)\\x_2'(t)\\x_3'(t)\\x_4'(t)\end{bmatrix} = \begin{bmatrix}1&0&0&0\\2&2&0&0\\0&3&3&0\\0&0&4&4\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\\x_4(t)\end{bmatrix}$$

We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}1&0&0&0\\2&2&0&0\\0&3&3&0\\0&0&4&4\end{bmatrix} - \lambda\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}1-\lambda&0&0&0\\2&2-\lambda&0&0\\0&3&3-\lambda&0\\0&0&4&4-\lambda\end{bmatrix}\right) = 0$$

Since this is a lower triangular matrix, then the determinant is the product of the entries on the diagonal Hence the characteristic equation is

$$(1-\lambda)(2-\lambda)(3-\lambda)(4-\lambda)=0$$

Therefore the roots are

$$\begin{aligned}\lambda_1&=1\\\lambda_2&=2\\\lambda_3&=3\\\lambda_4&=4\end{aligned}$$

Next, the eigenvectors for each eigenvalue are found.

eigenvalue 1

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}1&0&0&0\\2&2&0&0\\0&3&3&0\\0&0&4&4\end{bmatrix}-(1)\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}0&0&0&0\\2&1&0&0\\0&3&2&0\\0&0&4&3\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c}0&0&0&0&0\\2&1&0&0&0\\0&3&2&0&0\\0&0&4&3&0\end{array}\right]$$

current pivot $A(1,1)$ is zero. Hence we need to replace current pivot row with non-zero row. Replacing row 1 with row 2 gives

$$\left[\begin{array}{cccc|c}2&1&0&0&0\\0&0&0&0&0\\0&3&2&0&0\\0&0&4&3&0\end{array}\right]$$

current pivot $A(2,2)$ is zero. Hence we need to replace current pivot row with non-zero row. Replacing row 2 with row 3 gives

$$\left[\begin{array}{cccc|c}2&1&0&0&0\\0&3&2&0&0\\0&0&0&0&0\\0&0&4&3&0\end{array}\right]$$

current pivot $A(3,3)$ is zero. Hence we need to replace current pivot row with non-zero row. Replacing row 3 with row 4 gives

$$\left[\begin{array}{cccc|c}2&1&0&0&0\\0&3&2&0&0\\0&0&4&3&0\\0&0&0&0&0\end{array}\right]$$

Therefore the system is now in Echelon form

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_4 and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of the free variable gives equations $v_1 = -\frac{t}{4}, v_2 = \frac{t}{2}, v_3 = -\frac{3t}{4}$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -\frac{t}{4} \\ \frac{t}{2} \\ -\frac{3t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ -\frac{3}{4} \\ 1 \end{bmatrix}$$

By letting $t = 1$ the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ -\frac{3}{4} \\ 1 \end{bmatrix}$$

Normalizing gives

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$$

<u>eigenvalue 2</u>

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{array}\right]$$

current pivot $A(2,2)$ is zero. Hence we need to replace current pivot row with a non-zero row. Replacing row 2 with row 3 gives

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{array}\right]$$

current pivot $A(3,3)$ is still zero. Hence we need to replace current pivot row with non-zero row. Replacing row 3 with row 4 gives

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system is now in Echelon form. Hence

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_4 and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of free variable gives equations $v_1 = 0, v_2 = \frac{t}{6}, v_3 = -\frac{t}{2}$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{t}{6} \\ -\frac{t}{2} \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ \frac{1}{6} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{6} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Normalizing gives

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -3 \\ 6 \end{bmatrix}$$

<u>eigenvalue 3</u>

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -2 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cccc|c} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_2 \Longrightarrow \left[\begin{array}{cccc|c} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \end{array}\right]$$

current pivot $A(3,3)$ is zero. Hence we need to replace current pivot row with non-zero row. Replacing row 3 with row 4 gives

$$\left[\begin{array}{cccc|c} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system is now in Echelon form. Hence

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_4 and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of free variable gives equations $v_1 = 0, v_2 = 0, v_3 = -\frac{t}{4}$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

By letting $t = 1$ the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

Normalizing gives

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 4 \end{bmatrix}$$

<u>eigenvalue 4</u>

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -3 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{cccc|c} -3 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_2}{2} \Longrightarrow \left[\begin{array}{cccc|c} -3 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{array}\right]$$

$$R_4 = R_4 + 4R_3 \Longrightarrow \left[\begin{array}{cccc|c} -3 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_4 and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of the free variable gives equations $v_1 = 0, v_2 = 0, v_3 = 0$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue.

If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity: algebraic m	multiplicity: geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 0 \\ 1 \\ -3 \\ 6 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ -1 \\ 4 \end{bmatrix}$
4	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Now that the we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis.

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{t} = e^{t} \begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{2t} = e^{2t} \begin{bmatrix} 0 \\ 1 \\ -3 \\ 6 \end{bmatrix}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{3t} = e^{3t} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 4 \end{bmatrix}$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_4(t) = \vec{v}_4 e^{4t} = e^{4t} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t) + c_4 \vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 e^t \begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ -3 \\ 6 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 4 \end{bmatrix} + c_4 e^{4t} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Or

$$\begin{aligned} x_1(t) &= -c_1 e^t \\ x_2(t) &= 2c_1 e^t + c_2 e^{2t} \\ x_3(t) &= -3c_1 e^t - 3c_2 e^{2t} - c_3 e^{3t} \\ x_4(t) &= 4c_1 e^t + 6c_2 e^{2t} + 4c_3 e^{3t} + c_4 e^{4t} \end{aligned}$$

2.10.7 Additional problem 1

Consider the differential equation $x'''(t) + x''(t) - 2x'(t) = 0$. (a) Transform this into an equivalent system of first-order differential equations. (b) Write the system from (a) as $\vec{x}'(t) = A\vec{x}(t)$ (the matrix A should be 3×3). (c) Use the eigenvalue method to solve the system. (d) Using your solution to (c), what is the general solution $x(t)$ to the given differential equation?

Solution

2.10.7.1 Part (a)

Since this is a third order ODE, we need three state variables. Let

$$\begin{aligned} x_1 &= x \\ x_2 &= x' \\ x_3 &= x'' \end{aligned}$$

Taking derivatives of the above gives

$$\begin{aligned} x_1' &= x' \\ &= x_2 \\ x_2' &= x'' \\ &= x_3 \\ x_3' &= x''' \\ &= -x'' + 2x' \\ &= -x_3 + 2x_2 \end{aligned}$$

Therefore the equations are

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 2x_2 - x_3 \end{aligned}$$

2.10.7.2 Part (b)

The equations in part (a) in matrix form are

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

2.10.7.3 Part (c)

We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 2 & -1-\lambda \end{bmatrix}\right) = 0$$

Expansion along first column gives

$$\begin{aligned} -\lambda \begin{vmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{vmatrix} &= 0 \\ -\lambda((-\lambda)(-1-\lambda) - 2) &= 0 \\ -\lambda\left(\lambda^2 + \lambda - 2\right) &= 0 \\ -\lambda(\lambda+2)(\lambda-1) &= 0 \end{aligned}$$

Hence the roots are $\lambda_1 = 0, \lambda_2 = -2, \lambda_3 = 1$. For each eigenvalue we find the corresponding eigenvector.

$\underline{\lambda = 0}$

We need now to determine the eigenvector $\vec{v}$ where

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ A\vec{v} - \lambda\vec{v} &= \vec{0} \\ (A - \lambda I)\vec{v} &= \vec{0} \\ \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \implies \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \implies \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_1 and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of free variable gives equations $v_2 = 0, v_3 = 0$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = -2$

We need now to determine the eigenvector $\vec{v}$ where

$$\begin{aligned} A\vec{v} &= \lambda \vec{v} \\ A\vec{v} - \lambda \vec{v} &= \vec{0} \\ (A - \lambda I)\vec{v} &= \vec{0} \end{aligned}$$

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_3 and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of free variable gives equations $v_1 = \frac{t}{4}, v_2 = -\frac{t}{2}$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{t}{4} \\ -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\vec{v}_2 = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

$\underline{\lambda = 1}$

We need now to determine the eigenvector $\vec{v}$ where

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ A\vec{v} - \lambda\vec{v} &= \vec{0} \\ (A - \lambda I)\vec{v} &= \vec{0} \end{aligned}$$

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_3 and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of the free variable gives equations $v_1 = t, v_2 = t$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table summarizes the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
0	1	1	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
−2	1	1	No	$\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvectors, then the solution is

$$\begin{aligned} \vec{x}(t) &= c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) \\ &= c_1 e^{\lambda_1 t}\vec{v}_1(t) + c_2 e^{\lambda_2 t}\vec{v}_2(t) + c_3 e^{\lambda_3 t}\vec{v}_3(t) \\ &= c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + c_3 e^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned} \tag{1}$$

Or

$$\begin{aligned} x_1(t) &= c_1 + c_2 e^{-2t} + c_3 e^t \\ x_2(t) &= -2c_2 e^{-2t} + c_3 e^t \\ x_3(t) &= 4c_2 e^{-2t} + c_3 e^t \end{aligned}$$

2.10.7.4 Part (d)

From the solution we found in part(c), which is the general solution in vector form, this part is asking what is the solution to $x'''(t) + x''(t) - 2x'(t) = 0$. Since the solution to this ode is $x(t)$, and this is the same as $x_1(t)$, then the solution to the ODE is

$$x(t) = c_1 + c_2 e^{-2t} + c_3 e^t$$

Which is the first row in the vector solution found in part(c).

2.10.8 Additional problem 2

Consider the following system of brine tanks: There are three tanks. Tank 1 contains $20L$ of water, tank 2 contains $30L$ of water, and tank 3 contains $60L$ of water. Fresh water is pumped into tank 1 at a rate of 120 L/min. The well-mixed solution is pumped from tank 1 to tank 2, from tank 2 to tank 3, and out of tank 3 all at a rate of 120 L/min.

(a) Draw and label a diagram describing the system (b) Let $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ -x_2(t) \\ x_3(t) \end{bmatrix}$ be the vector function of the amount of salt in each tank at time t. Write a differential equation $\vec{x}'(t) = A\vec{x}(t)$ describing the system.(c) Find the general solution to the differential equation you wrote in (b) using the eigenvalue method. (d) Initially, there is 100 kg of salt in tank 1 and 20 kg of salt in tank 2. Find the particular solution corresponding to these initial conditions.

Solution

2.10.8.1 Part (a)

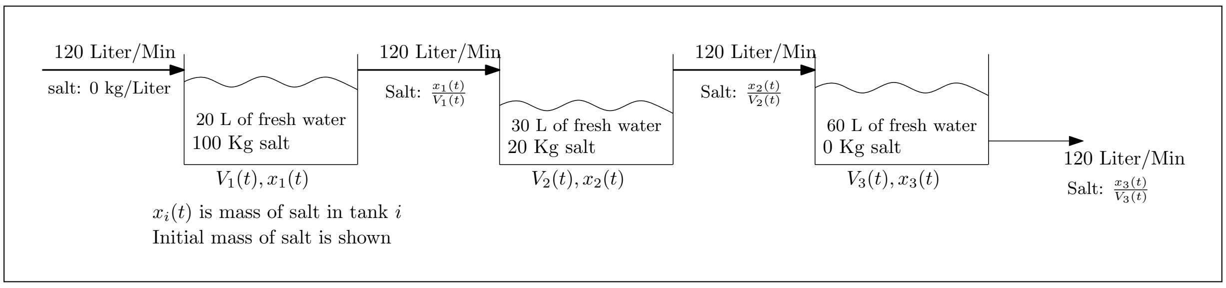


Figure 2.11: Diagram description of the problem

2.10.8.2 Part (b)

We notice that, since the rate of flow in and out from each tank is the same, then volume of water mix is constant and remain the same all the time in each tank. Hence

$$\begin{aligned} x_1'(t) &= \text{rate of flow in} - \text{ rate of flow out} \\ &= \left(120\left(\frac{\text{L}}{\text{min}}\right)0\left(\frac{\text{kg}}{\text{L}}\right)\right) - \left(120\left(\frac{\text{L}}{\text{min}}\right)\frac{x_1(t)}{V_1(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) \\ &= -120\frac{x_1(t)}{20(t)} \end{aligned} \tag{1}$$

And

$$\begin{aligned} x_2'(t) &= \text{rate of flow in} - \text{ rate of flow out} \\ &= \left(120\left(\frac{\text{L}}{\text{min}}\right)\frac{x_1(t)}{V_1(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) - \left(120\left(\frac{\text{L}}{\text{min}}\right)\frac{x_2(t)}{V_2(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) \\ &= 120\frac{x_1(t)}{20} - 120\frac{x_2(t)}{30} \end{aligned} \tag{2}$$

And

$$\begin{aligned} x_3'(t) &= \text{rate of flow in} - \text{ rate of flow out} \\ &= \left(120\left(\frac{\text{L}}{\text{min}}\right)\frac{x_2(t)}{V_2(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) - \left(120\left(\frac{\text{L}}{\text{min}}\right)\frac{x_3(t)}{V_3(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) \\ &= 120\frac{x_2(t)}{30} - 120\frac{x_3(t)}{60} \end{aligned} \tag{3}$$

Therefore the differential equations are

$$\begin{aligned} x_1'(t) &= -6x_1(t) \\ x_2'(t) &= 6x_1(t) - 4x_2(t) \\ x_3'(t) &= 4x_2(t) - 2x_3(t) \end{aligned}$$

In Matrix form

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -6 & 0 & 0 \\ 6 & -4 & 0 \\ 0 & 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$
$$\vec{x}' = A\vec{x}$$

2.10.8.3 Part (c)

We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -6 & 0 & 0 \\ 6 & -4 & 0 \\ 0 & 4 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -6-\lambda & 0 & 0 \\ 6 & -4-\lambda & 0 \\ 0 & 4 & -2-\lambda \end{bmatrix}\right) = 0$$

Since this is lower triangle matrix, then the determinant is the product of the elements along the diagonal. Hence

$$(-6-\lambda)(-4-\lambda)(-2-\lambda) = 0$$

Hence the roots are

$$\lambda_1 = -2$$
$$\lambda_2 = -6$$
$$\lambda_3 = -4$$

Next, the eigenvectors for each eigenvalue are found.

eigenvalue −6

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -6 & 0 & 0 \\ 6 & -4 & 0 \\ 0 & 4 & -2 \end{bmatrix} - (-6)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 & 0 \\ 6 & 2 & 0 \\ 0 & 4 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right]$$

current pivot $A(1,1)$ is zero. Hence we need to replace current pivot row with non-zero row. Replacing row 1 with row 2 gives

$$\left[\begin{array}{ccc|c} 6 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right]$$

current pivot $A(2,2)$ is zero. Hence we need to replace current pivot row with non-zero row. Replacing row 2 with row 3 gives

$$\left[\begin{array}{ccc|c} 6 & 2 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system is now in Echelon form

$$\begin{bmatrix} 6 & 2 & 0 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_3 and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of the free variables gives equations $v_1 = \frac{t}{3}, v_2 = -t$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{3} \\ -1 \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\vec{v}_1(t) = \begin{bmatrix} \frac{1}{3} \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$$

eigenvalue -4

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -6 & 0 & 0 \\ 6 & -4 & 0 \\ 0 & 4 & -2 \end{bmatrix} - (-4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 0 \\ 6 & 0 & 0 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + 3R_1 \implies \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \end{array}\right]$$

current pivot $A(2,2)$ is zero. Hence we need to replace current pivot row with one non-zero. Replacing row 2 with row 3 gives

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_3 and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of the free variable gives equations $v_1 = 0, v_2 = -\frac{t}{2}$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\vec{v}_2(t) = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

eigenvalue -2

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -6 & 0 & 0 \\ 6 & -4 & 0 \\ 0 & 4 & -2 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 0 & 0 \\ 6 & -2 & 0 \\ 0 & 4 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 6 & -2 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_3 and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of the free variable gives equations $v_1 = 0, v_2 = 0$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\vec{v}_3(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue.

If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
−2	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
−4	1	1	No	$\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$
−6	1	1	No	$\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$

Now that the we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis.

Since eigenvalue −2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= e^{-2t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Since eigenvalue −4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{-4t} \\ &= e^{-4t}\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \end{aligned}$$

Since eigenvalue −6 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_3(t) &= \vec{v}_3 e^{-6t} \\ &= e^{-6t}\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \end{aligned}$$

Therefore the final solution is

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 e^{-2t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-4t}\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + c_3 e^{-6t}\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$$

2.10.8.4 Part (d)

Initial conditions are

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 20 \\ 0 \end{bmatrix}$$

Hence the solution found in part(c) at $t = 0$ becomes

$$\begin{bmatrix} 100 \\ 20 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$$

Or in matrix form

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 20 \\ 0 \end{bmatrix}$$

From first row, $c_3 = 100$. From second row $-c_2 - 3c_3 = 20$ or $c_2 = -20 - 3c_3 = -20 - 300 = -320$ and from last row $c_1 + 2c_2 + 3c_3 = 0$ or $c_1 = -2c_2 - 3c_3 = -2(-320) - 3(100) = 340$. Hence

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 340 \\ -320 \\ 100 \end{bmatrix}$$

And the solution found at end of part (c) becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = 340e^{-2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 320e^{-4t} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + 100e^{-6t} \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$$

Or

$$\begin{aligned} x_1(t) &= 100e^{-6t} \\ x_2(t) &= 320e^{-4t} - 300e^{-6t} \\ x_3(t) &= 340e^{-2t} - 640e^{-4t} + 300e^{-6t} \end{aligned}$$

We see that as $t \to \infty$ then there will be no salt left in any tank, since each $x_i(t) \to 0$, and therefore only fresh water will remain in each tank, as expected.

2.10.9 key solution for HW10

Homework 10 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

7.3.4 We have $\vec{x}' = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} \vec{x}$. We need eigenvalues and eigenvectors.

$$\begin{aligned} \det \begin{bmatrix} 4-\lambda & 1 \\ 6 & -1-\lambda \end{bmatrix} &= (4-\lambda)(-1-\lambda) - 6 \\ &= \lambda^2 - 3\lambda - 10 \\ &= (\lambda - 5)(\lambda + 2) \end{aligned}$$

For $\lambda_1 = -2$, we have

$$\begin{bmatrix} 6 & 1 \\ 6 & 1 \end{bmatrix} \to \begin{bmatrix} 6 & 1 \\ 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_1 = (-1, 6)$.

For $\lambda_2 = 5$, we have

$$\begin{bmatrix} -1 & 1 \\ 6 & -6 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_2 = (1, 1)$.

Our general solution is

$$\vec{x}(t) = c_1 e^{-2t} \begin{bmatrix} -1 \\ 6 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

7.3.6 We have $\vec{x}' = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} \vec{x}$ and $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We need eigenvalues and eigenvectors.

$$\begin{aligned} \det \begin{bmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{bmatrix} &= (9-\lambda)(-2-\lambda) + 30 \\ &= \lambda^2 - 7\lambda + 12 \\ &= (\lambda - 3)(\lambda - 4) \end{aligned}$$

For $\lambda_1 = 3$, we have

$$\begin{bmatrix} 6 & 5 \\ -6 & -5 \end{bmatrix} \to \begin{bmatrix} 6 & 5 \\ 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_1 = (-5, 6)$.

1

For $\lambda_2 = 4$, we have

$$\begin{bmatrix} 5 & 5 \\ -6 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_2 = (-1, 1)$.

Our general solution is

$$\vec{x}(t) = c_1 e^{3t} \begin{bmatrix} -5 \\ 6 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

To apply our initial conditions, we set $t = 0$:

$$\vec{x}(0) = c_1 \begin{bmatrix} -5 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We solve the system by reducing the augmented matrix:

$$\begin{aligned} \begin{bmatrix} -5 & -1 & 1 \\ 6 & 1 & 0 \end{bmatrix} &\xrightarrow{R_1+R_2} \begin{bmatrix} -5 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{5R_2+R_1} \begin{bmatrix} 0 & -1 & 6 \\ 1 & 0 & 1 \end{bmatrix} \end{aligned}$$

So $c_1 = 1$ and $c_2 = -6$. Our particular solution is

$$\vec{x}(t) = e^{3t} \begin{bmatrix} -5 \\ 6 \end{bmatrix} + e^{4t} \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

7.3.8 We have $\vec{x}' = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \vec{x}$. We need eigenvalues and eigenvectors.

$$\begin{aligned} \det \begin{bmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{bmatrix} &= (1-\lambda)(-1-\lambda) + 5 \\ &= \lambda^2 + 4 \end{aligned}$$

Our eigenvalues are $\pm 2i$. For $\lambda_1 = 2i$, we have

$$\begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix} \xrightarrow{\frac{1}{1-2i}R_1+R_2} \begin{bmatrix} 1-2i & -5 \\ 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_1 = (5, 1-2i)$. The corresponding complex-valued solution is

$$\begin{aligned} \vec{v}_1 e^{\lambda_1 t} &= \begin{bmatrix} 5 \\ 1-2i \end{bmatrix} e^{2it} \\ &= \begin{bmatrix} 5 \\ 1-2i \end{bmatrix} (\cos 2t + i \sin 2t) \\ &= \begin{bmatrix} 5\cos 2t + 5i \sin 2t \\ \cos 2t + i \sin 2t - 2i \cos 2t + 2 \sin 2t \end{bmatrix} \\ &= \begin{bmatrix} 5\cos 2t \\ \cos 2t + 2\sin 2t \end{bmatrix} + \begin{bmatrix} 5 \sin 2t \\ \sin 2t - 2\cos 2t \end{bmatrix} i \end{aligned}$$

2

Having separated this into real and imaginary components, we now have two linearly independent real-valued solutions. So our general solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} 5\cos 2t \\ \cos 2t + 2\sin 2t \end{bmatrix} + c_2 \begin{bmatrix} 5\sin 2t \\ \sin 2t - 2\cos 2t \end{bmatrix}$$

7.3.18 We have $\vec{x}' = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} \vec{x}$. We need eigenvalues and eigenvectors.

$$\begin{aligned} \det \begin{bmatrix} 1-\lambda & 2 & 2 \\ 2 & 7-\lambda & 1 \\ 2 & 1 & 7-\lambda \end{bmatrix} &= (1-\lambda)\det \begin{bmatrix} 7-\lambda & 1 \\ 1 & 7-\lambda \end{bmatrix} - 2\det \begin{bmatrix} 2 & 2 \\ 1 & 7-\lambda \end{bmatrix} + 2\det \begin{bmatrix} 2 & 2 \\ 7-\lambda & 1 \end{bmatrix} \\ &= (1-\lambda)\left[(7-\lambda)^2 - 1\right] - 2(14 - 2\lambda - 2) + 2(2 - 14 + 2\lambda) \\ &= (1-\lambda)(\lambda^2 - 14\lambda + 48) + 8\lambda - 48 \\ &= (1-\lambda)(\lambda-6)(\lambda-8) + 8(\lambda-6) \\ &= (\lambda-6)\left[(1-\lambda)(\lambda-8) + 8\right] \\ &= (\lambda-6)(-\lambda^2 + 9\lambda) \\ &= -\lambda(\lambda-6)(\lambda-9) \end{aligned}$$

For $\lambda_1 = 0$, we have

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & -3 \\ -3 & 3 & \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_1 = (-4, 1, 1)$.

For $\lambda_2 = 6$, we have

$$\begin{bmatrix} -5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow[-R_2+R_3]{3R_2+R_1} \begin{bmatrix} 1 & 5 & 5 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 5 & 5 \\ 0 & -9 & -9 \\ 0 & 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_2 = (0, 1, -1)$.

For $\lambda_3 = 9$, we have

$$\begin{bmatrix} -8 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix} \xrightarrow[4R_2+R_1]{-R_2+R_3} \begin{bmatrix} 0 & -6 & 6 \\ 2 & -2 & 1 \\ 0 & 3 & -3 \end{bmatrix} \xrightarrow{2R_3+R_1} \begin{bmatrix} 0 & 0 & 0 \\ 2 & -2 & 1 \\ 0 & 3 & -3 \end{bmatrix}$$

An eigenvector is $\vec{v}_3 = (1, 2, 2)$.

Our general solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 e^{9t} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

3

7.3.38 We have the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix}$. We need eigenvalues and eigenvectors.

$$\det \begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 2 & 2-\lambda & 0 & 0 \\ 0 & 3 & 3-\lambda & 0 \\ 0 & 0 & 4 & 4-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda)(3-\lambda)(4-\lambda)$$

Here we used that the determinant of a lower triangular matrix is the product of the diagonal entries. Alternatively, you can expand along the first row several times.

For $\lambda_1 = 1$, we have

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 4 & 3 \end{bmatrix}$$

If we set $x_4 = 4$, we get $x_3 = -3$, $x_2 = 2$, and $x_1 = -1$. So an eigenvector is $\vec{v}_1 = (-1, 2, -3, 4)$.

For $\lambda_2 = 2$, we have

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_2 = (0, 1, -3, 6)$.

For $\lambda_3 = 3$, we have

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_3 = (0, 0, 1, -4)$.

For $\lambda_4 = 4$, we have

$$\begin{bmatrix} -3 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_4 = (0, 0, 0, 1)$.

Our general solution is

$$\vec{x}(t) = c_1 e^t \begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ -3 \\ 6 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix} + c_4 e^{4t} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

4

Additional Problems:

1. (a) Let $x_1 = x, x_2 = x', x_3 = x''$. We get the system

$$\begin{aligned} x_3' + x_3 - 2x_2 &= 0 \\ x_2' &= x_3 \\ x_1' &= x_2 \end{aligned}$$

(b) We have to rearrange some terms to write things in matrix form, but we get

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(c) We need eigenvalues and eigenvectors.

$$\begin{aligned} \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 2 & -1-\lambda \end{bmatrix} &= -\lambda \det \begin{bmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{bmatrix} \\ &= -\lambda(\lambda + \lambda^2 - 2) \\ &= -\lambda(\lambda+2)(\lambda-1) \end{aligned}$$

For $\lambda_1 = 0$, we have

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_1 = (1, 0, 0)$.
For $\lambda_2 = 1$, we have

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_2 = (1, 1, 1)$.
For $\lambda_3 = -2$, we have

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

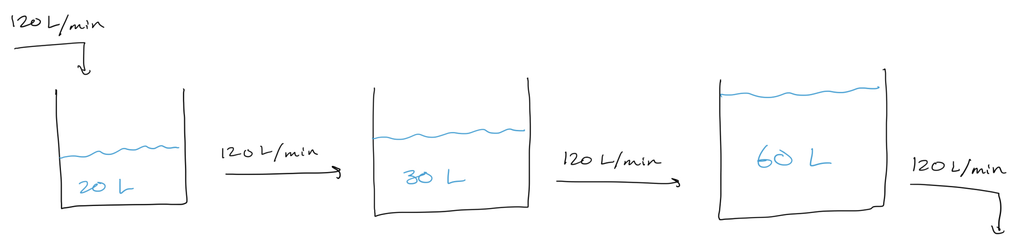
An eigenvector is $\vec{v}_3 = (1, -2, 4)$
Our general solution to this system is thus

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

(d) The first component of $\vec{x}$ is $x_1 = x$. So looking at that first component of our general solution we can say that the general solution of our original equation is

$$x(t) = c_1 + c_2 e^t + c_3 e^{-2t}$$

2. (a) Our system looks like:



(b) Note that each tank has constant volume. Our differential equations for the amount of salt in each tank are:

$$\begin{aligned} x_1' &= -120 * \frac{x_1}{20} \\ x_2' &= 120 * \frac{x_1}{20} - 120 * \frac{x_2}{30} \\ x_3' &= 120 * \frac{x_2}{30} - 120 * \frac{x_3}{60} \end{aligned}$$

Writing this as a matrix equation, we have

$$\vec{x}' = \begin{bmatrix} -6 & 0 & 0 \\ 6 & -4 & 0 \\ 0 & 4 & -2 \end{bmatrix} \vec{x}$$

(c) To solve, we need eigenvalues and eigenvectors:

$$\det \begin{bmatrix} -6-\lambda & 0 & 0 \\ 6 & -4-\lambda & 0 \\ 0 & 4 & -2-\lambda \end{bmatrix} = (-6-\lambda)(-4-\lambda)(-2-\lambda)$$

For $\lambda_1 = -6$, we have

$$\begin{bmatrix} 0 & 0 & 0 \\ 6 & 2 & 0 \\ 0 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have eigenvector $\vec{v}_1 = (1, -3, 3)$.
For $\lambda_2 = -4$, we have

$$\begin{bmatrix} -2 & 0 & 0 \\ 6 & 0 & 0 \\ 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have eigenvector $\vec{v}_1 = (0, 1, -2)$.
For $\lambda_3 = -2$, we have

$$\begin{bmatrix} -4 & 0 & 0 \\ 6 & -2 & 0 \\ 0 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We have eigenvector $\vec{v}_1 = (0, 0, 1)$.
So our general solution is

$$\vec{x}(t) = c_1 e^{-6t} \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(d) We have the initial condition $\vec{x}(0) = \begin{bmatrix} 100 \\ 20 \\ 0 \end{bmatrix}$. Plugging this in to our general solution, we have

$$\begin{bmatrix} 100 \\ 20 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ -3c_1 + c_2 \\ 3c_1 - 2c_2 + c_3 \end{bmatrix}$$

We can immediately back substitute to get $c_1 = 100$, $c_2 = 320$, and $c_3 = 340$. The particular solution is

$$\vec{x}(t) = e^{-6t} \begin{bmatrix} 100 \\ -300 \\ 300 \end{bmatrix} + e^{-4t} \begin{bmatrix} 0 \\ 320 \\ -640 \end{bmatrix} + e^{-2t} \begin{bmatrix} 0 \\ 0 \\ 340 \end{bmatrix}$$

2.11 HW 11

Local contents

2.11.1 Problems listing

Homework 11 - Due December 3

Homework instructions: Complete the assigned problems on your own paper. Once you are finished, scan or photograph your work and upload it to Gradescope. When prompted, tell Gradescope where to find each problem.
You are allowed (and in fact encouraged) to work with other students on homework assignments. If you do that, please indicate on each problem who you worked with. If you use sources other than your notes, the textbook, and any resources on Canvas for your homework, you must indicate the source on each problem. You are not permitted to view, request, or look for solutions to any of the homework problems from solutions manuals, homework help websites, online forums, other students, or any other sources.

Textbook Problems:

- §1.2: 6, 8, 24, 26

  For problems in this section, use 9.8 $\mathrm{m/s^2}$ or 32 $\mathrm{ft/s^2}$ as the acceleration due to gravity

- §1.3: 5, 9

  For the problems in section 1.3, ignore the instructions in the textbook and do the following:

  - Make a copy of the slope field given in the textbook. You may sketch it by hand, make a photocopy of the textbook page, generate your own using a computer, or do anything else to make a copy that you can write on.
  - On your slope field, sketch at least 3 different solution curves. Label each solution curve by the initial point (x, y) that you chose

Additional Problems:
For these problems, please use a calculator to compute approximate times, distances, and speeds. Round all numbers to two decimal places. **Be careful with your units!**

1. A racecar accelerates from stationary at a rate of 14 $\mathrm{m/s^2}$. How long does it take the car to reach its top speed of 300 km/h? How far does the car travel in that time?

2. The car is approaching a tight turn at 300 km/h. In order to safely make the corner, it must be traveling at 80 km/h when it enters the corner. The brakes on the car cause a deceleration of 39 $\mathrm{m/s^2}$. How far away from the corner must the driver begin braking to make the corner?

3. At the exit of the corner, two cars are traveling at 100 km/h, with car A 10 m behind car B. Out of the corner, car A accelerates at 14 $\mathrm{m/s^2}$ and car B accelerates at 13 $\mathrm{m/s^2}$. How much time does it take for car A to be right next to car B? How fast are the cars going when this happens? How far from the corner exit have they traveled?

1

2.11.2 Problem 6, section 1.2

Solve

$$\frac{dy}{dx} = x\sqrt{x^2+9}$$
$$y(-4) = 0$$

Solution

This is separable ODE. Integrating both sides gives

$$y(x) = \int x\sqrt{x^2+9}\,dx + c \tag{1}$$

Where c is constant of integration. To integrate $\int x\sqrt{x^2+9}\,dx$, let $u = x^2+9$. Hence $\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$. Therefore the integral becomes

$$\begin{aligned}\int x\sqrt{x^2+9}\,dx &= \int x\sqrt{u}\,\frac{du}{2x}\\ &= \frac{1}{2}\int u^{\frac{1}{2}}du\\ &= \frac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\\ &= \frac{1}{2}\frac{2}{3}u^{\frac{3}{2}}\\ &= \frac{1}{3}u^{\frac{3}{2}}\end{aligned}$$

But $u = x^2+9$, hence the above becomes

$$\int x\sqrt{x^2+9}\,dx = \frac{1}{3}\left(x^2+9\right)^{\frac{3}{2}}$$

Substituting the above in (1) gives

$$y(x) = \frac{1}{3}\left(x^2+9\right)^{\frac{3}{2}} + c \tag{2}$$

The constant c is from initial conditions. Since $y(-4) = 0$ then Eq (2) becomes

$$\begin{aligned}0 &= \frac{1}{3}(16+9)^{\frac{3}{2}} + c\\ &= \frac{1}{3}(25)^{\frac{3}{2}} + c\\ &= \frac{1}{3}\left(5^2\right)^{\frac{3}{2}} + c\\ &= \frac{1}{3}(5)^3 + c\\ &= \frac{125}{3} + c\end{aligned}$$

Hence $c = -\frac{125}{3}$. Therefore the solution (2) becomes

$$\begin{aligned}y(x) &= \frac{1}{3}\left(x^2+9\right)^{\frac{3}{2}} - \frac{125}{3}\\ &= \frac{1}{3}\left(\left(x^2+9\right)^{\frac{3}{2}} - 125\right)\end{aligned}$$

2.11.3 Problem 8, section 1.2

Solve

$$\frac{dy}{dx} = \cos 2x$$
$$y(0) = 1$$

Solution

This is separable ODE. Integrating both sides gives

$$\begin{aligned} y(x) &= \int \cos 2x dx + c \\ &= \frac{1}{2}\sin(2x) + c \qquad (1) \end{aligned}$$

The constant c is from initial conditions. Since $y(0) = 1$ then (1) becomes

$$\begin{aligned} 1 &= \frac{\sin(0)}{2} + c \\ &= c \end{aligned}$$

Hence the solution (1) becomes

$$y(x) = \frac{1}{2}\sin(2x) + 1$$

2.11.4 Problem 24, section 1.2

A ball is dropped from the top of a building 400 ft high. How long does it take to reach the ground? With what speed does the ball strike the ground?

Solution

Let the ground by level 0 (i.e. $y = 0$) and let up be positive and down negative. Therefore $y(0) = 400$ ft and assuming initial velocity is zero then $y'(0) = v(0) = 0$.. Therefore

$$v(t) = \int a(t)dt$$

Where $a(t)$ is the acceleration, which in this case is $g = -32$ ft/sec^2. The above becomes

$$\begin{aligned} v(t) &= -32t + v(0) \qquad (1) \\ &= -32t \end{aligned}$$

And

$$\begin{aligned} y(t) &= \int v(t)dt \\ &= \int -32t\ dt \\ &= -\frac{32}{2}t^2 + y(0) \end{aligned}$$

But $y(0) = 400$ ft. The above becomes

$$y(t) = -16t^2 + 400$$

To find the time it takes to hit the ground, the above is solved for $y(t) = 0$. This gives

$$\begin{aligned} 0 &= -16t^2 + 400 \\ t^2 &= \frac{400}{16} \\ &= 25 \end{aligned}$$

Therefore the time is $t = 5$ seconds. Now we know how long it takes to reach the ground, we can find the velocity when ball strike the ground from (1). Substituting $t = 5$ in (1) gives

$$\begin{aligned} v(5) &= -32(5) \\ &= -160 \text{ ft/sec} \end{aligned}$$

So it strikes the ground with speed 160 ft/sec in the downwards (negative) direction.

2.11.5 Problem 26, section 1.2

A projectile is fired straight upward with an initial velocity of 100 m/s from the top of a building 20 m high and falls to the ground at the base of the building. Find (a) its maximum height above the ground (b) when it passes the top of the building (c) its total time in the air.

Solution

2.11.5.1 Part a

Let the ground be level 0 . (i.e. $y = 0$) and let up be positive and down negative. Therefore $y(0) = 20$ m. Initial velocity is 100 m/s, hence $y'(0) = v(0) = 100$. The acceleration due to gravity is $g = -9.8$ m/s^2.

$$\begin{aligned} v(t) &= \int a(t)dt \\ &= -gt + v(0) \\ &= -gt + 100 \end{aligned}$$

When the ball reaches maximum high above the building, it must have zero velocity. From the above this means

$$\begin{aligned} 0 &= -9.8t + 100 \\ t &= \frac{100}{g} \text{ sec} \end{aligned}$$

The above is how long it takes for the ball to reach maximum high. Now

$$\begin{aligned} y(t) &= \int v(t)dt \\ &= \int \left(-gt + 100\right)dt + y(0) \\ y(t) &= -\frac{1}{2}gt^2 + 100t + y(0) \end{aligned}$$

But $y(0) = 20$. Therefore

$$y(t) = -\frac{1}{2}gt^2 + 100t + 20$$

Substituting $t = \frac{100}{g}$ in the above, gives the distance traveled above the ground until the ball reached maximum high. Therefore

$$\begin{aligned} y\left(\frac{100}{g}\right) &= -\frac{1}{2}g\left(\frac{100}{g}\right)^2 + 100\left(\frac{100}{g}\right) + 20 \\ &= -\frac{1}{2}\frac{100^2}{g} + \frac{100^2}{g} + 20 \\ &= \frac{1}{2}\frac{100^2}{g} + 20 \end{aligned}$$

Using $g = 9.8$ the above gives

$$\begin{aligned} y\left(\frac{100}{g}\right) &= \frac{1}{2}\frac{100^2}{9.8} + 20 \\ y_{\max} &= 530.2 \text{ meter} \end{aligned}$$

2.11.5.2 Part b

The ball will take the same amount of time to fall down back to top of building, as the time it took to reach the maximum high above the building, since the distance is the same, and the acceleration is the same (gravity acceleration). This time is $t_0 = \frac{100}{g}$ sec found in part (a). Therefore, twice this time gives

$$\begin{aligned} t_{\text{travel}} &= \frac{200}{g} \\ &= \frac{200}{9.8} \\ &= 20.408 \text{ sec} \end{aligned}$$

2.11.5.3 Part c

Now we find the time it take to reach the ground. We now take initial velocity as $v(0) = 0$, which is when the ball was at its maximum high above the building. And initial position is from part (a) was found $y_{\max} = 530.2$ meter. Hence $y(0) = 530.2$ m. Now we will find the time to reach the ground, starting from the maximum high.

$$
\begin{aligned}
v(t) &= \int g dt \\
&= gt + v(0) \\
&= gt
\end{aligned}
$$

And

$$
\begin{aligned}
y(t) &= \int v(t) dt \\
&= \int gt\ dt + y(0) \\
&= \frac{1}{2} g t^2 + 530.2
\end{aligned}
$$

When it hits the ground $y(t) = 0$,. Hence we now have an equation to solve for time

$$
0 = \frac{1}{2} g t^2 + 530.2
$$

But $g = -9.8$. The above becomes

$$
\begin{aligned}
0 &= \frac{1}{2}(-9.8) t^2 + 530.2 \\
t^2 &= \frac{2(530.2)}{9.8}
\end{aligned}
$$

Hence $t = \sqrt{\frac{2(530.2)}{9.8}} = 10.402$ sec. This is the time it takes to fall to the ground, starting from maximum high. Adding to this time, the time it took to reach maximum high from top of building, which is $\frac{100}{g}$ sec as found from part (a), gives total time in air

$$
\begin{aligned}
t_{\text{total}} &= 10.402 + \frac{100}{9.8} \\
&= 20.606 \text{ sec}
\end{aligned}
$$

2.11.6 Problem 5, section 1.3

<u>Solution</u>

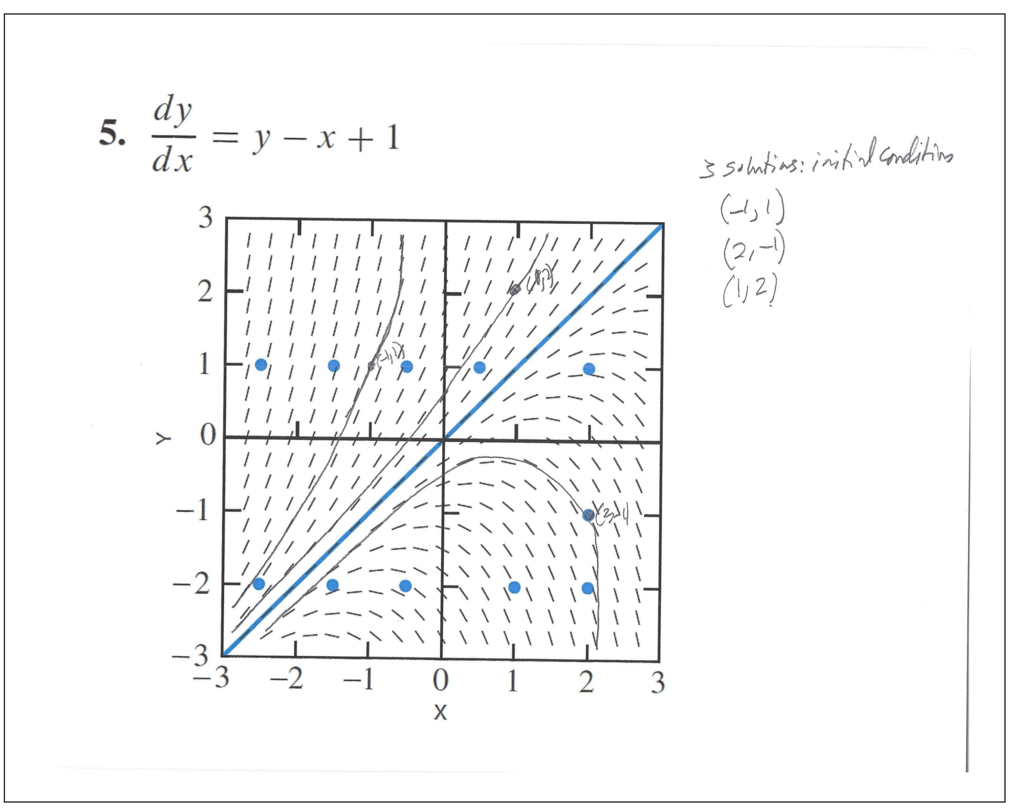


Figure 2.12: Shoiwng 3 solution curves with different initial conditions

2.11.7 Problem 9, section 1.3

Solution

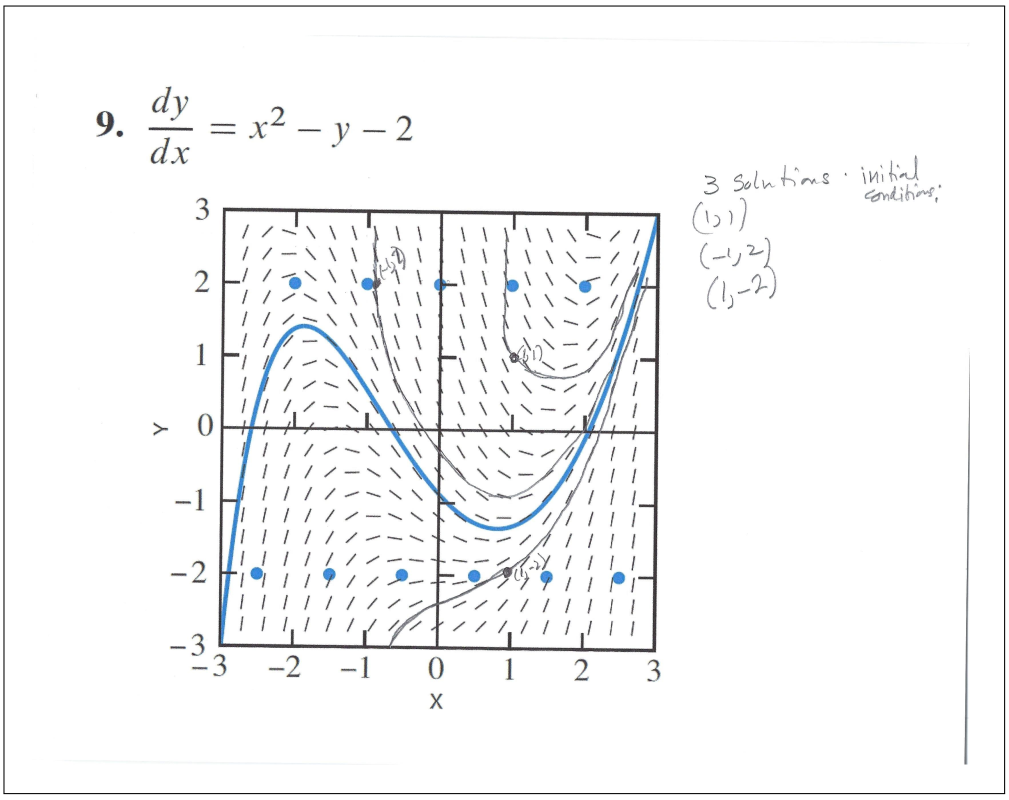


Figure 2.13: Shoiwng 3 solution curves with different initial conditions

2.11.8 Additional problem 1

A racecar accelerates from stationary at a rate of 14 m/s^2. How long does it take the car to reach its top speed of 300 km/h? How far does the car travel in that time?

Solution

Let $x(0) = 0, v(0) = 0$ and $a = 14$ m/s^2.

$$\begin{aligned} v(t) &= \int a(t)dt \\ &= \int 14dt \\ &= 14t + v(0) \\ &= 14t \end{aligned}$$

Since we want to find time to reach $v_{\max} = 300$ km/h which in SI units is $\frac{(300)(1000)}{(60)(60)} = \frac{250}{3}$ m/sec. Substituting this in the above gives

$$\begin{aligned} \frac{250}{3} &= 14t_{\max} \\ t_{\max} &= \frac{250}{3(14)} \\ &= \frac{125}{21} \\ &= 5.95 \text{ seconds} \end{aligned}$$

To find the distance traveled in this time, since

$$\begin{aligned} x(t) &= \int v(t)dt \\ &= \int 14tdt \\ &= \frac{14}{2}t^2 + x(0) \\ &= 7t^2 \end{aligned}$$

When $t = t_{\max}$ the above gives

$$\begin{aligned} x(t_{\max}) &= 7\left(\frac{125}{21}\right)^2 \\ &= 248.02 \text{ meters} \end{aligned}$$

2.11.9 Additional problem 2

The car is approaching a tight turn at 300 km/h. In order to safely make the corner, it must be traveling at 80 km/h when it enters the corner. The brakes on the car cause a deceleration of 39 m/s^2. How far away from the corner must the driver begin braking to make the corner?

Solution

In SI units 300 km/h is $\frac{(300)(1000)}{(60)(60)} = \frac{250}{3}$ m/s. And 80 km/h is $\frac{80(1000)}{(60)(60)} = \frac{200}{9}$ m/s. Therefore we have initial velocity $v(0) = \frac{250}{3}$ m/s and final velocity $v_f(t) = \frac{200}{9}$ m/s and have acceleration of -39 m/s^2.

We first find the time it takes to go from $v(0)$ to $v_f(t)$. Since

$$\begin{aligned} v(t) &= \int a(t)dt \\ &= \int -39dt \\ &= -39t + v(0) \end{aligned}$$

Therefore we have the equation

$$\begin{aligned} v_f(t) &= -39t + v(0) \\ \frac{200}{9} &= -39t + \frac{250}{3} \\ 39t &= \frac{250}{3} - \frac{200}{9} \\ t_f &= \frac{550}{351} \\ &= 1.567 \text{ sec} \end{aligned}$$

This is the time needed to decelerate from 300 km/h to 80 km/h. Now we find the distance traveled during this time. Since

$$\begin{aligned} x(t) &= \int v(t)dt \\ &= \int -39t + v(0)dt \\ &= \int -39t + \frac{250}{3}dt \\ &= -\frac{39}{2}t^2 + \frac{250}{3}t + x(0) \end{aligned}$$

Let $x(0) = 0$, by taking initial position as zero. Replacing t in the above with t_f found earlier gives

$$\begin{aligned} x(t) &= -\frac{39}{2}\left(1.567^2\right) + \frac{250}{3}(1.567) \\ &= 82.7 \text{ meter} \end{aligned}$$

Therefore the car needs to be 82.7 meter away from corner to begin the braking.

2.11.10 Additional problem 3

At the exit of the corner, two cars are traveling at 100 km/h, with car A being 10 m behind car B. Out of the corner, car A accelerates at 14 m/s^2 and car B accelerates at 13 m/s^2. How much time does it take for car A to be right next to car B? How fast are the cars going when this happens? How far from the corner exit have they traveled?

Solution

Using SI units, 100 km/h is $\frac{(100)(1000)}{(60)(60)} = \frac{250}{9}$ m/s. Let at $t=0$, $x_A(0)=0$ and therefore $x_B(0)=10$, since car B is ahead by 10 meters initially. Let $v_A(0)=\frac{250}{9}$ m/s and also $v_B(0)=\frac{250}{9}$ m/s. We now need to determine the time, say t_f, where $x_A(t_f)=x_B(t_f)$. But for car A we have

$$\begin{aligned} v_A(t) &= \int a_A(t)dt \\ &= \int 14dt \\ &= 14t + v_A(0) \\ &= 14t + \frac{250}{9} \end{aligned}$$

And

$$\begin{aligned} x_A(t) &= \int v_A(t)dt \\ &= \int\left(14t + \frac{250}{9}\right)dt \\ &= \frac{14}{2}t^2 + \frac{250}{9}t + x_A(0) \\ &= 7t^2 + \frac{250}{9}t \end{aligned} \tag{1}$$

Since $x_A(0)=0$. Now we do the same for car B

$$\begin{aligned} v_B(t) &= \int a_B(t)dt \\ &= \int 13dt \\ &= 13t + v_A(0) \\ &= 13t + \frac{250}{9} \end{aligned}$$

And

$$\begin{aligned} x_B(t) &= \int v_B(t)dt \\ &= \int\left(13t + \frac{250}{9}\right)dt \\ &= \frac{13}{2}t^2 + \frac{250}{9}t + x_B(0) \\ &= \frac{13}{2}t^2 + \frac{250}{9}t + 10 \end{aligned} \tag{2}$$

Since $x_B(0)=10$ m. Now we solve for t by equating (1) and (2)

$$\begin{aligned} 7t^2 + \frac{250}{9}t &= \frac{13}{2}t^2 + \frac{250}{9}t + 10 \\ 7t^2 + &= \frac{13}{2}t^2 + 10 \\ 7t^2 - \frac{13}{2}t^2 &= 10 \\ \frac{1}{2}t^2 &= 10 \\ t^2 &= 20 \\ t &= \sqrt{20} \\ t_f &= 4.47 \text{ sec} \end{aligned}$$

So it takes 4.47 sec for car A to be be next to car B. To find the speed at this time, we substitute this value of time back in the velocity equation above. For car A

$$\begin{aligned} v_A(t) &= 14t + \frac{250}{9} \\ v_A\left(t_f\right) &= 14(4.47) + \frac{250}{9} \\ &= 90.36 \text{ m/s} \end{aligned} \tag{3}$$

And for car B

$$\begin{aligned} v_B(t) &= 13t + \frac{250}{9} \\ v_B\left(t_f\right) &= 13(4.47) + \frac{250}{9} \\ &= 85.89 \text{ m/s} \end{aligned}$$

To find the distance traveled during this time, we substitute this time in the position equation. For car A, from Eq (1)

$$\begin{aligned} x_A(t) &= 7t^2 + \frac{250}{9}t \\ x_A\left(t_f\right) &= 7(4.47)^2 + \frac{250}{9}(4.47) \\ &= 264.03 \text{ meter} \end{aligned}$$

The distance traveled by car B is 10 meters less than this value, since it was ahead by 10 meters at the start at time $t = 0$.

2.11.11 key solution for HW11

HOMEWORK 11 – SOLUTIONS

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

1.2.6 We are given $y' = x\sqrt{x^2+9}$; $y(-4) = 0$. We integrate to solve the differential equation, using a u-substitution of $u = x^2 + 9$ so that $du = 2x\ dx$.

$$\begin{aligned} y &= \int x\sqrt{x^2+9}\ dx \\ &= \frac{1}{2}\int \sqrt{u}\ du \\ &= \frac{1}{2}\cdot\frac{2}{3}\cdot u^{\frac{3}{2}} + C \\ &= \frac{1}{3}(x^2+9)^{\frac{3}{2}} + C \end{aligned}$$

To solve for C, we plug in our initial condition:

$$\begin{aligned} y(-4) = 0 &= \frac{1}{3}\left((-4)^2+9\right)^{\frac{3}{2}} + C \\ &= \frac{1}{3}(25)^{\frac{3}{2}} + C \\ &= \frac{125}{3} + C \end{aligned}$$

So $C = -\frac{125}{3}$ and our particular solution is $y(x) = \frac{1}{3}(x^2+9)^{\frac{3}{2}} - \frac{125}{3}$.

1.2.8 We are given $y' = \cos(2x)$; $y(0) = 1$. We integrate to solve the differential equation using $u = 2x$ so that $du = 2\ dx$.

$$\begin{aligned} y &= \int \cos(2x)\ dx \\ &= \frac{1}{2}\int \cos u\ du \\ &= \frac{1}{2}\sin u + C \\ &= \frac{1}{2}\sin(2x) + C \end{aligned}$$

To solve for C, we plug in our intial condition:

$$\begin{aligned} y(0) = 1 &= \frac{1}{2}\sin(0) + C \\ &= C \end{aligned}$$

So $C = 1$ and our particular solution is $y(x) = \frac{1}{2}\sin(2x) + 1$.

1

1.2.24 The building is 400 ft high, so $y_0 = 400$. We drop the ball, so $v_0 = 0$. The acceleration is $a(t) = -32$ ft/s^2 due to gravity. Here, we are using the convention that movement downward is a negative velocity. This is consistent with the ball falling down from $y(0) = 400$ to $y(t) = 0$.

Since we have constant acceleration, we can use the formulas given in the textbook. So the velocity and position functions are

$$v(t) = -32t$$
$$y(t) = -16t^2 + 400$$

We want to know when $y(t) = 0$. So, we solve

$$\begin{aligned} -16t^2 + 400 &= 0 \\ 16t^2 &= 400 \\ t^2 &= 25 \end{aligned}$$

There are two solutions here, $t = \pm 5$. Since we are modeling a process that is going forward in time, we choose the solution $t = 5$. At $t = 5$, we have $v(t) = -160$.

So the ball hits the ground after 5 seconds, at which point it is traveling 160 ft/s downwards.

1.2.26 We start on top of a building 20 m high, so $y_0 = 20$. We fire the projectile upwards at 100 m/s, so $v_0 = 100$. The acceleration is constant $a(t) = -9.8$ m/s^2. We are using the same convention about positive and negative velocities as in the previous problem. Our velocity and position functions are

$$v(t) = -9.8t + 100$$
$$y(t) = -4.9t^2 + 100t + 20$$

(a) To find the maximum height above the ground, we maximize the function $y(t) = -4.9t^2 + 100t + 20$. This is a downward-facing parabola, with maximum value when $y'(t) = v(t) = 0$. So we solve $-9.8t + 100 = 0$ to get $t = 10.2$. At this time, $y(10.2) = 530.2$ m.

(b) The projectile passes the top of the building again when $y(t) = 20$. So we solve

$$\begin{aligned} -4.9t^2 + 100t + 20 &= 20 \\ t(-4.9t + 100) &= 0 \end{aligned}$$

The solutions here are $t = 0, 20.4$. The $t = 0$ is the moment we shoot the projectile up, and we already knew it was at the top of the building at that point. So the time we are looking for is $t = 20.4$ s.

2

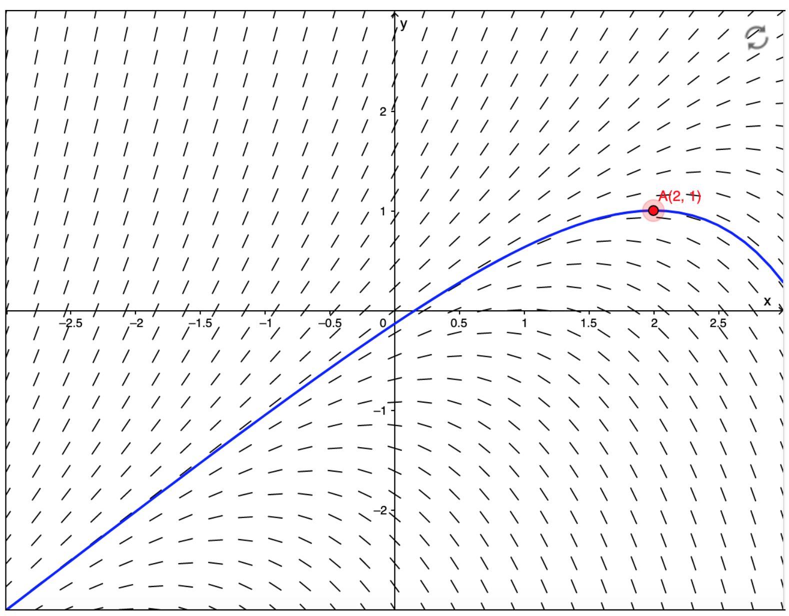
(c) The total time in the air is the time from $t = 0$ until $y(t) = 0$. So we solve

$$-4.9t^2 + 100t + 20 = 0$$

This quadratic has roots $t = 20.6, -0.2$. As usual, we choose the positive time $t = 20.6$ s. So the projectile is in the air for 20.6 seconds.

1.3.5 In order to give you the most accurate picture, I've given computer-generated solution curves. Your sketches don't need to be perfect, but they should look somewhat similar to the real thing.

The three solution curves I've given started from the initial points $(2, 1)$, $(1, -1)$, and $(-1, 1)$.



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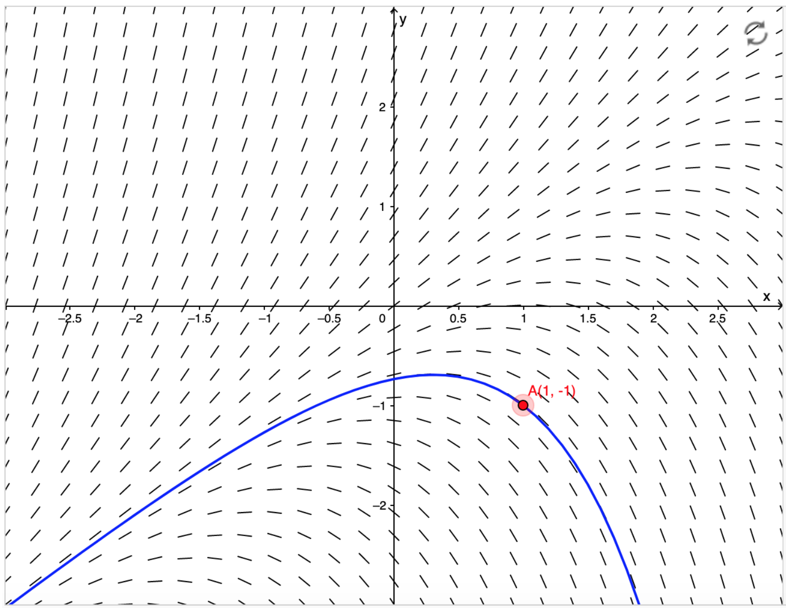
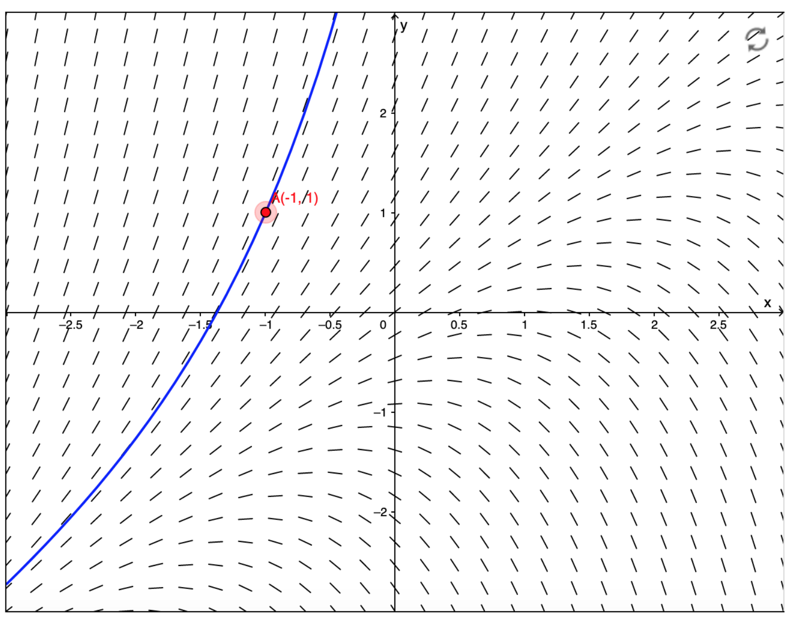
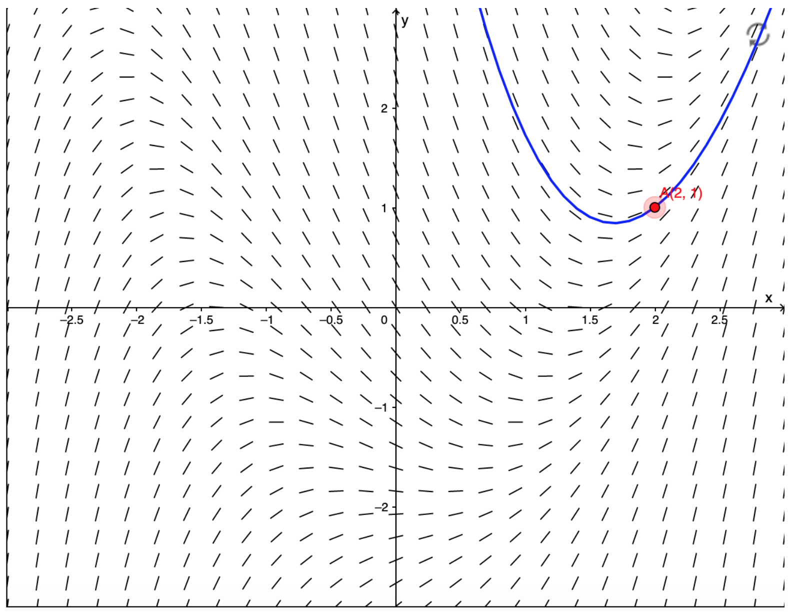


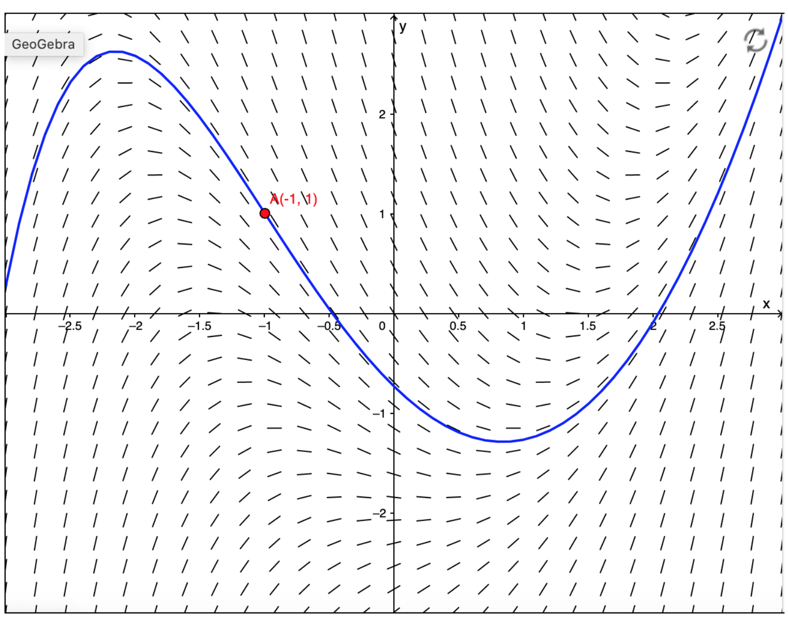


1.3.9 In order to give you the most accurate picture, I've given computer-generated solution curves. Your sketches don't need to be perfect, but they should look somewhat similar to the real thing.

4

The three solution curves I've given started from the initial points $(2, 1)$, $(-1, 1)$, and $(-1, -1)$.





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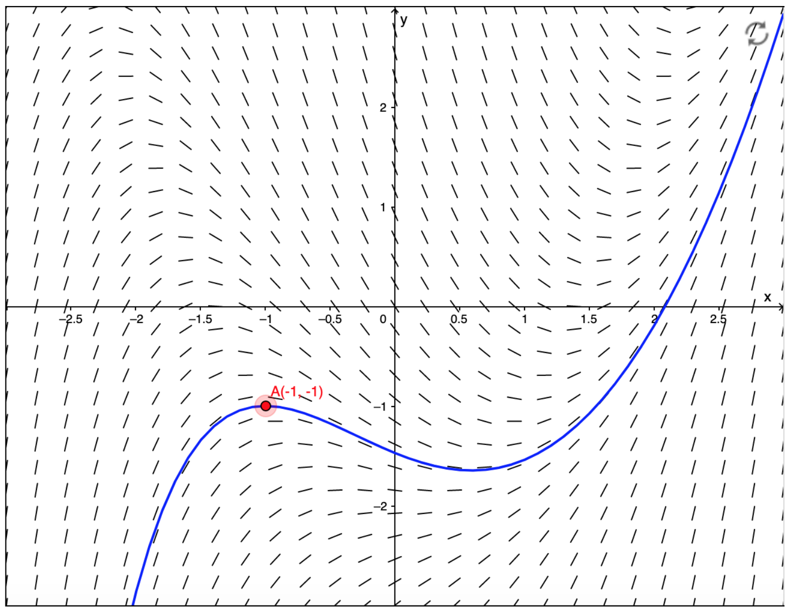

Additional Problems:

1. First, we convert our top speed to m/s:

$$\frac{300 \text{ km}}{1 \text{ h}} \cdot \frac{1000 \text{ m}}{1 \text{ h}} \cdot \frac{1 \text{ h}}{60 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ s}} = 83.33 \text{ m/s}$$

Now all of our units are consistent. Our acceleration is constant $a(t) = 14$ m/s^2, our initial velocity is $v_0 = 0$ m/s and our initial position is $x_0 = 0$ m. So, our velocity and position functions are $v(t) = 14t$ and $x(t) = 7t^2$. We want the time when $v(t) = 83.33$, so we solve $14t = 83.33$ to get $t = 5.95$ s. At that time, the position is $x(5.95) = 7 \cdot 5.95^2 = 247.82$ m.

So we reach top speed after 5.95 s, at which point we have traveled 247.82 m.

Cultural Aside: These acceleration and velocity numbers are as close as I could find to the real figures for modern Formula One cars. However, the fastest F1 cars take over 8 seconds to reach 300 kph. There are two primary factors that explain the difference between our calculation and the real-world data. First of all, F1 cars do not have enough traction to convert all of their power to forward motion at low speeds. You will often see cars spinning their wheels at the start of a race for just this reason. The second factor is that once the car gets above about 100 kph, there is a significant amount of drag due to air resistance. The aerodynamics of F1 cars slightly reduce top speed, but make the cars faster over the course of a lap by providing incredible grip through the corners.

2. We already calculated that 300 km/h is 83.33 m/s. We similarly calculate that

$$\frac{80 \text{ km}}{1 \text{ h}} \cdot \frac{1000 \text{ m}}{1 \text{ h}} \cdot \frac{1 \text{ h}}{60 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ s}} = 22.22 \text{ m/s}$$

6

Our units are now consistent. We are applying brakes which decreases velocity, so the acceleration is a negative constant $a(t) = -39$ m/s^2. The initial speed is $v_0 = 83.33$ m/s. At time $t = 0$ when we start braking, we set our position as $x_0 = 0$ m. So our velocity and position functions are $v(t) = -39t + 83.33$ and $x(t) = -\frac{39}{2}t^2 + 83.33t$. We want to know the time when $v(t) = 22.22$, so we solve $-39t + 83.33 = 22.22$ to get $t = 1.57$ s. At that time, the position is $x(1.57) = -\frac{39}{2}(1.57)^2 + 83.33 \cdot 1.57 = 82.76$ m.

So 82.76 m after we start braking, the car will reach the target speed of 80 km/h. The driver must brake 82.76 m before the corner entry.

3. To take care of units, we calculate

$$\frac{100 \text{ km}}{1 \text{ h}} \cdot \frac{1000 \text{ m}}{1 \text{ h}} \cdot \frac{1 \text{ h}}{60 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ s}} = 27.78 \text{ m/s}$$

We have two different cars' positions to model in this case. We have accelerations $a_A(t) = 14$ and $a_B(t) = 13$. Both cars have the same initial velocity $v_A(0) = v_B(0) = 27.78$. At time $t = 0$, we will set car B's position as $x_B(0) = 0$ and car A 10 m behind at $x_A(0) = -10$. We are looking for the time t where $x_A(t) = x_B(t)$. Our velocity and position functions are

$$v_A(t) = 14t + 27.78 \qquad v_B(t) = 13t + 27.78$$
$$x_A(t) = 7t^2 + 27.78t - 10 \qquad x_B(t) = 6.5t^2 + 27.78t$$

We want $x_A(t) = x_B(t)$, so we solve

$$\begin{aligned} 7t^2 + 27.78t - 10 &= 6.5t^2 + 27.78t \\ 0.5t^2 &= 10 \end{aligned}$$

There are two solutions here, one with t positive and one with t negative. It is reasonable to assume that time only moves forwards, so we choose the solution with $t > 0$, namely $t = 4.47$ s. At this time, we have

$$\begin{aligned} v_A(4.47) &= 14 \cdot 4.47 + 27.78 \\ &= 90.36 \\ v_B(4.47) &= 13 \cdot 4.47 + 27.78 \\ &= 85.89 \\ x_A(4.47) = x_B(4.47) &= 7 \cdot 4.47^2 + 27.78 \cdot 4.47 - 10 \\ &= 254.05 \end{aligned}$$

So after 4.47 s and 254.05 m from the corner exit, the two cars will be side-by-side. At that moment, car A is traveling 90.36 m/s (about 325 km/h) and car B is traveling 85.89 m/s (about 309 km/h).

If there is a long enough straight following the corner, we would expect car A to use its superior acceleration to move ahead of car B. However, if car B has a higher top speed or can brake much later into the next corner, it may be able to stay ahead.

2.12 HW 12

Local contents

2.12.1 Problems listing

Homework 12 - Due December 15

Homework instructions: Complete the assigned problems on your own paper. Once you are finished, scan or photograph your work and upload it to Gradescope. When prompted, tell Gradescope where to find each problem.
You are allowed (and in fact encouraged) to work with other students on homework assignments. If you do that, please indicate on each problem who you worked with. If you use sources other than your notes, the textbook, and any resources on Canvas for your homework, you must indicate the source on each problem. You are not permitted to view, request, or look for solutions to any of the homework problems from solutions manuals, homework help websites, online forums, other students, or any other sources.

Textbook Problems:

- §1.4: 4, 17, 19, 33, 43
- §1.5: 3, 17, 37
- §2.1: 15, 16, 17

Additional Problems:

1. This problem will discuss two different ways to solve the differential equation $y' + y = e^x$.

   (a) Using the methods of chapter 5, solve the homogeneous linear differential equation with constant coefficients
   $$y' + y = 0$$

   (b) Use the method of undetermined coefficients to find a particular solution to
   $$y' + y = e^x$$

   (c) Using (a) and (b), write the general solution of
   $$y' + y = e^x$$

   (d) We can also view this differential equation as a first-order linear differential equation of the type discussed in section 1.5. For the differential equation
   $$y' + y = e^x$$
   what are the functions $P(x)$ and $Q(x)$?

   (e) Use the method of integrating factors to solve
   $$y' + y = e^x$$

1

(f) Compare your answers in (c) and (e). Do they describe the same solutions to the differential equation?

2. This problem will explore the spread of Green's disease (a highly contagious illness that causes green skin and no other symptoms) in a city of 100,000 residents. There is currently no cure, and it appears that those who catch the disease remain infectious forever.

   (a) We will assume that the number $P(t)$ of positive cases satisfies a logistic equation of the form
   $$\frac{dP}{dt} = kP(M - P)$$
   This essentially says that the number of new infections each day depends on the number of currently infected individuals *and* on the number of remaining susceptible individuals.

   On day $t = 0$, there are 5,000 positive cases identified in the city. Leaving k as an unknown constant, what is the initial value problem (differential equation and initial condition) satisfied by $P(t)$?

   (b) Solve the initial value problem you wrote in (a). Show all steps – do not use the formula for solutions of logistic equations given in the textbook.

   (c) On day $t = 0$, there are 500 new cases being identified each day. Determine the value of k.

   (d) After how many days will half the population of this city have contracted Green's disease?

   (e) There's a saying in this area of mathematics that "all models are wrong, but some are useful." In the last year or so, we've seen that even the most sophisticated models of disease spread will never be perfectly accurate. Still, they remain useful tools for policy makers and public health officials.

   In a few sentences, reflect on the model for disease spread you explored in this problem. What useful information does it tell us, and in what ways is the model likely to be wrong?

2.12.2 Problem 4, section 1.4

Find general solutions (implicit if necessary, explicit if convenient) of the differential equations in Problems 1 through 18. Primes denote derivatives with respect to x.

$$(1+x)\frac{dy}{dx} = 4y$$

Solution

This is separable as it can be written as

$$\frac{dy}{dx} = F(x)G(y)$$

Where in this case $G(y) = y$ and $F(x) = \frac{4}{1+x}$. Assuming $x \neq 1$. Therefore we can now separate and write

$$\begin{aligned}\frac{dy}{dx}\frac{1}{G(y)} &= F(x)\\ \frac{dy}{G(y)} &= F(x)dx\end{aligned}$$

Integrating both sides gives

$$\int \frac{dy}{G(y)} = \int F(x)dx$$

Replacing $G(y) = y$ and $F(x) = \frac{4}{1+x}$, the above becomes

$$\begin{aligned}\int \frac{dy}{y} &= \int \frac{4}{1+x}dx\\ \ln|y| &= \ln\left|(1+x)^4\right| + c\end{aligned}$$

Taking the exponential of both sides

$$\begin{aligned}|y| &= e^{\ln\left|(1+x)^4\right|+c}\\ &= e^c e^{\ln\left|(1+x)^4\right|}\end{aligned}$$

Let $e^c = c_1$ and since $(1+x)^4$ can not be negative, therefore the above simplifies to

$$\begin{aligned}|y| &= c_1 e^{\ln(1+x)^4}\\ &= c_1(1+x)^4\end{aligned}$$

Let the sign $\pm$ be absorbed into the constant of integration. The above simplifies to

$$y(x) = c_1(1+x)^4 \qquad x \neq 1$$

2.12.3 Problem 17, section 1.4

Find general solutions (implicit if necessary, explicit if convenient) of the differential equations in Problems 1 through 18. Primes denote derivatives with respect to x.

$$\frac{dy}{dx} = 1 + x + y + xy$$

Solution

Writing the above as

$$\frac{dy}{dx} = (1+x)(1+y)$$

This is separable. It can be written as

$$\frac{dy}{dx} = F(x)G(y)$$

Where in this case $G(y) = (1+y)$ and $F(x) = (1+x)$. Therefore we can now separate and write

$$\frac{dy}{dx}\frac{1}{G(y)} = F(x)$$
$$\frac{dy}{G(y)} = F(x)dx$$

Integrating both sides gives

$$\int \frac{dy}{G(y)} = \int F(x)dx$$

Replacing $G(y) = (1+y)$ and $F(x) = (1+x)$, the above becomes

$$\int \frac{dy}{(1+y)} = \int (1+x)dx$$
$$\ln|1+y| = x + \frac{x^2}{2} + c$$

Taking the exponential of both sides

$$|1+y| = e^{x+\frac{x^2}{2}+c}$$
$$= e^c e^{x+\frac{x^2}{2}}$$

Let $e^c = c_1$ the above becomes

$$|1+y| = c_1 e^{x+\frac{x^2}{2}}$$

Let the sign ± be absorbed into the constant of integration. The above simplifies to

$$1+y = c_1 e^{x+\frac{x^2}{2}}$$
$$y = c_1 e^{x+\frac{x^2}{2}} - 1$$

2.12.4 Problem 19, section 1.4

Find explicit particular solutions of the initial value problems in Problems 19 through 28.

$$\frac{dy}{dx} = ye^x$$
$$y(0) = 2e$$

Solution

This is separable because it can be written as

$$\frac{dy}{dx} = F(x)G(y)$$

Where in this case $G(y) = y$ and $F(x) = e^x$. Therefore we can now separate and write

$$\frac{dy}{dx}\frac{1}{G(y)} = F(x)$$
$$\frac{dy}{G(y)} = F(x)dx$$

Integrating both sides gives

$$\int \frac{dy}{G(y)} = \int F(x)dx$$

Replacing $G(y) = y$ and $F(x) = e^x$, the above becomes

$$\int \frac{dy}{y} = \int e^x dx$$
$$\ln|y| = e^x + c$$

Taking the exponential of both sides

$$\begin{aligned} |y| &= e^{e^x + c} \\ &= e^c e^{e^x} \end{aligned}$$

Let $e^c = c_1$, therefore the above simplifies to

$$|y| = c_1 e^{e^x}$$

Let the sign $\pm$ be absorbed into the constant of integration. The above simplifies to

$$y(x) = c_1 e^{e^x} \tag{1}$$

Now we apply initial conditions to find c_1. Since $y(0) = 2e$ then the above solution becomes

$$\begin{aligned} 2e &= c_1 e \\ c_1 &= 2 \end{aligned}$$

Hence the general solution (1) now becomes

$$y(x) = 2e^{e^x}$$

2.12.5 Problem 33, section 1.4

A certain city had a population of 25,000 in 1960 and a population of 30,000 in 1970. Assume that its population will continue to grow exponentially at a constant rate. What population can its city planners expect in the year 2000?

Solution

The differential equation model is

$$\frac{dP}{dt} = kP$$

Where $P(t)$ is the population at time t. The initial conditions are $P(0) = 25000$ where $t = 0$ is taken as the year 1960. We are also given that $P(10) = 30000$. We are asked to determine $P(40)$ which is the year 2000. First we solve the ode. This is both linear and separable. Using the separable method, it can be written as

$$\frac{dP}{dt} = F(t)G(P)$$

Where in this case $G(P) = P$ and $F(t) = k$. Therefore we can now separate and write

$$\begin{aligned} \frac{dP}{dt}\frac{1}{G(P)} &= F(t) \\ \frac{dP}{G(P)} &= F(t)dt \end{aligned}$$

Integrating both sides gives

$$\int \frac{dP}{G(P)} = \int F(t)dt$$

Replacing $G(P) = P$ and $F(t) = k$, the above becomes

$$\begin{aligned} \int \frac{dP}{P} &= \int kdt \\ \ln P &= kt + c \end{aligned}$$

No need for absolute sign here, since P can not be negative. Taking exponential of both sides gives

$$P(t) = ce^{kt} \tag{1}$$

Applying initial conditions $P(0) = 25000$ the above gives

$$25000 = c$$

Hence (1) now becomes

$$P(t) = 25000e^{kt} \tag{2}$$

Applying second condition $P(10) = 30000$ to the above gives

$$\begin{aligned}
30000 &= 25000e^{10k} \\
\frac{30000}{25000} &= e^{10k} \\
\frac{6}{5} &= e^{10k}
\end{aligned}$$

Taking natural log of both sides

$$\begin{aligned}
\ln\left(\frac{6}{5}\right) &= 10k \\
k &= \frac{1}{10}\ln\left(\frac{6}{5}\right)
\end{aligned}$$

Hence (2) becomes

$$P(t) = 25000e^{\left(\frac{1}{10}\ln\left(\frac{6}{5}\right)\right)t}$$

At $t = 40$

$$P(40) = 25000e^{\left(\frac{1}{10}\ln\left(\frac{6}{5}\right)\right)40}$$

Using calculator it gives

$$P(40) = 51840$$

Hence the population in year 2000 is 51840.

2.12.6 Problem 43, section 1.4

Cooling. A pitcher of buttermilk initially at 25 C is to be cooled by setting it on the front porch, where the temperature is 0 C. Suppose that the temperature of the buttermilk has dropped to 15 C after 20 min. When will it be at 5 C?

Solution

Cooling of object is governed by the Newton's law cooling

$$\frac{dT}{dt} = k(T_{out} - T)$$

Where T_{out} is the ambient temperature, which is 0 C in this problem and k is positive constant. Hence the above becomes

$$\frac{dT}{dt} = -kT$$

This is separable (and also linear in T). Solving it as separable, it can be written as

$$\frac{dT}{dt} = F(t)G(T)$$

Where in this case $G(T) = T$ and $F(t) = -k$. Therefore we can now separate and write

$$\begin{aligned}
\frac{dT}{dt}\frac{1}{G(T)} &= F(t) \\
\frac{dT}{G(T)} &= F(t)dt
\end{aligned}$$

Integrating both sides gives

$$\int \frac{dT}{G(T)} = \int F(t)dt$$

Replacing $G(T) = T$ and $F(t) = -k$, the above becomes

$$\int \frac{dy}{T} = -k \int dt$$
$$\ln|T| = -kt + c$$

Taking the exponential of both sides

$$\begin{aligned}|T| &= e^{-kt+c} \\ &= e^{c} e^{-kt}\end{aligned}$$

Let $e^c = c_1$, therefore the above simplifies to

$$|T| = c_1 e^{-kt}$$

Let the sign $\pm$ be absorbed into the constant of integration. The above simplifies to

$$T(t) = c_1 e^{-kt} \tag{1}$$

Now initial conditions are used to determine c_1. At $t = 0$, we are given $T(0) = 25$. The above becomes

$$25 = c_1$$

Therefore (1) becomes

$$T(t) = 25e^{-kt} \tag{2}$$

Now the second condition $T(20) = 15$ is used to determine k. The above becomes

$$\begin{aligned}15 &= 25e^{-20k} \\ \frac{15}{25} &= e^{-20k} \\ \frac{3}{5} &= e^{-20k}\end{aligned}$$

Taking natural log of both sides gives (using property $\ln e^{f(x)} = f(x)$)

$$\begin{aligned}\ln\left(\frac{3}{5}\right) &= -20k \\ k &= \frac{-1}{20}\ln\left(\frac{3}{5}\right) \\ &= \frac{1}{20}\ln\frac{5}{3}\end{aligned}$$

Substituting the above value of k back into (2) gives

$$\begin{aligned}T(t) &= 25e^{\left(\frac{-1}{20}\ln\frac{5}{3}\right)t} \\ &= 25e^{\left(\frac{1}{20}\ln\frac{3}{5}\right)t}\end{aligned}$$

To answer the final part, let $T(t) = 5$ and we need to solve for t from the above.

$$\begin{aligned}5 &= 25e^{\left(\frac{1}{20}\ln\frac{3}{5}\right)t} \\ \frac{1}{5} &= e^{\left(\frac{1}{20}\ln\frac{3}{5}\right)t}\end{aligned}$$

Taking natural log of both sides gives

$$\begin{aligned}\ln\left(\frac{1}{5}\right) &= \left(\frac{1}{20}\ln\frac{3}{5}\right)t \\ t &= \frac{\ln\left(\frac{1}{5}\right)}{\ln\left(\frac{3}{5}\right)^{\frac{1}{20}}}\end{aligned}$$

Using the calculator gives

$$t = 63.013 \text{ min}$$

2.12.7 Problem 3, section 1.5

Find general solutions of the differential equations in Problems 1 through 25. If an initial condition is given, find the corresponding particular solution. Throughout, primes denote derivatives with respect to x.

$$y' + 3y = 2xe^{-3x} \tag{1}$$

<u>Solution</u>

This is of the form $y' + p(x)y = q(x)$. Hence it is linear in y. Where

$$\begin{aligned} p(x) &= 3 \\ q(x) &= 2xe^{-3x} \end{aligned}$$

The integrating factor is

$$\begin{aligned} \rho &= e^{\int p dx} \\ &= e^{\int 3dx} \\ &= e^{3x} \end{aligned}$$

Multiplying both sides of (1) by the integration factor gives

$$\begin{aligned} \frac{d}{dx}\left(y\rho\right) &= \rho\left(2xe^{-3x}\right) \\ \frac{d}{dx}\left(e^{3x}y\right) &= e^{3x}\left(2xe^{-3x}\right) \\ \frac{d}{dx}\left(e^{3x}y\right) &= 2x \end{aligned}$$

Integrating gives

$$\begin{aligned} \int d\left(e^{3x}y\right) &= \int 2xdx \\ e^{3x}y &= x^2 + c \\ y(x) &= e^{-3x}\left(x^2 + c\right) \end{aligned}$$

The above is the general solution.

2.12.8 Problem 17, section 1.5

Find general solutions of the differential equations in Problems 1 through 25. If an initial condition is given, find the corresponding particular solution. Throughout, primes denote derivatives with respect to x.

$$\begin{aligned} (1 + x)y' + y &= \cos x \qquad (1) \\ y(0) &= 1 \end{aligned}$$

<u>Solution</u>

Dividing both sides of (1) by $(1 + x)$ where $x \neq -1$ gives

$$y' + \frac{1}{1+x}y = \frac{\cos x}{1+x} \tag{2}$$

This is now in the form $y' + p(x)y = q(x)$. Hence it is linear in y. Where

$$\begin{aligned} p(x) &= \frac{1}{1+x} \\ q(x) &= \frac{\cos x}{1+x} \end{aligned}$$

The integrating factor is

$$\begin{aligned}\rho &= e^{\int p dx}\\ &= e^{\int \frac{1}{1+x} dx}\\ &= e^{\ln(1+x)}\\ &= 1+x\end{aligned}$$

Multiplying both sides of (2) by the above integration factor gives

$$\begin{aligned}\frac{d}{dx}\left(y\rho\right) &= \rho\left(\frac{\cos x}{1+x}\right)\\ \frac{d}{dx}\left((1+x)y\right) &= (1+x)\left(\frac{\cos x}{1+x}\right)\\ \frac{d}{dx}\left((1+x)y\right) &= \cos x\end{aligned}$$

Integrating gives

$$\begin{aligned}\int d\left((1+x)y\right) &= \int \cos x dx\\ (1+x)y &= \sin x + c\\ y(x) &= \frac{1}{1+x}(\sin x + c) \qquad x \neq -1 \end{aligned} \tag{3}$$

The above is the general solution. Now we use initial conditions to determine c. Since we are given that $y(0) = 1$ then (3) becomes

$$\begin{aligned}1 &= (\sin 0 + c)\\ c &= 1\end{aligned}$$

Therefore (3) becomes

$$y(x) = \frac{1}{1+x}(1+\sin x) \qquad x \neq -1$$

2.12.9 Problem 37, section 1.5

A 400-gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at the rate of 5 gal/s, and the well-mixed brine in the tank flows out at the rate of 3 gal/s. How much salt will the tank contain when it is full of brine?

Solution

Let $x(t)$ be mass of salt in lb at time t in the tank. The differential equation that describes how the mass of salt changes in time is therefore

$$\frac{dx}{dt} = (5)(1) - (3)\frac{x}{V(t)} \tag{1}$$

But

$$\begin{aligned}V(t) &= 100 + (5t - 3t)\\ &= 100 + 2t\end{aligned}$$

Therefore (1) becomes

$$\begin{aligned}\frac{dx}{dt} &= 5 - 3\frac{x}{100+2t}\\ \frac{dx}{dt} + \frac{3}{100+2t}x &= 5\end{aligned} \tag{2}$$

This is now in the form $x' + p(t)x = q(t)$. Hence it is linear in x. Where

$$\begin{aligned}p(t) &= \frac{3}{100+2t}\\ q(t) &= 5\end{aligned}$$

The integrating factor is

$$\begin{aligned}\rho &= e^{\int p dt} \\ &= e^{3\int \frac{1}{100+2t} dt}\end{aligned}$$

Let $100 + 2t = u$. Hence $\frac{du}{dt} = 2$. The integral becomes $\int \frac{1}{100+2t} dt$ becomes $\int \frac{1}{u}\frac{du}{2} = \frac{1}{2}\ln(u) = \frac{1}{2}\ln(100+2t)$. The above becomes

$$\begin{aligned}\rho &= e^{\frac{3}{2}\ln(100+2t)} \\ &= (100+2t)^{\frac{3}{2}}\end{aligned}$$

Multiplying both sides of (2) by the above integration factor gives

$$\begin{aligned}\frac{d}{dt}\left(x\rho\right) &= 5\rho \\ \frac{d}{dt}\left((100+2t)^{\frac{3}{2}}x\right) &= 5(100+2t)^{\frac{3}{2}}\end{aligned}$$

Integrating gives

$$(100+2t)^{\frac{3}{2}}x = 5\int (100+2t)^{\frac{3}{2}} dt$$

Let $100 + 2t = u$ hence $\frac{du}{dt} = 2$ and the integral on the right becomes $\int \frac{u^{\frac{3}{2}}}{2} du = \frac{1}{2}\frac{u^{\frac{5}{2}}}{\frac{5}{2}} = \frac{1}{5}u^{\frac{5}{2}}$. Hence the above now becomes

$$\begin{aligned}(100+2t)^{\frac{3}{2}}x &= 5\left(\frac{1}{5}u^{\frac{5}{2}}\right) + c \\ &= u^{\frac{5}{2}} + c \\ &= (100+2t)^{\frac{5}{2}} + c\end{aligned}$$

Solving for $x(t)$ gives

$$\begin{aligned}x &= (100+2t)^{\frac{5}{2}-\frac{3}{2}} + c(100+2t)^{\frac{-3}{2}} \\ &= (100+2t) + c(100+2t)^{\frac{-3}{2}} \qquad (3)\end{aligned}$$

Now we find c from initial conditions. At $t = 0$ we are told that $x = 50$. Hence

$$\begin{aligned}50 &= (100) + c(100)^{\frac{-3}{2}} \\ -50 &= \frac{c}{100^{\frac{3}{2}}} \\ c &= (-50)\left(100^{\frac{3}{2}}\right) \\ &= -50000\end{aligned}$$

Therefore (3) becomes

$$x(t) = (100+2t) - \frac{50000}{(100+2t)^{\frac{3}{2}}} \qquad (4)$$

The above gives the mass of salt as function of time. We now to find the time when the tank is full. From the volume function we know that $V(t) = 100 + 2t$. Since the tank size is 400 gal, then we solve for t from

$$\begin{aligned}400 &= 100 + 2t \\ t &= \frac{300}{2} \\ &= 150 \text{ sec}\end{aligned}$$

So the tank fills up after 150 seconds. Substituting this value of time in (4) gives

$$\begin{aligned} x(t) &= (100+2(150)) - \frac{50000}{(100+2(150))^{\frac{3}{2}}} \\ &= (100+300) - \frac{50000}{(100+300)^{\frac{3}{2}}} \\ &= 400 - \frac{50000}{400^{\frac{3}{2}}} \\ &= \frac{1575}{4} \\ &= 393.75 \text{ lb} \end{aligned}$$

2.12.10 Problem 15, section 2.1

Consider a population $P(t)$ satisfying the logistic equation $\frac{dP}{dt} = aP - bP^2$, where $B = aP$ is the time rate at which births occur and $D = bP^2$ is the rate at which deaths occur. If the initial population $P(0) = P_0$, and B_0 births per month and D_0 deaths per month are occurring at time $t = 0$, show that the limiting population is $M = \frac{B_0 P_0}{D_0}$

Solution

We are given the logistic equation in he form

$$\begin{aligned} \frac{dP}{dt} &= aP - bP^2 \\ &= a\left(P - \frac{b}{a}P^2\right) \\ &= aP\left(1 - \frac{b}{a}P\right) \end{aligned} \tag{1}$$

Comparing (1) to the other standard form given in textbook which is

$$\frac{dP}{dt} = kP(M - P) \tag{2}$$

Where in this form M is the limiting population. Factoring M out from (2) gives

$$\frac{dP}{dt} = (kM)P\left(1 - \frac{P}{M}\right) \tag{3}$$

Comparing (1) and (3) shows that, by inspection that

$$\begin{aligned} a &= kM \\ M &= \frac{a}{b} \end{aligned} \tag{4}$$

But we are told that $a = \frac{B}{P}$. At time $t = 0$ this gives

$$a = \frac{B_0}{P_0} \tag{5}$$

And we are told that $b = \frac{D}{P^2}$ which at $t = 0$ gives

$$b = \frac{D_0}{P_0^2} \tag{6}$$

Substituting (5,6) back in (4) gives

$$\begin{aligned} M &= \frac{\frac{B_0}{P_0}}{\frac{D_0}{P_0^2}} \\ &= \frac{B_0 P_0^2}{P_0 D_0} \end{aligned}$$

Or

$$M = \frac{B_0 P_0}{D_0}$$

Which is what we are asked to show.

2.12.11 Problem 16, section 2.1

Consider a rabbit population $P(t)$ satisfying the logistic equation as in Problem 15. If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 95% of the limiting population M ?

Solution

We have $P(0) = 120$ and $B = aP = 8$ per month and $D = bP^2 = 6$ per month. Hence

$$a = \frac{B}{P} = \frac{8}{P(0)} = \frac{8}{120} = \frac{1}{15}$$

The limiting population is

$$\begin{aligned} M &= \frac{B_0 P_0}{D_0} \\ &= \frac{(8)(120)}{6} \\ &= 160 \end{aligned}$$

Therefore, we need to find the time the population reaches 95% of the above value, or $\frac{95}{100}(160) = 152$ rabbits. The solution to the logistic equation is given in equation (7) page 77 as

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

This was derived from the form $\frac{dP}{dt} = kp(M - P)$. But as we found in the last problem, $k = \frac{a}{M}$ and $a = \frac{1}{15}$ in this problem. Hence $k = \frac{1}{15M}$. The above solution now becomes

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-\frac{1}{15}t}}$$

But $M = 160$ and $P_0 = 120$. The above becomes

$$\begin{aligned} P(t) &= \frac{(160)(120)}{120 + (160 - 120)e^{-\frac{1}{15}t}} \\ &= \frac{19200}{120 + 40e^{-\frac{1}{15}t}} \end{aligned}$$

We want to find t when $P(t) = 152$. Hence

$$152 = \frac{19200}{120 + 40e^{-\frac{1}{15}t}}$$

We need to solve the above for t.

$$\begin{aligned} 152\left(120 + 40e^{-\frac{1}{15}t}\right) &= 19200 \\ 6080e^{\frac{-1}{15}t} + 18240 &= 19200 \\ e^{\frac{-1}{15}t} &= \frac{19200 - 18240}{6080} \\ &= \frac{3}{19} \end{aligned}$$

Taking natural log gives

$$\begin{aligned} \frac{-1}{15}t &= \ln\left(\frac{3}{19}\right) \\ t &= -15\ln\left(\frac{3}{19}\right) \end{aligned}$$

Using the calculator the above gives

$$t = 27.687 \text{ months}$$

2.12.12 Problem 17, section 2.1

Consider a rabbit population $P(t)$satisfying the logistic equation as in Problem 15. If the initial population is 240 rabbits and there are 9 births per month and 12 deaths per month occurring at time $t = 0$. How many months does it take for $P(t)$ to reach 105% of the limiting population M ?

Solution

This is similar to the above problem. We have $P(0) = 240$ and $B = aP = 9$ per month and $D = bP^2 = 12$ per month. Hence

$$a = \frac{B}{P} = \frac{9}{P(0)} = \frac{9}{240} = \frac{3}{80}$$

The limiting population is

$$\begin{aligned} M &= \frac{B_0 P_0}{D_0} \\ &= \frac{(9)(240)}{12} \\ &= 180 \end{aligned}$$

Therefore, we need to find the time the population reaches 105% of the above value, or $\frac{105}{100}(180) = 189$ rabbits. The solution to the logistic equation is given in equation (7) page 77 as

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

This was derived from the form $\frac{dP}{dt} = kp(M - P)$. But as we found in the last problem, $k = \frac{a}{M}$ and $a = \frac{3}{80}$ in this problem. Hence $k = \frac{3}{80M}$. The above solution now becomes

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-\frac{3}{80}t}}$$

But $M = 180$ and $P_0 = 240$. The above becomes

$$\begin{aligned} P(t) &= \frac{(180)(240)}{240 + (180 - 240)e^{-\frac{3}{80}t}} \\ &= \frac{43200}{240 - 60e^{-\frac{3}{80}t}} \end{aligned}$$

We want to find t when $P(t) = 189$. Hence

$$189 = \frac{43\,200}{240 - 60e^{-\frac{9}{140}t}}$$

We need to solve the above for t.

$$\begin{aligned} 189\left(240 - 60e^{-\frac{3}{80}t}\right) &= 43200 \\ 45360 - 11\,340e^{-\frac{3}{80}t} &= 43200 \\ e^{-\frac{3}{80}t} &= -\frac{43200 - 45360}{11\,340} \\ &= \frac{4}{21} \end{aligned}$$

Taking natural log gives

$$\begin{aligned} -\frac{3}{80}t &= \ln\left(\frac{4}{21}\right) \\ t &= -\frac{80}{3}\ln\left(\frac{4}{21}\right) \end{aligned}$$

Using the calculator the above gives

$$t = 44.219 \text{ months}$$

2.12.13 Additional problem 1

Solution

2.12.13.1 Part a

$$y' + y = 0$$

The characteristic equation is

$$r + 1 = 0$$

The root is $r = -1$. Therefore the general solution is given by

$$\begin{aligned} y_h(x) &= Ce^{rx} \\ &= Ce^{-x} \end{aligned}$$

Where C is arbitrary constant.

2.12.13.2 Part b

$$y' + y = e^x \tag{1}$$

From part (a) we found that e^{-x} is basis solution for the homogeneous ODE. The RHS in this ode is e^x. No duplication. Therefore we let

$$y_p = Ae^x$$

Substituting this in (1) gives

$$\begin{aligned} Ae^x + Ae^x &= e^x \\ 2A &= 1 \\ A &= \frac{1}{2} \end{aligned}$$

Therefore

$$y_p = \frac{1}{2}e^x$$

2.12.13.3 Part c

The general solution is the sum of the homogeneous solution (part a) and the particular solution (part b). Therefore

$$\begin{aligned} y &= y_h + y_p \\ &= Ce^{-x} + \frac{1}{2}e^x \end{aligned}$$

2.12.13.4 Part d

The ODE

$$y' + y = e^x \tag{1}$$

Has the form

$$y' + P(x)y = Q(x)$$

Which implies that

$$\begin{aligned} P(x) &= 1 \\ Q(x) &= e^x \end{aligned}$$

2.12.13.5 Part e

The integrating factor is therefore $\rho = e^{\int P(x)dx} = e^{\int dx} = e^x$. Multiplying both sides of (1) by ρ results in

$$\begin{aligned}\frac{d}{dx}(\rho y) &= \rho e^x\\ d(\rho y) &= (\rho e^x)dx\\ d(e^x y) &= e^{2x}dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^x y &= \int e^{2x}dx\\ e^x y &= \frac{1}{2}e^{2x} + C\end{aligned}$$

Therefore

$$y = \frac{1}{2}e^x + Ce^{-x}$$

2.12.13.6 Part f

Comparing the solution obtained in part (c) and (e) shows they are the same solution.

2.12.14 Additional problem 2

Solution

2.12.14.1 Part (a)

$$\frac{dP}{dt} = kP(M-P) \tag{1}$$

The solution $P(t)$, where $P(t)$ is number of positive cases at time t should satisfy the above ODE, with $P(0) = 5000$.

2.12.14.2 Part (b)

The ODE in part(a) is separable. It has the form

$$\frac{dP}{dt} = F(t)G(P)$$

Where

$$\begin{aligned}F(t) &= 1\\ G(P) &= kP(M-P)\end{aligned}$$

Therefore the ODE (1) can be written as

$$\begin{aligned}\frac{dP}{G(P)} &= F(t)dt\\ \frac{dP}{kP(M-P)} &= dt\\ \int\frac{dP}{kP(M-P)} &= \int dt \qquad (2)\end{aligned}$$

To integrate the left side will use partial fractions. Let

$$\frac{1}{kP(M-P)} = \frac{A}{kP} + \frac{B}{M-P}$$

Therefore

$$
\begin{aligned}
A &= \left.\frac{1}{M-P}\right|_{P=0} \\
&= \frac{1}{M}
\end{aligned}
$$

And

$$
\begin{aligned}
B &= \left.\frac{1}{kP}\right|_{P=M} \\
&= \frac{1}{kM}
\end{aligned}
$$

Hence (2) becomes

$$
\begin{aligned}
\int \frac{1}{M}\frac{1}{kP} + \frac{1}{kM}\frac{1}{M-P} dP &= t + C \\
\frac{1}{kM}\ln|P| - \frac{1}{kM}\ln|M-P| &= t + C \\
\frac{1}{kM}\ln\left|\frac{P}{M-P}\right| &= t + C \\
\ln\left|\frac{P}{M-P}\right| &= kMt + C_2
\end{aligned}
$$

Where $C_2 = CkM$ a new constant. The above now can be written as

$$
\frac{P}{M-P} = C_3 e^{kMt}
$$

Where the $\pm$ sign it taken care of by the constant C_3. Hence

$$
\begin{aligned}
P &= C_3 e^{kMt}(M-P) \\
P &= C_3 M e^{kMt} - C_3 P e^{kMt} \\
P + C_3 P e^{kMt} &= C_3 M e^{kMt} \\
P\left(1 + C_3 e^{kMt}\right) &= C_3 M e^{kMt} \\
P(t) &= \frac{C_3 M e^{kMt}}{1 + C_3 e^{kMt}} \\
&= \frac{C_3 M}{e^{-kMt} + C_3} \qquad (3)
\end{aligned}
$$

When $t = 0, P = P_0$. Hence the above becomes

$$
\begin{aligned}
P_0 &= \frac{C_3 M}{1 + C_3} \\
P_0 + P_0 C_3 &= C_3 M \\
C_3(P_0 - M) &= -P_0 \\
C_3 &= \frac{P_0}{M - P_0}
\end{aligned}
$$

Substituting this back in (3) gives

$$
\begin{aligned}
P(t) &= \frac{\frac{P_0}{M-P_0} M}{e^{-kMt} + \frac{P_0}{M-P_0}} \\
&= \frac{P_0 M}{e^{-kMt}(M - P_0) + P_0}
\end{aligned}
$$

Or

$$
P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}
$$

Which is the solution given in the textbook. Now, using $P_0 = 5000$ given in this problem gives

$$
P(t) = \frac{5000M}{5000 + (M - 5000)e^{-kMt}}
$$

But $M = 100000$ which is the limiting capacity (total population). The above simplifies to

$$\begin{aligned} P(t) &= \frac{(5000)(100000)}{5000 + (100000 - 5000)e^{-100000kt}} \\ &= \frac{(5000)(100000)}{5000 + (95\,000)e^{-100000kt}} \\ &= \frac{100000}{1 + \left(\frac{95\,000}{5000}\right)e^{-100000kt}} \\ &= \frac{100000}{1 + 19e^{-100000kt}} \end{aligned} \tag{4}$$

The above is the solution we will use for the rest of the problem.

2.12.14.3 Part (c)

We are told there are 500 new cases on first day. This means $P(1) = 5000 + 500 = 5500$. Using the solution found above we now solve for k. Let $t = 1$, we obtain

$$\begin{aligned} 5500 &= \frac{100000}{1 + 19e^{-100000k}} \\ e^{-100000k} &= 100000 \\ &= \frac{100000 - 5500}{(19)(5500)} \\ &= \frac{189}{209} \end{aligned}$$

Hence

$$\begin{aligned} -k100000 &= \ln\left(\frac{189}{209}\right) \\ k &= -\frac{1}{100000}\ln\left(\frac{189}{209}\right) \\ &= 1 \times 10^{-6} \end{aligned}$$

2.12.14.4 Part (d)

We now need to find the time t where $P(t) = 50000$. Therefore, using (4)

$$P(t) = \frac{100000}{1 + 19e^{-100000kt}}$$

And replacing k by value found in part(c) and $P(t)$ by 50000 gives

$$\begin{aligned} 50000 &= \frac{100000}{1 + 19e^{-100000\left(1\times10^{-6}\right)t}} \\ 50000 &= \frac{100000}{1 + 19e^{-\frac{1}{10}t}} \\ 50000\left(1 + 19e^{-\frac{1}{10}t}\right) &= 100000 \\ 1 + 19e^{-\frac{1}{10}t} &= \frac{100000}{50000} \\ 1 + 19e^{-\frac{1}{10}t} &= 2 \\ e^{-\frac{1}{10}t} &= \frac{1}{19} \end{aligned}$$

Therefore

$$\begin{aligned} -\frac{1}{10}t &= \ln\left(\frac{1}{19}\right) \\ t &= -10\ln\left(\frac{1}{19}\right) \\ &= 29.444 \end{aligned}$$

Therefore it will take about $\underline{\text{29 days}}$ for the half the population to be infected.

2.12.14.5 Part (e)

The model

$$\frac{dP}{dt} = kP(M-P)$$

Says that the rate of infection depends on $M-P$ where P is current size of infected population and M is limiting size of the population that could become infected, which is assumed to be the total population, and this is assumed to remain constant all the time. Hence as more population is infected, the value $M-P$ becomes smaller and smaller, since $P(t)$ is increasing, but M is fixed. This means the rate at which people get infected becomes smaller as more people are infected. This is a good model, assuming people who get infected remain infected all the time, which is the case here, and assuming M remain constant. This model does not account for death or birth of the overall population and any migration from outside. A more accurate model would account for this.

This model gives useful information for predicting how many of the population will become infected in the future given initial conditions.

2.12.15 key solution for HW12

HOMEWORK 12 – SOLUTIONS

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

1.4.4 We separate variables and then integrate.

$$\begin{aligned}(1+x)\frac{dy}{dx} &= 4y\\ \frac{1}{y}\,dy &= \frac{4}{1+x}\,dx\\ \int\frac{1}{y}\,dy &= \int\frac{4}{1+x}\,dx\\ \ln y &= 4\ln(1+x)+C\end{aligned}$$

Now we exponentiate both sides to solve for y.

$$\begin{aligned}y &= e^{4\ln(1+x)+C}\\ y &= C_1(1+x)^4\end{aligned}$$

1.4.17 We can factor to write the differential equation as $y' = (1+x)(1+y)$. Now we separate variables and integrate.

$$\begin{aligned}\frac{dy}{dx} &= (1+x)(1+y)\\ \frac{1}{1+y}\,dy &= (1+x)\,dx\\ \int\frac{1}{1+y}\,dy &= \int(1+x)\,dx\\ \ln(1+y) &= x+\frac{x^2}{2}+C\end{aligned}$$

We exponentiate to solve for y.

$$\begin{aligned}1+y &= e^{x+\frac{1}{2}x^2+C}\\ y &= C_1e^{x+\frac{1}{2}x^2}-1\end{aligned}$$

In this case, solving for y gives us a bit of a mess, so it would be acceptable to leave it in the implicit form found above.

1

1.4.19 We separate variables and integrate.

$$\begin{aligned}\frac{dy}{dx} &= ye^x \\ \frac{1}{y}\,dy &= e^x\,dx \\ \int \frac{1}{y}\,dy &= \int e^x\,dx \\ \ln y &= e^x + C\end{aligned}$$

We have the initial condition $y(0) = 2e$, so we find the constant C.

$$\begin{aligned}\ln(2e) &= e^0 + C \\ \ln 2 + \ln e &= 1 + C \\ \ln 2 &= C\end{aligned}$$

Now we exponentiate to solve for y.

$$\begin{aligned}y &= e^{e^x + \ln 2} \\ y &= 2e^{e^x}\end{aligned}$$

1.4.33 Our population is modeled by $\frac{dP}{dt} = kP$, so that $P(t) = Ce^{kt}$. Let t be the years since 1960 and $P(t)$ the population in thousands. Then our initial conditions are $P(0) = 25$ and $P(10) = 30$. This lets us solve for the constants C and k:

$$\begin{aligned}25 &= Ce^{0k} \\ 25 &= C\end{aligned}$$

$$\begin{aligned}30 &= 25e^{10k} \\ \frac{6}{5} &= e^{10k} \\ 10k &= \ln(6/5) \\ k &= \frac{\ln(6/5)}{10} \approx 0.0182\end{aligned}$$

So $P(t) = 25e^{0.0182t}$ and in 2000 we predict $P(40) = 25e^{k \cdot 40} = 25(6/5)^4 \approx 51.8$ thousand residents.

1.4.43 The temperature is modeled by $\frac{dT}{dt} = k(0 - T) = -kT$ so that $T(t) = Ce^{-kt}$. Our initial conditions are $T(0) = 25$ and $T(20) = 15$. We can solve for the constants now.

$$25 = Ce^0 = C$$

2

$$
\begin{aligned}
15 &= 25e^{-20k} \\
\frac{3}{5} &= e^{-20k} \\
-20k &= \ln(3/5) \\
k &= -\frac{\ln(3/5)}{20} \approx 0.0255
\end{aligned}
$$

We want to know when $T(t) = 5$.

$$
\begin{aligned}
5 &= 25e^{-kt} \\
\frac{1}{5} &= e^{-kt} \\
-kt &= \ln(1/5) \\
t &= -\frac{\ln(1/5)}{k} = 20\frac{\ln(1/5)}{\ln(3/5)} \approx 63.01
\end{aligned}
$$

So it will take about 63 minutes for the buttermilk to cool to 5°.

1.5.3 We have $y'+3y = 2xe^{-3x}$, so that $P(x) = 3$. Our integrating factor is $\exp(\int 3\,dx) = e^{3x}$. After multiplying by e^{3x}, we have

$$
\begin{aligned}
\frac{d}{dx}\left[ye^{3x}\right] &= 2x \\
ye^{3x} &= x^2 + C \\
y &= x^2e^{-3x} + Ce^{-3x}
\end{aligned}
$$

1.5.17 We have $(1+x)y' + y = \cos x$, which after dividing by $1+x$ is

$$
y' + \frac{1}{1+x}y = \frac{\cos x}{1+x}
$$

So $P(x) = \frac{1}{1+x}$ and our integrating factor is $\exp(\int \frac{1}{1+x}\,dx) = \exp(\ln(1+x)) = 1+x$. After multiplication by $1+x$, we have

$$
\begin{aligned}
\frac{d}{dx}\left[y(1+x)\right] &= \cos x \\
y(1+x) &= \sin x + C \\
y &= \frac{\sin x + C}{1+x}
\end{aligned}
$$

We are given the initial condition that $y(0) = 1$, so we have

$$
\begin{aligned}
1 &= \frac{\sin 0 + C}{1} \\
1 &= C
\end{aligned}
$$

So our solution is $y = \frac{\sin x + 1}{1+x}$

3

1.5.37 After t seconds, the volume of liquid in the tank is $V(t) = 100 + 5t - 3t = 100 + 2t$. The differential equation that describes the amount $x(t)$ of salt in the tank at time t is

$$x' = 5 \cdot 1 - 3 \cdot \frac{x}{100 + 2t}$$

We rewrite this slightly to allow us to find the integrating factor.

$$x' + \frac{3}{100 + 2t} x = 5$$

So $P(t) = \frac{3}{100+2t}$ and our integrating factor is $\exp(\int \frac{3}{100+2t} dt) = \exp(\frac{3}{2} \ln(100 + 2t)) = (100 + 2t)^{3/2}$. After multiplying by the integrating factor, we have

$$\begin{aligned} \frac{d}{dt}\left[x(100 + 2t)^{3/2}\right] &= 5(100 + 2t)^{3/2} \\ x(100 + 2t)^{3/2} &= (100 + 2t)^{5/2} + C \\ x &= (100 + 2t) + \frac{C}{(100 + 2t)^{3/2}} \end{aligned}$$

Initially, we have $x(0) = 50$ pounds of salt. So we can solve for C.

$$\begin{aligned} 50 &= 100 + \frac{C}{100^{3/2}} \\ -50 \cdot 1000 &= C \end{aligned}$$

So, we have

$$x(t) = 100 + 2t - \frac{50,000}{(100 + 2t)^{3/2}}$$

We want the amount of salt when the tank is full. This happens when $V(t) = 400$, so when $t = 150$. At that time, we have

$$x(150) = 400 - \frac{50,000}{400^{3/2}} = 393.75 \text{ pounds of salt}$$

2.1.15 We are given that $\frac{dP}{dt} = aP - bP^2 = bP(\frac{a}{b} - P)$. We are given that the birth rate is aP, which at $t = 0$ is B_0 and the death rate is bP^2, which at $t = 0$ is D_0. Since the initial population is $P(0) = P_0$, this tells us that $aP_0 = B_0$ and $bP_0^2 = D_0$. So we have

$$\begin{aligned} \frac{a}{b} &= \frac{B_0/P_0}{D_0/P_0^2} \\ &= \frac{B_0 P_0}{D_0} \end{aligned}$$

So we have written our differential equation in the form $\frac{dP}{dt} = kP(M - P)$ where $k = b = \frac{D_0}{P_0^2}$ and $M = \frac{a}{b} = \frac{B_0 P_0}{D_0}$. Thus our limiting population is indeed $\frac{B_0 P_0}{D_0}$.

4

2.1.16 In this case we have $P_0 = 120$, $B_0 = 8$, and $D_0 = 6$. So our differential equation can be written as

$$\frac{dP}{dt} = \frac{6}{120^2} P \left(\frac{8 \cdot 120}{6} - P \right) = \frac{1}{2400} P\,(160 - P)$$

Knowing k, M, and P_0, we can use the formula for the general solution of a logistic equation to get

$$\begin{aligned} P(t) &= \frac{160 \cdot 120}{120 + (160 - 120)e^{-\frac{1}{2400} \cdot 160t}} \\ &= \frac{480}{3 + e^{-t/15}} \end{aligned}$$

We wish to know when $P(t) = 0.95 \cdot M = 152$. We solve:

$$\begin{aligned} 152 &= \frac{480}{3 + e^{-t/15}} \\ 456 + 152e^{-t/15} &= 480 \\ 152e^{-t/15} &= 24 \\ \frac{-t}{15} &= \ln(3/19) \\ t &= -15 \ln(3/19) \approx 27.7 \end{aligned}$$

So it will take nearly 28 months for the population to reach 95% of the limiting population.

2.1.17 In this case, we have $P_0 = 240$, $B_0 = 9$, and $D_0 = 12$. So our differential equation can be written as

$$\frac{dP}{dt} = \frac{12}{240^2} P \left(\frac{9 \cdot 240}{12} - P \right) = \frac{1}{4800} P\,(180 - P)$$

Knowing k, M, and P_0, we can use the formula for the general solution of a logistic equation to get

$$\begin{aligned} P(t) &= \frac{180 \cdot 240}{240 + (180 - 240)e^{-\frac{1}{4800} \cdot 180t}} \\ &= \frac{720}{4 - e^{-3t/80}} \end{aligned}$$

We wish to know when $P(t) = 1.05 \cdot M = 189$. We solve:

$$\begin{aligned} 189 &= \frac{720}{4 - e^{-3t/80}} \\ 756 - 189e^{-3t/80} &= 720 \\ 189e^{-3t/80} &= 36 \\ \frac{-3t}{80} &= \ln(4/21) \\ t &= \frac{-80 \ln(4/21)}{3} \approx 44.2 \end{aligned}$$

5

So it will take just more than 44 months for the population to fall to 105% of the limiting population.

Additional Problems:

1. (a) $y' + y = 0$ has characteristic equation $r + 1 = 0$ with root $r = -1$. So we have general solution $y = c_1 e^{-x}$.

   (b) Our particular solution has form $y_p = Ae^x$ since there is no repetition. We substitute into the equation to find the value of A:

   $$y_p' + y_p = Ae^x + Ae^x = e^x$$

   So $A = \frac{1}{2}$ and we have particular solution $y_p = \frac{1}{2}e^x$.

   (c) The general solution is $y = y_p + y_c = \frac{1}{2}e^x + c_1 e^{-x}$.

   (d) For $y' + y = e^x$ we have $P(x) = 1$ and $Q(x) = e^x$.

   (e) Our integrating factor is

   $$\begin{aligned} e^{\int P(x)\,dx} &= e^{\int 1\,dx} \\ &= e^x \end{aligned}$$

   So we have

   $$\begin{aligned} y'e^x + ye^x &= e^{2x} \\ \frac{d}{dx}\left[y \cdot e^x\right] &= e^{2x} \\ y \cdot e^x &= \int e^{2x}\,dx \\ y \cdot e^x &= \frac{1}{2}e^{2x} + C \\ y &= \frac{1}{2}e^x + Ce^{-x} \end{aligned}$$

   (f) The solutions we wrote in (e) and (c) are identical, except for the name of the constants.

2. (a) The initial value problem is

   $$\frac{dP}{dt} = kP(100 - P) \qquad P(0) = 5$$

   where $P(t)$ is the number of people with Green's disease (in thousands) on day t.

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(b) This differential equation is separable:

$$\frac{1}{P(100-P)}dP = k\ dt$$

After calculating a partial fraction decomposition, we can integrate both sides:

$$\begin{aligned}
\frac{1}{100}\int\left(\frac{1}{P}+\frac{1}{100-P}\right)dP &= \int k\ dt \\
\frac{1}{100}\left(\ln P - \ln(100-P)\right) &= kt + C \\
\ln\left(\frac{P}{100-P}\right) &= 100kt + C_0 \\
\frac{P}{100-P} &= C_1 e^{100kt}
\end{aligned}$$

At this point, it seems prudent to solve for the constant C_1. With $P(0) = 5$, we compute

$$\begin{aligned}
\frac{5}{100-5} &= C_1 e^0 \\
C_1 &= \frac{5}{95}
\end{aligned}$$

We will leave the symbol C_1 in our calculation for the time being as we solve for P:

$$\begin{aligned}
\frac{P}{100-P} &= C_1 e^{100kt} \\
P &= 100C_1e^{100kt} - PC_1e^{100kt} \\
P(1 + C_1e^{100kt}) &= 100C_1e^{100kt} \\
P &= \frac{100C_1e^{100kt}}{1 + C_1e^{100kt}} \\
P &= \frac{100 \cdot \frac{5}{95}e^{100kt}}{1 + \frac{5}{95}e^{100kt}}
\end{aligned}$$

Multiplying the fraction by $95e^{-100kt}$ in both numerator and denominator, we get a cleaner expression

$$P(t) = \frac{500}{95e^{-100kt} + 5}$$

(c) We are given that $P'(0) = 0.5$ and that $P(0) = 5$. We can put these values into the differential equation to get

$$\begin{aligned}
0.5 &= k(5)(100-5) \\
k &= \frac{0.5}{5 \cdot 95} \approx 0.00105
\end{aligned}$$

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(d) We need to know when $P(t) = 50$. So, we solve for t in

$$\begin{aligned}
\frac{500}{95e^{-100kt}+5} &= 50 \\
500 &= 50 \cdot 95e^{-100kt} + 5 \cdot 50 \\
\frac{500 - 5 \cdot 50}{50 \cdot 95} &= e^{-100kt} \\
\frac{1}{19} &= e^{-100kt} \\
-100kt &= \ln(\tfrac{1}{19}) \approx -2.944 \\
t &\approx \frac{-2.944}{-100 \cdot 0.00105} = 28.04
\end{aligned}$$

So it will take about 28 days for half the population to be infected.

(e) One way to view this kind of model is that it tells us what will happen if the system is left to run without intervention. This model will only be accurate if human behavior, the biology of the disease, and our treatment capability all stay the same. Factors like mask wearing, social distancing, curfews, hand washing, business closures, and holiday celebrations can all impact the level of transmission between individuals. There may be changes to the transmissibility of the disease itself, caused by mutations or by the weather. Medical intervention may allow us to make infected individuals no longer contagious or make some people immune through vaccines. There may also be further complicating factors such as travel to and from other cities.

Another relevant saying here is "garbage in, garbage out." This means that our model is only as good as the data we feed into it. If the count of total infections or daily infections is wrong due to inaccurate tests, insufficient testing, or incomplete reporting, our model has no hope of predicting the true numbers.

With all of that said, what utility can we still get from this model? Well, it does tell us about one possible scenario for how disease transmission could evolve. If we adjust our assumptions slightly, we can get other possible scenarios. In reality, most modeling of this kind gives a *range* of possible outcomes, rather than a single prediction. This model is one such possible outcome and is probably most useful when viewed in the context of other possible outcomes.

The predictions of this model also give us a benchmark to compare future data to. If we introduce public health interventions like mask mandates, stay-at-home orders, or messaging about hand washing, we can assess their effectiveness by comparing future data to our predictions. If there are fewer infections than our model predicted, that indicates that the public health interventions may be helping. If infection rates rise above our predictions, we will need to explore possible causes such as disease mutations, weather changes, or "superspreader" events. Having this model helps us understand whether the new data we get each day is as expected, a cause for concern, or a cause for celebration.

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2.13 HW 13

Local contents

2.13.1 Problems listing

HOMEWORK 13

As we are rapidly approaching the final exam, this homework will not be collected or graded. These problems are simply to aid you in studying for the exam. As such, I will not be providing a specific list of textbook problems beyond that on the review sheet. If you can comfortably do two or three of each of the problem types from the textbook, you are probably safe to skip the rest. I will only type up solutions for the additional problems, but I encourage you to check your answers for the textbook problems with the back of the book.

Textbook Problems:

- §2.2: 1-12, 20-22
- §2.3: 1-3, 9-12
- §2.4: 1-10

Additional Problems:

1. Write a differential equation of the form $\frac{dx}{dt} = f(x)$ that has a stable equilibrium at $x = 5$ and an unstable equilibrium at $x = 1$.
2. Write a differential equation of the form $\frac{dx}{dt} = f(x)$ that has stable equilibria at $x = 5$ and $x = 7$ and an unstable equilibrium at $x = 1$.
3. Formula One cars have a feature called the drag reduction system (DRS) which opens a flap on the rear wing to decrease drag at particular points in the race and facilitate overtaking. For the purposes of this problem, we will assume that the cars have constant acceleration and drag proportional to velocity, so that

$$\frac{dv}{dt} = a - \rho v$$

where ρ is a positive constant called the *drag coefficient*. The *top speed* of a car is defined to be $\lim_{t\to\infty} v(t)$, otherwise known as the *terminal velocity*.

(a) The differential equation $\frac{dv}{dt} = a - \rho v$ is separable. Solve it (leaving a and ρ as constants), and find the particular solution when $v(0) = v_0$. Find the top speed $\lim_{t\to\infty} v(t)$.

(b) Under normal conditions, the top speed of a car is 85 m/s. With the DRS active, the top speed increases to 90 m/s. If the car's engine provides a constant acceleration of 14 m/s^2, what is the drag coefficient with and without DRS?

(c) Two cars exit a corner at 25 m/s, with car A 10 meters behind car B. At the same time, they both begin accelerating at 14 m/s^2. However, car A has DRS enabled while car B does not. Using the drag coefficients calculated in the previous problem,

1

determine how much time it will take for car A to be right next to car B. *(The position equations will involve both exponential and polynomial terms. I recommend using a computer algebra system to solve for the time where $x_A(t) = x_B(t)$)*

There are 500 meters from the corner exit to where the drivers must begin braking for the next corner. Which car will be ahead when they brake for the next corner?

(d) Formula One cars are exceptionally "draggy," and at high speed can get more deceleration from just their aerodynamics than a road car gets from slamming on the brakes. With no throttle or brakes, we only have drag acting on the car, so that

$$\frac{dv}{dt} = -\rho v$$

Say that a driver's brakes have failed during a race. If they can slow the car to 10 m/s by the time they reach the pit lane, their mechanics will be able to safely bring the car to a stop. If the driver is currently traveling at 80 m/s, how far before the pit lane must they begin coasting? Use the non-DRS drag coefficient you calculated in part (b).

2

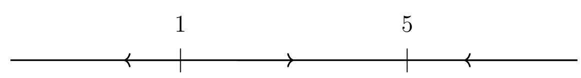
2.13.2 key solution for HW13

HOMEWORK 13 – SOLUTIONS

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Additional Problems:

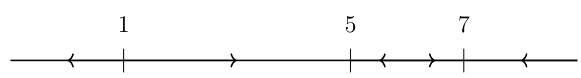
1. We need our function $f(x)$ to have zeros at $x = 1, 5$ and we need the phase diagram to be



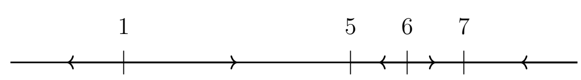
We take as a first guess $f(x) = (x-1)(x-5)$. We find then that $f(0) = (-1)(-5) = 5 > 0$, which would be the wrong sign. So we take as a next guess $f_1(x) = -(x-1)(x-5)$. Then we have $f_1(0) = -5$, $f_1(3) = 4$, and $f_1(6) = -5$ which is exactly what is needed. So our differential equation is

$$\frac{dx}{dt} = -(x-1)(x-5)$$

2. We need our function to have zeros at $x = 1, 5, 7$ and we need the phase diagram to be



The problem here is that we have two conflicting signs needed in the interval between 5 and 7. To rectify this, we will introduce another critical point at 6 where the sign of $f(x)$ will change. This means we are now looking to get the phase diagram



We take as a guess $f(x) = (x-1)(x-5)(x-6)(x-7)$. We see that $f(0) = (-1)(-5)(-6)(-7) > 0$, so the sign is wrong again. Our next guess is $f_1(x) = -(x-1)(x-5)(x-6)(x-7)$. You can check that now all of the signs are correct, so our differential equation is

$$\frac{dx}{dt} = -(x-1)(x-5)(x-6)(x-7)$$

If we add the additional requirement that $f(x)$ is continuous on all of $\mathbb{R}$, you can show that the only way to have both $x = 5$ and $x = 7$ as stable equilibria is if there is another equilibrium solution between 5 and 7. If we don't require that $f(x)$ is continuous, we can use a function like

$$f(x) = -\frac{(x-1)(x-5)(x-7)}{x-6}$$

which has only the three required equilibria and no others. This discontinuity at $x = 6$ makes this differential equation quite unpleasant to work with, however.

1

3. (a) We are solving $\frac{dv}{dt} = a - \rho v$ where a and ρ are some unknown constants. This is a separable equation, and we can write it as

$$\frac{1}{a - \rho v} dv = dt$$

Integrating both sides and solving for v, we get

$$\begin{aligned} -\frac{1}{\rho} \ln(a - \rho v) &= t + C_0 \\ \ln(a - \rho v) &= -\rho t + C_1 \\ a - \rho v &= C_2 e^{-\rho t} \\ -\rho v &= C_2 e^{-\rho t} - a \\ v &= C_3 e^{-\rho t} + \frac{a}{\rho} \end{aligned}$$

With the initial condition $v(0) = v_0$, we have

$$v_0 = C_3 + \frac{a}{\rho}$$

So our particular solution is

$$v(t) = \left(v_0 - \frac{a}{\rho}\right) e^{-\rho t} + \frac{a}{\rho}$$

Since ρ is a positive constant, the top speed is

$$\lim_{t \to \infty} v(t) = \frac{a}{\rho}$$

(b) If the top speed is 85 m/s and our acceleration is 14 m/s^2, then we can solve for ρ.

$$\begin{aligned} 85 &= \frac{14}{\rho} \\ \rho &= \frac{14}{85} \approx 0.165 \end{aligned}$$

In the case where DRS is active, we have a top speed of 90 m/s, so

$$\begin{aligned} 90 &= \frac{14}{\rho_{DRS}} \\ \rho_{DRS} &= \frac{14}{90} \approx 0.156 \end{aligned}$$

2

(c) Car A begins 10 meters behind car B so $x_A(0) = -10$ and $x_B(0) = 0$. Both cars have initial velocity 25 m/s, so using our solution to (a) we can write down the two velocity functions in terms of the drag coefficients.

$$\begin{aligned} v_A(t) &= \left(25 - \frac{14}{\rho_{DRS}}\right) e^{-\rho_{DRS} t} + \frac{14}{\rho_{DRS}} \\ v_B(t) &= \left(25 - \frac{14}{\rho}\right) e^{-\rho t} + \frac{14}{\rho} \end{aligned}$$

Integrating with respect to t, we get the position functions

$$\begin{aligned} x_A(t) &= \left(-\frac{25}{\rho_{DRS}} + \frac{14}{\rho_{DRS}^2}\right) e^{-\rho_{DRS} t} + \frac{14}{\rho_{DRS}} t + C_A \\ x_B(t) &= \left(-\frac{25}{\rho} + \frac{14}{\rho^2}\right) e^{-\rho t} + \frac{14}{\rho} t + C_B \end{aligned}$$

Our initial conditions now let us solve for the constants.

$$\begin{aligned} -10 &= \left(-\frac{25}{\rho_{DRS}} + \frac{14}{\rho_{DRS}^2}\right) + C_A \\ C_A &= \frac{25}{\rho_{DRS}} - \frac{14}{\rho_{DRS}^2} - 10 \\ 0 &= \left(-\frac{25}{\rho} + \frac{14}{\rho^2}\right) + C_B \\ C_B &= \frac{25}{\rho} - \frac{14}{\rho^2} \end{aligned}$$

We now want to know when $x_A(t) = x_B(t)$. So we set these equal to each other, substitute the values we found for ρ and ρ_{DRS}, and solve for t.

$$\begin{aligned} 417.857e^{\frac{-14t}{90}} + 90t - 427.857 &= 364.286e^{\frac{-14t}{85}} + 85t - 364.286 \\ 417.857e^{\frac{-14t}{90}} - 364.286e^{\frac{-14t}{85}} + 5t &= 63.571 \\ t &\approx 8.306 \end{aligned}$$

So it will take about 8.3 seconds for the two cars to be side by side. At time $t = 8.306$, we have $x_A(t) = x_B(t) = 434.472$ meters. So by the time the cars have driven 500 meters, car A should be ahead.

(d) Setting $a = 0$ m/s^2 and $v_0 = 80$ m/s in our solution found in (a), we have

$$v(t) = 80e^{-\rho t}$$

3

Given that $\rho = \frac{14}{85}$, we want to find when $v(t) = 10$.

$$\begin{aligned} 10 &= 80e^{\frac{-14t}{85}} \\ \frac{-14t}{85} &= \ln\left(\frac{10}{80}\right) \\ t &= -\frac{85}{14}\ln\left(\frac{10}{80}\right) \approx 12.625 \end{aligned}$$

The position function is obtained by integrating, so that

$$x(t) = -\frac{80}{\rho}e^{-\rho t} + C$$

Taking the place where we begin coasting as position 0, we have $x(0) = 0$ and $C = \frac{80}{\rho}$. Now we evaluate the position function at the calculated time to get

$$\begin{aligned} x\left(-\frac{85}{14}\ln\left(\frac{10}{80}\right)\right) &= -\frac{80}{14/85}e^{\frac{14}{85}\cdot\frac{85}{14}\ln\left(\frac{10}{80}\right)} + \frac{80}{14/85} \\ &= -\frac{80\cdot 85}{14}\cdot\frac{10}{80} + \frac{80\cdot 85}{14} = 425 \end{aligned}$$

So the driver will need to coast for 425 meters to reach the safe speed of 10 m/s.

An interesting thing to consider about this exponential decay model is that the car is not predicted to come to a complete stop after any finite amount of time, i.e. $v(t)$ is strictly positive for all $t \geq 0$. This doesn't match our real-world experience, indicating that forces like friction between the asphalt and the tires are playing a role that this model isn't accounting for.

One consequence of this model that does match our real-world experience can be seen if you run the numbers again for a target velocity of 5 m/s. In this case, it takes 16.8 seconds to slow down and we travel 455 meters. It takes more than 30% longer to decrease our speed by 75 m/s than it does to decrease our speed by 70 m/s. Despite the significant increase in time, we only increase distance traveled by 7%. This is significantly different from the behavior of the constant deceleration model we explored in Additional Problem 2 of Homework 11. But if you have played golf or billiards, you will have experienced how long it can take for the ball to slowly roll those last few inches before coming to a stop.

4

Chapter 3

study notes

3.1 How to solve some problems

1. Problem gives set S of vectors $\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \cdots\right\}$ and asks to find basis that span S consisting of elements from S. To answer this, write the vectors as columns of matrix A. Then convert the matrix to Echelon form (Row reduction). The pivot columns in A are the basis.

2. Problem gives set S of vectors $\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \cdots\right\}$ and just asks to find basis that span S. It does not say consisting of elements from S. This was not clear and I asked about it. The answer I got is to use same method as above, and that will work also.

3. Problem gives set S of vectors $\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}$ say in $\mathbb{R}^3$ and asks to find basis for $\mathbb{R}^3$ that contains S. Here, we also set up the matrix A where the first 3 columns are these vectors, but also add 3 more columns, which are $(1,0,0),(0,1,0),(0,0,1)$ and now convert A to Echelon form and the pivot columns of A are the basis. The difference between this and above, is that we append the elementary basis for $\mathbb{R}^3$ to the matrix A before starting.

4. Problem gives A matrix and asks to find is NULL space. This is asking for basis of solution space for $A\vec{x} = \vec{0}$. To solve, convert A to Echelon form. (no need to do reduced Echelon form). Then the number of the free variables is the dimension of the NULL space. So if we have 2 free variables, the NULL space is 2 dimensions. Call the free variables s, t and so on. Then solve for the leading variables in terms of the free variables. Then at end let $t = 1, r = 1$ and this gives the basis for the NULL space.

5. Problem gives set of vectors $S = \left\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \cdots\right\}$ and asks if they are linearly independent or not. If the dimension of each vector is the same as the number of the vectors, then make a square matrix A of the vectors as its columns and find the determinant. If $|A| = 0$ then the vectors are linearly independent, else they are not.

   Another way to do this is to write $c_1\vec{v}_1 + c_1\vec{v}_2 + c_1\vec{v}_3 = \vec{0}$ and solve for c_i and see if the only solution is $c_i = 0$. If so, then linearly independent, else not.

6. Problems gives A matrix and asks for its column space and its row space. To solve, reduce the matrix to Echelon form. The row space are those rows which are not all zeros. The column space are the pivot columns in the original A (not the pivot columns in the final Echelon form matrix).

7. Problem gives set of vectors $S = \left\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \cdots\right\}$ and one vector $\vec{w}$ and asks if $\vec{w}$ is linear combinations of the vectors in S. To solve, write $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{w}$ and set up $A\vec{x} = \vec{b}$. Then set up the augments matrix $[A|b]$. Reduce to Echelon form. Now see if it is consistent or not. If not consistent, then there is no solution and they it means $\vec{w}$ can not be written as linear combination. If consistent, then this means we can write $\vec{w}$ as linear combination (there can be infinite ways to do this).

8. Problem gives set of vectors $S = \left\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \cdots\right\}$ and asks if S spans all of $\mathbb{R}^n$? Let $n = 3$ for example. If we can find 3 of vectors from S that are linearly independent, then the answer is yes. Otherwise no.

9. Problem gives set of vectors $S = \left\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \cdots\right\}$ and asks if these vectors are basis for $\mathbb{R}^n$? This is similar to the above. The difference is that the set S must contain only 3 vectors and no more, which are linearly independent. These they are basis. This means any vector $\vec{w}$ can be expressed as linear combination of the basis in one unique way.

10. Problem gives square matrix A and asks to find its inverse A. To solve, set up the augmented matrix by appending to the right side the identity matrix I. Then convert the whole augmented matrix to Echelon form, and now convert this to reduced Echelon form. When done, the right side (which was I initially) is A^{-1}

11. Problem gives square matrix A and asks to find its determinant. To solve, look first if possible to do any row operations to increase the number of zeros in the matrix. Then expand along one row or one column that has most zeros in it. Remember the sign is found using $(-1)^{m+n}$ where m is row number and n is column number.

12. Problem gives $A\vec{x} = \vec{b}$ and ask what kind of solutions are possible? There are only three possible solutions: No solution, one unique solution, or an infinite number of solutions. So was can not have for example 2 or 3 solutions. This is not possible.

13. Problem gives matrix A, B and asks to find matrix X such that $AX = B$. To solve, premultiply both sides by A^{-1} to get $X = A^{-1}B$. So we need to find A^{-1} then do matrix multiplication to find X.

3.2 Some definitions

span of set of vectors given set $S = \left\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \cdots\right\}$, then the span of S is the set W of all possible linear combinations of elements of S.

Chapter 4

Exams

Local contents

4.1 Exam 1, Thursday Oct 15, 2020

Local contents

4.1.1 Practice problems

Local contents

4.1.1.1 Problem section 3.2 number 23

In Problems 23–27, determine for what values of k each system has (a) a unique solution; (b) no solution; (c) infinitely many solutions.

$$\begin{aligned}3x + 2y &= 1\\ 6x + 4y &= k\end{aligned}$$

Solution

$$\begin{bmatrix}3 & 2\\ 6 & 4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}1\\ k\end{bmatrix}$$

Augmented matrix is

$$\begin{bmatrix}3 & 2 & 1\\ 6 & 4 & k\end{bmatrix}$$

$R_2 \to -2R_1 + R_2$ gives

$$\begin{bmatrix}3 & 2 & 1\\ 0 & 0 & k-2\end{bmatrix}$$

The above is in Echelon form. x is the leading variable and y is the free variable. Let $y = t$. The system in Echelon form becomes

$$\begin{bmatrix}3 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\ t\end{bmatrix} = \begin{bmatrix}1\\ k-2\end{bmatrix}$$

Last row says that $0 = k - 2$. This means only $k = 2$ is possible. First row gives $3x + 2t = 1$. When $k = 2$, we have infinite number of solutions given by $\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}\frac{1-2t}{3}\\ t\end{bmatrix}$. For any t.

When $k \neq 2$ there is no solution. There is no unique solution for any k since y is free variable. Hence answer is

(a) None (b) $k \neq 2$ (c) $k = 2$.

4.1.1.2 Problem section 3.2 number 24

In Problems 23–27, determine for what values of k each system has (a) a unique solution; (b) no solution; (c) infinitely many solutions.

$$\begin{aligned}3x + 2y &= 0\\ 6x + ky &= 0\end{aligned}$$

Solution

$$\begin{bmatrix}3 & 2\\ 6 & k\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Augmented matrix is

$$\begin{bmatrix}3 & 2 & 0\\ 6 & k & 0\end{bmatrix}$$

$R_2 \to -2R_1 + R_2$ gives

$$\begin{bmatrix}3 & 2 & 0\\ 0 & k-4 & 0\end{bmatrix}$$

We see that when $k = 4$, then y is free variable giving ∞ number of solutions. When $k \neq 4$ then unique solution exist, which is the trivial solution. Hence answer is

(a) $k \neq 4$ (b) None (c) $k = 4$.

Notice, the answer in back of the book seems wrong. It says (a) is when $k \neq 2$. It should be $k \neq 4$.

4.1.1.3 Problem section 3.2 number 25

In Problems 23–27, determine for what values of k each system has (a) a unique solution; (b) no solution; (c) infinitely many solutions.

$$\begin{aligned}3x+2y&=11\\6x+ky&=21\end{aligned}$$

Solution

$$\begin{bmatrix}3&2\\6&k\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}11\\21\end{bmatrix}$$

Augmented matrix is

$$\begin{bmatrix}3&2&11\\6&k&21\end{bmatrix}$$

$R_2\rightarrow -2R_1+R_2$ gives

$$\begin{bmatrix}3&2&11\\0&k-4&-1\end{bmatrix}$$

We see that when $k=4$, then inconsistent, since it leads to $0=-1$, hence no solution in this case. When $k\neq 4$ then unique solution exist. These are the only two possible cases. Hence answer is

$$\text{(a) } k\neq 4 \text{ (b) } k=4 \text{ (c) None}$$

4.1.1.4 Problem section 3.2 number 26

In Problems 23–27, determine for what values of k each system has (a) a unique solution; (b) no solution; (c) infinitely many solutions.

$$\begin{aligned}3x+2y&=1\\7x+5y&=k\end{aligned}$$

Solution

$$\begin{bmatrix}3&2\\7&5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\k\end{bmatrix}$$

Augmented matrix is

$$\begin{bmatrix}3&2&1\\7&5&k\end{bmatrix}$$

$R_1\rightarrow 7R_1, R_2\rightarrow 3R_2$ gives

$$\begin{bmatrix}21&14&7\\21&15&3k\end{bmatrix}$$

$R_2\rightarrow R_2-R_1$ gives

$$\begin{bmatrix}21&14&7\\0&1&3k-7\end{bmatrix}$$

The Echelon form shows that there are no free variables. Hence unique solution exist for all k values. Hence the answer is

$$\text{(a) any } k \text{ (b) None (c) None}$$

4.1.1.5 Problem section 3.2 number 27

In Problems 23–27, determine for what values of k each system has (a) a unique solution; (b) no solution; (c) infinitely many solutions.

$$\begin{aligned}x+2y+z&=3\\2x-y-3z&=5\\4x+3y-z&=k\end{aligned}$$

Solution

$$\begin{bmatrix}1 & 2 & 1\\ 2 & -1 & -3\\ 4 & 3 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ 5\\ k\end{bmatrix}$$

Augmented matrix is

$$\begin{bmatrix}1 & 2 & 1 & 3\\ 2 & -1 & -3 & 5\\ 4 & 3 & -1 & k\end{bmatrix}$$

$R_2 \to -2R_1 + R_2$ gives

$$\begin{bmatrix}1 & 2 & 1 & 3\\ 0 & -5 & -5 & -1\\ 4 & 3 & -1 & k\end{bmatrix}$$

$R_3 \to -4R_1 + R_3$ gives

$$\begin{bmatrix}1 & 2 & 1 & 3\\ 0 & -5 & -5 & -1\\ 0 & -5 & -5 & k-12\end{bmatrix}$$

$R_3 \to -R_2 + R_3$ gives

$$\begin{bmatrix}1 & 2 & 1 & 3\\ 0 & -5 & -5 & -1\\ 0 & 0 & 0 & k-11\end{bmatrix}$$

The Echelon form shows that only when $k = 11$ we get consistent system. And in this case, z is the free variable, leading to ∞ solutions. If $k \neq 11$ then system is inconsistent and no solution exist.

(a) None (b) $k \neq 11$ (c) $k = 11$

4.1.1.6 Problem section 3.2 number 28

Under what condition on the constants a, b, and c does the system have a unique solution? No solution? Infinitely many solutions?

$$\begin{aligned}2x - y + 3z &= a\\ x + 2y + z &= b\\ 7x + 4y + 9z &= c\end{aligned}$$

Solution

$$\begin{bmatrix}2 & -1 & 3\\ 1 & 2 & 1\\ 7 & 4 & 9\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ b\\ c\end{bmatrix}$$

Augmented matrix is

$$\begin{bmatrix}2 & -1 & 3 & a\\ 1 & 2 & 1 & b\\ 7 & 4 & 9 & c\end{bmatrix}$$

$R_2 \to -R_1 + 2R_2$ gives

$$\begin{bmatrix}2 & -1 & 3 & a\\ 0 & 5 & -1 & 2b-a\\ 7 & 4 & 9 & c\end{bmatrix}$$

$R_3 \to 2R_3, R_1 \to 7R_1$ gives

$$\begin{bmatrix}14 & -7 & 21 & 7a\\ 0 & 5 & -1 & 2b-a\\ 14 & 8 & 18 & 2c\end{bmatrix}$$

$R_3 \to -R_1 + R_3$ gives

$$\begin{bmatrix} 14 & -7 & 21 & 7a \\ 0 & 5 & -1 & 2b-a \\ 0 & 15 & -3 & 2c-7a \end{bmatrix}$$

$R_3 \to -3R_2 + R_3$ gives

$$\begin{bmatrix} 14 & -7 & 21 & 7a \\ 0 & 5 & -1 & 2b-a \\ 0 & 0 & 0 & (2c-7a)-3(2b-a) \end{bmatrix}$$

Or

$$\begin{bmatrix} 14 & -7 & 21 & 7a \\ 0 & 5 & -1 & 2b-a \\ 0 & 0 & 0 & 2c-6b-4a \end{bmatrix}$$

The Echelon form shows that only when $2c - 6b - 4a = 0$ or

$$c = 3b + 2a$$

We get consistent system. And in this case, z is the free variable, leading to ∞ solutions. If $c \neq 3b + 2a$ then system is inconsistent and no solution exist.

4.1.1.7 Problem section 3.3 number 37

Show that the homogeneous system in problem 35 has non-trivial solution iff $ad - bc = 0$

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

Solution

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Augmented matrix is

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \end{bmatrix}$$

$R_2 \to -\frac{c}{a} R_1 + R_2$ gives

$$\begin{bmatrix} a & b & 0 \\ 0 & d - \frac{c}{a} b & 0 \end{bmatrix}$$

Or

$$\begin{bmatrix} a & b & 0 \\ 0 & \frac{ad-cb}{c} & 0 \end{bmatrix}$$

There are two cases. First case is when $\frac{ad-cb}{c} = 0$ or $ad - cb = 0$ then we get infinite number of solutions since y is the free variable. The second case is when $ad - cb \neq 0$ and in this case, we get unique solution which is the trivial solution $x = 0, y = 0$.

Hence only when $ad - cb = 0$ do we get non-trivial solution which is what we are asked to show.

4.1.1.8 Problem section 3.3 number 38

Use the result of problem 37 to find all values of c for which

$$\begin{aligned} (c+2)x + 3y &= 0 \\ 2x + (c-3)y &= 0 \end{aligned} \tag{1}$$

Has non-trivial solution.

Solution

From problem 37 we found that non-trivial solution exist when

$$ad - cb = 0 \tag{2}$$

Where

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned} \tag{3}$$

Comparing (1) to (3) shows that

$$\begin{aligned} a &\equiv (c+2) \\ b &\equiv 3 \\ c &\equiv 2 \\ d &\equiv (c-3) \end{aligned}$$

Hence (2) now becomes

$$\begin{aligned} (c+2)(c-3) - (2)(3) &= 0 \\ c^2 - c - 12 &= 0 \\ (c+3)(c-4) &= 0 \end{aligned}$$

Hence only possible values are $c = -3, c = 4$. These values give non-trivial solution.

4.1.1.9 Problem section 3.5 number 9

Use the method of example 7 to find A^{-1} for given A

$$A = \begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix}$$

Solution

Augmented matrix is

$$\begin{bmatrix} 5 & 6 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{bmatrix}$$

$R_1 \to 4R_1, R_2 \to 5R_2$ gives

$$\begin{bmatrix} 20 & 24 & 4 & 0 \\ 20 & 25 & 0 & 5 \end{bmatrix}$$

$R_2 \to -R_1 + R_2$

$$\begin{bmatrix} 20 & 24 & 4 & 0 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

$R_1 \to \frac{R_1}{20}$ gives

$$\begin{bmatrix} 1 & \frac{6}{5} & \frac{1}{5} & 0 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

$R_1 \to R_1 - \frac{6}{5}R_2$

$$\begin{bmatrix} 1 & 0 & 5 & -6 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

Since the left half is the identity matrix, then the inverse is the right side. Hence

$$A^{-1} = \begin{bmatrix} 5 & -6 \\ -4 & 5 \end{bmatrix}$$

4.1.1.10 Problem section 3.5 number 10

Use the method of example 7 to find A^{-1} for given A

$$A = \begin{bmatrix} 5 & 7 \\ 4 & 6 \end{bmatrix}$$

Solution

Augmented matrix is

$$\begin{bmatrix} 5 & 7 & 1 & 0 \\ 4 & 6 & 0 & 1 \end{bmatrix}$$

$R_1 \rightarrow 4R_1, R_2 \rightarrow 5R_2$ gives

$$\begin{bmatrix} 20 & 28 & 4 & 0 \\ 20 & 30 & 0 & 5 \end{bmatrix}$$

$R_2 \rightarrow -R_1 + R_2$ gives

$$\begin{bmatrix} 20 & 28 & 4 & 0 \\ 0 & 2 & -4 & 5 \end{bmatrix}$$

$R_1 \rightarrow \frac{R_1}{20}$ gives

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{1}{5} & 0 \\ 0 & 2 & -4 & 5 \end{bmatrix}$$

$R_2 \rightarrow \frac{R_2}{2}$ gives

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{1}{5} & 0 \\ 0 & 1 & -2 & \frac{5}{2} \end{bmatrix}$$

$R_1 \rightarrow R_1 - \frac{7}{5}R_2$ gives

$$\begin{bmatrix} 1 & 0 & 3 & -\frac{7}{2} \\ 0 & 1 & -2 & \frac{5}{2} \end{bmatrix}$$

Since the left half is the identity matrix, then the inverse is the right side. Hence

$$A^{-1} = \begin{bmatrix} 3 & -\frac{7}{2} \\ -2 & \frac{5}{2} \end{bmatrix}$$

4.1.1.11 Problem section 3.5 number 11

Use the method of example 7 to find A^{-1} for given A

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix}$$

Solution

Augmented matrix is

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 7 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$ gives

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 2 & 7 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_1$ gives

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{bmatrix}$$

$R_2 \to 3R_2, R_3 \to 5R_3$ gives

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -15 & -6 & -6 & 3 & 0 \\ 0 & -15 & -5 & -10 & 0 & 5 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ gives

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -15 & -6 & -6 & 3 & 0 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{bmatrix}$$

$R_2 \to -\frac{R_2}{15}$ gives

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{6}{15} & \frac{6}{15} & -\frac{3}{15} & 0 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{bmatrix}$$

$R_2 \to R_2 - \frac{6}{15}R_3$ gives

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{bmatrix}$$

$R_1 \to R_1 - R_3$ gives

$$\begin{bmatrix} 1 & 5 & 0 & 5 & 3 & -5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{bmatrix}$$

$R_1 \to R_1 - 5R_2$ gives

$$\begin{bmatrix} 1 & 0 & 0 & -5 & -2 & 5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{bmatrix}$$

Since the left half is the identity matrix, then the inverse is the right side. Hence

$$A^{-1} = \begin{bmatrix} -5 & -2 & 5 \\ 2 & 1 & -2 \\ -4 & -3 & 5 \end{bmatrix}$$

4.1.1.12 Problem section 3.5 number 12

Use the method of example 7 to find A^{-1} for given A

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 8 & 3 \\ 3 & 10 & 6 \end{bmatrix}$$

Solution

Augmented matrix is

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 2 & 8 & 3 & 0 & 1 & 0 \\ 3 & 10 & 6 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ gives

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 1 & 0 \\ 3 & 10 & 6 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$ gives

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{bmatrix}$$

$R_3 \to 2R_3 - R_2$ gives

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & -1 & 2 \end{bmatrix}$$

$R_2 \to \frac{1}{2}R_2$ gives

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -4 & -1 & 2 \end{bmatrix}$$

$R_2 \to \frac{1}{2}R_3 + R_2$ gives

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & -4 & -1 & 2 \end{bmatrix}$$

$R_1 \to R_1 - 2R_3$ gives

$$\begin{bmatrix} 1 & 3 & 0 & 9 & 2 & -4 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & -4 & -1 & 2 \end{bmatrix}$$

$R_1 \to R_1 - 3R_2$ gives

$$\begin{bmatrix} 1 & 0 & 0 & 18 & 2 & -7 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & -4 & -1 & 2 \end{bmatrix}$$

Since the left half is the identity matrix, then the inverse is the right side. Hence

$$A^{-1} = \begin{bmatrix} 18 & 2 & -7 \\ -3 & 0 & 1 \\ -4 & -1 & 2 \end{bmatrix}$$

4.1.1.13 Problem section 3.5 number 13

Use the method of example 7 to find A^{-1} for given A

$$A = \begin{bmatrix} 2 & 7 & 3 \\ 1 & 3 & 2 \\ 3 & 7 & 9 \end{bmatrix}$$

<u>Solution</u>

Augmented matrix is

$$\begin{bmatrix} 2 & 7 & 3 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{bmatrix}$$

Swap R_1, R_2

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 2 & 7 & 3 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ gives

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$ gives

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & -2 & 3 & 0 & -3 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 2R_2$ gives

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & -7 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 + R_3$ gives

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & -9 & 1 \\ 0 & 0 & 1 & 2 & -7 & 1 \end{bmatrix}$$

$R_1 \rightarrow R_1 - 2R_3$ gives

$$\begin{bmatrix} 1 & 3 & 0 & -4 & 15 & -2 \\ 0 & 1 & 0 & 3 & -9 & 1 \\ 0 & 0 & 1 & 2 & -7 & 1 \end{bmatrix}$$

$R_1 \rightarrow R_1 - 3R_2$ gives

$$\begin{bmatrix} 1 & 0 & 0 & -13 & 42 & -5 \\ 0 & 1 & 0 & 3 & -9 & 1 \\ 0 & 0 & 1 & 2 & -7 & 1 \end{bmatrix}$$

Since the left half is the identity matrix, then the inverse is the right side. Hence

$$A^{-1} = \begin{bmatrix} -13 & 42 & -5 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{bmatrix}$$

4.1.1.14 Problem section 3.5 number 23

Find matrix X such that $AX = B$

$$A = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -5 \\ -1 & -2 & 5 \end{bmatrix}$$

Solution

Pre-multiplying both sides of $AX = B$ by A^{-1} gives

$$X = A^{-1}B \tag{1}$$

But

$$\begin{aligned} A^{-1} &= \frac{1}{16-15}\begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix} \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} X &= \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 & -5 \\ -1 & -2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 18 & -35 \\ -9 & -23 & 45 \end{bmatrix} \end{aligned}$$

4.1.1.15 Problem section 3.5 number 24

Find matrix X such that $AX = B$

$$A = \begin{bmatrix} 7 & 6 \\ 8 & 7 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 5 & -3 \end{bmatrix}$$

Solution

Pre-multiplying both sides of $AX=B$ by A^{-1} and using $A^{-1}A=I$ results in

$$X = A^{-1}B \tag{1}$$

But

$$\begin{aligned} A^{-1} &= \frac{1}{49-48}\begin{bmatrix} 7 & -6 \\ -8 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 7 & -6 \\ -8 & 7 \end{bmatrix} \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} X &= \begin{bmatrix} 7 & -6 \\ -8 & 7 \end{bmatrix}\begin{bmatrix} 2 & 0 & 4 \\ 0 & 5 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 14 & -30 & 46 \\ -16 & 35 & -53 \end{bmatrix} \end{aligned}$$

4.1.1.16 Problem section 3.5 number 25

Find matrix X such that $AX=B$

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 8 & 3 \\ 2 & 7 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

Solution

Pre-multiplying both sides of $AX=B$ by A^{-1} and using $A^{-1}A=I$ results in

$$X = A^{-1}B \tag{1}$$

But

$$A^{-1} = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 8 & 3 \\ 2 & 7 & 4 \end{bmatrix}^{-1}$$

Augmented matrix is

$$\begin{bmatrix} 1 & 4 & 1 & 1 & 0 & 0 \\ 2 & 8 & 3 & 0 & 1 & 0 \\ 2 & 7 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$ gives

$$\begin{bmatrix} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 2 & 7 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 - 2R_1$ gives

$$\begin{bmatrix} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & -1 & 2 & -2 & 0 & 1 \end{bmatrix}$$

Swap R_2, R_3

$$\begin{bmatrix} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix}$$

$R_2 \rightarrow -R_2$

$$\begin{bmatrix} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_3$

$$\begin{bmatrix} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix}$$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & 4 & 0 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 4R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 11 & -9 & 4 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix}$$

Since the left half is the identity matrix, then the inverse is the right side. Hence

$$A^{-1} = \begin{bmatrix} -13 & 42 & -5 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{bmatrix}$$

Therefore (1) becomes

$$\begin{aligned} X &= \begin{bmatrix} 11 & -9 & 4 \\ -2 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ -1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 7 & -14 & 15 \\ -1 & 3 & -2 \\ -2 & 2 & -4 \end{bmatrix} \end{aligned}$$

4.1.1.17 Problem section 3.5 number 26

Find matrix X such that $AX = B$

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & -2 \\ 1 & 7 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ 1 & 0 & 2 \end{bmatrix}$$

Solution

Pre-multiplying both sides of $AX = B$ by A^{-1} and using $A^{-1}A = I$ results in

$$X = A^{-1}B \tag{1}$$

But

$$A^{-1} = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & -2 \\ 1 & 7 & 2 \end{bmatrix}^{-1}$$

Augmented matrix is

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 1 & 7 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -9 & -4 & -2 & 1 & 0 \\ 1 & 7 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -9 & -4 & -2 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow 2R_2, R_3 \rightarrow 9R_3$

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -18 & -8 & -4 & 2 & 0 \\ 0 & 18 & 9 & -9 & 0 & 9 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -18 & -8 & -4 & 2 & 0 \\ 0 & 0 & 1 & -13 & 2 & 9 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 8R_3$

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -18 & 0 & -108 & 18 & 72 \\ 0 & 0 & 1 & -13 & 2 & 9 \end{bmatrix}$$

$R_1 \rightarrow R_1 - R_3$

$$\begin{bmatrix} 1 & 5 & 0 & 14 & -2 & -9 \\ 0 & -18 & 0 & -108 & 18 & 72 \\ 0 & 0 & 1 & -13 & 2 & 9 \end{bmatrix}$$

$R_2 \rightarrow \frac{-1}{18} R_2$

$$\begin{bmatrix} 1 & 5 & 0 & 14 & -2 & -9 \\ 0 & 1 & 0 & 6 & -1 & -4 \\ 0 & 0 & 1 & -13 & 2 & 9 \end{bmatrix}$$

$R_1 \rightarrow R_1 - 5R_2$

$$\begin{bmatrix} 1 & 0 & 0 & -16 & 3 & 11 \\ 0 & 1 & 0 & 6 & -1 & -4 \\ 0 & 0 & 1 & -13 & 2 & 9 \end{bmatrix}$$

Since the left half is the identity matrix, then the inverse is the right side. Hence

$$A^{-1} = \begin{bmatrix} -16 & -3 & 11 \\ 6 & -1 & -4 \\ -13 & 2 & 9 \end{bmatrix}$$

Therefore (1) becomes

$$\begin{aligned} X &= \begin{bmatrix} -16 & -3 & 11 \\ 6 & -1 & -4 \\ -13 & 2 & 9 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ 1 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -21 & -9 & 0 \\ 8 & -3 & -4 \\ -17 & 6 & 9 \end{bmatrix} \end{aligned}$$

4.1.1.18 Problem section 3.6 number 7

In Problems 7–12, evaluate each given determinant after first simplifying the computation (as in Example 6) by adding an appropriate multiple of some row or column to another.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

Solution

We see before starting that the determinant must be zero, since its rows are linearly dependent. We now show this is the case.

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 3 & 3 & 3 \end{bmatrix}$$

Now we do expansion along R_2. This gives

$$\begin{aligned} \det(A) &= 0\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} \\ &= 0 \end{aligned}$$

4.1.1.19 Problem section 3.6 number 8

In Problems 7–12, evaluate each given determinant after first simplifying the computation (as in Example 6) by adding an appropriate multiple of some row or column to another.

$$A = \begin{bmatrix} 2 & 3 & 4 \\ -2 & -3 & 1 \\ 3 & 2 & 7 \end{bmatrix}$$

Solution

$R_1 \to R_1 + R_2$

$$A = \begin{bmatrix} 0 & 0 & 5 \\ -2 & -3 & 1 \\ 3 & 2 & 7 \end{bmatrix}$$

Expansion along first row gives

$$\begin{aligned} \det(A) &= 5\begin{vmatrix} -2 & -3 \\ 3 & 2 \end{vmatrix} \\ &= 5(-4 + 9) \\ &= 25 \end{aligned}$$

4.1.1.20 Problem section 3.6 number 9

In Problems 7–12, evaluate each given determinant after first simplifying the computation (as in Example 6) by adding an appropriate multiple of some row or column to another.

$$A = \begin{bmatrix} 3 & -2 & 5 \\ 0 & 5 & 17 \\ 6 & -4 & 12 \end{bmatrix}$$

Solution

$R_3 \to R_3 - 2R_1$

$$A = \begin{bmatrix} 3 & -2 & 5 \\ 0 & 5 & 17 \\ 0 & 0 & 2 \end{bmatrix}$$

Expansion along third row gives

$$\begin{aligned}\det(A) &= 2\begin{vmatrix}3 & -2\\ 0 & 5\end{vmatrix}\\ &= 2(15)\\ &= 30\end{aligned}$$

4.1.1.21 Problem section 3.6 number 10

In Problems 7–12, evaluate each given determinant after first simplifying the computation (as in Example 6) by adding an appropriate multiple of some row or column to another.

$$A = \begin{bmatrix}-3 & 6 & 5\\ 1 & -2 & -4\\ 2 & -5 & 12\end{bmatrix}$$

Solution

$R_1 \to R_1 + 3R_2$

$$A = \begin{bmatrix}0 & 0 & -7\\ 1 & -2 & -4\\ 2 & -5 & 12\end{bmatrix}$$

Expansion along first row gives

$$\begin{aligned}\det(A) &= -7\begin{vmatrix}1 & -2\\ 2 & -5\end{vmatrix}\\ &= -7(-5+4)\\ &= 7\end{aligned}$$

4.1.1.22 Problem section 3.6 number 11

In Problems 7–12, evaluate each given determinant after first simplifying the computation (as in Example 6) by adding an appropriate multiple of some row or column to another.

$$A = \begin{bmatrix}1 & 2 & 3 & 4\\ 0 & 5 & 6 & 7\\ 0 & 0 & 8 & 9\\ 2 & 4 & 6 & 9\end{bmatrix}$$

Solution

$R_3 \to R_3 - 2R_1$

$$A = \begin{bmatrix}1 & 2 & 3 & 4\\ 0 & 5 & 6 & 7\\ 0 & 0 & 8 & 9\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Expansion along last row

$$\det(A) = (-1)^{i+j}(1)\begin{vmatrix}1 & 2 & 3\\ 0 & 5 & 6\\ 0 & 0 & 8\end{vmatrix}$$

Where $i=4, j=4$ (since it is entry $(4,4)$). Hence $(-1)^{i+j}=(-1)^8=1$. So the sign is +. The above becomes

$$\det(A)=(1)\begin{vmatrix}1&2&3\\0&5&6\\0&0&8\end{vmatrix}$$

For the second determination, expansion along its third row gives

$$\begin{aligned}\det(A)&=1\left((-1)^{3+3}8\begin{vmatrix}1&2\\0&5\end{vmatrix}\right)\\&=1\left(8\begin{vmatrix}1&2\\0&5\end{vmatrix}\right)\\&=8(5)\\&=40\end{aligned}$$

4.1.1.23 Problem section 3.6 number 12

In Problems 7–12, evaluate each given determinant after first simplifying the computation (as in Example 6) by adding an appropriate multiple of some row or column to another.

$$A=\begin{bmatrix}2&0&0&-3\\0&1&11&12\\0&0&5&13\\-4&0&0&7\end{bmatrix}$$

Solution

$R_4 \to R_4+2R_1$

$$A=\begin{bmatrix}2&0&0&-3\\0&1&11&12\\0&0&5&13\\0&0&0&1\end{bmatrix}$$

Expansion along last row

$$\begin{aligned}\det(A)&=(-1)^{4+4}(1)\begin{vmatrix}2&0&0\\0&1&11\\0&0&5\end{vmatrix}\\&=\begin{vmatrix}2&0&0\\0&1&11\\0&0&5\end{vmatrix}\end{aligned}$$

Expansion along last row

$$\begin{aligned}\det(A)&=(-1)^{3+3}(5)\begin{vmatrix}2&0\\0&1\end{vmatrix}\\&=5(2)\\&=10\end{aligned}$$

4.1.1.24 Problem section 4.3 number 17

In Problems 17–22, three vectors $\vec{v}_1$,$\vec{v}_2$, and $\vec{v}_3$ are given. If they are linearly independent, show this; otherwise find a

nontrivial linear combination of them that is equal to the zero vector.

$$\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}2\\-3\\4\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}3\\5\\2\end{bmatrix}$$

Solution

The vectors are Linearly independent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

only when $c_1 = c_2 = c_3 = 0$. If we can find at least one $c_i \neq 0$ where the above is true, then the vectors are Linearly dependent.

Writing the above as $A\vec{c} = \vec{0}$ gives

$$\begin{bmatrix}1 & 2 & 3\\0 & -3 & 5\\1 & 4 & 2\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix}1 & 2 & 3 & 0\\0 & -3 & 5 & 0\\1 & 4 & 2 & 0\end{bmatrix}$$

$R_3 \to -R_1 + R_3$ gives

$$\begin{bmatrix}1 & 2 & 3 & 0\\0 & -3 & 5 & 0\\0 & 2 & -1 & 0\end{bmatrix}$$

$R_3 \to R_3, R_2 \to 2R_2$ gives

$$\begin{bmatrix}1 & 2 & 3 & 0\\0 & -6 & 10 & 0\\0 & 6 & -3 & 0\end{bmatrix}$$

$R_3 \to R_2 + R_3$ gives

$$\begin{bmatrix}1 & 2 & 3 & 0\\0 & -6 & 10 & 0\\0 & 0 & 7 & 0\end{bmatrix}$$

Hence the original system (1) in Echelon form becomes

$$\begin{bmatrix}1 & 2 & 3\\0 & -6 & 10\\0 & 0 & 7\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Leading variables are c_1, c_2, c_3. Since there are no free variables, then only the trivial solution exist. We see this by backsubstitution. Last row gives $c_3 = 0$. Second row gives $c_2 = 0$ and first row gives $c_1 = 0$.

Since all $c_i = 0$, then the vectors are Linearly independent.

4.1.1.25 Problem section 4.3 number 18

In Problems 17–22, three vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are given. If they are linearly independent, show this; otherwise find a nontrivial linear combination of them that is equal to the zero vector.

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 4 \\ -5 \\ -6 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

solution

The vectors are Linearly independent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

only when $c_1 = c_2 = c_3 = 0$. If we can find at least one $c_i \neq 0$ where the above is true, then the vectors are Linearly dependent.

Writing the above as $A\vec{c} = \vec{0}$ gives

$$\begin{bmatrix} 2 & 4 & -2 \\ 0 & -5 & 1 \\ -3 & -6 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 0 & -5 & 1 & 0 \\ -3 & -6 & 3 & 0 \end{bmatrix}$$

$R_1 \to 3R_1, R_3 \to 2R_3$ gives

$$\begin{bmatrix} 6 & 12 & -6 & 0 \\ 0 & -5 & 1 & 0 \\ -6 & -12 & 6 & 0 \end{bmatrix}$$

$R_3 \to R_1 + R_3$ gives

$$\begin{bmatrix} 6 & 12 & -6 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the system (1) becomes

$$\begin{bmatrix} 6 & 12 & -6 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The leading variables are c_1, c_2 and free variable is c_3. Since there is a free variable, then the vectors are Linearly dependent. To see this, let $c_3 = t$. From second row $-5c_2 + t = 0$ or $c_2 = \frac{1}{5}t$. From first row $6c_1 + 12c_2 - 6t = 0$. Or $c_1 = \frac{6t - 12\left(\frac{1}{5}t\right)}{6} = \frac{3}{5}t$. Hence

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{5}t \\ \frac{1}{5}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix} = \frac{1}{5}t\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

Taking $\tilde{t} = 5$ the above becomes

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

Therefore we found one solution where

$$\begin{aligned} c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 &= \vec{0} \\ 3\vec{v}_1 + \vec{v}_2 + 5\vec{v}_3 &= \vec{0} \end{aligned}$$

not all c_i zero. Hence linearly dependent vectors.

4.1.1.26 Problem section 4.3 number 19

In Problems 17–22, three vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are given. If they are linearly independent, show this; otherwise find a nontrivial linear combination of them that is equal to the zero vector.

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 4 \\ -2 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

solution

The vectors are Linearly independent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

only when $c_1 = c_2 = c_3 = 0$. If we can find at least one $c_i \neq 0$ where the above is true, then the vectors are Linearly dependent.

Writing the above as $A\vec{c} = \vec{0}$ gives

$$\begin{bmatrix} 2 & 5 & 2 \\ 0 & 4 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 2 & 5 & 2 & 0 \\ 0 & 4 & -1 & 0 \\ 3 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$R_3 \to 2R_3, R_1 \to 3R_1$

$$\begin{bmatrix} 6 & 15 & 6 & 0 \\ 0 & 4 & -1 & 0 \\ 6 & -4 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 6 & 15 & 6 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & -19 & -4 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$R_2 \to 19R_2, R_3 \to 4R_3$

$$\begin{bmatrix} 6 & 15 & 6 & 0 \\ 0 & 76 & -19 & 0 \\ 0 & -76 & -16 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 6 & 15 & 6 & 0 \\ 0 & 76 & -19 & 0 \\ 0 & 0 & -35 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$R_4 \to 76R_4$

$$\begin{bmatrix} 6 & 15 & 6 & 0 \\ 0 & 76 & -19 & 0 \\ 0 & 0 & -35 & 0 \\ 0 & 76 & -76 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$\begin{bmatrix} 6 & 15 & 6 & 0 \\ 0 & 76 & -19 & 0 \\ 0 & 0 & -35 & 0 \\ 0 & 0 & -57 & 0 \end{bmatrix}$$

$R_4 \to R_4 - \frac{57}{35} R_3$

$$\begin{bmatrix} 6 & 15 & 6 & 0 \\ 0 & 76 & -19 & 0 \\ 0 & 0 & -35 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is Echelon form. Hence

$$\begin{bmatrix} 6 & 15 & 6 \\ 0 & 76 & -19 \\ 0 & 0 & -35 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Lead variables are c_1, c_2, c_3. There are no free variables. Therefore unique solution exist and is $c_1 = 0, c_2 = 0, c_3 = 0$. Hence the vectors are linearly independent.

4.1.1.27 Problem section 4.3 number 20

In Problems 17–22, three vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are given. If they are linearly independent, show this; otherwise find a nontrivial linear combination of them that is equal to the zero vector.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix}$$

solution

The vectors are Linearly independent if

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

only when $c_1 = c_2 = c_3 = 0$. If we can find at least one $c_i \neq 0$ where the above is true, then the vectors are Linearly dependent.

Writing the above as $A\vec{c} = \vec{0}$ gives

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ -1 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 3 & 7 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 3 & 7 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is Echelon form. Hence

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Lead variables are c_1, c_2, c_3. There are no free variables. Therefore a unique solution exists and is the trivial solution $c_1 = 0, c_2 = 0, c_3 = 0$. Hence the vectors are linearly independent.

4.1.1.28 Problem section 4.3 number 21

In Problems 17–22, three vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are given. If they are linearly independent, show this; otherwise find a nontrivial linear combination of them that is equal to the zero vector.

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

solution

The vectors are Linearly independent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

only when $c_1 = c_2 = c_3 = 0$. If we can find at least one $c_i \neq 0$ where the above is true, then the vectors are Linearly dependent.

Writing the above as $A\vec{c} = \vec{0}$ gives

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 3 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

$R_3 \to 3R_3 - R_1$

$$\begin{bmatrix} 3 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

$R_1 \to 2R_1, R_4 \to 3R_4$

$$\begin{bmatrix} 6 & 2 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 6 & 3 & 0 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_1$

$$\begin{bmatrix} 6 & 2 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 6 & 2 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

$R_4 \to R_4 + R_2$

$$\begin{bmatrix} 6 & 2 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is Echelon form. Hence

$$\begin{bmatrix} 6 & 2 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Lead variables are c_1, c_2. And free variable is c_3. Since there is a free variable, then non-trivial solution exist. Hence the vectors are linearly dependent. Let $c_3 = t$. From second row

$$\begin{aligned} -c_2 + 2c_3 &= 0 \\ c_2 &= 2c_3 = 2t \end{aligned}$$

From first row

$$\begin{aligned} 6c_1 + 2c_2 + 2c_3 &= 0 \\ c_1 &= \frac{-2c_2 - 2c_3}{6} \\ &= \frac{-2(2t) - 2t}{6} \\ &= -t \end{aligned}$$

Hence solution is

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}-t\\2t\\t\end{bmatrix}$$
$$=t\begin{bmatrix}-1\\2\\1\end{bmatrix}$$

Let $t=1$

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}-1\\2\\1\end{bmatrix}$$

Therefore

$$c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3=\vec{0}$$
$$-\vec{v}_1+2\vec{v}_2+\vec{v}_3=\vec{0}$$

4.1.1.29 Problem section 4.3 number 22

In Problems 17–22, three vectors $\vec{v}_1,\vec{v}_2$, and $\vec{v}_3$ are given. If they are linearly independent, show this; otherwise find a nontrivial linear combination of them that is equal to the zero vector.

$$\vec{v}_1=\begin{bmatrix}3\\9\\0\\5\end{bmatrix},\vec{v}_2=\begin{bmatrix}3\\0\\9\\-7\end{bmatrix},\vec{v}_3=\begin{bmatrix}4\\7\\5\\0\end{bmatrix}$$

solution

The vectors are Linearly independent if

$$c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3=\vec{0}$$

only when $c_1=c_2=c_3=0$. If we can find at least one $c_i\neq 0$ where the above is true, then the vectors are Linearly dependent.

Writing the above as $A\vec{c}=\vec{0}$ gives

$$\begin{bmatrix}3&3&4\\9&0&7\\0&9&5\\5&-7&0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\tag{1}$$

The augmented matrix is

$$\begin{bmatrix}3&3&4&0\\9&0&7&0\\0&9&5&0\\5&-7&0&0\end{bmatrix}$$

$R_2\rightarrow R_2-3R_1$

$$\begin{bmatrix}3&3&4&0\\0&-9&-5&0\\0&9&5&0\\5&-7&0&0\end{bmatrix}$$

$R_1\rightarrow 5R_1,R_4\rightarrow 3R_4$

$$\begin{bmatrix}15&15&20&0\\0&-9&-5&0\\0&9&5&0\\15&-21&0&0\end{bmatrix}$$

$R_4 \to R_4 - R_1$

$$\begin{bmatrix} 15 & 15 & 20 & 0 \\ 0 & -9 & -5 & 0 \\ 0 & 9 & 5 & 0 \\ 0 & -36 & -20 & 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 15 & 15 & 20 & 0 \\ 0 & -9 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -36 & -20 & 0 \end{bmatrix}$$

$R_4 \to R_4 - 4R_2$

$$\begin{bmatrix} 15 & 15 & 20 & 0 \\ 0 & -9 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is Echelon form. Hence

$$\begin{bmatrix} 15 & 15 & 20 \\ 0 & -9 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Lead variables are c_1, c_2. And free variable is c_3. Since there is a free variable, then non-trivial solution exist. Hence the vectors are linearly dependent. Let $c_3 = t$. From second row

$$\begin{aligned} -9c_2 - 5c_3 &= 0 \\ c_2 &= -\frac{5}{9}c_3 \\ &= -\frac{5}{9}t \end{aligned}$$

From first row

$$\begin{aligned} 15c_1 + 15c_2 + 20c_3 &= 0 \\ c_1 &= \frac{-15c_2 - 20c_3}{15} \\ &= \frac{-15\left(-\frac{5}{9}t\right) - 20t}{15} \\ &= -\frac{7}{9}t \end{aligned}$$

Hence solution is

$$\begin{aligned} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} &= \begin{bmatrix} -\frac{7}{9}t \\ -\frac{5}{9}t \\ t \end{bmatrix} \\ &= t\begin{bmatrix} -\frac{7}{9} \\ -\frac{5}{9} \\ 1 \end{bmatrix} \\ &= \frac{1}{9}t\begin{bmatrix} -7 \\ -5 \\ 9 \end{bmatrix} \end{aligned}$$

Let $t=-9$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ -9 \end{bmatrix}$$

Therefore

$$\begin{aligned} c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 &= \vec{0} \\ 7\vec{v}_1 + 5\vec{v}_2 - 9\vec{v}_3 &= \vec{0} \end{aligned}$$

4.1.1.30 Problem section 4.4 number 15

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ 2x_1 - 3x_2 - x_3 &= 0 \end{aligned}$$

<u>Solution</u>

$A\vec{x} = \vec{0}$ gives

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & -3 & -1 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & -7 & 0 \end{bmatrix}$$

Hence the leading variables are x_1, x_2 and the free variable is $x_3 = t$. The system becomes

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From second row

$$\begin{aligned} x_2 - 7x_3 &= 0 \\ x_2 &= 7x_3 \\ &= 7t \end{aligned}$$

From first row

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ x_1 &= 2x_2 - 3x_3 \\ &= 2(7t) - 3t \\ &= 11t \end{aligned}$$

Therefore the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11t \\ 7t \\ t \end{bmatrix} = t \begin{bmatrix} 11 \\ 7 \\ 1 \end{bmatrix}$$

Let $t=1$, the basis is

$$\begin{bmatrix} 11 \\ 7 \\ 1 \end{bmatrix}$$

A one dimensional subspace.

4.1.1.31 Problem section 4.4 number 16

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$\begin{aligned} x_1+3x_2+4x_3&=0\\ 3x_1+8x_2+7x_3&=0 \end{aligned}$$

solution

$A\vec{x}=\vec{0}$ gives

$$\begin{bmatrix} 1 & 3 & 4 \\ 3 & 8 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & 4 & 0 \\ 3 & 8 & 7 & 0 \end{bmatrix}$$

$R_2 \to -3R_1+R_2$ gives

$$\begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix}$$

Hence the leading variables are x_1, x_2 and the free variable is $x_3=t$. The system becomes

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Last row gives $-x_2-5x_3=0$ or $-x_2=5t$. Hence $x_2=-5t$. From first row, $x_1+3x_2+4x_3=0$, or $x_1=-3x_2-4x_3$ or $x_1=-3(-5t)-4t=11t$. Therefore the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 11t \\ -5t \\ t \end{bmatrix}=t\begin{bmatrix} 11 \\ -5 \\ 1 \end{bmatrix}$$

Let $t=1$. The basis is

$$\begin{bmatrix} 11 \\ -5 \\ 1 \end{bmatrix}$$

A one dimensional subspace.

4.1.1.32 Problem section 4.4 number 17

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$\begin{aligned} x_1-3x_2+2x_3-4x_4&=0\\ 2x_1-5x_2+7x_3-3x_4&=0 \end{aligned}$$

solution

$A\vec{x} = \vec{0}$ gives

$$\begin{bmatrix} 1 & -3 & 2 & -4 \\ 2 & -5 & 7 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -3 & 2 & -4 & 0 \\ 2 & -5 & 7 & -3 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & -3 & 2 & -4 & 0 \\ 0 & 1 & 3 & 5 & 0 \end{bmatrix}$$

Leading variables are x_1, x_2 Free variables are $x_3 = t, x_4 = s$. The system becomes

$$\begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Second row gives

$$\begin{aligned} x_2 + 3x_3 + 5x_4 &= 0 \\ x_2 &= -3x_3 - 5x_4 \\ &= -3t - 5s \end{aligned}$$

First row gives

$$\begin{aligned} x_1 - 3x_2 + 2x_3 - 4x_4 &= 0 \\ x_1 &= 3x_2 - 2x_3 + 4x_4 \\ &= 3(-3t - 5s) - 2t + 4s \\ &= -11s - 11t \end{aligned}$$

Hence the solution is

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} -11s - 11t \\ -3t - 5s \\ t \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -11 \\ -3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -11 \\ -5 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Let $t = 1, s = 1$, the basis vectors are

$$\left\{ \begin{bmatrix} -11 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -11 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

A two dimensional subspace.

4.1.1.33 Problem section 4.4 number 18

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$\begin{aligned} x_1 + 3x_2 + 4x_3 + 5x_4 &= 0 \\ 2x_1 + 6x_2 + 9x_3 + 5x_4 &= 0 \end{aligned}$$

solution

$A\vec{x} = \vec{0}$ gives

$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 6 & 9 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & 4 & 5 & 0 \\ 2 & 6 & 9 & 5 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 3 & 4 & 5 & 0 \\ 0 & 0 & 1 & -5 & 0 \end{bmatrix}$$

Leading variables are x_1, x_3 Free variables are $x_2 = t, x_4 = s$. The system becomes

$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Second row gives

$$\begin{aligned} x_3 - 5x_4 &= 0 \\ x_3 &= 5x_4 \\ &= 5s \end{aligned}$$

First row gives

$$\begin{aligned} x_1 + 3x_2 + 4x_3 + 5x_4 &= 0 \\ x_1 &= -3x_2 - 4x_3 - 5x_4 \\ &= -3t - 4(5s) - 5s \\ &= -25s - 3t \end{aligned}$$

Hence the solution is

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} -25s - 3t \\ t \\ 5s \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -25 \\ 0 \\ 5 \\ 1 \end{bmatrix} \end{aligned}$$

Let $t = 1, s = 1$, the basis vectors are

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -25 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

4.1.1.34 Problem section 4.4 number 19

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$\begin{aligned} x_1 - 3x_2 - 9x_3 - 5x_4 &= 0 \\ 2x_1 + x_2 - 4x_3 + 11x_4 &= 0 \\ x_1 + 3x_2 + 3x_3 + 13x_4 &= 0 \end{aligned}$$

solution

$A\vec{x}=\vec{0}$ gives

$$\begin{bmatrix}1 & -3 & -9 & -5\\ 2 & 1 & -4 & 11\\ 1 & 3 & 3 & 13\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix}1 & -3 & -9 & -5 & 0\\ 2 & 1 & -4 & 11 & 0\\ 1 & 3 & 3 & 13 & 0\end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix}1 & -3 & -9 & -5 & 0\\ 0 & 7 & 14 & 21 & 0\\ 1 & 3 & 3 & 13 & 0\end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix}1 & -3 & -9 & -5 & 0\\ 0 & 7 & 14 & 21 & 0\\ 0 & 6 & 12 & 18 & 0\end{bmatrix}$$

$R_3 \to R_3 - \frac{6}{7}R_2$

$$\begin{bmatrix}1 & -3 & -9 & -5 & 0\\ 0 & 7 & 14 & 21 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Leading variables are x_1, x_2 Free variable is $x_3 = t, x_4 = s$. The system becomes

$$\begin{bmatrix}1 & -3 & -9 & -5\\ 0 & 7 & 14 & 21\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

second row gives $7x_2 + 14x_3 + 21x_4 = 0$ or $x_2 = \frac{-14x_3-21x_4}{7} = \frac{-14t-21s}{7} = -3s - 2t$. First row gives $x_1 - 3x_2 - 9x_3 - 5x_4 = 0$ or $x_1 = 3x_2 + 9x_3 + 5x_4$ or $x_1 = 3(-3s - 2t) + 9t + 5s = 3t - 4s$. Hence the solution is

$$\begin{aligned}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} &= \begin{bmatrix}3t-4s\\ -3s-2t\\ t\\ s\end{bmatrix}\\ &= t\begin{bmatrix}3\\ -2\\ 1\\ 0\end{bmatrix}+s\begin{bmatrix}-4\\ -3\\ 0\\ 1\end{bmatrix}\end{aligned}$$

Let $t = 1, s = 1$, hence the basis are

$$\left\{\begin{bmatrix}3\\ -2\\ 1\\ 0\end{bmatrix},\begin{bmatrix}-4\\ -3\\ 0\\ 1\end{bmatrix}\right\}$$

A two dimensional subspace.

4.1.1.35 Problem section 4.4 number 20

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$
\begin{aligned}
x_1 - 3x_2 - 10x_3 + 5x_4 &= 0\\
x_1 + 4x_2 + 11x_3 - 2x_4 &= 0\\
x_1 + 3x_2 + 8x_3 - x_4 &= 0
\end{aligned}
$$

<u>solution</u>

$A\vec{x} = \vec{0}$ gives

$$\begin{bmatrix}1 & -3 & -10 & 5\\ 1 & 4 & 11 & -2\\ 1 & 3 & 8 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix}1 & -3 & -10 & 5 & 0\\ 1 & 4 & 11 & -2 & 0\\ 1 & 3 & 8 & -1 & 0\end{bmatrix}$$

$R_2 \to -R_1 + R_2$ gives

$$\begin{bmatrix}1 & -3 & -10 & 5 & 0\\ 0 & 7 & 21 & -7 & 0\\ 1 & 3 & 8 & -1 & 0\end{bmatrix}$$

$R_3 \to -R_1 + R_3$ gives

$$\begin{bmatrix}1 & -3 & -10 & 5 & 0\\ 0 & 7 & 21 & -7 & 0\\ 0 & 6 & 18 & -6 & 0\end{bmatrix}$$

$R_3 \to 7R_3$ and $R_2 \to 6R_2$ gives

$$\begin{bmatrix}1 & -3 & -10 & 5 & 0\\ 0 & 42 & 126 & -42 & 0\\ 0 & 42 & 126 & -42 & 0\end{bmatrix}$$

$R_3 \to -R_2 + R_3$ gives

$$\begin{bmatrix}1 & -3 & -10 & 5 & 0\\ 0 & 42 & 126 & -42 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Leading variables are x_1, x_2 Free variables are $x_3 = t, x_4 = s$. The system becomes

$$\begin{bmatrix}1 & -3 & -10 & 5\\ 0 & 42 & 126 & -42\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

second row gives $42x_2 + 126x_3 - 42x_4 = 0$ or $42x_2 = -126t + 42s$ or $x_2 = -\frac{126}{42}t + \frac{42}{42}s = -3t + s$. First row gives $x_1 - 3x_2 - 10x_3 + 5x_4 = 0$ or $x_1 = 3x_2 + 10x_3 - 5x_4$ or $x_1 = 3(-3t + s) + 10t - 5s = t - 2s$. Hence the solution is

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}t-2s\\-3t+s\\t\\s\end{bmatrix} = t\begin{bmatrix}1\\-3\\1\\0\end{bmatrix} + s\begin{bmatrix}-2\\1\\0\\1\end{bmatrix}$$

Let $t = 1, s = 1$. The basis are

$$\begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

A two dimensional subspace.

4.1.1.36 Problem section 4.4 number 21

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$\begin{aligned} x_1 - 4x_2 - 3x_3 - 7x_4 &= 0 \\ 2x_1 - x_2 + x_3 + 7x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 11x_4 &= 0 \end{aligned}$$

solution

$A\vec{x} = \vec{0}$ gives

$$\begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -4 & -3 & -7 & 0 \\ 2 & -1 & 1 & 7 & 0 \\ 1 & 2 & 3 & 11 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & -4 & -3 & -7 & 0 \\ 0 & 7 & 7 & 21 & 0 \\ 1 & 2 & 3 & 11 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -4 & -3 & -7 & 0 \\ 0 & 7 & 7 & 21 & 0 \\ 0 & 6 & 6 & 18 & 0 \end{bmatrix}$$

$R_4 \to R_4 - \frac{6}{7} R_2$

$$\begin{bmatrix} 1 & -4 & -3 & -7 & 0 \\ 0 & 7 & 7 & 21 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading variables are x_1, x_2 Free variables are $x_3 = t, x_4 = s$. The system becomes

$$\begin{bmatrix} 1 & -4 & -3 & -7 \\ 0 & 7 & 7 & 21 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Second row gives $7x_2 + 7x_3 + 21x_4 = 0$ or $x_2 = \frac{-7x_3 - 21x_4}{7} = \frac{-7t - 21s}{7} = -3s - t$. First row gives

$x_1 - 4x_2 - 3x_3 - 7x_4 = 0$ or $x_1 = 4x_2 + 3x_3 + 7x_4 = 4(-3s - t) + 3t + 7s = -5s - t$. Hence solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5s - t \\ -3s - t \\ t \\ s \end{bmatrix}$$
$$= t\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -5 \\ -3 \\ 0 \\ s \end{bmatrix}$$

Let $t = 1, s = 2$, the Basis are

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

A two dimensional subspace.

4.1.1.37 Problem section 4.4 number 22

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$\begin{aligned} x_1 - 2x_2 - 3x_3 - 16x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + 17x_4 &= 0 \\ x_1 - 2x_2 + 3x_3 + 26x_4 &= 0 \end{aligned}$$

solution

$A\vec{x} = \vec{0}$ gives

$$\begin{bmatrix} 1 & -2 & -3 & -16 \\ 2 & -4 & 1 & 17 \\ 1 & -2 & 3 & 26 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -2 & -3 & -16 & 0 \\ 2 & -4 & 1 & 17 & 0 \\ 1 & -2 & 3 & 26 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & -2 & -3 & -16 & 0 \\ 0 & 0 & 7 & 49 & 0 \\ 1 & -2 & 3 & 26 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & -2 & -3 & -16 & 0 \\ 0 & 0 & 7 & 49 & 0 \\ 0 & 0 & 6 & 42 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - \frac{6}{7}R_2$

$$\begin{bmatrix} 1 & -2 & -3 & -16 & 0 \\ 0 & 0 & 7 & 49 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence leading variables are x_1, x_3 and free variables are $x_2 = t, x_4 = s$.The system becomes

$$\begin{bmatrix} 1 & -2 & -3 & -16 \\ 0 & 0 & 7 & 49 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Second row gives $7x_3 + 49x_4 = 0$ or $x_3 = -7s$. First row gives $x_1 - 2x_2 - 3x_3 - 16x_4 = 0$ or $x_1 = 2x_2 + 3x_3 + 16x_4$ or $x_1 = 2t + 3(-7s) + 16s = 2t - 5s$. Hence solution is

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} 2t - 5s \\ t \\ -7s \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -5 \\ 0 \\ -7 \\ 1 \end{bmatrix} \end{aligned}$$

Let $t = 1, s = 1$, therefore the basis are

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -7 \\ 1 \end{bmatrix} \right\}$$

A two dimensional subspace.

4.1.1.38 Problem section 4.4 number 23

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$\begin{aligned} x_1 + 5x_2 + 13x_3 + 14x_4 &= 0 \\ 2x_1 + 5x_2 + 11x_3 + 12x_4 &= 0 \\ 2x_1 + 7x_2 + 17x_3 + 19x_4 &= 0 \end{aligned}$$

solution

$A\vec{x} = \vec{0}$ gives

$$\begin{bmatrix} 1 & 5 & 13 & 14 \\ 2 & 5 & 11 & 12 \\ 2 & 7 & 17 & 19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 5 & 13 & 14 & 0 \\ 2 & 5 & 11 & 12 & 0 \\ 2 & 7 & 17 & 19 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 5 & 13 & 14 & 0 \\ 0 & -5 & -15 & -16 & 0 \\ 2 & 7 & 17 & 19 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 5 & 13 & 14 & 0 \\ 0 & -5 & -15 & -16 & 0 \\ 0 & -3 & -9 & -9 & 0 \end{bmatrix}$$

$R_3 \to R_3 - \frac{3}{5}R_2$

$$\begin{bmatrix} 1 & 5 & 13 & 14 & 0 \\ 0 & -5 & -15 & -16 & 0 \\ 0 & 0 & 0 & \frac{3}{5} & 0 \end{bmatrix}$$

Hence leading variables are x_1, x_2, x_4 and free variables are $x_3 = t$.The system becomes

$$\begin{bmatrix} 1 & 5 & 13 & 14 \\ 0 & -5 & -15 & -16 \\ 0 & 0 & 0 & \frac{3}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Last equation gives $x_4 = 0$. Second equation gives $-5x_2 - 15x_3 = 0$, or $x_2 = -3x_3 = -3t$. First equation gives $x_1 = -5x_2 - 13x_3$ or $x_1 = -5(-3t) - 13t = 2t$. Hence solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2t \\ -3t \\ t \\ 0 \end{bmatrix}$$
$$= t \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$,therefore the basis is

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

A one dimensional subspace.

4.1.1.39 Problem section 4.4 number 24

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$\begin{aligned} x_1 + 3x_2 - 4x_3 - 8x_4 + 6x_5 &= 0 \\ x_1 + 2x_3 + x_4 + 3x_5 &= 0 \\ 2x_1 + 7x_2 - 10x_3 - 19x_4 + 13x_5 &= 0 \end{aligned}$$

<u>solution</u>

$A\vec{x} = \vec{0}$ gives

$$\begin{bmatrix} 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \\ 2 & 7 & -10 & -19 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & -4 & -8 & 6 & 0 \\ 1 & 0 & 2 & 1 & 3 & 0 \\ 2 & 7 & -10 & -19 & 13 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 3 & -4 & -8 & 6 & 0 \\ 0 & -3 & 6 & 9 & -3 & 0 \\ 2 & 7 & -10 & -19 & 13 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 3 & -4 & -8 & 6 & 0 \\ 0 & -3 & 6 & 9 & -3 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 \end{bmatrix}$$

$R_3 \rightarrow 3R_3 + R_2$

$$\begin{bmatrix} 1 & 3 & -4 & -8 & 6 & 0 \\ 0 & -3 & 6 & 9 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence leading variables are x_1, x_3 and free variables are $x_3 = t, x_4 = s, x_5 = r$. The system becomes

$$\begin{bmatrix} 1 & 3 & -4 & -8 & 6 \\ 0 & -3 & 6 & 9 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ t \\ s \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Second equation gives

$$\begin{aligned} -3x_2 + 6t + 9s - 3r = 0 \\ x_2 &= \frac{-6t - 9s + 3r}{-3} \\ &= 3s - r + 2t \end{aligned}$$

First equation gives

$$\begin{aligned} x_1 + 3x_2 - 4t - 8s + 6r = 0 \\ x_1 &= -3x_2 + 4t + 8s - 6r \\ &= -3(3s - r + 2t) + 4t + 8s - 6r \\ &= -3r - s - 2t \end{aligned}$$

Hence solution is

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} &= \begin{bmatrix} -3r - s - 2t \\ 3s - r + 2t \\ t \\ s \\ r \end{bmatrix} \\ &= t \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Let $t = 1, s = 1, r = 1$, then the basis are

$$\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

A three dimensional subspace.

4.1.1.40 Problem section 4.5 number 1

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$A=\begin{bmatrix}1&2&3\\1&5&-9\\2&5&2\end{bmatrix}$$

solution

We start by converting the matrix to reduced Echelon form.

$R_2 \to R_2 - R_1$

$$\begin{bmatrix}1&2&3\\0&3&-12\\2&5&2\end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix}1&2&3\\0&3&-12\\0&1&-4\end{bmatrix}$$

$R_3 \to 3R_3 - R_2$

$$\begin{bmatrix}1&2&3\\0&3&-12\\0&0&0\end{bmatrix}$$

Now to start the reduce Echelon form phase. Notice that this is not needed. But if done, the row space basis found will be same each time. If we stop here, the row space basis can look different depending on the reduction was done. But both will work.

The pivots all needs to be 1.

$R_2 \to \frac{1}{3}R_2$

$$\begin{bmatrix}1&2&3\\0&1&-4\\0&0&0\end{bmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix}1&0&11\\0&1&-4\\0&0&0\end{bmatrix}$$

The above is in reduced Echelon form. The pivot columns are $1,2$. The non-zero rows are rows $1,2$,. Hence row space basis are first and second rows (I prefer to show all basis as column vectors, instead of row vectors. This just makes it easier to read them).

$$\left\{\begin{bmatrix}1\\0\\11\end{bmatrix},\begin{bmatrix}0\\1\\-4\end{bmatrix}\right\}$$

The dimension is 2. The column space correspond to pivot columns in original A. These are columns $1,2$. Hence basis for column space are

$$\left\{\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}2\\5\\5\end{bmatrix}\right\}$$

The dimension is 2. We notice that the dimension of the row space and the column space is equal as expected. (This is called the rank of A. Hence $\text{rank}(A)=2$.)

The Null space of A has dimension 1, since there is only one free variable (x_3). We see that the number of columns of A (which is 2) is therefore the sum of column space dimension (or the rank) and the null space dimension as expected.

4.1.1.41 Problem section 4.5 number 2

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$A=\begin{bmatrix}5&2&4\\2&1&1\\4&1&5\end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_1\to 2R_1, R_2\to 5R_2$

$$\begin{bmatrix}10&4&8\\10&5&5\\4&1&5\end{bmatrix}$$

$R_2\to R_2-R_1$

$$\begin{bmatrix}10&4&8\\0&1&-3\\4&1&5\end{bmatrix}$$

$R_1\to 4R_1, R_3\to 10R_3$

$$\begin{bmatrix}40&16&32\\0&1&-3\\40&10&50\end{bmatrix}$$

$R_3\to R_3-R_1$

$$\begin{bmatrix}40&16&32\\0&1&-3\\0&-6&18\end{bmatrix}$$

$R_3\to R_3+6R_2$

$$\begin{bmatrix}40&16&32\\0&1&-3\\0&0&0\end{bmatrix}$$

The above is in Echelon form. The pivot columns are $1,2$. The non-zero rows are rows $1,2,$. Hence row space basis are first and second rows

$$\left\{\begin{bmatrix}40\\16\\32\end{bmatrix},\begin{bmatrix}0\\1\\-3\end{bmatrix}\right\}$$

The dimension is 2. The column space correspond to pivot columns in original A. These are columns $1,2$. Hence basis for column space are

$$\left\{\begin{bmatrix}5\\2\\4\end{bmatrix},\begin{bmatrix}2\\1\\1\end{bmatrix}\right\}$$

4.1.1.42 Problem section 4.5 number 3

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$A=\begin{bmatrix}1&-4&-3&-7\\2&-1&1&7\\1&2&3&11\end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & -4 & -3 & -7 \\ 0 & 7 & 7 & 21 \\ 1 & 2 & 3 & 11 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -4 & -3 & -7 \\ 0 & 7 & 7 & 21 \\ 0 & 6 & 6 & 18 \end{bmatrix}$$

$R_3 \to R_3 - \frac{6}{7}R_2$

$$\begin{bmatrix} 1 & -4 & -3 & -7 \\ 0 & 7 & 7 & 21 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are $1, 2$. The non-zero rows are rows $1, 2,$. Hence row space basis are first and second rows

$$\left\{ \begin{bmatrix} 1 \\ -4 \\ -3 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 7 \\ 21 \end{bmatrix} \right\}$$

The dimension is 2. The column space correspond to pivot columns in original A. These are columns $1, 2$. Hence basis for column space are

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix} \right\}$$

4.1.1.43 Problem section 4.5 number 4

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$A = \begin{bmatrix} 1 & -3 & -9 & -5 \\ 2 & 1 & 4 & 11 \\ 1 & 3 & 3 & 13 \end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & -3 & -9 & -5 \\ 0 & 7 & 22 & 21 \\ 1 & 3 & 3 & 13 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -3 & -9 & -5 \\ 0 & 7 & 22 & 21 \\ 0 & 6 & 12 & 18 \end{bmatrix}$$

$R_3 \to R_3 - \frac{6}{7}R_2$

$$\begin{bmatrix} 1 & -3 & -9 & -5 \\ 0 & 7 & 22 & 21 \\ 0 & 0 & -\frac{48}{7} & 0 \end{bmatrix}$$

The pivot columns are $1, 2, 3$. The non-zero rows are rows $1, 2, 3$. Hence row space basis are

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ -9 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 22 \\ 21 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -\frac{48}{7} \\ 0 \end{bmatrix} \right\}$$

The dimension is 3. The column space correspond to pivot columns in original A. These are columns $1, 2, 3$. Hence basis for column space are

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -9 \\ 4 \\ 3 \end{bmatrix} \right\}$$

4.1.1.44 Problem section 4.5 number *5*

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -3 & 4 \\ 2 & 5 & 11 & 12 \end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_2 \to -3R_1 + R_2$ gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -6 & 3 \\ 2 & 5 & 11 & 12 \end{bmatrix}$$

$R_3 \to -2R_1 + R_3$ gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -6 & 3 \\ 0 & 3 & 9 & 10 \end{bmatrix}$$

$R_2 \to 3R_2$ and $R_3 \to 2R_3$ gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -6 & -18 & 9 \\ 0 & 6 & 18 & 20 \end{bmatrix}$$

$R_3 \to R_2 + R_3$ gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -6 & -18 & 9 \\ 0 & 0 & 0 & 29 \end{bmatrix}$$

The above is now in Echelon form. Now we can answer the question. The basis for the row space are all the rows which are not zero. Hence row space basis are (I prefer to show all basis as column vectors, instead of row vectors. This just makes it easier to read them).

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -6 \\ -18 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 29 \end{bmatrix}$$

The dimension is 3. The column space correspond to pivot columns in original A. These are column $1,2,4$. Hence basis for column space are

$$\begin{bmatrix}1\\3\\2\end{bmatrix},\begin{bmatrix}1\\1\\5\end{bmatrix},\begin{bmatrix}1\\4\\12\end{bmatrix}$$

The dimension is 3. We notice that the dimension of the row space and the column space is equal as expected. (This is called the rank of A. Hence $\operatorname{rank}(A)=3$.)

The Null space of A has dimension 1, since there is only one free variable (x_3). We see that the number of columns of A (which is 4) is therefore the sum of column space dimension (or the rank) and the null space dimension as expected.

4.1.1.45 Problem section 4.5 number 6

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix}1&4&9&2\\2&2&6&-3\\2&7&16&3\end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix}1&4&9&2\\0&-6&-12&-7\\2&7&16&3\end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix}1&4&9&2\\0&-6&-12&-7\\0&-1&-2&-1\end{bmatrix}$$

$R_3 \to R_3 - \frac{1}{6}R_2$

$$\begin{bmatrix}1&4&9&2\\0&-6&-12&-7\\0&0&0&\frac{1}{6}\end{bmatrix}$$

The above is in Echelon form. The pivot columns are $1,2,4$. The non-zero rows are rows $1,2,3$. Hence row space basis are

$$\left\{\begin{bmatrix}1\\4\\9\\2\end{bmatrix},\begin{bmatrix}0\\-6\\-12\\-7\end{bmatrix},\begin{bmatrix}0\\0\\0\\\frac{1}{6}\end{bmatrix}\right\}$$

The dimension is 3. The column space correspond to pivot columns in original A. These are columns $1,2,4$. Hence basis for column space are

$$\left\{\begin{bmatrix}1\\2\\2\end{bmatrix},\begin{bmatrix}4\\2\\7\end{bmatrix},\begin{bmatrix}2\\-3\\3\end{bmatrix}\right\}$$

4.1.1.46 Problem section 4.5 number 7

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 1 & 4 & 5 & 16 \\ 1 & 3 & 3 & 13 \\ 2 & 5 & 4 & 23 \end{bmatrix}$$

solution

We start by converting the matrix to reduced Echelon form.

$R_2 \to -R_1 + R_2$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 3 & 6 & 9 \\ 1 & 3 & 3 & 13 \\ 2 & 5 & 4 & 23 \end{bmatrix}$$

$R_3 \to -R_1 + R_3$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 3 & 6 & 9 \\ 0 & 2 & 4 & 6 \\ 2 & 5 & 4 & 23 \end{bmatrix}$$

$R_4 \to -2R_1 + R_4$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 3 & 6 & 9 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix}$$

$R_2 \to 2R_2$ and $R_3 \to 3R_3$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 6 & 12 & 18 \\ 0 & 6 & 12 & 18 \\ 0 & 3 & 6 & 9 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 6 & 12 & 18 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 9 \end{bmatrix}$$

$R_4 \to -\frac{1}{2}R_2 + R_4$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 6 & 12 & 18 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot (leading) columns are $1, 2$ and free variables go with $3, 4$ columns. The Null space of A is therefore have dimension 2. The above is reduced Echelon form. The basis for the row space are all the rows which are not zero. Hence row space basis are (dimension 2)

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 12 \\ 18 \end{bmatrix} \right\}$$

The column space correspond to pivot columns in original A. These are columns 1,2. Hence basis for column space are (dimension 2)

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \\ 5 \end{bmatrix} \right\}$$

We notice that the dimension of the row space and the column space is equal as expected.

The Null space of A has dimension 2, since there is two free variables. We see that the number of columns of A (which is 4) is therefore the sum of column space dimension and the null space dimension as expected.

4.1.1.47 Problem section 4.5 number 8

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix} 1 & -2 & -3 & -5 \\ 1 & 4 & 9 & 2 \\ 1 & 3 & 7 & 1 \\ 2 & 2 & 6 & -3 \end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & -2 & -3 & -5 \\ 0 & 6 & 12 & 7 \\ 1 & 3 & 7 & 1 \\ 2 & 2 & 6 & -3 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -2 & -3 & -5 \\ 0 & 6 & 12 & 7 \\ 0 & 5 & 10 & 6 \\ 2 & 2 & 6 & -3 \end{bmatrix}$$

$R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & -2 & -3 & -5 \\ 0 & 6 & 12 & 7 \\ 0 & 5 & 10 & 6 \\ 0 & 6 & 12 & 7 \end{bmatrix}$$

$R_2 \to 5R_2, R_3 \to 6R_3$

$$\begin{bmatrix} 1 & -2 & -3 & -5 \\ 0 & 30 & 60 & 35 \\ 0 & 30 & 60 & 36 \\ 0 & 6 & 12 & 7 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & -2 & -3 & -5 \\ 0 & 30 & 60 & 35 \\ 0 & 0 & 0 & 1 \\ 0 & 6 & 12 & 7 \end{bmatrix}$$

$R_4 \rightarrow R_4 - \frac{6}{30}R_2$

$$\begin{bmatrix} 1 & -2 & -3 & -5 \\ 0 & 30 & 60 & 35 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is in Echelon form. The pivot columns are $1, 2, 4$. The non-zero rows are rows $1, 2, 3$. Hence row space basis are

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 30 \\ 60 \\ 35 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dimension is 3. The column space correspond to pivot columns in original A. These are columns $1, 2, 4$. Hence basis for column space are

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 1 \\ -3 \end{bmatrix} \right\}$$

4.1.1.48 Problem section 4.5 number 9

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix} 1 & 3 & 3 & 9 \\ 2 & 7 & 4 & 8 \\ 2 & 7 & 5 & 12 \\ 2 & 8 & 3 & 12 \end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_2 \rightarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 3 & 3 & 9 \\ 0 & 1 & -2 & -10 \\ 2 & 7 & 5 & 12 \\ 2 & 8 & 3 & 12 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 3 & 3 & 9 \\ 0 & 1 & -2 & -10 \\ 0 & 1 & -1 & -6 \\ 2 & 8 & 3 & 12 \end{bmatrix}$$

$R_4 \rightarrow R_4 - 2R_1$

$$\begin{bmatrix} 1 & 3 & 3 & 9 \\ 0 & 1 & -2 & -10 \\ 0 & 1 & -1 & -6 \\ 0 & 2 & -3 & -6 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 3 & 3 & 9 \\ 0 & 1 & -2 & -10 \\ 0 & 0 & 1 & 4 \\ 0 & 2 & -3 & -6 \end{bmatrix}$$

$R_4 \to R_4 - 2R_2$

$$\begin{bmatrix} 1 & 3 & 3 & 9 \\ 0 & 1 & -2 & -10 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 14 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 3 & 3 & 9 \\ 0 & 1 & -2 & -10 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

The above is in Echelon form. The pivot columns are $1,2,3,4$. The non-zero rows are rows $1,2,3,4$. Hence row space basis are

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -20 \\ -10 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix} \right\}$$

The dimension is 4. The column space correspond to pivot columns in original A. These are columns $1,2,3,4$. Hence basis for column space are

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ 8 \\ 12 \\ 12 \end{bmatrix} \right\}$$

4.1.1.49 Problem section 4.5 number 10

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 1 & 3 & 4 & 3 & 6 \\ 2 & 2 & 4 & 3 & 5 \\ 2 & 1 & 3 & 2 & 3 \end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 1 & 2 & 3 \\ 2 & 2 & 4 & 3 & 5 \\ 2 & 1 & 3 & 2 & 3 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & -2 & -2 & 1 & -1 \\ 2 & 1 & 3 & 2 & 3 \end{bmatrix}$$

$R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & -2 & -2 & 1 & -1 \\ 0 & -3 & -3 & 0 & -3 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & -3 & -3 & 0 & -3 \end{bmatrix}$$

$R_4 \to R_4 + 3R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 6 & 6 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is in Echelon form. The pivot columns are $1, 2, 4$. The non-zero rows are rows $1, 2, 3$. Hence row space basis are

$$\left\{ \begin{pmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 5 \end{bmatrix} \end{pmatrix} \right\}$$

The dimension is 3. The column space correspond to pivot columns in original A. These are columns $1, 2, 4$. Hence basis for column space are

$$\left\{ \begin{pmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ 2 \end{bmatrix} \end{pmatrix} \right\}$$

4.1.1.50 Problem section 4.5 number 11

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 2 & 3 & 7 & 8 & 2 \\ 2 & 3 & 7 & 8 & 3 \\ 3 & 1 & 7 & 5 & 4 \end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 2 & 3 & 7 & 8 & 3 \\ 3 & 1 & 7 & 5 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 3 & 1 & 7 & 5 & 4 \end{bmatrix}$$

$R_4 \to R_4 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & -2 & -2 & -4 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & -2 & -4 & 1 \end{bmatrix}$$

$R_4 \to R_4 + 2R_2$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is in Echelon form. The pivot columns are $1, 2, 5$. The non-zero rows are rows $1, 2, 3$. Hence row space basis are

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dimension is 3. The column space correspond to pivot columns in original A. These are columns $1, 2, 5$. Hence basis for column space are

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

4.1.1.51 Problem section 4.5 number 12

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 0 \\ -1 & 0 & -2 & -1 & 1 \\ 2 & 3 & 7 & 8 & 1 \\ -2 & 4 & 0 & 6 & 7 \end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_2 \rightarrow R_2 + R_1$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 2 & 3 & 7 & 8 & 1 \\ -2 & 4 & 0 & 6 & 7 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ -2 & 4 & 0 & 6 & 7 \end{bmatrix}$$

$R_4 \rightarrow R_4 + 2R_1$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 6 & 6 & 12 & 7 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 6 & 12 & 7 \end{bmatrix}$$

$R_4 \rightarrow R_4 - 6R_2$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Swap R_4, R_3

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is in Echelon form. The pivot columns are $1,2,5$. The non-zero rows are rows $1,2,3$. Hence row space basis are

$$\left\{ \begin{bmatrix} 1\\1\\3\\3\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\1\\2\\1 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\1 \end{bmatrix} \right\}$$

The dimension is 3. The column space correspond to pivot columns in original A. These are columns $1,2,5$. Hence basis for column space are

$$\left\{ \begin{bmatrix} 1\\-1\\2\\-2 \end{bmatrix}, \begin{bmatrix} 1\\0\\3\\4 \end{bmatrix}, \begin{bmatrix} 0\\1\\1\\7 \end{bmatrix} \right\}$$

4.1.1.52 Problem section 4.5 number 13

In Problems 13–16, a set S of vectors in $\mathbb{R}^4$ is given. Find a subset of S that forms a basis for the subspace of $\mathbb{R}^4$ spanned by S

$$\vec{v}_1 = \begin{bmatrix} 1\\3\\-2\\4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2\\-1\\3\\2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 5\\1\\4\\8 \end{bmatrix}$$

solution

We set up a matrix made of the above vectors, then find the column space.

$$\begin{bmatrix} 1 & 2 & 5\\ 3 & -1 & 1\\ -2 & 3 & 4\\ 4 & 2 & 8 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 5\\ 0 & -7 & -14\\ -2 & 3 & 4\\ 4 & 2 & 8 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 2R_1$

$$\begin{bmatrix} 1 & 2 & 5\\ 0 & -7 & -14\\ 0 & 7 & 14\\ 4 & 2 & 8 \end{bmatrix}$$

$R_4 \rightarrow R_4 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 5\\ 0 & -7 & -14\\ 0 & 7 & 14\\ 0 & -6 & -12 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -7 & -14 \\ 0 & 0 & 0 \\ 0 & -6 & -12 \end{bmatrix}$$

$R_4 \to R_4 - \frac{6}{7}R_2$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -7 & -14 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Pivot vectors are $1,2$. Hence the column space basis are $\vec{v}_1, \vec{v}_2$

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix} \right\}$$

These are the basis that span the set S.

4.1.1.53 Problem section 4.5 number 14

In Problems 13–16, a set S of vectors in $\mathbb{R}^4$ is given. Find a subset of S that forms a basis for the subspace of $\mathbb{R}^4$ spanned by S

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 4 \\ 1 \\ 8 \\ 7 \end{bmatrix}$$

solution

We set up a matrix made of the above vectors, then find the column space.

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ -1 & 3 & 1 & 1 \\ 2 & 4 & 2 & 8 \\ 3 & 1 & 1 & 7 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 5 & 2 & 5 \\ 2 & 4 & 2 & 8 \\ 3 & 1 & 1 & 7 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 3 & 1 & 1 & 7 \end{bmatrix}$$

$R_4 \to R_4 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & -2 & -5 \end{bmatrix}$$

$R_4 \to R_4 + R_2$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot vectors are $1, 2$. Hence the column space basis are $\vec{v}_1, \vec{v}_2$

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} \right\}$$

These are the basis that span the set S.

4.1.1.54 Problem section 4.5 number 15

In Problems 13–16, a set S of vectors in $\mathbb{R}^4$ is given. Find a subset of S that forms a basis for the subspace of $\mathbb{R}^4$ spanned by S

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

solution

We set up a matrix made of the above vectors, then find the dimensions of the column space.

$$\begin{bmatrix} 3 & 2 & 4 & 1 \\ 2 & 1 & 3 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 1 & 3 & 4 \end{bmatrix}$$

$R_1 \to 2R_1$ and $R_2 \to 3R_2$ and $R_3 \to 2R_3$ and $R_4 \to 3R_4$. This gives

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 6 & 3 & 9 & 6 \\ 6 & 6 & 6 & 9 \\ 6 & 3 & 9 & 12 \end{bmatrix}$$

$R_2 \to -R_1 + R_2$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 6 & 6 & 6 & 9 \\ 6 & 3 & 9 & 12 \end{bmatrix}$$

$R_3 \rightarrow -R_1 + R_3$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 2 & -2 & 7 \\ 6 & 3 & 9 & 12 \end{bmatrix}$$

$R_4 \rightarrow -R_1 + R_4$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 2 & -2 & 7 \\ 0 & -1 & 1 & 10 \end{bmatrix}$$

$R_3 \rightarrow 2R_2 + R_3$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 15 \\ 0 & -1 & 1 & 10 \end{bmatrix}$$

$R_4 \rightarrow -R_2 + R_4$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$R_4 \rightarrow 15R_4$ and $R_3 \rightarrow 6R_3$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 90 \\ 0 & 0 & 0 & 90 \end{bmatrix}$$

$R_4 \rightarrow R_3 + R_4$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 90 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the pivot columns are $1, 2, 4$. Therefore the column space basis are $\vec{v}_1, \vec{v}_2, \vec{v}_4$ given by

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

The above are the basis that span S.

4.1.1.55 Problem section 4.5 number 16

In Problems 13–16, a set S of vectors in $\mathbb{R}^4$ is given. Find a subset of S that forms a basis for the subspace of $\mathbb{R}^4$ spanned by S

$$\vec{v}_1 = \begin{bmatrix} 5 \\ 4 \\ 2 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 7 \\ 7 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 5 \\ 4 \\ 6 \\ 7 \end{bmatrix}$$

solution

We set up a matrix made of the above vectors, then find the column space.

$$\begin{bmatrix} 5 & 3 & 7 & 1 & 5 \\ 4 & 1 & 7 & -1 & 4 \\ 2 & 2 & 2 & 2 & 6 \\ 2 & 3 & 1 & 4 & 7 \end{bmatrix}$$

$R_1 \rightarrow 4R_1, R_2 \rightarrow 5R_2$

$$\begin{bmatrix} 20 & 12 & 28 & 4 & 20 \\ 20 & 5 & 35 & -5 & 20 \\ 2 & 2 & 2 & 2 & 6 \\ 2 & 3 & 1 & 4 & 7 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$$\begin{bmatrix} 20 & 12 & 28 & 4 & 20 \\ 0 & -7 & 7 & -9 & 0 \\ 2 & 2 & 2 & 2 & 6 \\ 2 & 3 & 1 & 4 & 7 \end{bmatrix}$$

$R_3 \rightarrow 10R_3 - R_1$

$$\begin{bmatrix} 20 & 12 & 28 & 4 & 20 \\ 0 & -7 & 7 & -9 & 0 \\ 0 & 8 & -8 & 16 & 40 \\ 2 & 3 & 1 & 4 & 7 \end{bmatrix}$$

$R_4 \rightarrow 10R_4 - R_1$

$$\begin{bmatrix} 20 & 12 & 28 & 4 & 20 \\ 0 & -7 & 7 & -9 & 0 \\ 0 & 8 & -8 & 16 & 40 \\ 0 & 18 & -18 & 36 & 50 \end{bmatrix}$$

$R_3 \rightarrow 7R_3, R_2 \rightarrow 8R_2$

$$\begin{bmatrix} 20 & 12 & 28 & 4 & 20 \\ 0 & -56 & 56 & -72 & 0 \\ 0 & 56 & -56 & 112 & 280 \\ 0 & 18 & -18 & 36 & 50 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} 20 & 12 & 28 & 4 & 20 \\ 0 & -56 & 56 & -72 & 0 \\ 0 & 0 & 0 & 40 & 280 \\ 0 & 18 & -18 & 36 & 50 \end{bmatrix}$$

$R_4 \rightarrow 56(R_4), R_2 \rightarrow 18(R_2)$

$$\begin{bmatrix} 20 & 12 & 28 & 4 & 20 \\ 0 & -1008 & 1008 & -1296 & 0 \\ 0 & 0 & 0 & 40 & 280 \\ 0 & 1008 & -1008 & 2016 & 2800 \end{bmatrix}$$

$R_4 \rightarrow R_4 + R_2$

$$\begin{bmatrix} 20 & 12 & 28 & 4 & 20 \\ 0 & -1008 & 1008 & -1296 & 0 \\ 0 & 0 & 0 & 40 & 280 \\ 0 & 0 & 0 & 720 & 2800 \end{bmatrix}$$

$R_4 \to R_4 - 18R_3$

$$\begin{bmatrix} 20 & 12 & 28 & 4 & 20 \\ 0 & -1008 & 1008 & -1296 & 0 \\ 0 & 0 & 0 & 40 & 280 \\ 0 & 0 & 0 & 0 & -2240 \end{bmatrix}$$

Hence, the pivot columns are $1, 2, 4, 5$. Therefore the column space basis are $\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_5$ given by

$$\left\{ \begin{bmatrix} 5 \\ 4 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 6 \\ 7 \end{bmatrix} \right\}$$

The above are the basis that span S.

4.1.1.56 Problem section 4.7 number 5

In Problems 5–8, determine whether or not each indicated set of functions is a subspace of the space F of all real-valued

functions on $\mathbb{R}$.

The set of all f such that $f(0) = 0$

Solution

The only condition given is that $f(0) = 0$. This means the zero function is included. $cf(0) = c(0) = 0$ and $f(0) + g(0) = 0 + 0 = 0$. Hence closed under addition and under scalar multiplication. Hence subspace.

4.1.1.57 Problem section 4.7 number 6

In Problems 5–8, determine whether or not each indicated set of functions is a subspace of the space F of all real-valued

functions on $\mathbb{R}$.

The set of all f such that $f(x) \neq 0$ for all x.

Solution

Since the zero function is not included, then this can not be a subspace.

4.1.1.58 Problem section 4.7 number 7

In Problems 5–8, determine whether or not each indicated set of functions is a subspace of the space F of all real-valued

functions on $\mathbb{R}$.

The set of all f such that $f(0) = 0$ and $f(1) = 1$

Solution

$$5f(1) = 5 \times 1 = 5$$

Hence not closed under scalar multiplication. Therefore not a subspace.

4.1.1.59 Problem section 4.7 number 8

In Problems 5–8, determine whether or not each indicated set of functions is a subspace of the space F of all real-valued

functions on $\mathbb{R}$.

The set of all f such that $f(-x) = -f(x)$ for all x

Solution

This is the definition of an odd function such as $\sin x$. The odd function is zero at $x = 0$, since the zero is included. Also adding two odd functions gives an odd function, and scaling an odd function does not change its oddness. Hence closed. Therefore a subspace.

4.1.1.60 Problem section 4.7 number 9

In Problems 9–12, a condition on the coefficients of a polynomial $a_0 + a_1x + a_2x^2 + a_3x^3$ is given. Determine whether or not the set of all such polynomials satisfying this condition is a subspace of the space P of all polynomials.

$$a_3 \neq 0$$

Solution

Not a subspace, since we can not obtain the zero polynomial if $a_3 \neq 0$ all the time. Hence not a subspace

4.1.1.61 Problem section 4.7 number 9

In Problems 9–12, a condition on the coefficients of a polynomial $a_0 + a_1x + a_2x^2 + a_3x^3$ is given. Determine whether or not the set of all such polynomials satisfying this condition is a subspace of the space P of all polynomials.

$$a_3 \neq 0$$

Solution

Let $p_1 = 3x^3$ and let $p_2 = -3x^3$, hence $p_1 + p_2 = 3x^3 - 3x^3 = 0$ which does not satisfy the condition that $a_3 \neq 0$. Hence not closed under addition. not a subspace

4.1.1.62 Problem section 4.7 number 10

In Problems 9–12, a condition on the coefficients of a polynomial $a_0 + a_1x + a_2x^2 + a_3x^3$ is given. Determine whether or not the set of all such polynomials satisfying this condition is a subspace of the space P of all polynomials.

$$a_0 = a_1 = 0$$

Solution

These all polynomials that look like $3x^2 + 5x^3, -x^2 + x^3$ and so on. Let $p_1 = a_2x^2 + a_3x^3$ and let $p_2 = b_2x^2 + b_3x^3$.

$$\begin{aligned} p_1 + p_2 &= a_2x^2 + a_3x^3 + b_2x^2 + b_3x^3 \\ &= x^2(a_2 + a_3) + x^3(a_3 + b_3) \end{aligned}$$

Which satisfies the condition that $a_0 = a_1 = 0$. Also under scalar multiplication

$$\begin{aligned} Cp_1 &= C\left(a_2x^2 + a_3x^3\right) \\ &= Ca_2x^2 + Ca_3x^3 \end{aligned}$$

Which satisfies the condition that $a_0 = a_1 = 0$. Hence a subspace

4.1.1.63 Problem section 4.7 number 11

In Problems 9–12, a condition on the coefficients of a polynomial $a_0 + a_1x + a_2x^2 + a_3x^3$ is given. Determine whether or not the set of all such polynomials satisfying this condition is a subspace of the space P of all polynomials.

$$a_0 + a_1 + a_2 + a_3 = 0$$

Solution

Let $p_1 = a_0 + a_1x + a_2x^2 + a_3x^3$ such that $a_0 + a_1 + a_2 + a_3 = 0$ and let $p_2 = b_0 + b_1x + b_2x^2 + b_3x^3$ such that $b_0 + b_1 + b_2 + b_3 = 0$ then

$$\begin{aligned} p_1 + p_2 &= a_0 + a_1x + a_2x^2 + a_3x^3 + b_0 + b_1x + b_2x^2 + b_3x^3 \\ &= (a_0 + b_0)x + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 \end{aligned}$$

Now,

$$\begin{aligned} (a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) &= (a_0 + a_1 + a_2 + a_3) + (b_0 + b_1 + b_2 + b_3) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Hence closed under addition. Also

$$\begin{aligned} cp_1(x) &= c\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) \\ &= ca_0 + ca_1x + ca_2x^2 + ca_3x^3 \end{aligned}$$

Now

$$\begin{aligned} ca_0 + ca_1 + ca_2 + ca_3 &= c(a_0 + a_1 + a_2 + a_3) \\ &= c(0) \\ &= 0 \end{aligned}$$

Hence closed under scalar multiplication. And since the zero polynomial is also included (when $a_i = 0$), then this is a subspace

4.1.1.64 Problem section 4.7 number 12

In Problems 9–12, a condition on the coefficients of a polynomial $a_0 + a_1x + a_2x^2 + a_3x^3$ is given. Determine whether or not the set of all such polynomials satisfying this condition is a subspace of the space P of all polynomials.

$$a_0, a_1, a_2, a_3 \text{ are all integers.}$$

Solution

Not closed under scalar multiplication. For example

$$\frac{1}{2}\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) = \frac{1}{2}a_0 + \frac{1}{2}a_1x + \frac{1}{2}a_2x^2 + \frac{1}{2}a_3x^3$$

But $\frac{1}{2}a_0$ is not integer when a_0 is integer. Therefore not a subspace

4.1.2 Questions

MATH 2243 - SECTION 002 - MIDTERM 1 - COVER PAGE

Instructions:

You are required to be present on Zoom with your camera on from 6pm until your exam is submitted. While taking the exam, you must remain in the frame of the camera at all times. You may not ask for or receive help from notes, textbooks, online resources, or other people during the exam. You may use a calculator, but as always in this class you are expected to show all work.

There is an honor statement at the bottom of this page. You are required to copy the statement in your own handwriting, sign it, and submit it with your exam. Unless we have made prior arrangements, failure to comply with these requirements may result in a grade of 0 on the exam.

If you have questions during the exam or encounter technical difficulties, you can ask me using the Zoom chat or by email. The exam is written to take approximately 60 minutes. This file has been made available at 6pm and you will have until 7:15pm to submit your solutions on Gradescope. It is your responsibility to ensure that you have sufficient time to scan and upload your work before the deadline. It is your responsibility to ensure that your scan is legible and includes all of your work.

Honor Statement:

Please copy the following in your own handwriting and sign it. Your exam will not be accepted without a signed honor statement.

I certify that I have not accepted, sought, or received help from any source on this exam. I have not used any notes, textbooks, or online resources and I have not spoken to any person. I understand that signing this statement untruthfully will constitute a violation of the University of Minnesota's Academic Honesty policies.

1

Math 2243 - Section 002 - Midterm 1

Problem 1 (10 points): Suppose that $\mathbf{A}$ is a matrix such that

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 1 & a & b \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} c & -1 & -1 \\ -2 & 1 & d \\ -1 & 0 & 1 \end{bmatrix}$$

Find a, b, c and d.

Problem 2 (5 points): Find a matrix $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{B}$, given that

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -4 & 0 & 5 & 1 & 2 \\ 7 & 1 & 3 & 1 & -3 \end{bmatrix}$$

Problem 3 (10 points): Let k be any real number and let $\mathbf{A}$ be the matrix

$$\mathbf{A} = \begin{bmatrix} k+2 & 6 & 1 \\ 1 & 3 & -2 \\ 2 & 6 & k \end{bmatrix}$$

(a) Compute $\det \mathbf{A}$ (your answer should be in terms of the constant k).

(b) For what values of k is it true that the linear system $\mathbf{A}\vec{x} = \vec{0}$ has exactly one solution?

Problem 4 (8 points): Let $\vec{v}_1 = (1, -3, 3), \vec{v}_2 = (-1, -4, 2),\ \vec{v}_3 = (7, -7, 11), \vec{v}_4 = (2, 1, 1)$ and let $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$. Consider the subspace $W = \operatorname{Span} S$ of $\mathbb{R}^3$.

Find a subset of S which is a basis for W. Your answer should explain why you know the vectors you have chosen are a basis for W.

Problem 5 (7 points): Let $\mathcal{P}_2$ denote the vector space of all polynomials of degree at most 2. For which values of the real constant k do the polynomials

$$1 + x, x + x^2, kx + 3x^2$$

form a basis for $\mathcal{P}_2$?

Problem 6 (10 points): For the following matrix $\mathbf{A}$, the dimension of the solution space of $\mathbf{A}\vec{x} = \vec{0}$ depends on the constant k.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 3k+2 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & k & 2 \end{bmatrix}$$

(a) What are the possible dimensions of the solution space $\mathbf{A}\vec{x} = \vec{0}$? What values of k do they correspond to?

(b) Find a basis for the solution space of of $\mathbf{A}\vec{x} = \vec{0}$ for each of the possibilities found in part (a).

2

4.2 Exam 2, Thursday Nov 19, 2020

Local contents

4.3 Final exam, Thursday Dec 17, 2020

Local contents

4.3.1 Review matrial for final exam

Practice Final

1.1: #1 Write a differential equation that is a mathematical model of the following situation.
The time rate of change of the velocity v of a spaceship
is proportional to the square of v.

Differential Equation:
$v' = kv^2$ or $\frac{dv}{dt} = kv^2$ or $D_t(v) = kv^2$

1.2: #2 Find the position function $x(t)$ of a moving particle
with the given acceleration $a(t) = \frac{1}{(t+2)^3}$, initial position $x_0 = x(0) = 3$
and initial velocity $v_0 = v(0) = 0$.

$v(t) = \int (t+2)^{-3} dt = -\frac{1}{2}(t+2)^{-2} + C,$

$(v(0) = 0)$...

$0 = -\frac{1}{2}(0+2)^{-2} + C = -\frac{1}{8} + C, \qquad C = \frac{1}{8}.$

Hence $x(t) = \int \left[-\frac{1}{2}(t+2)^{-2} + \frac{1}{8}\right] dt = \frac{1}{2}(t+2)^{-1} + \frac{1}{8}t + C$

$(x(0) = 3)\ldots$

$3 = \frac{1}{2}(0+2)^{-1} + \left(\frac{1}{8}\cdot 0\right) + C = \frac{1}{4} + C, \qquad C = \frac{11}{4}.$

$x(t) = \frac{1}{2}(t+2)^{-1} + \frac{1}{8}t + \frac{11}{4}.$

1.3: #3 Find explicit particular solutions to the initial value problem: $x\frac{dy}{dx} - y = 2x^2y, \qquad y(1) = 1.$

$x\frac{dy}{dx} = 2x^2y + y = y(2x^2+1) \qquad \frac{1}{y}dy = \frac{2x^2+1}{x} = 2x + \frac{1}{x}$

$\int \frac{dy}{y} = \int (\frac{1}{x} + 2x)dx; \qquad \ln|y| = \ln|x| + x^2 + \ln C; \qquad y = Cxe^{x^2}$

$y(1) = 1$ implies $1 = Ce^1$ and $C = e^{-1}$ so $y(x) = xe^{\left(x^2-1\right)}$.

$y(x) = xe^{\left(x^2-1\right)} = \frac{x}{e}e^{x^2}$

1.4: #4 An accident at a nuclear power plant has left the surrounding area polluted with radioactive material that decays naturally. The initial amount of radioactive material present is 15 su (safe units), and 5 months later it is still 10 su. What amount of radioactive material will remain after one year? Note that any radioactive material has a decay rate proportional to the amount of radioactive material P(t) present at that time, $\frac{dP(t)}{dt} = -kP(t), \quad k > 0.$

$\int \frac{1}{P}dP = -k\int dt$, when $P \neq 0$. (but if $P = 0$, we see that this is a singular solution to the DEQ)

$\Rightarrow \quad \ln P = -kt + C \quad \Rightarrow \quad P = e^C e^{-kt}.$

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Substituting in the first initial condition:

$15 = e^C e^0 \quad \Rightarrow \quad P = 15e^{-kt},$

Substituting in the 2nd initial condition:

$10 = 15e^{-5k} \quad \Rightarrow \quad \ln\frac{2}{3} = -5k$

$\quad \Rightarrow \quad k = -\frac{1}{5}\ln\frac{2}{3}.$

$\Rightarrow \qquad P(12) = 15e^{-12\left(-\frac{1}{5}\ln\frac{2}{3}\right)} = 15e^{\frac{12}{5}\ln\frac{2}{3}} \approx 5.6686$ **su**.

1.4: **#5** **Find the general solution of the differential equation**: $(1+x)^2\frac{dy}{dx} = (1+y)^2.$

$\frac{1}{(1+y)^2}dy = \frac{1}{(1+x)^2}dx,$ when $y \neq -1.$ (although note that when $y \equiv -1,$ $\frac{dy}{dx} = 0,$ $(1+y)^2 = 0,$ and our differential equation is satisfied for all x).

Continuing on with the case $y \neq -1,$ we have: $\int \frac{1}{(1+y)^2}dy = \int \frac{1}{(1+x)^2}dx$

$-(1+y)^{-1} = -(1+x)^{-1} + c \qquad \Rightarrow \qquad 1+x = (1+y) + (1+x)(1+y)c$

$1+x = 1+y+(c+cx+cy+cxy) \qquad \Rightarrow \qquad x = c+cx+(c+cx+1)y$

$-(c+cx+1)y = c+cx-x \qquad \Rightarrow \qquad y = -\frac{c+cx-x}{c+cx+1}. \qquad \blacksquare$

1.5: **#6** **Solve linear first order differential equation**: $y' + y = \sin x.$

$\rho = e^{\int 1dx} = e^x$

$\Rightarrow \quad (ye^x)' = e^x \sin x$

$\Rightarrow \quad e^x y = \int e^x \sin x dx$

Now we have to use integration by parts, twice!

$\int e^x \sin x dx = -e^x\cos x + \int e^x \cos x dx = -e^x\cos x + \left(e^x\sin x - \int e^x \sin x dx\right)$

$\Rightarrow \quad 2\int e^x \sin x dx = e^x(\sin x - \cos x) + c$

$\Rightarrow \quad y = e^{-x}e^x\ (\sin x - \cos x) = \frac{1}{2}\sin x - \frac{1}{2}\cos x + \frac{c}{e^x}.$

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1.5: #7 **Solve the initial value problem**: $xy' - 2y = 2x^2 \ln x;\ y(1) = 3.$

$y' - \frac{2}{x}y = 2x,$ when $x \neq 0.$ However, note that we already preclude $x = 0,$ as this is not defined for $\ln x.$

Integrating factor: $\rho = e^{-\int \frac{2}{x}dx} = e^{-2\ln|x|} = x^{-2}.$

$x^{-2}(y' - \frac{2}{x}y) = 2x \cdot x^{-2}$

$(y \cdot x^{-2})' = \frac{2}{x}$

$yx^{-2} = 2\int \frac{1}{x}dx + c = 2\ln|x| + c$

$y = 2x^2 \ln|x| + cx^2$

$3 = 2 \cdot 1\ln(1) + c \cdot 1 \quad \Rightarrow \quad c = 3,$ so

$y = 2x^2 \ln|x| + 3x^2.$

2.1: #8 **During the period from 1790 to 1930, the U.S. population** $P(t)$ **(**t **in years) grew from 4 million to** 124 **million. Throughout this period,** $P(t)$ **remained close to the solution of the initial value problem**:

$\frac{dP}{dt} = 0.03P - 0.00015P^2, \quad P(0) = 4.$

a) What 1930 population does this logistic equation predict?

$\frac{dP}{dt} = P\left(\frac{3}{100} - \frac{15}{100000}P\right) = \frac{15}{100{,}000}P\left(\frac{3}{100}\frac{100{,}000}{15} - P\right) = \frac{3}{20{,}000}P(200 - P).$

$\int \frac{1}{P(200-P)}dP = \frac{3}{20{,}000}\int dt \quad \Rightarrow \quad \frac{1}{P(200-P)} = \frac{A}{P} + \frac{B}{200-P} \quad \Rightarrow \quad 1 = A(200 - P) + BP$

$\Rightarrow \quad 1 = (B - A)P + 200A \quad \Rightarrow \quad A = 1/200$ and $B = 1/200$

$\Rightarrow \quad \int \frac{1}{P} + \frac{1}{200-P}dP = \frac{3}{100}t + C \quad \Rightarrow \quad \ln P - \ln(200 - P) = \frac{3}{100}t + C$

$\ln\frac{P}{200-P} = \frac{3}{100}t + C \quad \Rightarrow \quad \frac{P}{200-P} = De^{\frac{3}{100}t} \quad \Rightarrow \quad \frac{1}{49} = D$

$\frac{P}{200-P} = \frac{1}{49}e^{\frac{3}{100}\cdot 140} \approx 1.361 \quad \Rightarrow \quad P = 1.361\left(200 - P\right) \quad \Rightarrow \quad 2.361P \approx 272.2 \Rightarrow P \approx 115.3$ million.

b) What limiting population does it predict?

Limiting population of 200 million since in the logistic form of the equation
$(P' = kP(M - P) = \frac{3}{20{,}000}P(200 - P)),\ M = 200$ is in the spot for the limiting population.

2.2: #9 **The equation** $y' = -y^2(y^2 - 4)$ **has** …

A) A stable critical point at 0. **B) A stable critical point at** 2.

C) If $y(0) = -1,$ **then** $y(t) \to 0$ **as** $t \to \infty.$ **D) If** $y(5) = -1,$ **then** $y(t) \to 0$ **as** $t \to \infty.$

E) If $y(0) = -1,$ **then** $y(t) \to 0$ **as** $t \to -\infty.$ **F) If** $y(0) = 6,$ **then** $y(t) \to 2$ **as** $t \to -\infty.$

Answers: B,C,D

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2.3: #10 A skydiver drops from an airplane at an altitude of 5000 m and dives freely, with negligible air resistance, for 30 seconds. He then opens his parachute and falls such that the acceleration due to air resistance is proportional to his velocity, $a_R = -2v$.

You may take the acceleration of gravity to be $9.8 m/s^2$.

(a) Find the altitude at which the skydiver opened the parachute.

Prior to opening the parachute, since a resistance is negligible, we have $a = \frac{dv}{dt} = 9.8.$ Integrating to find velocity, we have: $\int \frac{dv}{dt} dt = \int -9.8 dt \quad \Rightarrow \quad v = -9.8t + v_0 \quad \Rightarrow \quad v = -9.8t$ (because we set $t = 0$ to be the time when they jump out of the plane, and observe that at that time $v = 0$, so $v_0 = 0$). Integrating again to get the position function: $\int \frac{dx}{dt} dt = \int -9.8t\,dt \quad \Rightarrow \quad x = -4.9t^2 + c$. Observing that $x(0) = 5000,$ we have:

$5000 = -4.9 \cdot 0^2 + c \quad \Rightarrow \quad c = 5000.$ So $x = -4.9t^2 + 5000.$

Evaluating this at 30 sec, we have: $x(30) = -4.9(30)^2 + 5000 \approx 590\ m.$

(b) Find the velocity of the skydiver 1 minute after he jumped.

Let $\overline{v}$ represent a new velocity function with initial condition $\overline{v}(0) = v(30)$. In other words, it represents the velocity function after the skydiver deployed the parachute. Observe that the new acceleration function is: $\frac{d\overline{v}}{dt} = -9.8 - 2\overline{v}$. In other words, we are now taking into account the fact that air resistance is slowing the fall (also note that since velocity is in the opposite direction from our position function, that $-2\overline{v}$ is a positive number!). Using separation of variables (although it is also possible to use an integrating factor), we have:

$$-\int \frac{1}{9.8+2\overline{v}} d\overline{v} = \int dt \quad \Rightarrow \quad -\frac{1}{2} \ln|9.8 + 2\overline{v}| = t + c$$

$$\Rightarrow \quad \ln|9.8 + 2\overline{v}| = -2t - c \quad \Rightarrow \quad |9.8 + 2\overline{v}| = e^{-2t-c}$$

Observe from above we have: $v = -9.8t,$ so $\overline{v}(0) = v(30) = -9.8 \cdot 30 = -294.$ So plugging in this initial condition, we have:

$$|9.8 + 2 \cdot (-294)| = e^{-2 \cdot 0 - c} \quad \Rightarrow \quad e^{-c} = 578.2.$$ Plugging this in:

$\Rightarrow \quad |9.8 + 2\overline{v}| = 578.2e^{-2t} \quad \Rightarrow \quad \overline{v} = -289.1e^{-2t} - 4.9$ (0ur choice of sign is once again due to the fact that since $\overline{v}(0) = -294.$)

Therefore, the velocity at 60 seconds is: $\overline{v}(30) = -289.1e^{-2 \cdot 30} - 4.9 \approx -4.9\ m/s.$

(c) Find his terminal velocity, that is, find $\lim_{t \to \infty} v(t)$.

$$\overline{v}(\infty) = -289.1e^{-2 \cdot \infty} - 4.9 = -4.9\ m/s.$$

2.3: #11 You just bought a new car, and its acceleration is proportional to the difference between 250 km per hour and its velocity. If your new car can accelerate from rest to 100 km/hr in 10 seconds, how long will it take for the car to accelerate from rest to 200 km/hr?

$a(t) = \frac{dv}{dt} = k(250 - v), \qquad v(10) = 100, \qquad v(0) = 0, \qquad v(?) = 200.$

$\frac{dv}{250-v} = kdt \qquad \Rightarrow \qquad \int \frac{dv}{250-v} = \int kdt \qquad \Rightarrow \qquad -\ln|250 - v| = kt + c.$

$250 - v = e^{-kt-c} \qquad \Rightarrow \qquad v = 250 - Ce^{-kt}.$

$0 = 250 - Ce^0$, so $C = 250$.

$100 = 250 - 250e^{-10k} \qquad \Rightarrow \qquad e^{-10k} = \frac{150}{250} = \frac{3}{5}$

$-10k = \ln(\frac{3}{5}) \qquad \Rightarrow \qquad k = -\frac{1}{10}\ln(\frac{3}{5}) \approx 0.0511.$

$200 = 250 - 250e^{-kt} \qquad \Rightarrow \qquad e^{-0.0511t} = \frac{50}{250} = \frac{1}{5}$

$-0.0511t = \ln(\frac{1}{5}) = -\ln(5) \qquad \Rightarrow \qquad t = \frac{\ln(5)}{0.0511} \approx 31.5$ sec. ■

2.4: #12 Use the Euler method (NOT the improved Euler method) with step size $h = 0.2$ to approximate $y(0.4)$ where $y(x)$ is the solution of the differential equation $y' = -2xy$ with initial value $y(0) = 2$.

$y(0.2) \approx 2 + 0.2(0) = 2$

$y(0.4) \approx 2 + 0.2(-2 \cdot 0.2 \cdot 2) = 1.84.$ ■

3.1: #13 a) Write down the augmented matrix of the system:

$$\begin{aligned} x + 3y + 2z &= 2 \\ 2x + 7y + 7z &= -1 \\ 2x + 5y + 2z &= 7 \end{aligned}$$

$$\begin{bmatrix} 1 & 3 & 2 & | & 2 \\ 2 & 7 & 7 & | & -1 \\ 2 & 5 & 2 & | & 7 \end{bmatrix}$$

b) **Row reduce the matrix to echelon form (just echelon, not reduced echelon).**

$$\Rightarrow r_2 - 2r_1 \text{ and } r_3 - 2r_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & -1 & -2 & 3 \end{array}\right]$$

$$\Rightarrow r_3 + r_2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

c) **How many solutions does the system have? Justify your answer.**

The system has only one solution.

This is because each column representing a variable has a leading "1", this means we do not have an arbitrary parameter, and therefore there are not an infinite number of solutions. Also, the reduction did not give us a contradictory $0x + 0y + 0z = c \neq 0$ row (which would indicate an inconsistent matrix with no solution), but instead the reduction left us in the desirable position of being able to solve for z by just reading it off the matrix, and then using back substitution to determine the exact values of x and y.

The question didn't ask for the actual answer, but that would be:

$z = -2,$

$y = -3z - 5 = 1,$

$x = -3y - 2z + 2 = 3,$

So, $\vec{x} = (3, 1, -2)$. ■

3.2: #14 **Use the method of Gaussian elimination to solve the system of equations:**

$$\begin{cases} x_1 - x_2 - x_4 = 1, \\ 2x_1 + x_2 + 3x_3 + 7x_4 = -1, \\ 3x_1 - 2x_2 + x_3 = 2. \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 1 \\ 2 & 1 & 3 & 7 & -1 \\ 3 & -2 & 1 & 0 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 1 \\ 0 & 3 & 3 & 9 & -3 \\ 0 & 1 & 1 & 3 & -1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_4 = t_4,\ x_3 = t_3,\ x_2 = -t_3 - 3t_4 - 1,\ x_1 = -t_3 - 2t_4$

$(x_1, x_2, x_3, x_4) = \left(-t_3 - 2t_4, \quad -t_3 - 3t_4 - 1, \quad t_3, \quad t_4\right)$

$\quad = (0, -1, 0, 0) + t_3(-1, -1, 1, 0) + t_4(-2, -3, 0, 1),$ for any $t_3, t_4 \in \mathbb{R}$.

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3.2: #15 **Consider the matrices**: $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

Calculate, if possible, $(\mathbf{A}-2\mathbf{B})C$, $(\mathbf{AB}-3\mathbf{A})C$, **and** $AB-BA$. **If one (or more) of these expressions is not defined, state so and give the reason.**

Case: $(\mathbf{A}-2\mathbf{B})\mathbf{C}$, The matrices **A** and 2**B** are not compatible for subtraction, as they have different dimensions.

Case: $(\mathbf{AB}-3\mathbf{A})\mathbf{C}$, $(\mathbf{AB}-3\mathbf{A})\mathbf{C} = \mathbf{ABC}-3\mathbf{AC} = \mathbf{A}\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix} - 3\begin{bmatrix} -2 & 1 \\ 1 & -2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$$= \begin{bmatrix} -2 & 1 \\ 1 & -2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix} = \begin{bmatrix} -18 \\ 9 \\ 0 \end{bmatrix} + \begin{bmatrix} -9 \\ 18 \\ 9 \end{bmatrix} = \begin{bmatrix} 27 \\ 27 \\ 9 \end{bmatrix}.$$

Case: $\mathbf{AB}-\mathbf{BA}$, The matrix **B** cannot be multiplied by **A**, because the number of columns of **B** do not equal the number of rows for **A**. ■

3.3 #16 **Find the reduced echelon form of the following matrix**:

$$\begin{bmatrix} 5 & 2 & 18 \\ 0 & 1 & 4 \\ 4 & 1 & 12 \end{bmatrix}$$

$$\Rightarrow \tfrac{1}{4}r_3 \text{ and } \tfrac{1}{5}r_1 \Rightarrow \begin{bmatrix} 1 & \frac{2}{5} & \frac{18}{5} \\ 0 & 1 & 4 \\ 1 & \frac{1}{4} & 3 \end{bmatrix} \Rightarrow r_3 - r_1 \Rightarrow \begin{bmatrix} 1 & \frac{2}{5} & \frac{18}{5} \\ 0 & 1 & 4 \\ 0 & \frac{1}{4}-\frac{2}{5} & 3-\frac{18}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{2}{5} & \frac{18}{5} \\ 0 & 1 & 4 \\ 0 & -\frac{3}{20} & -\frac{3}{5} \end{bmatrix} \Rightarrow -\tfrac{20}{3}r_3 \Rightarrow \begin{bmatrix} 1 & \frac{2}{5} & \frac{18}{5} \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{bmatrix}.$$

Reduced Echelon Form: $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$

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3.5: #17 **Find the inverse of the matrix** $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix}$. (**show all your work**!).

We would like to calculate $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}[A_{ij}]^T$. So first to calculate the determinant:

$$|\mathbf{A}| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{vmatrix} \overset{c_3+c_2}{=} \begin{vmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ 1 & -1 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix} = 1.$$

$$\text{Therefore, } \mathbf{A}^{-1} = \frac{1}{1}[A_{ij}]^T = \begin{bmatrix} +0 & +1 & +1 \\ -1 & -1 & -0 \\ +3 & +1 & -1 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Alternatively, one could calculate this with: $\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -3 & 3 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$

$$\overset{r_3-r_1}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -3 & 3 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \overset{r_2-2r_1}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \overset{-r_2 \text{ and } r_1+r_2}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 3 & -1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\overset{r_2+r_3 \text{ and } r_1+3r_3}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \overset{-r_3}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \Rightarrow \mathbf{A}^{-1} = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

b) **Then use this inverse to solve the system** $\mathbf{A}\vec{x} = \vec{b}$, **where** $\vec{b} = \begin{bmatrix} 15 \\ -9 \\ -3 \end{bmatrix}$. **You will not receive credit if you solve the system by any other method**.

$$\vec{x} = \mathbf{A}^{-1}\vec{b} = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 15 \\ -9 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 21 \\ 18 \end{bmatrix}. \quad \blacksquare$$

3.6: #18 Use cofactors (not the identity matrix) to evaluate the inverse of…

$$A = \begin{bmatrix} 3 & 5 & 2 \\ -2 & 3 & -4 \\ -5 & 0 & -5 \end{bmatrix}.$$

Using the last row to calculate my determinant…

$\det A = -5 \cdot (5 \cdot (-4) - 2 \cdot 3) + 0 - 5(3 \cdot 3 - (5)(-2)) = -5 \cdot (-26) - 5 \cdot 19$

$= 130 - 95 = 35$. So, $\frac{1}{\det A} = \frac{1}{35}$.

$$[c_{mn}] = \begin{bmatrix} -15 & 10 & 15 \\ 25 & -5 & -25 \\ -26 & 8 & 19 \end{bmatrix}, \quad [c_{mn}]^T = \begin{bmatrix} -15 & 25 & -26 \\ 10 & -5 & 8 \\ 15 & -25 & 19 \end{bmatrix}$$

$$A^{-1} = \frac{1}{35}\begin{bmatrix} -15 & 25 & -26 \\ 10 & -5 & 8 \\ 15 & -25 & 19 \end{bmatrix} \quad \text{or} \quad A^{-1} = \begin{bmatrix} -\frac{3}{7} & \frac{5}{7} & -\frac{26}{35} \\ \frac{2}{7} & -\frac{1}{7} & \frac{8}{35} \\ \frac{3}{7} & -\frac{5}{7} & \frac{19}{35} \end{bmatrix}$$

3.6: #19 Use Cramer's Rule to solve the following system. First, construct a matrix A to be the matrix associated with the system:

$5x + 8y = 3$, $8x + 13y = 5$.

$$A = \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}, \quad \det A = 1, \quad \frac{1}{\det A} = 1.$$

$\det A = 1$.

$$x = \frac{1}{\det A} \det\begin{bmatrix} 3 & 8 \\ 5 & 13 \end{bmatrix} = -1.$$

$$y = \frac{1}{\det A} \det\begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix} = 1.$$

3.7: #20 **With the data points given, find the** nth **degree polynomial** $y = f(x)$ **that fits these points**: $(-1,1),(1,5),$ **and** $(2,16)$.

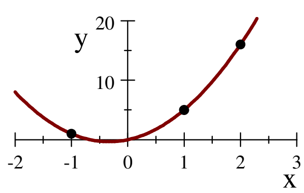
$ax^2 + bx + c = y$

$a(-1)^2 + b(-1) + c = 1, \quad a(1)^2 + b(1) + c = 5, \quad a(2)^2 + b(2) + c = 16$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 5 \\ 4 & 2 & 1 & 16 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 4 \\ 0 & 6 & -3 & 12 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \quad c = 0, \quad b = 2, \text{ and } d = 3.$$

Therefore, $y = 3x^2 + 2x$



4.1: #21 **Consider the matrix** $\begin{bmatrix} 1 & -1 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 3 & -1 \end{bmatrix}$. **Find a basis for each of the subspaces**:

a) $Null(\mathbf{A}) = \{x \in \mathbb{R}^4 : \mathbf{A}x = \mathbf{0}\}$.

b) The row space $Row(\mathbf{A})$.

c) The column space $Col(\mathbf{A})$.

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 3 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 2 & -2 \\ 0 & 2 & 3 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$x_4 = s, \; x_3 = -s, \; x_2 = 2s, \; x_1 = 0.$

$Null(\mathbf{A}) = span\{(0,2,-1,1)\}$.

$Row(\mathbf{A}) = span\left\{(1,0,0,0), \; (0,1,0,-2), \; (0,0,1,1)\right\}$.

$Col(\mathbf{A}) = span\left\{(1,1,0), \; (-1,1,2), \; (0,2,3)\right\}$.

4.2: #22 Let V be a vector space. Let W be a subset of V. There are three conditions you can check which are sufficient to show that W is a subspace of V. One of them is that W must not be empty. What are the other two conditions?

Condition 1: $W \neq \emptyset$, (or some other condition which implies the existence of at least one vector)

Condition 2: For all $\vec{w}_1, \vec{w}_2$ in W, $\vec{w}_1 + \vec{w}_2$ is also in W.

Condition 3: For all $\vec{w}_1$ in W, and $c \in \mathbb{R}$; $c\vec{w}_1$ is also in W.

4.4: #23 Find a basis for the solution space of the given homogeneous linear system:

$$x_1 - 4x_2 - 3x_3 - 7x_4 = 0$$
$$2x_1 - x_2 + x_3 + 7x_4 = 0$$
$$x_1 + 2x_2 + 3x_3 + 11x_4 = 0$$

$$\begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}, \text{Gaussian elimination: } \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow x_4 = s,\ x_3 = t,\ x_2 = -t - 3s,\ x_1 = -t - 5s$$

$(x_1, x_2, x_3, x_4) = (-t - 5s, -t - 3s, t, s) = s(-5, -3, 0, 1) + t(-1, -1, 1, 0)$.

Therefore, the basis for the solution space is $\{(-5, -3, 0, 1), (-1, -1, 1, 0)\}$

4.4: #24 a) Determine (with justification) if the vectors
$\vec{v}_1 = (1,1,1,1)$, $\vec{v}_2 = (1,2,3,0)$, $\vec{v}_3 = (3,6,0,0)$, **and** $\vec{v}_4 = (-1,0,0,0)$ **form a basis of $\mathbb{R}^4$ or not.**

$$\begin{vmatrix} 1 & 1 & 3 & -1 \\ 1 & 2 & 6 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 2 & 6 \\ 1 & 3 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 6 \\ 3 & 0 \end{vmatrix} = 0 - 12 \neq 0.$$ Therefore the column vectors we used were linearly independent. And since we used four vectors in $\mathbb{R}^4$, which is a vector space of four dimensions, we have a basis for $\mathbb{R}^4$.

b) Determine (with justification!) If the subset $W = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 x_2 = x_3 + x_4 + x_5\}$ is a subspace of $\mathbb{R}^5$ or not.

Since $\vec{0}$ is an element of any subspace, this would require that:

$\vec{0} = \left(\frac{1}{x_2}(x_3 + x_4 + x_5),\ \frac{1}{x_1}(x_3 + x_4 + x_5),\ x_1x_2 - x_4 - x_5,\ x_1x_2 - x_3 - x_5,\ x_1x_2 - x_3 + x_4\right)$.

From the first two components, we see we need $x_3 + x_4 + x_5 = 0$, and neither x_1 or x_2 are equal to zero. Plugging this into the third component, we have $x_1x_2 + x_3 = 0$. Similarly in the fourth component, we have $x_1x_2 + x_4 = 0$. Combining these, we see that $x_3 = x_4$. However, the fourth component and gives us that $x_1x_2 = 0$, implying that either x_1 or x_2, or both must be equal to zero. But this contradicts an earlier conclusion. Therefore, this must not be a subspace.

5.1 #25 **Solve the following equation**: $y''' + 2y'' + y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1.$

$r^3 + 2r^2 + r + 2 = 0$, note that $r = -2$ solves this. So, $(r+2)$ is a factor. Dividing:

$r^3 + 2r^2 + r + 2 = (r+2)r^2 + (r+2) = (r+2)(r^2+1)$, so our roots are $r \in \{-2, \pm i\}$, and our general equation is $y_g = c_1 e^{-2x} + c_2 \cos x + c_3 \sin x$.

$0 = c_1 + c_2$ or $c_1 = -c_2$ and $y = -c_2 e^{-2x} + c_2 \cos x + c_3 \sin x$.

$y' = 2c_2 e^{-2x} - c_2 \sin x + c_3 \cos x$.

$0 = 2c_2 + c_3$ or $c_2 = -\frac{1}{2}c_3$ and $y' = -c_3 e^{-2x} + \frac{1}{2}c_3 \sin x + c_3 \cos x$.

$y'' = 2c_3 e^{-2x} + \frac{1}{2}c_3 \cos x - c_3 \sin x$.

$1 = 2c_3 + \frac{1}{2}c_3$ or $c_3 = \frac{2}{5}$, $c_2 = -\frac{1}{5}$ and $c_1 = \frac{1}{5}$.

So, $y = \frac{1}{5}e^{-2x} - \frac{1}{5}\cos x + \frac{2}{5}\sin x$.

5.2: #26 **Show that the functions** $y_1(x) = e^x, \; y_2(x) = e^{2x}, \; y_3(x) = e^{3x}$ **are linearly independent on** $(-\infty, \infty)$.

$$\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x e^{2x} e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{vmatrix} = 2e^{6x} \neq 0 \text{ for any } x \in (-\infty, \infty) \,.$$

5.5: #27 **Find all solutions to**: $y' \cos x + y^2 \sin x = \sin x$. **Write the solution(s) in explicit form**.

$y' = \frac{\sin x}{\cos x}(1 - y^2) \quad \Rightarrow \quad \int \frac{1}{1-y^2} dy = \int \frac{\sin x}{\cos x} dx$, when $y \neq 1$.

However, observe that when $y = 1$, we have $0 \cdot \cos x + \sin x = \sin x$, which is true, so $y = 1$ is a solution. Continuing on with the case $y \neq 1$...

$\Rightarrow \quad \frac{1}{(1-y)(1+y)} = \frac{A}{1-y} + \frac{B}{1+y} \quad \Rightarrow \quad A(1+y) + B(1-y) = 1$

$\Rightarrow \quad y(A - B) + (A + B) = 1 \quad \Rightarrow \quad A = B$, and $A = \frac{1}{2}$.

$\Rightarrow \quad \frac{1}{2} \int \frac{1}{1-y} + \frac{1}{1+y} dy = \int \frac{\sin x}{\cos x} dx \quad \Rightarrow \quad -\ln|1-y| + \ln|1+y| = -2\ln|\cos x|$

$\Rightarrow \quad \ln\left|\frac{1+y}{1-y}\right| = -2\ln|\cos x| \quad \Rightarrow \quad \frac{1+y}{1-y} = \frac{1}{\cos^2 x} \quad \Rightarrow \quad \cos^2 x(1+y) = 1 - y$

$\Rightarrow \quad y \cos^2 x + y = 1 - \cos^2 x \quad \Rightarrow \quad y = -\frac{\cos^2 x - 1}{\cos^2 x + 1}$ or $y = 1$.

5.5: #28 **The roots of equation** $r^2 - 10r + 74 = 0$ **are** $r = 5 \pm 7i$.

Write down the general form of a particular solution with undetermined coefficients for the differential equation $y'' - 10y' + 74y = xe^{5x}\sin 7x + \cos 5x$. **Do not attempt to evaluate the coefficients**.

From the homogeneousversion of our equation,we get the characteristic equation: $r^2 - 10r + 74 = 0$, which we are told has the solutions $r = 5 \pm 7i$. Being complex conjugates, we get both linearly independent solutions from either one, so choosing $5 + 7i$, and putting it into the form e^{rt}, we get...

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$e^{(5+7i)x} = e^{5x}(\cos 7x + i \sin 7x)$

Taking the real and imaginary parts to be linearly independent solutions, we get the complementary solution:

$y_c = c_1 e^{5x} \cos 7x + c_2 e^{5x} \sin 7x.$

My pretrial solution is:

$y_0 = (A + Bx)e^{5x} \sin 7x + (C + Dx)e^{5x} \cos 7x + E \cos 5x + F \sin 5x$

Clearing up any linear dependence between y_0 and y_c (by multiplying terms by x as needed), I get:

$y_{trial} = (Ax + Bx^2)e^{5x} \sin 7x + (Cx + Dx^2)e^{5x} \cos 7x + E \cos 5x + F \sin 5x$

General form of a Particular Solution:

$$y_g = y_c + y_{trial}$$
$$= (c_1 e^{5x} \cos 7x + c_2 e^{5x} \sin 7x) + (Ax + Bx^2)e^{5x} \sin 7x + (Cx + Dx^2)e^{5x} \cos 7x + E \cos 5x + F \sin 5x$$

6.1: #29 Find an eigenvector associated to the eigenvalue $\lambda_1 = 2 + 2i$ of the matrix $A = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix}$.

$$A - \lambda_1 I = \begin{bmatrix} 1-\lambda_1 & -5 \\ 1 & 3-\lambda_1 \end{bmatrix} = \begin{bmatrix} 1-(2+2i) & -5 \\ 1 & 3-(2+2i) \end{bmatrix}$$
$$= \begin{bmatrix} -1-2i & -5 \\ 1 & 1-2i \end{bmatrix} \overset{r_1 \leftrightarrow r_2}{\Rightarrow} \begin{bmatrix} 1 & 1-2i \\ -1-2i & -5 \end{bmatrix} \overset{r_2+(1+2i)r_1}{\Rightarrow}$$

Observe that: $(1 + 2i)(1 - 2i) = 5$.

So we have: $\begin{bmatrix} 1 & 1-2i \\ 0 & 0 \end{bmatrix}$, and $y = s$, $x = -(1 - 2i)y = -s + 2is$. So, $\vec{v}_1 = \begin{bmatrix} -1+2i \\ 1 \end{bmatrix}$, when $s = 1$.

6.2: #30 The eigenvalues of the matrix $A = \begin{bmatrix} 6 & -5 & 2 \\ 4 & -3 & 2 \\ 2 & -2 & 3 \end{bmatrix}$ are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. Find a matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

$$\lambda_1 : \begin{bmatrix} 6-1 & -5 & 2 \\ 4 & -3-1 & 2 \\ 2 & -2 & 3-1 \end{bmatrix} = \begin{bmatrix} 5 & -5 & 2 \\ 4 & -4 & 2 \\ 2 & -2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 4 & -4 & 2 \\ 2 & -2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ y = t, \ z = 0, \text{ and } x = y = t. \text{ So, } \vec{v}_1 = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{where } t = 1.$$

$$\lambda_2 : \begin{bmatrix} 6-2 & -5 & 2 \\ 4 & -3-2 & 2 \\ 2 & -2 & 3-2 \end{bmatrix} = \begin{bmatrix} 4 & -5 & 2 \\ 4 & -5 & 2 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 4 & -5 & 2 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}, \ z = t, \ y = 0, \text{ and } x = y - \tfrac{1}{2}z = -\tfrac{1}{2}t. \text{ So,}$$

$$\vec{v}_2 = t\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \text{where } t = 2.$$

$$\lambda_3 : \begin{bmatrix} 6-3 & -5 & 2 \\ 4 & -3-3 & 2 \\ 2 & -2 & 3-3 \end{bmatrix} = \begin{bmatrix} 3 & -5 & 2 \\ 4 & -6 & 2 \\ 2 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -5 & 2 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \ z = t, \ y = z = t, \text{ and } x = y = t. \text{ So, } \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\text{Therefore, } P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

6.3: #31 **Use the diagonalization method to compute A^5 where** $A = \begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix}$.

Seeking eigenvalues, we calculate: $|A - \lambda I| = \begin{vmatrix} 6-\lambda & -10 \\ 2 & -3-\lambda \end{vmatrix}$

$= (6-\lambda)(-3-\lambda) + 20 = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0.$

Therefore, $\lambda \in \{1, 2\}$.

So we are guaranteed that A is diagonalizable and $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Seeking the eigenvectors, we calculate:

$\lambda = 1 : (A - I) = \begin{bmatrix} 5 & -10 \\ 2 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow y = b,\ x = 2b.$

$\vec{v}_1 = \begin{bmatrix} 2 & 1 \end{bmatrix}^T.$

$\lambda = 2 : (A - 2I) = \begin{bmatrix} 4 & -10 \\ 2 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -5 \end{bmatrix} \Rightarrow y = b,\ x = \frac{5}{2}b.$

$\vec{v}_2 = \begin{bmatrix} 5 & 2 \end{bmatrix}^T.$

Therefore, $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$ and $P^{-1} = \frac{1}{|A|}[A_{ij}]^T = \frac{1}{-1}\begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$.

$$A^5 = (PDP^{-1})(PDP^{-1})\ldots(PDP^{-1}) = PD(P^{-1}P)D(P^{-1}\ldots P)DP^{-1} = PDIDI\ldots IDP^{-1} = PD^5P^{-1}$$

$$= \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1^5 & 0 \\ 0 & 2^5 \end{bmatrix}\begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 32 \end{bmatrix}\begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 156 & -310 \\ 62 & -123 \end{bmatrix}.$$

7.1: #32 **Transform $x'' + 3x' + 7x = t^2$ into a system of first-order differential equations**.

$x_1 = x, \quad x_2 = x' = x_1'$. Below, we ideally want a system of first order equations (only in the new variables x_i), so that we are in a position to place them in a matrix equation $\vec{x}' = A\vec{x} + \vec{y}(t)$, for easy solving using our new techniques.

System of first order differential equations:
$x_1' = x_2$
$x_2' = t^2 - 3x_2 - 7x_1$

7.2: #33 **Write the given system in the vector/matrix form $\vec{x}' = A\vec{x} + \vec{f}(t)$. Then, find eigenvalues for A**.

$x' = x + 2y + 3e^t, \quad y' = x - 3y - t^2$

$$\vec{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3e^t \\ -t^2 \end{bmatrix}.$$

Eigenvalues for **A** : $\begin{vmatrix} 1-\lambda & 2 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 5. \qquad \lambda_{1,2} = \frac{-2\pm\sqrt{4+20}}{2} = -1 \pm \sqrt{6}.$

Observe that you could now solve for the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ of $\mathbf{A}$, then come up with the complementary solution ($\vec{x}_c = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$) to the associated homogeneous differential equation $\vec{x}' = \mathbf{A}\vec{x}$.

7.3 #34 **Apply the eigenvalue method to find a general solution to this system.**

$$x_1' = 2x_1 - 5x_2, \qquad x_2' = 4x_1 - 2x_2$$

Hint: **The characteristic equation is** $\lambda^2 + 16 = 0$, **and the eigenvalues are** $\lambda = \pm 4i$.

With $+4i$: $(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} 2-4i & -5 \\ 4 & -2-4i \end{bmatrix}$. Find $\vec{v} = \begin{bmatrix} 1+2i & 2 \end{bmatrix}^T$ from usual method, OR..

Alternative Method: Possible e-vector from first row (switch entries w/ sign change): $\begin{bmatrix} 5 & 2-4i \end{bmatrix}^T$.

Test on 2nd row…

$$\begin{bmatrix} 4 & -2-4i \end{bmatrix}\begin{bmatrix} 5 & 2-4i \end{bmatrix}^T = 4(5) + (-2-4i)(2-4i) = 20 - 4 - 8i + 8i - 16 = 0. \checkmark$$

Or with $-4i$: $(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} 2+4i & -5 \\ 4 & -2+4i \end{bmatrix}$. Find $\vec{v} = \begin{bmatrix} 1-2i & 2 \end{bmatrix}^T$ from usual method, OR..

Alternative Method: Possible e-vector from first row (switch entries w/ sign change):

$\begin{bmatrix} 5 & 2+4i \end{bmatrix}^T$.

Using $\vec{v} = \begin{bmatrix} 5 & 2-4i \end{bmatrix}^T$: $\quad \vec{v}e^{4it} = \begin{bmatrix} 5 \\ 2-4i \end{bmatrix}(\cos 4t + i\sin 4t) = \begin{bmatrix} 5(\cos 4t + i\sin 4t) \\ (2-4i)(\cos 4t + i\sin 4t) \end{bmatrix}$

$$= \begin{bmatrix} 5\cos 4t + 5i\sin 4t \\ 2(\cos 4t + i\sin 4t) - 4i(\cos 4t + i\sin 4t) \end{bmatrix} = \begin{bmatrix} 5\cos 4t + 5i\sin 4t \\ 2\cos 4t + 2i\sin 4t - 4i\cos 4t + 4\sin 4t \end{bmatrix}$$

$$= \begin{bmatrix} 5\cos 4t \\ 2\cos 4t + 4\sin 4t \end{bmatrix} + i\begin{bmatrix} 5\sin 4t \\ 2\sin 4t - 4\cos 4t \end{bmatrix}.$$

The general solution (from $\begin{bmatrix} 5 & 2 \pm 4i \end{bmatrix}^T$)…

$\vec{x} = c_1\begin{bmatrix} 5\cos 4t \\ 2\cos 4t + 4\sin 4t \end{bmatrix} + c_2\begin{bmatrix} 5\sin 4t \\ 2\sin 4t - 4\cos 4t \end{bmatrix}$, or some constant multiple.

From $\begin{bmatrix} 1 \pm 2i & 2 \end{bmatrix}^T$ ...

OR $\vec{x} = c_1\begin{bmatrix} \cos 4t - 2\sin 4t \\ 2\cos 4t \end{bmatrix} + c_2\begin{bmatrix} 2\sin 4t + 2\cos 4t \\ 2\sin 4t \end{bmatrix}$, or some constant multiple.

MIDTERM EXAM I, MATH 2243 (030), FALL 2020

This exam contains 6 problems. To receive full credit on a problem, you must show and explain your work.

1. Determine for what values of k the following system

$$3x + 2y = 1$$
$$6x + ky = 3$$

has

(a) (6 points) a unique solution

(b) (6 points) no solution

(c) (6 points) infinitely many solutions

2. Consider the system

$$4x_1 + 3x_2 + 2x_3 = 6$$
$$3x_1 + 5x_2 + 2x_3 = 10$$
$$5x_1 + 6x_2 + 3x_3 = 9$$

(a) (6 points) Write down the augmented coefficient matrix $\mathbf{M}$ of the system

(b) (6 points) Use the method of Gauss-Jordan elimination to transform the augmented coefficient matrix $\mathbf{M}$ to the reduced echelon form.

(c) (6 points) Use (b) to solve the system.

3. Consider the system

$$2x_1 + 3x_2 + 4x_3 = 2$$
$$4x_1 + 9x_2 + 16x_3 = 1$$
$$x_1 + x_2 + x_3 = 3$$

(a) (6 points) Write down the coefficient matrix $\mathbf{A}$ of the system and the corresponding matrix equation $\mathbf{Ax} = \mathbf{b}$.

(b) (10 points) Compute the determinant $\det(\mathbf{A})$ and the cofactor matrix $[\mathbf{A}_{ij}]$ of $\mathbf{A}$, and use the formula of the inverse for matrices to find $\mathbf{A}^{-1}$.

(c) (6 points) Use the formula $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ to solve the system.

1

4. (12 points) Consider the following four vectors in $\mathbf{R}^3$:

$$\mathbf{v}_1 = (2,1,3), \mathbf{v}_2 = (1,3,4), \mathbf{v}_3 = (2,5,4), \mathbf{v}_4 = (1,1,1)$$

If they are linearly independent, show this; otherwise find real numbers c_1, c_2, c_3, c_4 not all zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = 0$.

5. (12 points) Find a basis of the solution space of the homogenous linear system

$$\begin{aligned} 3x_1 + x_2 + 4x_3 + 18x_4 &= 0 \\ x_1 - 4x_2 - 3x_3 - 7x_4 &= 0 \\ 2x_1 - x_2 + x_3 + 7x_4 &= 0 \end{aligned}$$

6. Consider the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 1 & 0 & 3 & 4 \\ 3 & -2 & 7 & 0 \\ 3 & -1 & 8 & 6 \\ 0 & 1 & 1 & 7 \end{bmatrix}$$

(a) (6 points) Find a basis of the row space of $\mathbf{A}$ and use it to find the rank of $\mathbf{A}$.

(b) (6 points) Find a basis of the column space of $\mathbf{A}$

(c) (6 points) Find a subset of the vectors $\mathbf{v}_1 = (1,1,3,3,0), \mathbf{v}_2 = (-1,0,-2,-1,1), \mathbf{v}_3 = (2,3,7,8,1), \mathbf{v}_4 = (-2,4,0,6,7)$ that forms a basis for the subspace $\mathbf{W}$ of $\mathbf{R}^5$ spanned by these four vectors.

1) $\begin{bmatrix} 3 & 2 & 1 \\ 6 & k & 3 \end{bmatrix}$

$\xrightarrow{1/3 R_1} \begin{bmatrix} 1 & 2/3 & 1/3 \\ 6 & k & 3 \end{bmatrix}$

$\xrightarrow{R_2 - 6R_1} \begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & k-4 & 2 \end{bmatrix}$

$k - 4 = 2$

$\Rightarrow k = 6$

SO THERE IS

a) A UNIQUE SOLUTION FOR $k = 6$

b) NO SOLUTION FOR $k \neq 6$

c) INFINITELY MANY SOLUTIONS FOR NO VALUES OF k

---

2)

a) $M = \left[\begin{array}{ccc|c} 4 & 3 & 2 & 6 \\ 3 & 5 & 2 & 10 \\ 5 & 6 & 3 & 9 \end{array}\right]$

b) $\text{rref}(M) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -33 \end{array}\right]$

c) So $x_1 = 12$, $x_2 = 8$, $x_3 = -33$

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3) a) $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 9 & 16 \\ 1 & 1 & 1 \end{bmatrix}$ $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

b) $\det A = +1 \cdot \begin{vmatrix} 3 & 4 \\ 9 & 16 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 4 \\ 4 & 16 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix}$

$= +2$

$[A_{ij}] = \begin{bmatrix} -7 & 12 & -5 \\ 1 & -2 & 1 \\ 12 & -16 & 6 \end{bmatrix}$

$$A^{-1} = \frac{[A_{ij}]^T}{|A|} = \frac{1}{2}\begin{bmatrix} -7 & 12 & -5 \\ 1 & -2 & 1 \\ 12 & -16 & 6 \end{bmatrix}$$

c) $$\frac{1}{2}\begin{bmatrix} -7 & 12 & -5 \\ 1 & -2 & 1 \\ 12 & -16 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 23 \\ -26 \\ 9 \end{bmatrix}$$

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4) $v_1 = (2,1,3)$, $v_2 = (1,3,4)$, $v_3 = (2,5,4)$
$v_4 = (1,1,1)$

FOUR VECTORS IN $\mathbb{R}^3$ ARE ALWAYS LINEARLY DEPENDENT $\frac{2}{15}v_1 - \frac{1}{15}v_2 + \frac{4}{15}v_3 - v_4 = 0$

---

5 $$\begin{bmatrix} 3 & 1 & 4 & 18 & 0 \\ 1 & -4 & -3 & -7 & 0 \\ 2 & -1 & 1 & 7 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 & 5 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

SO LETTING $x_3 = s, x_4 = t$, WE SEE

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s-5t \\ -s-3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

SO A BASIS FOR THE SOLUTION SPACE IS

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

6)

$$A = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 1 & 0 & 3 & 4 \\ 3 & -2 & 7 & 0 \\ 3 & -1 & 8 & 6 \\ 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) THE ROW SPACE HAS BASIS $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

SO $\operatorname{rk} A = 3$

b) THE COLUMN SPACE HAS BASIS $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 0 \\ 6 \\ 7 \end{bmatrix} \right\}$

SINCE THOSE ARE THE COLS. CORRESPONDING TO PIVOTS.

c) SINCE THE COLUMN SPACE IS THE SPACE WE ARE SPANNING, THE BASIS FOR W IS THE SAME AS THE BASIS FOR COLUMN SPACE. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 0 \\ 6 \\ 7 \end{bmatrix} \right\}$

Midterm 2, Lecture 20

1. **Solve the initial value problem**: $xy' + 2y = 4x^2,\ y(1) = 2.$

This is a linear first order nonhomogeneous differential equation.

Putting it in standard form: $y' + \frac{2}{x}y = 4x \quad \Rightarrow \quad \rho = e^{2\int \frac{1}{x}dx} = e^{2\ln|x|} = x^2.$

Therefore, $y = x^{-2}\int x^2(4x)dx = 4x^{-2}\int x^3 dx = 4x^{-2}\left(\frac{1}{4}x^4 + c\right).$

Applying initial condition: $2 = 4\left(\frac{1}{4} + c\right), \quad \Rightarrow \quad c = \frac{1}{4}.$

Therefore, $y = 4x^{-2}\left(\frac{1}{4}x^4 + \frac{1}{4}\right) = x^2 + x^{-2}.$

---

2. **Find the initial value problem**: $y'' + y' + y = 0, \quad y(0) = 1, \quad y'(0) = 3.$

Characteristic equation: $r^2 + r + 1 \quad \Rightarrow \quad r = \frac{-1\pm\sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Euler Formula: $e^{rx} = e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x} = e^{-\frac{1}{2}x}e^{i\frac{\sqrt{3}}{2}x} = e^{-\frac{1}{2}x}\left(\cos\frac{\sqrt{3}}{2}x + i\sin\frac{\sqrt{3}}{2}x\right).$

Therefore: $y_g = e^{-\frac{1}{2}x}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right).$

The first initial condition gives: $1 = c_1.$

Taking the derivative for the next initial condition:
$y_g' = -\frac{1}{2}e^{-\frac{1}{2}x}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + \frac{\sqrt{3}}{2}e^{-\frac{1}{2}x}\left(-c_1\sin\frac{\sqrt{3}}{2}x + c_2\cos\frac{\sqrt{3}}{2}x\right).$

Applying the initial condition: $3 = -\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 = -\frac{1}{2}(1) + \frac{\sqrt{3}}{2}c_2 \quad \Rightarrow \quad c_2 = \left(3 + \frac{1}{2}\right)\frac{2}{\sqrt{3}} = \frac{7}{\sqrt{3}}.$

Therefore: $y_p = e^{-\frac{1}{2}x}\left(\cos\frac{\sqrt{3}}{2}x + \frac{7}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}x\right).$

---

3. **Find a particular solution to the nonhomogeneous equation**: $y'' + 4y' + 4y = e^{-2x}.$

Characteristic equation: $r^2 + 4r + 4 = (r+2)^2 \quad \Rightarrow \quad r \in \{-2, -2\}.$

Complementary solution: $y_c = c_1e^{-2x} + c_2xe^{-2x}.$

Pre-trial solution: $y_i = Ce^{-2x}.$

Clearing the dependencies, we get the trial solution: $y_{trial} = Cx^2e^{-2x}.$

Taking derivatives: $y_{trial}' = 2C(1-x)xe^{-2x}, \quad y_{trial}'' = 2C(2x^2 - 4x + 1)e^{-2x}.$

Substituting this into our differential equation: $y_{trial}'' + 4y_{trial}' + 4y_{trial}$

$= 2C(2x^2 - 4x + 1)e^{-2x} + 8C(1-x)xe^{-2x} + 4Cx^2e^{-2x} = 2Ce^{-2x}.$

Setting this equal to $f(x)$: $2Ce^{-2x} = e^{-2x}.$

Comparing sides of the equation gives us: $2C = 1$ or $C = \frac{1}{2}.$

Therefore: $y_p = \frac{1}{2}x^2e^{-2x}.$

---

4. **Consider the differential equation**: $xy' = 6y$.

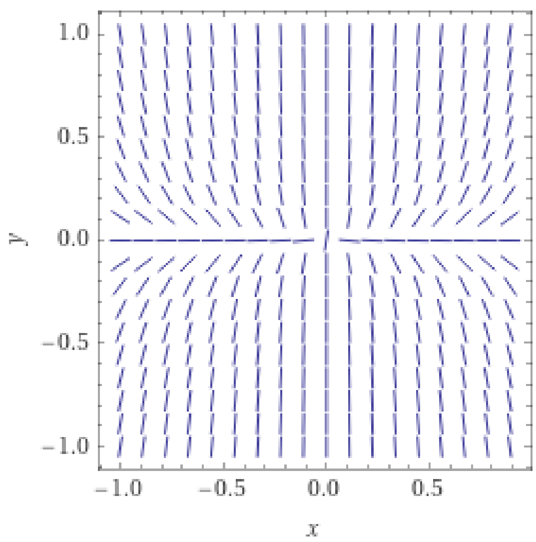
1) Find the singular solutions and the general solutions.

It is separable, so: $\int \frac{1}{y} dy = \int \frac{6}{x} dx$, when $y \neq 0$.

$\ln|y| = 6\ln|x| + c \quad \Rightarrow \quad |y| = e^{\ln x^6 + c} \quad \Rightarrow \quad y = Cx^6$, where $C \neq 0$.

But what about when $y \equiv 0$? Note that $xy' = 6y$ is satisfied for the function $y(x) = 0$. So, to include this singular solution, we say $y = Cx^6$, where $C \in \mathbb{R}$.

2) Sketch the direction field of the differential equation.



3) Show that there are infinitely many solutions of the differential equation with initial value $y(0) = 0$.

Note that with this initial value we have $0 = C0^6$ satisfied for any value of C. This gives us an infinite family of solutions going thru this initial value.

4) Explain why part 3 does not contradict the uniqueness theorem for differential equations.

Note that the uniqueness theorem requires that the coefficient functions be continuous around the initial value. However, in this case, in standard form our differential equation is $y' - \frac{6}{x}y = 0$, and the coefficient function $-\frac{6}{x}$ is not continuous around $(x, y) = (0, 0)$. Therefore, the uniqueness theorem does not apply, so there is no contradiction.

5. **Consider the functions** $f_1 = e^x$ **and** $f_2 = xe^x$ **on the real line.**

a) Compute the Wronskian of f_1 **and** f_2.

$$W(f_1,f_2) = \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)e^x \end{vmatrix} = e^x \begin{vmatrix} 1 & xe^x \\ 1 & e^x(1+x) \end{vmatrix} = e^{2x} \begin{vmatrix} 1 & x \\ 1 & 1+x \end{vmatrix} = e^{2x}(1+x-x) = e^{2x} \neq 0$$

b) Are the functions f_1,f_2 **linearly independent? If your answer is yes, please explain why. If your answer is no, please find constants** $c_1, c_2,$ **not all zero, such that** $c_1 f_1 + c_2 f_2 = 0$.

Using part a, since there is no interval on the real line in which the Wronskian e^{2x} is equivalent to the zero function, the functions f_1,f_2 are linearly independent

Alternatively, observe that f_1,f_2 are linearly independent if they are not scalar multiples of each other.

Using proof by contradiction, assume they are scalar multiples of each other, we get $f_1 = cf_2$ $\Rightarrow \quad e^x = cxe^x \quad \Rightarrow \quad \frac{1}{c} = x$. However, x is not a constant. So our assumption must be wrong, and it must be that the two functions are not scalar multiples of each other, and therefore linearly independent.

Midterm 2 Practice Sheet Solutions

1. Find the general solution of $xy' + 3y = \frac{\sin x}{x^2}$.

Putting it in standard format: $y' + \frac{3}{x}y = \frac{\sin x}{x^3}$. Note this is first-order linear.

$\rho = e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = e^{\ln|x|^3} = x^3$. Therefore, $y = \frac{1}{x^3}\int x^3 \frac{\sin x}{x^3}dx = -\frac{1}{x^3}\cos x + \frac{c}{x^3}$.

---

2. Find the general solution of $y' = \frac{3x^2 - e^x}{2y-5}$.

Note this is separable. $(2y-5)dy = (3x^2 - e^x)dx \quad \Rightarrow \quad \int(2y-5)dy = \int(3x^2 - e^x)dx$

$\Rightarrow \quad y^2 - 5y = x^3 - e^x + c.$

You could solve explicitly for y, using the quadratic equation: $y = \frac{5 \pm \sqrt{25 - 4\left(-x^3 + e^x - c\right)}}{2}$.

---

3. Use Euler's method w/step $h = 0.1$ to find a numeric solution of initial value problem at $x = 0.1,\ 0.2$.
$y' = x^2 + y^2, \quad y(0) = 1.$

$y_1 = y_0 + hf(x_0, y_0) = 1 + (0.1)(0^2 + 1^2) = 1.1$ at $x_1 = 0.1$.

$y_2 = y_1 + hf(x_1, y_1) = 1.1 + (0.1)(0.1^2 + 1.1^2) = 1.222$ at $x_2 = 0.2$.

---

4. Find the general solution of $y'' - 2y' + y = 0$.

$r^2 - 2r + 1 = (r-1)^2 \quad \Rightarrow \quad r \in \{1, 1\}$. Therefore, $y_g = c_1e^x + c_2xe^x$.

---

5. Solve the initial value problem: $y'' + y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0$.

$r^2 + r + 1 = 0 \quad \Rightarrow \quad r = \frac{-1 \pm \sqrt{1-4}}{2} \quad \Rightarrow \quad r \in \left\{-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right\}$.

$e^{\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x} = e^{-\frac{1}{2}x}e^{i\frac{\sqrt{3}}{2}x} = e^{-\frac{1}{2}x}\left(\cos\frac{\sqrt{3}}{2}x + i\sin\frac{\sqrt{3}}{2}x\right)$

$y_g = e^{-\frac{1}{2}x}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right)$.

Initial condition: $1 = e^0(c_1) \quad \Rightarrow \quad c_1 = 1$.

Initial condition: $y_g' = -\frac{1}{2}e^{-\frac{1}{2}x}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + \frac{\sqrt{3}}{2}e^{-\frac{1}{2}x}\left(-c_1\sin\frac{\sqrt{3}}{2}x + c_2\cos\frac{\sqrt{3}}{2}x\right)$

$$0 = -\frac{1}{2}e^0(c_1) + \frac{\sqrt{3}}{2}e^0(c_2) = -\frac{c_1}{2} + \frac{\sqrt{3}}{2}c_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}c_2 \quad \Rightarrow \quad c_2 = \frac{1}{\sqrt{3}}.$$

Therefore, $y_p = e^{-\frac{1}{2}x}\left(\cos\frac{\sqrt{3}}{2}x + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}x\right)$.

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6. Find the general solution to the inhomogeneous equation: $y'' + 2y' + y = 2e^t + \cos t + t$.

$r^2 + 2r + 1 = (r+1)^2 \Rightarrow r \in \{-1, -1\} \Rightarrow y_c = c_1 e^{-t} + c_2 t e^{-t}$.

$y_0 = Ae^t + (B\cos t + C\sin t) + (D + Et)$

Observe that the pretrialsolution is linearly independent from the complementary solution, therefore $y_0 = y_{trial} = Ae^t + (B\cos t + C\sin t) + (D + Et)$.

Taking derivatives: $y'_{trial} = Ae^t + (-B\sin t + C\cos t) + E$,

$y''_{trial} = Ae^t - (B\cos t + C\sin t)$.

Therefore: $y''_{trial} + 2y'_{trial} + y_{trial}$

$= Ae^t - (B\cos t + C\sin t) + 2(Ae^t + (-B\sin t + C\cos t) + E) + Ae^t + (B\cos t + C\sin t) + (D + Et)$

$= (A + 2A + A)e^t + ((B + 2C - B)\cos t + (C - C - 2B)\sin t) + Et + 2E + D$

$= 4Ae^t + 2(C\cos t - B\sin t) + Et + 2E + D$.

Comparing this to the right hand side of our given equation: $2e^t + \cos t + t$, we have $4A = 2,\quad 2C = 1,\quad -2B = 0,\quad E = 1,\quad 2E + D = 0$.

From this, we see that $A = \frac{1}{2},\ B = 0,\ C = \frac{1}{2},\ E = 1$, and $D = -2$.

Therefore, $y_p = \frac{1}{2}e^t + \frac{1}{2}\sin t - 2 + t$.

Finally, the general solution is $y_g = y_c + y_p = c_1 e^{-t} + c_2 t e^{-t} + \frac{1}{2}e^t + \frac{1}{2}\sin t - 2 + t$.

---

7. Given an example of a 2nd order nonlinear differential equation.

In general, linear 2nd order is of the form: $a(x)(y'')^i + b(x)(y')^j + c(x)y^k = d(x)$, where $i = j = k = 1$ and $a(x) \neq 0$.

So, if i, j, or k is not equal to one, we have nonlinear. An example is when $a(x) = 1,\ i = 2$, and $b(x) = c(x) = d(x) = 0$.

In other words: $(y'')^2 = 0$. Another way to have nonlinear is if $a(x, y, y')$, for example if $a(x, y, y') = xy$. If we use this in the previous example, but let $i = 1$, we have $xyy'' = 0$ as a nonlinear equation.

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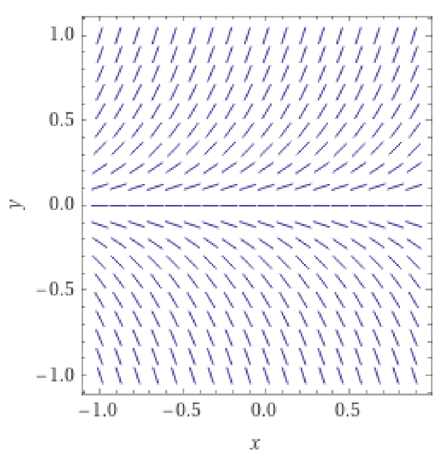
8. Sketch the slope fields of the differential equation: $y' = 3y$.



9. Find the general solution of $y^{(4)} + 2y^{(2)} + y = 0$.

$r^4 + 2r^2 + 1 = (r^2 + 1)^2 \quad \Rightarrow \quad r \in \{\pm i, \pm i\}$

$e^i = \cos x + i \sin x$

$y_g = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x.$

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10. Consider the logistic equation: $y' = y(3 - y)$.

a) Find the critical points and the corresponding equilibrium solutions.

$f(x) = y(3 - y) = 0$ when $y \in \{0, 3\}$.

b) Determine whether each critical point is stable or unstable.

Checking x-values on either side of the critical points, we find $f(-1) = -1(4) = -4 < 0$, $f(1) = 2 > 0$, $f(4) = 4(-1) = -4 < 0$. This gives us the phase diagram: $\leftarrow \ 0 \ \rightarrow \ 3 \ \leftarrow$

Therefore, $x = 0$ is an unstable critical point, and $x = 3$ is a stable critical point.

c) Find the general solution.

$\int \frac{1}{y(3-y)} dy = \int dx \qquad (*)$

Partial fractions: $\frac{1}{y(3-y)} = \frac{A}{y} + \frac{B}{3-y}$ when $1 = A(3 - y) + By = (B - A)y + 3A$, and comparing powers of y, we have: $3A = 1$ and $B - A = 0$. Therefore, $A = \frac{1}{3}$ and $B = \frac{1}{3}$.

$\int \frac{1}{y(3-y)} dy = \frac{1}{3} \int \frac{1}{y} + \frac{1}{3-y} dy = \frac{1}{3}(\ln|y| - \ln|3 - y|) + C_0 = \frac{1}{3} \ln\left|\frac{y}{3-y}\right| + C_0.$

Therefore from $(*)$, we have: $\frac{1}{3} \ln\left|\frac{y}{3-y}\right| = x + C_1$ or $\left|\frac{y}{3-y}\right| = e^{3x+3C_1} = C_2 e^{3x}$ where $C_2 > 0$.

Removing the absolute value: $\frac{y}{3-y} = Ce^{3x}$, where $C \neq 0$.

Solving explicitly: $y = Ce^{3x}(3 - y) = 3Ce^{3x} - yCe^{3x} \quad \Rightarrow \quad y(1 + Ce^{3x}) = 3Ce^{3x} \quad \Rightarrow \quad y = \frac{3Ce^{3x}}{1+Ce^{3x}}$.

---

11. Computer the Wronskian of the functions: $y_1 = e^{3x}$, $y_2 = \sin x$, $y_3 = \cos x$, and use it to show that y_1, y_2, and y_3 are linearly independent.

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^{3x} & \sin x & \cos x \\ 3e^{3x} & \cos x & -\sin x \\ 9e^{3x} & -\sin x & -\cos x \end{vmatrix} = e^{3x} \begin{vmatrix} 1 & \sin x & \cos x \\ 3 & \cos x & -\sin x \\ 9 & -\sin x & -\cos x \end{vmatrix} \stackrel{R_3+R_1}{=} e^{3x} \begin{vmatrix} 1 & \sin x & \cos x \\ 3 & \cos x & -\sin x \\ 10 & 0 & 0 \end{vmatrix}$$

$= 10e^{3x} \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = 10e^{3x}(-\sin^2 x - \cos^2 x) = -10e^{3x} \neq 0$, therefore y_1, y_2, and y_3 are linearly independent.

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Midterm 1 Practice Problems

1. Determine for what values of k the following system has (a) a unique solution, (b) no solution, (c) infinitely many solutions.

$$\mathbf{3x + 2y = 1}$$
$$\mathbf{7x + 5y = k}.$$

$$\begin{bmatrix} 3 & 2 & | & 1 \\ 7 & 5 & | & k \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 2 & | & 1 \\ 1 & 1 & | & k-2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & | & k-2 \\ 3 & 2 & | & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & | & k-2 \\ 0 & -1 & | & 7-3k \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & | & k-2 \\ 0 & 1 & | & 3k-7 \end{bmatrix}$$

$y = 3k - 7$ and $x = -y + k - 2 = -(3k - 7) + k - 2 = 5 - 2k$.

$\vec{v} = (5 - 2k, 3k - 7) = k(-2, 3) + (5, -7)$.

a) Unique solution for every value of k.

b) There are no values of k that will give no solution.

c) There are no values of k that will give infinitely many solution.

---

2. Consider the system:

$$\mathbf{4x_1+5x_2+3x_3= 6}$$
$$\mathbf{3x_1+6x_2+5x_3= 12}$$
$$\mathbf{2x_1+3x_2+2x_3= 18}$$

a) Write down the augmented coefficient matrix M of the system.

$$\begin{bmatrix} 4 & 5 & 3 & | & 6 \\ 3 & 6 & 5 & | & 12 \\ 2 & 3 & 2 & | & 18 \end{bmatrix}$$

b) Use the method of Gaussian elimination to transform the augmented coefficient matrix M to and echelon form matrix.

$$\Rightarrow \begin{bmatrix} 4 & 5 & 3 & | & 6 \\ 1 & 3 & 3 & | & -6 \\ 2 & 3 & 2 & | & 18 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 3 & | & -6 \\ 4 & 5 & 3 & | & 6 \\ 2 & 3 & 2 & | & 18 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 3 & | & -6 \\ 0 & -7 & -9 & | & 30 \\ 0 & -3 & -4 & | & 30 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 3 & | & -6 \\ 0 & -1 & -1 & | & -30 \\ 0 & -3 & -4 & | & 30 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 3 & | & -6 \\ 0 & -1 & -1 & | & -30 \\ 0 & 0 & -1 & | & 120 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 3 & | & -6 \\ 0 & 1 & 1 & | & 30 \\ 0 & 0 & 1 & | & -120 \end{bmatrix}$$

c) Use the method Gauss Jordan elimination to transform the augmented coefficient matrix M to the reduced echelon matrix.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & | & -96 \\ 0 & 1 & 0 & | & 150 \\ 0 & 0 & 1 & | & -120 \end{bmatrix}$$

d) Use either b or c to solve the system.

$x_3 = -120, \qquad x_2 = 150, \qquad x_1 = -96.$

---

3. List all possible reduced row echelon forms of a 3 x 3 matrix.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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4. Consider the system:

$$x_1 + 2x_2 + 4x_3 = 1$$
$$x_1 + 3x_2 + 9x_3 = 2$$
$$x_1 + 4x_2 + 16x_3 = 3$$

a) Write down the coefficient matrix A of the system and the corresponding matrix equation $Ax = b$.

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

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b) Use the algorithm explained in the class (see p181 **of the textbook) to find the inverse of** A.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 1 & 3 & 9 & 0 & 1 & 0 \\ 1 & 4 & 16 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 2 & 12 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -6 & 3 & -2 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -8 & 3 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -8 & 3 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -8 & 3 \\ 0 & 1 & 0 & -\frac{7}{2} & 6 & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right], \quad A^{-1} = \begin{bmatrix} 6 & -8 & 3 \\ -\frac{7}{2} & 6 & -\frac{5}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}.$$

c) Compute the determinant $\det(A)$ **and the cofactor matrix** $[A_{ij}]$ **of** A**, and use the formula of the inverse for matrices to find** A^{-1}.

$$\det(A) = \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 2 & 12 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{vmatrix} = 2.$$

$$[A_{ij}] = \begin{bmatrix} 12 & -7 & 1 \\ -16 & 12 & -2 \\ 6 & -5 & 1 \end{bmatrix}, \qquad [A_{ij}]^T = \begin{bmatrix} 12 & -16 & 6 \\ -7 & 12 & -5 \\ 1 & -2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 12 & -16 & 6 \\ -7 & 12 & -5 \\ 1 & -2 & 1 \end{bmatrix}.$$

d) Use the formula $x = A^{-1}b$ **to solve the system**.

$$x = A^{-1}b = \frac{1}{2}\begin{bmatrix} 12 & -16 & 6 \\ -7 & 12 & -5 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

e) Use Cramer's rule to solve the system.

$$x_1 = \frac{|B_1|}{|A|} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 9 \\ 3 & 4 & 16 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & -2 & 4 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = -1,$$

$$x_2 = \frac{|B_2|}{|A|} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 4 \\ 1 & 2 & 9 \\ 1 & 3 & 16 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 2 & 12 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{vmatrix} = 1,$$

$$x_3 = \frac{|B_3|}{|A|} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 4 & 3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0,$$

$x = (x_1, x_2, x_3) = (-1, 1, 0)$.

---

5. Consider the following 3 vectors in R^4:

$\mathbf{v}_1=(1,1,2,1)$, $\mathbf{v}_2=(1,0,3,4)$, $\mathbf{v}_3=(2,2,4,8)$.

If they are linearly independent, show this. Otherwise, find real numbers c_1, c_2, c_3 not all zero such that $c_1v_1 + c_2v_2 + c_3v_3 = 0$.

Observe that:

$$c_1v_1 + c_2v_2 + c_3v_3 = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 3 & 4 \\ 1 & 4 & 8 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{0}.$$

Gaussian Elimination:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 3 & 4 \\ 1 & 4 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 3 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

They are linearly independent.

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6. Consider the following 4 vectors in R^3:

$\mathbf{v}_1=(1,1,2)$, $\mathbf{v}_2=(1,3,4)$, $\mathbf{v}_3=(2,2,4)$, $\mathbf{v}_4=(0,0,1)$.

If they are linearly independent, show this. Otherwise, find real numbers c_1, c_2, c_3, c_4, not all zero such that $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$.

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 3 & 2 & 0 \\ 2 & 4 & 4 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \vec{0}.$$

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Gaussian Elimination:

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 3 & 2 & 0 \\ 2 & 4 & 4 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So: $c_3 = t, \quad c_1 = -2t, \quad c_2 = 0, \quad c_4 = 0.$

And: $\vec{c} = (c_1, c_2, c_3, c_4) = (-2t, 0, t, 0) = t(-2, 0, 1, 0).$

And observe that $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = -2v_1 + 0v_2 + v_3 + 0v_4 = 0.$

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7. **Find a basis for the following vector spaces**:

a) **The set of all vectors of the form** (a, b, c, d) **for which** $a + 2d = c + 3d = 0.$

So vectors have the form: $(-2d, b, -3d, d) = b(0, 1, 0, 0) + d(-2, 0, -3, 1).$

From this, we discover the vectors $\{(0, 1, 0, 0), (-2, 0, -3, 1)\}$ span the set of given vectors.

Now let's verify that they are linearly independent. Looking at the 3 x 3 submatrices, we calculate the sub determinants:

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 3 & 4 \end{vmatrix} = -1(4-6) = 2 \neq 0. \qquad \begin{vmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 1 & 4 & 8 \end{vmatrix} = 1(24-16) + 1(0-6) = 2 \neq 0.$$

So we see that these vectors are linearly independent. Therefore, we have a basis $\{(0, 1, 0, 0), (-2, 0, -3, 1)\}$ for our subspace of vectors.

b) **The solution space of the homogeneous linear system**:

$$x_1 - 3x_2 - 10x_3 + 5x_4 = 0$$
$$x_1 - 4x_2 + 11x_3 - 2x_4 = 0$$
$$x_1 - 3x_2 + 8x_3 - x_4 = 0.$$

Putting things into a matrix, and performing Gaussian reduction:

$$\begin{bmatrix} 1 & -3 & -10 & 5 \\ 1 & -4 & 11 & -2 \\ 1 & -3 & 8 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & -10 & 5 \\ 0 & -1 & 21 & -7 \\ 0 & 0 & 18 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & -10 & 5 \\ 0 & 1 & -21 & 7 \\ 0 & 0 & 3 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -73 & 26 \\ 0 & 1 & -21 & 7 \\ 0 & 0 & 1 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \frac{5}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} \end{bmatrix}$$

Applying an arbitrary parameter t to our free column x_4, gives us $x_3 = \frac{1}{3}t, \ x_2 = 0,$ and $x_1 = -\frac{5}{3}t.$

Therefore, $\vec{x} = \left(x_1,\ x_2,\ x_3,\ x_4\right) = \left(-\frac{5}{3}t,\ 0,\ \frac{1}{3}t,\ t\right) = t\left(-\frac{5}{3}, 0, \frac{1}{3}, 1\right)$.

And finally, we see a basis for our solution subspace by setting t to any value. To make our basis look simple, I will choose to set $t = 3$, so our basis becomes: $\{(-5, 0, 1, 3)\}$.

---

8. Consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & 2 \\ -3 & 4 & 1 & 2 \\ 0 & 1 & 7 & 4 \\ -5 & 7 & 4 & -2 \end{bmatrix}$$

a) Find a basis of the row space of A.

Doing our Gaussian reduction this:

$$\begin{bmatrix} 1 & -1 & 2 & 2 \\ -3 & 4 & 1 & 2 \\ 0 & 1 & 7 & 4 \\ -5 & 7 & 4 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 2 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 1 & 7 & 4 \\ 0 & 2 & 14 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 2 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbb{E}.$$

For the row space, we take at the nonzero rows of our reduced system:

$$\text{Basis} = \left\{ \begin{bmatrix} 1 & 0 & 9 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 7 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \right\}$$

b) Find a basis of the column space of A.

For the column space, we look back at the original matrix A, and the basis consists of the columns in A corresponding to the pivot columns in $\mathbb{E}$. Note that the columns in $\mathbb{E}$ that had the "leading ones" (the pivot columns) were columns $1, 2, 4$. So taking those columns from A, gives us:

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \\ -2 \end{bmatrix} \right\}.$$

9. **Find a subset of the vectors** $v_1 = (1,-3,0,5),\ v_2 = (-1,4,1,7),\ v_3 = (2,1,7,4),\ v_4 = (2,2,4,-2)$ **that forms a basis for the subspace** W **of** R^4 **spanned by these 4 vectors**.

Placing the vectors into columns of a matrix, we see:

$$\begin{vmatrix} 1 & -1 & 2 & 2 \\ -3 & 4 & 1 & 2 \\ 0 & 1 & 7 & 4 \\ -5 & 7 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 1 & 7 & 4 \\ 0 & 2 & 14 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 7 & 8 \\ 1 & 7 & 4 \\ 2 & 14 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 7 & 8 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

Therefore, the four vectors are not linearly independent. From the previous problem, we see that the 1st, 2nd, and 4th columns are linearly independent. But only being 3 vectors, we conclude that $\{v_1, v_2,\ v_4\}$ form a basis for a three-dimensional subspace W of $\mathbb{R}^4$.

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10. **Consider the following matrices**:

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}.$$

Calculate the following number of matrices: **a**) $\det(A^{-1})$, **b**) A^T, **c**) **BC**, **d**) **CB**, **e**) **AB**.

a) Recall that: $\det(A^{-1}) = \frac{1}{\det(A)}$. So let's calculate: $\det(A) = \begin{vmatrix} -1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix} = -1 \cdot 1 \cdot 4 = -4.$

Therefore, $\det(A^{-1}) = -\frac{1}{4}$.

b) $A^T = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 4 \end{bmatrix}$

c) $\mathbf{BC} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 5 & 10 & 15 \end{bmatrix}$

d) $\mathbf{CB} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = 22$

e) $\mathbf{AB} = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 13 \\ 20 \end{bmatrix}$