Homework 7 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

5.5.3 We have characteristic equation $r^2 - r - 6 = (r-3)(r+2)$. So the complementary solution is $y_c = c_1e^{3x} + c_2e^{-2x}$.

We have term $\sin 3x$ with derivative $\cos 3x$ and no repetition. So we set $y_p = A\sin 3x + B\cos 3x$. We compute

$$\begin{aligned} y_p'' - y' - 6y &= (-9A\sin 3x - 9B\cos 3x) - (3A\cos 3x - 3B\sin 3x) - 6(A\sin 3x + B\cos 3x) \\ &= (-3A - 15B)\cos 3x + (-15A + 3B)\sin 3x \end{aligned}$$

We now have the equations $-3A - 15B = 0$ and $-15A + 3B = 2$ which solve to $A = -5/39, B = 1/39$. So the particular solution is $y_p = \frac{-5}{39}\sin 3x + \frac{1}{39}\cos 3x$.

5.5.9 We have characteristic equation $r^2 + 2r - 3 = (r-1)(r+3)$. So the complementary solution is $y_c = c_1e^{x} + c_2e^{-3x}$.

We have term 1 with no derivatives and term xe^x with derivative e^x. We have repetition so we bump to x^2e^x, xe^x So we set $y_p = A + Bxe^x + Cx^2e^x$. We compute

$$\begin{aligned} y_p'' + 2y_p' - 3y_p &= ((2B + 2C)e^x + (B + 4C)xe^x + Cx^2e^x) \\ &\quad + 2(Be^x + (B + 2C)xe^x + Cx^2e^x) \\ &\quad - 3(A + Bxe^x + Cx^2e^x) \\ &= -3A + (4B + 2C)e^x + (8C)xe^x + (0)x^2e^x \end{aligned}$$

We now have the equations $-3A = 1$, $4B + 2C = 0$, and $8C = 1$ which solve to $A = -1/3, B = -1/16, C = 1/8$. So our particular solution is $y_p = \frac{-1}{3} - \frac{1}{16}xe^x + \frac{1}{8}x^2e^x$.

5.5.11 We have characteristic equation $r^3 + 4r = r(r^2 + 4)$ with roots $0, \pm 2i$. So our complementary solution is $y_c = c_1 + c_2\cos 2x + c_3\sin 2x$.

We have terms $x, 1$ with duplication, so we bump to x^2, x. So we set $y_p = Ax + Bx^2$. We compute

$$\begin{aligned} y_p^{(3)} + 4y_p' &= (0) + 4(A + 2Bx) \\ &= 4A + 8Bx \end{aligned}$$

We have equations $4A = -1$ and $8B = 3$ which solve to $A = -1/4, B = 3/8$. So the particular solution is $y_p = \frac{-1}{4}x + \frac{3}{8}x^2$.

5.5.23 We have characteristic equation r^2+4 with roots $\pm 2i$, so the complementary solution is $y_c = c_1\cos 2x + c_2\sin 2x$.

We have term $x\cos 2x$ with derivatives $x\sin 2x, \sin 2x, \cos 2x$. We have duplication, so we bump up to $x^2\sin 2x, x^2\cos 2x, x\sin 2x, x\cos 2x$. This gives us a particular solution of the form

$$y_p = Ax\cos 2x + Bx\sin 2x + Cx^2\cos 2x + Dx^2\sin 2x$$

5.5.32 We have characteristic equation $r^2+3r+2 = (r+2)(r+1)$, so we have complementary solution $y_c = c_1e^{-2x} + c_2e^{-x}$.

We have term e^x and no repetition, so we set $y_p = Ae^x$. We compute

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= Ae^x + 3Ae^x + 2Ae^x \\ &= 6Ae^x \end{aligned}$$

So we have the equation $6A = 1$ and thus our particular solution is $y_p = \frac{1}{6}e^x$.

The form of a general solution is now $y = y_c + y_p = c_1e^{-2x} + c_2e^{-x} + \frac{1}{6}e^x$. The initial conditions give us the following:

$$\begin{aligned} y(0) &= c_1e^0 + c_2e^0 + \frac{1}{6}e^0 \\ 0 &= c_1 + c_2 + \frac{1}{6} \\ y'(x) &= -2c_1e^{-2x} - c_2e^{-x} + \frac{1}{6}e^x \\ y'(0) &= -2c_1e^0 - c_2e^0 + \frac{1}{6}e^0 \\ 3 &= -2c_1 - c_2 + \frac{1}{6} \end{aligned}$$

So we have equations $c_1 + c_2 = -\frac{1}{6}$ and $-2c_1 - c_2 = \frac{17}{6}$. This system solves to $c_1 = -\frac{8}{3}$ and $c_2 = \frac{5}{2}$. So, our solution in this case is $y(x) = -\frac{8}{3}e^{-2x} + \frac{5}{2}e^{-x} + \frac{1}{6}e^x$.

6.1.7 First, we compute the characteristic polynomial:

$$\begin{aligned} \det(A-\lambda I) &= \det\begin{bmatrix} 10-\lambda & -8 \\ 6 & -4-\lambda \end{bmatrix} \\ &= (10-\lambda)(-4-\lambda) + 48 \\ &= \lambda^2 - 6\lambda + 8 \\ &= (\lambda-4)(\lambda-2) \end{aligned}$$

The eigenvalues are $\lambda_1 = 4, \lambda_2 = 2$.

For $\lambda_1 = 4$, we have the matrix $\begin{bmatrix} 6 & -8 \\ 6 & -8 \end{bmatrix}$ which reduces to $\begin{bmatrix} 3 & -4 \\ 0 & 0 \end{bmatrix}$. This is a dimension 1 eigenspace with eigenvector $\vec{v}_1 = (4,3)$.

For $\lambda_2 = 2$, we have the matrix $\begin{bmatrix} 8 & -8 \\ 6 & -6 \end{bmatrix}$ which reduces to $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. This is a dimension 1 eigenspace with eigenvector $\vec{v}_2 = (1, 1)$.

6.1.17 First, we compute the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 5 & -2 \\ 0 & 2-\lambda & 0 \\ 0 & 2 & 1-\lambda \end{bmatrix} \\ &= (3-\lambda) \det \begin{bmatrix} 2-\lambda & 0 \\ 2 & 1-\lambda \end{bmatrix} \\ &= (3-\lambda)(2-\lambda)(1-\lambda) \end{aligned}$$

The eigenvalues are $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1$.

For $\lambda_1 = 3$, we have the matrix $\begin{bmatrix} 0 & 5 & -2 \\ 0 & -1 & 0 \\ 0 & 2 & -2 \end{bmatrix}$ which reduces to $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. This is a dimension 1 eigenspace with eigenvector $\vec{v}_1 = (1, 0, 0)$.

For $\lambda_2 = 2$, we have the matrix $\begin{bmatrix} 1 & 5 & -2 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix}$ which reduces to $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. This is a dimension 1 eigenspace with eigenvector $\vec{v}_2 = (-1, 1, 2)$.

For $\lambda_3 = 1$, we have the matrix $\begin{bmatrix} 2 & 5 & -2 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ which reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This is a dimension 1 eigenspace with eigenvector $\vec{v}_3 = (1, 0, 1)$.

6.1.21 First, we compute the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 4-\lambda & -3 & 1 \\ 2 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} \\ &= (2-\lambda) \det \begin{bmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{bmatrix} \\ &= (2-\lambda)\left[(4-\lambda)(-1-\lambda) + 6\right] \\ &= (2-\lambda)\left[\lambda^2 - 3\lambda + 2\right] \\ &= (2-\lambda)(\lambda-2)(\lambda-1) \end{aligned}$$

The eigenvalues are $\lambda_1 = 1, \lambda_2 = 2$ with λ_2 occuring with multiplicity 2.

For $\lambda_1 = 1$, we have the matrix $\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ which reduces to $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. This is a 1-dimensional eigenspace with eigenvector $\vec{v}_1 = (1, 1, 0)$.

For $\lambda_2 = 2$, we have the matrix $\begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ which reduces to $\begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This is a 2-dimensional eigenspace with basis vectors $\vec{v}_2 = (1, 0, -2)$ and $\vec{v}_3 = (3, 2, 0)$.

6.1.25 We compute the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix} \\ &= (1-\lambda)^2(2-\lambda)^2 \end{aligned}$$

Here we are using that the determinant of an upper triangular matrix is the product of the diagonal entries, but you can also see this by successively expanding along the first column. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$, each of multiplicity 2.

For $\lambda_1 = 1$, we have the matrix $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ which reduces to $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ We have a dimension 2 eigenspace with basis vectors $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$.

For $\lambda_2 = 2$, we have the matrix $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which is already sufficiently reduced. This gives a dimension 2 eigenspace with basis vectors $(0, 0, 0, 1)$ and $(1, 1, 1, 0)$.

6.1.29 We compute the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -3 \\ 12 & -\lambda \end{bmatrix} \\ = \lambda^2 + 36 \end{aligned}$$

The eigenvalues are $\pm 6i$.

For $\lambda_1 = 6i$, we have matrix $\begin{bmatrix} -6i & -3 \\ 12 & -6i \end{bmatrix}$. We row reduce this matrix:

$$\begin{aligned} \begin{bmatrix} -6i & -3 \\ 12 & -6i \end{bmatrix} &\xrightarrow{-2iR_1 + R_2} \begin{bmatrix} -6i & -3 \\ 0 & 0 \end{bmatrix} \\ &\xrightarrow{-\frac{1}{3}R_1} \begin{bmatrix} 2i & 1 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

We have a dimension 1 eigenspace containing the eigenvector $\vec{v}_1 = (-1, 2i)$.

For the conjugate eigenvalue $\lambda_2 = -6i$, we have the conjugate eigenvector $\vec{v}_2 = (-1, -2i)$.

Additional Problems:

1. On homework 6, we found the general solution to the homogeneous equation, so we have the complementary solution in this case:

$$y_c(x) = c_1 + c_2 e^x + c_3 x e^x + c_4 \cos(2x) + c_5 \sin(2x) + c_6 x \cos(2x) + c_7 x \sin(2x)$$

The term e^{2x} is not duplicated. The term $x \sin x$ has derivatives $x \cos x, \sin x, \cos x$ and there is no duplication. The term x^2 has derivatives $x, 1$ and there is duplication. It is enough to bump up by a factor of x, so we get new terms x^3, x^2, x. So, the form of our particular solution is

$$y_p = Ae^{2x} + Bx \sin x + Cx \cos x + D \sin x + E \cos x + Fx^3 + Gx^2 + Hx$$

2. We compute the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} t_1 - \lambda & 0 & 0 \\ 0 & t_2 - \lambda & 0 \\ 0 & 0 & t_3 - \lambda \end{bmatrix} \\ &= (t_1 - \lambda)(t_2 - \lambda)(t_3 - \lambda) \end{aligned}$$

The computation of determinants like this was done in Additional Problem 4 of Homework 2. This has three distinct roots, t_1, t_2, t_3.

For $\lambda = t_1$, $A - \lambda I$ is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2 - t_1 & 0 \\ 0 & 0 & t_3 - t_1 \end{bmatrix}$. Notice that since t_1, t_2, t_3 are all distinct, $t_2 - t_1 \neq 0$ and $t_3 - t_1 \neq 0$ so we only have a free variable in the first column. A solution to this system is $\vec{v_1} = (1, 0, 0)$, so this is an eigenvector corresponding to t_1.

For $\lambda = t_2$, $A - \lambda I$ is $\begin{bmatrix} t_1 - t_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t_3 - t_2 \end{bmatrix}$. A solution to this system is $\vec{v_2} = (0, 1, 0)$, so this is an eigenvector corresponding to t_2.

For $\lambda = t_3$, $A - \lambda I$ is $\begin{bmatrix} t_1 - t_3 & 0 & 0 \\ 0 & t_2 - t_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. A solution to this system is $\vec{v_3} = (0, 0, 1)$, so this is an eigenvector corresponding to t_3.

3. Perhaps not surprisingly, the characteristic polynomial is

$$(t_1 - \lambda)(t_2 - \lambda) \cdots (t_n - \lambda) = \prod_{i=1}^{n} (t_i - \lambda)$$

So each t_i is an eigenvalue.

If we set $\lambda = t_i$, $A - \lambda I$ has a 0 in the i-th diagonal element. A solution vector is $\vec{e_i}$, the standard basis vector with 1 in the i-th position and 0 elsewhere.