HW 7

Math 2243
Linear Algebra and Differential Equations

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Contents

1 Problem 3 section 5.5

In Problems 1 through 20, find a particular solution y_p of the given equation.

$$y'' - y' - 6y = 2\sin 3x \tag{A}$$

Solution

The first step is to find the homogeneous solution y_h in order to determine the basis solutions to check for any duplication with basis solutions for the particular solution.

$$y'' - y' - 6y = 0$$

The characteristic equation is

$$\begin{aligned} r^2 - r - 6 &= 0 \\ (r-3)(r+2) &= 0 \end{aligned}$$

Hence the roots are $r_1 = 3, r_2 = -2$. Therefore the basis solutions are

$$\left\{e^{3x}, e^{-2x}\right\} \tag{1}$$

Which implies

$$y_h = c_1 e^{3x} + c_2 e^{-2x}$$

Now that we found the basis solution, we turn our attention to finding y_p. The RHS is $\sin 3x$. Looking at this function and all possible derivatives gives

$$\{\sin 3x, \cos 3x\} \tag{2}$$

Notice that we ignore any leading coefficients when doing this. Now we compare the above to the basis of the homogeneous solution found in (1) to check if there are duplication in basis or not. There is no duplication. Therefore we assume that particular solution y_p is a linear combination of the functions in (2). This implies that

$$\begin{aligned} y_p &= A\sin 3x + B\cos 3x \\ y_p' &= 3A\cos 3x - 3B\sin 3x \\ y_p'' &= -9A\sin 3x - 9B\cos 3x \end{aligned}$$

Substituting the above back in original ODE (A) gives

$$\begin{aligned} y_p'' - y_p' - 6y_p &= 2\sin 3x \\ (-9A\sin 3x - 9B\cos 3x) - (3A\cos 3x - 3B\sin 3x) - 6(A\sin 3x + B\cos 3x) &= 2\sin 3x \\ \sin(3x)(-9A + 3B - 6A) + \cos(3x)(-9B - 3A - 6B) &= 2\sin 3x \\ \sin(3x)(-15A + 3B) + \cos(3x)(-15B - 3A) &= 2\sin 3x \end{aligned}$$

Comparing coefficients gives

$$-15A + 3B = 2 \tag{3}$$

$$-15B - 3A = 0 \tag{4}$$

Multiplying first equation by 5 and adding result to second equation gives

$$\begin{aligned} (-75A + 15B) + (-15B - 3A) &= 10 \\ -78A &= 10 \\ A &= -\frac{10}{78} \\ &= -\frac{5}{39} \end{aligned}$$

From (3)

$$\begin{aligned}
-15\left(-\frac{5}{39}\right) + 3B &= 2 \\
\frac{25}{13} + 3B &= 2 \\
B &= \frac{2 - \frac{25}{13}}{3} \\
&= \frac{1}{39}
\end{aligned}$$

Hence the particular solution is

$$\begin{aligned}
y_p &= A \sin 3x + B \cos 3x \\
&= -\frac{5}{39} \sin 3x + \frac{1}{39} \cos 3x \\
&= \frac{1}{39}(\cos 3x - 5 \sin 3x)
\end{aligned}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= c_1 e^{3x} + c_2 e^{-2x} + \frac{1}{39}(\cos 3x - 5 \sin 3x)
\end{aligned}$$

2 Problem 9 section 5.5

In Problems 1 through 20, find a particular solution y_p of the given equation.

$$y'' + 2y' - 3y = 1 + xe^x \tag{A}$$

Solution

The first step is to find the homogeneous solution y_h in order to determine the basis solutions to check for any duplication with basis solutions for the particular solution.

$$y'' + 2y' - 3y = 0$$

The characteristic equation is

$$\begin{aligned} r^2 + 2r - 3 &= 0 \\ (r+3)(r-1) &= 0 \end{aligned}$$

Hence the roots are $r_1 = -3, r_2 = 1$. Therefore the basis solutions are

$$\left\{e^{-3x}, e^x\right\} \tag{1}$$

Which implies

$$y_h = c_1 e^{-3x} + c_2 e^x$$

Now that we found the basis solution, we turn our attention to finding y_p. The RHS is $1 + xe^x$. Hence it basis functions are

$$\{1, xe^x\}$$

taking derivatives of each basis gives

$$\{1, (xe^x, e^x)\} \tag{2}$$

Where we used () to group all basis generated from same one.

Now we compare the above to the basis of the homogeneous solution found in (1) to check if there are duplication in basis or not. We see duplication since e^x is basis in both (1) and (2). Therefore we multiply the group which generated e^x by x. The the above now becomes

$$\left\{1, \left(x^2 e^x, xe^x\right)\right\} \tag{2A}$$

We compare again (1) against (2A) and now we see no duplication. Therefore we assume that particular solution y_p is a linear combination of the functions in (2A). This implies that

$$\begin{aligned} y_p &= A + Bx^2 e^x + Cxe^x \\ y_p' &= 2Bxe^x + Bx^2 e^x + Ce^x + Cxe^x \\ y_p'' &= 2Be^x + 2Bxe^x + 2Bxe^x + Bx^2 e^x + Ce^x + Ce^x + Cxe^x \\ &= xe^x(2B + 2B + C) + x^2 e^x (B) + e^x(2B + 2C) \end{aligned}$$

Substituting the above back in original ODE (A) gives

$$\begin{aligned} y_p'' + 2y_p' - 3y_p &= 1 + xe^x \\ xe^x(2B + 2B + C) + x^2 e^x(B) + e^x(2B + 2C) + 2\left(2Bxe^x + Bx^2 e^x + Ce^x + Cxe^x\right) - 3\left(A + Bx^2 e^x + Cxe^x\right) &= 1 + xe^x \\ xe^x(2B + 2B + C + 4B + 2C - 3C) + e^x(2B + 2C + 2C) + x^2 e^x(B + 2B - 3B) - 3A &= 1 + xe^x \\ xe^x(8B) + e^x(2B + 4C) - 3A &= 1 + xe^x \end{aligned}$$

Comparing coefficients

$$\begin{aligned} -3A &= 1 \\ 2B + 4C &= 0 \\ 8B &= 1 \end{aligned}$$

Hence $B = \frac{1}{8}$ and from second equation $4C = -\frac{2}{8}$, or $C = -\frac{1}{16}$ and $A = -\frac{1}{3}$. Therefore the particular solution is

$$\begin{aligned} y_p &= A + Bx^2e^x + Cxe^x \\ &= \frac{-1}{3} + \frac{1}{8}x^2e^x - \frac{1}{16}xe^x \\ &= -\frac{1}{3} + \frac{1}{16}\left(2x^2 - x\right)e^x \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1e^{-3x} + c_2e^x - \frac{1}{3} + \frac{1}{16}\left(2x^2 - x\right)e^x \end{aligned}$$

3 Problem 11 section 5.5

In Problems 1 through 20, find a particular solution y_p of the given equation.

$$y^{(3)} + 4y' = 3x - 1 \tag{A}$$

Solution

The first step is to find the homogeneous solution y_h in order to determine the basis solutions to check for any duplication with basis solutions for the particular solution.

$$y^{(3)} + 4y' = 0$$

The characteristic equation is

$$\begin{aligned} r^3 + 4r &= 0 \\ r\left(r^2 + 4\right) &= 0 \end{aligned}$$

Hence the roots are $r_1 = 0, r_2 = \pm 2i$. Therefore the basis solutions are

$$\{1, \cos(2x), \sin(2x)\} \tag{1}$$

Which implies

$$y_h = c_1 + c_2 \cos(2x) + c_3 \sin(2x)$$

Now that we found the basis solution, we turn our attention to finding y_p. The RHS is $3x - 1$. Hence it basis functions are

$$\{1, x\} \tag{2}$$

Taking derivatives does not add any new basis. Now we compare the above to the basis of the homogeneous solution found in (1) to check if there are duplication in basis or not. We see duplication the constant is in both (1) and (2). Therefore we multiply the group by x. We took the whole basis as one group, since the constant 1 above is generated by taking derivative of x, so it is really in the same group. The above now becomes, after multiplying everything by x

$$\left\{x, x^2\right\} \tag{2A}$$

We compare again (1) against (2A) and now we see no duplication. Therefore we assume that particular solution y_p is a linear combination of the functions in (2A). This implies that

$$\begin{aligned} y_p &= Ax + Bx^2 \\ y_p' &= A + 2Bx \\ y_p'' &= 2B \\ y_p^{(3)} &= 0 \end{aligned}$$

Substituting the above back in original ODE (A) gives

$$\begin{aligned} y_p^{(3)} + y_p' &= 3x - 1 \\ 0 + 4(A + 2Bx) &= 3x - 1 \\ 4A + 8Bx &= 3x - 1 \end{aligned}$$

Comparing coefficients

$$\begin{aligned} 4A &= -1 \\ 8B &= 3 \end{aligned}$$

Hence $A = -\frac{1}{4}, B = \frac{3}{8}$. Therefore the particular solution is

$$\begin{aligned} y_p &= Ax + Bx^2 \\ &= -\frac{1}{4}x + \frac{3}{8}x^2 \\ &= \frac{1}{8}\left(3x^2 - 2x\right) \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 + c_2 \cos(2x) + c_3 \sin(2x) - \frac{1}{4}x + \frac{3}{8}x^2 \end{aligned}$$

4 Problem 23 section 5.5

In Problems 21 through 30, set up the appropriate form of a particular solution y_p, but do not determine the values of the coefficients.

$$y'' + 4y = 3x\cos(2x) \tag{A}$$

Solution

The first step is to find the homogeneous solution y_h in order to determine the basis solutions to check for any duplication with basis solutions for the particular solution.

$$y'' + 4y = 0$$

The characteristic equation is

$$r^2 + 4 = 0$$

Hence the roots are $r = \pm 2i$. Therefore the basis solutions are

$$\{\cos(2x), \sin(2x)\} \tag{1}$$

Which implies

$$y_h = c_1\cos(2x) + c_2\sin(2x)$$

Now that we found the basis solution, we turn our attention to finding y_p. The RHS is $3x\cos(2x)$. Hence it basis functions are

$$\{x\cos(2x)\} \tag{2}$$

Taking all possible derivatives of the above gives

$$\{x\cos(2x), \cos(2x), x\sin(2x), \sin(2x)\} \tag{2A}$$

Where in the above all signs and coefficients were ignored.

Now we compare the above to the basis of the homogeneous solution found in (1) to check if there are duplication in basis or not. We see duplication as $\cos(2x), \sin(2x)$ are in both. Therefore we multiply the group by x. We took the whole basis as one group since everything above was generated from (2). The above now becomes, after multiplying each term by x

$$\left\{x^2\cos(2x), x\cos(2x), x^2\sin(2x), x\sin(2x)\right\} \tag{2B}$$

Now we compare (2B) again with (1) and see no duplication. Hence

$$y_p = Ax^2\cos(2x) + Bx\cos(2x) + Cx^2\sin(2x) + Dx\sin(2x)$$

5 Problem 32 section 5.5

Solve the initial value problems in Problems 31 through 40.

$$
\begin{aligned}
y'' + 3y' + 2y &= e^x \qquad \text{(A)}\\
y(0) &= 0\\
y'(0) &= 3
\end{aligned}
$$

Solution

The first step is to find the homogeneous solution y_h in order to determine the basis solutions to check for any duplication with basis solutions for the particular solution.

$$y'' + 3y' + 2 = 0$$

The characteristic equation is

$$
\begin{aligned}
r^2 + 3r + 2 &= 0\\
(r+2)(r+1) &= 0
\end{aligned}
$$

Hence the roots are $r_1 = -2, r_2 = -1$. Therefore the basis solutions are

$$\left\{e^{-2x}, e^{-x}\right\} \tag{1}$$

Which implies

$$y_h = c_1 e^{-2x} + c_2 e^{-x}$$

Now that we found the basis solution, we turn our attention to finding y_p. The RHS is e^x. Hence it basis functions are

$$\{e^x\} \tag{2}$$

Taking all derivatives does not any terms. We also see no duplication between (2) and (1). Hence let

$$
\begin{aligned}
y_p &= Ae^x\\
y_p' &= Ae^x\\
y_p'' &= Ae^x
\end{aligned}
$$

Substituting these into (A) gives

$$
\begin{aligned}
y_p'' + 3y_p' + 2y_p &= e^x\\
Ae^x + 3Ae^x + 2Ae^x &= e^x\\
e^x(A + 3A + 2A) &= e^x
\end{aligned}
$$

Hence

$$
\begin{aligned}
6A &= 1\\
A &= \frac{1}{6}
\end{aligned}
$$

Therefore

$$y_p = \frac{1}{6}e^x$$

Therefore the complete solution is

$$
\begin{aligned}
y &= y_h + y_p\\
&= c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{6}e^x \qquad (3)
\end{aligned}
$$

We are now ready to apply the initial conditions. $y(0) = 0, y'(0) = 3$. Applying first IC to (3) gives

$$0 = c_1 + c_2 + \frac{1}{6} \tag{4}$$

Taking derivative of (3) gives

$$y' = -2c_1e^{-2x} - c_2e^{-x} + \frac{1}{6}e^x$$

Applying second IC to the above gives

$$3 = -2c_1 - c_2 + \frac{1}{6} \tag{5}$$

We now need to solve (4,5) for c_1, c_2. Adding (4,5) gives

$$\begin{aligned}
3 &= \left(c_1 + c_2 + \frac{1}{6}\right) + \left(-2c_1 - c_2 + \frac{1}{6}\right)\\
3 &= \frac{1}{3} - c_1\\
c_1 &= \frac{1}{3} - 3\\
&= -\frac{8}{3}
\end{aligned}$$

From (4)

$$\begin{aligned}
0 &= c_1 + c_2 + \frac{1}{6}\\
0 &= -\frac{8}{3} + c_2 + \frac{1}{6}\\
c_2 &= \frac{5}{2}
\end{aligned}$$

Therefore the complete solution (3) becomes

$$\begin{aligned}
y(x) &= c_1e^{-2x} + c_2e^{-x} + \frac{1}{6}e^x\\
&= -\frac{8}{3}e^{-2x} + \frac{5}{2}e^{-x} + \frac{1}{6}e^x\\
&= \frac{1}{6}\left(-16e^{-2x} + 15e^{-x} + e^x\right)
\end{aligned}$$

6 Problem 7 section 6.1

In Problems 1 through 26, find the (real) eigenvalues and associated eigenvectors of the given matrix A. Find a basis for each eigenspace of dimension 2 or larger.

$$A = \begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix}$$

Solution

We first need to find the eigenvalues. These are found by solving $|A - \lambda I| = 0$. Hence

$$\begin{aligned} \begin{vmatrix} 10-\lambda & -8 \\ 6 & -4-\lambda \end{vmatrix} &= 0 \\ (10-\lambda)(-4-\lambda) + 48 &= 0 \\ \lambda^2 - 6\lambda + 8 &= 0 \\ (\lambda-4)(\lambda-2) &= 0 \end{aligned}$$

Hence $\lambda_1 = 4, \lambda_2 = 2$. For each eigenvalue we find its associated eigenvectors.

$\lambda_1 = 4$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} 10-4 & -8 \\ 6 & -4-4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -8 \\ 6 & -8 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1}$$

Augmented matrix

$$\begin{bmatrix} 6 & -8 & 0 \\ 6 & -8 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$$\begin{bmatrix} 6 & -8 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore (1) becomes

$$\begin{bmatrix} 6 & -8 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1A}$$

v_2 is free variable. Let $v_2 = 1$. Then from first row $6v_1 - 8 = 0$ or $v_1 = \frac{8}{6}$. Hence

$$\vec{v}_1 = \begin{bmatrix} \frac{8}{6} \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$\lambda_1 = 2$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} 10-2 & -8 \\ 6 & -4-2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8 & -8 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1}$$

Augmented matrix

$$\begin{bmatrix} 8 & -8 & 0 \\ 6 & -6 & 0 \end{bmatrix}$$

$R_2 \to R_2 - \frac{6}{8}R_1$

$$\begin{bmatrix} 8 & -8 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore (1) becomes

$$\begin{bmatrix} 8 & -8 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1A}$$

v_2 is free variable. Let $v_2 = 1$. Then from first row $8v_1 - 8 = 0$ or $v_1 = 1$. Hence

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This table gives summary of the result

eigenvalue λ	associated eigenvector $\vec{v}$
4	$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$
2	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

7 Problem 17 section 6.1

In Problems 1 through 26, find the (real) eigenvalues and associated eigenvectors of the given matrix A. Find a basis for each eigenspace of dimension 2 or larger.

$$A = \begin{bmatrix} 3 & 5 & -2 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Solution

We first need to find the eigenvalues. These are found by solving $|A - \lambda I| = 0$. Hence

$$\begin{vmatrix} 3-\lambda & 5 & -2 \\ 0 & 2-\lambda & 0 \\ 0 & 2 & 1-\lambda \end{vmatrix} = 0$$

Expanding along the first columns.

$$(3-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix} = 0$$
$$(3-\lambda)(2-\lambda)(1-\lambda) = 0$$

Hence roots (eigenvalues) are $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1$. For each eigenvalue we find its associated eigenvectors.

$\lambda_1 = 3$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} 3-3 & 5 & -2 \\ 0 & 2-3 & 0 \\ 0 & 2 & 1-3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 5 & -2 \\ 0 & -1 & 0 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

$R_3 \to R_2 + 2R_2$

$$\begin{bmatrix} 0 & 5 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Free variable is v_1. Let $v_1 = 1$. Last row gives $v_3 = 0$. Second row gives $v_2 = 0$. Hence the eigenvector is

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_1 = 2$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} 3-2 & 5 & -2 \\ 0 & 2-2 & 0 \\ 0 & 2 & 1-2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 5 & -2 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

Swap R_2, R_3 (for clarify only)

$$\begin{bmatrix}1 & 5 & -2\\0 & 2 & -1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Free variable is v_3. Let $v_3 = 1$. From second row $2v_2 - v_3 = 0$. Hence $v_2 = \frac{1}{2}$. First row gives $v_1 + 5v_2 - 2v_3 = 0$. Hence $v_1 = -5\left(\frac{1}{2}\right) + 2 = -\frac{1}{2}$. Hence the eigenvector is

$$\vec{v}_2 = \begin{bmatrix}-\frac{1}{2}\\\frac{1}{2}\\1\end{bmatrix} = \begin{bmatrix}-1\\1\\2\end{bmatrix}$$

$\underline{\lambda_1 = 1}$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix}3-1 & 5 & -2\\0 & 2-1 & 0\\0 & 2 & 1-1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}2 & 5 & -2\\0 & 1 & 0\\0 & 2 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \tag{1}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix}2 & 5 & -2\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Free variable is v_3. Let $v_3 = 1$. From second row $v_2 = 0$. First row gives $2v_1 = 2v_3$. Hence $v_1 = 1$. Hence the eigenvector is

$$\vec{v}_3 = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

This table gives summary of the result

eigenvalue λ	associated eigenvector $\vec{v}$
3	$\begin{bmatrix}1\\0\\0\end{bmatrix}$
2	$\begin{bmatrix}-1\\1\\2\end{bmatrix}$
1	$\begin{bmatrix}1\\0\\1\end{bmatrix}$

8 Problem 21 section 6.1

In Problems 1 through 26, find the (real) eigenvalues and associated eigenvectors of the given matrix A. Find a basis for each eigenspace of dimension 2 or larger.

$$A = \begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution

We first need to find the eigenvalues. These are found by solving $|A - \lambda I| = 0$. Hence

$$\begin{vmatrix} 4-\lambda & -3 & 1 \\ 2 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

Expanding along the last row

$$\begin{aligned} (-1)^{3+3}(2-\lambda)\begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix} &= 0 \\ (2-\lambda)((4-\lambda)(-1-\lambda)+6) &= 0 \\ (2-\lambda)\left(\lambda^2 - 3\lambda + 2\right) &= 0 \\ (2-\lambda)(\lambda-2)(\lambda-1) &= 0 \end{aligned}$$

Hence the eigenvalues are $\lambda_1 = 2$ of algebraic multiplicity 2 and $\lambda_2 = 1$. For each eigenvalue we find its associated eigenvectors.

$\lambda_1 = 2$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} 4-2 & -3 & 1 \\ 2 & -1-2 & 1 \\ 0 & 0 & 2-2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

$R_2 \rightarrow R_2 - R_1$

$$\begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Free variables are $v_3.v_2$. This means this is a complete eigenvalue. Since it has algebraic multiplicity of 2 and have a geometric multiplicity of 2 as well. This means we can find two linearly independent eigenvectors from it. Let $v_2 = s, v_3 = t$. First row gives $2v_1 - 3s + t = 0$ or $v_1 = \frac{3}{2}s - \frac{1}{2}t$. Hence the solution is

$$\begin{aligned} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} \frac{3}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix} \\ &= s\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore the basis (eigenvectors) are

$$\left\{ \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\} \rightarrow \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

Now that we found the eigenvectors associated with $\lambda_1 = 2$, we will do the same for second eigenvalue.

$\underline{\lambda_2 = 1}$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} 4-1 & -3 & 1 \\ 2 & -1-1 & 1 \\ 0 & 0 & 2-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

$R_2 \rightarrow R_2 - \frac{2}{3} R_1$

$$\begin{bmatrix} 3 & -3 & 1 \\ 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_3 + \frac{3}{2} R_2$

$$\begin{bmatrix} 3 & -3 & 1 \\ 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Free variable is v_2, leading variables are v_1, v_3. Let $v_2 = 1$. From second row, $v_3 = 0$.First row gives $3v_1 = 3$. Hence $v_1 = 1$. Hence the eigenvector is

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

This table gives summary of the result

<table>
<tr><th>eigenvalue λ</th><th>associated eigenvector $\vec{v}$</th></tr>
<tr><td>2 (multiplicity 2)</td><td>$\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$</td></tr>
<tr><td>1</td><td>$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$</td></tr>
</table>

9 Problem 25 section 6.1

In Problems 1 through 26, find the (real) eigenvalues and associated eigenvectors of the given matrix A. Find a basis for each eigenspace of dimension 2 or larger.

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Solution

We first need to find the eigenvalues. These are found by solving $|A - \lambda I| = 0$. Hence

$$\begin{vmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

Since this is an upper triangle matrix, then the determinant is the product of the diagonal. Hence the above reduces to

$$(1-\lambda)^2(2-\lambda)^2 = 0$$

Therefore the eigenvalues are $\lambda_1 = 1$ of algebraic multiplicity 2 and $\lambda_2 = 2$ also of algebraic multiplicity 2. For each eigenvalue we find its associated eigenvectors.

$\lambda_1 = 1$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} 1-1 & 0 & 1 & 0 \\ 0 & 1-1 & 1 & 0 \\ 0 & 0 & 2-1 & 0 \\ 0 & 0 & 0 & 2-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Swapping R_3, R_2 (for clarify)

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Swapping R_4, R_2 (for clarify)

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence leading variables are v_3, v_4 free variables are v_1, v_2. Let $v_1 = s, v_2 = t$. Second row gives $v_4 = 0$. First row gives $v_3 = 0$. Therefore the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} s \\ t \\ 0 \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Therefore the two eigenvectors associated with this eigenvalues are

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$\underline{\lambda_2 = 2}$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} 1-2 & 0 & 1 & 0 \\ 0 & 1-2 & 1 & 0 \\ 0 & 0 & 2-2 & 0 \\ 0 & 0 & 0 & 2-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence leading variables are v_1, v_2 free variables are v_3, v_4 Let $v_3 = s, v_4 = t$. Second row gives $-v_2 + v_3 = 0$ or $v_2 = s$. First row gives $-v_1 + s = 0$ or $v_1 = s$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} s \\ s \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore the two eigenvectors associated with this eigenvalues are

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

This table gives summary of the result

eigenvalue λ	associated eigenvector $\vec{v}$
1 (multiplicity 2)	$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$
2 (multiplicity 2)	$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

10 Problem 29 section 6.1

Find the complex conjugate eigenvalues and corresponding eigenvectors of the matrices given in Problems 27 through 32

$$A = \begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix}$$

<u>Solution</u>

We first need to find the eigenvalues. These are found by solving $|A - \lambda I| = 0$. Hence

$$\begin{vmatrix} -\lambda & -3 \\ 12 & -\lambda \end{vmatrix} = 0$$
$$\lambda^2 + 36 = 0$$

Hence $\lambda = \pm 6i$. For each eigenvalue we find its associated eigenvectors.

<u>$\lambda_1 = 6i$</u>

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} -6i & -3 \\ 12 & -6i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + \frac{12}{6i} R_1$

$$\begin{bmatrix} -6i & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Leading variable is v_1, free variable is v_2. Let $v_2 = 1$. From first row $-6iv_1 - 3v_2 = 0$ or $v_1 = -\frac{3}{6i} = -\frac{1}{2i} = \frac{1}{2}i$. Hence the eigenvector is

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{2}i \\ 1 \end{bmatrix} = \begin{bmatrix} i \\ 2 \end{bmatrix}$$

<u>$\lambda_1 = -6i$</u>

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} +6i & -3 \\ 12 & +6i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - \frac{12}{6i} R_1$

$$\begin{bmatrix} 6i & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Leading variable is v_1, free variable is v_2. Let $v_2 = 1$. From first row $6iv_1 - 3v_2 = 0$ or $v_1 = \frac{3}{6i} = \frac{1}{2i} = -\frac{1}{2}i$. Hence the eigenvector is

$$\vec{v}_1 = \begin{bmatrix} -\frac{1}{2}i \\ 1 \end{bmatrix} = \begin{bmatrix} -i \\ 2 \end{bmatrix}$$

This table gives summary of the result

eigenvalue λ	associated eigenvector $\vec{v}$
$6i$	$\begin{bmatrix} i \\ 2 \end{bmatrix}$
$-6i$	$\begin{bmatrix} -i \\ 2 \end{bmatrix}$

11 Additional problem 1

Find particular solution to

$$y^{(7)}(x) - 2y^{(6)} + 9y^{(5)} - 16y^{(4)} + 24y^{(3)} - 32y'' + 16y' = e^{2x} + x\sin x + x^2 \tag{1}$$

<u>Solution</u>

From HW6, we found y_h as

$$y(x) = c_1 + (c_2 e^x + c_3 x e^x) + (c_4 \cos 2x + c_5 \sin 2x) + x(c_6 \cos 2x + c_7 \sin 2x)$$

Therefore, we see the basis functions for y_h are

$$\{1, e^x, xe^x, \cos 2x, \sin 2x, x\cos 2x, x\sin 2x\} \tag{2}$$

Looking at RHS of (1), we see the basis functions for y_p are

$$\left\{e^{2x}, x^2, x\sin x\right\}$$

Taking derivative e^{2x} does not generate new basis. Taking derivative of x^2 generates $x, 1$. And taking derivative of $x\sin x$ generates $\sin x, x\cos x, \cos x$. Hence the above becomes

$$\left\{e^{2x}, \left(x^2, x, 1\right), (x\sin x, \sin x, x\cos x, \cos x)\right\} \tag{3}$$

There are 3 groups. Comparing (2,3) we see there is one duplication, which is the constant term. Hence we need to multiply that one group by x. The above becomes

$$\left\{e^{2x}, x\left(x^2, x, 1\right), (x\sin x, \sin x, x\cos x)\right\} = \left\{e^{2x}, \left(x^3, x^2, x\right), (x\sin x, \sin x, x\cos x, \cos x)\right\} \tag{3A}$$

Now we again compare (3A) and (2). Now there is no duplication. Therefore the particular solution is

$$y_p = A_1 e^{2x} + A_2\left(x^3\right) + A_3\left(x^2\right) + A_4(x) + A_5(x\sin x) + A_6(\sin x) + A_7(x\cos x) + A_8\cos x$$

12 Additional problem 2

Let $A=\begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}$ where t_1, t_2, t_3 are distinct real numbers. Find the eigenvalues of A and the corresponding eigenvectors.

Solution

We first need to find the eigenvalues. These are found by solving $|A-\lambda I|=0$. Hence

$$\begin{vmatrix} t_1-\lambda & 0 & 0 \\ 0 & t_2-\lambda & 0 \\ 0 & 0 & t_3-\lambda \end{vmatrix}=0$$

Since this is a diagonal matrix, then the determinant is the product of the diagonal. Hence the above reduces to

$$(t_1-\lambda)(t_2-\lambda)(t_3-\lambda)=0$$

Hence the eigenvalues are $\lambda_1=t_1, \lambda_2=t_2, \lambda_3=t_3$. For each eigenvalue we find its associated eigenvectors.

$\lambda_1=t_1$

We nee to solve $A\vec{v}=\lambda\vec{v}$. This becomes $(A-\lambda I)\vec{v}=\vec{0}$. Therefore

$$\begin{bmatrix} t_1-t_1 & 0 & 0 \\ 0 & t_2-t_1 & 0 \\ 0 & 0 & t_3-t_1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2-t_1 & 0 \\ 0 & 0 & t_3-t_1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Leading variables are v_2, v_3 and free variables is v_1. Let $v_1=s$. Second and third rows give $v_2=0, v_3=0$, this is because t_1, t_2, t_3 are distinct real numbers therefore $t_2-t_1\neq 0, t_3-t_1\neq 0$. Therefore the solution is

$$\vec{v}_1=\begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix}=s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $s=1$ this gives the eigenvector

$$\vec{v}_1=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_2=t_2$

We nee to solve $A\vec{v}=\lambda\vec{v}$. This becomes $(A-\lambda I)\vec{v}=\vec{0}$. Therefore

$$\begin{bmatrix} t_1-t_2 & 0 & 0 \\ 0 & t_2-t_2 & 0 \\ 0 & 0 & t_3-t_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} t_1-t_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t_3-t_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Leading variables are v_1, v_3 and free variable is v_2. Let $v_2=s$. First and third rows give $v_1=0, v_3=0$, this is because t_1, t_2, t_3 are distinct real numbers therefore $t_1-t_2\neq 0, t_3-t_2\neq 0$. Therefore the solution is

$$\vec{v}_2=\begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix}=s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $s = 1$ this gives the eigenvector

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\underline{\lambda_3 = t_3}$

We nee to solve $A\vec{v} = \lambda\vec{v}$. This becomes $(A - \lambda I)\vec{v} = \vec{0}$. Therefore

$$\begin{bmatrix} t_1 - t_3 & 0 & 0 \\ 0 & t_2 - t_3 & 0 \\ 0 & 0 & t_3 - t_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} t_1 - t_3 & 0 & 0 \\ 0 & t_2 - t_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Leading variables are v_1, v_2 and free variable is v_3. Let $v_3 = s$. First and third rows give $v_1 = 0, v_2 = 0$, this is because t_1, t_2, t_3 are distinct real numbers therefore $t_1 - t_3 \neq 0, t_2 - t_3 \neq 0$. Therefore the solution is

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} = s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

For $s = 1$ this gives the eigenvector

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This table gives summary of the result

eigenvalue λ	associated eigenvector $\vec{v}$
t_1	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
t_2	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
t_2	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

13 Additional optional problem 3

Extend the result in problem 2 to the case of $n \times n$ matrices. That is, let A be a matrix with entries $t_1, t_2, \cdots, t_n$ on the main diagonal and 0s everywhere else, where the t_i are distinct real numbers. Find the eigenvalues and corresponding eigenvectors.

Solution

This follows immediately from the last problem. Therefore each eigenvalue will be $\lambda_1 = t_1, \lambda_2 = t_2, \cdots, \lambda_n = t_n$. And corresponding eigenvectors are (each eigenvector is $n \times 1$.

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \cdots, v_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

These eigenvectors are the standard basis for $\mathbb{R}^n$.