HW 6

Math 2243
Linear Algebra and Differential Equations

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Contents

1 Problem 9 section 5.2

In Problems 7 through 12, use the Wronskian to prove that the given functions are linearly independent on the indicated interval.

$$f(x) = e^x, g(x) = \cos x, h(x) = \sin x$$

On the real line.

Solution

$$W(x) = \begin{bmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{bmatrix}$$

Hence

$$W(x) = \begin{bmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{bmatrix}$$

The determinant is, expanding along first row is

$$\begin{aligned} |W(x)| &= e^x \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} - \cos x \begin{vmatrix} e^x & \cos x \\ e^x & -\sin x \end{vmatrix} + \sin x \begin{vmatrix} e^x & -\sin x \\ e^x & -\cos x \end{vmatrix} \\ &= e^x \left(\sin^2 x + \cos^2 x\right) - \cos x(-e^x \sin x - e^x \cos x) + \sin x(-e^x \cos x + e^x \sin x) \end{aligned}$$

But $\sin^2 x + \cos^2 x = 1$ and the above simplifies to

$$\begin{aligned} |W(x)| &= e^x - \left(-e^x \sin x \cos x - e^x \cos^2 x\right) + \left(-e^x \cos x \sin x + e^x \sin^2 x\right) \\ &= e^x + e^x \sin x \cos x + e^x \cos^2 x - e^x \cos x \sin x + e^x \sin^2 x \\ &= e^x + e^x \cos^2 x + e^x \sin^2 x \\ &= e^x + e^x \left(\sin^2 x + \cos^2 x\right) \\ &= 2e^x \end{aligned}$$

And since e^x is never zero on the real line, then $|W(x)| \neq 0$ Hence functions are linearly independent.

2 Problem 16 section 5.2

In Problems 13 through 20, a third-order homogeneous linear equation and three linearly independent solutions are given. Find a particular solution satisfying the given initial conditions.

$$
\begin{aligned}
y''' - 5y'' + 8y' - 4y &= 0 \\
y_1 &= e^x \\
y_2 &= e^{2x} \\
y_3 &= xe^{2x}
\end{aligned}
$$

I.C. are

$$y(0) = 1, y'(0) = 4, y''(0) = 0$$

Solution

The general solution is

$$
\begin{aligned}
y(x) &= c_1 y_2 + c_2 y_2 + c_3 y_3 \\
&= c_1 e^x + c_2 e^{2x} + c_3 x e^{2x} \qquad (1)
\end{aligned}
$$

At $y(0) = 0$ the above becomes

$$1 = c_1 + c_2 \qquad (2)$$

Taking derivative of (1) gives

$$y'(x) = c_1 e^x + 2c_2 e^{2x} + c_3\left(e^{2x} + 2xe^{2x}\right)$$

At $y'(0) = 4$ the above becomes

$$4 = c_1 + 2c_2 + c_3 \qquad (3)$$

Taking derivative of $y''(x)$ gives

$$
\begin{aligned}
y''(x) &= c_1 e^x + 4c_2 e^{2x} + c_3\left(2e^{2x} + 2\left(e^{2x} + 2xe^{2x}\right)\right) \\
&= c_1 e^x + 4c_2 e^{2x} + c_3\left(2e^{2x} + 2e^{2x} + 4xe^{2x}\right)
\end{aligned}
$$

At $y''(0) = 0$ the above becomes

$$0 = c_1 + 4c_2 + 4c_3 \qquad (4)$$

Equations (2,3,4) are now solved for c_1, c_2, c_3

$$
\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 4 & 4 \end{bmatrix}
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}
$$

Augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 4 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 4 & -1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -10 \end{bmatrix}$$

The above is Echelon form. Hence the system becomes

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -10 \end{bmatrix}$$

From last row, $c_3 = -10$. From second row $c_2 + c_3 = 3$ or $c_2 = 13$. From first row $c_1 + c_2 = 1$. Hence $c_1 = -12$. Therefore

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -12 \\ 13 \\ -10 \end{bmatrix}$$

Substituting these values back in general solution (1) gives the solution that satisfies these initial conditions as

$$\begin{aligned} y(x) &= c_1 e^x + c_2 e^{2x} + c_3 x e^{2x} \\ &= -12e^x + 13e^{2x} - 10xe^{2x} \end{aligned}$$

3 Problem 19 section 5.2

In Problems 13 through 20, a third-order homogeneous linear equation and three linearly independent solutions are given. Find a particular solution satisfying the given initial conditions.

$$
\begin{aligned}
x^3y''' - 3x^2y'' + 6xy' - 6y &= 0\\
y_1 &= x\\
y_2 &= x^2\\
y_3 &= x^3
\end{aligned}
$$

I.C. are

$$y(1) = 6, y'(1) = 14, y''(1) = 22$$

Solution

The general solution is

$$
\begin{aligned}
y(x) &= c_1y_2 + c_2y_2 + c_3y_3\\
&= c_1x + c_2x^2 + c_3x^3 \qquad (1)
\end{aligned}
$$

At $y(1) = 0$ the above becomes

$$6 = c_1 + c_2 + c_3 \tag{2}$$

Taking derivative of (1) gives

$$y'(x) = c_1 + 2c_2x + 3c_3x^2$$

At $y'(1) = 14$ the above becomes

$$14 = c_1 + 2c_2 + 3c_3 \tag{3}$$

Taking derivative of $y'(x)$ gives

$$y''(x) = 2c_2 + 6c_3x$$

At $y''(1) = 22$ the above becomes

$$22 = 2c_2 + 6c_3 \tag{4}$$

Equations (2,3,4) are now solved for c_1, c_2, c_3

$$\begin{bmatrix} 1 & 1 & 1\\ 1 & 2 & 3\\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} c_1\\ c_2\\ c_3 \end{bmatrix} = \begin{bmatrix} 6\\ 14\\ 22 \end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 6\\ 1 & 2 & 3 & 14\\ 0 & 2 & 6 & 22 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 6 & 22 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

The above is Echelon form. Hence the system becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 6 \end{bmatrix}$$

From last row, $2c_3 = 6$ or $c_3 = 3$. From second row $c_2 + 2c_3 = 8$ or $c_2 = 8 - 2(3) = 2$. From first row $c_1 + c_2 + c_3 = 6$. Hence $c_1 = 6 - 2 - 3 = 1$. Therefore

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Substituting these values back in general solution (1) gives the solution that satisfies these initial conditions as

$$\begin{aligned} y(x) &= c_1 x + c_2 x^2 + c_3 x^3 \\ &= x + 2x^2 + 3x^3 \end{aligned}$$

4 Problem 24 section 5.2

In Problems 21 through 24, a nonhomogeneous differential equation, a complementary solution y_c, and a particular solution

y_p are given. Find a solution satisfying the given initial conditions.

$$
\begin{aligned}
y'' - 2y' + 2y = 2x \\
y_c &= c_1 e^x \cos x + c_2 e^x \sin x \\
y_p &= x + 1
\end{aligned}
$$

I.C. are

$$y(0) = 4, y'(0) = 8$$

<u>Solution</u>

The general solution is

$$
\begin{aligned}
y(x) &= y_c + y_p \\
&= c_1 e^x \cos x + c_2 e^x \sin x + x + 1 \qquad (1)
\end{aligned}
$$

At $y(0) = 4$ the above becomes (using $e^0 = 1, \cos 0 = 1, \sin 0 = 0$)

$$4 = c_1 + 1 \qquad (2)$$

Taking derivative of (1) gives

$$y'(x) = c_1 (e^x \cos x - e^x \sin x) + c_2 e^x \cos x + 1$$

At $y'(0) = 8$ the above becomes

$$
\begin{aligned}
8 &= c_1 (1 - 0) + c_2 + 1 \\
8 &= c_1 + c_2 + 1 \qquad (3)
\end{aligned}
$$

We have two equations (2,3) to solve for c_1, c_2. From (3) we see that $c_1 = 3$. Hence from (3) $8 = 3 + c_2 + 1$ or $c_2 = 4$. Therefore the solution in (1) becomes

$$
\begin{aligned}
y(x) &= c_1 e^x \cos x + c_2 e^x \sin x + x + 1 \\
&= 3e^x \cos x + 4e^x \sin x + x + 1 \\
&= e^x (3 \cos x + 4 \sin x) + x + 1
\end{aligned}
$$

5 Problem 8 section 5.3

Find the general solutions of the differential equations in Problems 1 through 20.

$$y'' - 6y' + 13y = 0$$

Solution This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where here we see that $A = 1, B = -6, C = 13$.

Let the solution be $y(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 6\lambda e^{\lambda x} + 13e^{\lambda x} = 0 \tag{1}$$

Since $e^{\lambda x} \neq 0$, then dividing Eq. (1) throughout by$e^{\lambda x}$ results in

$$\lambda^2 - 6\lambda + 13 = 0 \tag{2}$$

Eq. (2) is the characteristic equation of the ODE. We need to determine its roots to find the general solution. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-6^2 - (4)(1)(13)} \\ &= 3 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 3 + 2i \\ \lambda_2 &= 3 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 3$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y(x) = e^{\alpha x}\Big(c_1 \cos(\beta x) + c_2 \sin(\beta x)\Big)$$

Which becomes

$$y(x) = e^{3x}(c_1 \cos(2x) + c_2 \sin(2x))$$

6 Problem 11 section 5.3

Find the general solutions of the differential equations in Problems 1 through 20.

$$y^{(4)}(x) - 8y^{(3)} + 16y'' = 0$$

Solution

We start by writing the characteristic equation of the ODE

$$\lambda^4 - 8\lambda^3 + 16\lambda^2 = 0$$

We now solve for the roots of the above equation. Writing the above as

$$\lambda^2\left(\lambda^2 - 8\lambda + 16\right) = 0$$

We see that $\lambda^2 = 0$ gives $\lambda = 0$ with multiplicity 2. The equation $\lambda^2 - 8\lambda + 16 = 0$ can be factored to $(\lambda - 4)(\lambda - 4) = 0$. Therefor $\lambda = 4$ with multiplicity 2.

Hence the roots are

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= 0\\ \lambda_3 &= 4\\ \lambda_4 &= 4\end{aligned}$$

This table summarizes the result

root	multiplicity	type of root
0	2	real root
4	2	real root

For a real root λ with multiplicity one, we obtain a basis solution of the form $e^{\lambda x}$ and real root λ with multiplicity two we obtain basis solutions $\left\{e^{\lambda x}, xe^{\lambda x}\right\}$. Therefore the solution is

$$\begin{aligned}y(x) &= c_2 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x} + c_2 e^{\lambda_3 x} + c_2 x e^{\lambda_3 x}\\ &= c_2 + c_2 x + c_2 e^{4x} + c_2 x e^{4x}\end{aligned}$$

7 Problem 14 section 5.3

Find the general solutions of the differential equations in Problems 1 through 20.

$$y^{(4)}(x) + 3y'' - 4y = 0$$

Solution

We start by writing the characteristic equation

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

Let

$$z = \lambda^2$$

The characteristic becomes

$$z^2 + 3z - 4 = 0$$

Factoring the above gives

$$(z + 4)(z - 1) = 0$$

Hence $z = -4, z = 1$. When $z = -4$, then $\lambda = \pm\sqrt{-4} = \pm 2i$. And when $z = 1$, then $\lambda = \pm\sqrt{1} = \pm 1$. Therefore the roots are

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\\ \lambda_3 &= 2i\\ \lambda_4 &= -2i\end{aligned}$$

This table summarizes the result

root	multiplicity	type of root
-1	1	real root
1	1	real root
$\pm 2i$	1	complex conjugate root

For a real root λ with multiplicity one, we obtain a basis of the form $c_1 e^{\lambda x}$ and for a complex conjugate root of the form $a \pm ib$ we obtain basis solution of the form $e^{ax}(c_1 \cos(bx) + c_2 \sin(bx))$. Therefore the final solution, using $a = 0, b = 2$ is

$$y(x) = c_1 e^{-x} + c_2 e^{x} + c_3 \cos(2x) + c_4 \sin(2x)$$

8 Problem 18 section 5.3

Find the general solutions of the differential equations in Problems 1 through 20.

$$y^{(4)}(x) = 16y$$

Solution

We start by writing the characteristic equation

$$\lambda^4 = 16$$

Let

$$z = \lambda^2$$

The characteristic becomes

$$z^2 = 16$$

Hence $z = \pm 4$. When $z = 4$ then $\lambda = \pm\sqrt{4} = \pm 2$. And when $z = -4$ then $\lambda = \pm\sqrt{-4} = \pm 2i$. Hence the roots are

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\\ \lambda_3 &= 2i\\ \lambda_4 &= -2i\end{aligned}$$

This table summarizes the result

root	multiplicity	type of root
-2	1	real root
2	1	real root
$\pm 2i$	1	complex conjugate root

As in the earlier problem, we now can write the general solution as

$$y(x) = e^{-2x}c_1 + c_2e^{2x} + c_3\cos(2x) + c_4\sin(2x)$$

9 Additional problem 1

Find the general solutions of the differential equations in Problems 1 through 20.

$$y^{(7)}(x) - 2y^{(6)} + 9y^{(5)} - 16y^{(4)} + 24y^{(3)} - 32y'' + 16y' = 0$$

Solution

9.1 Part a

The characteristic equation is

$$\begin{aligned} r^7 - 2r^6 + 9r^5 - 16r^4 + 24r^3 - 32r^2 + 16r &= 0 \\ r\left(r^6 - 2r^5 + 9r^4 - 16r^3 + 24r^2 - 32r + 16\right) &= 0 \end{aligned}$$

Hence one root is $r = 0$. And now we need to solve

$$r^6 - 2r^5 + 9r^4 - 16r^3 + 24r^2 - 32r + 16 = 0$$

9.2 Part b

Substituting $r = 1$ in the above gives

$$\begin{aligned} 1 - 2 + 9 - 16 + 24 - 32 + 16 &= 0 \\ 0 &= 0 \end{aligned}$$

Therefore $(r - 1)$ is a factor. Doing long division (do not know how type polynomial division in Latex, please see scanned hand solution in appendix of this problem).

$$\frac{r^6 - 2r^5 + 9r^4 - 16r^3 + 24r^2 - 32r + 16}{(r-1)} = r^5 - r^4 + 8r^3 - 8r^2 + 16r - 16$$

Hence

$$r^6 - 2r^5 + 9r^4 - 16r^3 + 24r^2 - 32r + 16 = (r-1)\left(r^5 - r^4 + 8r^3 - 8r^2 + 16r - 16\right)$$

Substituting $r = 1$ in $\left(r^5 - r^4 + 8r^3 - 8r^2 + 16r - 18\right)$ gives

$$r^5 - r^4 + 8r^3 - 8r^2 + 16r - 18 \to 1 - 1 + 8 - 8 + 16 - 16 = 0$$

Hence $(r - 1)$ is a factor of $\left(r^5 - r^4 + 8r^3 - 8r^2 + 16r - 16\right)$. Therefore we now need to do long division

$$\frac{r^5 - r^4 + 8r^3 - 8r^2 + 16r - 16}{(r-1)} = r^4 + 8r^2 + 16$$

Hence now we have

$$r^6 - 2r^5 + 9r^4 - 16r^3 + 24r^2 - 32r + 16 = (r-1)(r-1)\left(r^4 + 8r^2 + 16\right)$$

9.3 Part c

Looking at $r^4 + 8r^2 + 16 = 0$, let $z = r^2$. Therefore $r^4 + 8r^2 + 16$ becomes $z^2 + 8z + 16 = 0$, This can be factored to $(z+4)(z+4) = 0$. Hence roots are $z = -4$ which is double root.

9.4 Part d

Therefore when $z = -4$ then $r = \pm\sqrt{-4} = \pm 2i$ with multiplicity 2 since $z = -4$ is double root. Therefore the final factorization is

$$r^6 - 2r^5 + 9r^4 - 16r^3 + 24r^2 - 32r + 16 = (r-1)(r-1)(r-2i)(r+2i)(r-2i)(r+2i)$$

9.5 Part e

This table summarizes the result

root	multiplicity	type of root
0	1	real root
1	2	real root
$\pm 2i$	2	complex conjugate

Now we are above to write down the general solution.

$$\begin{aligned} y(x) &= c_1 e^{0x} + (c_2 e^x + c_3 x e^x) + \left(c_4 e^{2ix} + c_5 x e^{2ix}\right) + \left(c_6 e^{-2ix} + c_7 x e^{-2ix}\right) \\ &= c_1 + (c_2 e^x + c_3 x e^x) + \left(c_4 e^{2ix} + c_5 x e^{2ix}\right) + \left(c_6 e^{-2ix} + c_7 x e^{-2ix}\right) \\ &= c_1 + (c_2 e^x + c_3 x e^x) + \left(c_4 e^{2ix} + c_6 e^{-2ix}\right) + x\left(c_5 e^{2ix} + c_7 e^{-2ix}\right) \end{aligned}$$

We see the above has 7 terms. But using Euler relation, we can write $\left(e^{2ix} + e^{-2ix}\right)$ using trig functions. The above becomes

$$y(x) = c_1 + (c_2 e^x + c_3 x e^x) + (c_4 \cos 2x + c_5 \sin 2x) + x(c_6 \cos 2x + c_7 \sin 2x)$$

(constants of integrations kept the same as originally for simplicity, since it does not matter as these are found from initial conditions if given).

9.6 Appendix

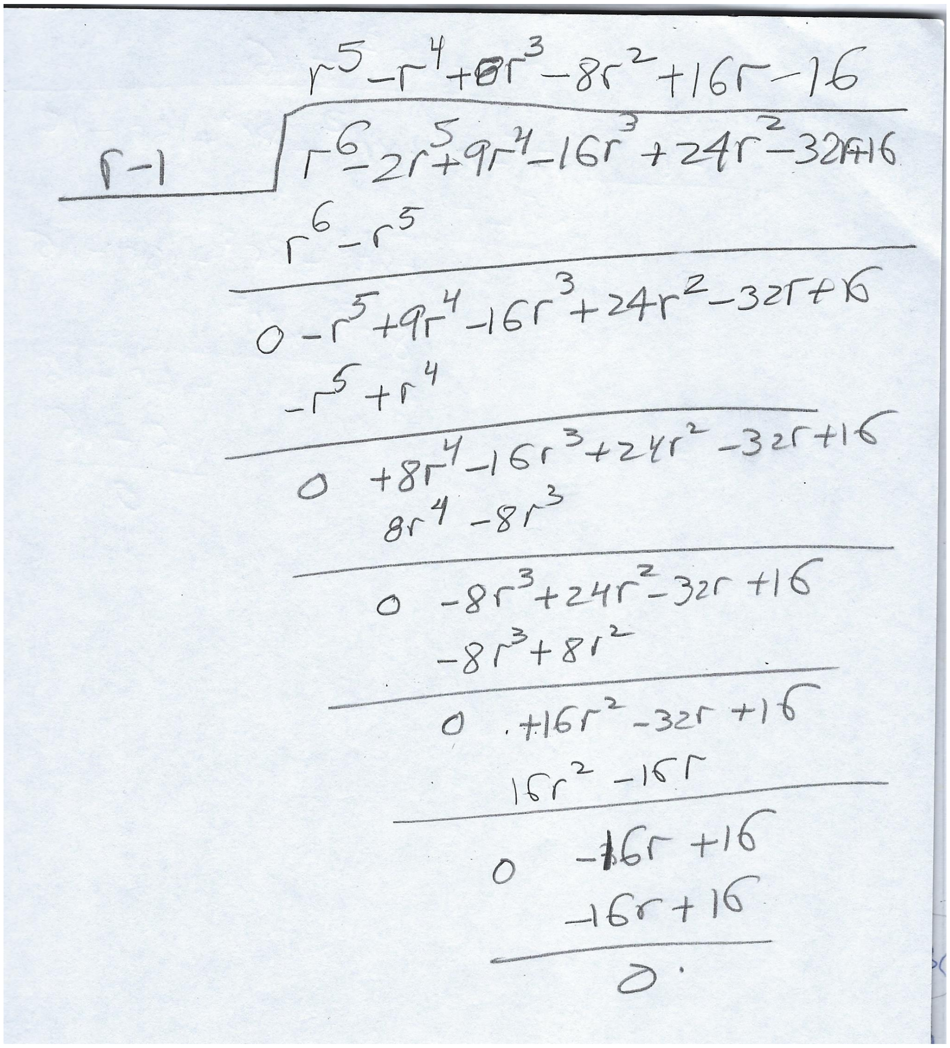

Figure 1: First long division

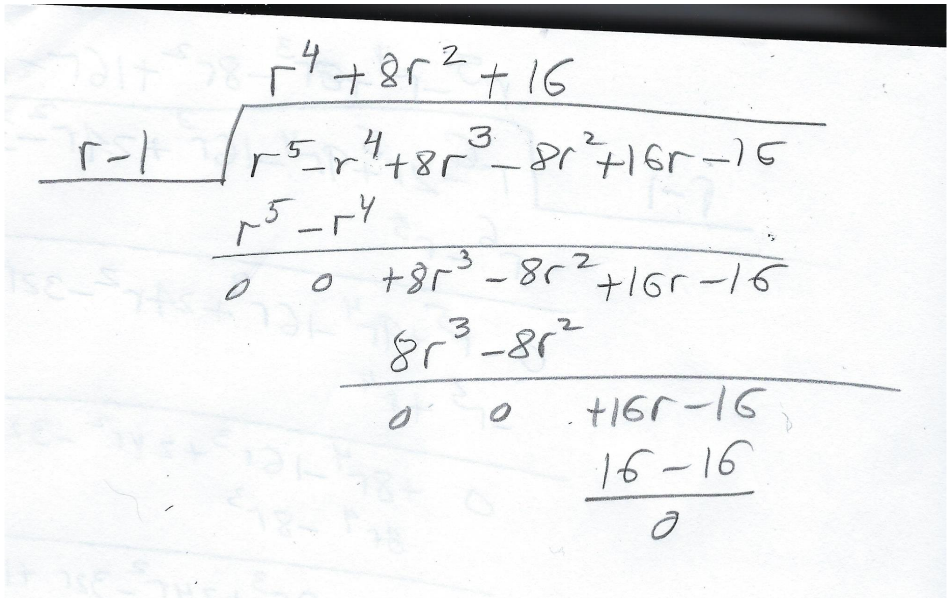

Figure 2: Second long division