Homework 5 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

4.7.7 This is not a subspace. It fails everything fairly badly, but I'll go through why it fails to be closed under addition. Suppose that f and g are functions where $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Then $(f+g)(1) = f(1) + g(1) = 1 + 1 = 2$ so $f + g$ is not in the set.

4.7.10 This is a subspace. Polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ where $a_0 = a_1 = 0$ are of the form $bx^2 + cx^3$. If we add two such polynomials, we have

$$(b_1x^2 + c_1x^3) + (b_2x^2 + c_2x^3) = (b_1 + b_2)x^2 + (c_1 + c_2)x^3$$

The result here is another polynomial of this form. Similarly, when we scale we get

$$k(bx^2 + cx^3) = (kb)x^2 + (kc)x^3$$

Again, the result is in the set. So we have a subspace.

1.1.5 We have the differential equation $y' = y + 2e^{-x}$. We need to check that $y = e^x - e^{-x}$ is a solution. We compute:

$$\begin{aligned} \frac{d}{dx}(e^x - e^{-x}) &= e^x - (-e^{-x}) \\ &= e^x + e^{-x} \\ &= e^x - e^{-x} + 2e^{-x} \\ &= y + 2e^{-x} \end{aligned}$$

So this is indeed a solution.

1.1.17 We have the differential equation $y' + y = 0$. First, we check that $y(x) = Ce^{-x}$ is a solution:

$$\begin{aligned} \frac{d}{dx}(Ce^{-x}) + (Ce^{-x}) &= -Ce^{-x} + Ce^{-x} \\ &= 0 \end{aligned}$$

We need to find the value of C so that $y(0) = 2$. We have $y(0) = Ce^0 = C$, so $C = 2$ is the necessary value.

5.1.3 We omit the verification that y_1 and y_2 are solutions. Our general solution is $y(x) = c_1 \cos 2x + c_2 \sin 2x$. This has derivative $y'(x) = -2c_1 \sin 2x + 2c_2 \cos 2x$. So, our initial conditions tell us that

$$\begin{aligned} y(0) &= c_1 \cos 0 + c_2 \sin 0 \\ 3 &= c_1 \\ y'(0) &= -2c_1 \sin 0 + 2c_2 \cos 0 \\ 8 &= 2c_2 \end{aligned}$$

So, our particular solution is $y(x) = 3 \cos 2x + 4 \sin 2x$.

5.1.5 We omit the verification that y_1 and y_2 are solutions. Our general solution is $y(x) = c_1 e^x + c_2 e^{2x}$. This has derivative $y'(x) = c_1 e^x + 2c_2 e^{2x}$. So, our initial conditions tell us that

$$\begin{aligned} y(0) &= c_1 e^0 + c_2 e^0 \\ 1 &= c_1 + c_2 \\ y'(0) &= c_1 e^0 + 2c_2 e^0 \\ 0 &= c_1 + 2c_2 \end{aligned}$$

Subtracting our equations gives $c_2 = -1$, and we substitute to get that $c_1 = 2$. So, our particular solution is $y(x) = 2e^x - e^{2x}$.

5.1.7 We omit the verification that y_1 and y_2 are solutions. Our general solution is $y(x) = c_1 + c_2 e^{-x}$. This has derivative $y'(x) = -c_2 e^{-x}$. So, our initial conditions tell us that

$$\begin{aligned} y(0) &= c_1 + c_2 e^0 \\ -2 &= c_1 + c_2 \\ y'(0) &= -c_2 e^0 \\ 8 &= -c_2 \end{aligned}$$

So, $c_2 = -8$ and thus $c_1 = 6$. So our particular solution is $y(x) = 6 - 8e^{-x}$.

5.1.33 We have characteristic equation $r^2 - 3r + 2 = (r-1)(r-2)$ so we have roots $r = 1, 2$. This gives us general solution $y(x) = c_1 e^x + c_2 e^{2x}$.

5.1.35 We have characteristic equation $r^2 + 5r = r(r+5)$ so we have roots $r = 0, -5$. This gives us general solution $y(x) = c_1 e^{0x} + c_2 e^{-5x} = c_1 + c_2 e^{-5x}$.

5.1.39 We have characteristic equation $4r^2 + 4r + 1 = (2r+1)(2r+1)$ so we have repeated root $r = -\frac{1}{2}$. This gives us general solution $y(x) = c_1 e^{-\frac{x}{2}} + c_2 x e^{-\frac{x}{2}}$.

Additional Problems:

1. We wish to show that $\{3+x, 1+x+x^2, x-2x^2\}$ is a basis for $\mathcal{P}_2$. So, we need to show that the equation

$$c_1(3+x) + c_2(1+x+x^2) + c_3(x-2x^2) = a_0 + a_1 x + a_2 x^2$$

has a unique solution for each value of a_0, a_1, a_2. Rearranging terms, we have

$$(3c_1 + c_2) + (c_1 + c_2 + c_3)x + (c_2 - 2c_3)x^2 = a_0 + a_1 x + a_2 x^2$$

Equating the coefficients of each power of x, we get the linear system

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

If we can show this matrix is invertible, we will be done. This matrix looks annoying to row reduce, so I'll compute the determinant by expanding along the first row:

$$\begin{aligned} \det \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} &= (+3) \det \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} + (-1) \det \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \\ &= 3(-2-1) - (-2-0) \\ &= -7 \end{aligned}$$

Since this determinant is nonzero, the matrix is invertible and the system we are considering always has a unique solution.

2. We have the initial value problem $y'' - 25y = 0$, $y(0) = a$, $y'(0) = b$.

The characteristic equaiton is $r^2 - 25$ which has roots $r = \pm 5$. So our general solution is $y(x) = c_1 e^{5x} + c_2 e^{-5x}$. We compute $y'(x) = 5c_1 e^{5x} - 5c_2 e^{-5x}$, so our initial conditions give us the system

$$\begin{aligned} c_1 + c_2 &= a \\ 5c_1 - 5c_2 &= b \end{aligned}$$

We can write this system in matrix form as

$$\begin{bmatrix} 1 & 1 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

We have the matrix inverse $\begin{bmatrix} 1 & 1 \\ 5 & -5 \end{bmatrix}^{-1} = \frac{1}{-5-5} \begin{bmatrix} -5 & -1 \\ -5 & 1 \end{bmatrix}$ so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{10} \\ \frac{1}{2} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{a}{2} + \frac{b}{10} \\ \frac{a}{2} - \frac{b}{10} \end{bmatrix}$$

This solves for the constants c_1 and c_2 in terms of the given initial values a and b.