Exam 1 Practice problems

Solved problems from Differential Equations and linear algebra, 4th ed., Edwards, Penney and Calvis. Pearson. 2017.

Math 2243
Linear Algebra and Differential Equations

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Contents

1 Problem section 3.2 number 23

In Problems 23–27, determine for what values of k each system has (a) a unique solution; (b) no solution; (c) infinitely many solutions.

$$\begin{aligned} 3x + 2y &= 1 \\ 6x + 4y &= k \end{aligned}$$

Solution

$$\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ k \end{bmatrix}$$

Augmented matrix is

$$\begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & k \end{bmatrix}$$

$R_2 \to -2R_1 + R_2$ gives

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & k-2 \end{bmatrix}$$

The above is in Echelon form. x is the leading variable and y is the free variable. Let $y = t$. The system in Echelon form becomes

$$\begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ k-2 \end{bmatrix}$$

Last row says that $0 = k - 2$. This means only $k = 2$ is possible. First row gives $3x + 2t = 1$. When $k = 2$, we have infinite number of solutions given by $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1-2t}{3} \\ t \end{bmatrix}$. For any t.

When $k \neq 2$ there is no solution. There is no unique solution for any k since y is free variable. Hence answer is

(a) None (b) $k \neq 2$ (c) $k = 2$.

2 Problem section 3.2 number 24

In Problems 23–27, determine for what values of k each system has (a) a unique solution; (b) no solution; (c) infinitely many solutions.

$$\begin{aligned} 3x + 2y &= 0 \\ 6x + ky &= 0 \end{aligned}$$

<u>Solution</u>

$$\begin{bmatrix} 3 & 2 \\ 6 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Augmented matrix is

$$\begin{bmatrix} 3 & 2 & 0 \\ 6 & k & 0 \end{bmatrix}$$

$R_2 \rightarrow -2R_1 + R_2$ gives

$$\begin{bmatrix} 3 & 2 & 0 \\ 0 & k-4 & 0 \end{bmatrix}$$

We see that when $k = 4$, then y is free variable giving ∞ number of solutions. When $k \neq 4$ then unique solution exist, which is the trivial solution. Hence answer is

$$\text{(a) } k \neq 4 \text{ (b) None (c) } k = 4.$$

Notice, the answer in back of the book seems wrong. It says (a) is when $k \neq 2$. It should be $k \neq 4$.

3 Problem section 3.2 number 25

In Problems 23–27, determine for what values of k each system has (a) a unique solution; (b) no solution; (c) infinitely many solutions.

$$
\begin{aligned}
3x + 2y &= 11\\
6x + ky &= 21
\end{aligned}
$$

Solution

$$
\begin{bmatrix} 3 & 2 \\ 6 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 21 \end{bmatrix}
$$

Augmented matrix is

$$
\begin{bmatrix} 3 & 2 & 11 \\ 6 & k & 21 \end{bmatrix}
$$

$R_2 \rightarrow -2R_1 + R_2$ gives

$$
\begin{bmatrix} 3 & 2 & 11 \\ 0 & k-4 & -1 \end{bmatrix}
$$

We see that when $k = 4$, then inconsistent, since it leads to $0 = -1$, hence no solution in this case. When $k \neq 4$ then unique solution exist. These are the only two possible cases. Hence answer is

(a) $k \neq 4$ (b) $k = 4$ (c) None

4 Problem section 3.2 number 26

In Problems 23–27, determine for what values of k each system has (a) a unique solution; (b) no solution; (c) infinitely many solutions.

$$\begin{aligned} 3x + 2y &= 1 \\ 7x + 5y &= k \end{aligned}$$

Solution

$$\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ k \end{bmatrix}$$

Augmented matrix is

$$\begin{bmatrix} 3 & 2 & 1 \\ 7 & 5 & k \end{bmatrix}$$

$R_1 \to 7R_1, R_2 \to 3R_2$ gives

$$\begin{bmatrix} 21 & 14 & 7 \\ 21 & 15 & 3k \end{bmatrix}$$

$R_2 \to R_2 - R_1$ gives

$$\begin{bmatrix} 21 & 14 & 7 \\ 0 & 1 & 3k - 7 \end{bmatrix}$$

The Echelon form shows that there are no free variables. Hence unique solution exist for all k values. Hence the answer is

(a) any k (b) None (c) None

5 Problem section 3.2 number 27

In Problems 23–27, determine for what values of k each system has (a) a unique solution; (b) no solution; (c) infinitely many solutions.

$$\begin{aligned} x + 2y + z &= 3 \\ 2x - y - 3z &= 5 \\ 4x + 3y - z &= k \end{aligned}$$

Solution

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & -3 \\ 4 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ k \end{bmatrix}$$

Augmented matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & -1 & -3 & 5 \\ 4 & 3 & -1 & k \end{bmatrix}$$

$R_2 \to -2R_1 + R_2$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -5 & -5 & -1 \\ 4 & 3 & -1 & k \end{bmatrix}$$

$R_3 \to -4R_1 + R_3$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -5 & -5 & -1 \\ 0 & -5 & -5 & k-12 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -5 & -5 & -1 \\ 0 & 0 & 0 & k-11 \end{bmatrix}$$

The Echelon form shows that only when $k = 11$ we get consistent system. And in this case, z is the free variable, leading to ∞ solutions. If $k \neq 11$ then system is inconsistent and no solution exist.

(a) None (b) $k \neq 11$ (c) $k = 11$

6 Problem section 3.2 number 28

Under what condition on the constants a, b, and c does the system have a unique solution? No solution? Infinitely many solutions?

$$\begin{aligned} 2x - y + 3z &= a \\ x + 2y + z &= b \\ 7x + 4y + 9z &= c \end{aligned}$$

<u>Solution</u>

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 1 \\ 7 & 4 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Augmented matrix is

$$\begin{bmatrix} 2 & -1 & 3 & a \\ 1 & 2 & 1 & b \\ 7 & 4 & 9 & c \end{bmatrix}$$

$R_2 \rightarrow -R_1 + 2R_2$ gives

$$\begin{bmatrix} 2 & -1 & 3 & a \\ 0 & 5 & -1 & 2b - a \\ 7 & 4 & 9 & c \end{bmatrix}$$

$R_3 \rightarrow 2R_3, R_1 \rightarrow 7R_1$ gives

$$\begin{bmatrix} 14 & -7 & 21 & 7a \\ 0 & 5 & -1 & 2b - a \\ 14 & 8 & 18 & 2c \end{bmatrix}$$

$R_3 \rightarrow -R_1 + R_3$ gives

$$\begin{bmatrix} 14 & -7 & 21 & 7a \\ 0 & 5 & -1 & 2b - a \\ 0 & 15 & -3 & 2c - 7a \end{bmatrix}$$

$R_3 \rightarrow -3R_2 + R_3$ gives

$$\begin{bmatrix} 14 & -7 & 21 & 7a \\ 0 & 5 & -1 & 2b - a \\ 0 & 0 & 0 & (2c - 7a) - 3(2b - a) \end{bmatrix}$$

Or

$$\begin{bmatrix} 14 & -7 & 21 & 7a \\ 0 & 5 & -1 & 2b - a \\ 0 & 0 & 0 & 2c - 6b - 4a \end{bmatrix}$$

The Echelon form shows that only when $2c - 6b - 4a = 0$ or

$$c = 3b + 2a$$

We get consistent system. And in this case, z is the free variable, leading to ∞ solutions. If $c \neq 3b + 2a$ then system is inconsistent and no solution exist.

7 Problem section 3.3 number 37

Show that the homogeneous system in problem 35 has non-trivial solution iff $ad - bc = 0$

$$
\begin{aligned}
ax + by &= 0\\
cx + dy &= 0
\end{aligned}
$$

<u>Solution</u>

$$
\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

Augmented matrix is

$$
\begin{bmatrix} a & b & 0 \\ c & d & 0 \end{bmatrix}
$$

$R_2 \rightarrow -\frac{c}{a}R_1 + R_2$ gives

$$
\begin{bmatrix} a & b & 0 \\ 0 & d - \frac{c}{a}b & 0 \end{bmatrix}
$$

Or

$$
\begin{bmatrix} a & b & 0 \\ 0 & \frac{ad-cb}{c} & 0 \end{bmatrix}
$$

There are two cases. First case is when $\frac{ad-cb}{c} = 0$ or $ad - cb = 0$ then we get infinite number of solutions since y is the free variable. The second case is when $ad - cb \neq 0$ and in this case, we get unique solution which is the trivial solution $x = 0, y = 0$.

Hence only when $ad - cb = 0$ do we get non-trivial solution which is what we are asked to show.

8 Problem section 3.3 number 38

Use the result of problem 37 to find all values of c for which

$$\begin{aligned}(c+2)x+3y&=0\\2x+(c-3)y&=0\end{aligned} \tag{1}$$

Has non-trivial solution.

Solution

From problem 37 we found that non-trivial solution exist when

$$ad-cb=0 \tag{2}$$

Where

$$\begin{aligned}ax+by&=0\\cx+dy&=0\end{aligned} \tag{3}$$

Comparing (1) to (3) shows that

$$\begin{aligned}a&\equiv(c+2)\\b&\equiv3\\c&\equiv2\\d&\equiv(c-3)\end{aligned}$$

Hence (2) now becomes

$$\begin{aligned}(c+2)(c-3)-(2)(3)&=0\\c^2-c-12&=0\\(c+3)(c-4)&=0\end{aligned}$$

Hence only possible values are $c=-3, c=4$. These values give non-trivial solution.

9 Problem section 3.5 number 9

Use the method of example 7 to find A^{-1} for given A

$$A = \begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix}$$

Solution

Augmented matrix is

$$\begin{bmatrix} 5 & 6 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{bmatrix}$$

$R_1 \to 4R_1, R_2 \to 5R_2$ gives

$$\begin{bmatrix} 20 & 24 & 4 & 0 \\ 20 & 25 & 0 & 5 \end{bmatrix}$$

$R_2 \to -R_1 + R_2$

$$\begin{bmatrix} 20 & 24 & 4 & 0 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

$R_1 \to \frac{R_1}{20}$ gives

$$\begin{bmatrix} 1 & \frac{6}{5} & \frac{1}{5} & 0 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

$R_1 \to R_1 - \frac{6}{5}R_2$

$$\begin{bmatrix} 1 & 0 & 5 & -6 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

Since the left half is the identity matrix, then the inverse is the right side. Hence

$$A^{-1} = \begin{bmatrix} 5 & -6 \\ -4 & 5 \end{bmatrix}$$

10 Problem section 3.5 number 10

Use the method of example 7 to find A^{-1} for given A

$$A = \begin{bmatrix} 5 & 7 \\ 4 & 6 \end{bmatrix}$$

Solution

Augmented matrix is

$$\begin{bmatrix} 5 & 7 & 1 & 0 \\ 4 & 6 & 0 & 1 \end{bmatrix}$$

$R_1 \to 4R_1, R_2 \to 5R_2$ gives

$$\begin{bmatrix} 20 & 28 & 4 & 0 \\ 20 & 30 & 0 & 5 \end{bmatrix}$$

$R_2 \to -R_1 + R_2$ gives

$$\begin{bmatrix} 20 & 28 & 4 & 0 \\ 0 & 2 & -4 & 5 \end{bmatrix}$$

$R_1 \to \frac{R_1}{20}$ gives

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{1}{5} & 0 \\ 0 & 2 & -4 & 5 \end{bmatrix}$$

$R_2 \to \frac{R_2}{2}$ gives

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{1}{5} & 0 \\ 0 & 1 & -2 & \frac{5}{2} \end{bmatrix}$$

$R_1 \to R_1 - \frac{7}{5}R_2$ gives

$$\begin{bmatrix} 1 & 0 & 3 & -\frac{7}{2} \\ 0 & 1 & -2 & \frac{5}{2} \end{bmatrix}$$

Since the left half is the identity matrix, then the inverse is the right side. Hence

$$A^{-1} = \begin{bmatrix} 3 & -\frac{7}{2} \\ -2 & \frac{5}{2} \end{bmatrix}$$

11 Problem section 3.5 number 11

Use the method of example 7 to find A^{-1} for given A

$$A=\begin{bmatrix}1&5&1\\2&5&0\\2&7&1\end{bmatrix}$$

Solution

Augmented matrix is

$$\begin{bmatrix}1&5&1&1&0&0\\2&5&0&0&1&0\\2&7&1&0&0&1\end{bmatrix}$$

$R_2\to R_2-2R_1$ gives

$$\begin{bmatrix}1&5&1&1&0&0\\0&-5&-2&-2&1&0\\2&7&1&0&0&1\end{bmatrix}$$

$R_3\to R_3-2R_1$ gives

$$\begin{bmatrix}1&5&1&1&0&0\\0&-5&-2&-2&1&0\\0&-3&-1&-2&0&1\end{bmatrix}$$

$R_2\to 3R_2, R_3\to 5R_3$ gives

$$\begin{bmatrix}1&5&1&1&0&0\\0&-15&-6&-6&3&0\\0&-15&-5&-10&0&5\end{bmatrix}$$

$R_3\to R_3-R_2$ gives

$$\begin{bmatrix}1&5&1&1&0&0\\0&-15&-6&-6&3&0\\0&0&1&-4&-3&5\end{bmatrix}$$

$R_2\to -\frac{R_2}{15}$ gives

$$\begin{bmatrix}1&5&1&1&0&0\\0&1&\frac{6}{15}&\frac{6}{15}&-\frac{3}{15}&0\\0&0&1&-4&-3&5\end{bmatrix}$$

$R_2\to R_2-\frac{6}{15}R_3$ gives

$$\begin{bmatrix}1&5&1&1&0&0\\0&1&0&2&1&-2\\0&0&1&-4&-3&5\end{bmatrix}$$

$R_1 \to R_1 - R_3$ gives

$$\begin{bmatrix} 1 & 5 & 0 & 5 & 3 & -5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{bmatrix}$$

$R_1 \to R_1 - 5R_2$ gives

$$\begin{bmatrix} 1 & 0 & 0 & -5 & -2 & 5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{bmatrix}$$

Since the left half is the identity matrix, then the inverse is the right side. Hence

$$A^{-1} = \begin{bmatrix} -5 & -2 & 5 \\ 2 & 1 & -2 \\ -4 & -3 & 5 \end{bmatrix}$$

12 Problem section 3.5 number 12

Use the method of example 7 to find A^{-1} for given A

$$A=\begin{bmatrix}1&3&2\\2&8&3\\3&10&6\end{bmatrix}$$

<u>Solution</u>

Augmented matrix is

$$\begin{bmatrix}1&3&2&1&0&0\\2&8&3&0&1&0\\3&10&6&0&0&1\end{bmatrix}$$

$R_2\rightarrow R_2-2R_1$ gives

$$\begin{bmatrix}1&3&2&1&0&0\\0&2&-1&-2&1&0\\3&10&6&0&0&1\end{bmatrix}$$

$R_3\rightarrow R_3-3R_1$ gives

$$\begin{bmatrix}1&3&2&1&0&0\\0&2&-1&-2&1&0\\0&1&0&-3&0&1\end{bmatrix}$$

$R_3\rightarrow 2R_3-R_2$ gives

$$\begin{bmatrix}1&3&2&1&0&0\\0&2&-1&-2&1&0\\0&0&1&-4&-1&2\end{bmatrix}$$

$R_2\rightarrow \frac{1}{2}R_2$ gives

$$\begin{bmatrix}1&3&2&1&0&0\\0&1&-\frac{1}{2}&-1&\frac{1}{2}&0\\0&0&1&-4&-1&2\end{bmatrix}$$

$R_2\rightarrow \frac{1}{2}R_3+R_2$ gives

$$\begin{bmatrix}1&3&2&1&0&0\\0&1&0&-3&0&1\\0&0&1&-4&-1&2\end{bmatrix}$$

$R_1\rightarrow R_1-2R_3$ gives

$$\begin{bmatrix}1&3&0&9&2&-4\\0&1&0&-3&0&1\\0&0&1&-4&-1&2\end{bmatrix}$$

$R_1 \to R_1 - 3R_2$ gives

$$\begin{bmatrix} 1 & 0 & 0 & 18 & 2 & -7 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & -4 & -1 & 2 \end{bmatrix}$$

Since the left half is the identity matrix, then the inverse is the right side. Hence

$$A^{-1} = \begin{bmatrix} 18 & 2 & -7 \\ -3 & 0 & 1 \\ -4 & -1 & 2 \end{bmatrix}$$

13 Problem section 3.5 number 13

Use the method of example 7 to find A^{-1} for given A

$$A=\begin{bmatrix}2&7&3\\1&3&2\\3&7&9\end{bmatrix}$$

Solution

Augmented matrix is

$$\begin{bmatrix}2&7&3&1&0&0\\1&3&2&0&1&0\\3&7&9&0&0&1\end{bmatrix}$$

Swap R_1, R_2

$$\begin{bmatrix}1&3&2&0&1&0\\2&7&3&1&0&0\\3&7&9&0&0&1\end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ gives

$$\begin{bmatrix}1&3&2&0&1&0\\0&1&-1&1&-2&0\\3&7&9&0&0&1\end{bmatrix}$$

$R_3 \to R_3 - 3R_1$ gives

$$\begin{bmatrix}1&3&2&0&1&0\\0&1&-1&1&-2&0\\0&-2&3&0&-3&1\end{bmatrix}$$

$R_3 \to R_3 + 2R_2$ gives

$$\begin{bmatrix}1&3&2&0&1&0\\0&1&-1&1&-2&0\\0&0&1&2&-7&1\end{bmatrix}$$

$R_2 \to R_2 + R_3$ gives

$$\begin{bmatrix}1&3&2&0&1&0\\0&1&0&3&-9&1\\0&0&1&2&-7&1\end{bmatrix}$$

$R_1 \to R_1 - 2R_3$ gives

$$\begin{bmatrix}1&3&0&-4&15&-2\\0&1&0&3&-9&1\\0&0&1&2&-7&1\end{bmatrix}$$

$R_1 \to R_1 - 3R_2$ gives

$$\begin{bmatrix} 1 & 0 & 0 & -13 & 42 & -5 \\ 0 & 1 & 0 & 3 & -9 & 1 \\ 0 & 0 & 1 & 2 & -7 & 1 \end{bmatrix}$$

Since the left half is the identity matrix, then the inverse is the right side. Hence

$$A^{-1} = \begin{bmatrix} -13 & 42 & -5 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{bmatrix}$$

14 Problem section 3.5 number 23

Find matrix X such that $AX = B$

$$A = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -5 \\ -1 & -2 & 5 \end{bmatrix}$$

<u>Solution</u>

Pre-multiplying both sides of $AX = B$ by A^{-1} gives

$$X = A^{-1}B \tag{1}$$

But

$$\begin{aligned} A^{-1} &= \frac{1}{16-15}\begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix} \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} X &= \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 & -5 \\ -1 & -2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 18 & -35 \\ -9 & -23 & 45 \end{bmatrix} \end{aligned}$$

15 Problem section 3.5 number 24

Find matrix X such that $AX = B$

$$A = \begin{bmatrix} 7 & 6 \\ 8 & 7 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 5 & -3 \end{bmatrix}$$

<u>Solution</u>

Pre-multiplying both sides of $AX = B$ by A^{-1} and using $A^{-1}A = I$ results in

$$X = A^{-1}B \tag{1}$$

But

$$\begin{aligned} A^{-1} &= \frac{1}{49-48}\begin{bmatrix} 7 & -6 \\ -8 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 7 & -6 \\ -8 & 7 \end{bmatrix} \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} X &= \begin{bmatrix} 7 & -6 \\ -8 & 7 \end{bmatrix}\begin{bmatrix} 2 & 0 & 4 \\ 0 & 5 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 14 & -30 & 46 \\ -16 & 35 & -53 \end{bmatrix} \end{aligned}$$

16 Problem section 3.5 number 25

Find matrix X such that $AX = B$

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 8 & 3 \\ 2 & 7 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

<u>Solution</u>

Pre-multiplying both sides of $AX = B$ by A^{-1} and using $A^{-1}A = I$ results in

$$X = A^{-1}B \tag{1}$$

But

$$A^{-1} = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 8 & 3 \\ 2 & 7 & 4 \end{bmatrix}^{-1}$$

Augmented matrix is

$$\begin{bmatrix} 1 & 4 & 1 & 1 & 0 & 0 \\ 2 & 8 & 3 & 0 & 1 & 0 \\ 2 & 7 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$ gives

$$\begin{bmatrix} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 2 & 7 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 - 2R_1$ gives

$$\begin{bmatrix} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & -1 & 2 & -2 & 0 & 1 \end{bmatrix}$$

Swap R_2, R_3

$$\begin{bmatrix} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix}$$

$R_2 \to -R_2$

$$\begin{bmatrix} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_3$

$$\begin{bmatrix} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix}$$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & 4 & 0 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 4R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 11 & -9 & 4 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix}$$

Since the left half is the identity matrix, then the inverse is the right side. Hence

$$A^{-1} = \begin{bmatrix} -13 & 42 & -5 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{bmatrix}$$

Therefore (1) becomes

$$X = \begin{bmatrix} 11 & -9 & 4 \\ -2 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -14 & 15 \\ -1 & 3 & -2 \\ -2 & 2 & -4 \end{bmatrix}$$

17 Problem section 3.5 number 26

Find matrix X such that $AX = B$

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & -2 \\ 1 & 7 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ 1 & 0 & 2 \end{bmatrix}$$

<u>Solution</u>

Pre-multiplying both sides of $AX = B$ by A^{-1} and using $A^{-1}A = I$ results in

$$X = A^{-1}B \tag{1}$$

But

$$A^{-1} = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & -2 \\ 1 & 7 & 2 \end{bmatrix}^{-1}$$

Augmented matrix is

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 1 & 7 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -9 & -4 & -2 & 1 & 0 \\ 1 & 7 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -9 & -4 & -2 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$R_2 \to 2R_2, R_3 \to 9R_3$

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -18 & -8 & -4 & 2 & 0 \\ 0 & 18 & 9 & -9 & 0 & 9 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -18 & -8 & -4 & 2 & 0 \\ 0 & 0 & 1 & -13 & 2 & 9 \end{bmatrix}$$

$R_2 \to R_2 + 8R_3$

$$\begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -18 & 0 & -108 & 18 & 72 \\ 0 & 0 & 1 & -13 & 2 & 9 \end{bmatrix}$$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & 5 & 0 & 14 & -2 & -9 \\ 0 & -18 & 0 & -108 & 18 & 72 \\ 0 & 0 & 1 & -13 & 2 & 9 \end{bmatrix}$$

$R_2 \to \frac{-1}{18} R_2$

$$\begin{bmatrix} 1 & 5 & 0 & 14 & -2 & -9 \\ 0 & 1 & 0 & 6 & -1 & -4 \\ 0 & 0 & 1 & -13 & 2 & 9 \end{bmatrix}$$

$R_1 \to R_1 - 5R_2$

$$\begin{bmatrix} 1 & 0 & 0 & -16 & 3 & 11 \\ 0 & 1 & 0 & 6 & -1 & -4 \\ 0 & 0 & 1 & -13 & 2 & 9 \end{bmatrix}$$

Since the left half is the identity matrix, then the inverse is the right side. Hence

$$A^{-1} = \begin{bmatrix} -16 & -3 & 11 \\ 6 & -1 & -4 \\ -13 & 2 & 9 \end{bmatrix}$$

Therefore (1) becomes

$$X = \begin{bmatrix} -16 & -3 & 11 \\ 6 & -1 & -4 \\ -13 & 2 & 9 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -21 & -9 & 0 \\ 8 & -3 & -4 \\ -17 & 6 & 9 \end{bmatrix}$$

18 Problem section 3.6 number 7

In Problems 7–12, evaluate each given determinant after first simplifying the computation (as in Example 6) by adding an appropriate multiple of some row or column to another.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

Solution

We see before starting that the determinant must be zero, since its rows are linearly dependent. We now show this is the case.

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 3 & 3 & 3 \end{bmatrix}$$

Now we do expansion along R_2. This gives

$$\begin{aligned} \det(A) &= 0\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} \\ &= 0 \end{aligned}$$

19 Problem section 3.6 number 8

In Problems 7–12, evaluate each given determinant after first simplifying the computation (as in Example 6) by adding an appropriate multiple of some row or column to another.

$$A = \begin{bmatrix} 2 & 3 & 4 \\ -2 & -3 & 1 \\ 3 & 2 & 7 \end{bmatrix}$$

<u>Solution</u>

$R_1 \rightarrow R_1 + R_2$

$$A = \begin{bmatrix} 0 & 0 & 5 \\ -2 & -3 & 1 \\ 3 & 2 & 7 \end{bmatrix}$$

Expansion along first row gives

$$\begin{aligned} \det(A) &= 5 \begin{vmatrix} -2 & -3 \\ 3 & 2 \end{vmatrix} \\ &= 5(-4 + 9) \\ &= 25 \end{aligned}$$

20 Problem section 3.6 number 9

In Problems 7–12, evaluate each given determinant after first simplifying the computation (as in Example 6) by adding an appropriate multiple of some row or column to another.

$$A = \begin{bmatrix} 3 & -2 & 5 \\ 0 & 5 & 17 \\ 6 & -4 & 12 \end{bmatrix}$$

<u>Solution</u>

$R_3 \rightarrow R_3 - 2R_1$

$$A = \begin{bmatrix} 3 & -2 & 5 \\ 0 & 5 & 17 \\ 0 & 0 & 2 \end{bmatrix}$$

Expansion along third row gives

$$\begin{aligned} \det(A) &= 2\begin{vmatrix} 3 & -2 \\ 0 & 5 \end{vmatrix} \\ &= 2(15) \\ &= 30 \end{aligned}$$

21 Problem section 3.6 number 10

In Problems 7–12, evaluate each given determinant after first simplifying the computation (as in Example 6) by adding an appropriate multiple of some row or column to another.

$$A = \begin{bmatrix} -3 & 6 & 5 \\ 1 & -2 & -4 \\ 2 & -5 & 12 \end{bmatrix}$$

<u>Solution</u>

$R_1 \to R_1 + 3R_2$

$$A = \begin{bmatrix} 0 & 0 & -7 \\ 1 & -2 & -4 \\ 2 & -5 & 12 \end{bmatrix}$$

Expansion along first row gives

$$\begin{aligned} \det(A) &= -7 \begin{vmatrix} 1 & -2 \\ 2 & -5 \end{vmatrix} \\ &= -7(-5+4) \\ &= 7 \end{aligned}$$

22 Problem section 3.6 number 11

In Problems 7–12, evaluate each given determinant after first simplifying the computation (as in Example 6) by adding an appropriate multiple of some row or column to another.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 2 & 4 & 6 & 9 \end{bmatrix}$$

Solution

$R_3 \to R_3 - 2R_1$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Expansion along last row

$$\det(A) = (-1)^{i+j}(1)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 8 \end{vmatrix}$$

Where $i = 4, j = 4$ (since it is entry $(4,4)$). Hence $(-1)^{i+j} = (-1)^8 = 1$. So the sign is +. The above becomes

$$\det(A) = (1)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 8 \end{vmatrix}$$

For the second determination, expansion along its third row gives

$$\begin{aligned} \det(A) &= 1\left((-1)^{3+3}8\begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix}\right) \\ &= 1\left(8\begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix}\right) \\ &= 8(5) \\ &= 40 \end{aligned}$$

23 Problem section 3.6 number 12

In Problems 7–12, evaluate each given determinant after first simplifying the computation (as in Example 6) by adding an appropriate multiple of some row or column to another.

$$A = \begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & 1 & 11 & 12 \\ 0 & 0 & 5 & 13 \\ -4 & 0 & 0 & 7 \end{bmatrix}$$

Solution

$R_4 \rightarrow R_4 + 2R_1$

$$A = \begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & 1 & 11 & 12 \\ 0 & 0 & 5 & 13 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Expansion along last row

$$\begin{aligned} \det(A) &= (-1)^{4+4}(1)\begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 11 \\ 0 & 0 & 5 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 11 \\ 0 & 0 & 5 \end{vmatrix} \end{aligned}$$

Expansion along last row

$$\begin{aligned} \det(A) &= (-1)^{3+3}(5)\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \\ &= 5(2) \\ &= 10 \end{aligned}$$

24 Problem section 4.3 number 17

In Problems 17–22, three vectors $\vec{v}_1$,$\vec{v}_2$, and $\vec{v}_3$ are given. If they are linearly independent, show this; otherwise find a

nontrivial linear combination of them that is equal to the zero vector.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

Solution

The vectors are Linearly independent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

only when $c_1 = c_2 = c_3 = 0$. If we can find at least one $c_i \neq 0$ where the above is true, then the vectors are Linearly dependent.

Writing the above as $A\vec{c} = \vec{0}$ gives

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & 5 \\ 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & 5 & 0 \\ 1 & 4 & 2 & 0 \end{bmatrix}$$

$R_3 \rightarrow -R_1 + R_3$ gives

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & 5 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3, R_2 \rightarrow 2R_2$ gives

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -6 & 10 & 0 \\ 0 & 6 & -3 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_2 + R_3$ gives

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -6 & 10 & 0 \\ 0 & 0 & 7 & 0 \end{bmatrix}$$

Hence the original system (1) in Echelon form becomes

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & 10 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Leading variables are c_1, c_2, c_3. Since there are no free variables, then only the trivial solution exist. We see this by backsubstitution. Last row gives $c_3 = 0$. Second row gives $c_2 = 0$ and first row gives $c_1 = 0$.

Since all $c_i = 0$, then the vectors are Linearly independent.

25 Problem section 4.3 number 18

In Problems 17–22, three vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are given. If they are linearly independent, show this; otherwise find a nontrivial linear combination of them that is equal to the zero vector.

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 4 \\ -5 \\ -6 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

solution

The vectors are Linearly independent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

only when $c_1 = c_2 = c_3 = 0$. If we can find at least one $c_i \neq 0$ where the above is true, then the vectors are Linearly dependent.

Writing the above as $A\vec{c} = \vec{0}$ gives

$$\begin{bmatrix} 2 & 4 & -2 \\ 0 & -5 & 1 \\ -3 & -6 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 0 & -5 & 1 & 0 \\ -3 & -6 & 3 & 0 \end{bmatrix}$$

$R_1 \to 3R_1, R_3 \to 2R_3$ gives

$$\begin{bmatrix} 6 & 12 & -6 & 0 \\ 0 & -5 & 1 & 0 \\ -6 & -12 & 6 & 0 \end{bmatrix}$$

$R_3 \to R_1 + R_3$ gives

$$\begin{bmatrix} 6 & 12 & -6 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the system (1) becomes

$$\begin{bmatrix} 6 & 12 & -6 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The leading variables are c_1, c_2 and free variable is c_3. Since there is a free variable, then the vectors are Linearly dependent. To see this, let $c_3 = t$. From second row $-5c_2 + t = 0$ or $c_2 = \frac{1}{5}t$. From first row $6c_1 + 12c_2 - 6t = 0$. Or $c_1 = \frac{6t - 12\left(\frac{1}{5}t\right)}{6} = \frac{3}{5}t$. Hence

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{5}t \\ \frac{1}{5}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix} = \frac{1}{5}t\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

Taking $\tilde{t} = 5$ the above becomes

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

Therefore we found one solution where

$$\begin{aligned} c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 &= \vec{0} \\ 3\vec{v}_1 + \vec{v}_2 + 5\vec{v}_3 &= \vec{0} \end{aligned}$$

not all c_i zero. Hence linearly dependent vectors.

26 Problem section 4.3 number 19

In Problems 17–22, three vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are given. If they are linearly independent, show this; otherwise find a nontrivial linear combination of them that is equal to the zero vector.

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 4 \\ -2 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

solution

The vectors are Linearly independent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

only when $c_1 = c_2 = c_3 = 0$. If we can find at least one $c_i \neq 0$ where the above is true, then the vectors are Linearly dependent.

Writing the above as $A\vec{c} = \vec{0}$ gives

$$\begin{bmatrix} 2 & 5 & 2 \\ 0 & 4 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 2 & 5 & 2 & 0 \\ 0 & 4 & -1 & 0 \\ 3 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$R_3 \rightarrow 2R_3, R_1 \rightarrow 3R_1$

$$\begin{bmatrix} 6 & 15 & 6 & 0 \\ 0 & 4 & -1 & 0 \\ 6 & -4 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 6 & 15 & 6 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & -19 & -4 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$R_2 \to 19R_2, R_3 \to 4R_3$

$$\begin{bmatrix} 6 & 15 & 6 & 0 \\ 0 & 76 & -19 & 0 \\ 0 & -76 & -16 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 6 & 15 & 6 & 0 \\ 0 & 76 & -19 & 0 \\ 0 & 0 & -35 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$R_4 \to 76R_4$

$$\begin{bmatrix} 6 & 15 & 6 & 0 \\ 0 & 76 & -19 & 0 \\ 0 & 0 & -35 & 0 \\ 0 & 76 & -76 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$\begin{bmatrix} 6 & 15 & 6 & 0 \\ 0 & 76 & -19 & 0 \\ 0 & 0 & -35 & 0 \\ 0 & 0 & -57 & 0 \end{bmatrix}$$

$R_4 \to R_4 - \frac{57}{35}R_3$

$$\begin{bmatrix} 6 & 15 & 6 & 0 \\ 0 & 76 & -19 & 0 \\ 0 & 0 & -35 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is Echelon form. Hence

$$\begin{bmatrix} 6 & 15 & 6 \\ 0 & 76 & -19 \\ 0 & 0 & -35 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Lead variables are c_1, c_2, c_3. There are no free variables. Therefore unique solution exist and is $c_1 = 0, c_2 = 0, c_3 = 0$. Hence the vectors are linearly independent.

27 Problem section 4.3 number 20

In Problems 17–22, three vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are given. If they are linearly independent, show this; otherwise find a nontrivial linear combination of them that is equal to the zero vector.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix}$$

solution

The vectors are Linearly independent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

only when $c_1 = c_2 = c_3 = 0$. If we can find at least one $c_i \neq 0$ where the above is true, then the vectors are Linearly dependent.

Writing the above as $A\vec{c} = \vec{0}$ gives

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ -1 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 3 & 7 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 3 & 7 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is Echelon form. Hence

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Lead variables are c_1, c_2, c_3. There are no free variables. Therefore a unique solution exists and is the trivial solution $c_1 = 0, c_2 = 0, c_3 = 0$. Hence the vectors are linearly independent.

28 Problem section 4.3 number 21

In Problems 17–22, three vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are given. If they are linearly independent, show this; otherwise find a nontrivial linear combination of them that is equal to the zero vector.

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

solution

The vectors are Linearly independent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

only when $c_1 = c_2 = c_3 = 0$. If we can find at least one $c_i \neq 0$ where the above is true, then the vectors are Linearly dependent.

Writing the above as $A\vec{c} = \vec{0}$ gives

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 3 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

$R_3 \rightarrow 3R_3 - R_1$

$$\begin{bmatrix} 3 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

$R_1 \rightarrow 2R_1, R_4 \rightarrow 3R_4$

$$\begin{bmatrix} 6 & 2 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 6 & 3 & 0 & 0 \end{bmatrix}$$

$R_4 \rightarrow R_4 - R_1$

$$\begin{bmatrix} 6 & 2 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 6 & 2 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

$R_4 \rightarrow R_4 + R_2$

$$\begin{bmatrix} 6 & 2 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is Echelon form. Hence

$$\begin{bmatrix} 6 & 2 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Lead variables are c_1, c_2. And free variable is c_3. Since there is a free variable, then non-trivial solution exist. Hence the vectors are linearly dependent. Let $c_3 = t$. From second row

$$\begin{aligned} -c_2 + 2c_3 &= 0 \\ c_2 &= 2c_3 = 2t \end{aligned}$$

From first row

$$\begin{aligned} 6c_1 + 2c_2 + 2c_3 &= 0 \\ c_1 &= \frac{-2c_2 - 2c_3}{6} \\ &= \frac{-2(2t) - 2t}{6} \\ &= -t \end{aligned}$$

Hence solution is

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -t \\ 2t \\ t \end{bmatrix}$$
$$= t\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Let $t = 1$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Therefore

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$
$$-\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3 = \vec{0}$$

29 Problem section 4.3 number 22

In Problems 17–22, three vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are given. If they are linearly independent, show this; otherwise find a nontrivial linear combination of them that is equal to the zero vector.

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 9 \\ 0 \\ 5 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 9 \\ -7 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 4 \\ 7 \\ 5 \\ 0 \end{bmatrix}$$

<u>solution</u>

The vectors are Linearly independent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

only when $c_1 = c_2 = c_3 = 0$. If we can find at least one $c_i \neq 0$ where the above is true, then the vectors are Linearly dependent.

Writing the above as $A\vec{c} = \vec{0}$ gives

$$\begin{bmatrix} 3 & 3 & 4 \\ 9 & 0 & 7 \\ 0 & 9 & 5 \\ 5 & -7 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 3 & 3 & 4 & 0 \\ 9 & 0 & 7 & 0 \\ 0 & 9 & 5 & 0 \\ 5 & -7 & 0 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 3R_1$

$$\begin{bmatrix} 3 & 3 & 4 & 0 \\ 0 & -9 & -5 & 0 \\ 0 & 9 & 5 & 0 \\ 5 & -7 & 0 & 0 \end{bmatrix}$$

$R_1 \rightarrow 5R_1, R_4 \rightarrow 3R_4$

$$\begin{bmatrix} 15 & 15 & 20 & 0 \\ 0 & -9 & -5 & 0 \\ 0 & 9 & 5 & 0 \\ 15 & -21 & 0 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_1$

$$\begin{bmatrix} 15 & 15 & 20 & 0 \\ 0 & -9 & -5 & 0 \\ 0 & 9 & 5 & 0 \\ 0 & -36 & -20 & 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 15 & 15 & 20 & 0 \\ 0 & -9 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -36 & -20 & 0 \end{bmatrix}$$

$R_4 \to R_4 - 4R_2$

$$\begin{bmatrix} 15 & 15 & 20 & 0 \\ 0 & -9 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is Echelon form. Hence

$$\begin{bmatrix} 15 & 15 & 20 \\ 0 & -9 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Lead variables are c_1, c_2. And free variable is c_3. Since there is a free variable, then non-trivial solution exist. Hence the vectors are linearly dependent. Let $c_3 = t$. From second row

$$\begin{aligned} -9c_2 - 5c_3 &= 0 \\ c_2 &= -\frac{5}{9}c_3 \\ &= -\frac{5}{9}t \end{aligned}$$

From first row

$$\begin{aligned} 15c_1 + 15c_2 + 20c_3 &= 0 \\ c_1 &= \frac{-15c_2 - 20c_3}{15} \\ &= \frac{-15\left(-\frac{5}{9}t\right) - 20t}{15} \\ &= -\frac{7}{9}t \end{aligned}$$

Hence solution is

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -\frac{7}{9}t \\ -\frac{5}{9}t \\ t \end{bmatrix}$$

$$= t\begin{bmatrix} -\frac{7}{9} \\ -\frac{5}{9} \\ 1 \end{bmatrix}$$

$$= \frac{1}{9}t\begin{bmatrix} -7 \\ -5 \\ 9 \end{bmatrix}$$

Let $t = -9$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ -9 \end{bmatrix}$$

Therefore

$$\begin{aligned} c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 &= \vec{0} \\ 7\vec{v}_1 + 5\vec{v}_2 - 9\vec{v}_3 &= \vec{0} \end{aligned}$$

30 Problem section 4.4 number 15

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system.

$$
\begin{aligned}
x_1 - 2x_2 + 3x_3 &= 0\\
2x_1 - 3x_2 - x_3 &= 0
\end{aligned}
$$

Solution

$A\vec{x} = \vec{0}$ gives

$$
\begin{bmatrix} 1 & -2 & 3\\ 2 & -3 & -1 \end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3 \end{bmatrix} = \begin{bmatrix} 0\\ 0 \end{bmatrix}
$$

The augmented matrix is

$$
\begin{bmatrix} 1 & -2 & 3 & 0\\ 2 & -3 & -1 & 0 \end{bmatrix}
$$

$R_2 \rightarrow R_2 - 2R_1$

$$
\begin{bmatrix} 1 & -2 & 3 & 0\\ 0 & 1 & -7 & 0 \end{bmatrix}
$$

Hence the leading variables are x_1, x_2 and the free variable is $x_3 = t$. The system becomes

$$
\begin{bmatrix} 1 & -2 & 3\\ 0 & 1 & -7 \end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3 \end{bmatrix} = \begin{bmatrix} 0\\ 0 \end{bmatrix}
$$

From second row

$$
\begin{aligned}
x_2 - 7x_3 &= 0\\
x_2 &= 7x_3\\
&= 7t
\end{aligned}
$$

From first row

$$
\begin{aligned}
x_1 - 2x_2 + 3x_3 &= 0\\
x_1 &= 2x_2 - 3x_3\\
&= 2(7t) - 3t\\
&= 11t
\end{aligned}
$$

Therefore the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11t \\ 7t \\ t \end{bmatrix} = t\begin{bmatrix} 11 \\ 7 \\ 1 \end{bmatrix}$$

Let $t = 1$, the basis is

$$\begin{bmatrix} 11 \\ 7 \\ 1 \end{bmatrix}$$

A one dimensional subspace.

31 Problem section 4.4 number 16

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$
\begin{aligned}
x_1 + 3x_2 + 4x_3 &= 0\\
3x_1 + 8x_2 + 7x_3 &= 0
\end{aligned}
$$

solution

$A\vec{x} = \vec{0}$ gives

$$
\begin{bmatrix} 1 & 3 & 4 \\ 3 & 8 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The augmented matrix is

$$
\begin{bmatrix} 1 & 3 & 4 & 0 \\ 3 & 8 & 7 & 0 \end{bmatrix}
$$

$R_2 \rightarrow -3R_1 + R_2$ gives

$$
\begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix}
$$

Hence the leading variables are x_1, x_2 and the free variable is $x_3 = t$. The system becomes

$$
\begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

Last row gives $-x_2 - 5x_3 = 0$ or $-x_2 = 5t$. Hence $x_2 = -5t$. From first row, $x_1 + 3x_2 + 4x_3 = 0$, or $x_1 = -3x_2 - 4x_3$ or $x_1 = -3(-5t) - 4t = 11t$. Therefore the solution is

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11t \\ -5t \\ t \end{bmatrix} = t \begin{bmatrix} 11 \\ -5 \\ 1 \end{bmatrix}
$$

Let $t = 1$. The basis is

$$
\begin{bmatrix} 11 \\ -5 \\ 1 \end{bmatrix}
$$

A one dimensional subspace.

32 Problem section 4.4 number 17

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$\begin{aligned} x_1 - 3x_2 + 2x_3 - 4x_4 &= 0 \\ 2x_1 - 5x_2 + 7x_3 - 3x_4 &= 0 \end{aligned}$$

solution

$A\vec{x} = \vec{0}$ gives

$$\begin{bmatrix} 1 & -3 & 2 & -4 \\ 2 & -5 & 7 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -3 & 2 & -4 & 0 \\ 2 & -5 & 7 & -3 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & -3 & 2 & -4 & 0 \\ 0 & 1 & 3 & 5 & 0 \end{bmatrix}$$

Leading variables are x_1, x_2 Free variables are $x_3 = t, x_4 = s$. The system becomes

$$\begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Second row gives

$$\begin{aligned} x_2 + 3x_3 + 5x_4 &= 0 \\ x_2 &= -3x_3 - 5x_4 \\ &= -3t - 5s \end{aligned}$$

First row gives

$$\begin{aligned} x_1 - 3x_2 + 2x_3 - 4x_4 &= 0 \\ x_1 &= 3x_2 - 2x_3 + 4x_4 \\ &= 3(-3t - 5s) - 2t + 4s \\ &= -11s - 11t \end{aligned}$$

Hence the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -11s - 11t \\ -3t - 5s \\ t \\ s \end{bmatrix}$$

$$= t \begin{bmatrix} -11 \\ -3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -11 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1, s = 1$, the basis vectors are

$$\left\{ \left(\begin{bmatrix} -11 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -11 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right) \right\}$$

A two dimensional subspace.

33 Problem section 4.4 number 18

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$\begin{aligned} x_1 + 3x_2 + 4x_3 + 5x_4 &= 0 \\ 2x_1 + 6x_2 + 9x_3 + 5x_4 &= 0 \end{aligned}$$

solution

$A\vec{x} = \vec{0}$ gives

$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 6 & 9 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & 4 & 5 & 0 \\ 2 & 6 & 9 & 5 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 3 & 4 & 5 & 0 \\ 0 & 0 & 1 & -5 & 0 \end{bmatrix}$$

Leading variables are x_1, x_3 Free variables are $x_2 = t, x_4 = s$. The system becomes

$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Second row gives

$$\begin{aligned} x_3 - 5x_4 &= 0 \\ x_3 &= 5x_4 \\ &= 5s \end{aligned}$$

First row gives

$$\begin{aligned} x_1 + 3x_2 + 4x_3 + 5x_4 &= 0 \\ x_1 &= -3x_2 - 4x_3 - 5x_4 \\ &= -3t - 4(5s) - 5s \\ &= -25s - 3t \end{aligned}$$

Hence the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -25s - 3t \\ t \\ 5s \\ s \end{bmatrix}$$

$$= t\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -25 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

Let $t = 1, s = 1$, the basis vectors are

$$\left\{ \left(\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -25 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right) \right\}$$

34 Problem section 4.4 number 19

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$\begin{aligned} x_1 - 3x_2 - 9x_3 - 5x_4 &= 0 \\ 2x_1 + x_2 - 4x_3 + 11x_4 &= 0 \\ x_1 + 3x_2 + 3x_3 + 13x_4 &= 0 \end{aligned}$$

solution

$A\vec{x} = \vec{0}$ gives

$$\begin{bmatrix} 1 & -3 & -9 & -5 \\ 2 & 1 & -4 & 11 \\ 1 & 3 & 3 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -3 & -9 & -5 & 0 \\ 2 & 1 & -4 & 11 & 0 \\ 1 & 3 & 3 & 13 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & -3 & -9 & -5 & 0 \\ 0 & 7 & 14 & 21 & 0 \\ 1 & 3 & 3 & 13 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -3 & -9 & -5 & 0 \\ 0 & 7 & 14 & 21 & 0 \\ 0 & 6 & 12 & 18 & 0 \end{bmatrix}$$

$R_3 \to R_3 - \frac{6}{7}R_2$

$$\begin{bmatrix} 1 & -3 & -9 & -5 & 0 \\ 0 & 7 & 14 & 21 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading variables are x_1, x_2 Free variable is $x_3 = t, x_4 = s$. The system becomes

$$\begin{bmatrix} 1 & -3 & -9 & -5 \\ 0 & 7 & 14 & 21 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

second row gives $7x_2 + 14x_3 + 21x_4 = 0$ or $x_2 = \frac{-14x_3 - 21x_4}{7} = \frac{-14t - 21s}{7} = -3s - 2t$. First row gives $x_1 - 3x_2 - 9x_3 - 5x_4 = 0$ or $x_1 = 3x_2 + 9x_3 + 5x_4$ or $x_1 = 3(-3s - 2t) + 9t + 5s = 3t - 4s$. Hence the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3t - 4s \\ -3s - 2t \\ t \\ s \end{bmatrix}$$
$$= t\begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1, s = 1$, hence the basis are

$$\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

A two dimensional subspace.

35 Problem section 4.4 number 20

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$
\begin{aligned}
x_1 - 3x_2 - 10x_3 + 5x_4 &= 0\\
x_1 + 4x_2 + 11x_3 - 2x_4 &= 0\\
x_1 + 3x_2 + 8x_3 - x_4 &= 0
\end{aligned}
$$

solution

$A\vec{x} = \vec{0}$ gives

$$
\begin{bmatrix} 1 & -3 & -10 & 5 \\ 1 & 4 & 11 & -2 \\ 1 & 3 & 8 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The augmented matrix is

$$
\begin{bmatrix} 1 & -3 & -10 & 5 & 0 \\ 1 & 4 & 11 & -2 & 0 \\ 1 & 3 & 8 & -1 & 0 \end{bmatrix}
$$

$R_2 \rightarrow -R_1 + R_2$ gives

$$
\begin{bmatrix} 1 & -3 & -10 & 5 & 0 \\ 0 & 7 & 21 & -7 & 0 \\ 1 & 3 & 8 & -1 & 0 \end{bmatrix}
$$

$R_3 \rightarrow -R_1 + R_3$ gives

$$
\begin{bmatrix} 1 & -3 & -10 & 5 & 0 \\ 0 & 7 & 21 & -7 & 0 \\ 0 & 6 & 18 & -6 & 0 \end{bmatrix}
$$

$R_3 \rightarrow 7R_3$ and $R_2 \rightarrow 6R_2$ gives

$$
\begin{bmatrix} 1 & -3 & -10 & 5 & 0 \\ 0 & 42 & 126 & -42 & 0 \\ 0 & 42 & 126 & -42 & 0 \end{bmatrix}
$$

$R_3 \rightarrow -R_2 + R_3$ gives

$$
\begin{bmatrix} 1 & -3 & -10 & 5 & 0 \\ 0 & 42 & 126 & -42 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

Leading variables are x_1, x_2 Free variables are $x_3 = t, x_4 = s$. The system becomes

$$\begin{bmatrix} 1 & -3 & -10 & 5 \\ 0 & 42 & 126 & -42 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

second row gives $42x_2 + 126x_3 - 42x_4 = 0$ or $42x_2 = -126t + 42s$ or $x_2 = -\frac{126}{42}t + \frac{42}{42}s = -3t + s$. First row gives $x_1 - 3x_2 - 10x_3 + 5x_4 = 0$ or $x_1 = 3x_2 + 10x_3 - 5x_4$ or $x_1 = 3(-3t + s) + 10t - 5s = t - 2s$. Hence the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t - 2s \\ -3t + s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1, s = 1$. The basis are

$$\begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

A two dimensional subspace.

36 Problem section 4.4 number 21

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$\begin{aligned} x_1 - 4x_2 - 3x_3 - 7x_4 &= 0 \\ 2x_1 - x_2 + x_3 + 7x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 11x_4 &= 0 \end{aligned}$$

solution

$A\vec{x} = \vec{0}$ gives

$$\begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -4 & -3 & -7 & 0 \\ 2 & -1 & 1 & 7 & 0 \\ 1 & 2 & 3 & 11 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & -4 & -3 & -7 & 0 \\ 0 & 7 & 7 & 21 & 0 \\ 1 & 2 & 3 & 11 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -4 & -3 & -7 & 0 \\ 0 & 7 & 7 & 21 & 0 \\ 0 & 6 & 6 & 18 & 0 \end{bmatrix}$$

$R_4 \to R_4 - \frac{6}{7} R_2$

$$\begin{bmatrix} 1 & -4 & -3 & -7 & 0 \\ 0 & 7 & 7 & 21 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading variables are x_1, x_2 Free variables are $x_3 = t, x_4 = s$. The system becomes

$$\begin{bmatrix} 1 & -4 & -3 & -7 \\ 0 & 7 & 7 & 21 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Second row gives $7x_2 + 7x_3 + 21x_4 = 0$ or $x_2 = \frac{-7x_3 - 21x_4}{7} = \frac{-7t - 21s}{7} = -3s - t$. First row gives $x_1 - 4x_2 - 3x_3 - 7x_4 = 0$ or $x_1 = 4x_2 + 3x_3 + 7x_4 = 4(-3s - t) + 3t + 7s = -5s - t$. Hence solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5s - t \\ -3s - t \\ t \\ s \end{bmatrix}$$
$$= t \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -5 \\ -3 \\ 0 \\ s \end{bmatrix}$$

Let $t = 1, s = 2$, the Basis are

$$\left\{ \left(\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right) \right\}$$

A two dimensional subspace.

37 Problem section 4.4 number 22

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$\begin{aligned} x_1 - 2x_2 - 3x_3 - 16x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + 17x_4 &= 0 \\ x_1 - 2x_2 + 3x_3 + 26x_4 &= 0 \end{aligned}$$

solution

$A\vec{x} = \vec{0}$ gives

$$\begin{bmatrix} 1 & -2 & -3 & -16 \\ 2 & -4 & 1 & 17 \\ 1 & -2 & 3 & 26 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -2 & -3 & -16 & 0 \\ 2 & -4 & 1 & 17 & 0 \\ 1 & -2 & 3 & 26 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & -2 & -3 & -16 & 0 \\ 0 & 0 & 7 & 49 & 0 \\ 1 & -2 & 3 & 26 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -2 & -3 & -16 & 0 \\ 0 & 0 & 7 & 49 & 0 \\ 0 & 0 & 6 & 42 & 0 \end{bmatrix}$$

$R_3 \to R_3 - \frac{6}{7}R_2$

$$\begin{bmatrix} 1 & -2 & -3 & -16 & 0 \\ 0 & 0 & 7 & 49 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence leading variables are x_1, x_3 and free variables are $x_2 = t, x_4 = s$.The system becomes

$$\begin{bmatrix} 1 & -2 & -3 & -16 \\ 0 & 0 & 7 & 49 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Second row gives $7x_3 + 49x_4 = 0$ or $x_3 = -7s$. First row gives $x_1 - 2x_2 - 3x_3 - 16x_4 = 0$ or $x_1 = 2x_2 + 3x_3 + 16x_4$ or $x_1 = 2t + 3(-7s) + 16s = 2t - 5s$. Hence solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2t - 5s \\ t \\ -7s \\ s \end{bmatrix}$$
$$= t\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -5 \\ 0 \\ -7 \\ 1 \end{bmatrix}$$

Let $t = 1, s = 1$, therefore the basis are

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -7 \\ 1 \end{bmatrix} \right\}$$

A two dimensional subspace.

38 Problem section 4.4 number 23

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$\begin{aligned} x_1+5x_2+13x_3+14x_4&=0\\ 2x_1+5x_2+11x_3+12x_4&=0\\ 2x_1+7x_2+17x_3+19x_4&=0 \end{aligned}$$

solution

$A\vec{x}=\vec{0}$ gives

$$\begin{bmatrix} 1 & 5 & 13 & 14\\ 2 & 5 & 11 & 12\\ 2 & 7 & 17 & 19 \end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{bmatrix}=\begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 5 & 13 & 14 & 0\\ 2 & 5 & 11 & 12 & 0\\ 2 & 7 & 17 & 19 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 5 & 13 & 14 & 0\\ 0 & -5 & -15 & -16 & 0\\ 2 & 7 & 17 & 19 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 5 & 13 & 14 & 0\\ 0 & -5 & -15 & -16 & 0\\ 0 & -3 & -9 & -9 & 0 \end{bmatrix}$$

$R_3 \to R_3 - \frac{3}{5}R_2$

$$\begin{bmatrix} 1 & 5 & 13 & 14 & 0\\ 0 & -5 & -15 & -16 & 0\\ 0 & 0 & 0 & \frac{3}{5} & 0 \end{bmatrix}$$

Hence leading variables are x_1, x_2, x_4 and free variables are $x_3 = t$.The system becomes

$$\begin{bmatrix} 1 & 5 & 13 & 14\\ 0 & -5 & -15 & -16\\ 0 & 0 & 0 & \frac{3}{5} \end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{bmatrix}=\begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix}$$

Last equation gives $x_4 = 0$. Second equation gives $-5x_2 - 15x_3 = 0$, or $x_2 = -3x_3 = -3t$. First equation gives $x_1 = -5x_2 - 13x_3$ or $x_1 = -5(-3t) - 13t = 2t$. Hence solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2t \\ -3t \\ t \\ 0 \end{bmatrix}$$
$$= t\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$,therefore the basis is

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

A one dimensional subspace.

39 Problem section 4.4 number 24

In Problems 15–26, find a basis for the solution space of the given homogeneous linear system

$$\begin{aligned} x_1 + 3x_2 - 4x_3 - 8x_4 + 6x_5 &= 0 \\ x_1 + 2x_3 + x_4 + 3x_5 &= 0 \\ 2x_1 + 7x_2 - 10x_3 - 19x_4 + 13x_5 &= 0 \end{aligned}$$

solution

$A\vec{x} = \vec{0}$ gives

$$\begin{bmatrix} 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \\ 2 & 7 & -10 & -19 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & -4 & -8 & 6 & 0 \\ 1 & 0 & 2 & 1 & 3 & 0 \\ 2 & 7 & -10 & -19 & 13 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 3 & -4 & -8 & 6 & 0 \\ 0 & -3 & 6 & 9 & -3 & 0 \\ 2 & 7 & -10 & -19 & 13 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 3 & -4 & -8 & 6 & 0 \\ 0 & -3 & 6 & 9 & -3 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 \end{bmatrix}$$

$R_3 \to 3R_3 + R_2$

$$\begin{bmatrix} 1 & 3 & -4 & -8 & 6 & 0 \\ 0 & -3 & 6 & 9 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence leading variables are x_1, x_3 and free variables are $x_3 = t, x_4 = s, x_5 = r$. The system becomes

$$\begin{bmatrix} 1 & 3 & -4 & -8 & 6 \\ 0 & -3 & 6 & 9 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ t \\ s \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Second equation gives

$$
\begin{aligned}
-3x_2 + 6t + 9s - 3r &= 0 \\
x_2 &= \frac{-6t - 9s + 3r}{-3} \\
&= 3s - r + 2t
\end{aligned}
$$

First equation gives

$$
\begin{aligned}
x_1 + 3x_2 - 4t - 8s + 6r &= 0 \\
x_1 &= -3x_2 + 4t + 8s - 6r \\
&= -3(3s - r + 2t) + 4t + 8s - 6r \\
&= -3r - s - 2t
\end{aligned}
$$

Hence solution is

$$
\begin{aligned}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} &= \begin{bmatrix} -3r - s - 2t \\ 3s - r + 2t \\ t \\ s \\ r \end{bmatrix} \\
&= t \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

Let $t = 1, s = 1, r = 1$, then the basis are

$$
\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}
$$

A three dimensional subspace.

40 Problem section 4.5 number 1

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$A=\begin{bmatrix}1 & 2 & 3\\ 1 & 5 & -9\\ 2 & 5 & 2\end{bmatrix}$$

solution

We start by converting the matrix to reduced Echelon form.

$R_2 \to R_2 - R_1$

$$\begin{bmatrix}1 & 2 & 3\\ 0 & 3 & -12\\ 2 & 5 & 2\end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix}1 & 2 & 3\\ 0 & 3 & -12\\ 0 & 1 & -4\end{bmatrix}$$

$R_3 \to 3R_3 - R_2$

$$\begin{bmatrix}1 & 2 & 3\\ 0 & 3 & -12\\ 0 & 0 & 0\end{bmatrix}$$

Now to start the reduce Echelon form phase. Notice that this is not needed. But if done, the row space basis found will be same each time. If we stop here, the row space basis can look different depending on the reduction was done. But both will work.

The pivots all needs to be 1.

$R_2 \to \frac{1}{3}R_2$

$$\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & -4\\ 0 & 0 & 0\end{bmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix}1 & 0 & 11\\ 0 & 1 & -4\\ 0 & 0 & 0\end{bmatrix}$$

The above is in reduced Echelon form. The pivot columns are $1,2$. The non-zero rows are rows $1,2,$. Hence row space basis are first and second rows (I prefer to show all basis as column vectors, instead of row vectors. This just makes it easier to read them).

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} \right\}$$

The dimension is 2. The column space correspond to pivot columns in original A. These are columns $1, 2$. Hence basis for column space are

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix} \right\}$$

The dimension is 2. We notice that the dimension of the row space and the column space is equal as expected. (This is called the rank of A. Hence $\text{rank}(A) = 2$.)

The Null space of A has dimension 1, since there is only one free variable (x_3). We see that the number of columns of A (which is 2) is therefore the sum of column space dimension (or the rank) and the null space dimension as expected.

41 Problem section 4.5 number 2

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$A = \begin{bmatrix} 5 & 2 & 4 \\ 2 & 1 & 1 \\ 4 & 1 & 5 \end{bmatrix}$$

<u>solution</u>

We start by converting the matrix to Echelon form.

$R_1 \to 2R_1, R_2 \to 5R_2$

$$\begin{bmatrix} 10 & 4 & 8 \\ 10 & 5 & 5 \\ 4 & 1 & 5 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 10 & 4 & 8 \\ 0 & 1 & -3 \\ 4 & 1 & 5 \end{bmatrix}$$

$R_1 \to 4R_1, R_3 \to 10R_3$

$$\begin{bmatrix} 40 & 16 & 32 \\ 0 & 1 & -3 \\ 40 & 10 & 50 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 40 & 16 & 32 \\ 0 & 1 & -3 \\ 0 & -6 & 18 \end{bmatrix}$$

$R_3 \to R_3 + 6R_2$

$$\begin{bmatrix} 40 & 16 & 32 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

The above is in Echelon form. The pivot columns are $1, 2$. The non-zero rows are rows $1, 2,$. Hence row space basis are first and second rows

$$\left\{ \begin{bmatrix} 40 \\ 16 \\ 32 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right\}$$

The dimension is 2. The column space correspond to pivot columns in original A. These are columns $1, 2$. Hence basis for column space are

$$\left\{ \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

42 Problem section 4.5 number 3

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$A=\begin{bmatrix}1 & -4 & -3 & -7\\ 2 & -1 & 1 & 7\\ 1 & 2 & 3 & 11\end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix}1 & -4 & -3 & -7\\ 0 & 7 & 7 & 21\\ 1 & 2 & 3 & 11\end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix}1 & -4 & -3 & -7\\ 0 & 7 & 7 & 21\\ 0 & 6 & 6 & 18\end{bmatrix}$$

$R_3 \to R_3 - \frac{6}{7}R_2$

$$\begin{bmatrix}1 & -4 & -3 & -7\\ 0 & 7 & 7 & 21\\ 0 & 0 & 0 & 0\end{bmatrix}$$

The pivot columns are $1,2$. The non-zero rows are rows $1,2,$. Hence row space basis are first and second rows

$$\left\{\begin{bmatrix}1\\-4\\-3\\-7\end{bmatrix},\begin{bmatrix}0\\7\\7\\21\end{bmatrix}\right\}$$

The dimension is 2. The column space correspond to pivot columns in original A. These are columns $1,2$. Hence basis for column space are

$$\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}-4\\-1\\2\end{bmatrix}\right\}$$

43 Problem section 4.5 number 4

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$A=\begin{bmatrix}1 & -3 & -9 & -5\\ 2 & 1 & 4 & 11\\ 1 & 3 & 3 & 13\end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_2 \rightarrow R_2 - 2R_1$

$$\begin{bmatrix}1 & -3 & -9 & -5\\ 0 & 7 & 22 & 21\\ 1 & 3 & 3 & 13\end{bmatrix}$$

$R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix}1 & -3 & -9 & -5\\ 0 & 7 & 22 & 21\\ 0 & 6 & 12 & 18\end{bmatrix}$$

$R_3 \rightarrow R_3 - \frac{6}{7}R_2$

$$\begin{bmatrix}1 & -3 & -9 & -5\\ 0 & 7 & 22 & 21\\ 0 & 0 & -\frac{48}{7} & 0\end{bmatrix}$$

The pivot columns are $1,2,3$. The non-zero rows are rows $1,2,3$. Hence row space basis are

$$\left\{\begin{bmatrix}1\\ -3\\ -9\\ -5\end{bmatrix},\begin{bmatrix}0\\ 7\\ 22\\ 21\end{bmatrix},\begin{bmatrix}0\\ 0\\ -\frac{48}{7}\\ 0\end{bmatrix}\right\}$$

The dimension is 3. The column space correspond to pivot columns in original A. These are columns $1,2,3$. Hence basis for column space are

$$\left\{\begin{bmatrix}1\\ 2\\ 1\end{bmatrix},\begin{bmatrix}-3\\ 1\\ 3\end{bmatrix},\begin{bmatrix}-9\\ 4\\ 3\end{bmatrix}\right\}$$

44 Problem section 4.5 number 5

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -3 & 4 \\ 2 & 5 & 11 & 12 \end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_2 \to -3R_1 + R_2$ gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -6 & 3 \\ 2 & 5 & 11 & 12 \end{bmatrix}$$

$R_3 \to -2R_1 + R_3$ gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -6 & 3 \\ 0 & 3 & 9 & 10 \end{bmatrix}$$

$R_2 \to 3R_2$ and $R_3 \to 2R_3$ gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -6 & -18 & 9 \\ 0 & 6 & 18 & 20 \end{bmatrix}$$

$R_3 \to R_2 + R_3$ gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -6 & -18 & 9 \\ 0 & 0 & 0 & 29 \end{bmatrix}$$

The above is now in Echelon form. Now we can answer the question. The basis for the row space are all the rows which are not zero. Hence row space basis are (I prefer to show all basis as column vectors, instead of row vectors. This just makes it easier to read them).

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -6 \\ -18 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 29 \end{bmatrix}$$

The dimension is 3. The column space correspond to pivot columns in original A. These are column $1, 2, 4$. Hence basis for column space are

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 12 \end{bmatrix}$$

The dimension is 3. We notice that the dimension of the row space and the column space is equal as expected. (This is called the rank of A. Hence $\text{rank}(A) = 3$.)

The Null space of A has dimension 1, since there is only one free variable (x_3). We see that the number of columns of A (which is 4) is therefore the sum of column space dimension (or the rank) and the null space dimension as expected.

45 Problem section 4.5 number 6

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix} 1 & 4 & 9 & 2 \\ 2 & 2 & 6 & -3 \\ 2 & 7 & 16 & 3 \end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 4 & 9 & 2 \\ 0 & -6 & -12 & -7 \\ 2 & 7 & 16 & 3 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 4 & 9 & 2 \\ 0 & -6 & -12 & -7 \\ 0 & -1 & -2 & -1 \end{bmatrix}$$

$R_3 \to R_3 - \frac{1}{6}R_2$

$$\begin{bmatrix} 1 & 4 & 9 & 2 \\ 0 & -6 & -12 & -7 \\ 0 & 0 & 0 & \frac{1}{6} \end{bmatrix}$$

The above is in Echelon form. The pivot columns are $1,2,4$. The non-zero rows are rows $1,2,3$. Hence row space basis are

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -6 \\ -12 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{6} \end{bmatrix} \right\}$$

The dimension is 3. The column space correspond to pivot columns in original A. These are columns $1,2,4$. Hence basis for column space are

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} \right\}$$

46 Problem section 4.5 number 7

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 1 & 4 & 5 & 16 \\ 1 & 3 & 3 & 13 \\ 2 & 5 & 4 & 23 \end{bmatrix}$$

<u>solution</u>

We start by converting the matrix to reduced Echelon form.

$R_2 \to -R_1 + R_2$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 3 & 6 & 9 \\ 1 & 3 & 3 & 13 \\ 2 & 5 & 4 & 23 \end{bmatrix}$$

$R_3 \to -R_1 + R_3$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 3 & 6 & 9 \\ 0 & 2 & 4 & 6 \\ 2 & 5 & 4 & 23 \end{bmatrix}$$

$R_4 \to -2R_1 + R_4$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 3 & 6 & 9 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix}$$

$R_2 \to 2R_2$ and $R_3 \to 3R_3$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 6 & 12 & 18 \\ 0 & 6 & 12 & 18 \\ 0 & 3 & 6 & 9 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 6 & 12 & 18 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 9 \end{bmatrix}$$

$R_4 \to -\frac{1}{2}R_2 + R_4$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 6 & 12 & 18 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot (leading) columns are $1, 2$ and free variables go with $3, 4$ columns. The Null space of A is therefore have dimension 2. The above is reduced Echelon form. The basis for the row space are all the rows which are not zero. Hence row space basis are (dimension 2)

$$\left\{ \left(\begin{bmatrix} 1 \\ 1 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 12 \\ 18 \end{bmatrix} \right) \right\}$$

The column space correspond to pivot columns in original A. These are columns $1, 2$. Hence basis for column space are (dimension 2)

$$\left\{ \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \\ 5 \end{bmatrix} \right) \right\}$$

We notice that the dimension of the row space and the column space is equal as expected.

The Null space of A has dimension 2, since there is two free variables. We see that the number of columns of A (which is 4) is therefore the sum of column space dimension and the null space dimension as expected.

47 Problem section 4.5 number 8

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix} 1 & -2 & -3 & -5 \\ 1 & 4 & 9 & 2 \\ 1 & 3 & 7 & 1 \\ 2 & 2 & 6 & -3 \end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_2 \rightarrow R_2 - R_1$

$$\begin{bmatrix} 1 & -2 & -3 & -5 \\ 0 & 6 & 12 & 7 \\ 1 & 3 & 7 & 1 \\ 2 & 2 & 6 & -3 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & -2 & -3 & -5 \\ 0 & 6 & 12 & 7 \\ 0 & 5 & 10 & 6 \\ 2 & 2 & 6 & -3 \end{bmatrix}$$

$R_4 \rightarrow R_4 - 2R_1$

$$\begin{bmatrix} 1 & -2 & -3 & -5 \\ 0 & 6 & 12 & 7 \\ 0 & 5 & 10 & 6 \\ 0 & 6 & 12 & 7 \end{bmatrix}$$

$R_2 \rightarrow 5R_2, R_3 \rightarrow 6R_3$

$$\begin{bmatrix} 1 & -2 & -3 & -5 \\ 0 & 30 & 60 & 35 \\ 0 & 30 & 60 & 36 \\ 0 & 6 & 12 & 7 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & -2 & -3 & -5 \\ 0 & 30 & 60 & 35 \\ 0 & 0 & 0 & 1 \\ 0 & 6 & 12 & 7 \end{bmatrix}$$

$R_4 \rightarrow R_4 - \frac{6}{30}R_2$

$$\begin{bmatrix} 1 & -2 & -3 & -5 \\ 0 & 30 & 60 & 35 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is in Echelon form. The pivot columns are $1,2,4$. The non-zero rows are rows $1,2,3$. Hence row space basis are

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 30 \\ 60 \\ 35 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dimension is 3. The column space correspond to pivot columns in original A. These are columns $1,2,4$. Hence basis for column space are

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 1 \\ -3 \end{bmatrix} \right\}$$

48 Problem section 4.5 number 9

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix}1&3&3&9\\2&7&4&8\\2&7&5&12\\2&8&3&12\end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix}1&3&3&9\\0&1&-2&-10\\2&7&5&12\\2&8&3&12\end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix}1&3&3&9\\0&1&-2&-10\\0&1&-1&-6\\2&8&3&12\end{bmatrix}$$

$R_4 \to R_4 - 2R_1$

$$\begin{bmatrix}1&3&3&9\\0&1&-2&-10\\0&1&-1&-6\\0&2&-3&-6\end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix}1&3&3&9\\0&1&-2&-10\\0&0&1&4\\0&2&-3&-6\end{bmatrix}$$

$R_4 \to R_4 - 2R_2$

$$\begin{bmatrix}1&3&3&9\\0&1&-2&-10\\0&0&1&4\\0&0&1&14\end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 3 & 3 & 9 \\ 0 & 1 & -2 & -10 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

The above is in Echelon form. The pivot columns are $1, 2, 3, 4$. The non-zero rows are rows $1, 2, 3, 4$. Hence row space basis are

$$\left\{ \begin{pmatrix} \begin{bmatrix} 1 \\ 3 \\ 3 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -20 \\ -10 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix} \end{pmatrix} \right\}$$

The dimension is 4. The column space correspond to pivot columns in original A. These are columns $1, 2, 3, 4$. Hence basis for column space are

$$\left\{ \begin{pmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ 8 \\ 12 \\ 12 \end{bmatrix} \end{pmatrix} \right\}$$

49 Problem section 4.5 number 10

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 1 & 3 & 4 & 3 & 6 \\ 2 & 2 & 4 & 3 & 5 \\ 2 & 1 & 3 & 2 & 3 \end{bmatrix}$$

<u>solution</u>

We start by converting the matrix to Echelon form.

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 1 & 2 & 3 \\ 2 & 2 & 4 & 3 & 5 \\ 2 & 1 & 3 & 2 & 3 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & -2 & -2 & 1 & -1 \\ 2 & 1 & 3 & 2 & 3 \end{bmatrix}$$

$R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & -2 & -2 & 1 & -1 \\ 0 & -3 & -3 & 0 & -3 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & -3 & -3 & 0 & -3 \end{bmatrix}$$

$R_4 \to R_4 + 3R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 6 & 6 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is in Echelon form. The pivot columns are $1, 2, 4$. The non-zero rows are rows $1, 2, 3$. Hence row space basis are

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 5 \end{bmatrix} \right\}$$

The dimension is 3. The column space correspond to pivot columns in original A. These are columns $1, 2, 4$. Hence basis for column space are

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ 2 \end{bmatrix} \right\}$$

50 Problem section 4.5 number 11

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 2 & 3 & 7 & 8 & 2 \\ 2 & 3 & 7 & 8 & 3 \\ 3 & 1 & 7 & 5 & 4 \end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 2 & 3 & 7 & 8 & 3 \\ 3 & 1 & 7 & 5 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 3 & 1 & 7 & 5 & 4 \end{bmatrix}$$

$R_4 \to R_4 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & -2 & -2 & -4 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & -2 & -4 & 1 \end{bmatrix}$$

$R_4 \to R_4 + 2R_2$

$$\begin{bmatrix}1&1&3&3&1\\0&1&1&2&0\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}$$

$R_4 \rightarrow R_4 - R_3$

$$\begin{bmatrix}1&1&3&3&1\\0&1&1&2&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}$$

The above is in Echelon form. The pivot columns are $1,2,5$. The non-zero rows are rows $1,2,3$. Hence row space basis are

$$\left\{\begin{bmatrix}1\\1\\3\\3\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\\2\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$$

The dimension is 3. The column space correspond to pivot columns in original A. These are columns $1,2,5$. Hence basis for column space are

$$\left\{\begin{bmatrix}1\\2\\2\\3\end{bmatrix},\begin{bmatrix}1\\3\\3\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\right\}$$

51 Problem section 4.5 number 12

In Problems 1–12, find both a basis for the row space and a basis for the column space of the given matrix A.

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 0 \\ -1 & 0 & -2 & -1 & 1 \\ 2 & 3 & 7 & 8 & 1 \\ -2 & 4 & 0 & 6 & 7 \end{bmatrix}$$

solution

We start by converting the matrix to Echelon form.

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 2 & 3 & 7 & 8 & 1 \\ -2 & 4 & 0 & 6 & 7 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ -2 & 4 & 0 & 6 & 7 \end{bmatrix}$$

$R_4 \to R_4 + 2R_1$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 6 & 6 & 12 & 7 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 6 & 12 & 7 \end{bmatrix}$$

$R_4 \to R_4 - 6R_2$

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Swap R_4, R_3

$$\begin{bmatrix} 1 & 1 & 3 & 3 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is in Echelon form. The pivot columns are $1, 2, 5$. The non-zero rows are rows $1, 2, 3$. Hence row space basis are

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dimension is 3. The column space correspond to pivot columns in original A. These are columns $1, 2, 5$. Hence basis for column space are

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 7 \end{bmatrix} \right\}$$

52 Problem section 4.5 number 13

In Problems 13–16, a set S of vectors in $\mathbb{R}^4$ is given. Find a subset of S that forms a basis for the subspace of $\mathbb{R}^4$ spanned by S

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 5 \\ 1 \\ 4 \\ 8 \end{bmatrix}$$

solution

We set up a matrix made of the above vectors, then find the column space.

$$\begin{bmatrix} 1 & 2 & 5 \\ 3 & -1 & 1 \\ -2 & 3 & 4 \\ 4 & 2 & 8 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -7 & -14 \\ -2 & 3 & 4 \\ 4 & 2 & 8 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 2R_1$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -7 & -14 \\ 0 & 7 & 14 \\ 4 & 2 & 8 \end{bmatrix}$$

$R_4 \rightarrow R_4 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -7 & -14 \\ 0 & 7 & 14 \\ 0 & -6 & -12 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -7 & -14 \\ 0 & 0 & 0 \\ 0 & -6 & -12 \end{bmatrix}$$

$R_4 \rightarrow R_4 - \frac{6}{7}R_2$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -7 & -14 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Pivot vectors are $1, 2$. Hence the column space basis are $\vec{v}_1, \vec{v}_2$

$$\left\{ \left(\begin{bmatrix} 1 \\ 3 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix} \right) \right\}$$

These are the basis that span the set S.

53 Problem section 4.5 number 14

In Problems 13–16, a set S of vectors in $\mathbb{R}^4$ is given. Find a subset of S that forms a basis for the subspace of $\mathbb{R}^4$ spanned by S

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 4 \\ 1 \\ 8 \\ 7 \end{bmatrix}$$

solution

We set up a matrix made of the above vectors, then find the column space.

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ -1 & 3 & 1 & 1 \\ 2 & 4 & 2 & 8 \\ 3 & 1 & 1 & 7 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 5 & 2 & 5 \\ 2 & 4 & 2 & 8 \\ 3 & 1 & 1 & 7 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 3 & 1 & 1 & 7 \end{bmatrix}$$

$R_4 \to R_4 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & -2 & -5 \end{bmatrix}$$

$R_4 \to R_4 + R_2$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot vectors are $1, 2$. Hence the column space basis are $\vec{v}_1, \vec{v}_2$

$$\left\{ \left(\begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} \right) \right\}$$

These are the basis that span the set S.

54 Problem section 4.5 number 15

In Problems 13–16, a set S of vectors in $\mathbb{R}^4$ is given. Find a subset of S that forms a basis for the subspace of $\mathbb{R}^4$ spanned by S

$$\vec{v}_1 = \begin{bmatrix}3\\2\\2\\2\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}2\\1\\2\\1\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}4\\3\\2\\3\end{bmatrix}, \vec{v}_4 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

solution

We set up a matrix made of the above vectors, then find the dimensions of the column space.

$$\begin{bmatrix}3 & 2 & 4 & 1\\2 & 1 & 3 & 2\\2 & 2 & 2 & 3\\2 & 1 & 3 & 4\end{bmatrix}$$

$R_1 \rightarrow 2R_1$ and $R_2 \rightarrow 3R_2$ and $R_3 \rightarrow 2R_3$ and $R_4 \rightarrow 3R_4$. This gives

$$\begin{bmatrix}6 & 4 & 8 & 2\\6 & 3 & 9 & 6\\6 & 6 & 6 & 9\\6 & 3 & 9 & 12\end{bmatrix}$$

$R_2 \rightarrow -R_1 + R_2$

$$\begin{bmatrix}6 & 4 & 8 & 2\\0 & -1 & 1 & 4\\6 & 6 & 6 & 9\\6 & 3 & 9 & 12\end{bmatrix}$$

$R_3 \rightarrow -R_1 + R_3$

$$\begin{bmatrix}6 & 4 & 8 & 2\\0 & -1 & 1 & 4\\0 & 2 & -2 & 7\\6 & 3 & 9 & 12\end{bmatrix}$$

$R_4 \rightarrow -R_1 + R_4$

$$\begin{bmatrix}6 & 4 & 8 & 2\\0 & -1 & 1 & 4\\0 & 2 & -2 & 7\\0 & -1 & 1 & 10\end{bmatrix}$$

$R_3 \to 2R_2 + R_3$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 15 \\ 0 & -1 & 1 & 10 \end{bmatrix}$$

$R_4 \to -R_2 + R_4$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$R_4 \to 15R_4$ and $R_3 \to 6R_3$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 90 \\ 0 & 0 & 0 & 90 \end{bmatrix}$$

$R_4 \to R_3 + R_4$

$$\begin{bmatrix} 6 & 4 & 8 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 90 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the pivot columns are $1, 2, 4$. Therefore the column space basis are $\vec{v}_1, \vec{v}_2, \vec{v}_4$ given by

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

The above are the basis that span S.

55 Problem section 4.5 number 16

In Problems 13–16, a set S of vectors in $\mathbb{R}^4$ is given. Find a subset of S that forms a basis for the subspace of $\mathbb{R}^4$ spanned by S

$$\vec{v}_1 = \begin{bmatrix}5\\4\\2\\2\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}3\\1\\2\\3\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}7\\7\\2\\1\end{bmatrix}, \vec{v}_4 = \begin{bmatrix}1\\-1\\2\\4\end{bmatrix}, \vec{v}_5 = \begin{bmatrix}5\\4\\6\\7\end{bmatrix}$$

solution

We set up a matrix made of the above vectors, then find the column space.

$$\begin{bmatrix}5 & 3 & 7 & 1 & 5\\4 & 1 & 7 & -1 & 4\\2 & 2 & 2 & 2 & 6\\2 & 3 & 1 & 4 & 7\end{bmatrix}$$

$R_1 \to 4R_1, R_2 \to 5R_2$

$$\begin{bmatrix}20 & 12 & 28 & 4 & 20\\20 & 5 & 35 & -5 & 20\\2 & 2 & 2 & 2 & 6\\2 & 3 & 1 & 4 & 7\end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix}20 & 12 & 28 & 4 & 20\\0 & -7 & 7 & -9 & 0\\2 & 2 & 2 & 2 & 6\\2 & 3 & 1 & 4 & 7\end{bmatrix}$$

$R_3 \to 10R_3 - R_1$

$$\begin{bmatrix}20 & 12 & 28 & 4 & 20\\0 & -7 & 7 & -9 & 0\\0 & 8 & -8 & 16 & 40\\2 & 3 & 1 & 4 & 7\end{bmatrix}$$

$R_4 \to 10R_4 - R_1$

$$\begin{bmatrix}20 & 12 & 28 & 4 & 20\\0 & -7 & 7 & -9 & 0\\0 & 8 & -8 & 16 & 40\\0 & 18 & -18 & 36 & 50\end{bmatrix}$$

$R_3 \to 7R_3, R_2 \to 8R_2$

$$\begin{bmatrix} 20 & 12 & 28 & 4 & 20 \\ 0 & -56 & 56 & -72 & 0 \\ 0 & 56 & -56 & 112 & 280 \\ 0 & 18 & -18 & 36 & 50 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 20 & 12 & 28 & 4 & 20 \\ 0 & -56 & 56 & -72 & 0 \\ 0 & 0 & 0 & 40 & 280 \\ 0 & 18 & -18 & 36 & 50 \end{bmatrix}$$

$R_4 \to 56(R_4), R_2 \to 18(R_2)$

$$\begin{bmatrix} 20 & 12 & 28 & 4 & 20 \\ 0 & -1008 & 1008 & -1296 & 0 \\ 0 & 0 & 0 & 40 & 280 \\ 0 & 1008 & -1008 & 2016 & 2800 \end{bmatrix}$$

$R_4 \to R_4 + R_2$

$$\begin{bmatrix} 20 & 12 & 28 & 4 & 20 \\ 0 & -1008 & 1008 & -1296 & 0 \\ 0 & 0 & 0 & 40 & 280 \\ 0 & 0 & 0 & 720 & 2800 \end{bmatrix}$$

$R_4 \to R_4 - 18R_3$

$$\begin{bmatrix} 20 & 12 & 28 & 4 & 20 \\ 0 & -1008 & 1008 & -1296 & 0 \\ 0 & 0 & 0 & 40 & 280 \\ 0 & 0 & 0 & 0 & -2240 \end{bmatrix}$$

Hence, the pivot columns are $1, 2, 4, 5$. Therefore the column space basis are $\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_5$ given by

$$\left\{ \begin{bmatrix} 5 \\ 4 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 6 \\ 7 \end{bmatrix} \right\}$$

The above are the basis that span S.

56 Problem section 4.7 number 5

In Problems 5–8, determine whether or not each indicated set of functions is a subspace of the space F of all real-valued

functions on $\mathbb{R}$.

The set of all f such that $f(0) = 0$

Solution

The only condition given is that $f(0) = 0$. This means the zero function is included. $cf(0) = c(0) = 0$ and $f(0) + g(0) = 0 + 0 = 0$. Hence closed under addition and under scalar multiplication. Hence subspace.

57 Problem section 4.7 number 6

In Problems 5–8, determine whether or not each indicated set of functions is a subspace of the space F of all real-valued

functions on $\mathbb{R}$.

The set of all f such that $f(x) \neq 0$ for all x.

Solution

Since the zero function is not included, then this can not be a subspace.

58 Problem section 4.7 number 7

In Problems 5–8, determine whether or not each indicated set of functions is a subspace of the space F of all real-valued

functions on $\mathbb{R}$.

The set of all f such that $f(0) = 0$ and $f(1) = 1$

Solution

$$5f(1) = 5 \times 1 = 5$$

Hence not closed under scalar multiplication. Therefore not a subspace.

59 Problem section 4.7 number 8

In Problems 5–8, determine whether or not each indicated set of functions is a subspace of the space F of all real-valued

functions on $\mathbb{R}$.

The set of all f such that $f(-x) = -f(x)$ for all x

Solution

This is the definition of an odd function such as $\sin x$. The odd function is zero at $x = 0$, since the zero is included. Also adding two odd functions gives an odd function, and scaling an odd function does not change its oddness. Hence closed. Therefore a subspace.

60 Problem section 4.7 number 9

In Problems 9–12, a condition on the coefficients of a polynomial $a_0 + a_1x + a_2x^2 + a_3x^3$ is given. Determine whether or not the set of all such polynomials satisfying this condition is a subspace of the space P of all polynomials.

$$a_3 \neq 0$$

Solution

Not a subspace, since we can not obtain the zero polynomial if $a_3 \neq 0$ all the time. Hence not a subspace

61 Problem section 4.7 number 9

In Problems 9–12, a condition on the coefficients of a polynomial $a_0 + a_1x + a_2x^2 + a_3x^3$ is given. Determine whether or not the set of all such polynomials satisfying this condition is a subspace of the space P of all polynomials.

$$a_3 \neq 0$$

Solution

Let $p_1 = 3x^3$ and let $p_2 = -3x^3$, hence $p_1 + p_2 = 3x^3 - 3x^3 = 0$ which does not satisfy the condition that $a_3 \neq 0$. Hence not closed under addition. not a subspace

62 Problem section 4.7 number 10

In Problems 9–12, a condition on the coefficients of a polynomial $a_0 + a_1x + a_2x^2 + a_3x^3$ is given. Determine whether or not the set of all such polynomials satisfying this condition is a subspace of the space P of all polynomials.

$$a_0 = a_1 = 0$$

Solution

These all polynomials that look like $3x^2 + 5x^3, -x^2 + x^3$ and so on. Let $p_1 = a_2x^2 + a_3x^3$ and let $p_2 = b_2x^2 + b_3x^3$.

$$\begin{aligned} p_1 + p_2 &= a_2x^2 + a_3x^3 + b_2x^2 + b_3x^3 \\ &= x^2(a_2 + a_3) + x^3(a_3 + b_3) \end{aligned}$$

Which satisfies the condition that $a_0 = a_1 = 0$. Also under scalar multiplication

$$\begin{aligned} Cp_1 &= C\left(a_2x^2 + a_3x^3\right) \\ &= Ca_2x^2 + Ca_3x^3 \end{aligned}$$

Which satisfies the condition that $a_0 = a_1 = 0$. Hence a subspace

63 Problem section 4.7 number 11

In Problems 9–12, a condition on the coefficients of a polynomial $a_0 + a_1x + a_2x^2 + a_3x^3$ is given. Determine whether or not the set of all such polynomials satisfying this condition is a subspace of the space P of all polynomials.

$$a_0 + a_1 + a_2 + a_3 = 0$$

Solution

Let $p_1 = a_0 + a_1x + a_2x^2 + a_3x^3$ such that $a_0 + a_1 + a_2 + a_3 = 0$ and let $p_2 = b_0 + b_1x + b_2x^2 + b_3x^3$ such that $b_0 + b_1 + b_2 + b_3 = 0$ then

$$\begin{aligned} p_1 + p_2 &= a_0 + a_1x + a_2x^2 + a_3x^3 + b_0 + b_1x + b_2x^2 + b_3x^3 \\ &= (a_0 + b_0)x + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 \end{aligned}$$

Now,

$$\begin{aligned} (a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) &= (a_0 + a_1 + a_2 + a_3) + (b_0 + b_1 + b_2 + b_3) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Hence closed under addition. Also

$$\begin{aligned} cp_1(x) &= c\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) \\ &= ca_0 + ca_1x + ca_2x^2 + ca_3x^3 \end{aligned}$$

Now

$$\begin{aligned} ca_0 + ca_1 + ca_2 + ca_3 &= c(a_0 + a_1 + a_2 + a_3) \\ &= c(0) \\ &= 0 \end{aligned}$$

Hence closed under scalar multiplication. And since the zero polynomial is also included (when $a_i = 0$), then this is a subspace

64 Problem section 4.7 number 12

In Problems 9–12, a condition on the coefficients of a polynomial $a_0 + a_1x + a_2x^2 + a_3x^3$ is given. Determine whether or not the set of all such polynomials satisfying this condition is a subspace of the space P of all polynomials.

$$a_0, a_1, a_2, a_3 \text{ are all integers.}$$

Solution

Not closed under scalar multiplication. For example

$$\frac{1}{2}\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) = \frac{1}{2}a_0 + \frac{1}{2}a_1x + \frac{1}{2}a_2x^2 + \frac{1}{2}a_3x^3$$

But $\frac{1}{2}a_0$ is not integer when a_0 is integer. Therefore not a subspace