Study notes

Math 5587
Elementary Partial Differential Equations I

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Contents

1 Linear and Nonlinear Waves (Chapter 2)

stationary waves such as $u_t + 3u = 0$

Transport and Traveling Waves such as $u_t + cu_x = u$. Uniform transport. Speed c is constant. Characteristics are $\frac{dx}{dt} = c$. When speed is not constant, we get Nonuniform Transport. Characteristics is $\frac{dx}{dt} = c(x)$. Nonlinear Transport: $u_t + u\, u_x = 0$ where wave speed depends not on position x but on u itself.

d'Almbert

$$u(x,t) = \frac{1}{2}\Big(f(x-ct) + f(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds$$

With extranl force $u_{tt} = c^2 u_{xx} + F(x,t)$ we add the term $\frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} F(s,y)\, dy\right) ds$. The limits are the same as above, but replace t by $t-s$. remeber ds goes with t and dy goes with x.

2 Fourier series (Chapter 3)

Just need to know the F.S. definition. Either complex one or standard.

3 Seperation of variables (Chapter 4)

Theorem 4.2. If $u(t,x)$ is a solution to the heat equation with piecewise continuous initial data $f(x) = u(t0,x)$, or, more generally, initial data satisfying (4.27), then, for any $t > t_0$, the solution $u(t,x)$ is an infinitely differentiable function of x. (page 128) .

"In other words, the heat equation instantaneously smoothes out any discontinuities and corners in the initial temperature profile by fast damping of the high-frequency modes"

Heat PDE in 1D.

Inhomogeneous Boundary Conditions convert to homogeneous by using reference function.

Wave PDE in 1D. Fixed ends. d'Alembert Formula for Bounded Intervals: For Dirichlet do odd extension of initial position. For Neumann (free) boundary conditions, do even extension.

Laplace PDE on disk and on recrangle. in polar Laplace becomes $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$. When doing seperations, rememebr to use the angular ODE for finding the eigenvalues first. The radial ODE becomes Euler ODE. Solve using assuming $R(r) = r^k$. For disk, the solution is $u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n \left(A_n \cos(n\theta + B_n \sin n\theta)\right)$

Laplace PDE maximum principle. Lots of theorem here.

Characteristics and the Cauchy Problem see HW 7, Problem 4.4.16. This is for second order pde. Write pde as $Au_{xx} + Bu_{xy} + Cu_{yy} = G$ and then Characteristics is $A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0$. This gives ode $\frac{dy}{dx}$ which is the Characteristics.

Laplacian in 3D with no angle dependencty is $u_{rr} + \frac{2}{r}u_r = 0$

4 Generalized functions and Green function (Chapter 6)

$\delta(x-\xi)$: "In general, a unit impulse at position $a < \xi < b$ will be described by something called the delta function".

Two ways to define $\delta(x-\xi)$. one based on limit of function as $n \to \infty$ and one based on how it acts inside integral. For limit, use this one

$$g_n(x) = \frac{n}{\pi\left(1+n^2x^2\right)}$$

Then $\lim_{n\to\infty} g_n(x) = \delta_0(x)$. And the above also meets the integral relation $\int_{-\infty}^{\infty} g_n(x)\,dx = \frac{1}{\pi}\left[\arctan(nx)\right]_{-\infty}^{\infty} = 1$.

For calculus, remember this: When taking derivative of a function with jump discontitty, we get an impulse at location of the jump with magnitude of the jump. Direction is negative if the jump is down and positive if the jump is up, this is when moving from left to right. For example derivative of unit step is $\delta(x)$. And the integral of $\delta(x)$ is unit step (or 1). Hence if $f(x) = g(x) + \sigma(x)$ where $\sigma(x)$ is unit step and $g(x)$ is continuous everywhere, then $f'(x) = g'(x) + \delta(x)$

Fourier series of $\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi}(\cos x + \cos 2x + \cos 3x + \cdots)$

Green function for 1D boundary value problems.

Remember when satisfying the jump discontinuity, it is $A + \frac{1}{p} = B$ where p is one which matches when the ODE is written as $py'' + q(x)y' + ry = f(x)$ in the original ODE. And A is the top term and B is the bottom term, as is

$$\left[\frac{d}{dx}G(x;\xi)\right]_{x=\xi} = \begin{cases} A & x < \xi \\ B & x > \xi \end{cases}$$

So the second equation is

$$A + \frac{1}{p} = B$$

That is really the only tricky part in finding Green function. Getting the sign right here. So if the ODE is $-cy'' = f(x)$ then here $p = -c$ (notice, sign is negative, i.e. $p = -c$ including the sign) and the jump is $\frac{1}{p} = \frac{1}{-c} = -\frac{1}{c}$ and hence the equation becomes

$$A + \frac{1}{p} = B$$
$$A - \frac{1}{c} = B$$

And if the ODE is given as $cy'' = f(x)$ then $p = c$ and the equation becomes

$$A + \frac{1}{p} = B$$
$$A + \frac{1}{c} = B$$

"Thus, the Neumann boundary value problem admits a solution if and only if there is no net force on the bar." (page 239). This means $-u'' = f(x)$ with $u'(0) = 0 = u'(1)$ has Green function and solution if $\int_0^1 f(x)\,dx = 0$. If this holds, the $-u'' = f(x)$ has solution (but the solution is not unique) and any constant value is a solution.

Green function for Laplace $-\Delta u = f(x,y)$

Some relations: $\nabla \cdot \nabla u = \Delta u = u_{xx} + u_{yy}$. i.e. divergence of the gradient of u is Laplacian of u. Green function in full space for Laplacian in 2D is

$$G(x,y;\xi,\eta) = \frac{-1}{2\pi} \ln r$$

where r is distance from (x,y) to where the pulse is (ξ,η), i.e. $\sqrt{(x-\xi)^2 + (y-\eta)^2}$. In 3D, it is $\frac{1}{4\pi r}$.

Method of images To find $G(x,y;\xi,\eta)$ in say upper half, put a negative pulse at $(\xi,-\eta)$ and then use $G_{upper}(x,y;\xi,\eta) = G_{full}(x,y;\xi,\eta) - G_{full}(x,y;\xi,-\eta)$

For disk

$$G(\boldsymbol{x};\boldsymbol{\xi}) = \frac{1}{2\pi} \ln\left(\frac{\left\| \|\boldsymbol{\xi}\|^2 \boldsymbol{x} - \boldsymbol{\xi} \right\|}{\|\boldsymbol{\xi}\| \, \|\boldsymbol{x} - \boldsymbol{\xi}\|}\right)$$

In polar it becomes

$$G(r,\theta;\rho,\phi) = \frac{1}{4\pi} \ln\left(\frac{1 + r^2\rho^2 - \beta}{r^2 + \rho^2 - \beta}\right)$$

Where $\beta = 2r\rho\cos(\theta - \phi)$ where (r,θ) is variable point and pulse fixed at (ρ,ϕ), all using polar coordinates.

5 Fourier transform (chapter 7)

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx} dx$$
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx} dk$$

Table of Fourier transforms on page 272 will be given in exam also. Remember the shift property

$$\hat{f}(k-a) \Leftrightarrow e^{iax} f(x)$$
$$f(x-a) \Leftrightarrow e^{-iax} \hat{f}(k)$$

Gaussian integrals, for any b is

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$
$$\int_{-\infty}^{\infty} e^{-(x+b)^2} dx = \sqrt{\pi}$$
$$\int_{-\infty}^{\infty} e^{-(x-b)^2} dx = \sqrt{\pi}$$
$$\int_{-\infty}^{\infty} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}} \qquad a > 0$$
$$\int_{-\infty}^{\infty} e^{-a(x-b)^2} dx = \sqrt{\frac{\pi}{a}} \qquad a > 0$$

Derivative and integrals

$$f(x) \Leftrightarrow \hat{f}(k)$$
$$f'(x) \Leftrightarrow (ik)\, \hat{f}(k)$$
$$f''(x) \Leftrightarrow (ik)^2\, \hat{f}(k) = -k^2 \hat{f}(k)$$

Remember this also $x f(x) \Leftrightarrow i\frac{d\hat{f}(k)}{dk}$. On smoothness of $f(x)$ and relation to decay of $\hat{f}(k)$. see book page 276 "the smoothness of the function $f(x)$ is manifested in the rate of decay of its Fourier transform $f(k)$." and "Thus, the smoother $f(x)$, the more rapid the decay of its Fourier transform" and "This result can be viewed as the Fourier transform version of the Riemann–Lebesgue Lemma 3.46.)"

Integration

$$\int_{-\infty}^{x} f(x)\, dx \Leftrightarrow \frac{1}{ik} \hat{f}(k) + \pi \hat{f}(0)\, \delta(k)$$

Easy to remember when comparing it to $f'(x) \Leftrightarrow (ik)\, \hat{f}(k)$. Just change (ik) from numerator to denominator and add $\pi \hat{f}(0)\, \delta(k)$.

In context of generalized functions, we write

$$\int_{-\infty}^{\infty} f(x)\, dx = \sqrt{2\pi} \hat{f}(0)$$

So if we know the F.T. of $f(x)$ we do the above integration by using the above relation directly by evaluating $\hat{f}(k)$ at $k=0$. This can be handy. For example let us apply this to the Gaussian,. $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{2\pi}\hat{f}(0)$ where $\hat{f}(k) = \mathscr{F}\left(e^{-x^2}\right) = \frac{1}{\sqrt{2}} e^{-\frac{k^2}{4}}$. Hence $\hat{f}(0) = \frac{1}{\sqrt{2}}$ and therefore $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{2\pi}\frac{1}{\sqrt{2}} = \sqrt{\pi}$

Green function

Using F.T, to find Green function. Used only for infinite space. Put a $\delta_y(x)$ in RHS, solve for $\hat{G}\left(y,t\right)$ then find the inverse Fourier transform to get $G(x,t)$. For example for heat pde.

Weyl's law for eigenvalues convergence for large n. For 2D

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{A}$$

Where here $\lambda_n = \frac{l^2\pi^2}{a^2} + \frac{k^2\pi^2}{b^2}, l = 1,2,3,\cdots, k = 1,2,3,\cdots$. So λ_n are sorted in order. This is for reactangle with width a and high b.