University Course

Math 5587
Elementary Partial Differential Equations I

University of Minnesota, Twin Cities

Fall 2019

My Class Notes

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Fall 2019

Contents

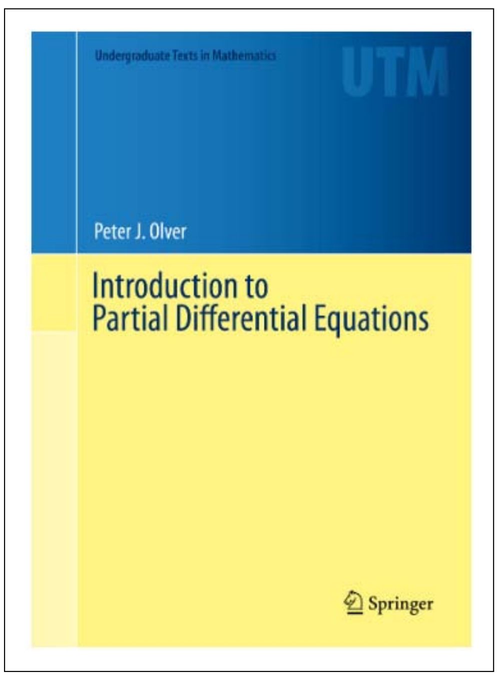
Chapter 1

Introduction

1.1 Links

1. class web page (scroll down the page) `http://www-users.math.umn.edu/~svitlana/PDE-F19-UMN.html`

1.2 Text book



1.3 syllabus

Math 5587, Elementary Partial Differential Equations, Fall 2019

- **Course Description:** Math 5587-8 is a year course that introduces the basics of partial differential equations, guided by applications in physics, engineering, biology, and finance. Both analytical and numerical solution techniques will be discussed.
- **Time and location:** TuTh 4:45PM - 6:00PM, @ Amundson Hall 156
- **Office hours:** Tu 01:40 P.M. - 02:30 P.M,, Th 2:30 P.M.- 3:20 P.M. and by appointment. @ Vincent Hall 510. Please try to let me know in advance if you are coming to office hours.
- **Prerequisites:** Strong background in linear algebra, multi-variable calculus, and ordinary differential equations. Some mathematical sophistication. Other topics will be introduced as needed. Basic familiarity with a programming language (Matlab preferrred) is required for numerical work.
- **Text:** The course will be based on the book by Peter Olver. I intend to cover chapters 1-4, 6, and parts of Chapter 7 in the fall semester.
- **Homework:** Homework will be assigned periodically throughout the semester and collected for grading. The assigned problems should be regarded as the **minimum** required for mastery of the material. No late homework will be accepted, but I will drop two lowest scores in the end of the semester.

  Homework 1 Solutions for Homework 1
  Homework 2 Solutions for Homework 2
  Homework 3 Solutions for Homework 3
  Homework 4 Solutions for Homework 4
  Homework 5 Solutions for Homework 5
  Homework 6 Solutions for Homework 6
  Homework 7 Solutions for Homework 7
  Homework 8 Solutions for Homework 8
  Homework 9 Solutions for Homework 9
  Homework 10 Solutions for Homework 10
  Homework 11 Solutions for Homework 11
  Homework 12 Solutions for Homework 12

- **Exams:** There will be three exams within the semester. Make-up exams will only be given in exceptional circumstances, and then only when notice is given to me prior to the exam and a suitable written excuse forthcoming.
  - **First Midterm: Thursday, October 8**
  - **Second Midterm: Thursday, November 7**
  - **Third Exam: Tuesday, December 10**
- **Final:** The third exam will serve as a Final.
- **Grading:**
  - Homework 30%
  - First and Second Midterm: 20% each
  - Third Midterm: 30% total

1.4 Review of lectures

Table 1.1: Class lectures review

#	date	book section	note
1	Sept 3, 2019	Chapter 1	Order of ODE, On Laplacian, why it shows up so frequently everywhere, review
2	Sept 5, 2019	Chapter 2	Transport PDE $u_t + cu_x = 0$, characteristic lines. Transport with decay $u_t + cu_x + au = 0$
3	Sept 10, 2019	Chapter 2	Continue with Transport PDE $u_t + cu_x = 0$, examples $u_t + (x^2 - 1)u_x = 0, u(0, x) = e^{-x^2}$
4	Sept 12, 2019	Chapter 2.4	Wave equation $u_{tt} = c^2 u_{xx}$, derivation of d'Alembert solution on infinite line. Example. Domain of influence. Also with external force. Resonance
5	Sept 17, 2019	Chapter 3	Starting Fourier series. Heat PDE $u_t = ku_{xx}$. Separation of variables. Periodic boundary conditions (ring). Obtain Fourier series solution. How to find coefficients, convergence, etc...
6	Sept 19, 2019	Chapter 3	More Fourier series. $f(x) \in L_2$, definition of norm of $f(x)$, basis functions. How to find Fourier coefficients. Example using $f(x) = x$. Definitions, jump discontinuity. Fourier series convergence theorem.
7	Sept 24, 2019	Chapter 3	even and odd functions. Complex Fourier series. Example.
8	Sept 26, 2019	Chapter 3	Integration of Fourier series. Find F.S. of $f(x)$ using integration of known F.S. for $g(x)$. Convergence of functions Uniform and piecewise. M test.
9	Oct 1, 2019	Chapter 3	More on convergence. Convergence in norm. Definitions and examples. More theories on Fourier series convergence. Bessel inequality. Proof (long). Riemann-Lebesgue Lemma
10	Oct 3, 2019	Chapter 3	Decay and smoothness of Fourier series. Proof of the Fourier series convergence theorem. Dirichlet kernel.
11	Oct 8, 2019	N/A	First exam

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#	date	book section	note
12	Oct 10, 2019	Chapter 4	Heat ODE $u_t = ku_{xx}$, going over instantaneous smoothness. Transport PDE we can go back and forward in time, but not with heat PDE. Heat PDE with non zero boundary conditions
13	Oct 15, 2019	Chapter 4	Root cellar problem. Solving heat PDE in complex domain example. Starting on wave equation. Fourier series solution
14	Oct 17, 2019	Chapter 4	Solving wave PDE on finite domain using d'Alembert. 2 cases. B.C. B.C. is Neumann and B.C. is Dirichlet (Even and Odd extension of initial position). Solving Laplace PDE $u_{xx} + u_{yy} = 0$ on rectangle.
15	Oct 22, 2019	Chapter 4.3	Laplace in disk. Polar coordinates. Separation of variables. Converting back the solution from polar to Cartesian coordinates. Closed form integral formula.
16	Oct 24, 2019	Chapter 4.4	Closed form integral solution for Laplace PDE inside disk. thm 4.6 and thm 4.9 (max or min of solution at boundary), thm 4.11. Classification of PDE's. General formula to find characteristic curves.
17	Oct 29, 2019	Chapter 6	Delta function. Definitions. Two cases, using limits and using integral. Integration of delta function, differentiation. Introduction to Green function
18	Oct 31, 2019	Chapter 6.2	Green function. Examples for $-u''(x) = f(x)$ with Dirichlet and Neumann B.C. Full derivation
19	Nov 5, 2019	Chapter 6.2	More Green function. Neumann B.C. Higher dimensions Green function. Laplace on square. Exam review
20	Nov 7, 2019		Second exam
21	Tuesday Nov 12, 2019	Chapter 6	Green function in higher dimensions. On whole plane. Green formula. Review of multivariable calculus. Derivation of Green function in 2D and 3D on whole space. Exam 2 returned.

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Table1.1 – continued from previous page

#	date	book section	note
22	Thursday Nov 14, 2019	Chapter 6	Green function. Method of images. half space and disk. Eigenfunctions
23	Tuesday Nov 19, 2019	Chapter 6	Eigenfunctions and eigenvalues for Laplacian in 2 and 3D. Behaviour of eigenvalues, Weyl law for eigenvalues. Solving PDE on 2D.
24	Thursday Nov 21, 2019	Chapter 6	Laplacian is energy minimizer. Equivelance between $E(u) = \int 1/2\lvert\triangle(u)\rvert^2 - f u \, dx$ and solution to $-\triangle(u) = f$ with Dirichlet B.C. Proofing that if u solves Laplace PDE then it minimizes the energy. And proofing that if u minimizes energy then it solves Laplace PDE
25	Tuesday Nov 26, 2019	Chapter 6	Fourier transform. Derivations and two examples using a box function and Gaussian e^{-x^2}

Chapter 2

HWs

Local contents

2.1 HW 1

Local contents

2.1.1 Problem 1.8a

Find all quadratic polynomial solutions of the 3D Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Solution

A quadratic polynomial in variables x, y, z is

$$u = a_1 + a_2 x + a_3 y + a_4 z + a_5 x^2 + a_6 y^2 + a_7 z^2 + a_8 xy + a_9 xz + a_{10} yz \tag{1}$$

Hence $u_x = a_2 + 2a_5 x + a_8 y + a_9 z$ which implies that $u_{xx} = 2a_5$. Similarly $u_y = a_3 + 2a_6 y + a_8 x + a_{10} z$, therefore $u_{yy} = 2a_6$. And finally $u_z = a_4 + 2a_7 z + a_9 x + a_{10} y$ and $u_{zz} = 2a_7$. Substituting these results in the Laplace equation gives above result in

$$\begin{aligned} 2a_5 + 2a_6 + 2a_7 &= 0 \\ a_5 + a_6 + a_7 &= 0 \end{aligned}$$

Therefore $a_5 = -(a_6 + a_7)$. Using this relation back in (1) gives

$$\begin{aligned} u &= a_1 + a_2 x + a_3 y + a_4 z - (a_6 + a_7) x^2 + a_6 y^2 + a_7 z^2 + a_8 xy + a_9 xz + a_{10} yz \\ &= a_1 + a_2 x + a_3 y + a_4 z + a_6 \left(-x^2 + y^2\right) + a_7 \left(-x^2 + z^2\right) + a_8 xy + a_9 xz + a_{10} yz \end{aligned}$$

Which can be written as

$$u\left(x,y,z\right) = A_1 + A_2 x + A_3 y + A_4 z + A_5 \left(y^2 - x^2\right) + A_6 \left(z^2 - x^2\right) + A_7 xy + A_8 xz + A_9 yz$$

2.1.2 Problem 1.7

Find all real solutions to 2D Laplace equation $u_{xx} + u_{yy} = 0$ of the form $u = \log\left(p\left(x,y\right)\right)$ where $p\left(x,y\right)$ is a quadratic polynomial.

Solution

A quadratic polynomial $p\left(x,y\right)$ in variables x,y is

$$p\left(x,y\right) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy$$

Therefore

$$u\left(x,y\right) = \log\left(a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy\right)$$

Hence

$$u_x = \frac{a_2 + 2a_4 x + a_6 y}{p\left(x,y\right)}$$

and

$$u_{xx} = \frac{2a_4}{p\left(x,y\right)} - \frac{\left(a_2 + 2a_4 x + a_6 y\right)^2}{p\left(x,y\right)^2} \tag{1}$$

Similarly

$$u_y = \frac{a_3 + 2a_5 y + a_6 x}{p\left(x,y\right)}$$

And

$$u_{yy} = \frac{2a_5}{p\left(x,y\right)} - \frac{\left(a_3 + 2a_5 y + a_6 x\right)^2}{p\left(x,y\right)^2} \tag{2}$$

Substituting (1,2) into $u_{xx} + u_{yy} = 0$ gives

$$\begin{aligned}
\left(\frac{2a_4}{p\left(x,y\right)} - \frac{\left(a_2 + 2a_4 x + a_6 y\right)^2}{p\left(x,y\right)^2}\right) + \left(\frac{2a_5}{p\left(x,y\right)} - \frac{\left(a_3 + 2a_5 y + a_6 x\right)^2}{p\left(x,y\right)^2}\right) &= 0 \\
2a_4 - \frac{\left(a_2 + 2a_4 x + a_6 y\right)^2}{p\left(x,y\right)} + 2a_5 - \frac{\left(a_3 + 2a_5 y + a_6 x\right)^2}{p\left(x,y\right)} &= 0 \\
2a_4 + 2a_5 - \frac{\left(a_2 + 2a_4 x + a_6 y\right)^2 + \left(a_3 + 2a_5 y + a_6 x\right)^2}{p\left(x,y\right)} &= 0
\end{aligned}$$

Or

$$(2a_4 + 2a_5)\, p\left(x,y\right) = \left(a_2 + 2a_4 x + a_6 y\right)^2 + \left(a_3 + 2a_5 y + a_6 x\right)^2$$

But $p\left(x,y\right) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy$. Hence the above becomes

$$(2a_4 + 2a_5)\left(a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy\right) = \left(a_2 + 2a_4 x + a_6 y\right)^2 + \left(a_3 + 2a_5 y + a_6 x\right)^2$$

Expanding and comparing coefficients gives

$$\begin{aligned}
&2x^2 a_4^2 + 2x^2 a_4 a_5 + 2a_6 a_4 xy + 2a_6 a_5 xy + 2a_2 a_4 x + 2a_2 x a_5 + 2y^2 a_4 a_5 + 2y^2 a_5^2 + 2a_3 a_4 y + 2a_3 a_5 y + 2a_1 a_4 + 2a_1 a_5 = \\
&\quad 4x^2 a_4^2 + x^2 a_6^2 + 4a_4 a_6 xy + 4a_5 a_6 xy + 4x a_2 a_4 + 2a_3 a_6 x + 4y^2 a_5^2 + y^2 a_6^2 + 2a_2 a_6 y + 4a_3 a_5 y + a_2^2 + a_3^2
\end{aligned}$$

Simplifying

$$2a_4a_5x^2 + 2a_2a_5x + 2a_4a_5y^2 + 2a_3a_4y + 2a_1a_4 + 2a_1a_5 =$$
$$2x^2a_4^2 + a_6^2x^2 + 2a_4a_6xy + 2a_5a_6xy + 2a_2a_4x + 2a_3a_6x + 2a_5^2y^2 + a_6^2y^2 + 2a_2a_6y + 2a_3a_5y + a_2^2 + a_3^2$$

Comparing coefficients of terms that contain no x, y and coefficients of x, y, xy, x^2, y^2 gives the following equations in order

$$\begin{aligned}
2a_1a_4 + 2a_1a_5 &= a_2^2 + a_3^2 \\
2a_2a_5 &= 2a_2a_4 + 2a_3a_6 \\
2a_3a_4 &= 2a_2a_6 + 2a_3a_5 \\
0 &= 4a_4a_6 \\
2a_4a_5 &= 2a_4^2 + a_6^2 \\
2a_4a_5 &= 2a_5^2 + a_6^2
\end{aligned}$$

Equation $0 = 4a_4a_6$ above implies that $a_4 = 0$ or $a_6 = 0$ or both are zero. But if both are zero, there is no solution. On the other hand, if $a_4 = 0$, then this also leads to no solution as all equations reduce to $0 = 0$. Therefore only choice left is $\underline{a_6 = 0}$. Now the above equations become

$$\begin{aligned}
2a_1a_4 + 2a_1a_5 &= a_2^2 + a_3^2 \\
2a_2a_5 &= 2a_2a_4 \\
2a_3a_4 &= 2a_3a_5 \\
0 &= 0 \\
2a_4a_5 &= 2a_4^2 \\
2a_4a_5 &= 2a_5^2
\end{aligned}$$

Or

$$\begin{aligned}
2a_1a_4 + 2a_1a_5 &= a_2^2 + a_3^2 \\
a_5 &= a_4 \\
a_4 &= a_5 \\
0 &= 0 \\
a_5 &= a_4 \\
a_4 &= a_5
\end{aligned}$$

Hence

$$\begin{aligned}
a_4 &= a_5 && (3) \\
a_6 &= 0 && (4) \\
2a_1a_4 + 2a_1a_5 &= a_2^2 + a_3^2
\end{aligned}$$

Since $a_4 = a_5$ then

$$\begin{aligned}
2a_1a_5 + 2a_1a_5 &= a_2^2 + a_3^2 \\
a_5 &= \frac{a_2^2 + a_3^2}{2a_1} && (5)
\end{aligned}$$

Using (3,4,5) in $p\left(x,y\right)=a_1+a_2x+a_3y+a_4x^2+a_5y^2+a_6xy$ gives

$$\begin{aligned}
p\left(x,y\right) &= a_1+a_2x+a_3y+a_5x^2+a_5y^2\\
&= a_1+a_2x+a_3y+a_5\left(x^2+y^2\right)\\
&= a_1+a_2x+a_3y+\frac{a_2^2+a_3^2}{2a_1}\left(x^2+y^2\right)
\end{aligned}$$

Only three arbitrary constants are needed. Let $a_1=a, a_2=b, a_3=c$ the above becomes

$$p\left(x,y\right)=a+bx+cy+\frac{b^2+c^2}{2a}\left(x^2+y^2\right)$$

And the solution becomes

$$u\left(x,y\right)=\log\left(a+bx+cy+\frac{b^2+c^2}{2a}\left(x^2+y^2\right)\right)$$

2.1.3 Problem 1.13

Find all solutions $u=f\left(r\right)$ of the 3D Laplace equation $u_{xx}+u_{yy}+u_{zz}=0$ that depends only on radial coordinates $r=\sqrt{x^2+y^2+z^2}$

Solution

The Laplacian in 3D in spherical coordinates is

$$\nabla^2 u\left(r,\theta,\phi\right)=u_{rr}+\frac{2}{r}u_r+\frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}u_\theta+u_{\theta\theta}\right)+\frac{1}{r^2\sin^2\theta}u_{\phi\phi}$$

The above shows that the terms that depend only on r makes the laplacian

$$\nabla^2 u\left(r\right)=u_{rr}+\frac{2}{r}u_r$$

Hence the PDE $\nabla^2 u\left(r\right)=0$ becomes an ODE now since there is only one dependent variable giving

$$u''\left(r\right)+\frac{2}{r}u'\left(r\right)=0$$

Let $v=u'\left(r\right)$ and the above becomes

$$v'\left(r\right)+\frac{2}{r}v\left(r\right)=0$$

This is linear first order ODE. The integrating factor is $I=e^{\int\frac{2}{r}dr}=e^{2\ln r}=r^2$. Therefore the above becomes $\frac{d}{dr}\left(vr^2\right)=0$ or $vr^2=C_1$ or $v\left(r\right)=\frac{C_1}{r^2}$. Therefore

$$\begin{aligned}
u' &= \frac{C_1}{r^2}\\
du &= \frac{C_1}{r^2}dr
\end{aligned}$$

Integrating gives the solution

$$u=-\frac{C_1}{r}+C_2$$

The above is the required solution. Hence

$$\boxed{f(r) = -\frac{C_1}{r} + C_2}$$

Where C_1, C_2 are arbitrary constants.

2.1.4 Problem 1.20

The displacement $u(t,x)$ of a forced violin string is modeled by the PDE $u_{tt} = 4u_{xx} + F(t,x)$. When the string is subjected to the external force $F(t,x) = \cos x$, the solution is $u(t,x) = \cos(x-2t) + \frac{1}{4}\cos x$, while when $F(t,x) = \sin x$, the solution is $u(t,x) = \sin(x-2t) + \frac{1}{4}\sin x$. Find a solution when the forcing function is (a) $\cos x - 5\sin x$, (b) $\sin(x-3)$

Solution

2.1.4.1 Part (a)

Since the PDE is linear, superposition can be used. When the input is $F(t,x) = \cos x - 5\sin x$ then the solution is

$$\begin{aligned} u(t,x) &= \left(\cos(x-2t) + \frac{1}{4}\cos x\right) - 5\left(\sin(x-2t) + \frac{1}{4}\sin x\right) \\ &= \cos(x-2t) + \frac{1}{4}\cos x - 5\sin(x-2t) - \frac{5}{4}\sin x \end{aligned}$$

2.1.4.2 Part (b)

Since the PDE is linear, superposition can be used. When the input is $F(t,x) = \sin(x-3)$ then the solution same as when the input is $\sin x$ but shifted by 3. Hence

$$u(t,x) = \sin((x-3)-2t) + \frac{1}{4}\sin(x-3)$$

2.1.5 Problem 1.27b

Solve the following inhomogeneous linear ODE $5u'' - 4u' + 4u = e^x \cos x$

Solution

First the homogeneous solution u_h is found, then a particular solution u_p is found. The general solution will be the sum of both $u = u_h + u_p$. Since this is a constant coefficient ODE, the characteristic equation is $5\lambda^2 - 4\lambda + 4 = 0$. The roots are $\lambda_1 = \frac{2}{5} + \frac{4}{5}i, \lambda_1 = \frac{2}{5} - \frac{4}{5}i$, which implies the solution is

$$u_h(x) = e^{\frac{2}{5}x}\left(c_1\cos\left(\frac{4}{5}x\right) + c_2\sin\left(\frac{4}{5}x\right)\right)$$

Using the method of undetermined coefficients, and since the forcing function is $e^x\cos x$, then let

$$u_p = Ae^x(B\cos x + C\sin x) \tag{1}$$

Hence

$$
\begin{aligned}
u_p' &= Ae^x(B\cos x + C\sin x) + Ae^x(-B\sin x + C\cos x) &(2)\\
u_p'' &= Ae^x(B\cos x + C\sin x) + Ae^x(-B\sin x + C\cos x) + Ae^x(-B\sin x + C\cos x) + Ae^x(-B\cos x - C\sin x)\\
&= Ae^x(B\cos x + C\sin x - B\sin x + C\cos x - B\sin x + C\cos x - B\cos x - C\sin x)\\
&= Ae^x(-B\sin x + C\cos x - B\sin x + C\cos x)\\
&= Ae^x(-2B\sin x + 2C\cos x) &(3)
\end{aligned}
$$

Substituting (1,2,3) back into the original ODE gives

$$
\begin{aligned}
5Ae^x(-2B\sin x + 2C\cos x) - 4\left(Ae^x(B\cos x + C\sin x) + Ae^x(-B\sin x + C\cos x)\right) + 4Ae^x(B\cos x + C\sin x) &= e^x\cos x\\
Ae^x(-10B\sin x + 10C\cos x) - Ae^x(4B\cos x + 4C\sin x) - Ae^x(-4B\sin x + 4C\cos x) + Ae^x(4B\cos x + 4C\sin x) &= e^x\cos x\\
Ae^x(-10B\sin x + 10C\cos x - 4B\cos x - 4C\sin x + 4B\sin x - 4C\cos x + 4B\cos x + 4C\sin x) &= e^x\cos x
\end{aligned}
$$

Hence

$$Ae^x(6C\cos x - 6B\sin x) = e^x\cos x$$

Comparing coefficients shows that

$$
\begin{aligned}
A &= 1\\
B &= 0\\
C &= \frac{1}{6}
\end{aligned}
$$

Hence from (1)

$$u_p = e^x\frac{\sin x}{6}$$

Therefore the general solution is

$$
\begin{aligned}
u(x) &= u_h(x) + u_p(x)\\
&= e^{\frac{2}{5}x}\left(c_1\cos\left(\frac{4}{5}x\right) + c_2\sin\left(\frac{4}{5}x\right)\right) + e^x\frac{\sin x}{6}
\end{aligned}
$$

2.1.6 Problem 2.1.6

Solve the PDE $\frac{\partial^2 u}{\partial x\partial y} = 0$ for $u(x,y)$

Solution

Integrating once w.r.t x gives

$$\frac{\partial u}{\partial y} = F(y)$$

Where $F(y)$ acts as the constant of integration, but since this is a PDE, it becomes an arbitrary function of y only. Integrating the above again w.r.t. y gives

$$u = \int F(y)\,dy + G(x)$$

Where $G(x)$ is an arbitrary function of x only. If we let $\int F(y)\,dy = H(y)$ where $H(y)$ is the antiderivative for the indefinite integral which depends on y only. Then the above can be written as

$$\boxed{u(x,y) = H(y) + G(x)}$$

To verify, from the above $\frac{\partial u}{\partial y} = H'(y)$ and hence

$$\begin{aligned}\frac{\partial^2 u}{\partial x \partial y} &= \frac{d}{dx}\left(H'(y)\right)\\ &= 0\end{aligned}$$

2.1.7 Problem 2.2.2

Solve the following initial value problems and graph the solutions at $t = 1, 2, 3$

a $u_t - 3u_x = 0, u(0,x) = e^{-x^2}$

b $u_t + 2u_x = 0, u(-1,x) = \frac{x}{1+x^2}$

c $u_t + u_x + \frac{1}{2}u = 0, u(0,x) = \arctan(x)$

d $u_t - 4u_x + u = 0, u(0,x) = \frac{1}{1+x^2}$

<u>Solution</u>

2.1.7.1 Part a

Let ξ be the characteristic variable defined such that $\xi = x - ct$. Where characteristic lines are given by $x = x_0 + ct$. But $c = -3$ in this problem. Hence characteristic lines are

$$x = x_0 - 3t$$

Where x_0 means the same as $x(0)$, i.e. $x(t)$ at time $t = 0$. Since $c = -3$ then

$$\xi = x + 3t$$

Let

$$u(t,x) \equiv v(t,\xi)$$

$u_t - 3u_x = 0$ is now transformed to $v(t,\xi)$ as follows

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial v}{\partial t}\frac{\partial t}{\partial t} + \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial t}\\ &= \frac{\partial v}{\partial t} + 3\frac{\partial v}{\partial \xi} \qquad (1)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x}\\ &= 0 + \frac{\partial v}{\partial \xi}\\ &= \frac{\partial v}{\partial \xi}\end{aligned} \tag{2}$$

Substituting (1,2) in $u_t - 3u_x = 0$ gives the transformed PDE as

$$\begin{aligned}\frac{\partial v}{\partial t} + 3\frac{\partial v}{\partial \xi} - 3\frac{\partial v}{\partial \xi} &= 0\\ \frac{\partial v}{\partial t} &= 0\end{aligned}$$

Integrating w.r.t ξ gives the solution in $v(t,\xi)$ space as

$$v(t,\xi) = F(\xi)$$

Where $F(\xi)$ is an arbitrary continuous function of ξ. Transforming back to $u(t,x)$ gives

$$u(t,x) = F(x+3t) \tag{3}$$

At $t=0$ the above becomes

$$e^{-x_0^2} = F(x_0)$$

This means that (3) becomes (since $x = x_0 + ct$ or $x = x_0 - 3t$ or $x_0 = x + 3t$)

$$u(t,x) = e^{-(x+3t)^2}$$

2.1.7.2 Part b

$$\begin{aligned}u_t + 2u_x &= 0\\ u(-1,x) &= \frac{x}{1+x^2}\end{aligned}$$

Let ξ be the characteristic variable defined such that $\xi = x - ct$. Where characteristic lines are given by $x = x_0 + ct$. But $c = 2$ in this problem. Hence characteristic lines are

$$x = x_0 + 2t$$

And

$$\xi = x - 2t$$

Let $u(t,x) \equiv v(t,\xi)$. Then $u_t + 2u_x = 0$ is transformed to $v(t,\xi)$ as was done in part (a) (will not be repeated) which results in

$$\frac{\partial v}{\partial t} = 0$$

Integrating w.r.t ξ gives the solution

$$v(t,\xi) = F(\xi)$$

Where $F(\xi)$ is an arbitrary continuous function of ξ. Transforming back to $u(t,x)$ results in

$$u(t,x) = F(x-2t) \tag{3}$$

At $t=-1$ the above becomes

$$\frac{x_0}{1+x_0^2} = F(x_0+2)$$

Let $x_0+2=z$. Then $x_0=z-2$. And the above becomes

$$\frac{z-2}{1+(z-2)^2} = F(z)$$

This means that (3) becomes

$$\begin{aligned} u(t,x) &= \frac{(x-2t)-2}{1+((x-2t)-2)^2} \\ &= \frac{x-2t-2}{1+(x-2t-2)^2} \end{aligned}$$

2.1.7.3 Part c

$$\begin{aligned} u_t+u_x+\frac{1}{2}u &= 0 \qquad (1) \\ u(0,x) &= \arctan(x) \end{aligned}$$

Let ξ be the characteristic variable defined such that $\xi = x-ct$. Where characteristic lines are given by $x=x_0+ct$. But $c=1$ in this problem. Hence characteristic lines are given by solution to

$$\begin{aligned} \frac{dx}{dt} &= 1 \\ x(t) &= x_0+t \end{aligned}$$

And

$$\begin{aligned} \xi &= x-ct \\ &= x-t \end{aligned}$$

Then u_t+u_x are transformed to $v(t,\xi)$ as was done in part (a) (will not be repeated) which results in

$$u_t+u_x = \frac{\partial v}{\partial t}$$

Substituting the above into (1) gives (where now v is used in place of u).

$$\frac{\partial v}{\partial t}+\frac{1}{2}v = 0$$

This is now first order ODE since it only depends on t. Therefore $v'+\frac{1}{2}v=0$. This is linear in v. Hence the solution is $\frac{d}{dt}\left(ve^{\int \frac{1}{2}dt}\right)=0$ or $ve^{\frac{1}{2}t}=F(\xi)$ where F is arbitrary function of ξ. Hence

$$v(t,\xi) = e^{\frac{-1}{2}t}F(\xi)$$

Converting back to $u(t,x)$ gives

$$u(t,x) = e^{\frac{-t}{2}} F(x-t) \tag{2}$$

At $t = 0$ the above becomes

$$\arctan(x_0) = F(x_0)$$

From the above then (2) can be written as

$$u(t,x) = e^{\frac{-t}{2}} \arctan(x-t)$$

2.1.7.4 Part d

$$\begin{aligned} u_t - 4u_x + u &= 0 \\ u(0,x) &= \frac{1}{1+x^2} \end{aligned}$$

Let ξ be the characteristic variable defined such that $\xi = x - ct$. Where characteristic lines are given by $x = x_0 + ct$. But $c = -4$ in this problem. Hence characteristic lines are

$$x = x_0 - 4t$$

And

$$\xi = x + 4t$$

Then $u_t - 4u_x$ are transformed to $v(t,\xi)$ as was done in part (a) (will not be repeated) which results in

$$u_t - 4u_x = \frac{\partial v}{\partial t}$$

Substituting the above into (1) gives (where now v is used in place of u).

$$\frac{\partial v}{\partial t} + v = 0$$

This is now first order ODE since it only depends on t. Therefore $v' + v = 0$. This is linear in v. Hence the solution is $\frac{d}{dt}\left(ve^{\int dt}\right) = 0$ or $ve^t = F(\xi)$ where F is arbitrary function of ξ. Hence

$$v(t,\xi) = e^{-t} F(\xi)$$

Converting to $u(t,x)$ gives

$$u(t,x) = e^{-t} F(x+4t) \tag{2}$$

At $u(0,x) = \frac{1}{1+x^2}$ the above becomes

$$\frac{1}{1+x_0^2} = F(x_0)$$

From the above then (2) can be written as

$$u(t,x) = \frac{e^{-t}}{1+(x+4t)^2}$$

2.1.8 Problem 2.2.3

Graph some of the characteristic lines for the following equation and write down the formula for the general solution

(b) $u_t + 5u_x = 0$, (d) $u_t - 4u_x + u = 0$

<u>Solution</u>

2.1.8.1 Part b

$$u_t + 5u_x = 0$$

Let ξ be the characteristic variable defined such that $\xi = x - ct$. Where characteristic lines are given by $x = x_0 + ct$. But $c = 5$ in this problem. Hence characteristic lines are

$$\boxed{x(t) = x_0 + 5t} \tag{1}$$

And

$$\xi = x - 5t$$

Then $u_t - 5u_x = 0$ is transformed to $v(t, \xi)$ as was done in earlier (will not be repeated) which results in

$$u_t - 5u_x = \frac{\partial v}{\partial t}$$

Therefore $\frac{\partial v}{\partial t} = 0$ which has the general solution $v(t, \xi) = F(\xi)$ where F is arbitrary function of ξ. Transforming back to $u(t, x)$ gives

$$\boxed{u(t, x) = F(x - 5t)}$$

On the characteristic lines given by (1) the solution $u(t, x)$ is constant. The slope of the characteristic lines is 5 and intercept is x_0. The following is a plot of few lines using different values of x_0.

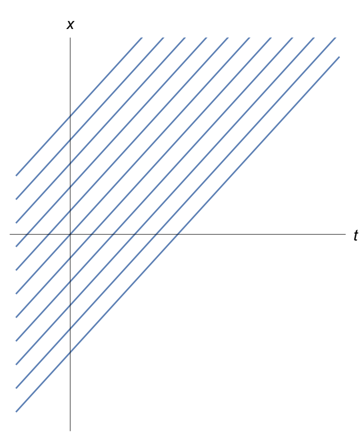


Figure 2.1: Showing some characteristic lines for part b

2.1.8.2 Part d

$$u_t - 4u_x + u = 0$$

Let ξ be the characteristic variable defined such that $\xi = x - ct$. Where characteristic lines are given by $x = x_0 + ct$. But $c = -4$ in this problem. Hence characteristic lines are

$$\boxed{x(t) = x_0 - 4t} \tag{1}$$

And

$$\xi = x + 4t$$

Then $u_t - 4u_x$ is transformed to $v(t, \xi)$ as was done in earlier (will not be repeated) which results in

$$u_t - 4u_x = \frac{\partial v}{\partial t}$$

Therefore the original PDE becomes $\frac{\partial v}{\partial t} + v = 0$, where u is replaced by v. This is linear first order ODE which has the solution $v(t, \xi) = e^{-t} F(\xi)$ where F is arbitrary function of ξ. Transforming back to $u(t, x)$ gives the general solution as

$$u(t, x) = e^{-t} F(x + 4t)$$

The following is a plot of few characteristic lines $x = x_0 - 4t$ using different values of x_0.

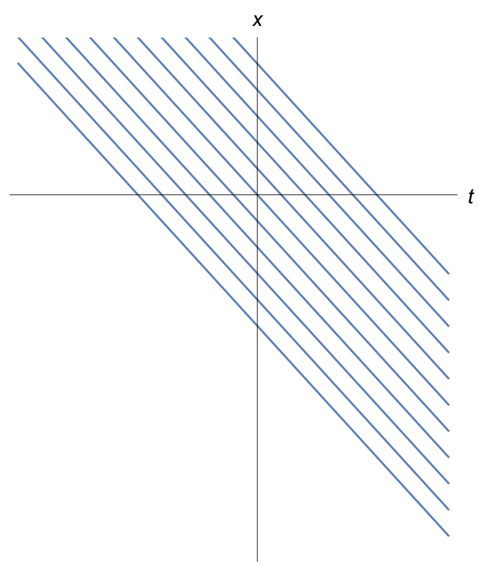


Figure 2.2: Showing some characteristic lines for part d

2.1.9 Problem 2.2.5

Solve $u_t + 2u_x = \sin x, u(0, x) = \sin x$

<u>Solution</u>

Let ξ be the characteristic variable defined such that $\xi = x - ct$. Where characteristic lines are given by $x = x_0 + ct$. But $c = 2$ in this problem. Hence characteristic lines are

$$\boxed{x = x_0 + 2t} \tag{1}$$

And

$$\xi = x - 2t$$

Then $u_t + 2u_x$ is transformed to $v(t,\xi)$ as was done in earlier (will not be repeated) which results in

$$u_t + 2u_x = \frac{\partial v}{\partial t}$$

Substituting this into the original PDE gives

$$\frac{\partial v(t,\xi)}{\partial t} = \sin(\xi + 2t)$$

Integrating w.r.t t gives

$$\begin{aligned} v(t,\xi) &= \int \sin(\xi + 2t)\,dt + F(\xi) \\ &= -\frac{\cos(\xi + 2t)}{2} + F(\xi) \end{aligned}$$

Transforming back to $u(t,x)$ gives

$$\begin{aligned} u(t,x) &= -\frac{\cos(x - 2t + 2t)}{2} + F(x - 2t) \\ &= \frac{-1}{2}\cos(x) + F(x - 2t) \end{aligned} \tag{1}$$

When $t = 0$, $u(0,x) = \sin x$, therefore the above becomes

$$\begin{aligned} \sin x_0 &= F(x_0) - \frac{1}{2}\cos x_0 \\ F(x_0) &= \sin x_0 + \frac{1}{2}\cos x_0 \end{aligned}$$

Therefore the solution (1) becomes

$$\begin{aligned} u(t,x) &= \left(\sin(x - 2t) + \frac{1}{2}\cos(x - 2t)\right) - \frac{1}{2}\cos x \\ &= \sin(x - 2t) + \frac{1}{2}\cos(x - 2t) - \frac{1}{2}\cos x \end{aligned}$$

2.1.10 Problem 2.2.9

(a) Prove that if the initial data is bounded, $|f(x)| \leq M$ for all $x \in \mathbb{R}$, then the solution to the damped transport equation (2.14) $u_t + cu_x + au = 0$ with $a > 0$ satisfies $u(t,x) \to 0$ as $t \to \infty$.
(b) Find a solution to (2.14) that is defined for all (t,x) but does not satisfy $u(t,x) \to 0$ as $t \to \infty$.

Solution

2.1.10.1 Part (a)

$u_t + cu_x + au = 0$ is solved to show what is required. Let ξ be the characteristic variable defined such that $\xi = x - ct$. Where characteristic lines are given by $x = x_0 + ct$. Hence characteristic lines are

$$x = x_0 + ct \tag{1}$$

And

$$\xi = x - ct$$

Then $u_t + cu_x$ is transformed to $v(t, \xi)$ as was done in earlier (will not be repeated) which results in

$$u_t + cu_x = \frac{\partial v}{\partial t}$$

Substituting this into the original PDE gives

$$\frac{\partial v}{\partial t} + av = 0$$

Where u is replaced by v. This can be viewed as first order linear ODE since it depends on t only. Its solution is $v(t, \xi) = e^{-at} F(\xi)$ where F is arbitrary function of ξ. Transforming back to $u(t, x)$ gives

$$u(t, x) = e^{-at} F(x - ct) \tag{1}$$

At $t = 0$ initial data is $f(x)$. Hence the above becomes at $t = 0$

$$f(x) = F(x)$$

Hence (1) now becomes

$$u(t, x) = e^{-at} f(x - ct) \tag{2}$$

But since $|f(x)|$ is bounded, and since $a > 0$ then $e^{-at} \to 0$ as $t \to \infty$. Which implies the solution itself $u(t, x)$ goes to zero as well. This is the reason why initial data needed to be bounded for this to happen.

2.1.10.2 Part (b)

Keeping $a > 0$. If initial data have the form $f(x) e^{-bx}$ where $|b| > a$, then at $t = 0$ the solution found in (1) becomes

$$f(x_0) e^{-bx_0} = F(x_0)$$

Then the solution (2) now becomes, after replacing x_0 by $x - ct$

$$\begin{aligned} u(t, x) &= e^{-at} e^{-b(x-ct)} f(x - ct) \\ &= e^{-at+bct} e^{-bx} f(x - ct) \\ &= e^{(bc-a)t} e^{-bx} f(x - ct) \end{aligned}$$

The problem is asking to show that this does not go to zero for all $x \in \mathbb{R}$ as $t \to \infty$. Since $|b| > a$ then $bc - a$ is positive quantity (c is assumed positive)[1].

Therefore $e^{(bc-a)t}$ will blow up as $t \to \infty$. And therefore the whole solution will not go to zero. For any x, no matter how large x is, a large enough t can be found to make the product $e^{(bc-a)t}e^{-bx}$ blow up.

[1] If c was negative then initial data could be choosen to be $f(x)e^{bx}$ where $|b| > a$ which will lead to same result.

2.1.11 Key solution for HW 1

Homework 1 Solutions

1.8 (a)

$c_0 + c_1 x + c_2 y + c_3 z + c_4(x^2 - y^2) + c_5(x^2 - z^2) + c_6 xy + c_7 xz + c_8 yz$,
where $c_0, \ldots, c_8$ are arbitrary constants.

1.7

$u = \log\left[c(x-a)^2 + c(y-b)^2\right]$, for a, b, c arbitrary constants.

1.13

$u = a + \dfrac{b}{r} = a + \dfrac{b}{\sqrt{x^2+y^2+z^2}}$, where a, b are arbitrary constants.

1.20

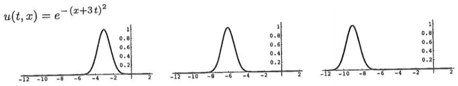
Solution: (a) $\cos(x-2t) + \frac{1}{4}\cos x - 5\sin(x-2t) - \frac{5}{4}\sin x$; (b) $-\sin 3\ \cos(x-2t) - \frac{1}{4}\sin 3\ \cos x + \cos 3\ \sin(x-2t) + \frac{1}{4}\cos 3\ \sin x = \sin(x-2t-3) + \frac{1}{4}\sin(x-3)$.

1.27

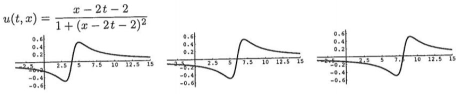
(b) $u(x) = \frac{1}{6}e^x \sin x + c_1 e^{2x/5}\cos\frac{4}{5}x + c_2 e^{2x/5}\sin\frac{4}{5}x$.

2.2.2

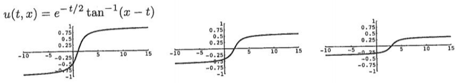
(a) $u(t,x) = e^{-(x+3t)^2}$



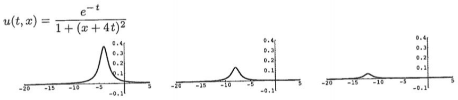
(b) $u(t,x) = \dfrac{x-2t-2}{1+(x-2t-2)^2}$



(c) $u(t,x) = e^{-t/2}\tan^{-1}(x-t)$

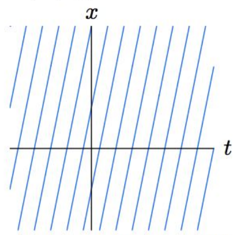


(d) $u(t,x) = \dfrac{e^{-t}}{1+(x+4t)^2}$

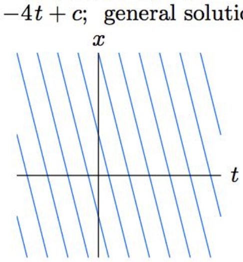


2.2.3

(b) Characteristic lines: $x = 5t + c$; general solution: $u(t,x) = f(x - 5t)$;



(d) Characteristic lines: $x = -4t + c$; general solution: $u(t,x) = e^{-t} f(x + 4t)$;



2.2.5

Solution: $u(t,x) = -\frac{1}{2}\cos x + \frac{1}{2}\cos(x - 2t) + \sin(x - 2t)$.

2.2.9

. (a) $|\,u(t,x)\,| = |\,f(x - ct)\,|\,e^{-a\,t} \le M\,e^{-a\,t} \to 0$ as $t \to \infty$ since $a > 0$.

(b) For example, if $c \ge a$, then the solution $u(t,x) = e^{(c-a)\,t - x} \not\to 0$ as $t \to \infty$.

2.2 HW 2

Local contents

2.2.1 Problem 2.2.17

(a) Solve the initial value problem $u_t - xu_x = 0, u(0,x) = \frac{1}{1+x^2}$. (b) Graph the solution at times $t = 0,1,2,3$. (c) What is $\lim_{t\to\infty} u(t,x)$?

Solution

2.2.1.1 Part a

The characteristic curves equations is given by

$$\frac{dx}{dt} = -x$$

Integrating this results in $\ln|x| = -t + C$ or $x = \xi e^{-t}$. Hence the characteristic variable is

$$\xi(x,t) = xe^t$$

u on the characteristic curves is an arbitrary function of the characteristic variable. Hence

$$u(t,\xi) = F(\xi)$$
$$u(t,x) = F\left(xe^t\right) \tag{1}$$

Where F is arbitrary function determined from initial conditions. Using initial conditions at $t = 0$, the above becomes

$$\frac{1}{1+x^2} = F(x)$$

Using the above in (1) gives the final solution as

$$u(t,x) = \frac{1}{1+(xe^t)^2} \tag{2}$$

2.2.1.2 Part b

The following are some plots and the code used.

```
In[ ]:= p = Grid[Partition[Table[Quiet@Plot[u[x, time], {x, -5, 5},
                        PlotRange → {All, {0, 1.1}},
                        AxesLabel → {Style["x", 12], Style["u", 14]},
                        BaseStyle → 12,
                        ImageSize → 400, PlotStyle → Red, GridLines → Automatic,
                        GridLinesStyle → LightGray,
                        PlotLabel → Row[{"time = ", padIt2[time, {1, 1}], " seconds"}]],
                  {time, {0, 1, 2, 3}}
              ], 2], Spacings → {1, 1}, Frame → All]
```

Figure 2.3: Source code

Out[ ]=

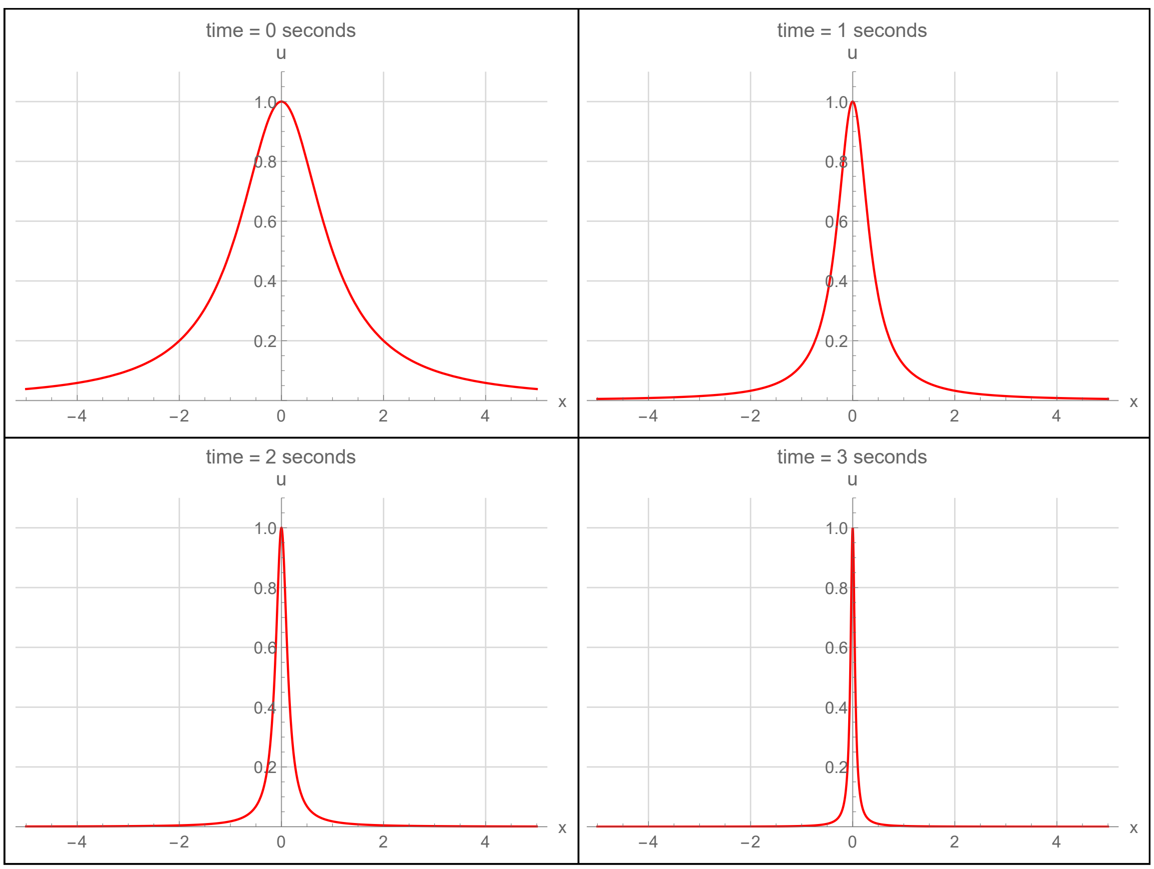


Figure 2.4: Solution at different times

2.2.1.3 Part c

From the solution in (2), when $x = 0$, then $\lim_{t\to\infty} u(t,0) = 1$. But when $x \neq 0$, then $\lim_{t\to\infty} u(t,x) = 0$. Therefore

$$\lim_{t\to\infty} u(t,x) = \begin{matrix} 1 & x = 0 \\ 0 & x \neq 0 \end{matrix}$$

Hence the solution is discontinuous at $x = 0$ in the limit as $t \to \infty$.

2.2.2 Problem 2.2.18

Suppose the initial data $u(0,x) = f(x)$ of the nonuniform transport equation (2.28), which is $u_t + \left(x^2 - 1\right) u_x = 0$ is continuous and satisfies $f(x) \to 0$ as $|x| \to \infty$. What is the limiting solution profile $u(t,x)$ as (a) $t \to \infty$ (b) $t \to -\infty$?

Solution

The characteristic curves equations is given by $\frac{dx}{dt} = \left(x^2 - 1\right)$. Integrating this results in

$$\begin{aligned} \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| &= t + C_3 \\ \ln\left|\frac{x-1}{x+1}\right| &= 2t + C_2 \\ \frac{x-1}{x+1} &= \xi e^{2t} \\ \xi &= \frac{x-1}{x+1}e^{-2t} \end{aligned}$$

u on the characteristic curves is an arbitrary function of the characteristic variable. Hence

$$\begin{aligned} u &= F(\xi) \\ &= F\left(\frac{x-1}{x+1}e^{-2t}\right) \end{aligned} \tag{1}$$

Where F is arbitrary function which is determined from initial conditions. From initial conditions the above becomes

$$f(x) = F\left(\frac{x-1}{x+1}\right)$$

Let $\frac{x-1}{x+1} = z$. Hence $(x-1) = z(x+1)$ or $x - 1 - z - zx = 0$ or $x(1-z) - 1 - z = 0$ or $x = \frac{1+z}{1-z}$. Therefore

$$f\left(\frac{1+z}{1-z}\right) = F(z)$$

Therefore (1) can now be written as

$$u(t,x) = f\left(\frac{1 + \left(\frac{x-1}{x+1}e^{-2t}\right)}{1 - \left(\frac{x-1}{x+1}e^{-2t}\right)}\right) \tag{2}$$

2.2.2.1 Part (a)

As $t \to \infty$ then solution (2) becomes

$$\begin{aligned}\lim_{t\to\infty} u(t,x) &= f\left(\frac{1+0}{1-0}\right)\\ &= f(1)\end{aligned}$$

2.2.2.2 Part (b)

And as $t \to -\infty$ then

$$\begin{aligned}\lim_{t\to-\infty} u(t,x) &= f\left(\frac{+\infty}{-\infty}\right)\\ &= f(-1)\end{aligned}$$

2.2.3 Problem 2.2.26

Consider the transport equation $\frac{\partial u}{\partial t} + c(t,x)\frac{\partial u}{\partial x} = 0$ with time varying wave speed. Define the corresponding characteristic ODE to be $\frac{dx}{dt} = c(t,x)$, the graphs of whose solutions $x(t)$ are the characteristic curves. (a) Prove that any solution $u(t,x)$ to the PDE is constant on each characteristic curve. (b) Suppose that the general solution to the characteristic equation is written in the form $\xi(t,x) = k$, where k is an arbitrary constant. Prove that $\xi(t,x)$ defines a characteristic variable, meaning that $u(t,x) = f(\xi(t,x))$ is a solution to the time-varying transport equation for any continuously differentiable scalar function $f \in C^1$.

Solution

2.2.3.1 Part (a)

Let $x(t)$ be the solution to characteristic ODE $\frac{dx}{dt} = c(t,x)$. Then

$$\begin{aligned}\frac{d}{dt}(u(t,x(t))) &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt}\\ &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}c(t,x)\end{aligned}$$

But $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}c(t,x) = 0$, since this is the given PDE above. The above now reduces to

$$\frac{d}{dt}(u(t,x(t))) = 0$$

Which implies that $u(t,x(t))$ is constant on the characteristic curves.

2.2.3.2 Part (b)

$$\begin{aligned}
\frac{\partial}{\partial t} f(\xi(t,x)) &= \frac{df}{d\xi(t,x)}\left(\frac{\partial}{\partial t}(\xi(t,x))\right) \\
&= \frac{df}{d\xi(t,x)}\left(\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}\frac{dx}{dt}\right) \\
&= \frac{df}{d\xi(t,x)}\left(\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x} c(t,x)\right)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial}{\partial x} f(\xi(t,x)) &= \frac{df}{d\xi(t,x)}\left(\frac{\partial}{\partial x}\xi(t,x)\right) \\
&= \frac{df}{d\xi(t,x)}\left(\frac{\partial \xi}{\partial t}\frac{dt}{dx} + \frac{\partial \xi}{\partial x}\frac{dx}{dx}\right) \\
&= \frac{df}{d\xi(t,x)}\left(\frac{\partial \xi}{\partial x}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial}{\partial t} f(\xi(t,x)) + c(t,x)\frac{\partial}{\partial x} f(\xi(t,x)) &= \frac{df}{d\xi(t,x)}\left(\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x} c(t,x)\right) + c(t,x)\frac{df}{d\xi(t,x)}\left(\frac{\partial \xi}{\partial x}\right) \\
&= \frac{df}{d\xi(t,x)}\left(\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x} c(t,x) + c(t,x)\frac{\partial \xi}{\partial x}\right) \\
&= \frac{df}{d\xi(t,x)}\left(\frac{\partial \xi}{\partial t} + 2\frac{\partial \xi}{\partial x} c(t,x)\right)
\end{aligned}$$

But $\xi(t,x)$ is constant k. Hence $\frac{df}{d\xi(t,x)} = 0$. Therefore RHS above is zero, and the above reduces to

$$\frac{\partial}{\partial t} f(\xi(t,x)) + c(t,x)\frac{\partial}{\partial x} f(\xi(t,x)) = 0$$

This shows that $f(\xi(t,x))$ satisfies the given transport PDE. Hence it is a solution. Or $u(t,x) = f(\xi(t,x))$.

2.2.4 Problem 2.2.29

Consider the first-order PDE $u_t + (1-2t)u_x = 0$. Use exercise 2.2.26 to: (a) Find and sketch the characteristic curves. (b) Write down the general solution. (c) Solve the initial value problem with $u(0,x) = \frac{1}{1+x^2}$. (d) Describe the behavior of your solution $u(t,x)$ from part (c) as $t \to \infty$. What about $t \to -\infty$?

Solution

2.2.4.1 Part (a)

The characteristic curves are given by $\frac{dx}{dt} = (1 - 2t)$. Therefore

$$
\begin{aligned}
x(t) &= t - t^2 + \xi \\
\xi &= x - \left(t - t^2\right)
\end{aligned}
$$

The following is plot of characteristic curves for different ξ values.

```
Table[Plot[t - t^2 + k, {t, 0, 4}, PlotRange → {All, {-10, 7}},
    AxesLabel → {"t", "x(t)"}, BaseStyle → 14],
  {k, 0, 5, 1}];
Show[%]
```

Out[∘]=

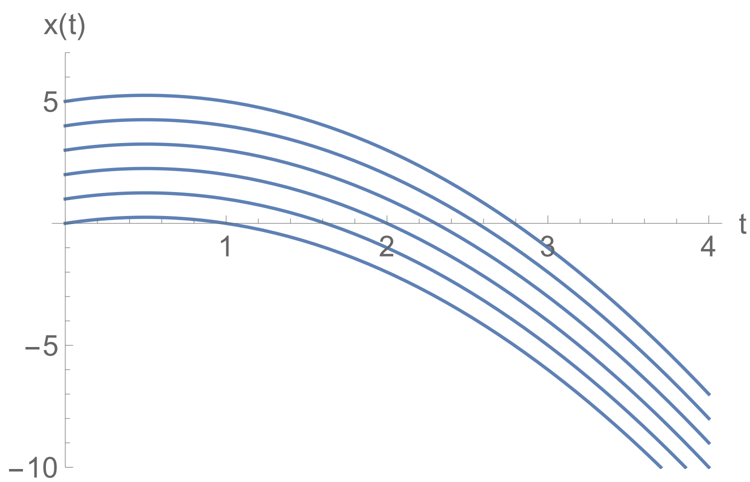


Figure 2.5: Plot of some characteristic curves

2.2.4.2 Part (b)

solution u on the characteristic curves is an arbitrary function of the characteristic variable. Hence

$$
\begin{aligned}
u(t,x) &= F(\xi) \\
&= F\left(x - \left(t - t^2\right)\right) \\
&= F\left(x - t + t^2\right) \qquad (1)
\end{aligned}
$$

Where F is arbitrarily function.

2.2.4.3 Part (c)

At $t = 0$ the above solution becomes

$$
\frac{1}{1 + x^2} = F(x) \qquad (2)
$$

Therefore using (2) in (1), then (1) becomes

$$u(t,x) = \frac{1}{1+\left(x-t+t^2\right)^2} \tag{3}$$

2.2.4.4 Part (d)

The solution in (3) shows that

$$\lim_{t\to\infty} u(t,x) = \frac{1}{\infty} = 0$$

Also

$$\lim_{t\to-\infty} u(t,x) = \frac{1}{\infty} = 0$$

Hence the solution vanishes for large t.

2.2.5 Problem 2.4.2

(a) Solve the wave equation $u_{tt} = u_{xx}$ when the initial displacement is the box function $u(0,x) = \begin{cases} 1 & 1<x<2 \\ 0 & \text{otherwise} \end{cases}$, while the initial velocity is zero. (b) Sketch the resulting solution at several times.

Solution

2.2.5.1 Part (a)

d'Alembert solution of the wave equation is given by

$$u(t,x) = \frac{1}{2}\left(f(x-ct)+f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

Where c is the wave speed which is $c=1$ in this problem and $f(x) = u(0,x)$ and $g(x) = u_t(0,x) = 0$. The above simplifies to

$$\begin{aligned}
u(t,x) &= \frac{1}{2}\left(f(x-t)+f(x+t)\right) \\
&= \frac{1}{2}\left(\begin{cases} 1 & 1<x-t<2 \\ 0 & \text{otherwise} \end{cases} + \begin{cases} 1 & 1<x+t<2 \\ 0 & \text{otherwise} \end{cases}\right) \\
&= \frac{1}{2}\left(\begin{cases} 1 & 1+t<x<2+t \\ 0 & \text{otherwise} \end{cases} + \begin{cases} 1 & 1-t<x<2-t \\ 0 & \text{otherwise} \end{cases}\right)
\end{aligned}$$

Complete split of the box function into two separate halves happens at $t=0.5$ because when $t=0.5$ in the above gives

$$u(t,x) = \frac{1}{2}\left(\begin{cases} 1 & 1.5<x<2.5 \\ 0 & \text{otherwise} \end{cases} + \begin{cases} 1 & 0.5<x<1.5 \\ 0 & \text{otherwise} \end{cases}\right)$$

This shows that just after $t = 0.5$, there is no longer a common region between $1.5 < x < 2.5$ and $0.5 < x < 1.5$.

Hence for $\underline{t > 0.5}$ the solution u will be $\frac{1}{2}$ when $1 + t < x < 2 + t$ or when $1 - t < x < 2 - t$ and will be zero otherwise.

But when $t < 0.5$, there will still be a common region before the full split. Some region is till common, and some region is not. For example, picking $t = 0.25$, then there is a common region between $1.25 < x < 2.25$ and $0.75 < x < 1.75$. In this case the common region is $1.25 < x < 1.75$. Over this region, $u = 1$. But over the non common region $u = \frac{1}{2}$ when $0.75 < x < 1.25$ and $u = \frac{1}{2}$ for $0.1.75 < x < 2.25$ and $u = 0$ otherwise. In terms of t the above can be written as

When $t \geq \frac{1}{2}$ then the solution is

$$u = \frac{1}{2}\begin{cases} \frac{1}{2} & 1 - t < x < 2 - t \\ \frac{1}{2} & 1 + t < x < 2 + t \\ 0 & \text{otherwise} \end{cases}$$

When $t < \frac{1}{2}$

$$u = \frac{1}{2}\begin{cases} 1 & 1 + t < x < 2 - t \\ \frac{1}{2} & 1 - t < x < 1 + t \\ \frac{1}{2} & 2 - t < x < 2 + t \\ 0 & \text{otherwise} \end{cases}$$

It it easier to do all of this using the computer by plotting the solution for different times.

2.2.5.2 Part (b)

The following are plots of the motion of the wave for several times.

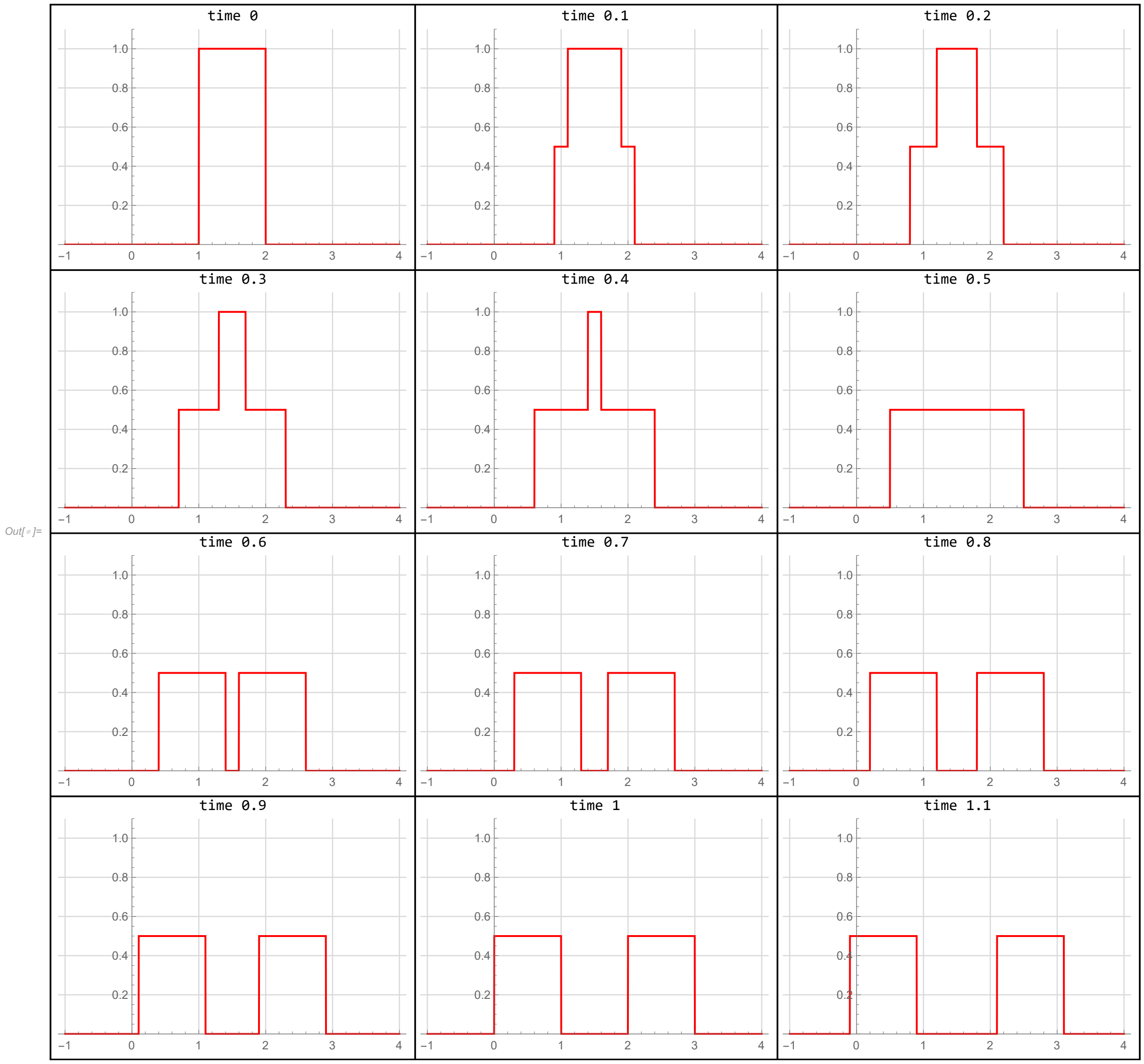


Figure 2.6: Plots for several times

```
In[∘]:= u[x_, t_] := 1/2 (Piecewise[{{1, 1 < x - t < 2}, {0, True}}] + Piecewise[{{1, 1 < x + t < 2}, {0, True}}]);
plots = Table[Grid[{{Row[{"time ", t}]},
                {Plot[u[x, t], {x, -1, 4}, Exclusions → None, ImageSize → 300,
                      PlotPoints → 40,
                      PerformanceGoal → "Quality", PlotStyle → Red,
                      GridLines → Automatic, GridLinesStyle → LightGray,
                      PlotRange → {All, {0, 1.1}}]}
             }], {t, {0, .1, .2, .3, .4, .5, .6, .7, .8, .9, 1, 1.1}}];
Grid[Partition[plots, 3], Frame → All]
```

Figure 2.7: Code used

2.2.6 Problem 2.4.3

Answer 2.4.2 when the initial velocity is the box function while the initial displacement is zero.

Solution

2.2.6.1 Part (a)

d'Alembert solution of the wave equation is

$$u(t,x)=\frac{1}{2}\Big(f(x-ct)+f(x+ct)\Big)+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)\,ds$$

Where c is the wave speed which is $c=1$ in this problem and $f(x)=0$ and $g(x)=u_t(0,x)=f(x)$ which is the box function given in the last problem. The above becomes

$$\begin{aligned}u(t,x)&=\frac{1}{2}\int_{x-t}^{x+t}f(s)\,ds\\&=\frac{1}{2}\int_{s=x-t}^{s=x+t}\begin{cases}1 & 1<s<2\\0 & \text{otherwise}\end{cases}ds\end{aligned}$$

2.2.6.2 Part (b)

The following are plots of the motion of the wave for several times of the above solution

```
In[◦]:= u[x_, t_] := 1/2 Integrate[Piecewise[{{1, 1 < s < 2}, {0, True}}], {s, x - t, x + t}];
plots = Table[Grid[{{Row[{"time ", t}]},
            {Plot[u[x, t], {x, -1, 4}, Exclusions → None, ImageSize → 300,
                PlotPoints → 40,
                PerformanceGoal → "Quality", PlotStyle → Red,
                GridLines → Automatic, GridLinesStyle → LightGray,
                PlotRange → {All, {0, 1.1}}]}
          }], {t, {0, .1, .2, .3, .4, .5, .6, .7, .8, .9, 1, 1.1}}];
Grid[Partition[plots, 3], Frame → All]
```

Figure 2.8: Code used

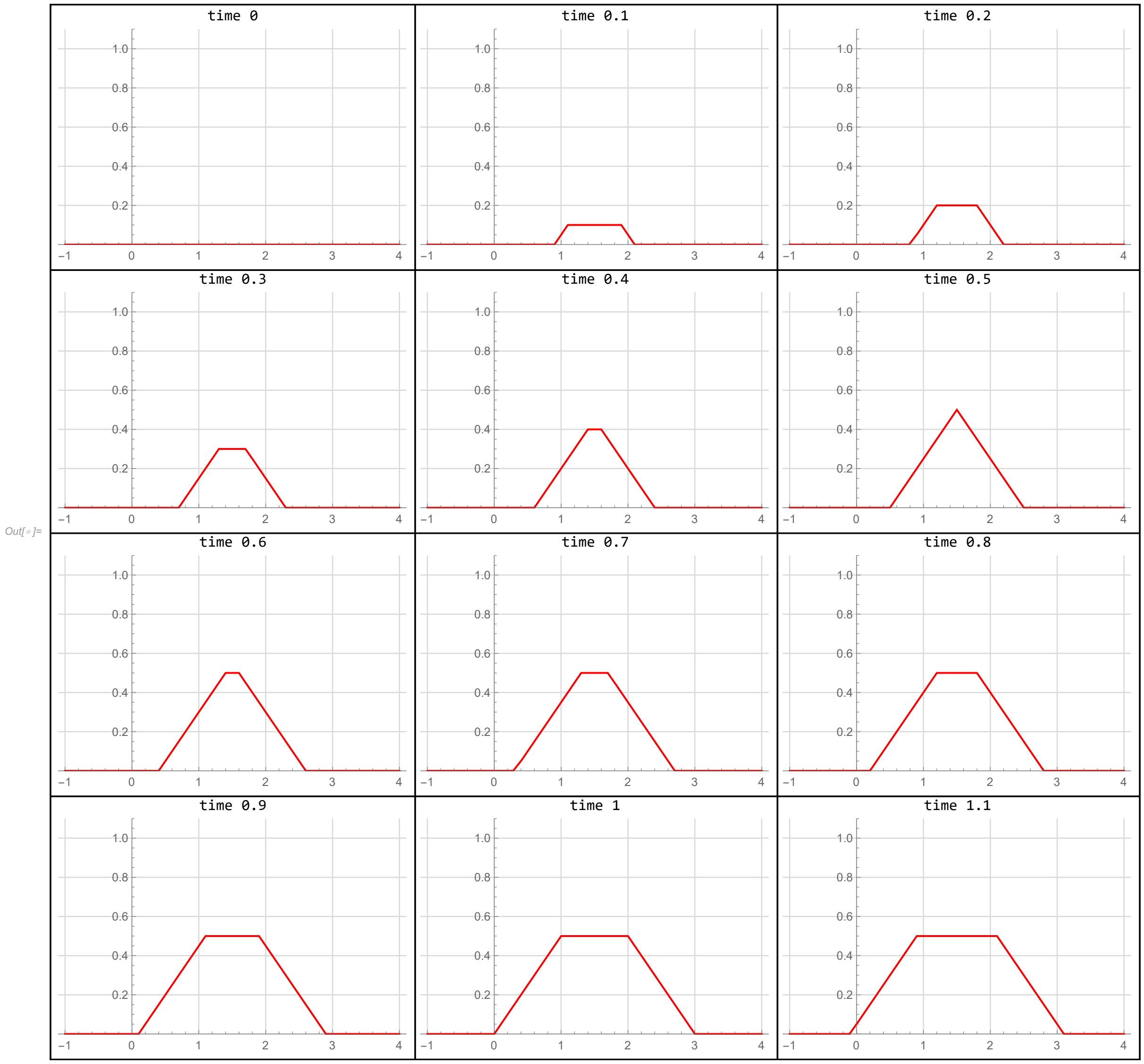


Figure 2.9: Plots for several times

2.2.7 Problem 2.4.4

Write the following solutions to the wave equation $u_{tt} = u_{xx}$ in d'Alembert form (2.82) which is $u(t,x) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$. Hint: What is the appropriate initial data? (b) $\cos 2x \sin 2t$. (d) $t^2 + x^2$

Solution

2.2.7.1 Part (b)

Since $c=1$, the solution becomes

$$\cos 2x \sin 2t = \frac{f(x-t)+f(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$

Let $f(x) = u(0,x) = 0$. The above solution simplifies to

$$\begin{aligned} 2\cos 2x \sin 2t &= \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \\ \cos 2x \sin 2t &= \frac{1}{4}\int_{x-t}^{x+t} g(s)\,ds \qquad (1) \end{aligned}$$

We now need to determine $g(s)$ to satisfy the above. By fundamental theorem of calculus

$$\frac{1}{4}\int_{x-t}^{x+t} g(s)\,ds = \frac{1}{4}\left[g'(x+t) - g'(x-t)\right] \qquad (2)$$

Let $g(x) = 2\cos 2x$. Now we need to verify that this will satisfy equation (1). Expanding RHS of (2) gives

$$\begin{aligned} g'(x+t) - g'(x-t) &= 2(-\sin(2(x+t)) + \sin(2(x-t))) \\ &= 2(\sin(2x-2t) - \sin(2x+2t)) \end{aligned}$$

But $\sin(A-B) = \sin A\cos B - \cos A \sin B$ and $\sin(A+B) = \sin A \cos B + \cos A \sin B$. Substituting these in the above, where $A = 2x, B = 2t$, the above becomes

$$\begin{aligned} g'(x+t) - g'(x-t) &= 2(\sin 2x\cos 2t - \cos 2x \sin 2t - (\sin 2x \cos 2t + \cos 2x \sin 2t)) \\ &= 2(\sin 2x \cos 2t - \cos 2x \sin 2t - \sin 2x \cos 2t - \cos 2x \sin 2t) \\ &= 4\cos 2x \sin 2t \qquad (3) \end{aligned}$$

Substituting (3) into (1) gives

$$\begin{aligned} \cos 2x \sin 2t &= \frac{1}{4}(4\cos 2x \sin 2t) \\ &= \cos 2x \sin 2t \end{aligned}$$

Verified.

Hence if initial condition is $f(x) = 0$ and if $g(x) = 2\cos 2x$, then the solution using d'Alembert form will be the one given $u(t,x) = 2\cos 2x \sin 2t$ which is what we are asked to show. Therefore

$$\begin{aligned} \cos 2x \sin 2t &= \frac{1}{2}\big(f(x-t) + f(x+t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \\ u(0,x) &= 0 \\ u_t(0,x) &= 2\cos 2x \end{aligned}$$

2.2.7.2 Part (d)

Since $c=1$, the solution becomes

$$t^2 + x^2 = \frac{1}{2}\big(f(x-t) + f(x+t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$

Let $g(x) = u_t(0,x) = 0$. The above reduces to

$$t^2 + x^2 = \frac{1}{2}\left(f(x-t) + f(x+t)\right)$$

Assuming $f(x) = x^2$, now we will see if this assumption generates the solution needed. The RHS above now becomes

$$\begin{aligned}
\frac{1}{2}\left(f(x-t) + f(x+t)\right) &= \frac{1}{2}\left((x-t)^2 + (x+t)^2\right) \\
&= \frac{1}{2}\left(\left(x^2+t^2-2xt\right) + \left(x^2+t^2+2xt\right)\right) \\
&= \frac{1}{2}\left(x^2+t^2+x^2+t^2\right) \\
&= t^2 + x^2
\end{aligned}$$

Verified.

Hence by setting $g(x) = 0$ and $f(x) = x^2$ the given solution is obtained. Therefore

$$\begin{aligned}
t^2 + x^2 &= \frac{1}{2}\left(f(x-t) + f(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \\
u(0,x) &= x^2 \\
u_t(0,x) &= 0
\end{aligned}$$

2.2.8 Problem 2.4.10

Suppose $u(t,x)$ solves the initial value problem $u_{tt} = 4u_{xx} + \sin(\omega t)\cos(x), u(0,x) = 0, u_t(0,x) = 0$. Is $h(t) = u(t,0)$ a periodic function?

<u>Solution</u>

The solution is given by eq (2.96) in the textbook (since $f(x) = 0$ and $g(x) = 0$ and $c^2 = 4$ or $c = 2$) as the following

$$u(t,x) = \frac{1}{4}\int_0^t \int_{x-(t-s)}^{x+(t-s)} F(s,y)\,dyds$$

But here $F(s,y) = \sin(\omega s)\cos(y)$. Therefore, using the book example 2.19, where we just need to change $\sin x$ to $\cos x$ in the solution shown, then the above integral gives

$$\begin{aligned}
u(t,x) &= \frac{1}{4}\int_0^t \int_{x-(t-s)}^{x+(t-s)} \sin(\omega s)\cos(y)\,dyds \\
&= \begin{cases} \frac{\sin(\omega t)-\omega\sin t}{1-\omega^2}\cos x & 0<\omega\neq 1 \\ \frac{\sin t - t\cos t}{2}\cos x & \omega = 1 \end{cases}
\end{aligned}$$

At $x = 0$, then

$$h(t) = u(t,0) = \begin{cases} \frac{\sin(\omega t)-\omega\sin t}{1-\omega^2} & 0<\omega\neq 1 \\ \frac{\sin t - t\cos t}{2} & \omega = 1 \end{cases}$$

Therefore $h(t)$ is periodic only if $\omega = \frac{p}{q} \neq 1$ is a rational number.

2.2.9 Problem 2.4.11

(a) Write down an explicit formula for the solution to initial value problem $u_{tt} = 4u_{xx}, u(0,x) = \sin x, u_t(0,x) = \cos x$ for $-\infty < x < \infty, t \geq 0$. (b) True of False: The solution is a periodic function of t. (c) Now solve the forced initial value problem $u_{tt} = 4u_{xx} + \cos 2t, u(0,x) = \sin x, u_t(0,x) = \cos x$ for $-\infty < x < \infty, t \geq 0$. (d) True of False: The forced equation exhibits resonance. Explain. (e) Does the answer to part (d) change if the forcing function is $\sin 2t$?

Solution

2.2.9.1 Part (a)

Using d'Alembert formula where $u(0,x) = f(x) = \sin x$ and $u_t(0,x) = g(x) = \cos x$, then the solution is

$$u(t.x) = \frac{1}{2}\Big(f(x-ct) + f(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

But $c = 2, f(x) = \sin x, g(x) = \cos x$, then the above becomes

$$\begin{aligned}
u(t,x) &= \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \frac{1}{4}\int_{x-2t}^{x+2t} \cos(s)\,ds \\
&= \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \frac{1}{4}[\sin(s)]_{x-2t}^{x+2t} \\
&= \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \frac{1}{4}(\sin(x+2t) - \sin(x-2t)) \\
&= \frac{1}{2}\sin(x-2t) + \frac{1}{2}\sin(x+2t) + \frac{1}{4}\sin(x+2t) - \frac{1}{4}\sin(x-2t) \\
&= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t)
\end{aligned}$$

2.2.9.2 Part (b)

True.

If we can find a common multiple between $x - 2t$ and $x + 2t$ then the solution is periodic. i.e. if $F_1(z)$ has period p_1 and $F_2(z)$ has period p_2, then if we can find positive integers a_1, a_2 such that $a_1p_1 = a_2p_2 = r$, then r is the period of $F_1(x) + F_2(x)$.

In this problem, $F_1 = \sin(x-2t), F_2 = \sin(x+2t)$. But both of these have period 2π. Hence $p_1 = 2\pi, p_2 = 2\pi$. Therefore choosing $a_1 = 1, a_2 = 1$, then $r = 2\pi$. The period of sum.

2.2.9.3 Part (c)

When the PDE becomes $u_{tt} = 4u_{xx} + \cos 2t$, then we need to add forcing solution part of the solution. Hence the solution now becomes, using 2.97 in the book as (using $c = 2$)

$$u(t,x) = \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \frac{1}{4}\int_{x-2t}^{x+2t} \cos(s)\,ds + \frac{1}{4}\int_0^t \int_{x-2(t-s)}^{x+2(t-s)} F\left(s,y\right) dyds$$

Where $F\left(s,y\right) = \cos(2t)$. Hence the above becomes (using result from part (a) for the non forcing part) as

$$\begin{aligned}
u(t,x) &= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4}\int_0^t \int_{x-2(t-s)}^{x+2(t-s)} \cos(2s)\,dyds \\
&= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4}\int_0^t \cos(2s)\int_{x-2(t-s)}^{x+2(t-s)} dyds \\
&= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4}\int_0^t \cos(2s)\left((x+2(t-s)) - (x-2(t-s))\right) ds \\
&= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4}\int_0^t \cos(2s)\left(x+2t-2s-(x-2t+2s)\right) ds \\
&= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4}\int_0^t \cos(2s)\left(x+2t-2s-x+2t-2s\right) ds \\
&= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4}\int_0^t \cos(2s)(4t-4s)\,ds
\end{aligned}$$

But $\frac{1}{4}\int_0^t \cos(2s)(4t-4s)\,ds = \frac{\sin^2 t}{2}$. Hence the above solution becomes

$$u(t,x) = \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{\sin^2 t}{2}$$

Which can also be written as

$$\begin{aligned}
u(t,x) &= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\cos(2t)\right) \\
&= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4} - \frac{1}{4}\cos(2t)
\end{aligned}$$

2.2.9.4 Part (d)

False. No resonance. Solution is periodic. There is no term in the solution which is being multiplied by t. Hence solution do not grow with time which indicates no resonance.

2.2.9.5 Part (e)

If the PDE now becomes $u_{tt} = 4u_{xx} + \sin 2t, u(0,x) = \sin x, u_t(0,x) = \cos x$, then the solution becomes

$$\begin{aligned}
u(t,x) &= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4}\int_0^t \int_{x-2(t-s)}^{x+2(t-s)} \sin(2s)\,dyds \\
&= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4}\int_0^t \sin(2s)\int_{x-2(t-s)}^{x+2(t-s)} dyds \\
&= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4}\int_0^t \sin(2s)\left((x+2(t-s)) - (x-2(t-s))\right)ds \\
&= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4}\int_0^t \sin(2s)\left(x+2t-2s-(x-2t+2s)\right)ds \\
&= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4}\int_0^t \sin(2s)\left(x+2t-2s-x+2t-2s\right)ds \\
&= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4}\int_0^t \sin(2s)(4t-4s)\,ds
\end{aligned}$$

But $\frac{1}{4}\int_0^t \sin(2s)(4t-4s)\,ds = \frac{1}{4}(2t - \sin(2t))$. Hence the solution now becomes

$$u(t,x) = \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4}(2t - \sin(2t))$$

We see now that <u>resonance now occurs</u> due to above term $\frac{1}{2}t$ in the solution. This means as t increases, the solution will keep increasing with no limit.

2.2.10 Problem 2.4.13

Let $u(t,x)$ be a classical solution to the wave equation $u_{tt} = c^2 u_{xx}$. The total energy

$$E(t) = \int_{-\infty}^{\infty} \frac{1}{2}\left(\left(\frac{\partial u}{\partial t}\right)^2 + c^2\left(\frac{\partial u}{\partial x}\right)^2\right)dx$$

Represents the sum of kinetic and potential energies of the displacement $u(t,x)$ at time t. Suppose that $\Delta u \to 0$ sufficiently rapidly as $x \to \pm\infty$; more precisely, one can find $\alpha > \frac{1}{2}$ and $C(t) > 0$ such that $|u_t(t,x)|, |u_x(t,x)| \leq \frac{C(t)}{|x|^\alpha}$ for each fixed t and all sufficiently large $|x| \gg 0$. For such solutions establish the law of conservation of energy by showing that $E(t)$ is finite and constant. Hint: You do not need the formula for the solution.

<u>Solution</u>

To show $E(t)$ is constant, it is sufficient to show that $\frac{d}{dt}E(t) = 0$. From above

$$\frac{d}{dt}E(t) = \frac{d}{dt}\int_{-\infty}^{\infty}\frac{1}{2}\left(u_t^2 + c^2 u_x^2\right)dx$$

Moving $\frac{d}{dt}$ inside the integral (assuming solution is piecewise smooth), the above becomes

$$\frac{d}{dt}E(t) = \int_{-\infty}^{\infty} \frac{1}{2}\left(\frac{d}{dt}u_t^2 + c^2\frac{d}{dt}u_x^2\right)dx$$

But $\frac{d}{dt}u_t^2 = 2u_t u_{tt}$ and $\frac{d}{dt}u_x^2 = 2u_x u_{xt}$. The above becomes

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_{-\infty}^{\infty} \frac{1}{2}\left(2u_t u_{tt} + 2c^2 u_x u_{xt}\right)dx \\ &= \int_{-\infty}^{\infty} u_t u_{tt} + c^2 u_x u_{xt} dx\end{aligned}$$

But $u_{tt} = c^2 u_{xx}$ from the PDE itself. The above now simplifies to

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_{-\infty}^{\infty} c^2 u_t u_{xx} + c^2 u_x u_{xt} dx \\ &= c^2 \int_{-\infty}^{\infty} u_t u_{xx} + u_x u_{xt} dx\end{aligned}$$

But $u_t u_{xx} + u_x u_{xt} = \frac{d}{dx}(u_t u_x)$. The above becomes

$$\begin{aligned}\frac{d}{dt}E(t) &= c^2 \int_{-\infty}^{\infty} \frac{d}{dx}(u_t u_x)\,dx \\ &= c^2 \int_{-\infty}^{\infty} d(u_t u_x) \\ &= c^2 [u_t u_x]_{-\infty}^{\infty}\end{aligned}$$

But the problem says that as $x \to \pm\infty$ then $u_x \to 0$. It also say that $|u_t|$ is bounded. This shows that the RHS above is zero. Therefore $\frac{d}{dt}E(t) = 0$ or <u>$E(t)$ is constant</u>. The fact constant is bounded is seen by noting that the problems says that $|u_x|$ and $|u_t|$ are bounded. This completes the proof.

2.2.11 Problem 2.4.15

The telegraph equation $u_{tt} + au_t = c^2 u_{xx}$ with $a > 0$, models the vibration of a string under frictional damping. (a) Show that, under the decay assumption of exercise 2.4.13, the wave energy (2.98)

$$E(t) = \int_{-\infty}^{\infty} \frac{1}{2}\left(\left(\frac{\partial u}{\partial t}\right)^2 + c^2\left(\frac{\partial u}{\partial x}\right)^2\right)dx$$

of a classical solution is a nonincreasing function of t. (b) Prove uniqueness of such solutions to the initial value problem for the telegraph equation.

<u>Solution</u>

2.2.11.1 Part (a)

$$\frac{d}{dt}E(t)=\frac{d}{dt}\int_{-\infty}^{\infty}\frac{1}{2}\left(u_t^2+c^2u_x^2\right)dx$$

Moving $\frac{d}{dt}$ inside the integral (assuming solution is smooth), the above becomes

$$\frac{d}{dt}E(t)=\int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{d}{dt}u_t^2+c^2\frac{d}{dt}u_x^2\right)dx$$

But $\frac{d}{dt}u_t^2=2u_tu_{tt}$ and $\frac{d}{dt}u_x^2=2u_xu_{xt}$. The above becomes

$$\begin{aligned}\frac{d}{dt}E(t)&=\int_{-\infty}^{\infty}\frac{1}{2}\left(2u_tu_{tt}+2c^2u_xu_{xt}\right)dx\\&=\int_{-\infty}^{\infty}u_tu_{tt}+c^2u_xu_{xt}dx\end{aligned}$$

But $u_{tt}=c^2u_{xx}-au_t$ from the PDE itself, hence the above simplifies to

$$\begin{aligned}\frac{d}{dt}E(t)&=\int_{-\infty}^{\infty}u_t\left(c^2u_{xx}-au_t\right)+c^2u_xu_{xt}dx\\&=\int_{-\infty}^{\infty}c^2u_tu_{xx}-au_t^2+c^2u_xu_{xt}dx\\&=c^2\int_{-\infty}^{\infty}u_tu_{xx}+u_xu_{xt}dx-a\int_{-\infty}^{\infty}u_t^2dx\end{aligned}$$

But $\int_{-\infty}^{\infty}u_tu_{xx}+u_xu_{xt}dx=\frac{d}{dx}(u_tu_x)$, then the above becomes

$$\begin{aligned}\frac{d}{dt}E(t)&=c^2\int_{-\infty}^{\infty}\frac{d}{dx}(u_tu_x)\,dx-a\int_{-\infty}^{\infty}u_t^2dx\\&=c^2\int_{-\infty}^{\infty}d(u_tu_x)\,dx-a\int_{-\infty}^{\infty}u_t^2dx\\&=c^2\left[u_tu_x\right]_{-\infty}^{\infty}-a\int_{-\infty}^{\infty}u_t^2dx\end{aligned}$$

As in the previous problem $\left[u_tu_x\right]_{-\infty}^{\infty}=0$ since $u_x\to 0$ for $x\to\pm\infty$. Then the above now reduces to

$$\frac{d}{dt}E(t)=-a\int_{-\infty}^{\infty}u_t^2dx$$

But $\int_{-\infty}^{\infty}u_t^2dx$ is either zero or positive because the integrand is always positive.

Hence $\frac{d}{dt}E(t)$ is negative quantity because $a>0$. This shows that rate of change of energy is either zero or negative and can not be positive. This means $E(t)$ is non increasing which is what we are asked to show.

2.2.11.2 Part (b)

Let $u_1(t,x)$ and $u_2(t,x)$ be two different solutions to same $u_{tt}+au_t=c^2u_{xx}$ with same initial data. Let $w(t,x)=u_1(t,x)-u_2(t,x)$. Therefore

$$w_{tt}+aw_t=c^2w_{xx}$$

Applying the energy formula to $w(t,x)$ shows that

$$\begin{aligned} E(t) &= \int_{-\infty}^{\infty}\frac{1}{2}\left((w_t)^2+c^2(w_x)^2\right)dx \\ \frac{dE}{dt} &= \frac{d}{dt}\int_{-\infty}^{\infty}\frac{1}{2}\left((w_t)^2+c^2(w_x)^2\right)dx \end{aligned}$$

Following same steps in problem 2.4.13, the above becomes zero. Which means that $\frac{dE}{dt}=0$ or $E(t)$ is constant. But $E(-\infty)=E(\infty)=0$ which means that $E(t)=0$. In other words

$$\int_{-\infty}^{\infty}\frac{1}{2}\left((w_t)^2+c^2(w_x)^2\right)dx=0$$

But since the integrand is positive, then this means $w_t=0$ and $w_x=0$. But this implies that $w(t,x)$ is itself a constant.

We now need to show that this constant is zero. i.e. to show that $w(t,x)=0$ to finish the proof.

Since $w(0,x)=0$, because this is the initial data, which is the difference between the initial data of the two solutions u_1,u_2 which is the same, hence the difference of the initial data is zero.

But if $w(0,x)=0$ and $w(t,x)$ is constant, it must be that $w(t,x)=0$ for all time and space.

But since $w(t,x)=u_1(t,x)=u_2(t,x)$ then

$$u_1(t,x)=u_2(t,x)$$

Which mean that the solution to the telegraph PDE is unique.

2.2.12 Key solution for HW 2

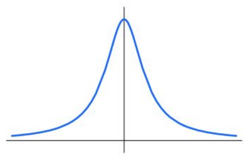
Homework 2 Solutions

2.2.17

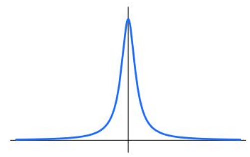
2.2.17. (a) $u(t,x) = \dfrac{1}{(x\,e^t)^2 + 1} = \dfrac{e^{-2t}}{x^2 + e^{-2t}}$.

(b)

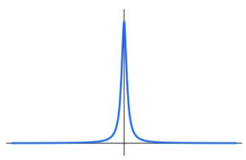
$t = 0$:



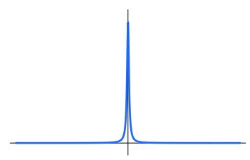
$t = 1$:



$t = 2$:



$t = 3$:



(c) The limit is discontinuous: $\displaystyle\lim_{t\to\infty} u(t,x) = \begin{cases} 1, & x = 0, \\ 0, & \text{otherwise.} \end{cases}$

2.2.18

(a) $\displaystyle\lim_{t\to\infty} u(t,x) = \begin{cases} 0, & x < -1, \\ f(-1), & x = -1, \\ f(1), & x > -1. \end{cases}$ (b) $\displaystyle\lim_{t\to-\infty} u(t,x) = \begin{cases} f(-1), & x < 1, \\ f(1), & x = 1, \\ 0, & x > 1. \end{cases}$

2.2.26

(a) Suppose $x = x(t)$ solves $\dfrac{dx}{dt} = c(t,x)$. Then, by the chain rule,

$$\frac{d}{dt}\,u\big(t,x(t)\big) = \frac{\partial u}{\partial t}\big(t,x(t)\big) + \frac{\partial u}{\partial x}\big(t,x(t)\big)\,\frac{dx}{dt} = \frac{\partial u}{\partial t}\big(t,x(t)\big) + c\big(t,x(t)\big)\,\frac{\partial u}{\partial x}\big(t,x(t)\big) = 0,$$

since we are assuming that $u(t,x)$ is a solution to the transport equation for all (t,x). We conclude that $u\big(t,x(t)\big)$ is constant.

(b) Since $\xi(t,x) = k$ implicitly defines a solution $x(t)$ to the characteristic equation,

$$0 = \frac{d}{dt}\,\xi\big(t,x(t)\big) = \frac{\partial \xi}{\partial t}\big(t,x(t)\big) + \frac{\partial \xi}{\partial x}\big(t,x(t)\big)\,\frac{dx}{dt} = \frac{\partial \xi}{\partial t}(t,x) + c(t,x)\,\frac{\partial \xi}{\partial x}(t,x),$$

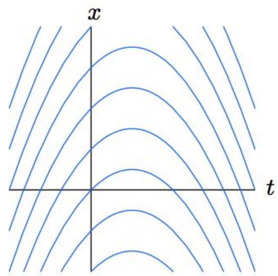
and hence $u = \xi(t,x)$ is a solution to the transport equation. Moreover, if $u(t,x) = f\big(\xi(t,x)\big)$, by the chain rule,

$$\frac{\partial u}{\partial t}(t,x) + c(t,x)\,\frac{\partial u}{\partial x}(t,x) = f'\big(\xi(t,x)\big)\left(\frac{\partial \xi}{\partial t}(t,x) + c(t,x)\,\frac{\partial \xi}{\partial x}(t,x)\right) = 0$$

according to the previous computation.

2.2.29

2.29. (a) Solving the characteristic equation $\dfrac{dx}{dt} = 1 - 2t$ produces the characteristic curves $x = t - t^2 + k$, where k is an arbitrary constant.



(b) The general solution is $u(t, x) = v(x - t + t^2)$, where $v(\xi)$ is an arbitrary C^1 function of the characteristic variable $\xi = x - t + t^2$.

(c) $u(t, x) = \dfrac{1}{1 + (x - t + t^2)^2}$.

(d) The solution is a hump of fixed shape that, as t increases, first moves to the right, slowing down and stopping at $t = \frac{1}{2}$, and then moving back to the left, at an ever accelerating speed. As $t \to -\infty$, the hump moves back to the left, accelerating.

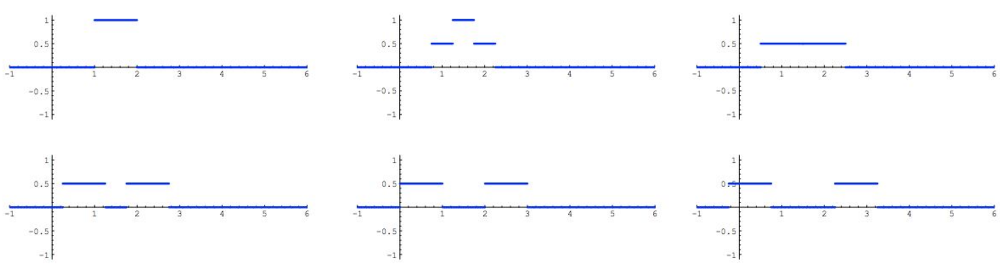
2.4.2

(a) The initial displacement splits into two half sized replicas, moving off to the right and to the left with unit speed.

For $t < \frac{1}{2}$, we have $u(t,x) = \begin{cases} 1, & 1+t < x < 2-t, \\ \frac{1}{2}, & 1-t < x < 1+t \quad \text{or} \quad 2-t < x < 2+t, \\ 0, & \text{otherwise,} \end{cases}$

For $t \geq \frac{1}{2}$, we have $u(t,x) = \begin{cases} \frac{1}{2}, & 1-t < x < 2-t \quad \text{or} \quad 1+t < x < 2+t, \\ 0, & \text{otherwise,} \end{cases}$

(b) Plotted at times $t = 0, .25, .5, .75, 1., 1.25$:



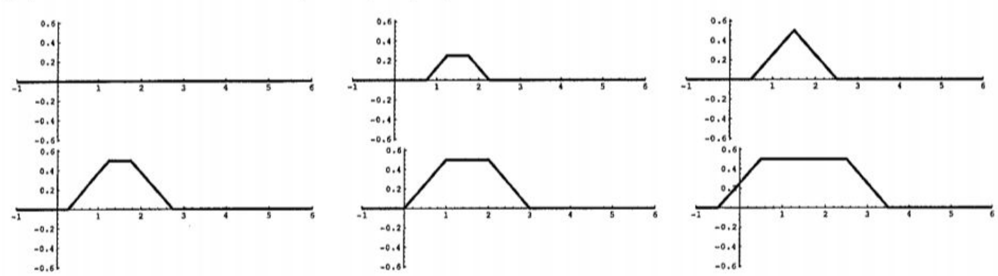
2.4.3

(a) The solution initially forms a trapezoidal displacement, with linearly growing height and sides of slope $\pm.5$ expanding in both directions from 1 and 2 at unit speed. At time $t = .5$, the height reaches .5, and it momentarily forms a triangle. After this the diagonal sides propagate to the right and to the left with unit speed, as the .5 displacement between then grows in extent.

$$\text{For } t < \tfrac{1}{2}\text{, we have } u(t,x) = \begin{cases} \frac{1}{2}(x-1+t), & 1-t<x<1+t, \\ t & 1+t<x<2-t, \\ \frac{1}{2}(2+t-x), & 2-t<x<2+t, \\ 0, & \text{otherwise}, \end{cases}$$

$$\text{For } t \geq \tfrac{1}{2}\text{, we have } u(t,x) = \begin{cases} \frac{1}{2}(x-1+t), & 1-t<x<2-t, \\ \frac{1}{2} & 2-t<x<1+t, \\ \frac{1}{2}(2+t-x), & 1+t<x<2+t, \\ 0, & \text{otherwise}, \end{cases}$$

(b) Plotted at times $t = 0, ..25.5, .75, 1., 1.5$:



2.4.4b, d

$$(b)\ \ \frac{1}{2}\int_{x-t}^{x+t} 2\cos(2\,z)\,dz = \frac{\sin 2\,(x+t) - \sin 2\,(x-t)}{2}; \quad \star\ (d)\ \ \frac{(x+t)^2 + (x-t)^2}{2}.$$

2.4.10

Solution: The solution to the initial value problem is

$$u(t,x) = \begin{cases} \dfrac{\omega\sin 2t - 2\sin\omega t}{2(\omega^2-4)}\cos x, & \omega \neq \pm 2, \\ \frac{1}{8}(\sin 2t - 2t\cos 2t)\cos x, & \omega = \pm 2. \end{cases}$$

Thus,

$$g(t) = u(t,0) = \begin{cases} \dfrac{\omega\sin 2t - 2\sin\omega t}{2(\omega^2-4)}, & \omega \neq \pm 2, \\ \frac{1}{8}(\sin 2t - 2t\cos 2t), & \omega = \pm 2, \end{cases}$$

is periodic when $\omega \neq \pm 2$ is a rational number, quasi-periodic when ω is irrational, and non-periodic and resonant when $\omega = \pm 2$.

2.4.11

Solution: (a) $u(t,x) = \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t)$; (b) True;
(c) $u(t,x) = \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4} - \frac{1}{4}\cos 2t$.
(d) The solution remains bounded and periodic, and hence is not resonant.
(e) Now the solution is $u(t,x) = \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4}t - \frac{1}{4}\sin 2t$. In this case, the solution is no longer periodic or bounded, and hence a form of resonance is exhibited.

2.4.13

First of all, the decay assumption implies that $E(t) < \infty$ for all t. To show $E(t)$ is constant, we prove that its derivative is 0. Using the smoothness of the solution to justify bringing the derivative under the integral sign, we compute

$$\frac{dE}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} \left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}c^2u_x^2\right)dx = \int_{-\infty}^{\infty}\left(u_t u_{tt} + c^2 u_x u_{xt}\right)dx$$
$$= c^2\int_0^{\ell}\left(u_t u_{xx} + u_x u_{xt}\right)dx = c^2\int_{-\infty}^{\infty}\frac{d}{dx}\left(u_t u_x\right)dx = 0,$$

since $u_t, u_x \to 0$ as $x \to \infty$. *Q.E.D.*

2.4.15

(a) As in Exercise 2.4.13, we compute

$$\frac{dE}{dt} = \int_{-\infty}^{\infty}\left(u_t u_{tt} + c^2 u_x u_{xt}\right)dx = \int_{-\infty}^{\infty}\left[c^2(u_t u_{xx} + u_x u_{xt}) - a\,u_t^2\right]dx$$
$$= c^2\int_{-\infty}^{\infty}\frac{d}{dx}\left(u_t u_x\right)dx - a\int_{-\infty}^{\infty}u_t^2\,dx = -\beta\int_{-\infty}^{\infty}u_t^2\,dx \le 0,$$

since $a > 0$. Thus, $E(t)$ is a nonincreasing function of t.

(b) First, let $u(t,x)$ be the solution to the initial-boundary value problem with zero initial conditions, and hence zero initial energy: $E(0) = 0$. Since $0 \le E(t) \le E(0)$ is decreasing, and nonnegative, we conclude that $E(t) \equiv 0$. But since the energy integrand is nonnegative, this can only happen if $u_t = u_x = 0$ for all (t,x), and hence $u(t,x)$ must be a constant function. Moreover, its initial value is $u(0,x) = 0$, and hence $u(t,x) \equiv 0$. With this in hand, in order to prove uniqueness, suppose $u_1(t,x)$ and $u_2(t,x)$ are two solutions to the initial-boundary value problem. Then, by linearity, their difference $u(t,x) = u_1(t,x) - u_2(t,x)$ solves the homogeneous initial-boundary value problem analyzed in part (a), and so must be identically zero: $u(t,x) \equiv 0$. This implies $u_1(t,x) = u_2(t,x)$ for all (t,x), and hence there is at most one solution.

2.3 HW 3

Local contents

2.3.1 Problem 3.1.2

Find all separable eigensolutions to the heat equation $u_t = u_{xx}$ on $0 \leq x \leq \pi$ subject to (a) homogeneous boundary conditions $u(t,0) = 0, u(t,\pi) = 0$. (b) mixed boundary conditions $u(t,0) = 0, u_x(t,\pi) = 0$

solution

Using separation of variables, let $u(t,x) = T(t)X(x)$. Substituting this into $u_t = u_{xx}$ gives $T'X = TX''$. Dividing by $XT \neq 0$ results in

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

Where λ is the seperation constant. The above gives the following ODE's to solve

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \\ T'(t) + \lambda T(t) &= 0 \end{aligned}$$

The boundary and initial conditions are transfered from the PDE to the ODE as shown below.

2.3.1.1 Part (a)

Using $u(t,0) = 0, u(t,\pi) = 0$. Starting with the spatial ODE, and transferring the boundary conditions to the ODE results in

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \\ X(0) &= 0 \\ X(\pi) &= 0 \end{aligned}$$

This is an eigenvalue boundary value ODE. The solution to the spatial ODE is

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \tag{1}$$

<u>case $\lambda < 0$</u>

Since $\lambda < 0$, then $-\lambda$ is positive. Let $\mu = -\lambda$, where μ is positive. The above solution becomes

$$X(x) = c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x}$$

Which can be written as

$$X(x) = c_1 \cosh\left(\sqrt{\mu}x\right) + c_2 \sinh\left(\sqrt{\mu}x\right)$$

At $x = 0$ this gives

$$0 = c_1$$

The solution now reduces to $X(x) = c_2 \sinh\left(\sqrt{\mu}x\right)$. At $x = \pi$ this gives

$$0 = c_2 \sinh\left(\sqrt{\mu}\pi\right)$$

But $\sinh$ is only zero when its argument is zero. Since $\mu \neq 0$, then the only choice is that $c_2 = 0$ also. But this gives trivial solution therefore $\lambda < 0$ is not an eigenvalue.

<u>case $\lambda = 0$</u>

In this case the solution is $X(x) = c_1 + c_2 x$. At $x = 0$ this gives $0 = c_1$. The solution becomes $X(x) = c_2 x$. At $x = \pi$, this gives $0 = c_2 \pi$. Therefore $c_2 = 0$ also. This also gives the trivial solution. Hence $\lambda = 0$ is not an eigenvalue.

<u>case $\lambda > 0$</u>

The solution in this case is

$$\begin{aligned} X(x) &= c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \\ &= c_1 e^{i\sqrt{\lambda}x} + c_2 e^{-i\sqrt{\lambda}x} \end{aligned}$$

Which can be rewritten as (the constants c_1, c_2 below will be different than the above c_1, c_2, but kept the same name for simplicity).

$$X(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$$

At $x = 0$ this gives

$$0 = c_1$$

The solution now reduces to

$$X(x) = c_2 \sin\left(\sqrt{\lambda}x\right)$$

At $x = \pi$ this gives

$$0 = c_2 \sin\left(\sqrt{\lambda}\pi\right)$$

non-trivial solution requires that $\sin\left(\sqrt{\lambda}\pi\right) = 0$ which implies that $\sqrt{\lambda}\pi = n\pi, n = 1, 2, 3, \cdots$. Hence eigenvalues are

$$\lambda_n = n^2 \qquad n = 1, 2, 3, \cdots$$

And corresponding eigenfunctions are

$$\boxed{X_n(x) = \sin(nx) \qquad n = 1, 2, 3, \cdots}$$

Now that the eigenvalues and eigenfunction are found, the time ODE can be solved. The time ODE now becomes

$$T'(t) + n^2 T(t) = 0$$

This is linear first order ode. The solution is $T_n(t) = C_n e^{-n^2 t}$. Therefore the fundamental solution is

$$\begin{aligned} u_n(t, x) &= C_n T_n(t) X_n(x) \\ &= C_n e^{-n^2 t} \sin(nx) \end{aligned}$$

Since this is a linear PDE, a linear combination of all fundamental solutions is a solution. Hence the general solution is

$$u(t, x) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} \sin(nx)$$

The constant C_n can be found if initial conditions are given.

2.3.1.2 Part (b)

Using $u(t, 0) = 0, u_x(t, \pi) = 0$. Starting with the spatial ODE, and transferring the boundary condition to X, it becomes

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \\ X(0) &= 0 \\ X'(\pi) &= 0 \end{aligned}$$

This is an eigenvalue boundary value problem. The solution to the spatial ODE is

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \tag{1}$$

<u>case $\lambda < 0$</u>

Since $\lambda < 0$, then $-\lambda$ is positive. Let $\mu = -\lambda$, where μ is positive. The solution becomes

$$X(x) = c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x}$$

The above can be written as

$$X(x) = c_1 \cosh\left(\sqrt{\mu}x\right) + c_2 \sinh\left(\sqrt{\mu}x\right)$$

At $x = 0$ this gives

$$0 = c_1$$

Hence the solution now becomes

$$X(x) = c_2 \sinh\left(\sqrt{\mu}x\right)$$

Taking derivative gives

$$X'(x) = c_2\sqrt{\mu} \cosh\left(\sqrt{\mu}x\right)$$

And at $x = \pi$ the above gives

$$0 = c_2\sqrt{\mu} \cosh\left(\sqrt{\mu}\pi\right)$$

But $\mu \neq 0$ and $\cosh$ is never zero for any argument. Hence the only choice is that $c_2 = 0$. This gives the trivial solution. Hence $\lambda < 0$ is not an eigenvalue.

case $\lambda = 0$

In this case the solution is $X(x) = c_1 + c_2 x$. At $x = 0$ this results in $0 = c_1$. The solution becomes $X(x) = c_2 x$. Hence $X'(x) = c_2$. At $x = \pi$, this implies $0 = c_2\pi$. Therefore $c_2 = 0$ also. This gives the trivial solution. Hence $\lambda = 0$ is not an eigenvalue.

case $\lambda > 0$

The solution in this case is

$$\begin{aligned} X(x) &= c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \\ &= c_1 e^{i\sqrt{\lambda}x} + c_2 e^{-i\sqrt{\lambda}x} \end{aligned}$$

Which can be rewritten as (the constants c_1, c_2 below will be different than the above c_1, c_2, but kept the same name for simplicity).

$$X(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$$

At $x = 0$ this gives

$$0 = c_1$$

The solution now reduces to

$$X(x) = c_2 \sin\left(\sqrt{\lambda}x\right)$$

Therefore

$$X'(x) = \sqrt{\lambda} c_2 \cos\left(\sqrt{\lambda} x\right)$$

At $x = \pi$

$$0 = \sqrt{\lambda} c_2 \cos\left(\sqrt{\lambda} \pi\right)$$

Non-trivial solution requires that $\cos\left(\sqrt{\lambda}\pi\right) = 0$, which implies $\sqrt{\lambda}\pi = \frac{n\pi}{2}, n = 1, 3, 5, \cdots$. or $\sqrt{\lambda} = \frac{n}{2}, n = 1, 3, 5, \cdots$. Therefore the eigenvalues are

$$\lambda_n = \left(\frac{n}{2}\right)^2 \qquad n = 1, 3, 5, \cdots$$

Or

$$\lambda_n = \left(\frac{2n-1}{2}\right)^2 \qquad n = 1, 2, 3, \cdots$$

Few eigenvalues are $\lambda = \left\{\frac{1}{4}, \frac{9}{4}, \frac{25}{4}, \cdots\right\}$. The corresponding eigenfunctions are

$$\boxed{X_n(x) = \sin\left(\frac{2n-1}{2} x\right) \qquad n = 1, 2, 3, \cdots}$$

Now that the eigenvalues and eigenfunction are found, the time ODE is solved. The time ODE now becomes

$$T'(t) + \left(\frac{2n-1}{2}\right)^2 T(t) = 0$$

This is linear first order ode. The solution is $T_n(t) = C_n e^{-\left(\frac{2n-1}{2}\right)^2 t}$. Therefore the fundamental solution is

$$\begin{aligned} u_n(t, x) &= C_n T_n(t) X_n(x) \\ &= C_n e^{-\left(\frac{2n-1}{2}\right)^2 t} \sin\left(\frac{2n-1}{2} x\right) \end{aligned}$$

A linear combination of all fundamental solution is a solution (due to linearity). Hence the general solution is

$$u(t, x) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{2n-1}{2}\right)^2 t} \sin\left(\frac{2n-1}{2} x\right)$$

2.3.2 Problem 3.1.5

(a) Find the real eigensolutions to the damped heat equation $u_t = u_{xx} - u$. (b) Which solutions satisfy the periodic boundary conditions $u(t, -\pi) = u(t, \pi), u_x(t, -\pi) = u_x(t, \pi)$?

solution

2.3.2.1 Part (a)

Using separation of variables, Let $u(t,x) = T(t)X(x)$. Substituting this into $u_t + u = u_{xx}$ gives $T'X + TX = TX''$. Dividing by $XT \neq 0$ gives

$$\frac{T'}{T} + 1 = \frac{X''}{X} = -\lambda$$

Where λ is the separation constant. This gives the following ODE's to solve

$$X''(x) + \lambda X(x) = 0$$
$$T'(t) + (\lambda + 1)T(t) = 0$$

Eigenfunctions are solutions to the spatial ODE.

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \tag{1}$$

To determine the actual eigenfunctions and eigenvalues, boundary conditions are used. This is part b below.

2.3.2.2 Part (b)

Using $u(t,-\pi) = u(t,\pi), u_x(t,-\pi) = u_x(t,\pi)$. Starting with the spatial ODE above, and transferring the boundary condition to X gives

$$X''(x) + \lambda X(x) = 0$$
$$X(-\pi) = X(\pi)$$
$$X'(-\pi) = X'(\pi)$$

This is an eigenvalue boundary value problem. The solution is

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \tag{1}$$

case $\lambda < 0$

Since $\lambda < 0$, then $-\lambda$ is positive. Let $\mu = -\lambda$, where μ is now positive. The solution (1) becomes

$$X(x) = c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x}$$

The above can be written as

$$X(x) = c_1 \cosh\left(\sqrt{\mu}x\right) + c_2 \sinh\left(\sqrt{\mu}x\right) \tag{2}$$

Applying first B.C. $X(-\pi) = X(\pi)$ using (2) gives

$$c_1 \cosh\left(\sqrt{\mu}\pi\right) + c_2 \sinh\left(-\sqrt{\mu}\pi\right) = c_1 \cosh\left(\sqrt{\mu}\pi\right) + c_2 \sinh\left(\sqrt{\mu}\pi\right)$$
$$c_2 \sinh\left(-\sqrt{\mu}\pi\right) = c_2 \sinh\left(\sqrt{\mu}\pi\right)$$

But $\sinh$ is only zero when its argument is zero which is not the case here. Therefore the above implies that $c_2 = 0$ as only possibility to satisfy the above equation. The solution (2) now reduces to

$$X(x) = c_1 \cosh\left(\sqrt{\mu}x\right) \tag{3}$$

Taking derivative

$$X'(x) = c_1\sqrt{\mu}\sinh\left(\sqrt{\mu}x\right) \tag{4}$$

Applying the second BC $X'(-\pi) = X'(\pi)$ gives

$$c_1\sqrt{\mu}\sinh\left(-\sqrt{\mu}\pi\right) = c_1\sqrt{\mu}\sinh\left(\sqrt{\mu}x\right)$$

But $\sinh$ is only zero when its argument is zero which is not the case here. Therefore the above implies that $c_1 = 0$. This means a trivial solution. Therefore $\lambda < 0$ is not an eigenvalue.

case $\lambda = 0$

In this case the solution is $X(x) = c_1 + c_2 x$. Applying first BC $X(-\pi) = X(\pi)$ gives

$$\begin{aligned} c_1 - c_2\pi &= c_1 + c_2\pi \\ -c_2\pi &= c_2\pi \end{aligned}$$

This gives $c_2 = 0$. The solution now becomes

$$X(x) = c_1$$

Therefore $X'(x) = 0$. Applying the second boundary conditions $X'(-\pi) = X'(\pi)$ is now satisfied for any c_1, since it gives $(0 = 0)$. Therefore $\underline{\lambda = 0}$ is an eigenvalue with eigenfunction $X_0(0) = 1$ (selecting $c_1 = 1$ since any arbitrary constant will work).

case $\lambda > 0$

The solution in this case is

$$\begin{aligned} X(x) &= c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \\ &= c_1 e^{i\sqrt{\lambda}x} + c_2 e^{-i\sqrt{\lambda}x} \end{aligned}$$

Which can be rewritten as (the constants c_1, c_2 below will be different than the above c_1, c_2, but kept the same name for simplicity).

$$X(x) = c_1\cos\left(\sqrt{\lambda}x\right) + c_2\sin\left(\sqrt{\lambda}x\right) \tag{5}$$

Applying first B.C. $X(-\pi) = X(\pi)$ using the above gives

$$\begin{aligned} c_1\cos\left(\sqrt{\lambda}\pi\right) + c_2\sin\left(-\sqrt{\lambda}\pi\right) &= c_1\cos\left(\sqrt{\lambda}\pi\right) + c_2\sin\left(\sqrt{\lambda}\pi\right) \\ c_2\sin\left(-\sqrt{\lambda}\pi\right) &= c_2\sin\left(\sqrt{\lambda}\pi\right) \end{aligned}$$

There are two choices here. Either $c_2 = 0$ or $\sqrt{\lambda}\pi = n\pi, n = 1,2,3,\cdots$. Using the second choice for now, which implies that

$$\lambda_n = n^2 \qquad n = 1,2,3,\cdots$$

And now we will now look to see what happens using the second BC with the above choice. The solution (5) now becomes

$$X(x) = c_1\cos(nx) + c_2\sin(nx) \qquad n = 1,2,3,\cdots$$

Therefore

$$X'(x) = -c_1 n\sin(nx) + c_2 n\cos(nx)$$

Applying the second BC $X'(-\pi) = X'(\pi)$ using the above gives

$$\begin{aligned} c_1 n \sin(n\pi) + c_2 n \cos(n\pi) &= -c_1 n \sin(n\pi) + c_2 n \cos(n\pi) \\ c_1 n \sin(n\pi) &= -c_1 n \sin(n\pi) \\ 0 &= 0 \end{aligned}$$

Since n is integer.

Therefore this means that using the choice $\lambda_n = n^2$ satisfied both boundary conditions with $c_2 \neq 0, c_1 \neq 0$. This means the solution (5) is

$$X_n(x) = A_n \cos(nx) + B_n \sin(nx) \qquad n = 1, 2, 3, \cdots$$

The above says that there are two eigenfunctions in this case. They are

$$X_n(x) = \begin{cases} \cos(nx) \\ \sin(nx) \end{cases}$$

Recalling that there is also a zero eigenvalue with constant as its eigenfunction, then the complete set of eigenfunctions is

$$X_n(x) = \begin{cases} 1 \\ \cos(nx) \\ \sin(nx) \end{cases}$$

Now that the eigenvalues are found, the solution to the time ODE can be found. The time ODE from above was found to be

$$T'(t) + (\lambda + 1) T(t) = 0$$

For the zero eigenvalue case, the above reduces to $T'(t) + T(t) = 0$ which has the solution $T_0(t) = C_0 e^{-t}$. For non zero eigenvalues $\lambda_n = n^2$, the ODE becomes $T'(t) + \left(n^2 + 1\right) T(t) = 0$, whose solution is $T_0(t) = C_n e^{-\left(n^2+1\right)t}$.

Putting all the above together, gives the fundamental solution as

$$u_n(t,x) = \begin{cases} C_0 e^{-t} & \\ C_n \cos(nx) e^{-\left(n^2+1\right)t} & n = 1, 2, 3, \cdots \\ B_n \sin(nx) e^{-\left(n^2+1\right)t} & n = 1, 2, 3, \cdots \end{cases}$$

The complete solution is the sum of the above solutions

$$u(t,x) = C_0 e^{-t} + \sum_{n=1}^{\infty} e^{-\left(n^2+1\right)t} \left(C_n \cos(nx) + B_n \sin(nx)\right)$$

The constants C_0, C_n, B_n can be found from initial conditions.

2.3.3 Problem 3.2.1

(d) Find the Fourier series of the following functions $f(x) = x^2$ (using $-\pi \leq x \leq \pi$)

solution

The Fourier series is given by

$$x^2 \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T} nx\right) + a_n \sin\left(\frac{2\pi}{T} nx\right)$$

Where T is the period of $f(x)$. Taking this period to be 2π, the above simplifies to

$$x^2 \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

The function x^2 is even, hence all b_n are zero. The above becomes

$$x^2 \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \tag{1}$$

But

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx \\
&= \frac{2}{\pi} \int_{0}^{\pi} x^2 dx \\
&= \frac{2}{\pi} \left[\frac{x^3}{3}\right]_0^{\pi} \\
&= \frac{2}{3\pi} \pi^3 \\
&= \frac{2}{3} \pi^2
\end{aligned}$$

And

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx)\, dx \\
&= \frac{2}{\pi} \int_{0}^{\pi} x^2 \cos(nx)\, dx
\end{aligned} \tag{1A}$$

Let $I = \int_0^{\pi} x^2 \cos(nx)\, dx$. Using integration by parts $\int u dv = uv - \int v du$. Let $u = x^2, dv = \cos(nx)$. Then $du = 2x, v = \frac{\sin(nx)}{n}$. Hence

$$\begin{aligned}
I &= \left[x^2 \frac{\sin(nx)}{n}\right]_0^{\pi} - 2 \int_0^{\pi} x \frac{\sin(nx)}{n} dx \\
&= \overbrace{\frac{1}{n} \left[x^2 \sin(nx)\right]_0^{\pi}}^{0} - \frac{2}{n} \int_0^{\pi} x \sin(nx)\, dx \\
&= -\frac{2}{n} \int_0^{\pi} x \sin(nx)\, dx
\end{aligned}$$

Integration by parts again. $u = x, dv = \sin(nx)$, then $du = 1, v = -\frac{\cos(nx)}{n}$. The above becomes

$$\begin{aligned}
I &= -\frac{2}{n}\left(\left[-x\frac{\cos(nx)}{n}\right]_0^\pi - \int_0^\pi -\frac{\cos(nx)}{n}dx\right)\\
&= -\frac{2}{n}\left(-\frac{1}{n}\left[x\cos(nx)\right]_0^\pi + \frac{1}{n}\int_0^\pi \cos(nx)\,dx\right)\\
&= \frac{2}{n^2}\left(\left[x\cos(nx)\right]_0^\pi - \int_0^\pi \cos(nx)\,dx\right)\\
&= \frac{2}{n^2}\left(\left[\pi\cos(n\pi)\right] - \left[\frac{\sin(nx)}{n}\right]_0^\pi\right)\\
&= \frac{2\pi}{n^2}\cos(n\pi)\\
&= \frac{2\pi}{n^2}(-1)^n
\end{aligned}$$

The above is I. Substituting this result back in (1A) gives

$$\begin{aligned}
a_n &= \frac{2}{\pi}I\\
&= \frac{2}{\pi}\frac{2\pi}{n^2}(-1)^n\\
&= \frac{4}{n^2}(-1)^n
\end{aligned}$$

Therefore (1) becomes

$$x^2 \sim \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$$

To verify this result, the Fourier series was compared to x^2 for an increasing number of terms to see if it converged to x^2. Here is the result. This shows the convergence is fast, after 6 terms only the approximation (in red color) is almost the same as the original function x^2.

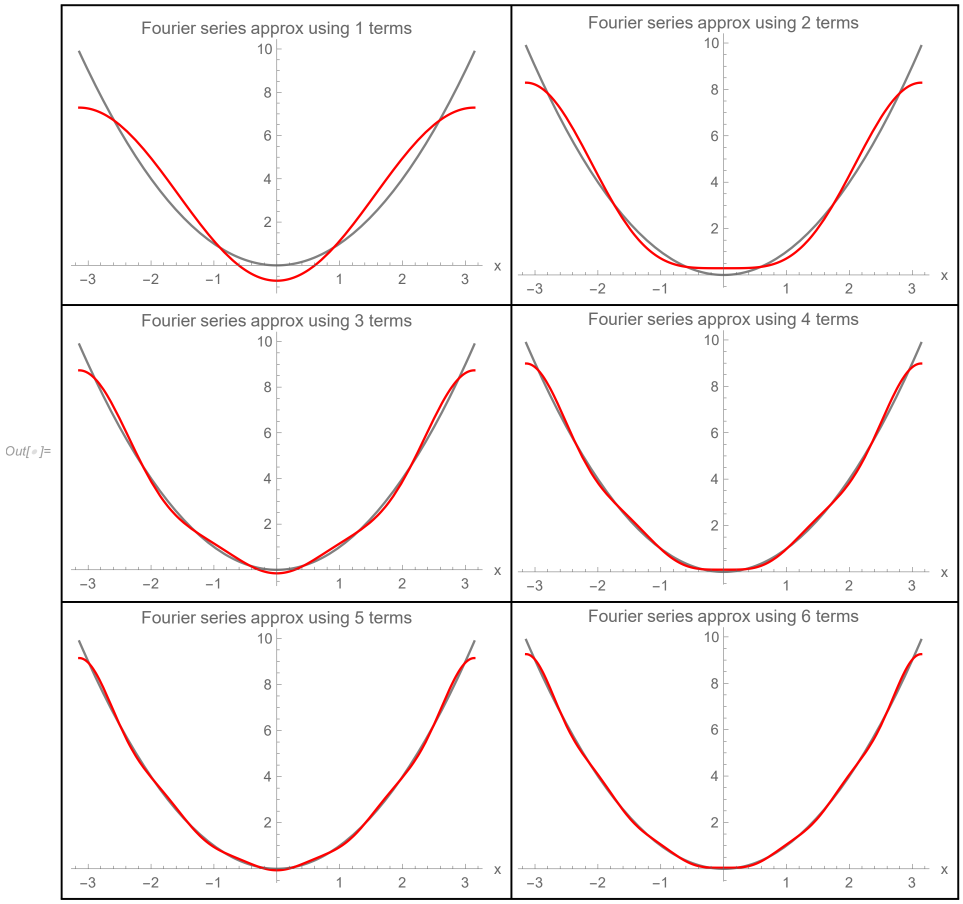


Figure 2.10: Fourier series of x^2

```
fs[x_, max_] := 1/3 π^2 + 4 Sum[(-1)^n/n^2 Cos[n x], {n, 1, max}]
makePlot[n_] := Plot[{x^2, fs[x, n]}, {x, -Pi, Pi},
          PlotStyle → {Gray, Red}, AxesLabel → {"x", None},
          PlotLabel → Row[{"Fourier series approx using ", n, " terms"}],
          ImageSize → 300
       ];
Grid[Partition[Table[makePlot[n], {n, {1, 2, 3, 4, 5, 6}}], 2],
   Frame → All]
```

Figure 2.11: Code used for the above plot

the following plot shows how the Fourier series approximation to x^2 when it is periodically extended to outside $[-\pi, \pi]$. This uses the range $[-3\pi, 3\pi]$ by adding one period to left and one period to the right.

```
In[]:= fs[x_, max_] := 1/3 π^2 + 4 Sum[(-1)^n/n^2 Cos[n x], {n, 1, max}]
fx[x_] := Piecewise[{
          {(x + 2 Pi)^2, x < -Pi},
          {x^2, -Pi < x < Pi},
          {(x - 2 Pi)^2, x > Pi}}];
makePlot[n_] := Plot[{fx[x], fs[x, n]}, {x, -3 Pi, 3 Pi},
        PlotStyle → {Gray, Red}, AxesLabel → {"x", None},
        PlotLabel → Row[{"Fourier series approx using ", n, " terms"}],
        ImageSize → 300
      ];
Grid[Partition[Table[makePlot[n], {n, {1, 2, 3, 4, 5, 6}}], 2],
  Frame → All]
```

Figure 2.12: Code used for the above plot

2.3.4 Problem 3.2.2

(d) Find the Fourier series of the following function $f(x) = \begin{cases} x & |x| < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$

<u>solution</u>

This is plot showing $f(x)$

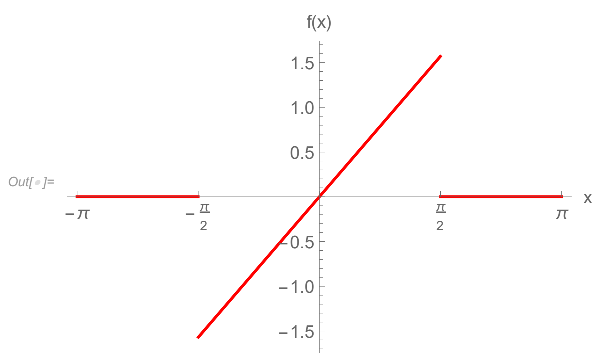


Figure 2.13: Plot of $f(x)$

The Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T} nx\right) + a_n \sin\left(\frac{2\pi}{T} nx\right)$$

Where T is the period of the function to be approximated. Taking this period to be 2π, the above simplifies to

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

The function $f(x)$ is odd then all a_n will zero. The above simplifies to

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx)$$

Where

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\
&= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin(nx)\,dx
\end{aligned}$$

But x is odd and $\sin(x)$ is odd, hence the product is even. The above simplifies to

$$b_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin(nx)\,dx$$

Using integration by parts $\int udv = uv - \int vdu$. Let $x = u, du = 1, dv = \sin(nx), v = \frac{-\cos(nx)}{n}$, the above gives

$$\begin{aligned}
b_n &= \frac{2}{\pi}\left(\frac{-1}{n}\left[x\cos(nx)\right]_0^{\frac{\pi}{2}} + \frac{1}{n}\int_0^{\frac{\pi}{2}}\cos(nx)\,dx\right) \\
&= \frac{2}{\pi n}\left(-\left[x\cos(nx)\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\cos(nx)\,dx\right) \\
&= \frac{2}{\pi n}\left(-\left[\frac{\pi}{2}\cos\left(n\frac{\pi}{2}\right)\right] + \frac{1}{n}\left[\sin(nx)\right]_0^{\frac{\pi}{2}}\right) \\
&= \frac{2}{\pi n}\left(-\left[\frac{\pi}{2}\cos\left(n\frac{\pi}{2}\right)\right] + \frac{1}{n}\left[\sin\left(n\frac{\pi}{2}\right)\right]\right) \\
&= \frac{2}{\pi n^2}\left(\sin\left(n\frac{\pi}{2}\right) - \frac{n\pi}{2}\cos\left(n\frac{\pi}{2}\right)\right)
\end{aligned}$$

Therefore the Fourier series becomes

$$f(x) \sim \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left(\sin\left(\frac{n\pi}{2}\right) - \frac{1}{2}n\pi\cos\left(\frac{n\pi}{2}\right)\right)\sin(nx)$$

To verify this result, the Fourier series was compared to $f(x)$ for increasing number of terms to see if it converges to x^2. Here is the result. This shows the convergence is fast, but not as fast as last problem due to jump discontinuity in $f(x)$. 10 terms are used below.

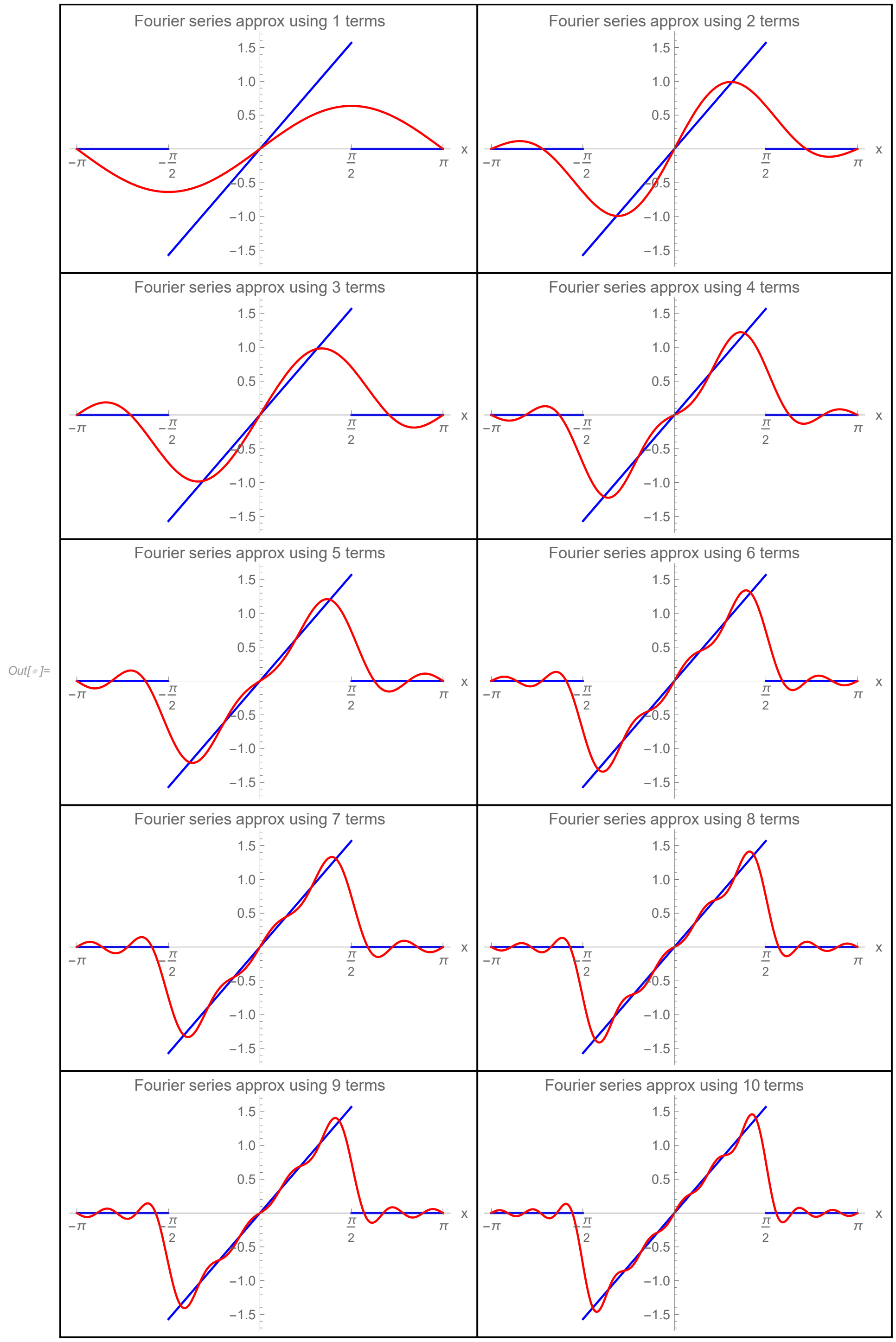


Figure 2.14: Fourier series approximation of $f(x)$

```
In[•]:= fs[x_, max_] := Sum[2/(n^2 π) (Sin[n π/2] - 1/2 n π Cos[n π/2]) Sin[n x], {n, 1, max}];
f[x_] := Piecewise[{{x, Abs[x] < Pi / 2}, {0, True}}];
makePlot[n_] := Plot[{f[x], fs[x, n]}, {x, -Pi, Pi},
        PlotStyle → {Blue, Red}, AxesLabel → {"x", None},
        PlotLabel → Row[{"Fourier series approx using ", n, " terms"}],
        ImageSize → 300,
        Ticks → {Range[-Pi, Pi, Pi / 2], Automatic}
      ];
Grid[Partition[Table[makePlot[n], {n, {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}}], 2],
  Frame → All]
```

Figure 2.15: Code used for the above plot

the following plot shows how the Fourier series approximate $f(x)$ when it is periodically extended to outside $[-\pi, \pi]$. This uses the range $[-3\pi, 3\pi]$ by adding one more period to left and to the right.

Out[ ]=

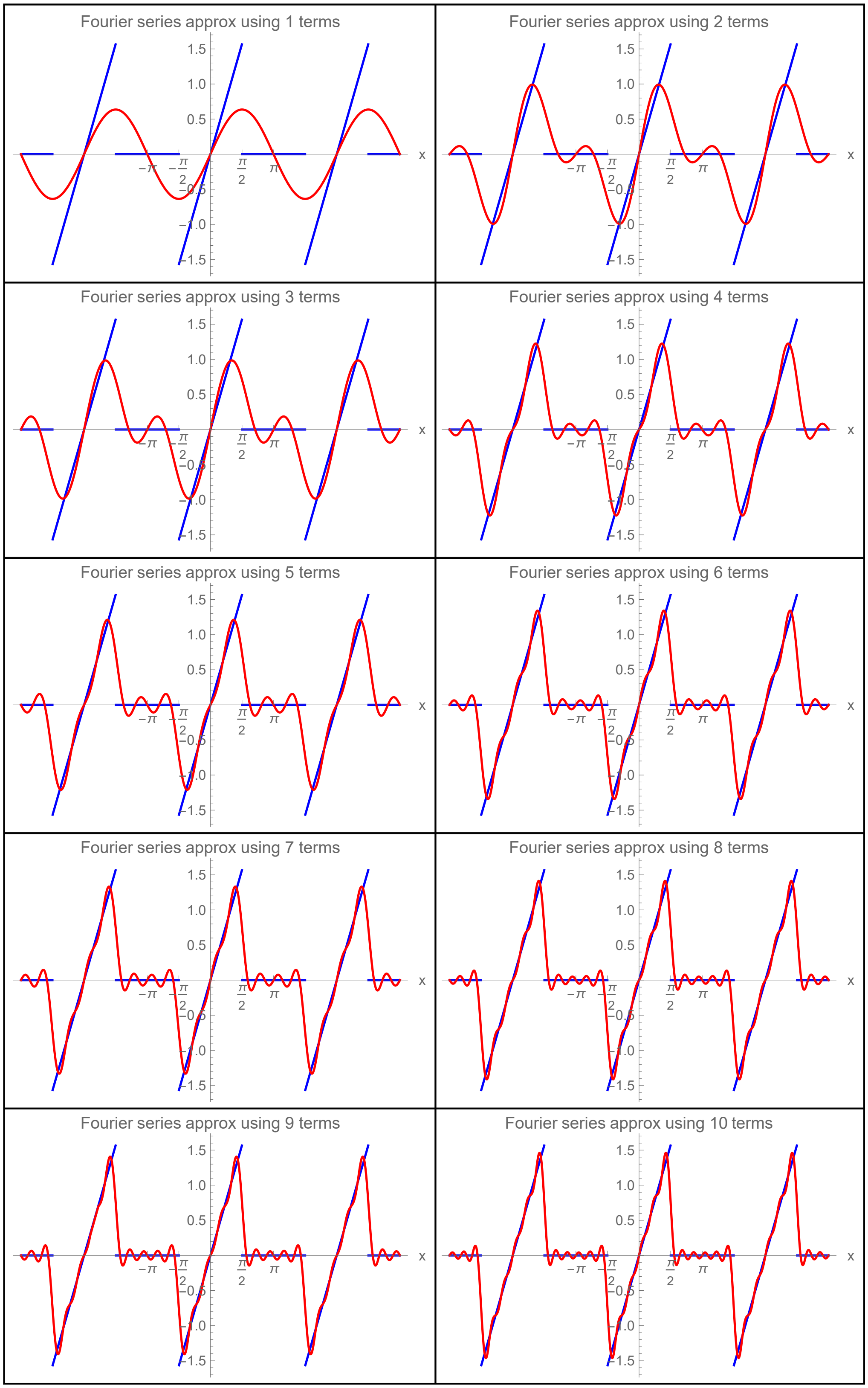


Figure 2.16: Fourier series of periodic extension $f(x)$

```
In[]:= fs[x_, max_] := Sum[2/(n^2 π) (Sin[n π/2] - 1/2 n π Cos[n π/2]) Sin[n x], {n, 1, max}];
f[x_] := Piecewise[{
        {0, x < -5/2 Pi},
        {x + 2 Pi, -5/2 Pi < x < -3/2 Pi},
        {0, -3/2 Pi < x < -Pi/2},
        {x, -Pi/2 < x < Pi/2},
        {0, Pi/2 < x < 3/2 Pi},
        {x - 2 Pi, 2/3 Pi < x < 5/2 Pi},
        {0, 5/2 Pi < x < 3 Pi}}];
makePlot[n_] := Plot[{f[x], fs[x, n]}, {x, -3 Pi, 3 Pi},
      PlotStyle → {Blue, Red}, AxesLabel → {"x", None},
      PlotLabel → Row[{"Fourier series approx using ", n, " terms"}],
      ImageSize → 300,
      Ticks → {Range[-Pi, Pi, Pi/2], Automatic}
    ];
Grid[Partition[Table[makePlot[n], {n, {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}}], 2],
  Frame → All]
```

Figure 2.17: Code used for the above plot

2.3.5 Problem 3.2.3

Find the Fourier series of $\sin^2 x$ and $\cos^2 x$ without directly calculating the Fourier coefficients.

solution

Using the known trig identity

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x) \tag{1}$$

And comparing the the above to the Fourier series expansion

$$\sin^2 x = \frac{a_0}{2} + (a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + \cdots) + (b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) + \cdots) \tag{A}$$

Shows that $\frac{a_0}{2} = \frac{1}{2}$ and $a_2 = \frac{-1}{2}$ and all other terms are zero. Because the Fourier series is unique for a function, then (1) is the Fourier series for $\sin^2 x$.

Similarly, Using the known trig identity

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x) \tag{2}$$

And comparing the the above to the Fourier series expansion (A), shows that $\frac{a_0}{2} = \frac{1}{2}$ and $a_2 = \frac{1}{2}$ and all other terms are zero. Therefore (2) is the Fourier series expansion for $\cos^2 x$.

2.3.6 Problem 3.2.6

Graph the 2π periodic extension of each of the following functions (h) $f(x) = \frac{1}{x}$. Which extension are continuous? Differentiable?

solution

2.3.6.1 Part (h)

The original function $f(x) = \frac{1}{x}$ is always taken from $-\pi \leq x \leq \pi$ (before extending it periodically). At $x = 0$ the function is not defined.

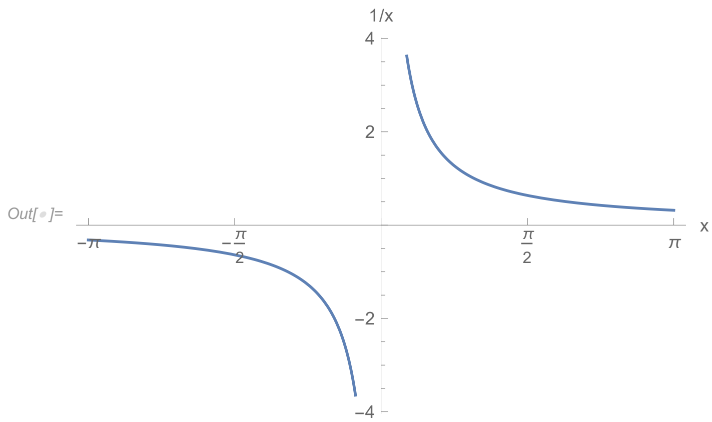


Figure 2.18: Plot of $f(x) = \frac{1}{x}$

Periodically extending it, it becomes (showing one extra period to the left and right) then following

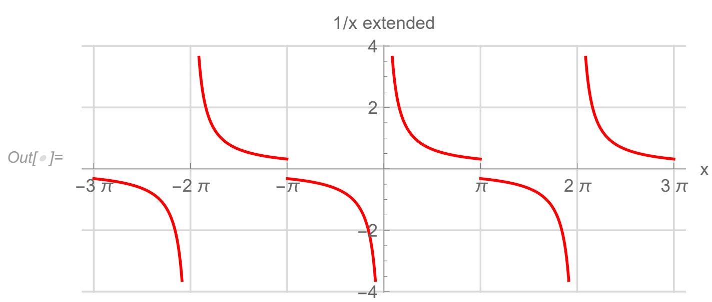


Figure 2.19: Plot of periodic extension of $f(x) = \frac{1}{x}$

```
In[•]:= f[x_] := Piecewise[{
                {1 / (x + 2 Pi), x < -Pi},
                {1 / x, -Pi < x < Pi},
                {1 / (x - 2 Pi), Pi < x}
            }];
        Plot[f[x], {x, -3 Pi, 3 Pi}, Ticks → {Range[-3 Pi, 3 Pi, Pi], Automatic},
          AxesLabel → {"x", "1/x extended"},
          GridLines → {Range[-3 Pi, 3 Pi, Pi], Automatic},
          GridLinesStyle → LightGray, PlotStyle → Red, AspectRatio → Automatic]
```

Figure 2.20: Code for the above plot

Looking at the above plot shows the extension is not continuous and also not Differentiable due to jump discontinuities.

2.3.7 Problem 3.2.9

Suppose that $f(x)$ is periodic with period T (using T instead of l as in book as it is more clear). Prove that for any a (a) $\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$. (b) $\int_0^T f(x+a)\,dx = \int_0^T f(x)\,dx$

solution

2.3.7.1 Part (a)

$$\int_a^{a+T} f(x)\,dx - \int_0^T f(x)\,dx = \overbrace{\left(\int_a^T f(x)\,dx + \int_T^{a+T} f(x)\,dx\right)}^{\int_a^{a+T} f(x)dx} - \overbrace{\left(\int_0^a f(x)\,dx + \int_a^T f(x)\,dx\right)}^{\int_0^T f(x)dx}$$

Simplifying the RHS above gives

$$\int_a^{a+T} f(x)\,dx - \int_0^T f(x)\,dx = \int_T^{a+T} f(x)\,dx - \int_0^a f(x)\,dx \tag{1}$$

But

$$\int_T^{a+T} f(x)\,dx = \int_0^a f(x+T)\,dx \tag{2}$$

To show how Eq(2) was derived: Let $u = x - T$. Then $du = dx$. When $x = T$ then $u = 0$. When $x = a + T$ then $u = a$. Hence $\int_T^{a+T} f(x)\,dx = \int_0^a f(u+T)\,du$. But u is arbitrary integral variable. Renaming it back to x gives that $\int_T^{a+T} f(x)\,dx = \int_0^a f(x+T)\,dx$.

Now, substituting (2) back into RHS of (1) gives

$$\begin{aligned}\int_a^{a+T} f(x)\,dx - \int_0^T f(x)\,dx &= \int_0^a f(x+T)\,dx - \int_0^a f(x)\,dx \\ &= \int_0^a f(x+T) - f(x)\,dx\end{aligned}$$

But since $f(x)$ is periodic, then $f(x+T)=f(x)$. Therefore the RHS above is zero.

$$\int_a^{a+T} f(x)\,dx - \int_0^T f(x)\,dx = 0$$
$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$$

Which is what the problem is asking to show.

2.3.7.2 Part (b)

Starting by rewriting $\int_0^T f(x+a)\,dx$ as the following. Let $u=x+a$. Hence $du=dx$. When $x=0,u=a$ and when $x=T,u=a+T$. The integral becomes $\int_a^{a+T} f(u)\,du$. But now u is arbitrary integration variable. Renaming is back to x then we obtain that

$$\int_0^T f(x+a)\,dx = \int_a^{a+T} f(x)\,dx \tag{1}$$

Now, to show that main result, considering

$$\int_0^T f(x+a)\,dx - \int_0^T f(x)\,dx = \int_a^{a+T} f(x)\,dx - \int_0^T f(x)\,dx$$

Where in the above, (1) was used to obtain RHS. The above can now be written as

$$\int_0^T f(x+a)\,dx - \int_0^T f(x)\,dx = \overbrace{\left(\int_a^T f(x)\,dx + \int_T^{T+a} f(x)\,dx\right)}^{\int_a^{a+T} f(x)dx} - \int_0^T f(x)\,dx$$

But $\int_T^{T+a} f(x)\,dx = \int_0^a f(x)\,dx$ since $f(x)$ is periodic with period T. The above now becomes

$$\begin{aligned}\int_0^T f(x+a)\,dx - \int_0^T f(x)\,dx &= \left(\int_a^T f(x)\,dx + \int_0^a f(x)\,dx\right) - \int_0^T f(x)\,dx\\ &= \int_0^T f(x)\,dx - \int_0^T f(x)\,dx\\ &= 0\end{aligned}$$

Therefore $\int_0^T f(x+a)\,dx = \int_0^T f(x)\,dx$ which is what the problem is asking to show.

2.3.8 Problem 3.2.25

(a) Sketch the 2π periodic half-wave $f(x)=\begin{cases}\sin x & 0<x\leq\pi\\ 0 & -\pi\leq x<0\end{cases}$. (b) Find its Fourier series. (c) Graph the first five Fourier sums and compare the function. (d) Discuss convergence of the Fourier series.

solution

2.3.8.1 Part (a)

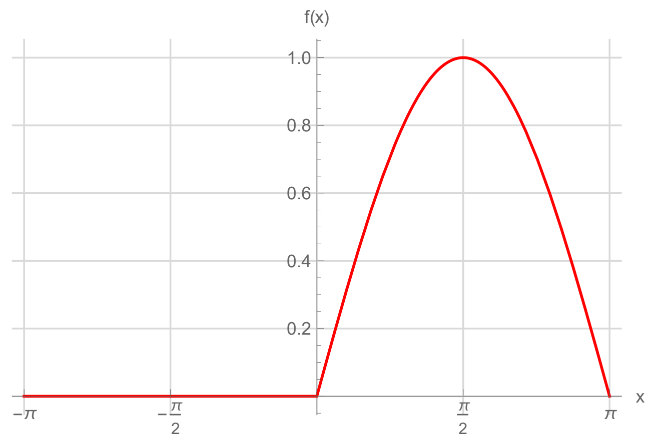


Figure 2.21: Plot of $f(x)$

```
f[x_] := Piecewise[{{Sin[x], 0 < x ≤ Pi}, {0, -Pi ≤ x < 0}}];
Plot[f[x], {x, -Pi, Pi}, Ticks → {Range[-Pi, Pi, Pi / 2], Automatic},
  AxesLabel → {"x", "f(x)"},
  GridLines → {Range[-Pi, Pi, Pi / 2], Automatic},
  GridLinesStyle → LightGray, PlotStyle → Red]
```

Figure 2.22: Code for the above plot

2.3.8.2 Part (b)

The Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T} n x\right) + a_n \sin\left(\frac{2\pi}{T} n x\right)$$

Where T is the period of the function to be approximated. Taking this period to be 2π, the above simplifies to

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Hence

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin(x)\,dx \\
&= \frac{1}{\pi}\left[-\cos(x)\right]_0^{\pi} \\
&= \frac{-1}{\pi}\left[\cos(x)\right]_0^{\pi} \\
&= \frac{-1}{\pi}\left[\cos(\pi)-1\right] \\
&= \frac{-1}{\pi}\left[-1-1\right] \\
&= \frac{2}{\pi}
\end{aligned}
$$

And

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin(x)\cos(nx)\,dx
\end{aligned}
$$

For $n = 1$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{0}^{\pi} \sin(x)\cos(x)\,dx \\
&= 0
\end{aligned}
$$

And for $n > 1$

$$
a_n = \frac{1}{\pi}\int_{0}^{\pi} \sin(x)\cos(nx)\,dx
$$

Using $\sin A\cos B = \frac{1}{2}(\sin(A-B)+\sin(A+b))$, then $\sin(x)\cos(nx) = \frac{1}{2}(\sin(x-nx)+\sin(x+nx))$. The above becomes

$$
\begin{aligned}
a_n &= \frac{1}{2\pi}\int_{0}^{\pi} \sin(x-nx)+\sin(x+nx)\,dx \\
&= \frac{1}{2\pi}\left(\int_{0}^{\pi} \sin(x-nx)\,dx + \int_{0}^{\pi} \sin(x+nx)\,dx\right) \\
&= \frac{1}{2\pi}\left(-\frac{1}{1-n}\left[\cos(x-nx)\right]_0^{\pi} - \frac{1}{1+n}\left[\cos(x+nx)\right]_0^{\pi}\right) \\
&= \frac{-1}{2\pi}\left(\frac{1}{1-n}\left[\cos(\pi-n\pi)-1\right] + \frac{1}{1+n}\left[\cos(\pi-n\pi)-1\right]\right)
\end{aligned}
$$

But $\cos(\pi - n\pi) = -\cos(n\pi)$. The above becomes

$$\begin{aligned}
a_n &= \frac{-1}{2\pi}\left(\frac{1}{1-n}\left[-\cos(n\pi)-1\right] + \frac{1}{1+n}\left[-\cos(n\pi)-1\right]\right)\\
&= \frac{1}{2\pi}\left(\frac{\cos(n\pi)+1}{1-n} + \frac{\cos(n\pi)+1}{1+n}\right)\\
&= \frac{1}{2\pi}\left(\frac{(1+n)(\cos(n\pi)+1) + (1-n)(\cos(n\pi)+1)}{(1-n)(1+n)}\right)\\
&= \frac{1}{2\pi}\left(\frac{(1+n)(\cos(n\pi)+1) + (1-n)(\cos(n\pi)+1)}{\left(1-n^2\right)}\right)\\
&= \frac{1}{2\pi\left(1-n^2\right)}\left((1+n)(\cos(n\pi)+1) + (1-n)(\cos(n\pi)+1)\right)\\
&= \frac{1}{2\pi\left(1-n^2\right)}\left(2\cos(\pi n)+2\right)\\
&= \frac{1}{\pi\left(1-n^2\right)}\left(\cos(\pi n)+1\right)\\
&= \frac{1+(-1)^n}{\pi\left(1-n^2\right)}
\end{aligned}$$

For odd $n = 3,5,\cdots$ then $a_n = 0$. For even n the above can be written as

$$a_n = \frac{2}{\pi\left(1-n^2\right)} \qquad n = 2,4,6,\cdots$$

Now b_n is found

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx\\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin(x)\sin(nx)\,dx
\end{aligned}$$

Consider case $n = 1$ first. The above gives

$$\begin{aligned}
b_1 &= \frac{1}{\pi}\int_0^{\pi} \sin^2(x)\,dx\\
&= \frac{1}{\pi}\int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos(2x)\,dx\\
&= \frac{1}{\pi}\left(\int_0^{\pi}\frac{1}{2}dx - \frac{1}{2}\int_0^{\pi}\cos(2x)\,dx\right)\\
&= \frac{1}{\pi}\left(\frac{1}{2}\pi - \frac{1}{2}\left[\frac{\sin(2x)}{2}\right]_0^{\pi}\right)\\
&= \frac{1}{2}
\end{aligned}$$

For $n > 1$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{\pi} \sin x \sin(nx)\,dx \\
&= \frac{1}{\pi}\frac{\sin(n\pi)}{n^2-1} \\
&= 0
\end{aligned}
$$

Therefore the Fourier series is

$$
\begin{aligned}
f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \\
&= \frac{1}{\pi} + \frac{1}{2}\sin(x) + \frac{2}{\pi}\sum_{n=2,4,6,\cdots}^{\infty} \frac{1}{1-n^2}\cos(nx) \\
&= \frac{1}{\pi} + \frac{1}{2}\sin(x) + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{1-(2n)^2}\cos(2nx)
\end{aligned}
$$

2.3.8.3 Part (c)

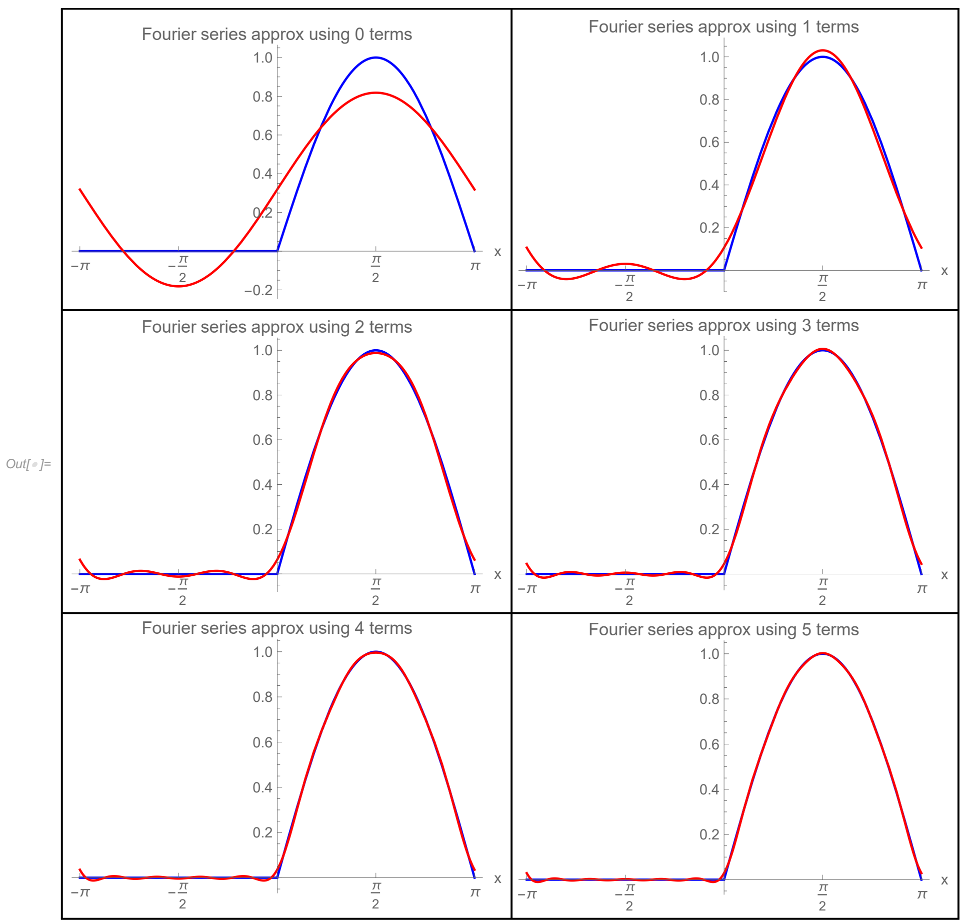


Figure 2.23: Plot of Fourier series approximation and $f(x)$

```
In[•]:= fs[x_, max_] := 1/π + 1/2 Sin[x] + 2/π Sum[1/(1 - (2 n)^2) Cos[2 n x], {n, 1, max}];
f[x_] := Piecewise[{{Sin[x], 0 < x ≤ Pi}, {0, -Pi ≤ x < 0}}];
makePlot[n_] := Plot[{f[x], fs[x, n]}, {x, -Pi, Pi},
          PlotStyle → { Blue, Red}, AxesLabel → {"x", None},
          PlotLabel → Row[{"Fourier series approx using ", n, " terms"}],
          ImageSize → 300,
          Ticks → {Range[-Pi, Pi, Pi / 2], Automatic}
       ];
Grid[Partition[Table[makePlot[n], {n, 0, 5}], 2],
   Frame → All]
```

Figure 2.24: Code for the above plot

2.3.8.4 Part (d)

The function $f(x)$ is piecewise C^1 continuous over $-\pi \leq x \leq \pi$. Therefore the 2π periodic extension is also piecewise C^1 continuous over all x. There are no jump discontinues (only corner points). The Fourier series converges to $f(x)$ at each $x \in \Re$. (If there was a jump discontinuity at some x, then the Fourier series will converge to the average of $f(x)$ at that x, but this is not the case here).

2.3.9 Problem 3.2.27

(a) Find the Fourier series of $f(x) = e^x$. (b) For which values of x does the Fourier series converges? Is the convergence uniform? (c) Graph the function it converges to.

solution

2.3.9.1 Part (a)

For generality, the Fourier series for e^{ax} is found, then at the end a is set to be one. It is assumed the period is 2π.

$$e^{ax} \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T} nx\right) + b_n \sin\left(\frac{2\pi}{T} nx\right)$$

But $T = 2\pi$ and the above becomes

$$e^{ax} \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Where

$$
\begin{aligned}
a_0 &= \frac{1}{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} dx \\
&= \frac{1}{\pi} \left[\frac{e^{ax}}{a} \right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi a} \left(e^{a\pi} - e^{-a\pi} \right)
\end{aligned}
$$

But $\frac{e^{a\pi}-e^{-a\pi}}{2} = \sinh(a\pi)$ hence the above simplifies to

$$
a_0 = \frac{2}{\pi a} \sinh(a\pi)
$$

And for $n > 0$

$$
\begin{aligned}
a_n &= \frac{1}{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \cos\left(\frac{2\pi}{T} nx\right) dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \cos(nx)\,dx \qquad (1)
\end{aligned}
$$

Let $I = \int_{-\pi}^{\pi} e^{ax} \cos(nx)\,dx$. Using integration by parts, $\int u dv = uv - \int v du$. Let $u = \cos nx, dv = e^{ax}$ then $v = \frac{e^{ax}}{a}, du = -n \sin(nx)$. Hence

$$
\begin{aligned}
I &= uv - \int v du \\
&= \left[\cos(nx) \frac{e^{ax}}{a} \right]_{-\pi}^{\pi} + \frac{n}{a} \int_{-\pi}^{\pi} e^{ax} \sin(nx)\,dx \\
&= \left[\cos(n\pi) \frac{e^{a\pi}}{a} - \cos(n\pi) \frac{e^{-a\pi}}{a} \right] + \frac{n}{a} \int_{-\pi}^{\pi} e^{ax} \sin(nx)\,dx \\
&= (-1)^n \left[\frac{e^{a\pi} - e^{-a\pi}}{a} \right] + \frac{n}{a} \int_{-\pi}^{\pi} e^{ax} \sin(nx)\,dx \\
&= \frac{2(-1)^n}{a} \left[\frac{e^{a\pi} - e^{-a\pi}}{2} \right] + \frac{n}{a} \int_{-\pi}^{\pi} e^{ax} \sin(nx)\,dx \\
&= \frac{2(-1)^n}{a} \sinh(a\pi) + \frac{n}{a} \int_{-\pi}^{\pi} e^{ax} \sin(nx)\,dx
\end{aligned}
$$

Applying integration by parts again on the integral above. Let $u = \sin nx, dv = e^{ax}$ then $v = \frac{e^{ax}}{a}, du = n\cos(nx)$ and the above becomes

$$\begin{aligned}
I &= \frac{2(-1)^n}{a}\sinh(a\pi) + \frac{n}{a}\left(\left(\sin nx\frac{e^{ax}}{a}\right)_{-\pi}^{\pi} - \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}\cos(nx)\,dx\right)\\
&= \frac{2(-1)^n}{a}\sinh(a\pi) + \frac{n}{a}\left(\frac{1}{a}\overbrace{(\sin(n\pi)e^{a\pi} + \sin(n\pi)e^{-a\pi})}^{0} - \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}\cos(nx)\,dx\right)\\
&= \frac{2(-1)^n}{a}\sinh(a\pi) - \frac{n^2}{a^2}\int_{-\pi}^{\pi} e^{ax}\cos(nx)\,dx
\end{aligned}$$

But $\int_{-\pi}^{\pi} e^{ax}\cos(nx)\,dx = I$, the original integral we are solving for. Hence solving for I from the above gives gives

$$\begin{aligned}
I &= \frac{2(-1)^n}{a}\sinh(a\pi) - \frac{n^2}{a^2}I\\
I + \frac{n^2}{a^2}I &= \frac{2(-1)^n}{a}\sinh(a\pi)\\
I\left(1 + \frac{n^2}{a^2}\right) &= \frac{2(-1)^n}{a}\sinh(a\pi)\\
I &= \frac{\frac{2(-1)^n}{a}\sinh(a\pi)}{1 + \frac{n^2}{a^2}}\\
&= \frac{2a(-1)^n\sinh(a\pi)}{a^2 + n^2} \qquad (2)
\end{aligned}$$

Using (2) in (1) gives

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos(nx)\,dx\\
&= \frac{a}{\pi}\frac{2(-1)^n\sinh(a\pi)}{a^2 + n^2} \qquad (3)
\end{aligned}$$

Now we will do the same to find b_n

$$\begin{aligned}
b_n &= \frac{1}{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin\left(\frac{2\pi}{T}nx\right)dx\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx \qquad (4)
\end{aligned}$$

Let $I=\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx$. Using integration by parts, $\int u dv=uv-\int v du$. Let $u=\sin(nx), dv=e^{ax}$ then $v=\frac{e^{ax}}{a}, du=n\cos(nx)$. Hence

$$\begin{aligned}
I &= uv-\int v du\\
&= \left[\sin(nx)\frac{e^{ax}}{a}\right]_{-\pi}^{\pi}-\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\cos(nx)\,dx\\
&= \overbrace{\left[\sin(n\pi)\frac{e^{a\pi}}{a}-\sin(n\pi)\frac{e^{-a\pi}}{a}\right]}^{0}-\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\cos(nx)\,dx\\
&= -\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\cos(nx)\,dx
\end{aligned}$$

Now we apply integration by parts again on the integral above. Let $u=\cos nx, dv=e^{ax}$ then $v=\frac{e^{ax}}{a}, du=-n\sin(nx)$ and the above becomes

$$\begin{aligned}
I &= -\frac{n}{a}\left(\left(\cos(nx)\frac{e^{ax}}{a}\right)_{-\pi}^{\pi}+\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx\right)\\
&= -\frac{n}{a}\left(\frac{1}{a}\left(\cos(n\pi)e^{a\pi}-\cos(n\pi)e^{-a\pi}\right)+\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx\right)\\
&= -\frac{n}{a}\left(\frac{1}{a}\cos(n\pi)\left(e^{a\pi}-e^{-a\pi}\right)+\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx\right)\\
&= -\frac{n}{a}\left(\frac{2}{a}\cos(n\pi)\left(\frac{e^{a\pi}-e^{-a\pi}}{2}\right)+\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx\right)\\
&= -\frac{n}{a}\left(\frac{2}{a}\cos(n\pi)\sinh(a\pi)+\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx\right)\\
&= -\frac{2n}{a^2}(-1)^n\sinh(a\pi)-\frac{n^2}{a^2}\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx
\end{aligned}$$

But $\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx=I$. Hence solving for I gives

$$\begin{aligned}
I &= -\frac{2n}{a^2}(-1)^n\sinh(a\pi)-\frac{n^2}{a^2}I\\
I+\frac{n^2}{a^2}I &= -\frac{2n}{a^2}(-1)^n\sinh(a\pi)\\
I\left(1+\frac{n^2}{a^2}\right) &= -\frac{2n}{a^2}(-1)^n\sinh(a\pi)\\
I &= -\frac{\frac{2n}{a^2}(-1)^n\sinh(a\pi)}{1+\frac{n^2}{a^2}}\\
I &= -\frac{2n(-1)^n}{a^2+n^2}\sinh(a\pi) \qquad (5)
\end{aligned}$$

Using (5) in (4) gives

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx\\
&= -\frac{1}{\pi}\frac{2n(-1)^n}{a^2+n^2}\sinh(a\pi)
\end{aligned}$$

Now that we found a_0, a_n, b_n then the Fourier series is

$$\begin{aligned}
e^{ax} &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)\\
&\sim \frac{\frac{2}{\pi a}\sinh(a\pi)}{2} + \sum_{n=1}^{\infty}\frac{a}{\pi}\frac{2(-1)^n\sinh(a\pi)}{a^2+n^2}\cos(nx) - \frac{1}{\pi}\frac{2n(-1)^n}{a^2+n^2}\sinh(a\pi)\sin(nx)\\
&\sim \frac{\sinh(a\pi)}{\pi a} + \frac{1}{\pi}\sinh(a\pi)\sum_{n=1}^{\infty}\frac{2(-1)^n}{a^2+n^2}(a\cos(nx) - n\sin(nx))\\
&\sim \sinh(a\pi)\left(\frac{1}{\pi a} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{2(-1)^n}{a^2+n^2}(a\cos(nx) - n\sin(nx))\right)\\
&\sim \frac{2\sinh(a\pi)}{\pi}\left(\frac{1}{2a} + \sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}(a\cos(nx) - n\sin(nx))\right)
\end{aligned}$$

When $a = 1$ the above becomes

$$e^x \sim \frac{2\sinh(\pi)}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx))\right)$$

2.3.9.2 Part (b)

The 2π periodic extended function shows it piecewise C^1 over all points except at the points $x = \cdots, -5\pi, -3\pi, \pi, 3\pi, 5\pi, \cdots$. These are points at the ends of the original domain. At these points, there is a jump discontinuity. Therefore the Fourier series at these points will converge to the average of the 2π periodic extended e^x. Due to the jump discontinuity Gibbs phenomena shows up at these points. This also means that the convergence is <u>not uniform</u>.

2.3.9.3 Part (c)

The following is a plot showing the convergence using different number of terms in the above sum. This shows the Fourier series converges to e^x at all points inside the interval, except at the end points $x = -\pi, \pi$ where it converges to the average of $f(x)$.

Out[•]=

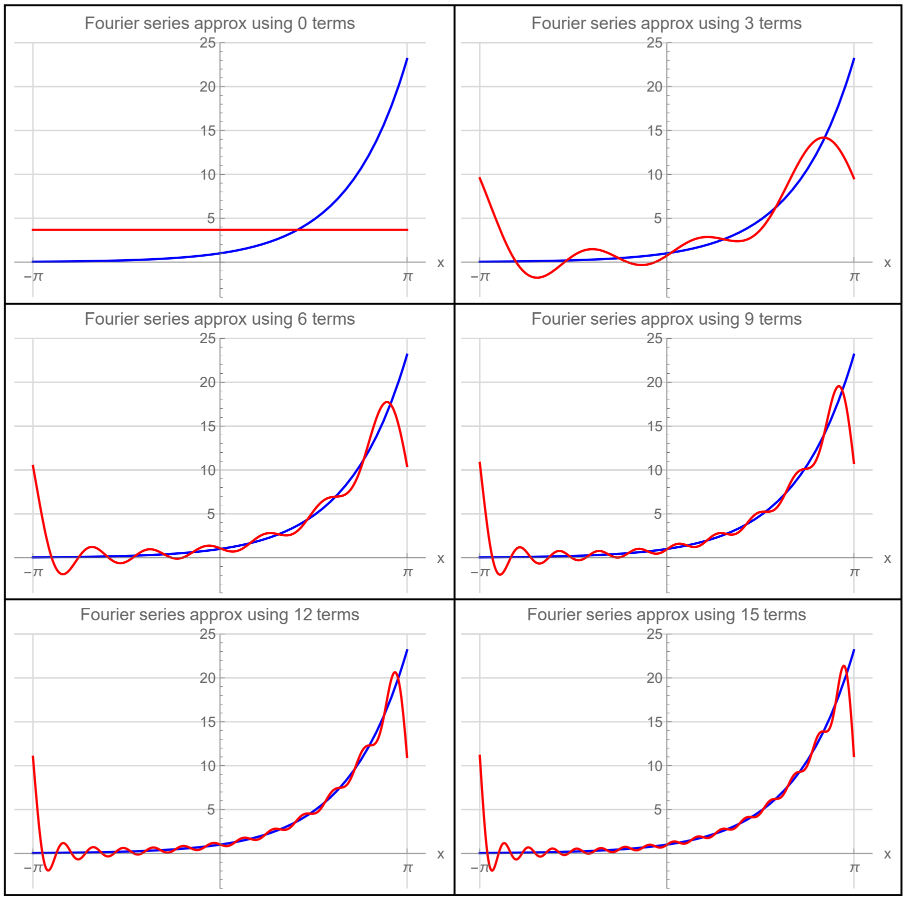


Figure 2.25: Plot of Fourier series approximation and $f(x)$

In[•]:=

```
padIt2[v_, f_List] := AccountingForm[v, f, NumberSigns → {"", ""},
          NumberPadding → {" ", " "}, SignPadding → True];
fs[x_, max_] := (2 Sinh[Pi]/Pi) (1/2 + Sum[((-1)^n/(1 + n^2)) (Cos[n x] - n Sin[n x]), {n, 1, max}]);
f[x_] := Exp[x];
fp[x_] := Piecewise[{{f[x + 2 Pi], x ≤ -Pi}, {f[x], -Pi < x < Pi}, {f[x - 2 Pi], x > Pi}}];
makePlot[n_] := Plot[{f[x], fs[x, n]}, {x, -Pi, Pi},
          PlotStyle → { Blue, Red}, AxesLabel → {"x", None},
          PlotLabel → Row[{"Fourier series approx using ", n, " terms"}],
          ImageSize → 300,
          Ticks → {Range[-Pi, Pi, Pi], Automatic},
          PlotRange → {{-1.1 Pi, 1.1 Pi}, {-4, 25}},
          GridLines → {Range[-Pi, Pi, Pi], Automatic}, GridLinesStyle → LightGray
       ];
Grid[Partition[Table[makePlot[n], {n, {0, 3, 6, 9, 12, 15}}], 2],
   Frame → All]
```

Figure 2.26: Code for the above plot

2.3.10 Problem 3.2.30

Suppose a_k, b_k are the Fourier coefficients of the function $f(x)$. (a) To which function does the Fourier series

$$\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k \cos(2kx)+b_k \sin(2kx)$$

Converge? (b) Test your answer with the Fourier series (3.37) for $f(x)=x$.

$$x\sim 2\left(\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\cdots\right) \tag{3.37}$$

solution

2.3.10.1 Part (a)

Let

$$\begin{aligned} g(x) &\sim \frac{a_0}{2}+\sum_{k=1}^{\infty} a_k \cos(2kx)+b_k \sin(2kx)\\ f(x) &\sim \frac{a_0}{2}+\sum_{k=1}^{\infty} a_k \cos(kx)+b_k \sin(kx)\end{aligned}$$

Then $g(x)$ has as its period half the period of $f(x)$. This is because when $2kx=\frac{2\pi}{T}kx$ then $T=\pi$ and when $kx=\frac{2\pi}{T}kx$ then $T=2\pi$.

Therefore, if $f(x)$ has fundamental period as $-\pi<x<\pi$, then $g(x)$ has a fundamental period as $-\frac{\pi}{2}<x<\frac{\pi}{2}$. And since $f(x), g(x)$ have the same Fourier series coefficients, then $g(x)$ converges to $2f(x)$ but only over $-\frac{\pi}{2}<x<\frac{\pi}{2}$.

2.3.10.2 Part (b)

Let $f(x)=x$ whose we are given that its Fourier series is

$$\begin{aligned} f(x) &\sim 2\left(\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\cdots\right)\\ &= 2\sin x-\sin 2x+\frac{2}{3}\sin 3x-\frac{1}{2}\sin 4x+\cdots\end{aligned}$$

The above says that $a_k=0$ and $b_k=\frac{2(-1)^{k+1}}{k}$. Hence

$$f(x)\sim\sum_{k=1}^{\infty}\frac{2(-1)^{k+1}}{k}\sin(kx)$$

Therefore $g(x)$ will converge to

$$\begin{aligned}
g(x) &\sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(2kx) + b_k \sin(2kx) \\
&= \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin(2kx) \\
&= 2(+1)\sin(2x) + \frac{-2}{2}\sin(4x) + \frac{2(+1)}{3}\sin(6x) + \frac{-2}{4}\sin(8x) + \cdots \\
&= 2\sin(2x) - \sin(4x) + \frac{2}{3}\sin(6x) - \frac{1}{2}\sin(8x) + \cdots
\end{aligned}$$

Over $-\frac{\pi}{2} < x < \frac{\pi}{2}$. To verify the above, we will now find a_k, b_k directly for x but using $T = \pi$ and not $T = 2\pi$ to see if the above Fourier series is obtained.

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x dx \\
&= 0
\end{aligned}$$

And

$$a_k = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos(2kx)\, dx$$

Since x is odd function and $\cos$ is even, the product is odd. Hence $a_k = 0$.

$$\begin{aligned}
b_k &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin(2kx)\, dx \\
&= \frac{4}{\pi}\int_{0}^{\frac{\pi}{2}} x\sin(2kx)\, dx \\
&= \frac{4}{\pi}\left(\frac{-k\pi\cos(k\pi) + \sin(k\pi)}{4k^2}\right) \\
&= \frac{1}{\pi k^2}\left(-k\pi\cos(k\pi)\right) \\
&= \frac{-1}{k}\cos(k\pi) \\
&= \frac{-1}{k}(-1)^k \\
&= \frac{(-1)^{k+1}}{k}
\end{aligned}$$

Therefore

$$\begin{aligned}
g(x) &\sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(2kx) + b_k \sin(2kx) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin(2kx)
\end{aligned}$$

2.3.11 Key solution for HW 3

Homework 3 Solutions

3.1.2

(a) $\exp(-n^2 t)\sin n x$ for $n = 1, 2, \ldots$.

(b) $\exp\left[-\left(n+\frac{1}{2}\right)^2 t\right]\sin\left(n+\frac{1}{2}\right)x$ for $n = 0, 1, 2, \ldots$.

3.1.5

5. (a)

λ	Eigenfunctions	Eigensolutions
$\lambda = -\omega^2 - 1 < -1$	$\cos\omega x,\ \sin\omega x$	$e^{-(\omega^2+1)t}\cos\omega x,\ e^{-(\omega^2+1)t}\sin\omega x$
$\lambda = -1$	$1,\ x$	$e^{-t},\ e^{-t}x$
$\lambda = \omega^2 - 1 > -1$	$e^{\omega x},\ e^{-\omega x}$	$e^{(\omega^2-1)t+\omega x},\ e^{(\omega^2-1)t-\omega x}$

(b) $e^{-t},\ e^{-(n^2+1)t}\cos n x,\ e^{-(n^2+1)t}\sin n x,$ for $n = 1, 2, 3, \ldots$.

3.2.1d

★ (d) $\dfrac{\pi^2}{3} + 4\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{\cos k x}{k^2}$

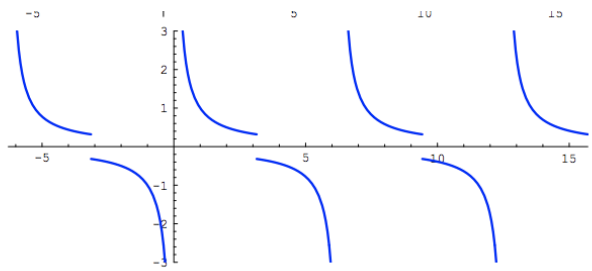
3.2.2d

(d) $\dfrac{2}{\pi}\displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j\sin(2j+1)x}{(2j+1)^2}$

3.2.3

Solution: $\sin^2 x \sim \frac{1}{2} - \frac{1}{2}\cos 2x$ and $\cos^2 x \sim \frac{1}{2} + \frac{1}{2}\cos 2x$.

3.2.6h

(h)  not continuous.

3.2.9

(a)

$$\int_a^{a+\ell} f(x)\,dx = \int_0^{\ell} f(x)\,dx - \int_0^{a} f(x)\,dx + \int_{\ell}^{a+\ell} f(x)\,dx. \qquad (*)$$

But, applying the change of variables $y = x - \ell$,

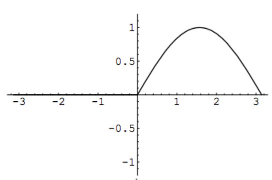
$$\int_{\ell}^{a+\ell} f(x)\,dx = \int_0^{a} f(y+\ell)\,dy = \int_0^{a} f(y)\,dy,$$

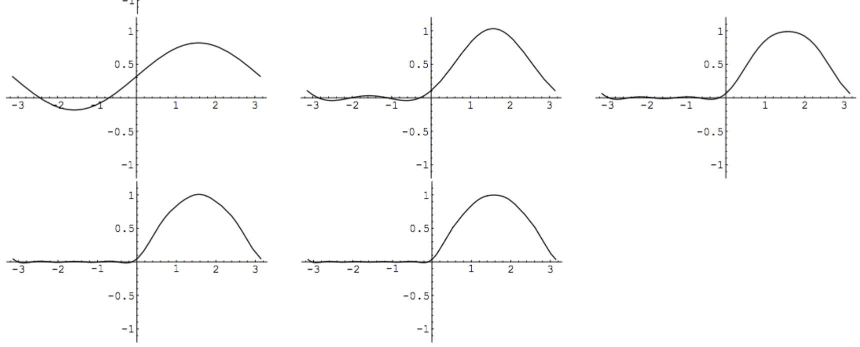
which follows from the periodicity of f. Thus, the second and third integrals in $(*)$ cancel, which establishes the result. *Q.E.D.*

(b) Using the change of variables $y = x + a$ and part (a),

$$\int_0^{\ell} f(x+a)\,dx = \int_a^{a+\ell} f(y)\,dy = \int_0^{\ell} f(x)\,dx.$$

3.2.25

(a)  (b) $f(x) \sim \dfrac{1}{\pi} + \dfrac{1}{2}\sin x - \dfrac{2}{\pi}\displaystyle\sum_{j=1}^{\infty} \dfrac{\cos 2jx}{4j^2-1}$.

(c) 

The maximal errors on $[-\pi,\pi]$ are, respectively $.3183, .1061, .06366, .04547, .03537, .02894$.

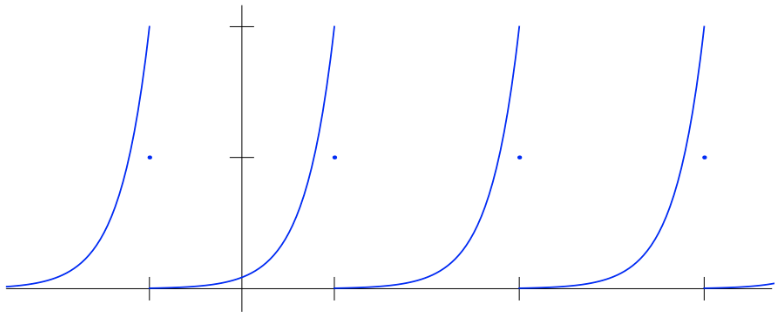
(d) The Fourier series converges (uniformly) to $\sin x$ when $2k\pi \le x \le (2k+1)\pi$ and to 0 when $(2k-1)\pi \le x \le 2k\pi$ for $k = 0, \pm 1, \pm 2, \ldots$.

3.2.27

(a) $\quad e^x \sim \dfrac{\sinh\pi}{\pi} + \dfrac{2\sinh\pi}{\pi}\displaystyle\sum_{k=1}^{\infty}(-1)^k\,\frac{\cos k\,x - k\sin k\,x}{1+k^2}.$

(b) The Fourier series converges for all real x to the 2π–periodic extension of e^x, with values $\cosh\pi = \frac{1}{2}(e^\pi + e^{-\pi})$ at the discontinuities at $x = \pm\pi, \pm 3\pi, \ldots$. The convergence is not uniform because the limiting sum is not continuous.

(c)



3.2.30

(a) If $\widetilde{f}(x)$ is the 2π–periodic extension of $f(x)$, then the Fourier series converges to $\widetilde{f}(2x)$, which is the π–periodic extension of $f(2x)$.

(b) The Fourier series $2\left(\sin 2x - \frac{1}{2}\sin 4x + \frac{1}{3}\sin 6x - \cdots\right)$ converges to the π–periodic extension of the function $\widehat{f}(x) = 2x$ for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, or, equivalently, the 2π–periodic extension of $f(x) = \begin{cases} 2(x+\pi), & -\pi < x < -\frac{1}{2}\pi, \\ 0, & x = \pm\frac{1}{2}\pi, \\ 2x, & -\frac{1}{2}\pi < x < \frac{1}{2}\pi, \\ 2(x-\pi), & \frac{1}{2}\pi < x < \pi. \end{cases}$

2.4 HW 4

Local contents

2.4.1 Problem 3.2.34

If $f(x)$ is odd, is $f'(x)$ (i) even? (ii) odd? (iii) neither? (iv) could be either?

solution

Answer is (i), even.

Proof: Since $f(x)$ is odd, then by definition

$$f(x) = -f(-x)$$

For all x in the domain of $f(x)$. Taking derivatives w.r.t. gives

$$\left[f(x)\right]' = \left[-f(-x)\right]'$$

Applying the chain rule to RHS gives $-f'(-x)(-1) = f'(-x)$ and the LHS gives $f'(x)$. Hence the above becomes

$$f'(x) = f'(-x)$$

But by definition $g(-x) = g(x)$ implies an even function. Hence the says that $f'(x)$ is an even function.

2.4.2 Problem 3.2.37

True or False. (a) If $f(x)$ is odd, its 2π periodic extension is odd. (b) if the 2π periodic extension of $f(x)$ is odd, then $f(x)$ is odd.

solution

2.4.2.1 Part a

True.

To show this, will use an illustration. In this illustration, and to reduce confusion, let $f(x)$ represents the original odd function defined over $-\pi \leq x \leq \pi$ and let $g(x)$ represents the 2π periodic extension of $f(x)$. For illustration, used the odd function $f(x) = x$.

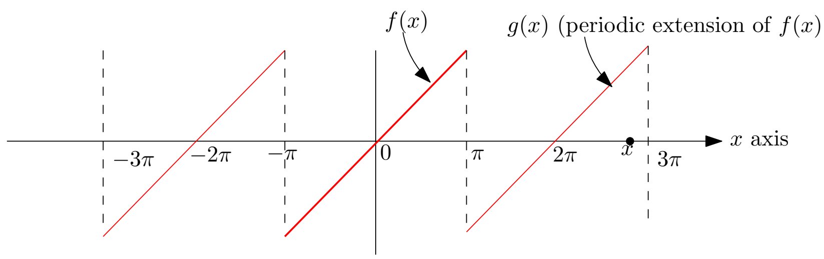


Figure 2.27: Showing $f(x)$ and its 2π extension

To show that $g(x)$ is odd, we pick any point x and now we need to show that $g(-x) = -g(x)$ or $g(x) = -g(-x)$.

On the right side of the x axis, $g(x) = f(x - n(2\pi))$ where n is positive integer due to the 2π extension. In the above illustration $n = 1$ but it can be any positive n. Let the point $x - n(2\pi) = z$. Hence now we have the following diagram

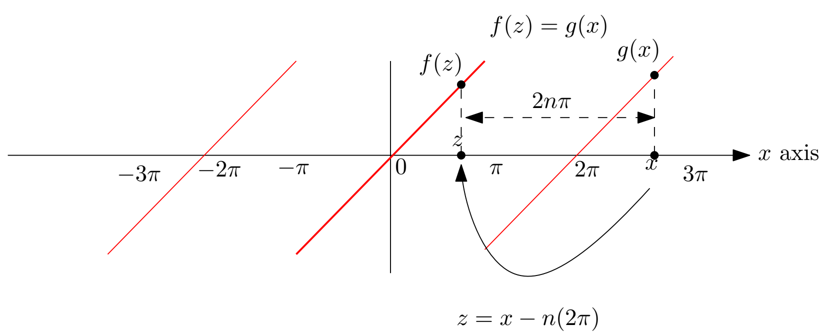


Figure 2.28: showing $g(x) = f(x - 2n\pi)$

Where $f(z) = g(x)$. But we are given that $f(x)$ is odd. Hence $f(z) = -f(-z)$. On the negative side of the x axis, we do the same we did on the positive side. Since the left side of $f(x)$ was also 2π extended, then $g(-x) = f(-x + n(2\pi)) = f(-z)$

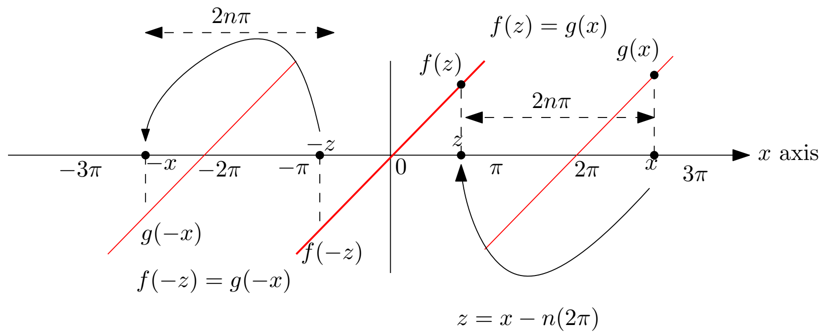


Figure 2.29: showing $g(-x) = f(-z)$

In conclusion, from the above we see that

$$g(-x) = f(-z)$$

But $f(-z) = -f(z)$ since f is odd. Hence the above becomes

$$g(-x) = -f(z)$$

But $f(z) = g(x)$ as shown in the first illustration, hence the above becomes

$$g(-x) = -g(x)$$

Which shows that $g(x)$ is odd.

Since $g(x)$ is the 2π periodic extension of $f(x)$. This is what we asked to show.

2.4.2.2 Part b

(b) True. Proof by contradiction. Since $g(x)$ is odd, then we know that

$$g(-x) = -g(x)$$

We also know that by the 2π extension of $f(x)$ that

$$f(z) = g(x)$$

Where we are using the same diagrams from part (a). Where $z = x - 2n\pi$. Now, let us assume that $f(x)$ is even. Then this means that

$$f(z) = f(-z)$$

But the 2π extension on the left side of the x axis, then we conclude that

$$g(-x) = f(-z)$$

Which means that

$$\begin{aligned} g(-x) &= f(z) \\ &= g(x) \end{aligned}$$

But this means $g(x)$ is even, which is a contradiction, since $g(x)$ is odd. Hence $f(x)$ can now now be even.

Only other choice is that $f(x)$ is neither odd or even, or an odd function. Let us now assume is neither. For example, take $f(x) = \begin{cases} x & 0 < x < \pi \\ 0 & -\pi < x < 0 \end{cases}$. Then following the above argument, we see that

$$g(x) = f(z)$$

But now $f(-z) = 0$, then by 2π extension of the left, then $g(-x) = f(-z) = 0$. But this means that $g(-x) \neq -g(x)$ which is not possible since $g(x)$ is odd. The only other choice left is that $f(x)$ is odd. Which is what we are asked to show.

2.4.3 Problem 3.2.40a

Find the Fourier series and discuss convergence for (a) the box function $b(x) = \begin{cases} 1 & |x| < \frac{1}{2}\pi \\ 0 & \frac{1}{2}\pi < |x| < \pi \end{cases}$

solution

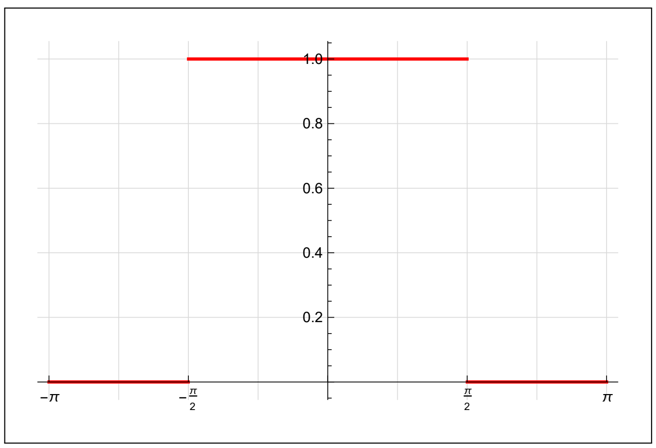


Figure 2.30: plot of $b(x)$

$$b(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

But $\frac{a_0}{2}$ is the average of the function over its 2π domain. Hence $\frac{a_0}{2} = \frac{\text{area}}{2\pi} = \frac{\pi}{2\pi} = \frac{1}{2}$, and

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\
&= \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos(nx)\,dx \\
&= \frac{2}{\pi}\int_{0}^{\frac{1}{2}\pi} \cos(nx)\,dx \\
&= \frac{2}{\pi n}\left[\sin(nx)\right]_0^{\frac{\pi}{2}} \\
&= \frac{2}{\pi n}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}
$$

And since the function is even, then all $b_n = 0$. Hence

$$
b(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n}\sin\left(\frac{n\pi}{2}\right)\cos nx
$$

To verify the above solution, it is plotted against $b(x)$ for increasing number of terms.

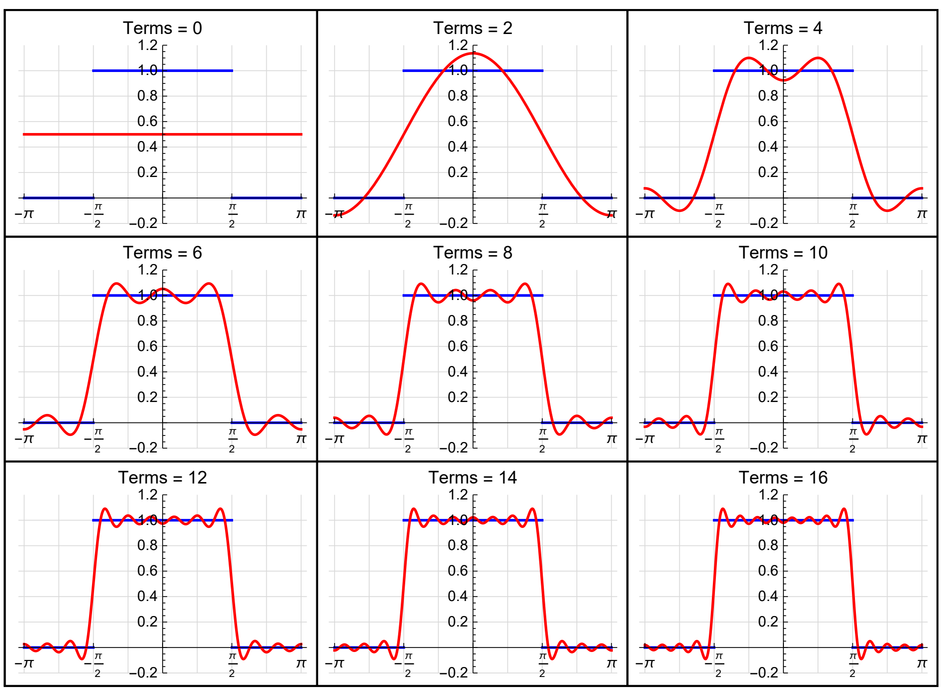


Figure 2.31: plot of Fourier series approximation to $b(x)$

```
In[ ]:= p = Table[
            Plot[{f[x], fs[x, max]}, {x, -Pi, Pi}, PlotStyle → {Blue, Red},
              Ticks → {Range[-Pi, Pi, Pi / 2], Automatic},
              GridLines → {Range[-Pi, Pi, Pi / 4], Automatic},
              GridLinesStyle → LightGray,
              PlotRange → {Automatic, {-.2, 1.2}},
              PlotLabel → Row[{"Terms = ", max}]
            ]
            ,
            {max, 0, 16, 2}
          ];
     p = Grid[Partition[p, 3], Frame → All]
```

Figure 2.32: Code used for the above plot

Since there are jump discontinuities in the function $b(x)$, this will cause Gibbs effect at those points. This also implies that the convergence is not uniform. Fourier series will converge to each x where the function is continuous, but it will converge to the average of $b(x)$ at those points where there is a jump discontinuity.

In this case at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ in the fundamental domain given as shown in the plots above. At those points, Fourier series converges to $\frac{1}{2}$.

2.4.4 Problem 3.2.54

Prove that $\coth \pi = \frac{1}{\pi} + \frac{2}{\pi}\left(\frac{1}{1+1^2} + \frac{1}{1+2^2} + \frac{1}{1+3^2} + \cdots\right)$, where $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x+e^{-x}}{e^x-e^{-x}}$

solution

The complex Fourier series of e^x is

$$e^x = \lim_{N\to\infty}\sum_{n=-N}^{N} c_n e^{inx} \tag{1}$$

Where

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx} dx\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x(1-in)} dx\\
&= \frac{1}{2\pi}\left[\frac{e^{x(1-in)}}{1-in}\right]_{-\pi}^{\pi}\\
&= \frac{1}{2\pi}\frac{1}{1-in}\left[e^x e^{-inx}\right]_{-\pi}^{\pi}\\
&= \frac{1}{2\pi}\frac{1}{1-in}\left[e^{\pi}e^{-in\pi} - e^{-\pi}e^{in\pi}\right]
\end{aligned}$$

But $e^{in\pi} = \cos(n\pi)$ and also $e^{-in\pi} = \cos(n\pi)$ since n is integer. The above simplifies to

$$c_n = \frac{1}{2\pi}\frac{\cos(n\pi)}{1-in}\left[e^{\pi} - e^{-\pi}\right]$$

But $e^{\pi} - e^{-\pi} = 2\sinh(\pi)$. Therefore

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\frac{\cos(n\pi)}{1-in}\left[2\sinh(\pi)\right]\\
&= \frac{1}{\pi}\frac{\cos(n\pi)\sinh(\pi)}{1-in}\\
&= \frac{1}{\pi}\frac{\cos(n\pi)\sinh(\pi)}{1-in}\frac{(1+in)}{1+in}\\
&= \frac{1}{\pi}\cos(n\pi)\sinh(\pi)\frac{(1+in)}{1+n^2}\\
&= \frac{(-1)^n}{\pi}\sinh(\pi)\frac{(1+in)}{1+n^2}
\end{aligned}
$$

Substituting this back into (1) gives

$$
\begin{aligned}
e^x &= \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{(-1)^n}{\pi}\sinh(\pi)\frac{(1+in)}{1+n^2}e^{inx}\\
&= \frac{\sinh(\pi)}{\pi}\lim_{N\to\infty}\sum_{n=-N}^{N}(-1)^n\frac{(1+in)}{1+n^2}e^{inx}
\end{aligned}
$$

At $x=\pi$

$$
\begin{aligned}
\frac{1}{2}\left(e^{\pi}+e^{-\pi}\right) &= \cosh(\pi)\\
\frac{1}{2}\left(\frac{\sinh(\pi)}{\pi}\lim_{N\to\infty}\sum_{n=-N}^{N}(-1)^n\frac{(1+in)}{1+n^2}e^{in\pi} + \frac{\sinh(\pi)}{\pi}\lim_{N\to\infty}\sum_{n=-N}^{N}(-1)^n\frac{(1+in)}{1+n^2}e^{-in\pi}\right) &= \cosh(\pi)\\
\frac{1}{2}\frac{\sinh(\pi)}{\pi}\left(\lim_{N\to\infty}\sum_{n=-N}^{N}(-1)^n\frac{(1+in)}{1+n^2}e^{in\pi} + \lim_{N\to\infty}\sum_{n=-N}^{N}(-1)^n\frac{(1+in)}{1+n^2}e^{-in\pi}\right) &= \cosh(\pi)
\end{aligned}
$$

But $e^{in\pi} = \cos n\pi = (-1)^n$ and $e^{-in\pi} = \cos\pi = (-1)^n$. The above becomes

$$
\begin{aligned}
\frac{1}{2}\frac{\sinh(\pi)}{\pi}\left(\lim_{N\to\infty}\sum_{n=-N}^{N}(-1)^{2n}\frac{(1+in)}{1+n^2} + \lim_{N\to\infty}\sum_{n=-N}^{N}(-1)^{2n}\frac{(1+in)}{1+n^2}\right) &= \cosh(\pi)\\
\frac{1}{2}\frac{\sinh(\pi)}{\pi}\left(\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{(1+in)}{1+n^2} + \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{(1+in)}{1+n^2}\right) &= \cosh(\pi)\\
\frac{\sinh(\pi)}{\pi}\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{(1+in)}{1+n^2} &= \cosh(\pi)\\
\frac{\sinh(\pi)}{\pi}\left(\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{1+n^2} + i\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{n}{1+n^2}\right) &= \cosh(\pi)
\end{aligned}
$$

But $\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{n}{1+n^2} = 0$ by symmetry. The above simplifies to

$$
\begin{aligned}
\frac{\sinh(\pi)}{\pi}\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{1+n^2} &= \cosh(\pi)\\
\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{1+n^2} &= \frac{\cosh(\pi)}{\sinh(\pi)}\\
&= \coth(\pi)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\coth(\pi) &= \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{1+n^2} \\
&= \frac{1}{\pi}\left(1+\sum_{n=-\infty}^{-1}\frac{1}{1+n^2}+\sum_{n=1}^{\infty}\frac{1}{1+n^2}\right) \\
&= \frac{1}{\pi}\left(1+2\sum_{n=1}^{-\infty}\frac{1}{1+n^2}\right) \\
&= \frac{1}{\pi}+\frac{2}{\pi}\sum_{n=1}^{-\infty}\frac{1}{1+n^2} \\
&= \frac{1}{\pi}+\frac{2}{\pi}\left(\frac{1}{1+1^2}+\frac{1}{1+2^2}+\frac{1}{1+3^2}+\cdots\right)
\end{aligned}
$$

Which is what the problem asked to show.

2.4.5 Problem 3.2.60

Can you recognize whether a function is real by looking at its complex Fourier coefficients?

solution

Yes. If complex Fourier coefficients come in conjugate pairs such that $c_{-n} = \overline{c_n}$ and c_0 is real. (c_0 should always be real, since this represents the average energy at the zero (D.C.) frequency, hence must be real quantity).

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx} \\
&= c_0 + \sum_{n=-\infty}^{-1} c_n e^{inx} + \sum_{n=1}^{\infty} c_n e^{inx} \\
&= c_0 + \sum_{n=1}^{\infty} c_{-n} e^{-inx} + \sum_{n=1}^{\infty} c_n e^{inx} \\
&= c_0 + \sum_{n=1}^{\infty}\left(c_{-n}e^{-inx} + c_n e^{inx}\right)
\end{aligned}
$$

Now, If $c_{-n} = \overline{c_n}$ then the above becomes

$$
f(x) = c_0 + \sum_{n=1}^{\infty} \overline{c_n} e^{-inx} + c_n e^{inx}
$$

But $\left(\overline{c_n}e^{-inx} + c_n e^{inx}\right)$ is real. (This could also be written as $\overline{c_n e^{-inx}} + c_n e^{inx}$ which now looks like standard $\overline{z} + z$ in complex numbers). Hence $f(x)$ is real.

To show this, here is an example. Let $c_n = a + ib$, then $c_{-n} = a - ib$. Therefore

$$\begin{aligned}
\overline{c_n}e^{-inx} + c_n e^{inx} &= \overline{(a+ib)}e^{-inx} + (a+ib)\,e^{inx} \\
&= (a-ib)\,e^{-inx} + (a+ib)\,e^{inx} \\
&= \left(ae^{-inx} - ibe^{-inx}\right) + \left(ae^{inx} + ibe^{inx}\right) \\
&= a\left(e^{inx} + e^{-inx}\right) + bi\left(e^{inx} - e^{-inx}\right) \\
&= a\,(\cos nx + i\sin nx + \cos nx - i\sin nx) + bi\,(\cos nx + i\sin nx - \cos nx + i\sin nx) \\
&= a\,(2\cos nx) + bi\,(2i\sin nx) \\
&= 2a\cos nx - 2b\sin nx
\end{aligned}$$

Which is real value. Therefore if each c_n is a complex conjugate of c_{-n} (with c_0 real) then $f(x)$ will be a real function.

2.4.6 Problem 3.3.2

Find the Fourier series for the function $f(x) = x^3$. If you differentiate your series, do you recover the Fourier series for $f'(x) = 3x^2$? If not, explain why not.

solution

The function $f(x)$ over $-\pi \leq x \leq \pi$ is

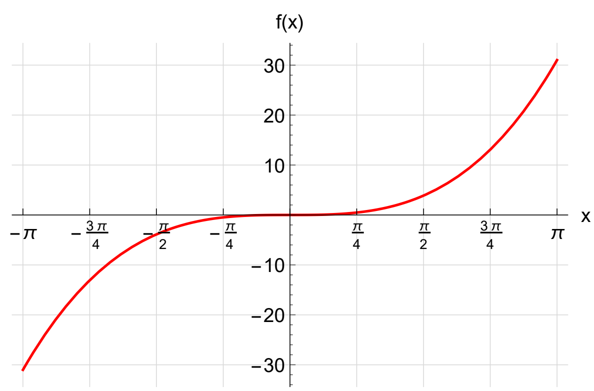


Figure 2.33: Plot of x^3

We see right away that differentiating term by term the Fourier series for the above function could not be justified. Even though the function x^3 has no jump discontinuity inside $-\pi \leq x \leq \pi$, which is good, it still fails the other test which requires that $f(-\pi) = f(\pi)$ for the term by term differentiation to be justified. This is because the 2π extension will now have jump discontinuities in it. The conditions under which the Fourier series for a function can be term by term differentiated are

1. $f(x)$ is piecewise continuous between $-\pi \leq x \leq \pi$ with no jump discontinuities.
2. $f(-\pi) = f(\pi)$

The function given fails condition (2) above. This explains why differentiating the Fourier series of x^3 will not give the Fourier series of $3x^2$. Now we will show this as required by the problem.

To find the Fourier series of x^3, since it is an odd function, then we only need to find b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin(nx)\,dx$$

Since x^3 is odd, and $\sin$ is odd, then the product is even, and the above simplifies to

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} x^3 \sin(nx)\,dx$$

Integration by parts., Let $u = x^3, \sin(nx) = dv$. Then $du = 3x^2, v = -\frac{1}{n}\cos(nx)$. Then $\int udv = uv - \int vdu$ gives

$$\begin{aligned}
b_n &= \frac{2}{\pi}\left(-\frac{1}{n}\left[x^3\cos(nx)\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi} 3x^2\cos(nx)\,dx\right)\\
&= \frac{2}{\pi}\left(-\frac{1}{n}\left[\pi^3\cos(n\pi)\right] + \frac{3}{n}\int_0^{\pi} x^2\cos(nx)\,dx\right)\\
&= \frac{2}{\pi}\left(-\frac{1}{n}\left[\pi^3(-1)^n\right] + \frac{3}{n}\int_0^{\pi} x^2\cos(nx)\,dx\right)\\
&= -\frac{2(-1)^n\pi^2}{n} + \frac{6}{n\pi}\int_0^{\pi} x^2\cos(nx)\,dx
\end{aligned}$$

Integration by parts again. Let $u = x^2, \cos(nx) = dv$. Then $du = 2x, v = \frac{1}{n}\sin(nx)$. Then using $\int udv = uv - \int vdu$ the above becomes

$$\begin{aligned}
b_n &= -\frac{2(-1)^n\pi^2}{n} + \frac{6}{n\pi}\left(\frac{1}{n}\left[x^2\sin(nx)\right]_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin(nx)\,dx\right)\\
&= -\frac{2(-1)^n\pi^2}{n} + \frac{6}{n\pi}\left(-\frac{2}{n}\int_0^{\pi} x\sin(nx)\,dx\right)\\
&= -\frac{2(-1)^n\pi^2}{n} - \frac{12}{n^2\pi}\int_0^{\pi} x\sin(nx)\,dx
\end{aligned}$$

Integration by parts again. Let $u = x, \sin(nx) = dv$. Then $du = 1, v = \frac{-1}{n}\cos(nx)$. Then using

$\int udv = uv - \int vdu$ the above becomes

$$
\begin{aligned}
b_n &= -\frac{2(-1)^n \pi^2}{n} - \frac{12}{n^2\pi}\left(\frac{-1}{n}\left[x\cos(nx)\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\right) \\
&= -\frac{2(-1)^n \pi^2}{n} - \frac{12}{n^2\pi}\left(\frac{-1}{n}\left[\pi\cos(n\pi)\right] + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right) \\
&= -\frac{2(-1)^n \pi^2}{n} - \frac{12}{n^2\pi}\left(\frac{-1}{n}\left[\pi(-1)^n\right]\right) \\
&= -\frac{2(-1)^n \pi^2}{n} + \frac{12}{n^3\pi}\pi(-1)^n \\
&= \frac{-2(-1)^n n^2\pi^2 + 12(-1)^n}{n^3} \\
&= \frac{-2(-1)^n\left(-6 + n^2\pi^2\right)}{n^3}
\end{aligned}
$$

Hence

$$
x^3 \sim \sum_{n=1}^{\infty} -\frac{2(-1)^n\left(-6+n^2\pi^2\right)}{n^3}\sin(nx) \tag{1}
$$

Now we apply Term by term differentiation to the RHS above and obtain

$$
\begin{aligned}
\left(\sum_{n=1}^{\infty} -\frac{2(-1)^n\left(-6+n^2\pi^2\right)}{n^3}\sin(nx)\right)' &= \sum_{n=1}^{\infty} -\frac{2(-1)^n\left(-6+n^2\pi^2\right)}{n^2}\cos(nx) \\
&= \sum_{n=1}^{\infty} \frac{12(-1)^n - 2(-1)^n n^2\pi^2}{n^2}\cos(nx) \\
&= \sum_{n=1}^{\infty}\left(\frac{12(-1)^n}{n^2} - 2(-1)^n\pi^2\right)\cos(nx)
\end{aligned} \tag{2}
$$

And differentiation of LHS of (1) gives

$$
\left(x^3\right)' = 3x^2
$$

Let us now find the Fourier series for $3x^2$ and see if it matches (2). Since x^2 is even, it will only have a_n terms

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} 3x^2 dx \\
&= \frac{3}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\left[x^3\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\left[\pi^3 - (-\pi)^3\right] \\
&= \frac{1}{\pi}\left[\pi^3 + \pi^3\right] \\
&= 2\pi^2
\end{aligned}
$$

And

$$\begin{aligned} a_n &= \frac{3}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\, dx \\ &= \frac{12(-1)^n}{n^2} \end{aligned}$$

Therefore the Fourier series for $3x^2$ is

$$3x^2 \sim \pi^2 + \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^2} \cos(nx) \tag{3}$$

Comparing (2,3) shows they are not the same. (2) has an extra term $-2(-1)^n \pi^2$ inside the sum and it also do not have the added π^2 term outside the sum. The explanation of why that is, is given earlier in the solution.

2.4.7 Problem 3.4.3 (b,d)

Find the Fourier series for the following functions on the indicated intervals, and graph the functions that it converges to. (b) $x^2 - 4$ over $-2 \le x \le 2$. (d) $\sin x$ over $-1 \le x \le 1$.

solution

2.4.7.1 Part (b)

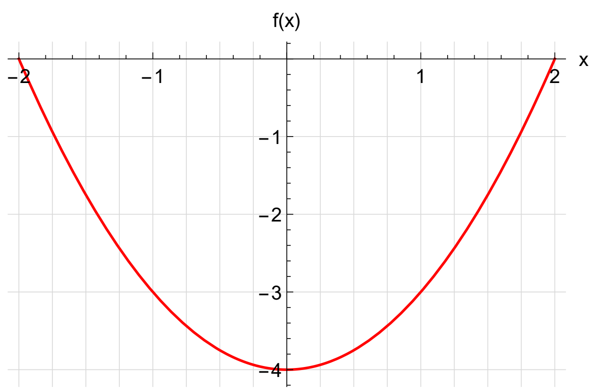


Figure 2.34: Plot of $x^2 - 4$

The function $x^2 - 4$ is even. Hence all b_n terms are zero. The period now is $T = 4$.

$$\begin{aligned}
a_0 &= \frac{1}{2}\int_{-2}^{2}\left(x^2-4\right)dx\\
&= \int_{0}^{2}\left(x^2-4\right)dx\\
&= \left[\frac{x^3}{3}-4x\right]_0^2\\
&= \frac{8}{3}-8\\
&= -\frac{16}{3}
\end{aligned}$$

And

$$a_n = \frac{1}{2}\int_{-2}^{2}\left(x^2-4\right)\cos\left(\frac{2\pi}{T}nx\right)dx$$

But $T = 4$, hence the above becomes

$$\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2}\left(x^2-4\right)\cos\left(\frac{\pi}{2}nx\right)dx\\
&= \int_{0}^{2}\left(x^2-4\right)\cos\left(\frac{\pi}{2}nx\right)dx\\
&= \int_{0}^{2}x^2\cos\left(\frac{\pi}{2}nx\right)dx - \int_{0}^{2}4\cos\left(\frac{\pi}{2}nx\right) \qquad \text{(1A)}
\end{aligned}$$

Looking at the term $\int_0^2 x^2\cos\left(\frac{\pi}{2}nx\right)dx$, applying integration by parts. Let $u = x^2, dv = \cos\left(\frac{\pi}{2}nx\right)$. Then $du = 2x, v = \frac{2}{\pi n}\sin\left(\frac{\pi}{2}nx\right)$. Then using $\int udv = uv - \int vdu$ gives

$$\begin{aligned}
\int_0^2 x^2\cos\left(\frac{\pi}{2}nx\right)dx &= \left[x^2\frac{2}{\pi n}\sin\left(\frac{\pi}{2}nx\right)\right]_0^2 - \int_0^2 2x\frac{2}{\pi n}\sin\left(\frac{\pi}{2}nx\right)dx\\
&= \frac{2}{n\pi}\left[\overbrace{4\sin\left(\pi n\right)}^{0}-0\right] - \frac{4}{\pi n}\int_0^2 x\sin\left(\frac{\pi}{2}nx\right)dx\\
&= -\frac{4}{\pi n}\int_0^2 x\sin\left(\frac{\pi}{2}nx\right)dx
\end{aligned}$$

Applying integration by parts again. Let $u = x, dv = \sin\left(\frac{\pi}{2}nx\right)$. Then $du = 1, v = \frac{-2}{\pi n}\cos\left(\frac{\pi}{2}nx\right)$ then the above becomes

$$
\begin{aligned}
\int_0^2 x^2\cos\left(\frac{\pi}{2}nx\right)dx &= -\frac{4}{\pi n}\left[\frac{-2}{\pi n}\left[x\cos\left(\frac{\pi}{2}nx\right)\right]_0^2 - \int_0^2 \frac{-2}{\pi n}\cos\left(\frac{\pi}{2}nx\right)dx\right] \\
&= -\frac{4}{\pi n}\left[\frac{-2}{\pi n}\left[2\cos(\pi n) - 0\right] + \frac{2}{\pi n}\int_0^2 \cos\left(\frac{\pi}{2}nx\right)dx\right] \\
&= -\frac{4}{\pi n}\frac{-4}{\pi n}(-1)^n + \frac{2}{\pi n}\left(\frac{2}{\pi n}\right)\left[\overbrace{\sin\left(\frac{\pi}{2}nx\right)}^{0}\right]_0^2 \\
&= -\frac{4}{\pi n}\left[\frac{-4}{\pi n}(-1)^n\right] \\
&= \frac{16}{\pi^2 n^2}(-1)^n \qquad \text{(1B)}
\end{aligned}
$$

The above takes care of the first term in (1A). The second term in (1A) is

$$
\begin{aligned}
4\int_0^2 \cos\left(\frac{\pi}{2}nx\right) &= 4\left[\frac{2}{n\pi}\sin\left(\frac{\pi}{2}nx\right)\right]_0^2 \\
&= \frac{8}{n\pi}\left[\sin\left(\frac{\pi}{2}nx\right)\right]_0^2 \\
&= 0 \qquad \text{(1C)}
\end{aligned}
$$

Using (1B,1C) results back in (1A) gives a_n as

$$
a_n = \frac{16}{\pi^2 n^2}(-1)^n
$$

Therefore the Fourier series is

$$
\begin{aligned}
x^2 - 4 &\sim -\frac{8}{3} + \sum_{n=1}^{\infty}\frac{16}{n^2\pi^2}(-1)^n\cos\left(\frac{\pi}{2}nx\right) \\
&\sim -\frac{8}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{\pi}{2}nx\right)
\end{aligned}
$$

The following shows how the above Fourier series converges for increasing number of terms. The convergence is uniform convergence.

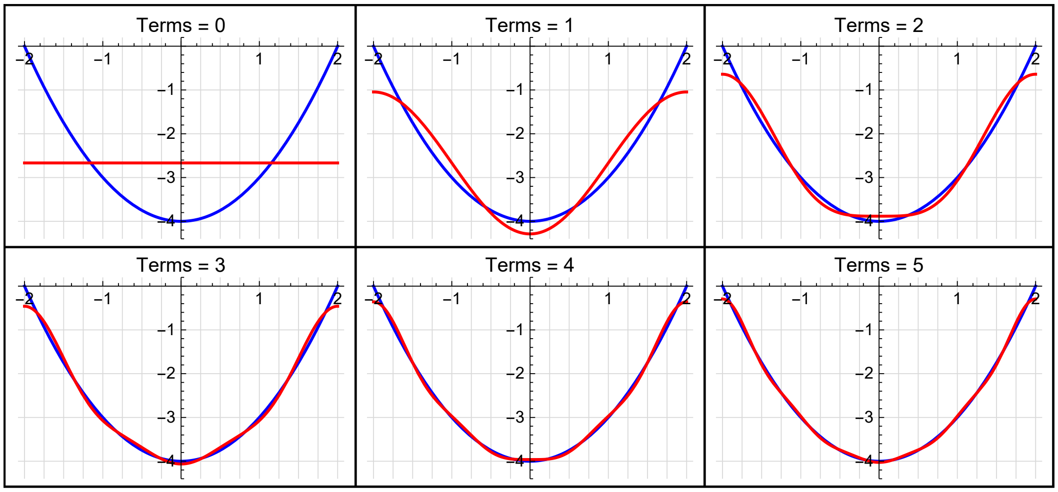


Figure 2.35: Plot of Fourier series for $x^2 - 4$

```
ClearAll[f, x, n, max];
f[x_] := x^2 - 4
fs[x_, max_] := -8 / 3 + 16/π^2 Sum[(-1)^n/n^2 Cos[π/2 n x], {n, 1, max}]
p = Table[
        Plot[{f[x], fs[x, max]}, {x, -2, 2},
           PlotStyle → {Blue, Red},
           GridLines → {Range[-2, 2, 1 / 4], Automatic},
           GridLinesStyle → LightGray,
           PlotRange → {Automatic, {-4.4, .2}},
           PlotLabel → Row[{"Terms = ", max}]
        ]
        ,
        {max, 0, 6, 1}
     ];
p = Grid[Partition[p, 3], Frame → All]
```

Figure 2.36: Code used for the above Plot

2.4.7.2 Part d

The function $\sin x$ is odd. Hence all a_n terms are zero. The period now is $T = 2$.

$$
\begin{aligned}
b_n &= \frac{1}{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sin(x) \sin\left(\frac{2\pi}{T} n x\right) dx \\
&= \int_{-1}^{1} \sin(x) \sin(\pi n x)\, dx
\end{aligned}
$$

But the integrand is even, then the above becomes

$$
b_n = 2\int_0^1 \sin(x) \sin(\pi n x)\, dx
$$

Integration by parts. Let $u = \sin x, dv = \sin(\pi n x)$, then $du = \cos x, v = -\frac{1}{\pi n}\cos(\pi n x)$ and the above becomes

$$
\begin{aligned}
b_n &= 2\left(-\frac{1}{\pi n}\left[\sin x \cos(\pi n x)\right]_0^1 + \frac{1}{\pi n}\int_0^1 \cos x \cos(\pi n x)\, dx\right) \\
&= 2\left(-\frac{1}{\pi n}\left[\sin(1)\cos(\pi n)\right] + \frac{1}{\pi n}\int_0^1 \cos x \cos(\pi n x)\, dx\right) \\
&= \frac{2}{\pi n}\left(-\sin(1)(-1)^n + \int_0^1 \cos x \cos(\pi n x)\, dx\right) \\
&= \frac{2}{\pi n}\left(\sin(1)(-1)^{n+1} + \int_0^1 \cos x \cos(\pi n x)\, dx\right)
\end{aligned}
$$

Integration by parts again. Let $u = \cos x, dv = \cos(\pi n x)$, then $du = -\sin x, v = \frac{1}{\pi n}\sin(\pi n x)$ and the above becomes

$$
\begin{aligned}
b_n &= \frac{2}{\pi n}\left(\sin(1)(-1)^{n+1} + \left(\frac{1}{\pi n}\left[\overbrace{\cos x \sin(\pi n x)}^{0}\right]_0^1 + \frac{1}{\pi n}\int_0^1 \sin x \sin(\pi n x)\right)\right) \\
&= \frac{2}{\pi n}\left(\sin(1)(-1)^{n+1} + \frac{1}{\pi n}\int_0^1 \sin x \sin(\pi n x)\right) \\
&= \frac{2}{\pi n}\sin(1)(-1)^{n+1} + \frac{2}{\pi^2 n^2}\int_0^1 \sin x \sin(\pi n x)
\end{aligned}
$$

But $2\int_0^1 \sin x \sin(\pi n x) = b_n$. Hence the above simplifies to

$$
\begin{aligned}
b_n - \frac{b_n}{\pi^2 n^2} &= \frac{2}{\pi n}\sin(1)(-1)^{n+1}\\
b_n\left(1-\frac{1}{\pi^2 n^2}\right) &= \frac{2}{\pi n}\sin(1)(-1)^{n+1}\\
b_n &= \frac{\frac{2}{\pi n}\sin(1)(-1)^{n+1}}{1-\frac{1}{\pi^2 n^2}}\\
&= \frac{\left(\pi^2 n^2\right)\frac{2}{\pi n}\sin(1)(-1)^{n+1}}{\pi^2 n^2 - 1}\\
&= \frac{2n\pi\sin(1)(-1)^{n+1}}{\pi^2 n^2 - 1}
\end{aligned}
$$

Hence the Fourier series is

$$
\begin{aligned}
\sin x &\sim \sum_{n=1}^{\infty} b_n \sin(\pi n x)\\
&\sim \sum_{n=1}^{\infty} \frac{2n\pi\sin(1)(-1)^{n+1}}{\pi^2 n^2 - 1}\sin(\pi n x)
\end{aligned}
$$

The following shows how the above Fourier series converges for increasing number of terms. The convergence is not uniform since the function is odd. Hence there will be a jump discontinuity when periodic extended leading to Gibbs effect at the edges.

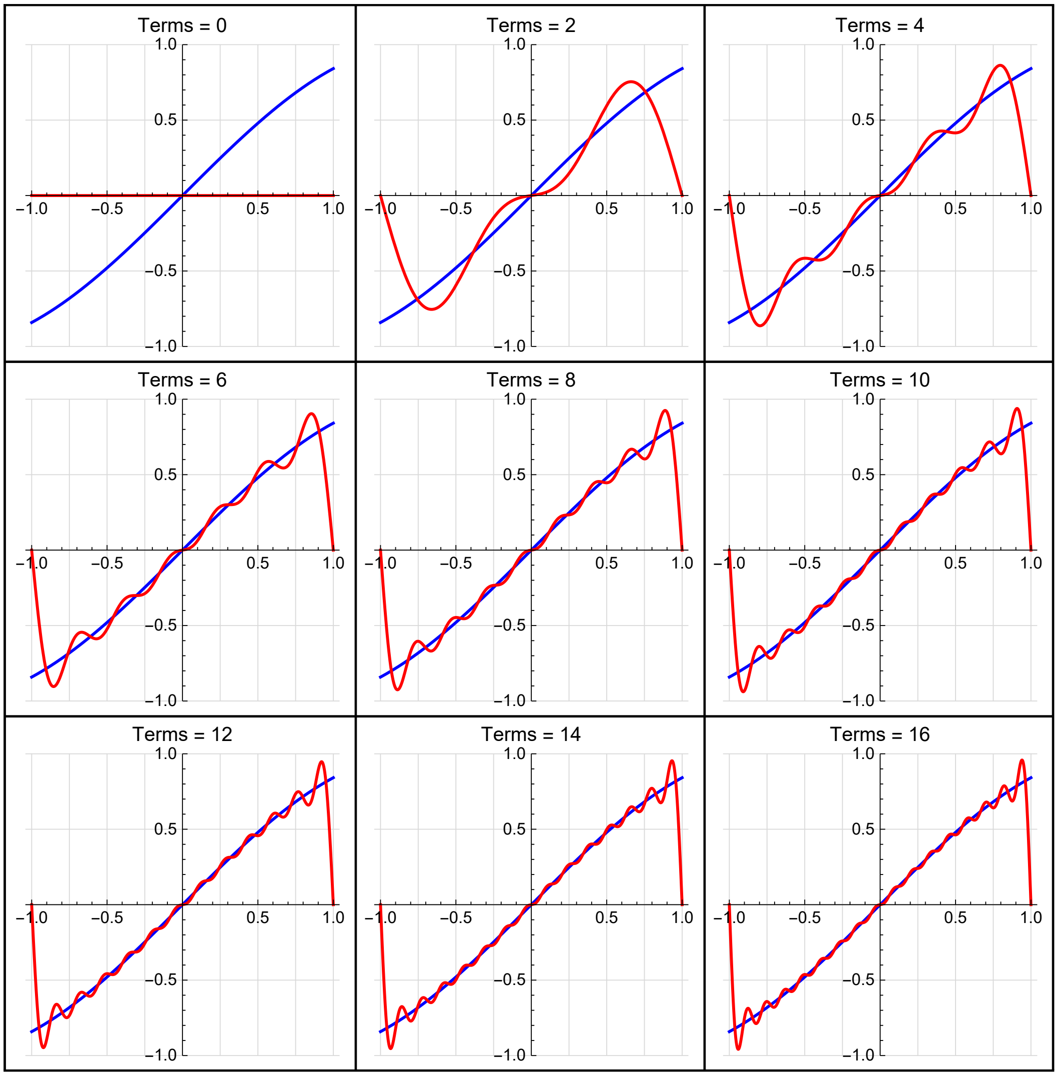


Figure 2.37: Plot of Fourier series for $\sin(x)$

```
In[•]:= ClearAll[f, x, n, max];
f[x_] := Sin[x]
fs[x_, max_] := Sum[-(2 (-1)^n n π Sin[1])/(n^2 π^2 - 1) Sin[π n x], {n, 1, max}]
p = Table[
          Plot[{f[x], fs[x, max]}, {x, -1, 1},
             PlotStyle → {Blue, Red},
             GridLines → {Range[-1, 1, 1/4], Automatic},
             GridLinesStyle → LightGray,
             PlotRange → {Automatic, {-1, 1}},
             PlotLabel → Row[{"Terms = ", max}], AspectRatio -> Automatic
          ]
          ,
          {max, 0, 18, 2}
       ];
p = Grid[Partition[p, 3], Frame → All]
```

Figure 2.38: Code used for the above Plot

2.4.8 Problem 3.4.4

For (b) $x^2 - 4$ over $-2 \leq x \leq 2$. (d) $\sin x$ over $-1 \leq x \leq 1$ write out the differentiated Fourier series and determine whether it converges to the derivative of the original function.

solution

2.4.8.1 Part b

From Problem 3.4.3

$$x^2 - 4 \sim -\frac{8}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{\pi}{2} n x\right) \tag{1}$$

Since the function $x^2 - 4$ is uniform convergent, then we expect that the differentiated Fourier series will converge to the derivative of the original function. The following calculations confirms this.

Taking derivative of the RHS of (1) gives

$$\begin{aligned}\left(-\frac{8}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{\pi}{2} n x\right)\right)' &= \frac{16}{\pi^2} \sum_{n=1}^{\infty} -\left(\frac{\pi}{2} n\right) \frac{(-1)^n}{n^2} \sin\left(\frac{\pi}{2} n x\right) \\ &= \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{\pi}{2} n x\right) \end{aligned} \tag{2}$$

And taking derivative of the LHS of (1) gives

$$\left(x^2-4\right)' = 2x \tag{3}$$

We now need to show if the Fourier series of $2x$ gives the RHS of (2). Let us now find the Fourier series for x, over $-2 \leq x \leq 2$ (period $T=4$). Since x is odd, then all $a_n = 0$.

$$\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} x\sin\left(\frac{2\pi}{T}nx\right)dx \\
&= \frac{1}{2}\int_{-2}^{2} x\sin\left(\frac{\pi}{2}nx\right)dx
\end{aligned}$$

But $x\sin\left(\frac{\pi}{2}nx\right)$ is even. The above becomes

$$b_n = \int_{0}^{2} x\sin\left(\frac{\pi}{2}nx\right)dx$$

Integration by parts. Let $u=x, dv=\sin\left(\frac{\pi}{2}nx\right)$, then $du=1, v=\frac{-2}{n\pi}\cos\left(\frac{\pi}{2}nx\right)$ and the above becomes

$$\begin{aligned}
b_n &= \frac{-2}{n\pi}\left[x\cos\left(\frac{\pi}{2}nx\right)\right]_0^2 + \frac{2}{n\pi}\int_0^2 \cos\left(\frac{\pi}{2}nx\right)dx \\
&= \frac{-2}{n\pi}\left[2\cos(n\pi)\right] + \frac{2}{n\pi}\int_0^2 \cos\left(\frac{\pi}{2}nx\right)dx \\
&= \frac{2}{n\pi}\left(-2\cos(n\pi) + \int_0^2 \cos\left(\frac{\pi}{2}nx\right)dx\right) \\
&= \frac{2}{n\pi}\left(-2\cos(n\pi) + \frac{2}{n\pi}\overbrace{\left[\sin\left(\frac{\pi}{2}nx\right)\right]_0^2}^{0}\right) \\
&= \frac{2}{n\pi}\left(-2\cos(n\pi)\right) \\
&= \frac{-4}{n\pi}(-1)^n
\end{aligned}$$

Hence the Fourier series for x over $-2 \leq x \leq 2$ is

$$x \sim \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{\pi}{2}nx\right)$$

Therefore the Fourier series for $2x$ is

$$2x \sim \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{\pi}{2}nx\right) \tag{4}$$

Comparing (4) and (2) shows they are the same. Hence term by term differentiation is valid in this case.

2.4.8.2 Part d

From Problem 3.4.3, the Fourier series for $\sin x$ over $-1 \leq x \leq 1$ is

$$\sin x \sim \sum_{n=1}^{\infty} -\frac{2n\pi(-1)^n}{n^2\pi^2-1} \sin(1)\sin(\pi n x) \tag{1}$$

Since the Fourier series for $\sin x$ over $-1 \leq x \leq 1$ is not uniform convergent, then we expect that the differentiated Fourier series will not converge to the derivative of the original function. The following calculations confirms this.

Taking derivative of the RHS of (1) gives

$$\begin{aligned}\left(\sum_{n=1}^{\infty} -\frac{2n\pi(-1)^n}{n^2\pi^2-1} \sin(1)\sin(\pi n x)\right)' &= \sum_{n=1}^{\infty} -\pi n\frac{2n\pi(-1)^n}{n^2\pi^2-1} \sin(1)\cos(\pi n x)\\ &= \sum_{n=1}^{\infty} \frac{2n^2\pi^2(-1)^{n+1}}{n^2\pi^2-1} \sin(1)\cos(\pi n x) \qquad (2)\end{aligned}$$

And Taking derivative of the LHS of (1) gives

$$(\sin x)' = \cos x \tag{3}$$

So now we need to show that the Fourier series for $\cos x$, over $-1 \leq x \leq 1$ (period $T=2$) agrees with (2).

Since $\cos x$ is even, then all $b_n = 0$.

$$\begin{aligned}a_0 &= \frac{1}{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(x)\,dx\\ &= \int_{-1}^{1}\cos(x)\,dx\\ &= 2\int_0^1 \cos x dx\\ &= 2\left[\sin(x)\right]_0^1\\ &= 2\sin(1)\end{aligned}$$

And

$$\begin{aligned}a_n &= \frac{1}{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos x\cos\left(\frac{2\pi}{T}nx\right)dx\\ &= \int_{-1}^{1}\cos x\cos(\pi n x)\,dx\\ &= 2\int_0^1 \cos x\cos(\pi n x)\,dx\end{aligned}$$

Integration by parts. Let $u=\cos x, dv = \cos(n\pi x)$, then $du = -\sin x, v = \frac{1}{\pi n}\sin(n\pi x)$ and the above becomes

$$
\begin{aligned}
a_n &= 2\left(\frac{1}{\pi n}\left[\cos x \sin(n\pi x)\right]_0^1 + \frac{1}{\pi n}\int_0^1 \sin x \sin(n\pi x)\,dx\right)\\
&= \frac{2}{\pi n}\left(\left[\cos x \sin(n\pi x)\right]_0^1 + \int_0^1 \sin x \sin(n\pi x)\,dx\right)\\
&= \frac{2}{\pi n}\left(\left[\cos(1)\sin(n\pi)\right] + \int_0^1 \sin x \sin(n\pi x)\,dx\right)\\
&= \frac{2}{\pi n}\int_0^1 \sin x \sin(n\pi x)\,dx
\end{aligned}
$$

Integration by parts again. Let $u = \sin x, dv = \sin(n\pi x)$, then $du = \cos x, v = \frac{-1}{\pi n}\cos(n\pi x)$ and the above becomes

$$
\begin{aligned}
a_n &= \frac{2}{\pi n}\left(\frac{-1}{\pi n}\left[\sin x \cos(n\pi x)\right]_0^1 + \frac{1}{\pi n}\int_0^1 \cos x \cos(n\pi x)\,dx\right)\\
&= \frac{2}{\pi^2 n^2}\left(-\left[\sin(1)\cos(n\pi)\right] + \int_0^1 \cos x \cos(n\pi x)\,dx\right)\\
&= -\frac{2}{\pi^2 n^2}\left(\sin(1)(-1)^n\right) + \frac{2}{\pi^2 n^2}\int_0^1 \cos x \cos(n\pi x)\,dx
\end{aligned}
$$

But $2\int_0^1 \cos x \cos(n\pi x)\,dx = a_n$. Hence the above becomes

$$
\begin{aligned}
a_n - \frac{a_n}{\pi^2 n^2} &= -\frac{2}{\pi^2 n^2}\left(\sin(1)(-1)^n\right)\\
a_n\left(1 - \frac{1}{\pi^2 n^2}\right) &= \frac{2\sin(1)(-1)^{n+1}}{\pi^2 n^2}\\
a_n\left(\frac{\pi^2 n^2 - 1}{\pi^2 n^2}\right) &= \frac{2\sin(1)(-1)^{n+1}}{\pi^2 n^2}\\
a_n &= \frac{2\sin(1)(-1)^{n+1}}{\pi^2 n^2 - 1}
\end{aligned}
$$

Hence the Fourier series for $\cos(x)$ over $-1 \le x \le 1$ is

$$
\begin{aligned}
\cos x &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\pi n x)\\
&\sim \sin(1) + \sum_{n=1}^{\infty} \frac{2\sin(1)(-1)^{n+1}}{\pi^2 n^2 - 1}\cos(\pi n x) \qquad (4)
\end{aligned}
$$

Comparing (4) and (2) shows they are not the same. Hence taking derivatives term by term of the Fourier series was not justified as expected.

2.4.9 Problem 3.4.5

For (b) $x^2 - 4$ over $-2 \le x \le 2$. (d) $\sin x$ over $-1 \le x \le 1$ find the Fourier series for the integral of the function.

solution

2.4.9.1 Part b

From Problem 3.4.3

$$x^2-4\sim -\frac{8}{3}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{\pi}{2}nx\right) \tag{1}$$

Integrating the RHS of (1) gives

$$\begin{aligned}
\int_0^x\left(-\frac{8}{3}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{\pi}{2}ns\right)\right)ds &= -\frac{8}{3}\int_0^x ds+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^x\cos\left(\frac{\pi}{2}ns\right)ds\\
&= -\frac{8}{3}x+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left[\frac{\sin\left(\frac{\pi}{2}ns\right)}{\frac{\pi}{2}n}\right]_0^x\\
&= -\frac{8}{3}x+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{2}{n\pi}\frac{(-1)^n}{n^2}\left[\sin\left(\frac{\pi}{2}ns\right)\right]_0^x\\
&= -\frac{8}{3}x+\frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\left(\frac{\pi}{2}nx\right) \qquad (2)
\end{aligned}$$

Integrating the LHS of (1) gives

$$\begin{aligned}
\int_0^x\left(s^2-4\right)ds &= \left(\frac{s^3}{2}-4s\right)_0^x\\
&= \frac{x^3}{2}-4x \qquad (3)
\end{aligned}$$

Now we find Fourier series for $\frac{x^3}{2}-4x$ and compare it with the (2) to see they match in order to see if term by term integration was justified or not above.

Let $f(x)=\frac{x^3}{2}-4x$ for $-2\leq x\leq 2$. This is an odd function. Hence only b_n exist.

$$\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2}\left(\frac{x^3}{2}-4x\right)\sin\left(\frac{\pi}{2}nx\right)dx\\
&= \frac{1}{4}\int_{-2}^{2}x^3\sin\left(\frac{\pi}{2}nx\right)dx-2\int_{-2}^{2}x\sin\left(\frac{\pi}{2}nx\right)dx \qquad (4)
\end{aligned}$$

Looking at the first integral above, $\frac{1}{4}\int_{-2}^{2}x^3\sin\left(\frac{\pi}{2}nx\right)dx$. Since the integrand is even, then $\frac{1}{4}\int_{-2}^{2}x^3\sin\left(\frac{\pi}{2}nx\right)dx=\frac{1}{2}\int_{0}^{2}x^3\sin\left(\frac{\pi}{2}nx\right)dx$. Integration by parts. $u=x^3, dv=\sin\left(\frac{\pi}{2}nx\right)$ then $du=3x^2, v=-\frac{2}{n\pi}\cos\left(\frac{\pi}{2}nx\right)$. Therefore

$$\begin{aligned}
\frac{1}{2}\int_0^2 x^3\sin\left(\frac{\pi}{2}nx\right)dx &= \frac{1}{2}\left(-\frac{2}{n\pi}\left[x^3\cos\left(\frac{\pi}{2}nx\right)\right]_0^2+\frac{2}{n\pi}\int_0^2 3x^2\cos\left(\frac{\pi}{2}nx\right)dx\right)\\
&= \frac{1}{n\pi}\left(-\left[8\cos(\pi n)\right]+3\int_0^2 x^2\cos\left(\frac{\pi}{2}nx\right)dx\right)\\
&= \frac{1}{n\pi}\left(8(-1)^{n+1}+3\int_0^2 x^2\cos\left(\frac{\pi}{2}nx\right)dx\right)
\end{aligned}$$

Integration by parts again. $u=x^2, dv=\cos\left(\frac{\pi}{2}nx\right)$ then $du=2x, v=\frac{2}{n\pi}\sin\left(\frac{\pi}{2}nx\right)$ and the above becomes

$$\frac{1}{2}\int_0^2 x^3\sin\left(\frac{\pi}{2}nx\right)dx = \frac{1}{n\pi}\left(8(-1)^{n+1} + 3\left(\frac{2}{n\pi}\left[x^2\sin\left(\frac{\pi}{2}nx\right)\right]_0^2 - \frac{2}{n\pi}\int_0^2 2x\sin\left(\frac{\pi}{2}nx\right)dx\right)\right)$$

$$= \frac{1}{n\pi}\left(8(-1)^{n+1} + \frac{6}{n\pi}\left(\overbrace{[4\sin(\pi n)]_0^2}^{0} - 2\int_0^2 x\sin\left(\frac{\pi}{2}nx\right)dx\right)\right)$$

$$= \frac{1}{n\pi}\left(8(-1)^{n+1} - \frac{12}{n\pi}\int_0^2 x\sin\left(\frac{\pi}{2}nx\right)dx\right)$$

Integration by parts again. $u = x, dv = \sin\left(\frac{\pi}{2}nx\right)$ then $du = 1, v = \frac{-2}{n\pi}\cos\left(\frac{\pi}{2}nx\right)$ and the above becomes

$$\frac{1}{2}\int_0^2 x^3\sin\left(\frac{\pi}{2}nx\right)dx = \frac{1}{n\pi}\left(8(-1)^{n+1} - \frac{12}{n\pi}\left(\frac{-2}{n\pi}\left[x\cos\left(\frac{\pi}{2}nx\right)\right]_0^2 - \int_0^2 \frac{-2}{n\pi}\cos\left(\frac{\pi}{2}nx\right)dx\right)\right)$$

$$= \frac{1}{n\pi}\left(8(-1)^{n+1} - \frac{12}{n\pi}\left(\frac{-2}{n\pi}[2\cos(\pi n)] + \frac{2}{n\pi}\int_0^2 \cos\left(\frac{\pi}{2}nx\right)dx\right)\right)$$

$$= \frac{1}{n\pi}\left(8(-1)^{n+1} - \frac{24}{n^2\pi^2}\left(-2(-1)^n + \int_0^2 \cos\left(\frac{\pi}{2}nx\right)dx\right)\right)$$

$$= \frac{1}{n\pi}\left(8(-1)^{n+1} - \frac{24}{n^2\pi^2}\left(-2(-1)^n + \frac{2}{n\pi}\overbrace{\left[\sin\left(\frac{\pi}{2}nx\right)\right]_0^2}^{0}\right)\right)$$

$$= \frac{1}{n\pi}\left(8(-1)^{n+1} + \frac{48}{n^2\pi^2}(-1)^n\right)$$

$$= \frac{1}{n\pi}\left(-8(-1)^n + \frac{48}{n^2\pi^2}(-1)^n\right)$$

$$= \frac{-8}{n\pi}(-1)^n + \frac{48}{n^3\pi^3}(-1)^n \tag{5}$$

The above takes care of the first term in (4). The second integral $2\int_{-2}^2 x\sin\left(\frac{\pi}{2}nx\right)dx$ in (4) is now found. Since integrand is even then

$$2\int_{-2}^2 x\sin\left(\frac{\pi}{2}nx\right)dx = 4\int_0^2 x\sin\left(\frac{\pi}{2}nx\right)dx$$

Integration by parts. Let $u = x, dv = \sin\left(\frac{\pi}{2}nx\right)$, then $du = 1, v = \frac{-2}{n\pi}\cos\left(\frac{\pi}{2}nx\right)$, therefore

$$
\begin{aligned}
4\int_0^2 x\sin\left(\frac{\pi}{2}nx\right)dx &= 4\left(\frac{-2}{n\pi}\left[x\cos\left(\frac{\pi}{2}nx\right)\right]_0^2 + \frac{2}{n\pi}\int_0^2\cos\left(\frac{\pi}{2}nx\right)dx\right)\\
&= 4\left(\frac{-2}{n\pi}\left[2\cos\left(\pi n\right)\right] + \frac{4}{n^2\pi^2}\overbrace{\left[\sin\left(\frac{\pi}{2}nx\right)\right]_0^2}^{0}\right)\\
&= 4\left(\frac{-2}{n\pi}\left[2\left(-1\right)^n\right]\right)\\
&= \frac{-16}{n\pi}\left(-1\right)^n\\
&= \frac{16}{n\pi}\left(-1\right)^{n+1} \qquad (6)
\end{aligned}
$$

Substituting (5,6) back into (4) gives

$$
\begin{aligned}
b_n &= \frac{1}{4}\int_{-2}^2 x^3\sin\left(\frac{\pi}{2}nx\right)dx - 2\int_{-2}^2 x\sin\left(\frac{\pi}{2}nx\right)dx\\
&= \left(\frac{-8}{n\pi}\left(-1\right)^n + \frac{48}{n^3\pi^3}\left(-1\right)^n\right) - \frac{16}{n\pi}\left(-1\right)^{n+1}\\
&= \left(\frac{-8}{n\pi}\left(-1\right)^n + \frac{48}{n^3\pi^3}\left(-1\right)^n\right) + \frac{16}{n\pi}\left(-1\right)^{n}\\
&= \frac{48}{n^3\pi^3}\left(-1\right)^n + \frac{8}{n\pi}\left(-1\right)^n
\end{aligned}
$$

Hence the Fourier series for $\frac{x^3}{2} - 4x$ is

$$
\begin{aligned}
\frac{x^3}{2} - 4x &\sim \sum_{n=1}^\infty b_n\sin\left(\frac{\pi}{2}nx\right)\\
&\sim \sum_{n=1}^\infty\left(\frac{48}{n^3\pi^3}\left(-1\right)^n + \frac{8}{n\pi}\left(-1\right)^n\right)\sin\left(\frac{\pi}{2}nx\right)\\
&\sim \sum_{n=1}^\infty\left(\frac{48 + 8\left(n^2\pi^2\right)}{n^3\pi^3}\right)\left(-1\right)^n\sin\left(\frac{\pi}{2}nx\right)\\
&\sim \sum_{n=1}^\infty\left(\frac{8\left(6 + n^2\pi^2\right)}{n^3\pi^3}\right)\left(-1\right)^n\sin\left(\frac{\pi}{2}nx\right) \qquad (7)
\end{aligned}
$$

Comparing (7,2) shows they are not the same. Hence integration term by term was not justified. This is because the function $x^2 - 4$ is not odd, hence its mean is not zero.

2.4.9.2 Part d

From Problem 3.4.3, the Fourier series for $\sin x$ over $-1 \leq x \leq 1$ is

$$
\sin x \sim \sum_{n=1}^\infty -\frac{2n\pi\left(-1\right)^n}{n^2\pi^2 - 1}\sin\left(1\right)\sin\left(\pi nx\right) \qquad (1)
$$

Integrating the LHS of (1) gives

$$\begin{aligned}\int_0^x \sin(s)\,ds &= -\left[\cos(s)\right]_0^x\\ &= -\left[\cos(x) - 1\right]\\ &= 1 - \cos x \end{aligned}\tag{2}$$

Integrating the RHS of (1) gives

$$\begin{aligned}\int_0^x \sum_{n=1}^{\infty} -\frac{2n\pi(-1)^n}{n^2\pi^2-1}\sin(1)\sin(\pi n s)\,ds &= \sum_{n=1}^{\infty}\frac{2n\pi(-1)^n}{n^2\pi^2-1}\sin(1)\left[\frac{\cos(n\pi s)}{n\pi}\right]_0^x\\ &= \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2\pi^2-1}\sin(1)\left[\cos(n\pi s)\right]_0^x\\ &= \sum_{n=1}^{\infty}\frac{2(-1)^n\sin(1)}{n^2\pi^2-1}\left(\cos(n\pi x)-1\right)\\ &= \sum_{n=1}^{\infty}\frac{2(-1)^n\sin(1)}{n^2\pi^2-1}\cos(n\pi x) - \sin(1)\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2\pi^2-1}\end{aligned}\tag{3}$$

Let $-\sin(1)\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2\pi^2-1} = m$, which is a constant. The above becomes

$$\int_0^x \sum_{n=1}^{\infty} -\frac{2n\pi(-1)^n}{n^2\pi^2-1}\sin(1)\sin(\pi n s)\,ds = \sum_{n=1}^{\infty}\frac{2(-1)^n\sin(1)}{n^2\pi^2-1}\cos(n\pi x) + m \tag{4}$$

But m is the average of the integral of (2) which is, where T the period is 2, gives

$$\begin{aligned} m &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}(1-\cos x)\,dx\\ &= \frac{1}{2}\int_{-1}^{1}(1-\cos x)\,dx\\ &= \frac{1}{2}(x-\sin x)_{-1}^{1}\\ &= \frac{1}{2}(1-\sin(1)-(-1-\sin(-1)))\\ &= \frac{1}{2}(1-\sin(1)-(-1+\sin(1)))\\ &= \frac{1}{2}(2-2\sin(1))\\ &= 1-\sin(1)\end{aligned}$$

Substituting this value for m back into (4) gives

$$\int_0^x \sum_{n=1}^{\infty} -\frac{2n\pi(-1)^n}{n^2\pi^2-1}\sin(1)\sin(\pi n s)\,ds = (1-\sin(1)) + \sum_{n=1}^{\infty}\frac{2(-1)^n\sin(1)}{n^2\pi^2-1}\cos(n\pi x) \tag{5}$$

Now the Fourier series for (2) which is $1-\cos(x)$ is found to compare it to (5) above to see they match in order to see if term by term integration was justified or not above. Since

$1-\cos(x)$ is even, then only a_n are not zero.

$$\begin{aligned}
a_0 &= \frac{1}{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} 1-\cos(x)\,dx\\
&= \int_{-1}^{1} 1-\cos(x)\,dx\\
&= 2-2\sin(1)
\end{aligned}$$

And

$$\begin{aligned}
a_n &= \int_{-1}^{1}(1-\cos(x))\cos(n\pi x)\,dx\\
&= \int_{-1}^{1}\cos(n\pi x)\,dx - \int_{-1}^{1}\cos(x)\cos(n\pi x)\,dx\\
&= 2\int_{0}^{1}\cos(n\pi x)\,dx - 2\int_{0}^{1}\cos(x)\cos(n\pi x)\,dx \qquad (6)
\end{aligned}$$

The first integral in (6)

$$\begin{aligned}
2\int_{0}^{1}\cos(n\pi x)\,dx &= \left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1\\
&= \frac{1}{n\pi}\sin(n\pi)\\
&= 0 \qquad (7)
\end{aligned}$$

The second integral in (6) is $2\int_0^1 \cos(x)\cos(n\pi x)\,dx$. Integration by parts. $u=\cos x, dv=\cos(n\pi x), du=-\sin x, v=\frac{\sin(n\pi x)}{n\pi}$. Therefore

$$\begin{aligned}
2\int_{0}^{1}\cos(x)\cos(n\pi x)\,dx &= 2\left(\left[\cos x\frac{\sin(n\pi x)}{n\pi}\right]_0^1 + \int_0^1 \sin x\frac{\sin(n\pi x)}{n\pi}dx\right)\\
&= 2\left(\frac{1}{n\pi}\overbrace{[\cos x\sin(n\pi x)]_0^1}^{0} + \frac{1}{n\pi}\int_0^1 \sin x\sin(n\pi x)\,dx\right)\\
&= \frac{2}{n\pi}\int_0^1 \sin x\sin(n\pi x)\,dx
\end{aligned}$$

Integration by parts. $u=\sin x, dv=\sin(n\pi x), du=\cos x, v=\frac{-\cos(n\pi x)}{n\pi}$. The above becomes

$$\begin{aligned}
2\int_{0}^{1}\cos(x)\cos(n\pi x)\,dx &= \frac{2}{n\pi}\left(\frac{-1}{n\pi}[\sin x\cos(n\pi x)]_0^1 + \frac{1}{n\pi}\int_0^1 \cos x\cos(n\pi x)\,dx\right)\\
&= \frac{2}{n\pi}\left(\frac{-1}{n\pi}[\sin(1)\cos(n\pi)] + \frac{1}{n\pi}\int_0^1 \cos x\cos(n\pi x)\,dx\right)\\
&= \frac{-2}{n^2\pi^2}\left[\sin(1)(-1)^n\right] + \frac{2}{n^2\pi^2}\int_0^1 \cos x\cos(n\pi x)\,dx
\end{aligned}$$

Moving the integral in the RHS to the left side gives

$$
\begin{aligned}
2\int_0^1 \cos(x)\cos(n\pi x)\,dx - \frac{2}{n^2\pi^2}\int_0^1 \cos x\cos(n\pi x) &= \frac{-2}{n^2\pi^2}\left[\sin(1)(-1)^n\right] \\
\left(2-\frac{2}{n^2\pi^2}\right)\int_0^1 \cos(x)\cos(n\pi x)\,dx &= \frac{-2}{n^2\pi^2}\left[\sin(1)(-1)^n\right] \\
\int_0^1 \cos(x)\cos(n\pi x)\,dx &= \frac{\frac{-1}{n^2\pi^2}\left[\sin(1)(-1)^n\right]}{\left(1-\frac{1}{n^2\pi^2}\right)} \\
&= \frac{-\left(\sin(1)(-1)^n\right)}{n^2\pi^2-1} \qquad (8)
\end{aligned}
$$

Substituting (7,8) back into (6) gives

$$a_n = \frac{2\sin(1)(-1)^n}{n^2\pi^2-1}$$

Hence the Fourier series for $1-\cos(x)$ over $-1\leq x\leq 1$ is

$$
\begin{aligned}
1-\cos(x) &\sim \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos(n\pi x) \\
&\sim \frac{(2-2\sin(1))}{2}+\sum_{n=1}^{\infty}\frac{2(-1)^n\sin(1)}{n^2\pi^2-1}\cos(n\pi x) \\
&\sim (1-\sin(1))+\sum_{n=1}^{\infty}\frac{2(-1)^n\sin(1)}{n^2\pi^2-1}\cos(n\pi x) \qquad (9)
\end{aligned}
$$

Comparing (9) and (5), shows they are the same. This shows that integration term by term was justified. This is because $\sin x$ is continuous and odd, hence its mean is zero. Then by Theorem 3.20 it can be integrated term by term.

2.4.10 Problem 3.5.5 (a,f,i)

Which of the following sequence of functions converge pointwise to the zero function for all $x\in\Re$? Which converges uniformly ?

(a) $-\frac{x^2}{n^2}$ (f) $|x-n|$ (i) $\begin{cases}\frac{x}{n} & |x|<1\\ \frac{1}{nx} & |x|\geq 1\end{cases}$

solution

2.4.10.1 Part a

Let $f_n(x) = -\frac{x^2}{n^2}$. At $x=0$ then $f_n(0)=0$. And for $x\neq 0$ then, if we fix x at say x^* and increase n, then $\lim_{n\to\infty} f_n(x^*)=0$. Hence it converges pointwise for the zero function for all x because for any x, we fix it and do the same as above, which goes to zero for that x.

For uniform convergence, it means that for any x we can find large enough n such that all $f_n(x)$ are inside a tube, of some diameter $<\varepsilon$ around the zero function. But since x is not bounded, then $f_n(x)$ can be as large as we want. So not possible to find n larger enough to bound all $f_n(x)$ for all x $x \in \Re$ to be $< \varepsilon$ from the zero function.

Hence not uniform convergent. The difference between this and the pointwise case earlier, is that here n we find, should work for all x at the same time.

2.4.10.2 Part f

Let $f_n(x) = |x - n|$. At any x, $\lim_{n\to\infty} |x - n|$ is positive. By fixing $x = x^*$, then $f_n(x^*)$ this will keep increasing as n increases. Hence not pointwise convergent to the zero function. Therefore also not uniform convergent since uniform convergence implies pointwise convergence.

2.4.10.3 Part i

At $x = 0, f_n(x) = 0$. And for $|x| < 1, \lim_{n\to\infty} \frac{x}{n} \to 0$ since $|x| < 1$. Hence for $|x| < 1$ it converges pointwise to zero. For $|x| \geq 1$, by fixing $x = x^*$, then $\lim_{n\to\infty} \frac{1}{nx^*} \to 0$ also. Hence converges pointwise to zero for all $x \in \Re$.

For uniform convergence, $\max |f_n(x)| = \frac{1}{n}$ which is at $x = 1$. And $\max |f_n(x)| \to 0$ as $n \to \infty$. Hence we could always find n which will make all $f_n(x)$ within ε from each others at any x by increasing n. Hence uniform convergent

2.4.11 Problem 3.5.7 (b,d,f)

Does the convergence of $v_n(x)$ converges pointwise to the zero function for all $x \in \Re$? Does it converge uniformly?

(b) $v_n(x) = \begin{cases} 1 & n < x < n+1 \\ 0 & \text{otherwise} \end{cases}$ (d) $v_n(x) = \begin{cases} \frac{1}{n} & n < x < 2n \\ 0 & \text{otherwise} \end{cases}$

(f) $v_n(x) = \begin{cases} n^2x^2 - 1 & -\frac{1}{n} < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$

solution

2.4.11.1 Part b

This is a pulse width 1 that keeps moving to the right as n increases. All other values are zero. Hence as $n \to \infty$, the pulse will move to ∞ and all values will be zero. Therefore converges pointwise. Since Max of $v_n(x)$ is 1, then it is not not uniform convergent since for $\varepsilon < 1$, we can not bound $v_n(x)$ for all values for all x to be inside the tube around zero function with width $\varepsilon < 1$.

2.4.11.2 Part d

$n < x < 2n$ is a pulse that moves to the right, but its width also increases as it moves. It height also decreases as it moves, keeping the area of the pulse 1 all the time. Fixing x at x^* the pulse will eventually become zero height at that x. Therefore converges pointwise to the zero function.

For uniform convergence, $\max |v_n(x)| = \frac{1}{n}$ and $\max |v_n(x)| \to 0$ as $n \to \infty$. Hence we could always find n which will make all $f_n(x)$ within ε from each others at any x by increasing n. Hence uniform convergent

2.4.11.3 Part f

As n increases, the range where x is not zero becomes smaller around $x = 0$. The value of $v_n(x)$ can be written as

$$v_n(x) = n^2 e^{2\ln x} - 1$$

As $x \to 0$ from either side, which what happens when $n \to \infty$, then $v_n(x) \to -1$. Hence it does not go to zero at $x = 0$. Therefore not pointwise convergent. It follows that not uniform convergent since uniform convergent implies pointwise convergent.

2.4.12 Key solution for HW 4

Homework 4 Solutions

3.2.34

$f'(x)$ is even.

3.2.37

(a) True. ★ (b) False. Only the restriction of $f(x)$ to $[-\pi, \pi]$ is odd. Its values outside that range are irrelevant as far as its periodic extension is concerned.

3.2.40a

(a) $\frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j \cos(2j+1)x}{2j+1}$; the Fourier series converges non-uniformly to the periodically extended box function, namely to 1 when $\left(2k - \frac{1}{2}\right)\pi < x < \left(2k + \frac{1}{2}\right)\pi$; to $\frac{1}{2}$ when $x = \left(k + \frac{1}{2}\right)\pi$; and to 0 when $\left(2k + \frac{1}{2}\right)\pi \le x \le \left(2k + \frac{3}{2}\right)\pi$ for $k = 0, \pm 1, \pm 2, \ldots$.

3.2.54

We substitute $x = \pi$ into the Fourier series (3.68) for e^x:

$$\tfrac{1}{2}(e^{\pi} + e^{-\pi}) = \frac{\sinh \pi}{2\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k (1 + \mathrm{i}\,k)}{1 + k^2}\, e^{\,\mathrm{i}\,k\pi} = \frac{e^{\pi} - e^{-\pi}}{2\pi} \left(1 + \sum_{k=1}^{\infty} \frac{2}{1 + k^2} \right),$$

which gives the result.

3.2.60

$$\sum_{k=-\infty}^{\infty} c_k\, e^{\,\mathrm{i}\,k x} \;\sim\; f(x) = \overline{f(x)} \;\sim\; \sum_{k=-\infty}^{\infty} \overline{c_k}\, e^{-\,\mathrm{i}\,k x}$$ is real if and only if $c_{-k} = \overline{c_k}$.

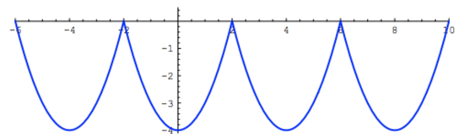
3.3.2

$$x^3 \;\sim\; \sum_{k=1}^{\infty} (-1)^k \left(\frac{12}{k^3} - \frac{2\pi^2}{6k} \right) \sin kx.$$

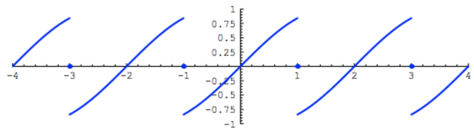
Differentiation does not produce the series for $3x^2$ because the periodic extension of x^3 is not continuous, and so Theorem 3.22 doesn't apply.

3.4.3 b,d

$(b)\ -\frac{8}{3} + \frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos\frac{k\pi x}{2};$



$(d)\ 2\pi\sin 1\sum_{k=1}^{\infty}(-1)^k\frac{k\sin k\pi x}{1-k^2\pi^2};$



3.4.4 (for 3.4.3 b,d)

.4. The differentiated Fourier series only converges when the periodic extension of the function is continuous:

$(b)\ \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sin\frac{k\pi x}{2}$: converges to the 4–periodic extension of $2x$;

$(d)\ 2\pi^2\sin 1\sum_{k=1}^{\infty}(-1)^k\frac{k^2}{1-k^2\pi^2}\cos k\pi x$:

does not converge to the 20–periodic extension of $\cos x$.

3.4.5 (for 3.4.3 b,d)

$$(b)\ \frac{x^3}{3}-4x \sim -\frac{8}{3}x + \frac{32}{\pi^3}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}\sin\frac{k\pi x}{2} \sim \frac{32}{3\pi}\sum_{k=1}^{\infty}\left(\frac{\pi^2k^2+3}{\pi^2k^3}\right)\sin\frac{k\pi x}{2}$$

$$(d)\ \cos x \sim \sin 1 + 2\sin 1\sum_{k=1}^{\infty}(-1)^k\frac{\cos k\pi x}{1-k^2\pi^2}.$$

3.5.5a,f,i

(a) Pointwise, but not uniformly; ★ (f) neither; ★ (i) both.

3.5.7b,d,f

(b) pointwise; ★ (d) pointwise and uniformly; ★ (f) neither pointwise nor uniformly.

2.5 HW 5

Local contents

2.5.1 Problem 3.5.11(e,f)

Which of the following series satisfy the M–test and hence converge uniformly on the interval $I = [0,1]$?

(e) $\sum_{k=1}^{\infty} \frac{e^{kx}}{k^2}$, (f) $\sum_{k=1}^{\infty} \frac{e^{-kx}}{k^2}$

Solution

2.5.1.1 Part e

Using theorem 3.27, we need to find $|u_k(x)| \leq m_k$ for all $x \in I$ such that $\sum_{k=1}^{\infty} m_k < \infty$ to show that series $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly. In this case $u_k(x) = \frac{e^{kx}}{k^2}$. At $x = 0, u_k(0) = \frac{1}{k^2}$ and at $x = 1, u_k(1) = \frac{e^k}{k^2}$. Hence if we pick $m_k = \frac{e^k}{k^2}$ then this will satisfy the condition $|u_k(x)| \leq m_k$. But

$$\sum_{k=1}^{\infty} m_k = \sum_{k=1}^{\infty} \frac{e^k}{k^2}$$

does not converge. This can be shown by ratio test. $\frac{m_{k+1}}{m_k} = \frac{\frac{e^{k+1}}{(k+1)^2}}{\frac{e^k}{k^2}} = \frac{e^{k+1}k^2}{e^k(k+1)^2} = e\frac{k^2}{(k+1)^2}$ and as $k \to \infty$ this goes to e. Which is larger than 1. Therefore $\sum_{k=1}^{\infty} \frac{e^{kx}}{k^2}$ is not uniform convergent.

2.5.1.2 Part e

In this case $u_k(x) = \frac{e^{-kx}}{k^2}$. At $x = 0, u_k(0) = \frac{1}{k^2}$ and at $x = 1, u_k(1) = \frac{1}{e^k k^2}$. Hence if we pick $m_k = \frac{1}{e^k k^2}$ then this will satisfy the condition $|u_k(x)| \leq m_k$.

$$\sum_{k=1}^{\infty} m_k = \sum_{k=1}^{\infty} \frac{1}{e^k k^2}$$

Using the ratio test $\frac{m_{k+1}}{m_k} = \frac{\frac{1}{e^{k+1}(k+1)^2}}{\frac{1}{e^k k^2}} = \frac{e^k k^2}{e^{k+1}(k+1)^2} = \frac{1}{e}\frac{k^2}{(k+1)^2}$and as $k \to \infty$ this goes to $\frac{1}{e}$. Which is smaller than 1. Hence by the ratio test $\sum_{k=1}^{\infty} m_k$ converges. Therefore $\sum_{k=1}^{\infty} \frac{e^{-kx}}{k^2}$ is uniform convergent.

2.5.2 Problem 3.5.21(a,c,e)

First, without explicitly evaluating them, how fast do you expect the Fourier coefficients of the following functions to go to zero as $k \to \infty$? Then prove your claim by evaluating the coefficients. (a) $x - \pi$, (c) x^2, (e) $\sin^2 x$.

Solution

2.5.2.1 Part a

$f(x) = x - \pi$. This is an odd function. Hence $f(-\pi) \neq f(\pi)$. Because of this, there will be a jump discontinuity in the 2π periodic extension. This also implies that the Fourier series is not uniform convergent.

Due to the jump discontinuity the convergence will be slow relative to a Fourier series which converges uniformly, and therefore we expect the b_n terms to be of the form $\frac{1}{n}$ instead of $\frac{1}{n^r}$ with $r > 1$, as would be the case with the faster uniform convergence.

Now we will find the Fourier series to confirm this.

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} (x-\pi)\sin nx dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx - \frac{1}{\pi}\int_{-\pi}^{\pi} \pi \sin nx dx
\end{aligned}
$$

But $\int_{-\pi}^{\pi} \pi \sin nx dx = 0$ since this is an integration over one period. Therefore the above becomes

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx \\
&= \frac{1}{\pi}\left(-\frac{1}{n}\left[x\cos nx\right]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos nx dx\right)
\end{aligned}
$$

But $\int_{-\pi}^{\pi} \cos nx dx = 0$ since this is an integration over one period. The above becomes

$$
\begin{aligned}
b_n &= \frac{-1}{n\pi}\left[x\cos nx\right]_{-\pi}^{\pi} \\
&= \frac{-1}{n\pi}\left[\pi\cos n\pi + \pi\cos n\pi\right] \\
&= \frac{-1}{n\pi}\left[2\pi(-1)^n\right] \\
&= \frac{-2(-1)^n}{n}
\end{aligned}
$$

Hence the Fourier series is

$$x - \pi \sim \sum_{n=1}^{\infty} \frac{-2\,(-1)^n}{n} \sin nx$$

The coefficient is $b_n = \frac{-2(-1)^n}{n}$. We see now that

$$\sum_{n=1}^{\infty} \sqrt{b_n^2} = 4 \sum_{n=1}^{\infty} \frac{1}{n}$$

But $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge, which implies it is not uniform convergent as expected. Piecewise convergence is of order $O\left(\frac{1}{n}\right)$ (slow).

2.5.2.2 Part c

$f(x) = x^2$. This is an even function and $f(-\pi) = f(\pi)$. Hence there will be no jump discontinuity in the 2π periodic extension. Therefore this is uniform convergent. Hence we expect the coefficient to have $\frac{1}{n^r}$ where $r > 1$. For example $\frac{1}{n^2}$. This is because $\sum_{n=1}^{\infty} \sqrt{a_n^2}$ should now converge. This is considered fast convergence. Now we will find the Fourier series to confirm this. Since $f(x)$ is an even function then only a_n exist.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi} \left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{3\pi}\left(\pi^3 + \pi^3\right) = \frac{2}{3}\pi^2$$

And

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx$$

Integration by parts. Let $u = x^2, \cos(nx) = dv$. Then $du = 2x, v = \frac{1}{n}\sin(nx)$. Then using $\int udv = uv - \int vdu$ the above becomes

$$\begin{aligned}
b_n &= \frac{1}{\pi}\left(\frac{1}{n}\left[x^2 \sin(nx)\right]_0^{\pi} - \frac{2}{n}\int_0^{\pi} x \sin(nx)\, dx\right) \\
&= \frac{1}{\pi}\left(-\frac{2}{n}\int_0^{\pi} x \sin(nx)\, dx\right) \\
&= -\frac{2}{n\pi}\int_0^{\pi} x \sin(nx)\, dx
\end{aligned}$$

Integration by parts again. Let $u = x, \sin(nx) = dv$. Then $du = 1, v = \frac{-1}{n}\cos(nx)$. Then using $\int udv = uv - \int vdu$ the above becomes

$$\begin{aligned}
b_n &= -\frac{2}{n\pi}\left(\frac{-1}{n}\left[x\cos(nx)\right]_0^\pi + \frac{1}{n}\int_0^\pi \cos(nx)\right) \\
&= --\frac{2}{n\pi}\left(\frac{-1}{n}\left[\pi\cos(n\pi)\right] + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^\pi\right) \\
&= -\frac{2}{n\pi}\left(\frac{-1}{n}\left[\pi(-1)^n\right]\right) \\
&= \frac{2}{n^2\pi}\left[\pi(-1)^n\right] \\
&= \frac{2}{n^2}(-1)^n
\end{aligned}$$

Hence

$$x^2 \sim \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{2}{n^2}(-1)^n\cos(nx) \tag{1}$$

We see that the coefficient is $a_n = \frac{2}{n^2}(-1)^n$, therefore

$$\sum_{n=1}^{\infty}\sqrt{a_n^2} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

But now $\sum_{n=1}^{\infty}\frac{1}{n^2}$ now converges since the power on n is larger than 1, which implies uniform convergent. Piecewise convergence is of order $O\left(\frac{1}{n^2}\right)$ (fast).

2.5.2.3 Part e

$f(x) = \sin^2 x$. This is an even function and $f(-\pi) = f(\pi)$. This is the same as part c. There will be no jump discontinuity in the 2π periodic extension. Therefore this is uniform convergent. Hence we expect the coefficient to have $\frac{1}{n^r}$ where $r > 1$. For example $\frac{1}{n^2}$ this is because $\sum_{n=1}^{\infty}\sqrt{a_n^2}$ should converge. This is fast convergence. Now we will find the Fourier series to confirm this.

But $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$, hence this is the Fourier series for $\sin^2 x$. If we need to show this explicitly, then since even function only a_n exist.

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2 x dx\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)dx\\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi}dx - \overbrace{\int_{-\pi}^{\pi}\cos 2x dx}^{0}\right)\\
&= \frac{1}{2\pi}2\pi\\
&= 1
\end{aligned}$$

And

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2 x\cos nx dx\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)\cos nx dx\\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi}\cos nx dx - \int_{-\pi}^{\pi}\cos 2x\cos nx dx\right)
\end{aligned}$$

But $\int_{-\pi}^{\pi}\cos nx dx = 0$ since integration over one period, and $\int_{-\pi}^{\pi}\cos 2x\cos nx dx = 0$ for all values other than $n = 2$ by orthogonality. Hence the above simplifies to

$$\begin{aligned}
a_2 &= \frac{1}{2\pi}\left(-\int_{-\pi}^{\pi}\cos^2 2x dx\right)\\
&= \frac{1}{2\pi}(-\pi)\\
&= -\frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\sin^2 x &= \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx\\
&= \frac{1}{2}-\frac{1}{2}\cos 2x
\end{aligned}$$

We see that $\sum_{n=1}^{\infty}\sqrt{a_n^2} = \frac{1}{2} < \infty$. Uniform convergence. Only 2 terms are needed. Very fast convergence.

2.5.3 Problem 3.5.22(a,f)

Using the criteria of Theorem 3.31, determine how many continuous derivatives the functions represented by the following Fourier series have (a) $\sum_{k=-\infty}^{\infty}\frac{e^{ikx}}{1+k^4}$, (f) $\sum_{k=1}^{\infty}\left(1-\cos\frac{1}{k^2}\right)e^{ikx}$

Theorem 3.31. Let $0 \leq n \in \mathbb{Z}$. If the Fourier coefficients of $f(x)$ satisfy

$$\sum_{k=-\infty}^{\infty} |k|^m |c_k| < \infty$$

Then the Fourier series $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ converges uniformly to an n–times continuously differentiable function $\tilde{f}(x) \in C^n$, which is the 2π periodic extension of $f(x)$.

Solution

2.5.3.1 Part a

$$f(x) \sim \sum_{k=-\infty}^{\infty} \frac{e^{ikx}}{1+k^4}$$

Therefore $c_k = \frac{1}{1+k^4}$, hence the series to consider is

$$\begin{aligned}\sum_{k=-\infty}^{\infty} |k|^n |c_k| &= \sum_{k=-\infty}^{\infty} \left|\frac{k^n}{1+k^4}\right| \\ &= \sum_{k=-\infty}^{\infty} \left|\frac{1}{\frac{1}{k^n}+k^{4-n}}\right|\end{aligned}$$

As $k \to \infty$ the term $\frac{1}{k^n} \to 0$. Then we just need to consider k^{4-n}. We want $4-n > 1$ for uniform convergence. Hence

$$\begin{aligned}4-n &> 1 \\ n &< 4\end{aligned}$$

Therefore $n = 3$. The Fourier series converges uniformly to an 3–times continuously differentiable function

2.5.3.2 Part f

$$f(x) \sim \sum_{k=1}^{\infty} \left(1 - \cos\frac{1}{k^2}\right) e^{ikx}$$

Therefore $c_k = 1 - \cos\frac{1}{k^2}$, hence the series to consider is

$$\sum_{k=-\infty}^{\infty} |k|^m |c_k| = \sum_{k=-\infty}^{\infty} |k^m| \left|\left(1 - \cos\frac{1}{k^2}\right)\right|$$

But $\left|\cos\frac{1}{k^2}\right| \leq 1$, hence

$$\sum_{k=-\infty}^{\infty} |k^m| \left|\left(1 - \cos\frac{1}{k^2}\right)\right| \leq 2 \sum_{k=-\infty}^{\infty} |k^m|$$

There is no $n \geq 0$ which will make $\sum_{k=-\infty}^{\infty} |k^n| < \infty$. The Fourier series does not converges uniformly to any continuously differentiable function.

2.5.4 Problem 3.5.26(c,e)

Which of the following sequences converge in norm to the zero function for $x \in \mathbb{R}$? (c) $v_n(x) = \begin{cases} 1 & n < x < n + \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$, (e) $v_n(x) = \begin{cases} \frac{1}{\sqrt{n}} & n < x < 2n \\ 0 & \text{otherwise} \end{cases}$

solution

2.5.4.1 Part c

Using definition 3.35: A sequence $v_n(x)$ is said to converge in the norm to f if $\|v_n - f\| \to 0$ as $n \to \infty$. Therefore, we need to show, since $f = 0$ here, that

$$\lim_{n\to\infty} \|v_n\| \to 0$$

The norm is L^2 which is defined as $\|v_n\| = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} |v_n(x)|^2 \, dx}$, hence

$$\begin{aligned}
\|v_n\| &= \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\begin{cases} 1 & n < x < n + \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}\right|^2 dx} \\
&= \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}\begin{cases} 1 & n < x < n + \frac{1}{n} \\ 0 & \text{otherwise} \end{cases} dx}
\end{aligned}$$

Let us look at the integral $\int_{-\pi}^{\pi}\begin{cases} 1 & n < x < n + \frac{1}{n} \\ 0 & \text{otherwise} \end{cases} dx$. The maximum value of top branch integral is $\int_{-\pi}^{\pi} dx$ which will occur when $x = n > 0$ and $x = n + \frac{1}{n} < \pi$. As this is when the whole pulse is between $[-\pi, \pi]$. When $x = n + \frac{1}{n} > \pi$ the area will be smaller as part of the above will be outside $[-\pi, \pi]$. So we could now consider the integral (its maximum) to be

$$\begin{aligned}
\int_{-\pi}^{\pi} dx &\leq \int_{n}^{n+\frac{1}{n}} dx \\
&= \left(n + \frac{1}{n}\right) - n \\
&= \frac{1}{n}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{n\to\infty} \|v_n\| &\leq \sqrt{\frac{1}{2\pi}\begin{cases} \frac{1}{n} & 0 < n, \frac{n^2+1}{n} < \pi \\ 0 & \text{otherwise} \end{cases}} \\
&= \begin{cases} \sqrt{\frac{1}{2\pi n}} & 0 < n, \frac{n^2+1}{n} < \pi \\ 0 & \text{otherwise} \end{cases} \\
&= 0
\end{aligned}$$

Hence this sequence converges to 0 function in the norm

2.5.4.2 Part e

Using definition 3.35: A sequence $v_n(x)$ is said to converge in the norm to f if $\left\|v_n - f\right\| \to 0$ as $n \to \infty$. Therefore, we need to show, since $f = 0$ here, that

$$\lim_{n\to\infty} \|v_n\| \to 0$$

The norm is L^2 which is defined as $\|v_n\| = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} |v_n(x)|^2\, dx}$, hence

$$\begin{aligned}
\|v_n\| &= \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\begin{cases} \frac{1}{\sqrt{n}} & n < x < 2n \\ 0 & \text{otherwise} \end{cases}\right|^2 dx} \\
&= \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} \begin{cases} \frac{1}{n} & n < x < 2n \\ 0 & \text{otherwise} \end{cases} dx}
\end{aligned}$$

Let us look at the integral $\int_{-\pi}^{\pi} \begin{cases} \frac{1}{n} & n < x < 2n \\ 0 & \text{otherwise} \end{cases} dx$. The maximum value of this integral is $\frac{1}{n}\int_{-\pi}^{\pi} dx$ which will occur when $x = n > 0$ and $x = 2n < \pi$ As this is when the whole pulse is between $[-\pi, \pi]$. So we could now consider the integral (its maximum) to be

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{1}{n} dx &\leq \frac{1}{n}\int_{n}^{2n} dx \\
&= \frac{1}{n}(2n - n) \\
&= \frac{n}{n} \\
&= 1
\end{aligned}$$

Therefore

$$\begin{aligned}
\|v_n\| &\leq \sqrt{\frac{1}{2\pi}} \begin{cases} 1 & 0 < n, 2n < \pi \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \sqrt{\frac{1}{2\pi}} & 0 < n < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Therefore as $n \to \infty$ then $\|v_n\| \to 0$ as the top branch will not be consider as it is limited to $0 < n, 2n < \pi$ or $0 < n < \frac{\pi}{2}$ only. Hence this sequence converges to 0 function in the norm

2.5.5 Problem 3.5.43

For each $n = 1, 2, \cdots$, define the function $f_n(x) = \begin{cases} 1 & \frac{k}{m} \leq x \leq \frac{k+1}{m} \\ 0 & \text{otherwise} \end{cases}$, where $n = \frac{1}{2}m(m+1) + k$ and $0 \leq k \leq m$. (a) Show first that m, k are uniquely determined by n. (b) Then prove that, on the interval $[0, 1]$ the sequence $f_n(x)$ converges in norm to 0 but does not converge pointwise anywhere.

solution

2.5.5.1 Part a

Proof by contradiction. Assuming there exist $m_1, m_2 \geq 0$ where $m_1 \neq m_2$ such that

$$n = \frac{1}{2}m_1(m_1+1)+k$$
$$n = \frac{1}{2}m_2(m_2+1)+k$$

Therefore

$$\frac{1}{2}m_1(m_1+1)+k = \frac{1}{2}m_2(m_2+1)+k$$
$$\frac{1}{2}m_1(m_1+1) = \frac{1}{2}m_2(m_2+1)$$
$$m_1(m_1+1) = m_2(m_2+1)$$

The above is true if $m_1 = m_2$ or if $m_2 = -m_1 - 1$. But m has to be positive. Hence we take the case $m_1 = m_2$. Therefore assumption is not valid. Hence m is unique.

Same proof for k. Assuming there exist $k_1, k_2 \geq 0$ where $k_1 \neq k_2$ such that

$$n = \frac{1}{2}m(m+1)+k_1$$
$$n = \frac{1}{2}m(m+1)+k_2$$

Then

$$\frac{1}{2}m(m+1)+k_1 = \frac{1}{2}m(m+1)+k_2$$

Hence $k_1 = k_2$. Therefore assumption is not valid. Hence k is unique.

2.5.5.2 Part b

$$f_n(x) = \begin{cases} 1 & \frac{k}{m} \leq x \leq \frac{k+1}{m} \\ 0 & \text{otherwise} \end{cases}$$

On the interval [0,1], the norm is L^2 which is defined as $\|f_n\| = \sqrt{\frac{1}{\frac{1}{2}}\int_0^1 |v_n(x)|^2\,dx}$, hence

$$\|f_n\| = \sqrt{2\int_0^1 \left| \begin{cases} 1 & \frac{k}{m} \leq x \leq \frac{k+1}{m} \\ 0 & \text{otherwise} \end{cases} \right|^2 dx}$$
$$= \sqrt{2\int_0^1 \begin{cases} 1 & \frac{k}{m} \leq x \leq \frac{k+1}{m} \\ 0 & \text{otherwise} \end{cases} dx}$$

Let us look at few values of n and see what happens.

For $n = 1, n = \frac{1}{2}m(m+1)+k$. Hence if $m = 1$ then $n = \frac{1}{2}(2)+0 = 1$, Hence $m = 1, k = 0$.

Therefore $\frac{k}{m} \leq x \leq \frac{k+1}{m}$ becomes $0 \leq x \leq 1$.

For $n = 2, n = \frac{1}{2}m(m+1) + k$. Hence if $m = 1$ then $n = \frac{1}{2}(2) + 1 = 1$, Hence $m = 1, k = 1$. Therefore $\frac{k}{m} \leq x \leq \frac{k+1}{m}$ becomes $1 \leq x \leq 2$.

For $n = 3, n = \frac{1}{2}m(m+1) + k$. Hence if $m = 1$ then $n = \frac{1}{2}(2) + 2 = 1$, But $k \leq m$. Try $m = 2$ then $n = \frac{1}{2}(2)(3) + 0 = 1$. Hence $m = 2, k = 0$. Therefore $\frac{k}{m} \leq x \leq \frac{k+1}{m}$ becomes $0 \leq x \leq \frac{1}{2}$.

It looks like the width is becoming smaller as n increases. To verify this, I wrote a small program which determines the width (we only need the width which remains inside [0,1]. Here is the code

```
#problem 3.5.43

f:= proc(num_terms)
local data,m,k,n;
    data:=Array(1..num_terms);
    for n from 1 to num_terms do
        for m from 1 to num_terms do
            if (m/2)*(m+1) = n then
                k:=0;
                data(n):=[m,k];
                break;
            else
                for k from 1 to m do
                    if (m/2)*(m+1)+k=n then
                        data(n):=[m,k];
                        break;
                    fi;
                od;
            fi;
        od;
    od;
    return data;
end proc:

data:=f(50):
#process the k,m found to see how the width changes as n increases.
out_file_name := cat(currentdir(),"/output.txt"):
file_id := fopen(out_file_name,WRITE):
for n from 1 to numelems(data) do
    item:=data(n);
    if item[2]/item[1]<1 then
        the_width:=(item[2]+1)/item[1] - item[2]/item[1];
        the_values:=cat("k=",convert(item[2],string),
                        " m=",convert(item[1],string));
        the_string:=cat(convert(item[2]/item[1],string),
           "<= x <=",convert((item[2]+1)/item[1],string)
           );
        the_width:=cat("Width=",convert(the_width,string));
        print(the_string);
        fprintf(file_id,"n=%-5d%-10s%-15s%-20s\n",
              n,the_values,the_string,the_width);
    fi;
od:
fclose(file_id);
```

And the output obtained

```
n=1     k=0 m=1   0<= x <=1      Width=1
n=3     k=0 m=2   0<= x <=1/2    Width=1/2
n=4     k=1 m=2   1/2<= x <=1    Width=1/2
n=6     k=0 m=3   0<= x <=1/3    Width=1/3
n=7     k=1 m=3   1/3<= x <=2/3  Width=1/3
n=8     k=2 m=3   2/3<= x <=1    Width=1/3
n=10    k=0 m=4   0<= x <=1/4    Width=1/4
n=11    k=1 m=4   1/4<= x <=1/2  Width=1/4
n=12    k=2 m=4   1/2<= x <=3/4  Width=1/4
n=13    k=3 m=4   3/4<= x <=1    Width=1/4
n=15    k=0 m=5   0<= x <=1/5    Width=1/5
n=16    k=1 m=5   1/5<= x <=2/5  Width=1/5
n=17    k=2 m=5   2/5<= x <=3/5  Width=1/5
n=18    k=3 m=5   3/5<= x <=4/5  Width=1/5
n=19    k=4 m=5   4/5<= x <=1    Width=1/5
n=21    k=0 m=6   0<= x <=1/6    Width=1/6
n=22    k=1 m=6   1/6<= x <=1/3  Width=1/6
n=23    k=2 m=6   1/3<= x <=1/2  Width=1/6
n=24    k=3 m=6   1/2<= x <=2/3  Width=1/6
n=25    k=4 m=6   2/3<= x <=5/6  Width=1/6
n=26    k=5 m=6   5/6<= x <=1    Width=1/6
n=28    k=0 m=7   0<= x <=1/7    Width=1/7
n=29    k=1 m=7   1/7<= x <=2/7  Width=1/7
n=30    k=2 m=7   2/7<= x <=3/7  Width=1/7
n=31    k=3 m=7   3/7<= x <=4/7  Width=1/7
n=32    k=4 m=7   4/7<= x <=5/7  Width=1/7
n=33    k=5 m=7   5/7<= x <=6/7  Width=1/7
n=34    k=6 m=7   6/7<= x <=1    Width=1/7
n=36    k=0 m=8   0<= x <=1/8    Width=1/8
n=37    k=1 m=8   1/8<= x <=1/4  Width=1/8
n=38    k=2 m=8   1/4<= x <=3/8  Width=1/8
n=39    k=3 m=8   3/8<= x <=1/2  Width=1/8
n=40    k=4 m=8   1/2<= x <=5/8  Width=1/8
n=41    k=5 m=8   5/8<= x <=3/4  Width=1/8
n=42    k=6 m=8   3/4<= x <=7/8  Width=1/8
n=43    k=7 m=8   7/8<= x <=1    Width=1/8
n=45    k=0 m=9   0<= x <=1/9    Width=1/9
n=46    k=1 m=9   1/9<= x <=2/9  Width=1/9
n=47    k=2 m=9   2/9<= x <=1/3  Width=1/9
n=48    k=3 m=9   1/3<= x <=4/9  Width=1/9
n=49    k=4 m=9   4/9<= x <=5/9  Width=1/9
n=50    k=5 m=9   5/9<= x <=2/3  Width=1/9
```

We see from the above that as n increases the range $\frac{k}{m} \leq x \leq \frac{k+1}{m}$ either goes outside the $[0,1]$ domain as in the case of $n = 2,5,9$ or stays inside $[0,1]$ but it becomes smaller with $n = 10$ giving $0 \leq x \leq \frac{1}{4}$ while $n = 1$ it was $0 \leq x \leq 1$.

Since we are integrating 1 over this range, and the width of integration is getting smaller and smaller, then for very large n the integral goes to zero as the width goes to zero.

In other words, we can bound the integral from above as

$$\begin{aligned}
\sqrt{2\int_0^1 \begin{cases} 1 & \frac{k}{m} \leq x \leq \frac{k+1}{m} \\ 0 & \text{otherwise} \end{cases} dx} &\leq \lim_{n\to\infty} \sqrt{2\int_0^{\frac{1}{n}} dx} \\
&= \lim_{n\to\infty} \sqrt{2}\frac{1}{n} \\
&= 0
\end{aligned}$$

Hence the sequence $f_n(x)$ converges in norm to 0. For piecewise convergence. The definition is that for any $\varepsilon > 0$, there exist $N(\varepsilon, x)$ such that $|f_n(x)| < \varepsilon$ for all $n \geq N$ for $x \in [0,1]$. This means if we fix x then $\lim_{n\to\infty} |f_n(x)| = 0$. But this does not happen here. Since the pulse shifts left and right all the time as the width gets smaller as n increases. For example, if we look at $x = \frac{1}{2}$ and then increase n, we see that $f_n\left(\frac{1}{2}\right)$ do not go to zero there as the function moves around due to changing of the domain. Hence it is not piecewise convergent.

2.5.6 Problem 4.1.7

The convection-diffusion equation $u_t + cu_x = \gamma u_{xx}$ is a simple model for the diffusion of a pollutant in a fluid flow moving with constant speed c. Show that $v(t,x) = u(t, x+ct)$ solves the heat equation. What is the physical interpretation of this change of variables?

solution

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt}$$

But $\frac{dx}{dt} = c$, the speed of fluid. Hence the above becomes

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}$$

But $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \gamma u_{xx}$, hence the above becomes

$$\frac{\partial v}{\partial t} = \gamma u_{xx}$$

But $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial t}\frac{dt}{dx} + \frac{\partial v}{\partial x}\frac{dx}{dx} = \frac{\partial v}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial^2 t}\frac{dt}{dx} + \frac{\partial^2 v}{\partial x^2}\frac{dx}{dx} = \frac{\partial^2 v}{\partial x^2}$. Hence the above gives

$$\frac{\partial v}{\partial t} = \gamma v_{xx}$$

Which is the heat equation. The change of variable puts the observer as moving with the same speed as fluid instead of stationary observer. It is a coordinates transformation.

2.5.7 Problem 4.1.10(a,c)

For each of the following initial temperature distributions, (i) write out the Fourier series solution to the heated ring (4.30–32), and (ii) find the resulting equilibrium temperature (a) $f(x) = \cos x$, (c) $f(x) = |x|$.

The heated ring problem (4.30–32) is: Solve for $u(x,t)$ in

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad -\pi < x < \pi, t > 0$$

With periodic BC $u(-\pi,t) = u(\pi,t), u_x(-\pi,t) = u_x(\pi,t)$ for $t \geq 0$. With initial conditions $u(x,0) = f(x)$

solution

2.5.7.1 Part a

Starting with the series solution as given in (4.34)

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-n^2 t}(a_n \cos nx + b_n \sin nx) \tag{1}$$

At $t = 0$ the above becomes (using $u(x,0) = \cos x$)

$$\cos x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

Hence a_n, b_n are the Fourier series coefficients of $\cos x$. Therefore $a_1 = 1$ and all other a_n, b_n are zero in order to match the left side with the right side.

The solution in (1) now becomes

$$u(x,t) = e^{-t} \cos x$$

The above is the Fourier series solution. To answer (ii), we let $t \to \infty$ in the above. This shows that equilibrium temperature will be zero.

2.5.7.2 Part b

Starting with the series solution as given in (4.34)

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-n^2 t}(a_n \cos nx + b_n \sin nx) \tag{1}$$

At $t = 0$ the above becomes (using $u(x,0) = |x|$)

$$|x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

Hence a_n, b_n are the Fourier series coefficients of $|x|$. But $|x|$ is even. Hence $b_n = 0$. So we only need to find a_0, a_n

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

Because $f(x)$ is even the above simplifies to

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} x dx \\ &= \frac{1}{\pi}\left[x^2\right]_0^{\pi} \\ &= \frac{1}{\pi}\left[\pi^2\right] \\ &= \pi \end{aligned}$$

And

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

But $f(x)$ is even and $\cos nx$ is even, hence product is even. The above simplifies to

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\cos nx dx \end{aligned}$$

Integration by parts gives

$$\begin{aligned} a_n &= \frac{2}{\pi}\left(\overbrace{\left[x\frac{\sin nx}{n}\right]_0^{\pi}}^{0} - \int_0^{\pi}\frac{\sin nx}{n}dx\right) \\ &= \frac{2}{\pi}\left(\frac{1}{n}\left[\frac{\cos nx}{n}\right]_0^{\pi}\right) \\ &= \frac{2}{\pi n^2}\left(\cos n\pi - 1\right) \\ &= \frac{2}{\pi n^2}\left((-1)^n - 1\right) \end{aligned}$$

Therefore (1) becomes

$$u(x,t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} e^{-n^2 t}\left(\frac{2}{\pi n^2}\left((-1)^n - 1\right)\cos nx\right) \tag{1A}$$

The above is the Fourier series solution. To answer (ii), we let $t \to \infty$ in the above. This shows that equilibrium temperature will become

$$u_{eq}(x,t) = \frac{\pi}{2}$$

2.5.8 Problem 4.1.16(a,b)

The cable equation $v_t = \gamma v_{xx} - \alpha v$ with $\gamma, v > 0$, also known as the lossy heat equation,was derived by the nineteenth-century Scottish physicist William Thomson to model propagation of signals in a transatlantic cable. Later, in honor of his work on thermodynamics, including determining the value of absolute zero temperature, he was named Lord Kelvin by Queen Victoria. The cable equation was later used to model the electrical activity of neurons. (a) Show that the general solution to the cable equation is given by $v(x,t) = e^{-\alpha t}u(x,t)$ where $u(x,t)$ solves the heat equation $u_t = \gamma u_{xx}$.

(b) Find a Fourier series solution to the Dirichlet initial-boundary value problem $v_t = \gamma v_{xx} - \alpha v$, with initial conditions $v(x,0) = f(x)$ and boundary conditions $v(0,t) = 0, v(1,t) = 0$ for $0 \le x \le 1, t > 0$. Does your solution approach an equilibrium value? If so, how fast?

solution

2.5.8.1 Part a

Given

$$v(x,t) = e^{-\alpha t}u(x,t) \tag{1}$$

Hence

$$\frac{\partial v}{\partial t} = -\alpha e^{-\alpha t}u + e^{-\alpha t}\frac{\partial u}{\partial t} \tag{2}$$

And

$$\begin{aligned} \frac{\partial v}{\partial x} &= e^{-\alpha t}\frac{\partial u}{\partial x} \\ \frac{\partial^2 v}{\partial x^2} &= e^{-\alpha t}\frac{\partial^2 u}{\partial x^2} \end{aligned} \tag{3}$$

Substituting (1,2,3) into $v_t = \gamma v_{xx} - \alpha v$ gives

$$-\alpha e^{-\alpha t}u + e^{-\alpha t}\frac{\partial u}{\partial t} = \gamma e^{-\alpha t}\frac{\partial^2 u}{\partial x^2} - \alpha e^{-\alpha t}u$$

Canceling $e^{-\alpha t} \neq 0$ from all the terms gives

$$\begin{aligned} -\alpha u + \frac{\partial u}{\partial t} &= \gamma\frac{\partial^2 u}{\partial x^2} - \alpha u \\ \frac{\partial u}{\partial t} &= \gamma\frac{\partial^2 u}{\partial x^2} \end{aligned}$$

Which is what problem asked to show.

2.5.8.2 Part b

Now we need to solve

$$v_t = \gamma v_{xx} - \alpha v \tag{1}$$

With initial and boundary conditions given. Using separation of variable, let $v = T(t)X(x)$ where $T(t)$ is function that depends on time only and $X(x)$ is a function that depends on x only. Using this substitution in (1) gives

$$T'X = \gamma X''T - \alpha XT$$

Dividing by $XT \neq 0$ gives

$$\frac{1}{\gamma}\frac{T'}{T}+\frac{\alpha}{\gamma}=\frac{X''}{X}=-\lambda$$

Where λ is the separation constant. The above gives two ODE's to solve

$$\begin{aligned} X''+\lambda X &= 0\\ X(0) &= 0\\ X(1) &= 0 \end{aligned} \tag{2}$$

And

$$\begin{aligned} \frac{1}{\gamma}\frac{T'}{T}+\frac{\alpha}{\gamma} &= -\lambda\\ T'+\alpha T &= -\lambda\gamma T\\ T'+\alpha T+\lambda\gamma T &= 0\\ T'+\left(\alpha+\lambda\gamma\right)T &= 0 \end{aligned} \tag{3}$$

ODE (2) is the boundary value ODE which will generate the eigenvalues and eigenfunctions.

<u>case $\lambda<0$</u>

Let $-\lambda=\mu^2$. The solution to (2) becomes

$$X=c_1\cosh\left(\mu x\right)+c_2\sinh\left(\mu x\right)$$

At $x=0$

$$0=c_1$$

Hence the solution becomes $X=c_2\sinh\left(\mu x\right)$. At $x=1$ this gives $0=c_2\sinh\left(\mu\right)$. But $\sinh\left(\mu\right)=0$ only when $\mu=0$ which is not the case here. Hence $c_2=0$ leading to trivial solution. Therefore $\lambda<0$ is not eigenvalue.

<u>case $\lambda=0$</u>

The solution is $X(x)=c_1x+c_2$. At $x=0$ this becomes $0=c_2$. Hence solution is $X=c_1x$. At $x=1$ this gives $0=c_1$. Therefore trivial solution. Hence $\lambda=0$ is not eigenvalue.

<u>case $\lambda>0$</u>

Solution is

$$X(x)=c_1\cos\left(\sqrt{\lambda}x\right)+c_2\sin\left(\sqrt{\lambda}x\right)$$

At $x=0$ this results in $0=c_1$. The above now becomes

$$X(x)=c_2\sin\left(\sqrt{\lambda}x\right)$$

At $x=1$

$$0=c_2\sin\left(\sqrt{\lambda}\right)$$

For non-trivial solution we want $\sin\left(\sqrt{\lambda}\right)=0$ or $\sqrt{\lambda}=n\pi, n=1,2,\cdots$. Hence

$$\lambda_n=n^2\pi^2 \qquad n=1,2,\cdots$$

And the corresponding eigenfunctions

$$X_n(x) = \sin(n\pi x) \tag{4}$$

Now we can solve (3)

$$T' + \left(\alpha + \lambda\gamma\right) T = 0$$
$$T_n' + \left(\alpha + n^2\pi^2\gamma\right) T_n = 0$$

The solution is

$$T_n(t) = b_n e^{-\left(\alpha+n^2\pi^2\gamma\right)t} \tag{5}$$

Where b_n is arbitrary constant of integration that depends on b. From (4,5) we obtain the fundamental solution

$$v_n(x,t) = b_n e^{-\left(\alpha+n^2\pi^2\gamma\right)t} \sin(n\pi x)$$

The general solution is linear combination of the above

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\alpha+n^2\pi^2\gamma\right)t} \sin(n\pi x) \tag{6}$$

At $t = 0$ the above becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

We see that b_n are the Fourier coefficients of $f(x)$, after odd extending it from $[-1,1]$. Therefore, the period of $f(x)$ becomes 2.

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx$$

Since $f(x)$ is odd (we did odd extension) and since sin is odd, then the product is even, and the above becomes

$$b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

Using the above in (6) gives

$$v(x,t) = \sum_{n=1}^{\infty} 2\left(\int_0^1 f(x)\sin(n\pi x)\,dx\right) e^{-\left(\alpha+n^2\pi^2\gamma\right)t} \sin(n\pi x)$$

To find equilibrium, we let $t \to \infty$ then $e^{-\left(\alpha+n^2\pi^2\gamma\right)t} \to 0$ because $\alpha,\gamma > 0$ and the above becomes

$$v_{eq}(x,t) = 0$$

2.5.9 Key solution for HW 5

3.5.11e,f

(*e*) doesn't pass test; ★ (*f*) uniformly convergent.

3.5.21 a,c,e

(*a*) The periodic extension is not continuous, and so the best one could hope for is $a_k, b_k \to 0$ like $1/k$. Indeed, $a_0 = -2\pi$, $a_k = 0$, $b_k = (-1)^{k+1}2/k$, for $k > 0$.

(*c*) The periodic extension is C^0, and so we expect $a_k, b_k \to 0$ like $1/k^2$. Indeed,

$$a_0 = \tfrac{2}{3}\pi^2, \quad a_k = (-1)^k 4/k^2, \quad b_k = 0, \quad \text{for } k > 0.$$

(*e*) The periodic extension is C^∞, and so we expect $a_k, b_k \to 0$ faster than any (negative) power of k. Indeed, $a_0 = 1$, $a_2 = -\frac{1}{2}$, and all other $a_k = b_k = 0$.

3.5.22 a,f

3.5.26 c,e

★ (*c*) converges in norm; ★ (*e*) does not converge in norm.

3.5.43

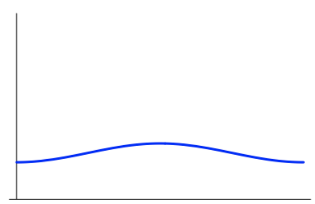
4.1.7

(*a*) $u(t,x) = \dfrac{1}{4} - \dfrac{2}{\pi^2}\displaystyle\sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} \exp\left(-(4j+2)^2\pi^2 t\right)\cos(4j+2)\pi x$; (*b*) $\frac{1}{4}$;

(*c*) At an exponential rate of $e^{-4\pi^2 t}$;

(*d*) As $t \to \infty$, the solution becomes a vanishingly small cosine wave centered around $u = \frac{1}{4}$, namely

$$u(t,x) \approx \frac{1}{4} - \frac{2}{\pi^2}e^{-4\pi^2 t}\cos 2\pi x:$$



4.1.10 a, c

4.1.10. (a) $u(t,x) = e^{-t}\cos x$; equilibrium temperature: $u(t,x) \to 0$.

★ (c) $u(t,x) = \frac{1}{2}\pi - \dfrac{4}{\pi}\displaystyle\sum_{k=0}^{\infty} \dfrac{e^{-(2k+1)^2 t}\cos(2k+1)x}{(2k+1)^2}$; equilibrium temperature: $u(t,x) \to \frac{1}{2}\pi$.

4.1.16 a, b

(a) If $u(t,x) = e^{\alpha t} v(t,x)$, then

$$\frac{\partial u}{\partial t} = \alpha e^{\alpha t} v(t,x) + e^{\alpha t}\frac{\partial v}{\partial t}(t,x) = \gamma e^{\alpha t}\frac{\partial^2 v}{\partial x^2} = \gamma \frac{\partial^2 u}{\partial x^2}.$$

(b) $v(t,x) = e^{-\alpha t}\displaystyle\sum_{n=1}^{\infty} b_n e^{-(\alpha+\gamma n^2\pi^2)t}\sin n\pi x$, where $b_n = 2\displaystyle\int_0^1 f(x)\sin n\pi x\,dx$ are the Fourier sine coefficients of the initial data. All solutions tend to the equilibrium value $u(t,x) \to 0$ as $t \to \infty$ at an exponential rate. For most initial data, i.e., those with $b_1 \neq 0$, the decay rate is $e^{-a t}$, where $a = \alpha + \gamma\pi^2$; other solutions decay at a faster rate.

2.6 HW 6

Local contents

2.6.1 Problem 4.1.4

Find a series solution to the initial-boundary value problem for the heat equation $u_t = u_{xx}$ for $0 < x < 1$ when one the end of the bar is held at 0 degree and the other is insulated. Discuss the asymptotic behavior of the solution as $t \to \infty$

Solution

The problem did not say which end is insulated. So assuming the left end is at 0 degree and the right end is the one that is insulated.

Using L for the length to make the solution more general and at the end L is replaced by 1. Assuming the initial conditions is $u(x,0) = f(x)$. Therefore the problem to solve is to solve for $u(x,t)$ in

$$u_t = u_{xx} \qquad 0 < x < L, t > 0$$

With boundary conditions

$$u(0,t) = 0$$
$$u_x(L,t) = 0$$

And initial conditions

$$u(x,0) = f(x)$$

Let $u(x,t) = T(t) X(x)$, then the PDE becomes

$$T'X = X''T$$

Dividing by XT

$$\frac{T'}{T} = \frac{X''}{X}$$

Since each side depends on different independent variable and both are equal, they must be both equal to same constant, say $-\lambda$. Where λ is real.

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

The two ODE's are

$$T' + \lambda T = 0 \tag{1}$$

And the eigenvalue ODE

$$\begin{aligned} X'' + \lambda X &= 0 \\ X(0) &= 0 \\ X'(L) &= 0 \end{aligned} \tag{2}$$

Now we solve (2) to find the eigenvalues and eigenfunctions.

<u>Case $\lambda < 0$</u>

Let $-\lambda = \omega^2$. Hence the ODE is $X'' - \omega^2 X = 0$ and the solution becomes

$$X(x) = C_1 \cosh(\omega x) + C_2 \sinh(\omega x)$$

At $x = 0$ the above gives

$$0 = C_1$$

Hence the solution now becomes

$$X(x) = C_2 \sinh(\omega x)$$

Taking derivative gives

$$X'(x) = \omega C_2 \sinh(\omega x)$$

At $x = L$

$$0 = \omega C_2 \cosh(\omega L)$$

But $\cosh(\omega L)$ is never zero. Therefore $C_2 = 0$ which leads to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.

<u>Case $\lambda = 0$</u>

The space equation becomes $X'' = 0$ with the solution

$$X = Ax + B$$

At $x = 0$ the above gives $0 = B$. Therefore the solution is $X = Ax$. Taking derivative gives $X' = A$. At $x = L$ this gives $0 = A$. Which leads to trivial solutions. Therefore <u>$\lambda = 0$ is not an eigenvalue.</u>

<u>Case $\lambda > 0$</u>

Starting with the space ODE, the solution is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

Left B.C. gives

$$0 = A$$

The solution becomes

$$X(x) = B\sin\left(\sqrt{\lambda}x\right)$$

Taking derivative gives

$$X'(x) = \sqrt{\lambda}B\cos\left(\sqrt{\lambda}x\right)$$

Applying right B.C. gives

$$0 = \sqrt{\lambda}B\cos\left(\sqrt{\lambda}L\right)$$

For non trivial solution we want $\cos\left(\sqrt{\lambda}L\right) = 0$ or

$$\sqrt{\lambda} = \frac{n\pi}{2L} \qquad n = 1,3,5,\cdots$$

Hence the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{2L}\right)^2 \qquad n = 1,3,5,\cdots$$

Therefore the eigenfunctions are

$$X_n(x) = \sin\left(\frac{n\pi}{2L}x\right) \qquad n = 1,3,5,\cdots$$

Now that we found the eigenvalues, we can solve the time ODE (1).

$$\begin{aligned} T_n' + \lambda_n T &= 0 \\ T_n &= B_n e^{-\lambda_n t} \\ &= B_n e^{-\left(\frac{n\pi}{2L}\right)^2 t} \end{aligned}$$

Hence the fundamental solution is

$$\begin{aligned} u_n(x,t) &= X_n T_n \\ u(x,t) &= \sum_{n=1,3,5,\cdots}^{\infty} B_n \sin\left(\frac{n\pi}{2L}x\right) e^{-\left(\frac{n\pi}{2L}\right)^2 t} \end{aligned} \tag{3}$$

From initial conditions

$$f(x) = \sum_{n=1,3,5,\cdots}^{\infty} B_n \sin\left(\frac{n\pi}{2L}x\right)$$

Multiplying both sides by $\sin\left(\frac{m\pi}{2L}x\right)$ and integrating

$$\int_0^L f(x)\sin\left(\frac{m\pi}{2L}x\right)dx = \int_0^L \left(\sum_{n=1,3,5,\cdots}^{\infty} B_n \sin\left(\frac{m\pi}{2L}xx\right)\sin\left(\frac{n\pi}{2L}x\right)\right)dx$$

Interchanging order of summation and integration and applying orthogonality between $\cos$ functions results in

$$\begin{aligned} \int_0^L f(x)\sin\left(\frac{m\pi}{2L}x\right)dx &= \int_0^L B_m \sin^2\left(\frac{m\pi}{2L}x\right)dx \\ &= B_m \frac{L}{2} \end{aligned}$$

Therefore

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{2L}x\right)dx$$

Therefore the solution is (3) becomes

$$u(x,t) = \frac{2}{L}\sum_{n=1,3,5,\cdots}^{\infty}\left(\int_0^L f(x)\sin\left(\frac{n\pi}{2L}x\right)dx\right)\sin\left(\frac{n\pi}{2L}x\right)e^{-\left(\frac{n\pi}{2L}\right)^2 t}$$

For $L = 1$ the above becomes

$$u(x,t) = 2\sum_{n=1,3,5,\cdots}^{\infty}\left(\int_0^1 f(x)\sin\left(\frac{n\pi}{2}x\right)dx\right)\sin\left(\frac{n\pi}{2}x\right)e^{-\left(\frac{n\pi}{2}\right)^2 t}$$

The above can be rewritten as

$$u(x,t) = 2\sum_{n=0}^{\infty}\left(\int_0^1 f(x)\sin\left(\frac{(2n+1)\pi}{2}x\right)dx\right)\sin\left(\frac{(2n+1)\pi}{2}x\right)e^{-\left(\frac{(2n+1)\pi}{2}\right)^2 t}$$

As $t\to\infty$ and since $\left(\frac{(2n-1)\pi}{2}\right)^2$ is positive and assuming the integral is finite which is valid for well behaved $f(x)$ the solution then $\lim_{t\to\infty} e^{-\left(\frac{(2n-1)\pi}{2}\right)^2 t} \to 0$ and the solution above becomes

$$\lim_{t\to\infty} u(x,t) = 0$$

This makes sense, since the right side of the bar is insulated, meaning no heat will escape from that side, and the left side is at kept a zero temperature. Therefore after long time the initial temperature distribution given by $f(x)$ will reach equilibrium which is zero temperature due to the left side kept at zero and since there are no external heat sources or heat sinks.

2.6.2 Problem 4.1.7

A metal bar of length $L = 1$ and thermal diffusivity $\gamma = 1$ is fully insulated, including its ends. Suppose the initial temperature distribution is

$$u(x,0) = f(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ 1-x & \frac{1}{2} \leq x \leq 1 \end{cases}$$

(a) Use Fourier series to write down the temperature distribution at time $t > 0$. (b) What is the equilibrium temperature distribution in the bar, i.e., for $t \gg 0$? (c) How fast does the solution go to equilibrium? (d) Just before the temperature distribution reaches equilibrium, what does it look like? Sketch a picture and discuss

Solution

2.6.2.1 Part (a)

Using L for the length to make the solution more general and at the end L is replaced 1.

$$\begin{aligned}\frac{\partial u}{\partial t} &= \gamma\frac{\partial^2 u}{\partial x^2}\\ u_x(0,t) &= 0\\ u_x(L,t) &= 0\\ u(x,0) &= f(x)\end{aligned}$$

Let $u(x,t) = T(t)X(x)$, then the PDE becomes

$$\frac{1}{\gamma}T'X = X''T$$

Dividing by $XT \neq 0$

$$\frac{1}{\gamma}\frac{T'}{T} = \frac{X''}{X}$$

Since each side depends on different independent variable and both are equal, they must be both equal to same constant, say $-\lambda$. Where λ is assumed real.

$$\frac{1}{\gamma}\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

The two ODE's generated are

$$T' + \gamma\lambda T = 0 \tag{1}$$

And the eigenvalue ODE

$$\begin{aligned}X'' + \lambda X &= 0 \qquad (2)\\ X'(0) &= 0\\ X'(L) &= 0\end{aligned}$$

Starting with the eigenvalue ODE equation (2). The following cases are considered.

<u>case $\lambda < 0$</u>

In this case, $-\lambda$ is positive. Let $-\lambda = \omega^2$. Hence the ODE is $X'' - \omega^2 X = 0$ and the solution becomes

$$X(x) = C_1\cosh(\omega x) + C_2\sinh(\omega x)$$

Therefore

$$X' = C_1\sinh(\omega x) + C_2\cosh(\omega x)$$

Applying the left B.C. gives

$$0 = C_2$$

Therefore the solution becomes $X(x) = C_1\cosh(\omega x)$ and $X'(x) = C_1\sinh(\omega x)$. Applying the right B.C. gives

$$0 = C_1\sinh(\omega L)$$

For non-trivial solution we want $\sinh(\omega L) = 0$. But this is not possible since $\sinh$ is zero when its argument is zero, which is not the case here. Hence only trivial solution results from this case. $\lambda < 0$ is not an eigenvalue.

case $\lambda = 0$

The solution is

$$\begin{aligned} X(x) &= c_1 x + c_2 \\ X'(x) &= c_1 \end{aligned}$$

Applying left boundary conditions gives

$$0 = c_1$$

Hence the solution becomes $X(x) = c_2$. Therefore $\frac{dX}{dx} = 0$. Applying the right B.C. provides no information. Any c_2 will work. Therefore this case leads to the solution $X(x) = c_2$. Associated with this one eigenvalue, the time equation becomes $T_0'(t) = 0$ hence $T_0(t)$ is a constant. Hence the solution $u_0(x,t)$ associated with this $\lambda = 0$ is

$$\begin{aligned} u_0(x,t) &= X_0 T_0 \\ &= A_0 \end{aligned}$$

where constant $c_2 T_0$ was renamed to $\frac{A_0}{2}$ to indicate it is associated with $\lambda = 0$. $\lambda = 0$ is an eigenvalue with eigenfunction constant $\frac{A_0}{2}$.

case $\lambda > 0$

The solution is

$$\begin{aligned} X(x) &= c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right) \\ X'(x) &= -c_1\sqrt{\lambda} \sin\left(\sqrt{\lambda}x\right) + c_2\sqrt{\lambda} \cos\left(\sqrt{\lambda}x\right) \end{aligned}$$

Applying the left B.C. gives

$$0 = c_2\sqrt{\lambda}$$

Therefore $c_2 = 0$ as $\lambda > 0$. The solution becomes

$$X(x) = c_1 \cos\left(\sqrt{\lambda}x\right)$$

And $X'(x) = -c_1\sqrt{\lambda} \sin\left(\sqrt{\lambda}x\right)$. Applying the right B.C. gives

$$0 = -c_1\sqrt{\lambda} \sin\left(\sqrt{\lambda}L\right)$$

$c_1 = 0$ gives a trivial solution. Selecting $\sin\left(\sqrt{\lambda}L\right) = 0$ gives

$$\sqrt{\lambda}L = n\pi \qquad n = 1,2,3,\cdots$$

Or

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots$$

Therefore the eigenfunctions are

$$X_n(x) = \cos\left(\frac{n\pi}{L}x\right) \qquad n = 1,2,3,\cdots$$

The time solution is found by solving

$$T_n'(t) + \gamma\lambda_n T_n(t) = 0$$

This has the solution

$$\begin{aligned} T_n(t) &= A_n e^{-\gamma\lambda_n t} \\ &= A_n e^{-\gamma\left(\frac{n\pi}{L}\right)^2 t} \qquad n = 1,2,3,\cdots \end{aligned}$$

The solution to the PDE is

$$u_n(x,t) = T_n(t) X_n(x) \qquad n = 0,1,2,3,\cdots$$

But for linear system sum of eigenfunctions is also a solution. Hence

$$\begin{aligned} u(x,t) &= u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) \\ &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\gamma\left(\frac{n\pi}{L}\right)^2 t} \end{aligned} \tag{1}$$

From the solution found above, setting $t = 0$ gives

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

Hence A_0, A_n are the Fourier $\cos$ coefficients for the function $f(x)$. Doing an even extension of $f(x)$ from $[-L, L]$, then $\frac{A_0}{2}$ is the average of the function $f(x)$ over $[-L, L]$. But this average is seen as $\frac{2\left(\frac{1}{2}\times\frac{1}{2}\right)}{2} = \frac{1}{4}$. The term $\frac{1}{2}\times\frac{1}{2}$ is the area of $f(x)$ from $[0, L]$.

$$\frac{A_0}{2} = \frac{1}{4}$$

For A_n

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right) dx$$

Replacing $L = 1$ and using the definition of $f(x)$ given above gives

$$A_n = \int_{-1}^{1} f(x)\cos\left(\frac{n\pi}{L}x\right) dx$$

But $f(x)$ is even (after even extending) and $\cos$ is even, hence the above becomes

$$\begin{aligned} A_n &= 2\int_0^1 f(x)\cos(n\pi x)\,dx \\ &= 2\left(\int_0^{\frac{1}{2}} x\cos(n\pi x)\,dx + \int_{\frac{1}{2}}^1 (1-x)\cos(n\pi x)\,dx\right) \\ &= 2\left(\int_0^{\frac{1}{2}} x\cos(n\pi x)\,dx + \int_{\frac{1}{2}}^1 \cos(n\pi x)\,dx - \int_{\frac{1}{2}}^1 x\cos(n\pi x)\,dx\right) \end{aligned} \tag{2}$$

But

$$
\begin{aligned}
\int_a^b x\cos(n\pi x)\,dx &= \frac{1}{n\pi}\left[x\sin(n\pi x)\right]_a^b - \frac{1}{n\pi}\int_a^b \sin(n\pi x)\,dx\\
&= \frac{1}{n\pi}\left[x\sin(n\pi x)\right]_a^b + \frac{1}{n^2\pi^2}\left[\cos(n\pi x)\right]_a^b \qquad (3)
\end{aligned}
$$

When $a=0, b=\frac{1}{2}$ the above gives

$$
\begin{aligned}
\int_0^{\frac{1}{2}} x\cos(n\pi x)\,dx &= \frac{1}{n\pi}\left[x\sin(n\pi x)\right]_0^{\frac{1}{2}} + \frac{1}{n^2\pi^2}\left[\cos(n\pi x)\right]_0^{\frac{1}{2}}\\
&= \frac{1}{n\pi}\left(\frac{1}{2}\sin\left(\frac{n\pi}{2}\right)\right) + \frac{1}{n^2\pi^2}\left(\cos\left(\frac{n\pi}{2}\right)-1\right)\\
&= \frac{1}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2}\left(\cos\left(\frac{n\pi}{2}\right)-1\right)\\
&= \frac{1}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2\pi^2} \qquad (4)
\end{aligned}
$$

And when $a=\frac{1}{2}, b=1$ (3) gives

$$
\begin{aligned}
\int_{\frac{1}{2}}^1 x\cos(n\pi x)\,dx &= \frac{1}{n\pi}\left[x\sin(n\pi x)\right]_{\frac{1}{2}}^1 + \frac{1}{n^2\pi^2}\left[\cos(n\pi x)\right]_{\frac{1}{2}}^1\\
&= \frac{1}{n\pi}\left[\sin(n\pi) - \frac{1}{2}\sin\left(\frac{n\pi}{2}\right)\right] + \frac{1}{n^2\pi^2}\left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right]\\
&= -\frac{1}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2}\cos(n\pi) - \frac{1}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) \qquad (5)
\end{aligned}
$$

Substituting (4,5) into (2) gives

$$
\begin{aligned}
\frac{A_n}{2} = &\frac{1}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2\pi^2}\\
&+ \int_{\frac{1}{2}}^1 \cos(n\pi x)\,dx\\
&- \left(-\frac{1}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2}\cos(n\pi) - \frac{1}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right)\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
\frac{A_n}{2} = &\frac{1}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2\pi^2}\\
&+ \frac{1}{n\pi}\overbrace{\sin(n\pi)}^{0} - \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)\\
&+ \frac{1}{2n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n^2\pi^2}\cos(n\pi) + \frac{1}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right)
\end{aligned}
$$

Or

$$\begin{aligned}\frac{A_n}{2} &= \left(\frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2\pi^2}\right) - \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n^2\pi^2}\cos(n\pi)\\ &= \frac{2\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n}{n^2\pi^2}\end{aligned}$$

Therefore the solution (1) becomes, replacing $L = 1$

$$\begin{aligned}u(x,t) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi x)\, e^{-\gamma n^2\pi^2 t}\\ &= \frac{1}{4} + 2\sum_{n=1}^{\infty} \frac{2\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n}{n^2\pi^2}\cos(n\pi x)\, e^{-\gamma n^2\pi^2 t} \qquad (6)\end{aligned}$$

2.6.2.2 Part b

From the solution (6) in part (a), since $\gamma > 0$ then $\lim_{t\to\infty} e^{-\gamma n^2\pi^2 t} = 0$ and the solution becomes

$$\lim_{t\to\infty} u(x,t) = \frac{1}{4}$$

This is the average of the initial temperature distribution. This makes sense, since there are no sources or sinks, and both ends are insulated. So all of the initial heat will remain in the bar but will average evenly over the bar length at the average which is $\frac{1}{4}$.

2.6.2.3 Part c

due to the exponential decay term $e^{-\gamma n^2\pi^2 t}$ and also having $\frac{1}{n^2}$ term, the decay of the sum is very fast. High frequency terms decay very fast since $e^{-\gamma n^2\pi^2 t} \lll 1$ for large n. Using $\gamma = 1$ only few terms are needed to show this. The solution goes to the average (the constant term in the Fourier series) at exponential rate.

This will be shown explicitly in the next part by plotting the solution using $\gamma = 1$ for illustration.

2.6.2.4 Part d

The following shows how fast the initial temperature reach equilibrium $\frac{1}{4}$ degree over the whole bar. Using only 4 terms in the Fourier series, and using $\gamma = 1$, it took only 0.1 seconds. Looking at the middle of the bar, where the initial temperature was highest at 0.5, we first see that initial temperature which was not smooth, become instantaneously smooth. Then it took 0.5 seconds for the temperature in the middle of the bar to go down to 0.3 degrees. And the next 0.5 second to go down to 0.25. This shows that the initial decay was rapid, then it slows down relatively until it reaches 0.25 degree which is the average then stops there.

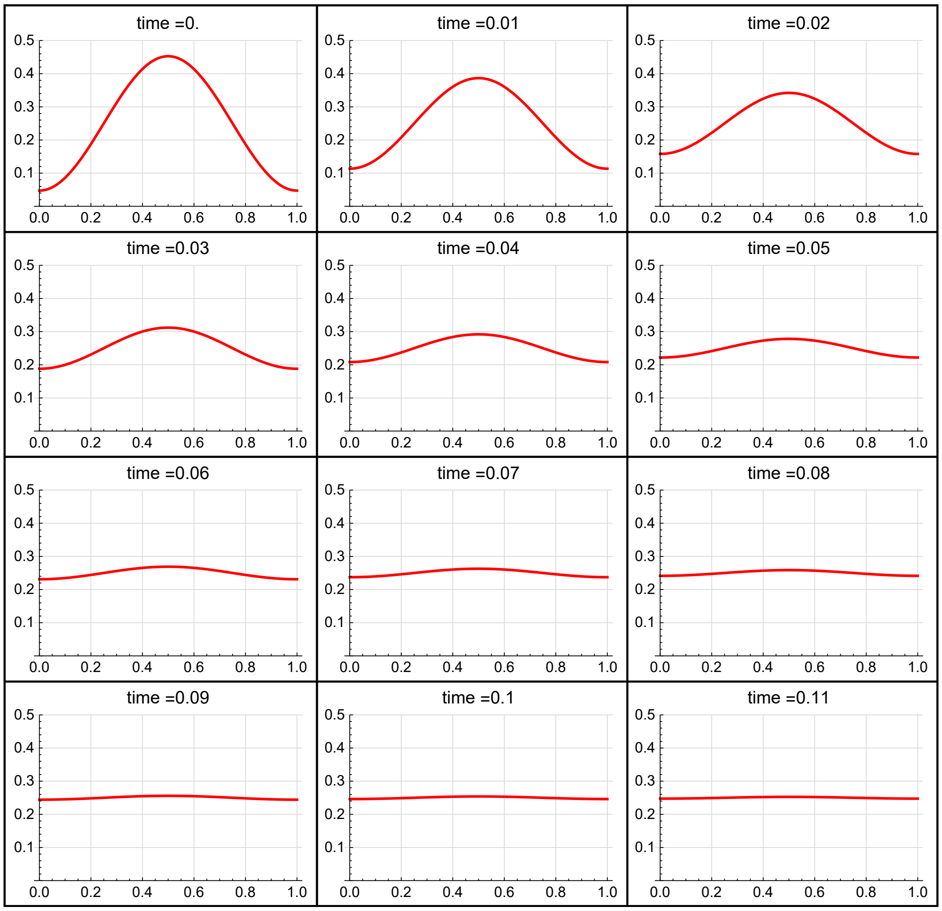


Figure 2.39: Plot showing solution in time

```
u[x_, t_, max_] :=
    1/4 + 2 Sum[ 1/(n^2 π^2) (2 Cos[n π/2] - 1 - (-1)^n) Cos[n π x] Exp[- n^2 π^2 t], {n, 1, max}];
p = Grid[Partition[Table[Quiet@Plot[u[x, t, 4], {x, 0, 1}, PlotRange → {Automatic, {0, 0.5}},
                GridLines → Automatic, GridLinesStyle → LightGray, PlotStyle → Red,
                PlotLabel → Row[{"time =", t}]], {t, 0, .11, 0.01}], 3], Frame → All];
```

Figure 2.40: Code used for the above plot

2.6.3 Problem 4.1.10c

For each of the following initial temperature distributions, (i) write out the Fourier series solution to the heated ring (4.30–32), and (ii) find the resulting equilibrium temperature as $t \to \infty$ (c) $u(x, 0) = |x|$

Solution

2.6.3.1 Part I

The heated ring is given by 4.30-4.32 as solving for $u(x,t)$ in

$$u_t = u_{xx} \qquad -\pi < x < \pi, t > 0$$

With periodic BC

$$\begin{aligned}
u(-\pi,t) &= u(\pi,t)\\
u_x(-\pi,t) &= u_x(\pi,t)
\end{aligned}$$

And initial conditions $u(x,0) = f(x) = |x|$. As given in the text, the Fourier series solution is (4.35)

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) e^{-n^2 t}$$

Since $f(x)$ is even, then all $b_n = 0$.

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx\\
&= \frac{2}{\pi}\int_0^{\pi} x dx\\
&= \frac{2}{\pi}\frac{1}{2}\left[x^2\right]_0^{\pi}\\
&= \pi
\end{aligned}$$

And

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx\\
&= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx\\
&= \frac{2}{\pi}\left(\overbrace{\left[\frac{x\sin nx}{n\pi}\right]_0^{\pi}}^{0} - \int_0^{\pi}\frac{\sin nx}{n\pi}dx\right)\\
&= \frac{2}{\pi}\left(\frac{1}{n\pi}\left[\cos nx\right]_0^{\pi}\right)\\
&= \frac{2}{n\pi^2}(\cos n\pi - 1)\\
&= \frac{2}{n\pi^2}\left((-1)^n - 1\right)
\end{aligned}$$

Hence the solution becomes

$$u(x,t) = \frac{\pi}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n}\cos(nx)\,e^{-n^2 t}$$

2.6.3.2 Part II

From the solution above, we see that

$$\lim_{t\to\infty} u(x,t) = \frac{\pi}{2}$$

Which is the average of the original temperature distribution.

2.6.4 Problem 4.1.16

The cable equation $v_t = \gamma v_{xx} - \alpha v$ with $\gamma, \alpha > 0$, also known as the lossy heat equation,was derived by the nineteenth-century Scottish physicist William Thomson to model propagation of signals in a transatlantic cable. Later, in honor of his work on thermodynamics, including determining the value of absolute zero temperature, he was named Lord Kelvin by Queen Victoria. The cable equation was later used to model the electrical activity of neurons. (a) Show that the general solution to the cable equation is given by $v(x,t) = e^{-\alpha t}u(x,t)$ where $u(x,t)$ solves the heat equation $u_t = \gamma u_{xx}$.

(b) Find a Fourier series solution to the Dirichlet initial-boundary value problem $v_t = \gamma v_{xx} - \alpha v$, with initial conditions $v(x,0) = f(x)$ and boundary conditions $v(0,t) = 0, v(1,t) = 0$ for $0 \leq x \leq 1, t > 0$. Does your solution approach an equilibrium value? If so, how fast? (c) Answer part (b) for the Neumann problem

$$v_t = \gamma v_{xx} - \alpha v \qquad 0 \leq x \leq 1, t > 0$$

With initial conditions

$$v(x,0) = f(x)$$

And B.C.

$$v_x(0,t) = 0$$
$$v_x(1,t) = 0$$

<u>Solution</u>

2.6.4.1 Part c

Part (a,b) were solved in HW5 so we only need to solve part c here.

Using separation of variable, let $v = T(t)X(x)$ where $T(t)$ is function that depends on time only and $X(x)$ is a function that depends on x only. Using this substitution in (1) gives

$$T'X = \gamma X''T - \alpha XT$$

Dividing by $XT \neq 0$ gives

$$\frac{1}{\gamma}\frac{T'}{T} + \frac{\alpha}{\gamma} = \frac{X''}{X} = -\lambda$$

Where λ is the separation constant. The above gives two ODE's to solve

$$\begin{aligned} X'' + \lambda X &= 0 \\ X'(0) &= 0 \\ X'(1) &= 0 \end{aligned} \tag{2}$$

And

$$\begin{aligned} \frac{1}{\gamma}\frac{T'}{T} + \frac{\alpha}{\gamma} &= -\lambda \\ T' + \alpha T &= -\lambda\gamma T \\ T' + \alpha T + \lambda\gamma T &= 0 \\ T' + \left(\alpha + \lambda\gamma\right) T &= 0 \end{aligned} \tag{3}$$

ODE (2) is the boundary value ODE which will generate the eigenvalues and eigenfunctions.

<u>case $\lambda < 0$</u>

Let $-\lambda = \omega^2$. The solution to (2) becomes

$$\begin{aligned} X &= c_1 \cosh(\omega x) + c_2 \sinh(\omega x) \\ X' &= \omega c_1 \sinh(\omega x) + \omega c_2 \cosh(\omega x) \end{aligned}$$

At $x = 0$

$$0 = \omega c_2$$

Therefore $c_2 = 0$. The solution becomes

$$\begin{aligned} X &= c_1 \cosh(\omega x) \\ X' &= \omega c_1 \sinh(\omega x) \end{aligned}$$

At $x = 1$ this gives $0 = \omega c_1 \sinh(\omega)$. But $\sinh(\omega) = 0$ only when $\omega = 0$ which is not the case here. Hence $c_1 = 0$ leading to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.

<u>case $\lambda = 0$</u>

The solution is $X(x) = c_1 x + c_2$ and $X' = c_1$. At $x = 0$ this gives $0 = c_1$. Hence solution is $X = c_2$ and $X' = 0$. At $x = 1$ this gives $0 = 0$. Therefore any c_2 will work. Taking $c_2 = 1$ the eigenfunction is $X_0(x) = 1$ and $\lambda = 0$ is eigenvalue.

<u>case $\lambda > 0$</u>

Solution is

$$\begin{aligned} X(x) &= c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right) \\ X'(x) &= \sqrt{\lambda}c_1 \sin\left(\sqrt{\lambda}x\right) + c_2\sqrt{\lambda} \cos\left(\sqrt{\lambda}x\right) \end{aligned}$$

At $x = 0$ this results in $0 = c_2\sqrt{\lambda}$. Hence $c_2 = 0$. The solution now becomes The above now becomes

$$X(x) = c_1 \cos\left(\sqrt{\lambda}x\right)$$
$$X'(x) = -c_1 \sin\left(\sqrt{\lambda}x\right)$$

At $x = 1$

$$0 = -c_1 \sin\left(\sqrt{\lambda}\right)$$

For non-trivial solution we want $\sin\left(\sqrt{\lambda}\right) = 0$ or $\sqrt{\lambda} = n\pi, n = 1, 2, \cdots$. Hence

$$\lambda_n = n^2\pi^2 \qquad n = 1, 2, \cdots$$

And the corresponding eigenfunctions

$$X_n(x) = \cos(n\pi x) \tag{4}$$

Now we can solve the time ODE (3). For the zero eigenvalue, (3) becomes

$$T' + \alpha T = 0$$

With solution

$$T_0(t) = \frac{A_0}{2} e^{-\alpha t}$$

And for the non zero eigenvalues $\lambda_n = n^2\pi^2$ the ODE (3) becomes

$$T' + \left(\alpha + n^2\pi^2\gamma\right) T = 0$$

With solution

$$T_n(t) = A_n e^{-\left(\alpha + n^2\pi^2\gamma\right)t}$$

The general solution is linear combination of the above

$$v(x,t) = \frac{A_0}{2} e^{-\alpha t} + \sum_{n=1}^{\infty} A_n e^{-\left(\alpha + n^2\pi^2\gamma\right)t} \cos(n\pi x) \tag{6}$$

At $t = 0$ the above becomes

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi x)$$

We see that A_n are the cosine Fourier coefficients of $f(x)$, after even extending $f(x)$ to $[-1, 1]$, the period of $f(x)$ becomes 2 giving

$$A_0 = \int_{-1}^{1} f(x)\,dx$$
$$= 2\int_{0}^{1} f(x)\,dx$$

And

$$A_n = \int_{-1}^{1} f(x)\cos(nx)\,dx$$
$$= 2\int_{0}^{1} f(x)\cos(nx)\,dx$$

Using the above in solution (6) gives

$$v(x,t) = \left(\int_0^1 f(x)\,dx\right)e^{-\alpha t} + 2\sum_{n=1}^{\infty}\left(\int_0^1 f(x)\cos(nx)\,dx\right)e^{-\left(\alpha+n^2\pi^2\gamma\right)t}\cos(n\pi x)$$

To find equilibrium, we let $t\to\infty$ then $e^{-\left(\alpha+n^2\pi^2\gamma\right)t}\to 0$ and also $e^{-\alpha t}$ because $\alpha,\gamma>0$ and the above becomes

$$v_{eq}(x,t) = 0$$

The decay is fast due to $e^{-\left(\alpha+n^2\pi^2\gamma\right)t}\gg 1$ for large n. Hence it is exponential decay. Solution each equilibrium value of 0 where it remains there.

2.6.5 Problem 4.2.3d

Write down the solutions to the following initial-boundary value problems for the wave equation in the form of a Fourier series

$$u_{tt} = 4u_{xx} \tag{1}$$

With boundary conditions

$$\begin{aligned} u(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned}$$

And initial conditions

$$\begin{aligned} u(x,0) &= x \\ u_t(x,0) &= -x \end{aligned}$$

Solution

To make the solution more general and useful, the length is taken as L and initial conditions $u(x,0)=f(x)$ and $u_t(x,0)=g(x)$ and $c^2=4$, and then at the end these are replaced by the actual values given in this problem which are $L=1, f(x)=x, g(x)=-x, c^2=4$.

Hence the PDE to solve is $u_{tt} = c^2u_{xx}$ with BC $u(0,t) = 0, u(L,0) = 0$ and $u(x,0) = f(x), u_t(x,0) = g(x)$.

Using separation of variables, let $u=X(x)T(t)$. The PDE becomes

$$\begin{aligned} \frac{T''X}{c^2} &= X''T \\ \frac{1}{c^2}\frac{T''}{T} &= \frac{X''}{X} = -\lambda \end{aligned}$$

Where λ is separation constant. Hence the eigenvalue ODE is

$$\begin{aligned} X''+\lambda X &= 0 \\ X(0) &= 0 \\ X(L) &= 0 \end{aligned} \tag{2}$$

And the time ODE is

$$T''+c^2\lambda T = 0 \tag{3}$$

Starting by the eigenvalue ODE to determine the eigenvalues and eigenfunctions.

Case $\lambda < 0$

Let $-\lambda = \omega^2$. Hence the ODE is $X'' - \omega^2 X = 0$ and the solution becomes

$$X(x) = C_1 \cosh(\omega x) + C_2 \sinh(\omega x)$$

At $x = 0$ the above gives

$$0 = C_1$$

Hence the solution now becomes

$$X(x) = C_2 \sinh(\omega x)$$

At $x = L$ the above gives

$$0 = C_2 \sinh(\omega L)$$

But $\sinh$ is zero only when its argument is zero which is not the case here. Therefore $C_2 = 0$ which leads to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.

Case $\lambda = 0$

The space equation becomes $X''(x) = 0$ with the solution

$$X = Ax + B$$

At $x = 0$ the above gives $0 = B$. Therefore the solution is $X = Ax$. At $x = L$ this gives $0 = AL$. Hence $A = 0$, which leads to trivial solutions. Therefore $\lambda = 0$ is not an eigenvalue.

case $\lambda > 0$

The solution to the above ODE now is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

Since $X(0) = 0$ then $A = 0$ and the solution becomes

$$X(x) = B\sin\left(\sqrt{\lambda}x\right)$$

Since $X(L) = 0$ then for non trivial solution we want $\sin\left(\sqrt{\lambda}L\right) = 0$ or $\sqrt{\lambda}L = n\pi$ or

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \cdots$$

Hence the eigenfunctions are

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right) \qquad n = 1, 2, 3, \cdots$$

The time ODE (3) now becomes

$$T'' + c^2\left(\frac{n\pi}{L}\right)^2 T = 0$$

Which has the solution

$$T(t) = B_n \cos\left(c\frac{n\pi}{L}t\right) + A_n \sin\left(c\frac{n\pi}{L}t\right)$$

Therefore the complete solution becomes

$$u(x,t)=\sum_{n=1}^{\infty}\left(B_n\cos\left(c\frac{n\pi}{L}t\right)+A_n\sin\left(c\frac{n\pi}{L}t\right)\right)\sin\left(\frac{n\pi}{L}x\right) \tag{4}$$

Now we can replace the given values in the above solution which gives

$$u(x,t)=\sum_{n=1}^{\infty}\left(B_n\cos(2n\pi t)+A_n\sin(2n\pi t)\right)\sin(n\pi x) \tag{4A}$$

At $t=0$ the above becomes

$$f(x)=\sum_{n=1}^{\infty}B_n\sin(n\pi x)\,dx$$

Hence B_n are the Fourier sine coefficients of $f(x)=x$. After odd extending $f(x)$ to $[-1,1]$ we obtain

$$\begin{aligned}
B_n &= \int_{-1}^{1} f(x)\sin(n\pi x)\,dx\\
&= 2\int_0^1 f(x)\sin(n\pi x)\,dx\\
&= 2\int_0^1 x\sin(n\pi x)\,dx\\
&= 2\left(\frac{-1}{n\pi}\left[x\cos n\pi x\right]_0^1+\frac{1}{n\pi}\int_0^1\cos n\pi x dx\right)\\
&= 2\left(\frac{-1}{n\pi}(\cos n\pi)+\frac{1}{n^2\pi^2}\left[\sin n\pi x\right]_0^1\right)\\
&= \frac{-2(-1)^n}{n\pi}
\end{aligned}$$

To find A_n, taking time derivative of (4A) gives

$$u_t(x,t)=\sum_{n=1}^{\infty}\left(-B_n 2n\pi\sin(2n\pi t)+2n\pi A_n\cos(2n\pi t)\right)\sin(n\pi x)$$

At $t=0$ the above becomes, using the initial conditions where $g(x)=-x$

$$g(x)=\sum_{n=1}^{\infty}(2n\pi A_n)\sin(n\pi x)$$

The above is the Fourier sine series for $g(x)$. By odd extending $-x$ to $[-1,1]$ then

$$
\begin{aligned}
2n\pi A_n &= \int_{-1}^{1} g(x)\sin(n\pi x)\,dx \\
&= 2\int_{0}^{1} g(x)\sin(n\pi x)\,dx \\
&= -2\int_{0}^{1} x\sin(n\pi x)\,dx \\
&= -2\left(-\frac{1}{n\pi}\left[x\cos(n\pi x)\right]_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx\right) \\
&= -2\left(-\frac{1}{n\pi}\cos(n\pi) + \frac{1}{n^2\pi^2}\overbrace{\left[\sin(n\pi x)\right]_0^1}^{0}\right) \\
&= \frac{2}{n\pi}\left[\cos(n\pi)\right] \\
&= \frac{2(-1)^n}{n\pi}
\end{aligned}
$$

Therefore

$$
A_n = \frac{(-1)^n}{n^2\pi^2}
$$

Now that we found A_n, B_n, then the solution (4A) is

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}\left(\frac{-2(-1)^n}{n\pi}\cos(2n\pi t) + \frac{(-1)^n}{n^2\pi^2}\sin(2n\pi t)\right)\sin(n\pi x) \\
&= \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\pi^2}\left(\sin(2n\pi t) - 2n\pi\cos(2n\pi t)\right)\sin(n\pi x)
\end{aligned}
$$

2.6.6 Problem 4.2.4b

Find all separable solutions to the wave equation $u_{tt} = u_{xx}$ on the interval $0 \le x \le \pi$ subject to (b) Neumann boundary conditions $u_x(0,t) = 0, u_x(\pi,t) = 0$.

Solution

Using separation of variables, let $u = X(x)T(t)$. The PDE becomes

$$
\begin{aligned}
T''X &= X''T \\
\frac{T''}{T} &= \frac{X''}{X} = -\lambda
\end{aligned}
$$

Where λ is separation constant. Hence the eigenvalue ODE is

$$
\begin{aligned}
X'' + \lambda X &= 0 \qquad (2) \\
X'(0) &= 0 \\
X'(\pi) &= 0
\end{aligned}
$$

And the time ODE is

$$T'' + \lambda T = 0 \tag{3}$$

Starting by the eigenvalue ODE to determine the eigenvalues and eigenfunctions.

Case $\lambda < 0$

Let $-\lambda = \omega^2$. Hence the ODE is $X'' - \omega^2 X = 0$ and the solution becomes

$$X(x) = C_1 \cosh(\omega x) + C_2 \sinh(\omega x)$$
$$X'(0) = C_1 \omega \sinh(\omega x) + C_2 \omega \cosh(\omega x)$$

At $x = 0$ the above gives

$$0 = C_2$$

Hence the solution now becomes

$$X(x) = C_1 \cosh(\omega x)$$
$$X'(x) = C_1 \omega \sinh(\omega x)$$

At $x = \pi$ the above gives

$$0 = C_1 \omega \sinh(\omega \pi)$$

But $\sinh$ is zero only when its argument is zero which is not the case here. Therefore $C_1 = 0$ which leads to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.

Case $\lambda = 0$

The space equation becomes $X''(x) = 0$ with the solution

$$X = Ax + B$$
$$X'(x) = A$$

At $x = 0$ the above gives $0 = A$. Therefore the solution is $X = B$. Therefore $X' = 0$. At $x = \pi$ this gives $0 = 0$. Therefore any value of B will work. Using the constant as 1, then the $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $X_0 = 1$.

case $\lambda > 0$

The solution to the above ODE now is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$
$$X'(x) = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)$$

Since $X'(0) = 0$ then $B = 0$ and the solution becomes

$$X(x) = A\cos\left(\sqrt{\lambda}x\right)$$
$$X'(x) = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)$$

Since $X'(\pi) = 0$ then for non trivial solution we want $\sin\left(\sqrt{\lambda}\pi\right) = 0$ or $\sqrt{\lambda}\pi = n\pi$ or

$$\lambda_n = n^2 \qquad n = 1, 2, 3, \cdots$$

Hence the eigenfunctions are

$$X_n(x) = \cos(nx) \qquad n = 1, 2, 3, \cdots$$

The time ODE (3) is now solved. For $\lambda = 0$ it becomes $T'' = 0$. Hence the solution is $T_0(t) = \frac{B_0}{2}t + \frac{A_0}{2}$ and for $\lambda_n = n^2$ it becomes

$$T_n'' + n^2 T_n = 0$$

Which has the solution

$$T_n(t) = A_n \cos(nt) + B_n \sin(nt)$$

Therefore the complete solution becomes

$$\begin{aligned} u(x,t) &= \overbrace{X_0}^{1} \overbrace{T_0}^{\frac{B_0}{2}t+\frac{A_0}{2}} + \sum_{n=1}^{\infty} X_n T_n \\ &= \frac{B_0}{2}t + \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos(nt) + B_n \sin(nt)) \cos(nx) \end{aligned} \tag{4}$$

To find A_0, B_0, A_n, B_n we need initial conditions which are not given. I was not sure if we are supposed to assume such initial conditions or not in order to continue. If so, then assuming $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, then at $t = 0$ the above becomes

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx)$$

Hence A_n are the Fourier cosine coefficients of $f(x)$. After even extending $f(x)$ to $[-\pi, \pi]$ we obtain

$$\begin{aligned} A_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{2}{\pi} \int_0^{\pi} f(x)\,dx \end{aligned}$$

And

$$\begin{aligned} A_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx \\ &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\,dx \end{aligned}$$

To find B_n, taking time derivative of (4) gives

$$u_t(x,t) = \frac{B_0}{2} + \sum_{n=1}^{\infty} (-nA_n \sin(nt) + nB_n \cos(nt)) \cos(nx)$$

At $t = 0$ the above gives

$$g(x) = \frac{B_0}{2} + \sum_{n=1}^{\infty} nB_n \cos(nx)$$

Hence was done above for A_0, A_n we obtain

$$B_0 = \frac{2}{\pi} \int_0^{\pi} g(x)\,dx$$

And

$$nB_n = \frac{2}{\pi}\int_0^{\pi} g(x)\cos(nx)\,dx$$
$$B_n = \frac{2}{n\pi}\int_0^{\pi} g(x)\cos(nx)\,dx$$

Now that we found A_n, B_n, then the solution (4) is

$$\begin{aligned} u(x,t) &= t\left(\frac{1}{\pi}\int_0^{\pi} g(x)\,dx\right) + \left(\frac{1}{\pi}\int_0^{\pi} f(x)\,dx\right) \\ &\quad + \sum_{n=1}^{\infty}\left[\left(\frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx\right)\cos(nt) + \left(\frac{2}{n\pi}\int_0^{\pi} g(x)\cos(nx)\,dx\right)\sin(nt)\right]\cos(nx) \end{aligned}$$

Or

$$\begin{aligned} u(x,t) &= t\left(\frac{1}{\pi}\int_0^{\pi} g(x)\,dx\right) + \left(\frac{1}{\pi}\int_0^{\pi} f(x)\,dx\right) \\ &\quad + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(n\cos(nt)\int_0^{\pi} f(x)\cos(nx)\,dx + \sin(nt)\int_0^{\pi} g(x)\cos(nx)\,dx\right)\cos(nx) \end{aligned}$$

2.6.7 Problem 4.2.6

(a) Formulate the periodic initial-boundary value problem for the wave equation on the interval $-\pi \le x \le \pi$, modeling the vibrations of a circular ring. (b) Write out a formula for the solution to your problem in the form of a Fourier series. (c) Is the solution a periodic function of t? If so, what is the period? (d) Suppose the initial displacement coincides with that in Figure 4.6, while the initial velocity is zero. Describe what happens to the solution as time evolves.

Solution

2.6.7.1 Part a

Solving for $u(x,t)$ in

$$u_{tt} = c^2 u_{xx} \tag{1}$$

With periodic boundary conditions

$$u(-\pi,t) = u(\pi,t)$$
$$u_x(-\pi,t) = u_x(\pi,t)$$

And initial conditions

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

2.6.7.2 Part b

Using separation of variables, let $u = X(x)T(t)$. Substituting in (1) gives

$$\frac{1}{c^2}T''X = X''T$$
$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = -\lambda$$

Where λ is the separation variable. This gives two ODE's to solve. The time ODE

$$T'' + c^2\lambda T = 0 \tag{2}$$

And the eigenvalue ODE

$$X'' + \lambda X = 0 \tag{3}$$

case $\lambda < 0$

Since $\lambda < 0$, then $-\lambda$ is positive. Let $\mu = -\lambda$, where μ is now positive. The solution to (3) becomes

$$X(x) = c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x}$$

The above can be written as

$$X(x) = c_1 \cosh\left(\sqrt{\mu}x\right) + c_2 \sinh\left(\sqrt{\mu}x\right) \tag{4}$$

Applying first B.C. $X(-\pi) = X(\pi)$ using (4) gives

$$c_1 \cosh\left(\sqrt{\mu}\pi\right) + c_2 \sinh\left(-\sqrt{\mu}\pi\right) = c_1 \cosh\left(\sqrt{\mu}\pi\right) + c_2 \sinh\left(\sqrt{\mu}\pi\right)$$
$$c_2 \sinh\left(-\sqrt{\mu}\pi\right) = c_2 \sinh\left(\sqrt{\mu}\pi\right)$$

But sinh is only zero when its argument is zero which is not the case here. Therefore the above implies that $c_2 = 0$. The solution (4) now reduces to

$$X(x) = c_1 \cosh\left(\sqrt{\mu}x\right)$$

Taking derivative gives

$$X'(x) = c_1\sqrt{\mu}\sinh\left(\sqrt{\mu}x\right)$$

Applying the second BC $X'(-\pi) = X'(\pi)$ the above gives

$$c_1\sqrt{\mu}\sinh\left(-\sqrt{\mu}\pi\right) = c_1\sqrt{\mu}\sinh\left(\sqrt{\mu}x\right)$$

But sinh is only zero when its argument is zero which is not the case here. Therefore the above implies that $c_1 = 0$. This means a trivial solution. Therefore $\lambda < 0$ is not an eigenvalue.

case $\lambda = 0$

In this case the solution is $X(x) = c_1 + c_2 x$. Applying first BC $X(-\pi) = X(\pi)$ gives

$$c_1 - c_2\pi = c_1 + c_2\pi$$
$$-c_2\pi = c_2\pi$$

This gives $c_2 = 0$. The solution now becomes $X(x) = c_1$ and $X'(x) = 0$. Applying the second boundary conditions $X'(-\pi) = X'(\pi)$ is not satisfies $(0 = 0)$. Therefore $\underline{\lambda = 0}$ is an eigenvalue with eigenfunction $X_0(0) = 1$ (selected $c_1 = 1$ since an arbitrary constant).

case $\lambda > 0$

The solution in this case is

$$X(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right) \tag{5}$$

Applying first B.C. $X(-\pi) = X(\pi)$ using the above gives

$$\begin{aligned} c_1 \cos\left(\sqrt{\lambda}\pi\right) + c_2 \sin\left(-\sqrt{\lambda}\pi\right) &= c_1 \cos\left(\sqrt{\lambda}\pi\right) + c_2 \sin\left(\sqrt{\lambda}\pi\right) \\ c_2 \sin\left(-\sqrt{\lambda}\pi\right) &= c_2 \sin\left(\sqrt{\lambda}\pi\right) \end{aligned}$$

There are two choices here. If $\sin\left(-\sqrt{\lambda}\pi\right) \neq \sin\left(\sqrt{\lambda}\pi\right)$, then this implies that $c_2 = 0$. If $\sin\left(-\sqrt{\lambda}\pi\right) = \sin\left(\sqrt{\lambda}\pi\right)$ then $c_2 \neq 0$. Assuming for now that $\sin\left(-\sqrt{\lambda}\pi\right) = \sin\left(\sqrt{\lambda}\pi\right)$. This happens when $\sqrt{\lambda}\pi = n\pi, n = 1, 2, 3, \cdots$, or

$$\lambda_n = n^2 \qquad n = 1, 2, 3, \cdots$$

Using this choice, we will now look to see what happens using the second BC. The solution (5) now becomes

$$X(x) = c_1 \cos(nx) + c_2 \sin(nx) \qquad n = 1, 2, 3, \cdots$$

Therefore

$$X'(x) = -c_1 n \sin(nx) + c_2 n \cos(nx)$$

Applying the second BC $X'(-\pi) = X'(\pi)$ using the above gives

$$\begin{aligned} c_1 n \sin(n\pi) + c_2 n \cos(n\pi) &= -c_1 n \sin(n\pi) + c_2 n \cos(n\pi) \\ c_1 n \sin(n\pi) &= -c_1 n \sin(n\pi) \\ 0 &= 0 \end{aligned}$$

Since n is integer.

Therefore this means that using $\lambda_n = n^2$ has satisfied both boundary conditions with $c_2 \neq 0, c_1 \neq 0$. This means the solution (5) becomes

$$X_n(x) = A_n \cos(nx) + B_n \sin(nx) \qquad n = 1, 2, 3, \cdots$$

The above says that there are two eigenfunctions in this case. They are

$$X_n(x) = \begin{cases} \cos(nx) \\ \sin(nx) \end{cases}$$

Since there is also zero eigenvalue, then the complete set of eigenfunctions become

$$X_n(x) = \begin{cases} 1 \\ \cos(nx) \\ \sin(nx) \end{cases}$$

Now that the eigenvalues are found, we go back and solve the time ODE. Recalling that the time ODE (2) from above was found to be

$$T'' + c^2 \lambda T = 0$$

When $\lambda = 0$ this becomes $T'' = 0$ with solution $T_0(t) = At + B$. When $\lambda_n = n^2$ the ODE becomes $T'' + c^2 n^2 T = 0$ with solution

$$T_n(t) = C_n \cos(cnt) + E_n \sin(cnt)$$

Adding all the above solutions using $u_n(x,t) = X_n(x) T_n(t)$ gives the final solution as

$$\begin{aligned} u(x,t) &= X_0(x) T_0(t) + \sum_{n=1}^{\infty} X_n(x) T_n(t) \\ &= At + B + \sum_{n=1}^{\infty} (\cos(nx) + \sin(nx)) (C_n \cos(cnt) + E_n \sin(cnt)) \end{aligned}$$

Or

$$\begin{aligned} u(x,t) &= At + B \\ &\quad + \sum_{n=1}^{\infty} (C_n \cos(cnt) + E_n \sin(cnt)) \cos(nx) \\ &\quad + \sum_{n=1}^{\infty} (C_n \cos(cnt) + E_n \sin(cnt)) \sin(nx) \end{aligned}$$

2.6.7.3 Part c

The solution is periodic in time. To find the period, solving $ct = \frac{2\pi}{T} t$ for T gives

$$T = \frac{2\pi}{c}$$

2.6.7.4 Part d

The solution will behave similar to the one on page 148 initially, where initial conditions splits in half, one half moving left and one moving right until each half reach the boundary conditions. But now, each half wave reflects off the boundary staying upside and starts moving back toward the middle again, until the two halves reunite again to reproduce the same initial conditions shape. This process then repeats again and again.

So the difference between periodic boundary conditions, and having ends fixed as the case in Figure 4.6, is that when ends are fixed, the two half waves reflect upside down at the boundaries, while here they do not not. The solution above was animated and plotted showing this. Initial conditions used is small triangle similar to one used in Figure 4.6 with zero initial conditions and using $c = 1$ for speed. The following is the result

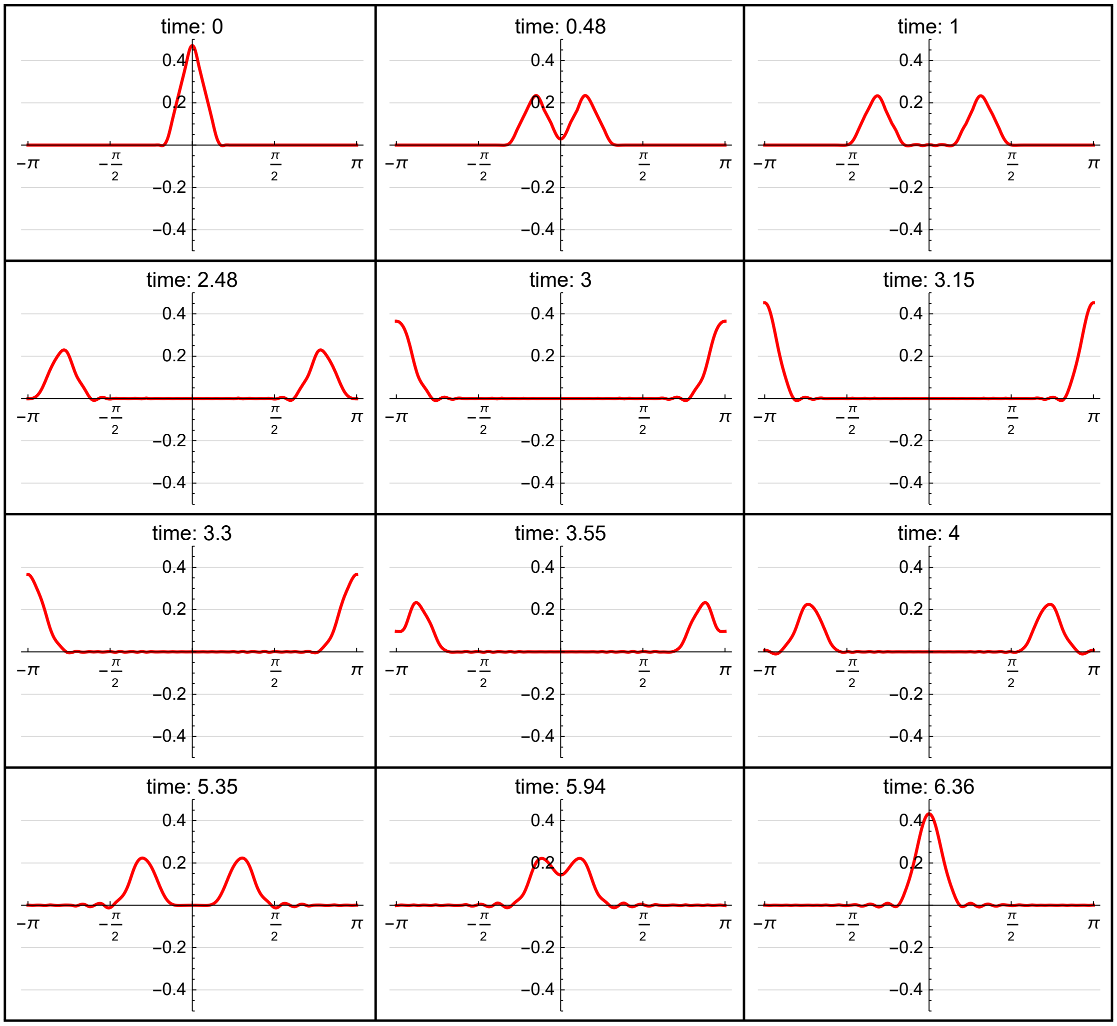


Figure 2.41: Plot showing solution in time, Periodic B.C.

In the above at $t = 3.15$ sec. is when each half wave reaches the boundary at $x = -\pi$ and $x = \pi$. At $t > 3.3$ the waves half reflects and are starting to moving back towards the center. At $t = 6.36$ the initial conditions shape is reconstructed again. For higher times, the above motion repeats.

2.6.8 Problem 4.2.14c

Sketch the solution of the wave equation $u_{tt} = u_{xx}$ and describe its behavior when the initial displacement is the box function $u(x,0) = \begin{cases} 1 & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$ while the initial velocity is 0 in each of the following scenarios (c) on the half-line $0 \leq x < \infty$, with homogeneous Neumann boundary condition at the end.

Solution

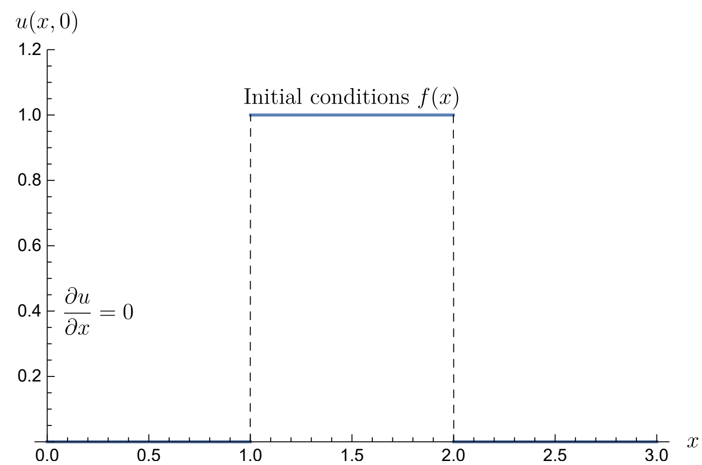


Figure 2.42: Initial conditions

Let $f(x) = u(x,0)$ and let $g(x) = u_t(x,0) = 0$. Since the boundary condition is homogeneous Neumann, then $f(x)$ is even extended to make it periodic with period 4. This is done so we can use d'Alembert solution which is valid for unbounded domain. Let $\tilde{f}(x)$ be the new periodic initial conditions as shown the in the following diagram.

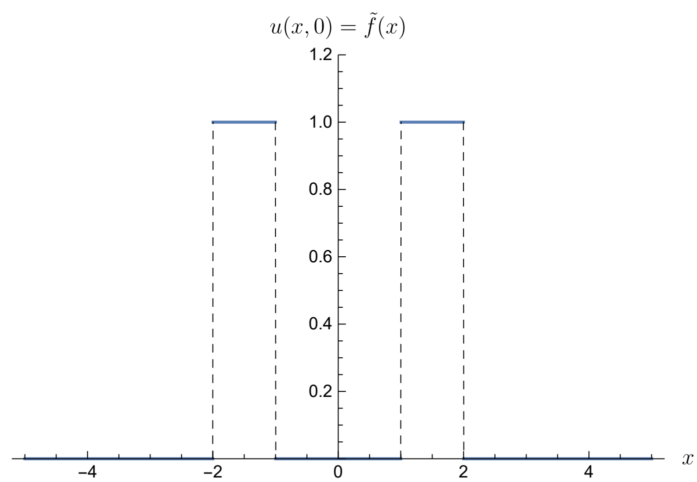


Figure 2.43: Initial conditions

With the new periodic initial conditions, we now can apply d'Alembert solution

$$\tilde{u}(x,t) = \frac{1}{2}\left(\tilde{f}(x-ct) + \tilde{f}(x+ct)\right)$$

Since $c = 1$ then above becomes

$$\tilde{u}(x,t) = \frac{1}{2}\left(\tilde{f}(x-t) + \tilde{f}(x+t)\right)$$

We will use the solution from above only for $x > 0$ since that is the physical domain.

The solution will starts by splitting each packet into 2 halves. One that move to the right and one that move to the left. When the half that moves to the left reach $x = 0$, at that same time the half wave that was moving to the right from $x < 0$ arrives. And they pass through each others. This appears as the wave half deflecting off $x = 0$ turning around, remaining upright, and starts to move to the right behind the half that was moving to the right from the start. So we end up with 2 half waves moving to the right after that. This is sketch of what happens in time.

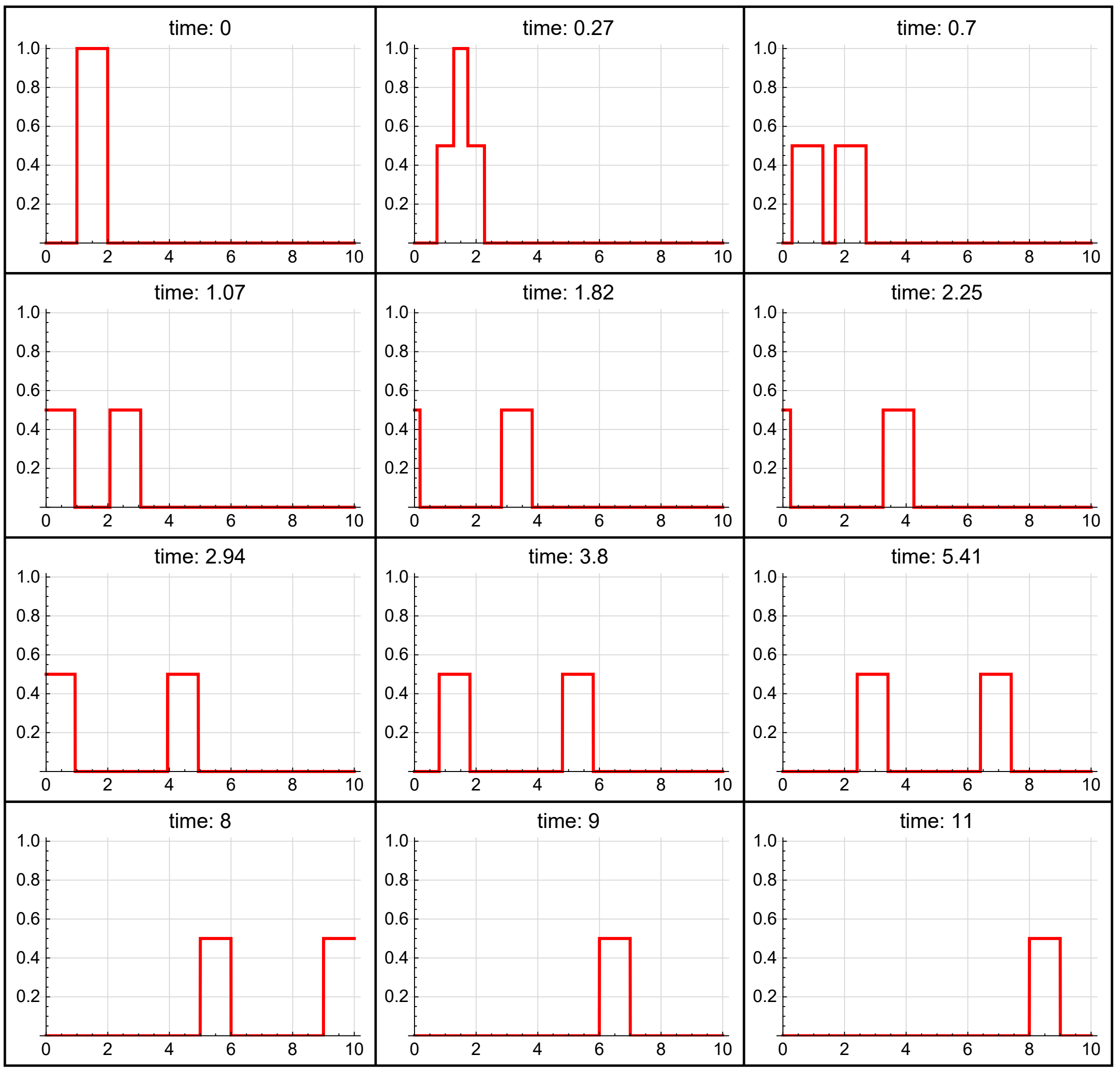


Figure 2.44: sketch of solution over time

```
pde = D[u[x, t], {t, 2}] == D[u[x, t], {x, 2}];
f[x_] := Piecewise[{{1, 1 < x < 2}, {0, True}}];
fbar[x_] := If[-3 < x < -1, fbar[x + 4], f[x]];
u[x_, t_] := 1/2 (fbar[x - t] + fbar[x + t]);
Table[Plot[u[x, t0], {x, 0, 10}, PlotRange → {Automatic, {0, 1.02}},
          GridLines → Automatic, GridLinesStyle → LightGray,
          PlotStyle → Red, PlotLabel → Row[{"time: ", t0}],
          PlotPoints → 40, Exclusions → None],
      {t0, {0, 0.27, 0.7, 1.07, 1.82, 2.25, 2.94, 3.8, 5.41, 8, 9, 11}}];
p = Grid[Partition[%, 3], Frame → All];
```

Figure 2.45: Code used for the above

2.6.9 Problem 4.2.22

Under what conditions is the solution to the Neumann boundary value problem for the wave equation on a bounded interval [0,1] periodic in time? What is the period?

Solution

By even-extending the initial displacement and initial velocity to [−1,1] and then repeating this again for the whole line $-\infty < x < \infty$, and then using the d'Alembert solution, then the resulting solution $u(x,t)$ will always be periodic since initial conditions are periodic. The period of the solution will $2L$ in x, where $L = 1$ here. Hence period is 2 in x.

2.6.10 Problem 4.2.25

Write down a formula for the solution $u(x,t)$ to the initial-boundary value problem $u_{tt} = 4u_{xx}$ with boundary conditions

$$u_x(0,t) = 0$$
$$u_x(\pi,t) = 0$$

And initial conditions

$$u(x,0) = \sin x$$
$$u_t(x,0) = 0$$

For $0 < x < \pi, t > 0$

Solution

Since boundary conditions are Neumann, then to use d'Alembert solution, we start by even extending both initial position $u(x,0) = \sin(x)$ and initial velocity (which is zero here) to be even over $[-\pi,\pi]$. Next we duplicate this over the whole line $-\infty < x < \infty$. Now we are able to use d'Alembert solution to solve the wave equation. The solution will be periodic with period 2π in x. Let $f(x) = \sin x$ and let $\tilde{f}(x)$ be its even periodic extension such that

$$\tilde{f}(-x) = f(x)$$
$$\tilde{f}(x + 2\pi) = f(x)$$
$$\tilde{f}(x - 2\pi) = f(x)$$

Hence the solution is

$$\tilde{u}(x,t) = \frac{1}{2}\left(\tilde{f}(x - ct) + \tilde{f}(x + ct)\right)$$

But $c = 2$ therefore the above becomes

$$\tilde{u}(x,t) = \frac{1}{2}(\sin(x - 2t) + \sin(x + ct))$$

The actual solution we want is over $[0,\pi]$ from the above since that is the physical domain of the original problem.

2.6.11 Key solution for HW 6

4.1.4

The solution is

$$u(t,x) = \sum_{n=1}^{\infty} d_n \exp\left[-\left(n+\tfrac{1}{2}\right)^2 \pi^2 t\right] \sin\left(n+\tfrac{1}{2}\right)\pi x$$

where

$$d_n = 2\int_0^1 f(x) \sin\left(n+\tfrac{1}{2}\right)\pi x \, dx$$

are the "mixed" Fourier coefficients of the initial temperature $u(0,x) = f(x)$. All solutions decay exponentially fast to zero: $u(t,x) \to 0$ as $t \to \infty$. For most initial conditions, i.e., those for which $d_1 \neq 0$, the decay rate is $e^{-\pi^2 t/4} \approx e^{-2.4674\,t}$. The solution profile eventually looks like a rapidly decaying version of the first eigenmode $\sin \frac{1}{2}\pi x$.

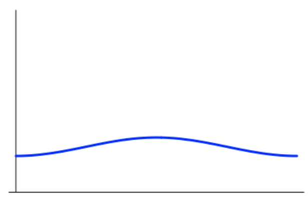
4.1.7

(*a*) $u(t,x) = \dfrac{1}{4} - \dfrac{2}{\pi^2} \displaystyle\sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} \exp\left(-(4j+2)^2 \pi^2 t\right) \cos(4j+2)\pi x$; (*b*) $\frac{1}{4}$;

(*c*) At an exponential rate of $e^{-4\pi^2 t}$;

(*d*) As $t \to \infty$, the solution becomes a vanishingly small cosine wave centered around $u = \frac{1}{4}$, namely

$$u(t,x) \approx \frac{1}{4} - \frac{2}{\pi^2} e^{-4\pi^2 t} \cos 2\pi x :$$



4.1.10c

(*c*) $u(t,x) = \frac{1}{2}\pi - \dfrac{4}{\pi} \displaystyle\sum_{k=0}^{\infty} \frac{e^{-(2k+1)^2 t} \cos(2k+1)x}{(2k+1)^2}$; equilibrium temperature: $u(t,x) \to \frac{1}{2}\pi$.

4.1.16

(a) If $u(t,x) = e^{\alpha t}\, v(t,x)$, then

$$\frac{\partial u}{\partial t} = \alpha e^{\alpha t}\, v(t,x) + e^{\alpha t}\, \frac{\partial v}{\partial t}\,(t,x) = \gamma e^{\alpha t}\, \frac{\partial^2 v}{\partial x^2} = \gamma\, \frac{\partial^2 u}{\partial x^2}\,.$$

(b) $v(t,x) = e^{-\alpha t} \sum_{n=1}^{\infty} b_n\, e^{-(\alpha+\gamma n^2 \pi^2)t} \sin n\pi x$, where $b_n = 2\int_0^1 f(x) \sin n\pi x\, dx$ are the Fourier sine coefficients of the initial data. All solutions tend to the equilibrium value $u(t,x) \to 0$ as $t \to \infty$ at an exponential rate. For most initial data, i.e., those with $b_1 \neq 0$, the decay rate is $e^{-a t}$, where $a = \alpha + \gamma\pi^2$; other solutions decay at a faster rate.

(c)

$$v(t,x) = \tfrac{1}{2}\, a_0\, e^{-\alpha t} + e^{-\alpha t} \sum_{n=1}^{\infty} a_n\, e^{-(\alpha+\gamma n^2\pi^2)t} \sin n\pi x,$$

where

$$a_n = 2\int_0^1 f(x) \cos n\pi x\, dx,$$

are the Fourier cosine coefficients of the initial data. All solutions tend to zero equilibrium value $u(t,x) \to 0$ as $t \to \infty$ at an exponential rate. For most initial data, i.e., those of non-zero mean, $\frac{1}{2}\, a_0 = \int_0^1 f(x)\, dx \neq 0$, the decay rate is $e^{-\alpha t}$; other solutions decay at a faster rate.

4.2.3d

(d) $u(t,x) = \dfrac{2}{\pi} \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \left(\cos 2n\pi t + \frac{\sin 2n\pi t}{2n\pi} \right) \frac{\sin n\pi x}{n}$

4.2.4b

(b) $1,\ t,\ \cos nt \cos nx,\ \sin nt \cos nx,\ $ for $n = 0, 1, 2, \ldots.$

4.2.6

(a)
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(t,-\pi) = u(t,\pi), \quad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi), \quad \begin{array}{l} -\pi < x < \pi, \\ -\infty < t < \infty, \end{array}$$

subject to the initial conditions

$$u(0,x) = f(x), \qquad \frac{\partial u}{\partial t}(0,x) = g(x), \qquad -\pi < x < \pi.$$

(b) The series solution is

$$u(t,x) = \tfrac{1}{2} a_0 + \tfrac{1}{2} c_0 t + \sum_{n=1}^{\infty} \Big(a_n \cos nct \ \cos nx + b_n \cos nct \ \sin nx$$
$$+ \frac{c_n}{nc} \sin nct \ \cos nx + \frac{d_n}{nc} \sin nct \ \sin nx \Big),$$

where a_n, b_n are the Fourier coefficients of $f(x)$, while c_n, d_n are the Fourier coefficients of $g(x)$.

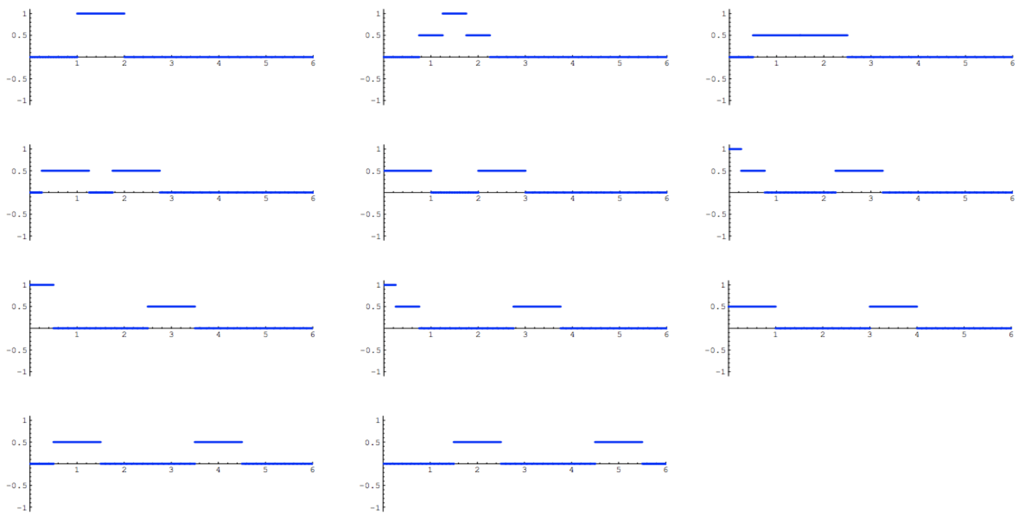
(c) The solution is periodic, with period $\frac{2\pi}{c}$, if and only if $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx = 0$, i.e., the average initial velocity is zero. Otherwise, it includes an unstable, linearly growing mode. *Note*: special solutions may have a shorter period. For example, if all odd coefficients vanish, $a_{2j+1} = b_{2j+1} = c_{2j+1} = d_{2j+1} = 0$, and $c_0 = 0$, then the solution has period π/c.

(d) The initial displacement breaks up into two half size replicas traveling with speed c in opposite directions. When the right moving wave arrives at the end point $-\pi$, it reappears unchanged and still moving to the right at the other end π. Similarly, when the left moving wave arrives at the left end, it reappears on the right end still moving left. The waves recombine into the original displacement after a time of $2\pi/c$, and then the process repeats periodically.

4.2.14c

(*c*) The initial displacement splits into two half sized replicas, initially moving off to the right and to the left with unit speed. When the left moving box collides with the origin, it reverses its direction, eventually following its right moving counterpart with the same unit speed at a fixed distance of 3 units. During the collision, the box temporarily increases its height before disengaging in its original upright form, but now moving to the right.

Plotted at times $t = 0, .25, .5, .75, 1., 1.25, 1.75, 2., 2.5, 3.5$:



4.2.22

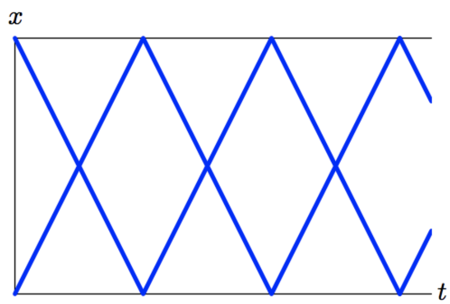
The solution is periodic if and only if the initial velocity has mean zero: $\int_0^\ell g(x)\,dx = 0$. For generic solutions, the period is $2\ell/c$, although some special solutions oscillate more rapidly.

4.2.25

(*a*) The even, 2π periodic extension of the initial data is $f(x) = |\sin x|$. Thus, by d'Alembert's formula, $u(t,x) = \frac{1}{2}|\sin(x-2t)| + \frac{1}{2}|\sin(x+2t)|$.

(*b*) $u\left(\frac{\pi}{2},\frac{\pi}{2}\right) = \frac{1}{2}|\sin\frac{\pi}{2}| + \frac{1}{2}|\sin(\frac{3}{2}\pi)| = 1$. (*c*) $h(t) = |\cos 2t|$ is periodic of period $\frac{1}{2}\pi$.

(*d*) Yes. On the interval $0 \le x \le \pi$, discontinuities initially appear at $x = 0$ and $x = \pi$, and then propagate into the interval at speed 2, reflecting whenever they reach one of the ends, as sketched in the following figure:



2.7 HW 7

Local contents

2.7.1 Problem 4.3.24

Use the method in Exercise 4.3.23 to solve an Euler equation whose characteristic equation has a double root $r_1 = r_2 = r$

Solution

2.7.1.1 Part (a)

Euler ODE is

$$ax^2u''(x) + bxu'(x) + cu(x) = 0$$

By assuming $u = x^r$ then $u' = rx^{r-1}, u'' = r(r-1)x^{r-2}$. Substituting back into the above ODE gives

$$\begin{aligned}
ax^2r(r-1)x^{r-2} + bxrx^{r-1} + cx^r &= 0\\
ar(r-1) + br + c &= 0\\
ar^2 - ar + br + c &= 0\\
ar^2 + r(b-a) + c &= 0
\end{aligned}$$

Solving for r gives the roots

$$r_{1,2} = -\frac{b-a}{2a} \pm \frac{1}{2a}\sqrt{(b-a)^2 - 4ac} \tag{1}$$

Double root means that $r = r_1 = r_2 = -\frac{b-a}{2a}$. Hence the first solution of the ODE is

$$u_1 = x^{r_1}$$

And now we need to find the second solution. Using reduction of order method, we assume the second solution is

$$u_2(x) = v(x)u_1(x) \tag{2}$$

And we need to determine the function $v(x)$. Therefore

$$\begin{aligned} u_2' &= v'u_1 + vu_1' \\ u_2'' &= v''u_1 + v'u_1' + v'u_1' + vu_1'' \\ &= v''u_1 + 2v'u_1' + vu_1'' \end{aligned}$$

Substituting the above into the ODE gives

$$\begin{aligned} ax^2\left(v''u_1 + 2v'u_1' + vu_1''\right) + bx\left(v'u_1 + vu_1'\right) + cvu_1 &= 0 \\ v''\left(ax^2u_1\right) + v'\left(2ax^2u_1' + bxu_1\right) + v\left(ax^2u_1'' + bxu_1' + cu_1\right) &= 0 \end{aligned}$$

But $ax^2u_1'' + bxu_1' + cu_1 = 0$ since u_1 is a solution. The above now simplifies to

$$v''\left(ax^2u_1\right) + v'\left(2ax^2u_1' + bxu_1\right) = 0$$

But $u_1 = x^r$, hence $u_1' = rx^{r-1}$ and the above becomes

$$\begin{aligned} v''\left(ax^2x^r\right) + v'\left(2arx^2x^{r-1} + bxx^r\right) &= 0 \\ av''x^{r+2} + v'\left(2arx^{r+1} + bx^{r+1}\right) &= 0 \\ av''x^{r+2} + v'(2ar+b)x^{r+1} &= 0 \\ \left(av''x + v'(2ar+b)\right)x^{r+1} &= 0 \\ av''x + v'(2ar+b) &= 0 \end{aligned}$$

But $r = r_1 = -\frac{b-a}{2a}$ from (1) since double root. The above simplifies to

$$\begin{aligned} av''x + v'\left(2a\left(-\frac{b-a}{2a}\right) + b\right) &= 0 \\ av''x + v'\left((-b+a) + b\right) &= 0 \\ av''x + av' &= 0 \\ v''x + v' &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \frac{d}{dx}(xv') &= 0 \\ xv' &= C_1 \\ v' &= \frac{C_1}{x} \\ v &= C_1 \ln x + C_2 \end{aligned}$$

Now that we found $v(x)$, then using (2) we find the second solution to the ODE as

$$\begin{aligned} u_2 &= vu_1 \\ &= (C_1 \ln x + C_2)x^2 \end{aligned}$$

Therefore the complete solution is

$$u = C_0x^r + (C_1 \ln x + C_2)x^r$$

By combining constants, the above simplifies to

$$u(x) = Ax^r + Bx^r \ln x$$

2.7.2 Problem 4.3.25

Solve the following boundary value problems (c) $\nabla^2 u = 0, x^2 + y^2 < 4, u = x^4, x^2 + y^2 = 4$ (d) $\nabla^2 u = 0, x^2 + y^2 < 1, \frac{\partial u}{\partial n} = x, x^2 + y^2 = 1$

Solution

2.7.2.1 Part c

In polar coordinates, where $x = r\cos\theta, y = r\sin\theta$, we need to solve for $u(r,\theta)$ inside disk of radius $r_0 = 4$. The Laplace PDE in polar coordinates is

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0 \qquad 0 < r < r_0, -\pi < \theta < \pi \\ u(r_0,\theta) &= f(\theta) = (r_0\cos\theta)^4 \\ u(-\pi) &= u(\pi) \\ u_\theta(-\pi) &= u_\theta(\pi) \end{aligned}$$

The solution to Laplace PDE of radius r_0 can be found using separation of variables and derived in the textbook (full derivation is also given in this HW in problem 4.3.33 below). The Fourier series solution is

$$u(r,\theta) = \frac{a_0}{2} + \sum a_n \left(\frac{r}{r_0}\right)^n \cos(n\theta) + b_n \left(\frac{r}{r_0}\right)^n \sin(n\theta)$$

Since $r_0 = 4$ the above becomes

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{r}{4}\right)^n \cos(n\theta) + b_n \left(\frac{r}{4}\right)^n \sin(n\theta) \tag{1A}$$

Where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta, b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta$.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} 256\cos^4\theta d\theta \\ &= \frac{256}{\pi}\int_{-\pi}^{\pi}\cos^4\theta d\theta \\ &= \frac{256}{\pi}\left(\frac{3\theta}{8} + \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta)\right)_{-\pi}^{\pi} \\ &= \frac{256}{\pi}\left(\frac{3\pi}{8} + \frac{3\pi}{8}\right) \\ &= \frac{256}{\pi}\left(\frac{3\pi}{4}\right) \\ &= 192 \end{aligned}$$

And

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} 256\cos^4(\theta)\cos(n\theta)\,d\theta \\
&= \frac{256}{\pi}\int_{-\pi}^{\pi} \cos^4(\theta)\cos(n\theta)\,d\theta
\end{aligned}
$$

To evaluate the above integral, we will start by using the identity

$$
\cos^4(\theta) = \frac{3}{8} + \frac{1}{8}\cos(4\theta) + \frac{1}{2}\cos(2\theta)
$$

Therefore the integral now becomes

$$
\begin{aligned}
a_n &= \frac{256}{\pi}\int_{-\pi}^{\pi}\left(\frac{3}{8} + \frac{1}{8}\cos(4\theta) + \frac{1}{2}\cos(2\theta)\right)\cos(n\theta)\,d\theta \\
&= \frac{256}{\pi}\left[\frac{3}{8}\int_{-\pi}^{\pi}\cos(n\theta)\,d\theta + \frac{1}{8}\int_{-\pi}^{\pi}\cos(4\theta)\cos(n\theta)\,d\theta + \frac{1}{2}\int_{-\pi}^{\pi}\cos(2\theta)\cos(n\theta)\,d\theta\right] \qquad (1)
\end{aligned}
$$

But $\int_{-\pi}^{\pi}\cos(n\theta)\,d\theta = 0$ and $\int_{-\pi}^{\pi}\cos(4\theta)\cos(n\theta)\,d\theta$ is not zero, only for $n = 4$ by orthogonality of cosine functions. Hence

$$
\begin{aligned}
\int_{-\pi}^{\pi}\cos(4\theta)\cos(n\theta)\,d\theta &= \int_{-\pi}^{\pi}\cos^2(4\theta)\,d\theta \\
&= \pi
\end{aligned}
$$

And similarly, $\int_{-\pi}^{\pi}\cos(2\theta)\cos(n\theta)\,d\theta$ is not zero, only for $n = 2$ by orthogonality of cosine functions. Hence

$$
\begin{aligned}
\int_{-\pi}^{\pi}\cos(2\theta)\cos(n\theta)\,d\theta &= \int_{-\pi}^{\pi}\cos^2(2\theta)\,d\theta \\
&= \pi
\end{aligned}
$$

Using these results in (1) gives, for $n = 2$

$$
\begin{aligned}
a_2 &= \frac{256}{\pi}\left[\frac{1}{2}\int_{-\pi}^{\pi}\cos^2(2\theta)\,d\theta\right] \\
&= \frac{256}{\pi}\left(\frac{\pi}{2}\right) \\
&= 128
\end{aligned}
$$

And for $n = 4$

$$
\begin{aligned}
a_4 &= \frac{256}{\pi}\left[\frac{1}{8}\int_{-\pi}^{\pi}\cos^2(4\theta)\,d\theta\right] \\
&= \frac{256}{\pi}\left(\frac{\pi}{8}\right) \\
&= 32
\end{aligned}
$$

And all other a_n are zero. Now that we found all a_n, and since $b_n = 0$ for all n (because $f(\theta)$ is even function) then the solution (1A) becomes

$$\begin{aligned} u(r,\theta) &= \frac{192}{2} + a_2\left(\frac{r}{4}\right)^2 \cos(2\theta) + a_4\left(\frac{r}{4}\right)^4 \cos(4\theta) \\ &= 96 + 128\left(\frac{r^2}{16}\right)\cos(2\theta) + 32\frac{r^4}{256}\cos(4\theta) \end{aligned}$$

Therefore

$$\boxed{u(r,\theta) = 96 + 8r^2\cos(2\theta) + \frac{1}{8}r^4\cos(4\theta)}$$

Here is plot of the above solution.

```
sol = 96 + 8 r^2 Cos[2 θ] + 1/8 r^4 Cos[4 θ];
ParametricPlot3D[{r Cos[θ], r Sin[θ], sol}, {r, 0, 4}, {θ, 0, 2 Pi},
   AxesLabel → {x, y, "u(x,y"}, ImageSize → 400, BoxRatios → {1,  1,  1}, BaseStyle → 14]
```

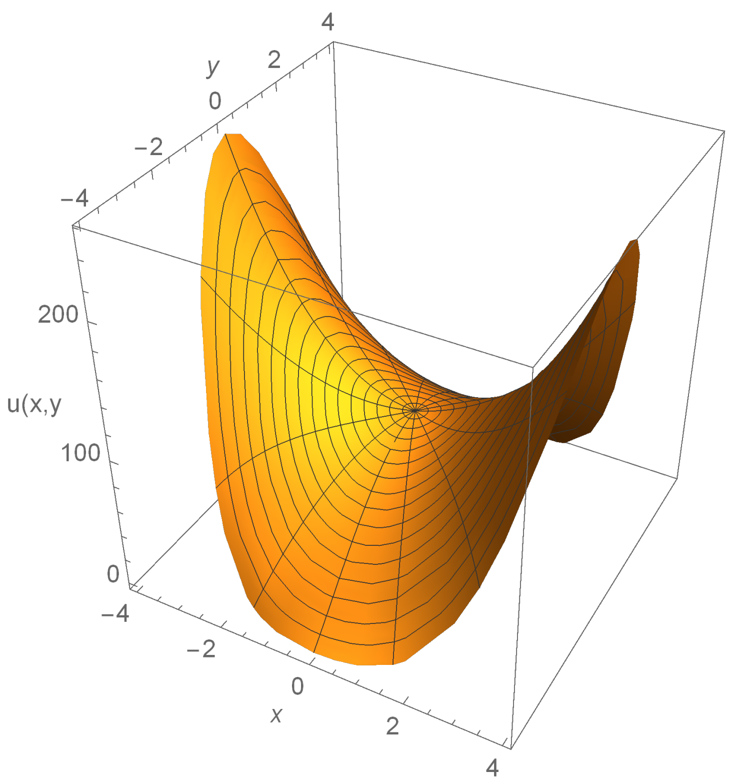


Figure 2.46: Solution plot to the above problem with code used

It is also possible to use, as shown in textbook, the closed form sum as given in theorem 4.6

as

$$u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\phi)\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}d\phi$$

Notice that theorem 4.6 is for a unit disk. Since the disk here has radius 4 then r is changed to $\frac{r}{4}$ in 4.126 as given in book. Here $f(\theta)=(4\cos\theta)^4$. Hence the above becomes

$$\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}256\cos^4(\phi)\frac{1-\left(\frac{r}{4}\right)^2}{1+\left(\frac{r}{4}\right)^2-2\left(\frac{r}{4}\right)\cos(\theta-\phi)}d\phi\\
&= \frac{128}{\pi}\int_{-\pi}^{\pi}\cos^4(\phi)\frac{1-\frac{r^2}{16}}{1+\frac{r^2}{16}-\frac{r}{2}\cos(\theta-\phi)}d\phi\\
&= \frac{128}{\pi}\int_{-\pi}^{\pi}\cos^4(\phi)\frac{16-r^2}{16+r^2-8r\cos(\theta-\phi)}d\phi
\end{aligned}$$

But evaluating the above integral was hard to do by hand. It should of course give the same solution as found above using Fourier series.

2.7.2.2 Part d

In polar coordinates, where $x=r\cos\theta, y=r\sin\theta$, we need to solve for $u(r,\theta)$ inside disk of radius $r_0=1$. The Laplace PDE in polar coordinates is

$$\begin{aligned}
u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta} &= 0 \qquad 0<r<1, -\pi<\theta<\pi\\
u_r(1,\theta) &= f(\theta)=\cos\theta\\
u(-\pi) &= u(\pi)\\
u_\theta(-\pi) &= u_\theta(\pi)
\end{aligned}$$

Using separation of variables, let $u(r,\theta)=R(r)\Theta(\theta)$ the solution is given by

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_nr^n\cos(n\theta)+b_nr^n\sin(n\theta) \tag{1}$$

At $r=r_0=1$ we have that $\frac{\partial u(r,\theta)}{\partial r}=\cos\theta$ (since $x=r\cos\theta$ but $r=1$ at boundary). The above becomes

$$\cos\theta=\sum_{n=1}^{\infty}na_nr^{n-1}\cos(n\theta)+nb_nr^{n-1}\sin(n\theta)$$

Therefore $n=1$ is only term that survives in the sum. Hence $a_1=1$ and all others are zero. The solution (1) becomes

$$u(r,\theta)=\frac{a_0}{2}+r\cos(\theta)$$

The solution is not unique as there is a_0 arbitrary constant.

2.7.3 Problem 4.3.33

Write out the series solution to the boundary value problem $u(1,\theta) = 0, u(2,\theta) = h(\theta)$ for the Laplace equation on an annulus $1 < r < 2$.

Solution

Using a for the inner radius and b for the outer radius to keep the solution more general. At the end these are replaced with $a = 1, b = 2$.

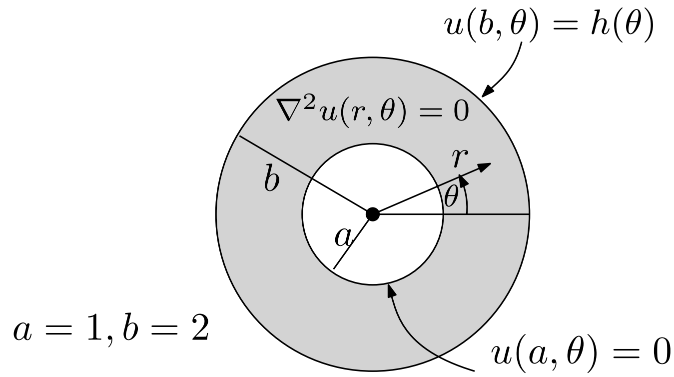


Figure 2.47: PDE to solve using polar coordinates

The Laplace PDE in polar coordinates is

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \tag{A}$$

With

$$\begin{aligned} u(a,\theta) &= 0 \\ u(b,\theta) &= h(\theta) \end{aligned} \tag{B}$$

Let the solution be

$$u(r,\theta) = R(r)\Theta(\theta)$$

Substituting this assumed solution back into the (A) gives

$$r^2 R''\Theta + rR'\Theta + R\Theta'' = 0$$

Dividing the above by $R\Theta$ gives

$$\begin{aligned} r^2 \frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} &= 0 \\ r^2 \frac{R''}{R} + r\frac{R'}{R} &= -\frac{\Theta''}{\Theta} \end{aligned}$$

Since each side depends on different independent variable and they are equal, they must be equal to the same constant. say λ.

$$r^2 \frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

This results in the following two ODE's. The boundaries conditions in (B) are also transferred to each ODE. Hence

$$\begin{aligned}\Theta'' + \lambda\Theta &= 0 \\ \Theta(-\pi) &= \Theta(\pi) \\ \Theta'(-\pi) &= \Theta'(\pi)\end{aligned} \tag{1}$$

And

$$\begin{aligned}r^2R'' + rR' - \lambda R &= 0 \\ R(a) &= 0\end{aligned} \tag{2}$$

Starting with ODE (1) with periodic boundary conditions.

Case $\lambda < 0$ The solution is

$$\Theta(\theta) = A\cosh\left(\sqrt{|\lambda|}\theta\right) + B\sinh\left(\sqrt{|\lambda|}\theta\right)$$

First B.C. gives

$$\begin{aligned}\Theta(-\pi) &= \Theta(\pi) \\ A\cosh\left(-\sqrt{|\lambda|}\pi\right) + B\sinh\left(-\sqrt{|\lambda|}\pi\right) &= A\cosh\left(\sqrt{|\lambda|}\pi\right) + B\sinh\left(\sqrt{|\lambda|}\pi\right) \\ A\cosh\left(\sqrt{|\lambda|}\pi\right) - B\sinh\left(\sqrt{|\lambda|}\pi\right) &= A\cosh\left(\sqrt{|\lambda|}\pi\right) + B\sinh\left(\sqrt{|\lambda|}\pi\right) \\ 2B\sinh\left(\sqrt{|\lambda|}\pi\right) &= 0\end{aligned}$$

But $\sinh = 0$ only at zero and $\lambda \neq 0$, hence $B = 0$ and the solution becomes

$$\begin{aligned}\Theta(\theta) &= A\cosh\left(\sqrt{|\lambda|}\theta\right) \\ \Theta'(\theta) &= A\sqrt{\lambda}\cosh\left(\sqrt{|\lambda|}\theta\right)\end{aligned}$$

Applying the second B.C. gives

$$\begin{aligned}\Theta'(-\pi) &= \Theta'(\pi) \\ A\sqrt{|\lambda|}\cosh\left(-\sqrt{|\lambda|}\pi\right) &= A\sqrt{|\lambda|}\cosh\left(\sqrt{|\lambda|}\pi\right) \\ A\sqrt{|\lambda|}\cosh\left(\sqrt{|\lambda|}\pi\right) &= A\sqrt{|\lambda|}\cosh\left(\sqrt{|\lambda|}\pi\right) \\ 2A\sqrt{|\lambda|}\cosh\left(\sqrt{|\lambda|}\pi\right) &= 0\end{aligned}$$

But $\cosh$ is never zero, hence $A = 0$. Therefore trivial solution and $\lambda < 0$ is not an eigenvalue.

Case $\lambda = 0$ The solution is $\Theta = A\theta + B$. Applying the first B.C. gives

$$\begin{aligned}\Theta(-\pi) &= \Theta(\pi) \\ -A\pi + B &= \pi A + B \\ 2\pi A &= 0 \\ A &= 0\end{aligned}$$

And the solution becomes $\Theta = B_0$. A constant. Hence $\lambda = 0$ is an eigenvalue.

Case $\lambda > 0$

The solution becomes

$$\Theta = A\cos\left(\sqrt{\lambda}\theta\right) + B\sin\left(\sqrt{\lambda}\theta\right)$$
$$\Theta' = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\theta\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\theta\right)$$

Applying first B.C. gives

$$\begin{aligned}\Theta(-\pi) &= \Theta(\pi)\\ A\cos\left(-\sqrt{\lambda}\pi\right) + B\sin\left(-\sqrt{\lambda}\pi\right) &= A\cos\left(\sqrt{\lambda}\pi\right) + B\sin\left(\sqrt{\lambda}\pi\right)\\ A\cos\left(\sqrt{\lambda}\pi\right) - B\sin\left(\sqrt{\lambda}\pi\right) &= A\cos\left(\sqrt{\lambda}\pi\right) + B\sin\left(\sqrt{\lambda}\pi\right)\\ 2B\sin\left(\sqrt{\lambda}\pi\right) &= 0 \qquad (3)\end{aligned}$$

Applying second B.C. gives

$$\begin{aligned}\Theta'(-\pi) &= \Theta'(\pi)\\ -A\sqrt{\lambda}\sin\left(-\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(-\sqrt{\lambda}\pi\right) &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right)\\ A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right)\\ A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right)\\ 2A\sin\left(\sqrt{\lambda}\pi\right) &= 0 \qquad (4)\end{aligned}$$

Equations (3,4) can be both zero only if $A = B = 0$ which gives trivial solution, or when $\sin\left(\sqrt{\lambda}\pi\right) = 0$. Therefore taking $\sin\left(\sqrt{\lambda}\pi\right) = 0$ gives a non-trivial solution. Hence

$$\begin{aligned}\sqrt{\lambda}\pi &= n\pi \qquad n = 1,2,3,\cdots\\ \lambda_n &= n^2 \qquad n = 1,2,3,\cdots\end{aligned}$$

Hence the eigenfunctions are

$$\{1, \cos(n\theta), \sin(n\theta)\} \qquad n = 1,2,3,\cdots \qquad (5)$$

Now the R equation is solved

The case for $\lambda = 0$ gives from (2)

$$\begin{aligned}r^2R'' + rR' &= 0\\ R'' + \frac{1}{r}R' &= 0 \qquad r \neq 0\end{aligned}$$

As was done in last problem, the solution to this is

$$R_0(r) = A\ln r + C$$

Applying the B.C. $R(a) = 0$ gives

$$\begin{aligned}0 &= A\ln a + C\\ C &= -A\ln a\end{aligned}$$

Hence the solution becomes

$$\begin{aligned} R_0(r) &= A\ln r - A\ln a \\ &= A\ln\frac{r}{a} \end{aligned}$$

<u>Case $\lambda > 0$</u> The ODE (2) becomes

$$r^2R'' + rR' - n^2R = 0 \qquad n = 1,2,3,\cdots$$

Let $R = r^p$, the above becomes

$$\begin{aligned} r^2p\left(p-1\right)r^{p-2} + rpr^{p-1} - n^2r^p &= 0 \\ p\left(p-1\right)r^p + pr^p - n^2r^p &= 0 \\ p\left(p-1\right) + p - n^2 &= 0 \\ p^2 &= n^2 \\ p &= \pm n \end{aligned}$$

Hence the solution is

$$R_n(r) = Cr^n + D\frac{1}{r^n} \qquad n = 1,2,3,\cdots$$

Applying the boundary condition $R(a) = 0$ gives

$$\begin{aligned} 0 &= Ca^n + D\frac{1}{a^n} \\ -Ca^n &= D\frac{1}{a^n} \\ D &= -Ca^{2n} \end{aligned}$$

The solution becomes

$$\begin{aligned} R_n(r) &= Cr^n - Ca^{2n}\frac{1}{r^n} \qquad n = 1,2,3,\cdots \\ &= C_n\left(r^n - \frac{a^{2n}}{r^n}\right) \end{aligned}$$

Hence the complete solution for $R(r)$ is

$$R(r) = A\ln\frac{r}{a} + \sum_{n=1}^{\infty} C_n\left(r^n - \frac{a^{2n}}{r^n}\right) \tag{6}$$

Using (5),(6) gives

$$\begin{aligned} u_n(r,\theta) &= R_n\Theta_n \\ u(r,\theta) &= \left(A\ln\frac{r}{a} + \sum_{n=1}^{\infty} C_n\left(r^n - \frac{a^{2n}}{r^n}\right)\right)\left(A_0 + \sum_{n=1}^{\infty} A_n\cos(n\theta) + B_n\sin(n\theta)\right) \end{aligned}$$

Combining constants to simplify things gives

$$u(r,\theta) = A\ln\frac{r}{a} + \sum_{n=1}^{\infty}\left(r^n - \frac{a^{2n}}{r^n}\right)\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)$$

But $a = 1$, then above simplifies to

$$u(r,\theta) = A\ln r + \sum_{n=1}^{\infty}\left(r^n - \frac{1}{r^n}\right)(A_n\cos(n\theta) + B_n\sin(n\theta)) \tag{7}$$

At $r = b$ we use $u(b,\theta) = h(\theta)$ to find A_0, A_n, B_n.

$$\begin{aligned} u(b,\theta) &= h(\theta) \\ h(\theta) &= A_0\ln b + \sum_{n=1}^{\infty}\left(b^n + \frac{1}{b^n}\right)(A_n\cos(n\theta) + B_n\sin(n\theta)) \end{aligned}$$

Hence

$$\begin{aligned} A_0\ln b &= \frac{2}{\pi}\int_{-\pi}^{\pi} h(\theta)\,d\theta \\ A_n\left(b^n + \frac{1}{b^n}\right) &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos(n\theta)\,d\theta \\ B_n\left(b^n + \frac{1}{b^n}\right) &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\sin(n\theta)\,d\theta \end{aligned}$$

The solution (7) becomes

$$u(r,\theta) = \left(\frac{2}{\pi}\int_{-\pi}^{\pi} h(\theta)\,d\theta\right)\frac{\ln r}{\ln b} + \sum_{n=1}^{\infty}\frac{\left(r^n - \frac{1}{r^n}\right)}{b^n + \frac{1}{b^n}}\left(\left(\frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos(n\theta)\,d\theta\right)\cos(n\theta) + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos(n\theta)\,d\theta\right)\sin(n\theta)\right)$$

But $b = 2$ and the above becomes

$$\begin{aligned} u(r,\theta) &= \left(\frac{2}{\pi}\int_{-\pi}^{\pi} h(\theta)\,d\theta\right)\frac{\ln r}{\ln 2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\left(r^n - \frac{1}{r^n}\right)}{2^n + \frac{1}{2^n}}\left(\left(\int_{-\pi}^{\pi} h(\theta)\cos(n\theta)\,d\theta\right)\cos(n\theta) + \left(\int_{-\pi}^{\pi} h(\theta)\cos(n\theta)\,d\theta\right)\sin(n\theta)\right) \\ &= \left(\frac{2}{\pi}\int_{-\pi}^{\pi} h(\theta)\,d\theta\right)\frac{\ln r}{\ln 2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{2^n}{r^n}\frac{\left(r^{2n} - 1\right)}{2^{2n} + 1}\left(\left(\int_{-\pi}^{\pi} h(\theta)\cos(n\theta)\,d\theta\right)\cos(n\theta) + \left(\int_{-\pi}^{\pi} h(\theta)\cos(n\theta)\,d\theta\right)\sin(n\theta)\right) \end{aligned}$$

2.7.4 Problem 4.3.38

Suppose $\int_{-\pi}^{\pi} |h(\theta)|\,d\theta < \infty$. Prove that (4.115) converges uniformly to the solution to the boundary value problem (4.101) on any smaller disk $D_{r_*} = \{r \leq r_* < 1\} \subsetneq D_1$

<u>Solution</u>

4.115 is solution for $u(r,\theta)$ inside unit disk $0 < r < 1$ and $u = h(\theta)$ at $r = 1$.

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n\cos(n\theta) + b_n\sin(n\theta)) \tag{4.115}$$

This problem is asking to show that the Fourier series solution 4.115 converges uniformly to solution of Laplace PDE $\nabla^2 u = 0$ inside disk with radius less than unity with above boundary conditions.

Let $f_n = r^n(a_n\cos(n\theta) + b_n\sin(n\theta))$, then to show uniform convergence, we need to show that for any $\varepsilon > 0$, there exist integer $N(\varepsilon)$ such that for all $n > N$ the following is true

$$|u_n - u_*| < \varepsilon$$

Where

$$u_* = \left(\frac{r}{r_*}\right)^n (a_n \cos(n\theta) + b_n \sin(n\theta))$$

Hence we need to show, we can find N such that for all $n > N$

$$\left| r^n (a_n \cos(n\theta) + b_n \sin(n\theta)) - \left(\frac{r}{r_*}\right)^n (a_n \cos(n\theta) + b_n \sin(n\theta)) \right| < \varepsilon$$

But

$$\begin{aligned} \left| r^n (a_n \cos(n\theta) + b_n \sin(n\theta)) - \left(\frac{r}{r_*}\right)^n (a_n \cos(n\theta) + b_n \sin(n\theta)) \right| &= \left| \left(r^n - \left(\frac{r}{r_*}\right)^n\right) (a_n \cos(n\theta) + b_n \sin(n\theta)) \right| \\ &= \left| \left(r^n - \left(\frac{r}{r_*}\right)^n\right) \right| |a_n \cos(n\theta) + b_n \sin(n\theta)| \end{aligned} \tag{1}$$

But $|a_n \cos(n\theta) + b_n \sin(n\theta)|$ can be made as small as we want by increasing n. This is because

$$|a_n \cos(n\theta) + b_n \sin(n\theta)| \leq |a_n \cos(n\theta)| + |b_n \sin(n\theta)|$$

And since $\int_{-\pi}^{\pi} |h(\theta)| \, d\theta < \infty$ it implies the Fourier series coefficients $a_n, b_n \to 0$ as $n \to \infty$ per Lemma 3.40 on page 112. Hence (1) can be made as small as we want for large n and it will remain smaller as n increases because $\left| \left(r^n - \left(\frac{r}{r_*}\right)^n\right) \right| < 1$.

Therefore there exist such an $N(\varepsilon)$. Hence u converges uniformly to u_*.

2.7.5 Problem 4.3.42

Complete the proof of Theorem 4.9 by showing that $u(x,y) = M^*$ for all $(x,y) \in \Omega$. Hint: Join (x_0, y_0) to (x,y) by curve $C \subset \Omega$ of finite length, and use the preceding part of the proof to inductively deduce the existence of a finite sequence of points $(x_i, y_i) \in C, i = 0, \cdots, n$ with $(x_n, y_n) = (x,y)$ and such that $u(x_i, y_i) = M^*$

Solution

Theorem 4.9 : Let u be a nonconstant harmonic function defined on a bounded domain Ω and continuous on $\partial\Omega$. Then u achieves its maximum and minimum values only at boundary points of the domain. In other words, if $m = \min\{u(x,y) \mid (x,y) \in \partial\Omega\}$, $M = \max\{u(x,t) \mid (x,y) \in \partial\Omega\}$ are respectively, its maximum and minimum values on the boundary, then $m < u(x,y) < M$ at all interior points $(x,y) \in \Omega$.

The book gives the proof showing that maximum M^* occurs on the boundary $\partial\Omega$. We are asked here to show that once we determined that given a circle inside Ω and assuming the maximum is at it center meaning all points inside this disk are $u = M^*$ then this implies that all points inside Ω must also be $u = M^*$ leading to contradiction of the nonconstant requirement. Hence the starting point is this diagram

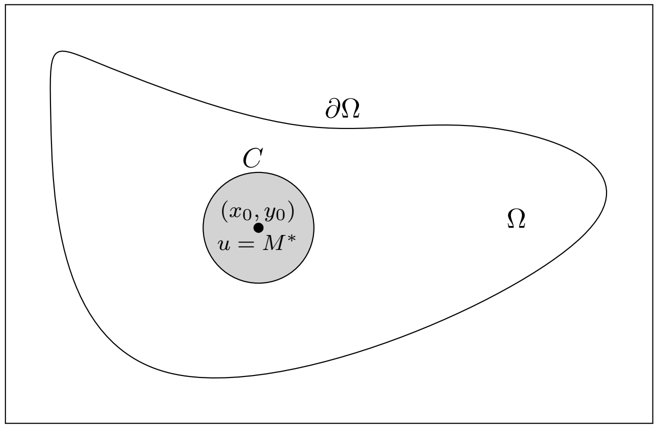


Figure 2.48: All points inside C have same value M^*

Now, we pick a new point from inside the disk C near the edge and apply the first part of the proof to show that all points inside the new disk C_2 also have $u = M^*$ there. So we have this new diagram.

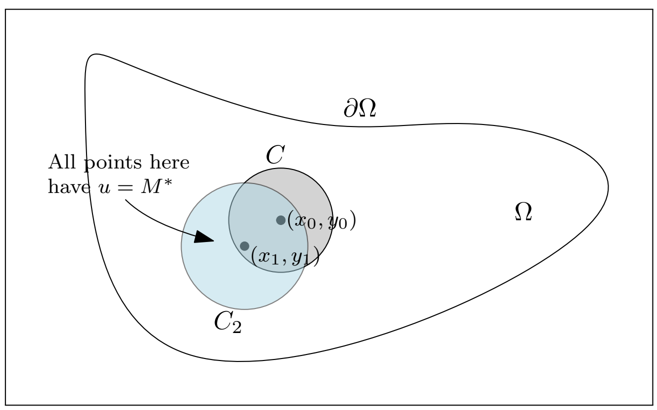


Figure 2.49: All points inside C_2 have same value M^*

We continue this way connecting points and adding the domain where all points have $u = M^*$ values.

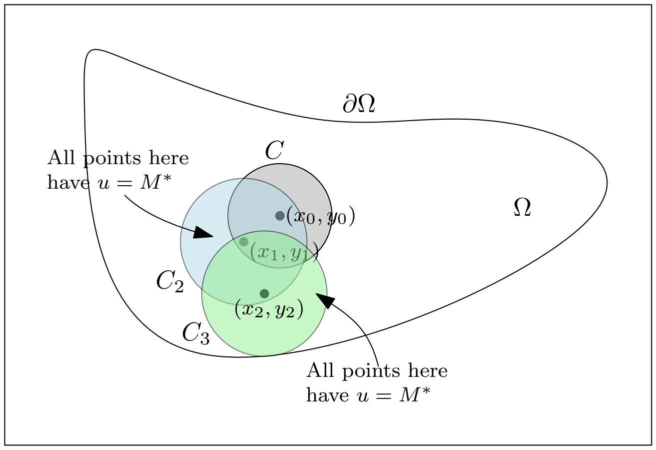


Figure 2.50: All points inside C_3 have same value M^*

Since Ω is connected then we can cover the whole region Ω this way all the way to the boundary $\partial\Omega$. This complete the proof given in the book.

2.7.6 Problem 4.3.46

Write down an integral formula for the solution to the Dirichlet boundary value problem on a disk of radius $R > 0$, namely, $\nabla^2 u = 0$, $x^2 + y^2 < R^2, u = h, x^2 + y^2 = R^2$

Solution

The closed form sum as given in theorem 4.6 in the book as the Poisson kernel integral formula

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}d\phi$$

Theorem 4.6 is for a unit disk. Since the disk here has radius R then r is changed to $\frac{r}{R}$ in the above giving

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\frac{1-\left(\frac{r}{R}\right)^2}{1+\left(\frac{r}{R}\right)^2-2\left(\frac{r}{R}\right)\cos(\theta-\phi)}d\phi$$

Which can be simplified to

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\frac{R^2-r^2}{R^2+r^2-2rR\cos(\theta-\phi)}d\phi$$

2.7.7 Problem 4.4.4

Consider the following partial differential equations. At what points of the plane is the equation elliptic? hyperbolic? parabolic? degenerate?

(a) $x^2 u_{xx} + x u_x + u_{yy} = 0$ (c) $u_t = \frac{\partial}{\partial x}((x+t)u_x)$

Solution

2.7.7.1 Part a

The general form of two variables $\left(x,y\right)$ PDE is

$$L\left[u\right] = Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{1}$$

The type of PDE depends on value of the discriminant

$$\Delta = B^2 - 4AC$$

Comparing the PDE $x^2u_{xx} + xu_x + u_{yy}$ to (1) shows that $A = x^2, B = 0, C = 1$. Hence

$$\Delta = -4x^2$$

This is always negative ($x = 0$ is not possible, since this would made the PDE not a PDE any more). Therefore using definition 4.12 this means the PDE is elliptic.

2.7.7.2 Part b

$$\begin{aligned} u_t &= \frac{\partial}{\partial x}\left(\left(x+t\right)u_x\right) \\ &= \left(\frac{\partial}{\partial x}\left(x+t\right)\right)u_x + \left(x+t\right)\frac{\partial}{\partial x}u_x \\ &= u_x + \left(x+t\right)u_{xx} \end{aligned}$$

Hence

$$u_x + \left(x+t\right)u_{xx} - u_t = 0 \tag{2}$$

The general form of two variables $\left(t,x\right)$ PDE is

$$L\left[u\right] = Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G \tag{3}$$

Comparing (2) to (3) shows that $C = \left(x+t\right), A = 0, B = 0$. Hence

$$\begin{aligned} \Delta &= B^2 - 4AC \\ &= 0 \end{aligned}$$

Hence PDE is parabolic.

2.7.8 Problem 4.4.11

Prove that the complex change of variables $x = x, t = iy$, maps the Laplace equation $u_{xx} + u_{yy} = 0$ to the wave equation $u_{tt} = u_{xx}$. Explain why the type of a partial differential equation is not necessarily preserved under a complex change of variables.

Solution

Given $u_{xx}+u_{yy}=0$, let $x=x,t=iy$. Hence we are to go from $u\left(x,y\right)$ to $v\left(t,x\right)$. Therefore

$$\begin{aligned}\frac{\partial u\left(x,y\right)}{\partial x} &= \overbrace{\frac{\partial u}{\partial t}\frac{dt}{dx}}^{0}+\frac{\partial u}{\partial x}\frac{dx}{dx}\\ &= \frac{\partial u}{\partial x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\\ &= \frac{\partial^2 u}{\partial x\partial t}\frac{dt}{dx}+\frac{\partial^2 u}{\partial x^2}\frac{dx}{dx}\\ &= \frac{\partial^2 u}{\partial x^2} \qquad (1)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial t}\frac{dt}{dy}+\overbrace{\frac{\partial u}{\partial x}\frac{dx}{dy}}^{0}\\ &= i\frac{\partial u}{\partial t}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\\ &= i\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right)\\ &= i\left(\frac{\partial^2 u}{\partial t^2}\frac{dt}{dy}+\overbrace{\frac{\partial^2 u}{\partial t\partial x}\frac{dx}{dy}}^{0}\right)\\ &= i\left(i\frac{\partial^2 u}{\partial t^2}\right)\\ &= -\frac{\partial^2 u}{\partial t^2} \qquad (2)\end{aligned}$$

Substituting (1,2) into $u_{xx}+u_{yy}=0$ gives

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2} &= 0\\ u_{tt} &= u_{xx}\end{aligned}$$

Which is the wave equation.

When change of variables contains only real quantities, then no sign change will occur. Only stretching (scaling) can occur, so the type of the PDE do not change. But with complex variables, a sign change can occur as in this example due to multiplying i with i. And this is what causes the PDE type to change.

2.7.9 Problem 4.4.16

True or false: The characteristic curves of the Helmholtz equation $u_{xx}+u_{yy}-u=0$ are circles.

Solution

Comparing the above to $L[u]=Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=G$ shows that

$$
\begin{aligned}
A &= 1\\
B &= 0\\
C &= 1
\end{aligned}
$$

Hence the characteristic curves are given by (4.151) as (where we choose $y\equiv y(x)$ and hence $s=x$ here)

$$
\begin{aligned}
A(x,y)\left(\frac{dy}{dx}\right)^2 - B(x,y)\frac{dy}{dx} + C(x,y) &= 0\\
\left(\frac{dy}{dx}\right)^2 + 1 &= 0\\
\left(\frac{dy}{dx}\right)^2 &= -1\\
\frac{dy}{dx} &= \pm i
\end{aligned}
$$

There are no real characteristic curves. Therefore the answer is false.

2.7.10 Key solution for HW 7

4.3.24(a)

(a) $v(y) = u(e^y)$ solves a constant coefficient second-order ordinary differential equation with a double root r, and hence $v(y) = c_1 e^{ry} + c_2 y e^{ry}$. Therefore,

$$u(x) = c_1 |x|^r + c_2 |x|^r \log |x|.$$

4.3.25(c)(d)

(c) $u(x,y) = \frac{1}{8} r^4 \cos 4\theta + 2r^2 \cos 2\theta + 6 = \frac{1}{8} x^4 - \frac{3}{4} x^2 y^2 + \frac{1}{8} y^4 + 2x^2 - 2y^2 + 6$

(d) $u(x,y) = r \cos\theta = x$.

4.3.33

$$u(r,\theta) = \frac{a_0}{2} \frac{\log r}{\log 2} + \sum_{n=1}^{\infty} \frac{r^n - r^{-n}}{2^n - 2^{-n}} (a_n \cos n\theta + b_n \sin n\theta),$$

where a_n, b_n are the usual Fourier coefficients of $h(\theta)$.

4.3.38

First, if $C = \frac{1}{\pi} \int_{-\pi}^{\pi} | h(\theta) | \, d\theta$, then the Fourier coefficients are bounded by

$$\left.\begin{aligned} |a_n| &\le \frac{1}{\pi} \int_{-\pi}^{\pi} | h(\theta) \cos n\theta | \, d\theta \\ |b_n| &\le \frac{1}{\pi} \int_{-\pi}^{\pi} | h(\theta) \sin n\theta | \, d\theta \end{aligned}\right\} \le \frac{1}{\pi} \int_{-\pi}^{\pi} | h(\theta) | \, d\theta = C.$$

Thus, the summands in (4.115) are bounded by

$$| a_n r^n \cos n\theta + b_n r^n \sin n\theta | \le r^n \big(|a_n| + |b_n| \big) \le 2 C r_\star^n.$$

According to the Weierstrass M test, since the geometric series $\sum_{n=1}^{\infty} 2 C r_\star^n < \infty$ converges, the series (4.115) converges uniformly. *Q.E.D.*

4.3.42

Given such a curve, let $\delta > 0$ be the minimum distance between C and the boundary $\partial\Omega$, which is positive since C is assumed to lie in the interior of Ω and both curves are compact (closed and bounded). Let $(x_i, y_i) \in C$, $i = 0, \ldots, n$, be a finite sequence of points on the curve with $(x_n, y_n) = (x, y)$ and such that the distance from (x_i, y_i) to (x_{i+1}, y_{i+1}) is $\le \frac{1}{2}\delta$, which implies that the disk centered at (x_i, y_i) whose boundary circle passes through (x_{i+1}, y_{i+1}) is contained in Ω. Using the preceding argument, a straightforward induction then shows that

$$M^\star = u(x_0, y_0) = u(x_1, y_1) = u(x_2, y_2) = \cdots = u(x_n, y_n) = u(x, y),$$

as desired. *Q.E.D.*

4.3.46

The rescaled function $\widehat{u}(x,y) = u(Rx, Ry)$ satisfies the boundary value problem (4.101) on the unit disk, and hence by (4.126)

$$u(r,\theta) = \widehat{u}(r/R,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\,\frac{1 - r^2/R^2}{1 + r^2/R^2 - 2\,(r/R)\cos(\theta-\phi)}\,d\phi.$$

4.4.4(a)(c)

(a) Elliptic when $x \neq 0$; parabolic when $x = 0$.

(c) Parabolic when $x + t \neq 0$; degenerate when $t = -x$.

4.4.11

By the chain rule,

$$\frac{\partial u}{\partial y} = \mathrm{i}\,\frac{\partial u}{\partial t}, \qquad \frac{\partial^2 u}{\partial y^2} = \mathrm{i}^2\,\frac{\partial^2 u}{\partial t^2} = -\,\frac{\partial^2 u}{\partial t^2}, \quad \text{and hence} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2}.$$

Thus, this complex change of variables maps the elliptic Laplace equation to the hyperbolic wave equation, and the type is not preserved.

4.4.16

False. The equation is elliptic and so has no real characteristics.

2.8 HW 8

Local contents

2.8.1 Problem 6.1.4c

Find and sketch a graph of the derivative (in the context of generalized functions) of the following functions

(c) $h(x) = \begin{cases} \sin(\pi x) & x > 1 \\ 1 - x^2 & -1 < x < 1 \\ e^x & x < -1 \end{cases}$

Solution

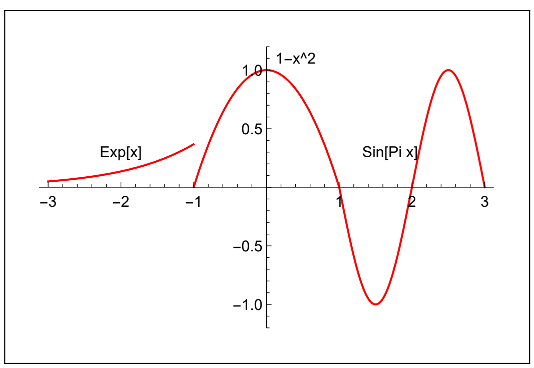


Figure 2.51: Sketch of the function $h(x)$

There is only one jump discontinuity at $x = -1$. The amount of jump [2] at $x = -1$ is $\frac{-1}{e}$. Hence

$$h'(x) = -e^{-1}\delta(x+1) + \begin{cases} \pi\cos(\pi x) & x > 1 \\ -2x & -1 < x < 1 \\ e^x & x < -1 \end{cases}$$

[2] When determining the sign of the jump, we go from left to right always. Dropping down means negative sign and moving higher means positive sign.

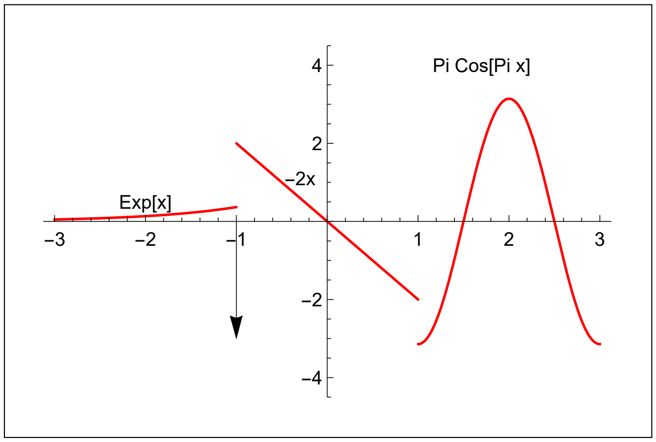


Figure 2.52: Sketch of the function $h'(x)$

2.8.2 Problem 6.1.5b

Find the first and second derivatives of the functions

(b) $k(x) = \begin{cases} |x| & -2 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

<u>Solution</u>

First, the function $k(x)$ is shown below

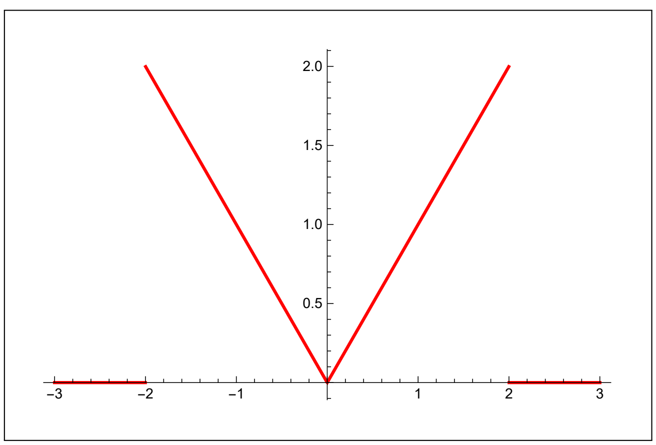


Figure 2.53: Sketch of the function $k(x)$

We see there is a jump discontinuity at $x = -2$ of value 2 and at $x = 2$ of value -2. Now, when $-2 < x < 0$, then $k(x) = -x$ and when $0 < x < 2$, then $k(x) = x$. Hence

$$k'(x) = 2\delta(x+2) - 2\delta(x-2) + \begin{cases} 0 & x < -2 \\ -1 & -2 < x < 0 \\ 1 & 0 < x < 2 \\ 0 & x > 2 \end{cases}$$

The derivative is not defined at $x = 0$. A plot of the above gives

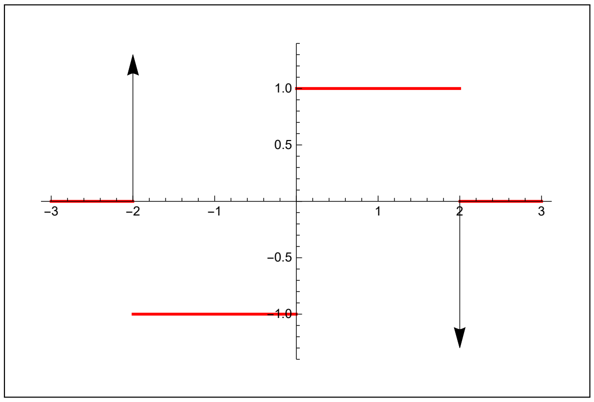


Figure 2.54: Sketch of the function $k'(x)$

We see that there is now a jump discontinuity at $x = -2$ of value -1 and jump discontinuity at $x = 0$ of value 2 and jump discontinuity at $x = 2$ of value -1. Hence

$$k''(x) = 2\delta'(x+2) - 2\delta'(x-2) - \delta(x+2) + 2\delta(x) - \delta(x-2)$$

Where $\delta'(x+2)$ and $\delta'(x-2)$ are called "doublets" at $x = -2$ and at $x = 2$ respectively.

2.8.3 Problem 6.1.9

For each positive integer n, let $g_n(x) = \begin{cases} \frac{1}{2}n & |x| < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$ (a) Sketch a graph of $g_n(x)$. (b) Show that $\lim_{n\to\infty} g_n(x) = \delta(x)$. (c) Evaluate $f_n(x) = \int_{-\infty}^{x} g_n(y)\, dy$ and sketch a graph. Does the sequence $f_n(x)$ converge to the step function $\sigma(x)$ as $n \to \infty$? (d) Find the derivative $h_n(x) = g_n'(x)$. (e) Does the sequence $h_n(x)$ converge to $\delta'(x)$ as $n \to \infty$?

Solution

2.8.3.1 Part a

Lets try few values of n.

$\underline{n=1}$ $g_1(x) = \begin{cases} \frac{1}{2} & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$

$\underline{n=2}$ $g_2(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$

$\underline{n=3}$ $g_2(x) = \begin{cases} \frac{3}{2} & |x| < \frac{1}{3} \\ 0 & \text{otherwise} \end{cases}$

And so on. We see that as n increases, the function value increases and the domain it is not zero on becomes smaller. As $n \to \infty$ this becomes a $\delta(x)$ function. Here is a plot of few values of increasing n.

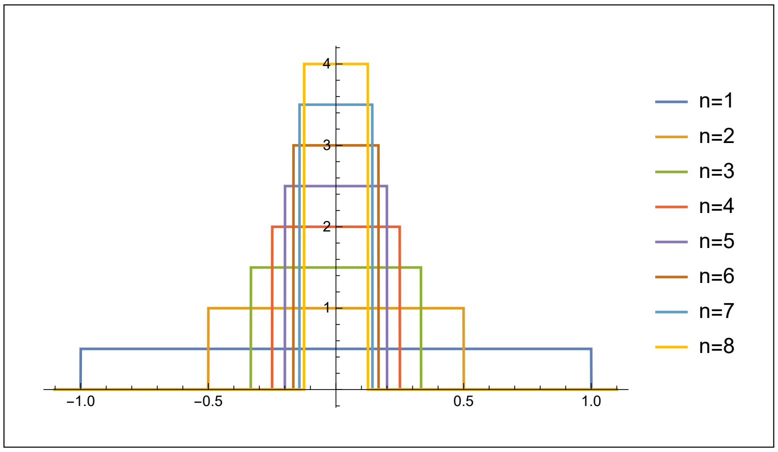


Figure 2.55: $g_n(x)$ for increasing n

2.8.3.2 Part b

$$
\begin{aligned}
\lim_{n\to\infty} g_n(x) &= \begin{cases} \lim_{n\to\infty} \frac{1}{2}n & \lim_{n\to\infty} |x| < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \infty & |x| \to 0 \\ 0 & \text{otherwise} \end{cases} \\
&= \delta(x)
\end{aligned}
$$

2.8.3.3 Part c

We want to integrate this function

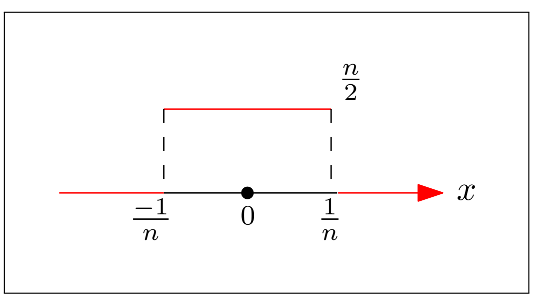


Figure 2.56: Integrating $g_n(x)$

Therefore

$$f_n(x)=\int_{-\infty}^{x} g_n(y)\,dy=\begin{cases}0 & x<\frac{-1}{n}\\ \left(\frac{1}{n}+x\right)\frac{n}{2} & \frac{-1}{n}<x<0\\ \left(\frac{1}{n}-x\right)\frac{n}{2} & 0<x<\frac{1}{n}\\ 1 & x>\frac{1}{n}\end{cases}$$

This is a sketch of the above We see that as $n\to\infty$ then $f_n(x)$ becomes

$$\lim_{n\to\infty} f_n(x)=\begin{cases}0 & x<0\\ \frac{1}{2} & x=0\\ 1 & x>0\end{cases}$$

Which is the step function $\sigma(x)$

2.8.3.4 Part d

From the plot of $g_n(x)$ above, we see there is a jump discontinuity at $x=-\frac{1}{n}$ of value $\frac{n}{2}$ and a jump discontinuity at $x=\frac{1}{n}$ of value $-\frac{n}{2}$. And since $g_n(x)$ is constant everywhere else, then

$$h_n(x)=g_n'(x)=\frac{n}{2}\delta\left(x+\frac{1}{n}\right)-\frac{n}{2}\delta\left(x-\frac{1}{n}\right)$$

2.8.3.5 Part e

Yes, $\lim_{n\to\infty} h_n(x)=\delta'(x)$. By definition, and as shown in figure 6.6 in textbook, $\delta'(x)$ is "doublets". Which is an impulse in positive direction just to the left of x and another impulse in negative direction just to the right of x and this is what happens when $\lim_{n\to\infty} h_n(x)$ as seen from the result in part d.

2.8.4 Problem 6.1.30

(a) Find the complex Fourier series for the derivative of the delta function $\delta'(x)$ by direct evaluation of the coefficient formulas (b) Verify that your series can be obtained by term-by-term differentiation of the series for $\delta(x)$. (c) Write a formula for the n^{th} partial sum of your series. (d) Use a computer graphics package to investigate the convergence of the series.

Solution

2.8.4.1 Part a

By first doing 2π periodic extension (similar to Dirac comb) we can calculate the coefficients. First we find the Fourier series for $\delta(x)$

$$\delta(x)\sim\sum_{k=-\infty}^{k=\infty} c_k e^{ikx}$$

Where $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(x)\,e^{-ikx}dx = \frac{1}{2\pi}$. Hence

$$
\begin{aligned}
\delta(x) &\sim \frac{1}{2\pi}\sum_{k=-\infty}^{k=\infty} e^{ikx}\\
&\sim \frac{1}{2\pi}\left(\cdots + e^{-2ix} + e^{-ix} + 1 + e^{ix} + e^{2ix} + \cdots\right) \qquad (1)
\end{aligned}
$$

Now

$$
\delta'(x) \sim \sum_{k=-\infty}^{k=\infty} d_k e^{ikx} \qquad (2)
$$

Where

$$
\begin{aligned}
d_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta'(x)\,e^{-ikx}dx\\
&= \frac{1}{2\pi}\left[\left(e^{-ikx}\right)'\right]_{x=0}\\
&= \frac{1}{2\pi}\left[-ike^{-ikx}\right]_{x=0}\\
&= \frac{1}{2\pi}[-ik]\\
&= -i\frac{k}{2\pi}
\end{aligned}
$$

Hence from (2) we obtain the Fourier series for $\delta'(x)$ as

$$
\begin{aligned}
\delta'(x) &\sim \frac{-i}{2\pi}\sum_{k=-\infty}^{k=\infty} ke^{ikx}\\
&\sim \frac{-i}{2\pi}\left(\cdots - 2e^{-2ix} - e^{-ix} + e^{ix} + 2e^{2ix} + \cdots\right)\\
&\sim \frac{1}{2\pi}\left(\cdots + 2ie^{-2ix} + ie^{-ix} - ie^{ix} - 2ie^{2ix} + \cdots\right) \qquad (3)
\end{aligned}
$$

2.8.4.2 Part b

To do term by term differentiation of $\delta(x)$, we first have to note the use of the following relation and the sign change needed to add

$$
\begin{aligned}
\lim_{n\to\infty} g_n(x) &\to \delta(x)\\
-\lim_{n\to\infty} g_n'(x) &\to \delta'(x)
\end{aligned}
$$

The above means we need to add a minus sign to the RHS when taking derivative of $\delta(x)$. Therefore, term by term differentiation of the Fourier series for $\delta(x)$ given in (1) now gives

$$
\begin{aligned}
\delta'(x) &\sim (-)\frac{1}{2\pi}\frac{d}{dx}\left(\cdots + e^{-2ix} + e^{-ix} + 1 + e^{ix} + e^{2ix} + \cdots\right)\\
&\sim (-)\frac{1}{2\pi}\left(\cdots - 2ie^{-2ix} - ie^{-ix} + ie^{ix} + 2ie^{2ix} + \cdots\right)\\
&\sim \frac{1}{2\pi}\left(\cdots + 2ie^{-2ix} + ie^{-ix} - ie^{ix} - 2ie^{2ix} + \cdots\right) \qquad (4)
\end{aligned}
$$

Comparing (4) and (3) shows they are the same.

2.8.4.3 Part c

It is easier to use normal Fourier series for this.

$$\begin{aligned}a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi}\delta'(x)\cos(kx)\,dx\\ &= \frac{1}{\pi}\left[(\cos kx)'\right]_{x=0}\\ &= \frac{1}{\pi}\left[-k\sin kx\right]_{x=0}\\ &= 0\end{aligned}$$

And

$$\begin{aligned}b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi}\delta'(x)\sin(kx)\,dx\\ &= \frac{1}{\pi}\left[(\sin kx)'\right]_{x=0}\\ &= \frac{1}{\pi}\left[k\cos kx\right]_{x=0}\\ &= \frac{k}{\pi}\end{aligned}$$

Hence

$$\delta'(x) \sim \frac{1}{\pi}\sum_{k=1}^{\infty} k\sin(kx) \tag{1}$$

Therefore the n^{th} partial sum is

$$\delta_n'(x) \sim \frac{1}{\pi}\sum_{k=1}^{n} k\sin(kx)$$

Since $|\sin(kx)| \leq 1$, used partial sum formula for the above given by

$$\sum_{k=1}^{n} k\sin(kx) = \frac{n\sin((n+1)x) - (n+1)\sin(nx)}{2\cos(x) - 2} \tag{2}$$

Hence

$$\delta_n'(x) \sim \frac{1}{\pi}\frac{n\sin((n+1)x) - (n+1)\sin(nx)}{2\cos(x) - 2}$$

It is possible to obtain the above formula by writing $\sin(kx) = \operatorname{Im}\left(e^{ikx}\right)$ and then using $\operatorname{Im}\sum_{k=1}^{n} ke^{ikx} = \operatorname{Im}\sum_{k=1}^{n} kz^k$ where $z = e^{ix}$. Since $|z| \leq 1$ then using the partial sum formula

$$
\begin{aligned}
\sum_{k=1}^{n} kz^k &= \frac{z(1-z^n)}{(1-z)^2} - \frac{nz^{n+1}}{1-z}\\
&= \frac{z(1-z^n) - nz^{n+1}(1-z)}{(1-z)^2}\\
&= \frac{z - z^{n+1} - nz^{n+1} + nz^{n+2}}{(1-z)^2}\\
&= \frac{z - (1+n)z^{n+1} + nz^{n+2}}{(1-z)^2}
\end{aligned}
$$

Then replacing z back by e^{ix} in the above, and using $e^{ix} = \cos x + i\sin x$ and simplifying and taking the imaginary part to obtain (2).

2.8.4.4 Part d

Using computer graphics, the following is plot of (2) for increasing values of n. This shows that as n increases $\delta_n'(x)$ approaches "doublets", which is a pulse to the left of $x = 0$ and one to the right of $x = 0$.

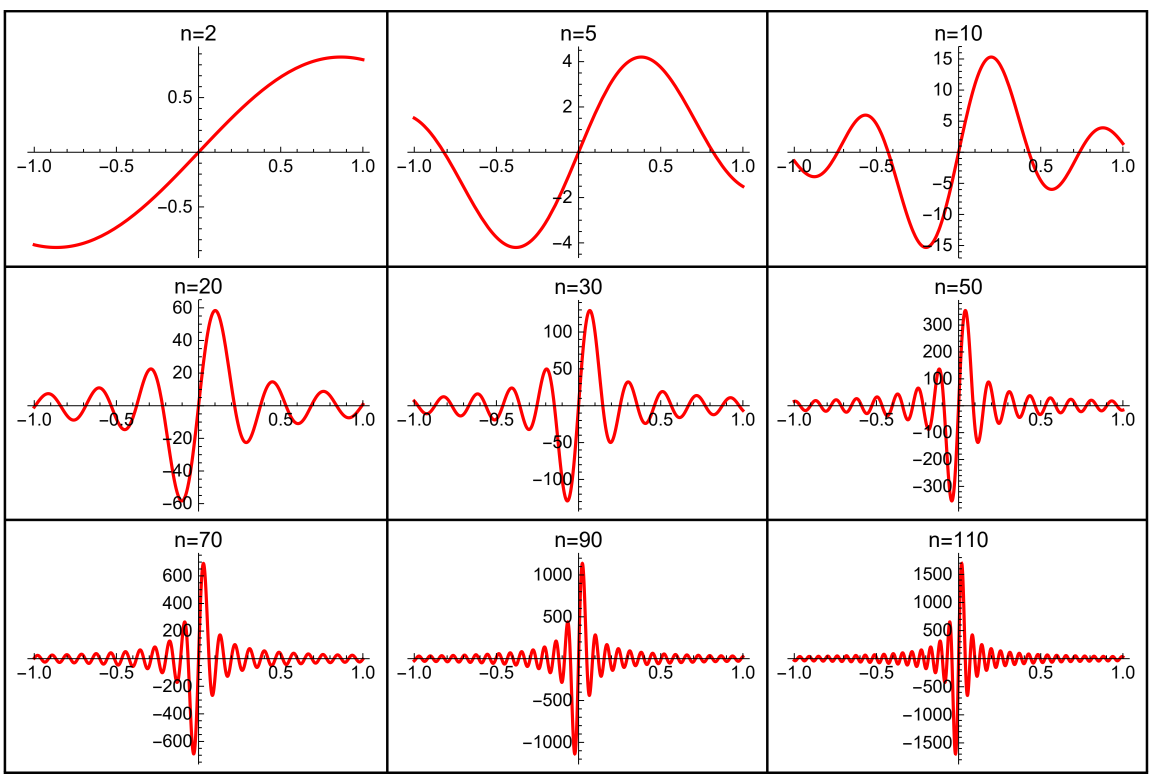


Figure 2.57: Convergence of Fourier series of $\delta'(x)$ as n increases

```
f[x_, n_] := 1/π (n Sin[(n + 1) x] - (n + 1) Sin[n x])/(2 Cos[x] - 2)
data = Table[Plot[f[x, n], {x, -1, 1}, PlotRange → All, PlotStyle → Red,
        PlotLabel → Row[{"n=", n}]], {n, {2, 5, 10, 20, 30, 50, 70, 90, 110}}];
p = Grid[Partition[data, 3], Frame → All];
```

Figure 2.58: Code used for the above plot

2.8.5 Problem 6.1.36

True or false: If you integrate the Fourier series for the delta function $\delta(x)$ term by term, you obtain the Fourier series for the step function $\sigma(x)$.

Solution

The Fourier series for delta function $\delta(x)$ is (assuming 2π periodic extension)

$$\delta(x) \sim \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \cos nx$$

Integrating RHS term by term gives

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{1}{2\pi} dx + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} \cos nx dx &= 1 + \frac{1}{\pi}\sum_{n=1}^{\infty} \overbrace{\left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi}}^{0} \\ &= 1 \end{aligned} \tag{1}$$

The step function $\sigma(x)$ is defined as

$$\sigma(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

Its Fourier series was already found on page 83 (assuming 2π periodic extension) in Example 3.9 as

$$\begin{aligned}\sigma(x) &\sim \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin((2n-1)x) \\ &= \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)\end{aligned} \tag{2}$$

Comparing (1) and (2), the answer is false.

2.8.6 Problem 6.2.4

The boundary value problem $-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x), u(0) = u(1) = 0$, models the displacement $u(x)$ of a nonuniform elastic bar with stiffness $c(x) = \frac{1}{1+x^2}$ for $0 \leq x \leq 1$. (a) Find the displacement when the bar is subjected to a constant external force, $f = 1$. (b) Find the

Green's function for the boundary value problem (c) Use the resulting superposition formula to check your solution to part (a). (d) Which point $0 < \xi < 1$ on the bar is the "weakest", i.e., the bar experiences the largest displacement under a unit impulse concentrated at that point?

Solution

2.8.6.1 Part a

The ode to solve is

$$\frac{d}{dx}\left(\frac{1}{1+x^2}\frac{du}{dx}\right) = -1$$

Integrating once gives

$$\begin{aligned}\frac{1}{1+x^2}\frac{du}{dx} &= -x + C_1\\ \frac{du}{dx} &= \left(1+x^2\right)(-x+C_1)\\ &= C_1 - x + C_1x^2 - x^3\end{aligned}$$

Integrating once more gives

$$\begin{aligned}u(x) &= C_1x - \frac{x^2}{2} + C_1\frac{x^3}{3} - \frac{x^4}{4} + C_2\\ &= -\frac{x^4}{4} + C_1\frac{x^3}{3} - \frac{x^2}{2} + C_1x + C_2 \qquad (1)\end{aligned}$$

Applying left B.C. $u(0) = 0$ gives

$$0 = C_2$$

Hence solution (1) becomes

$$u(x) = -\frac{x^4}{4} + C_1\frac{x^3}{3} - \frac{x^2}{2} + C_1x \qquad (2)$$

Applying left B.C. $u(1) = 0$ gives

$$\begin{aligned}0 &= -\frac{1}{4} + C_1\frac{1}{3} - \frac{1}{2} + C_1\\ C_1 &= \frac{9}{16}\end{aligned}$$

Hence the solution (2) becomes

$$\begin{aligned}u(x) &= -\frac{x^4}{4} + \frac{3}{16}x^3 - \frac{x^2}{2} + \frac{9}{16}x\\ &= \frac{1}{16}\left(-4x^4 + 3x^3 - 8x^2 + 9x\right)\end{aligned}$$

```
u[x_] := -x^4/4 + 3/16 x^3 - x^2/2 + 9/16 x;
Plot[u[x], {x, 0, 1}, PlotStyle → Red,
  GridLines → Automatic, GridLinesStyle → LightGray]
```

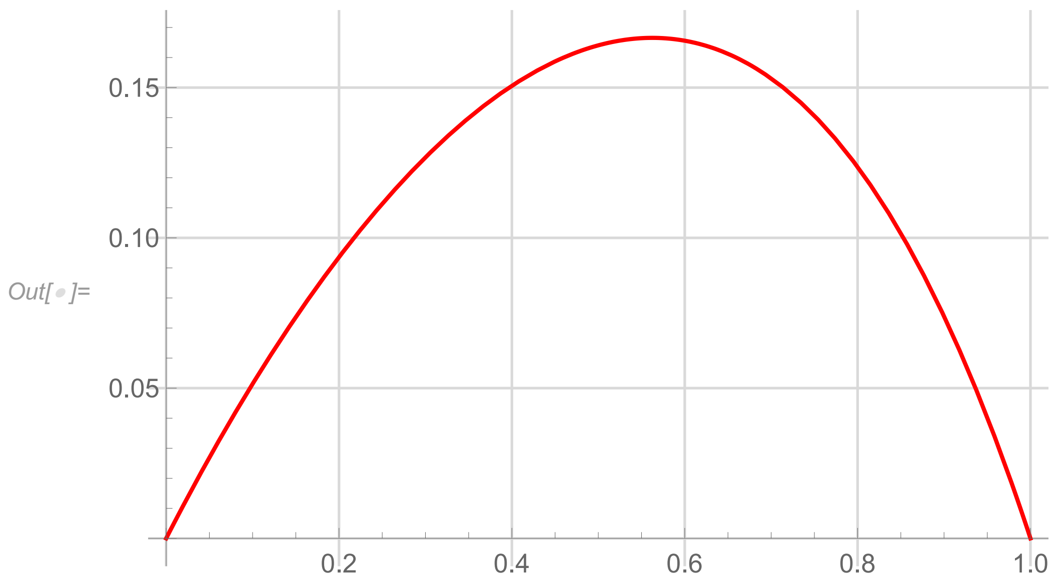


Figure 2.59: Plot of the above solution

2.8.6.2 Part b

When $x \neq y$, then Green function satisfies $\frac{d}{dx}\left(c(x)\frac{dG(x,y)}{dx}\right) = 0$. This means that

$$c(x)\frac{dG(x,y)}{dx} = A_1$$

But $c(x) = \frac{1}{1+x^2}$, therefore

$$\frac{dG(x,y)}{dx} = A_1\left(1+x^2\right)$$

Integrating gives

$$G(x,y) = A_1 x + A_1\frac{x^3}{3} + A_2$$

Therefore Green function is

$$G(x,y) = \begin{cases} A_1 x + A_1\frac{x^3}{3} + A_2 & x<y \\ B_1 x + B_1\frac{x^3}{3} + B_2 & x>y \end{cases} \tag{1}$$

Notice we used different constants of integrations for each side of the delta location y. Now we use boundary conditions on the left and right end to find these unknowns. Since Green function satisfies same boundary conditions as the solution, then at $x=0$ we need

$$\begin{aligned} G(0,y) &= 0 \\ &= A_2 \end{aligned}$$

And at $x=1$

$$\begin{aligned} G(1,y) &= 0 \\ &= B_1 + B_1\frac{1}{3} + B_2 \end{aligned}$$

Which means $-\frac{4}{3}B_1 = B_2$. Using these results in (1) gives

$$\begin{aligned} G(x,y) &= \begin{cases} A_1\left(x+\frac{x^3}{3}\right) & x<y \\ B_1x + B_1\frac{x^3}{3} - \frac{4}{3}B_1 & x>y \end{cases} \\ &= \begin{cases} A_1\left(x+\frac{x^3}{3}\right) & x<y \\ B_1\left(x+\frac{x^3}{3}-\frac{4}{3}\right) & x>y \end{cases} \end{aligned} \tag{1A}$$

We now need to determine A_1, B_1. From continuity condition of $G(x,y)$ at $x=y$ we obtain the first equation

$$A_1\left(y+\frac{y^3}{3}\right) = B_1\left(y+\frac{y^3}{3}-\frac{4}{3}\right) \tag{2}$$

And

$$\frac{dG(x,y)}{dx} = \begin{cases} A_1\left(1+x^2\right) & x<y \\ B_1\left(1+x^2\right) & x>y \end{cases}$$

Evaluated at $x=y$

$$\frac{dG(x,y)}{dx} = \begin{cases} A_1\left(1+y^2\right) & x<y \\ B_1\left(1+y^2\right) & x>y \end{cases}$$

There is a jump discontinuity in $\frac{dG(x,y)}{dx}$ of value $\frac{1}{p}$ where $-\left(py''\right)=0$. Comparing this with $-\frac{d}{dx}\left(c(x)\frac{dG(x,y)}{dx}\right) = f(x)$ shows that $p = \frac{1}{c(x)} = \left(1+x^2\right)$ or $\left(1+y^2\right)$ at $x=y$. Therefore this condition gives the second equation we need

$$A_1\left(1+y^2\right) - B_1\left(1+y^2\right) = \frac{1}{p} \tag{3}$$

$$= \left(1+y^2\right) \tag{2.1}$$

We now have the two equations we want (2,3) to solve for A_1, B_1. Solving for A_1, B_1 gives

$$\begin{aligned} A_1 &= \frac{1}{4}\left(4-3y-y^3\right) \\ B_1 &= \frac{1}{4}\left(-3y-y^3\right) \end{aligned}$$

Substituting the above into (1A) gives the Green function

$$
\begin{aligned}
G\left(x,y\right) &= \begin{cases} \frac{1}{4}\left(4-3y-y^3\right)\left(x+\frac{x^3}{3}\right) & x<y \\ \frac{1}{4}\left(-3y-y^3\right)\left(x+\frac{x^3}{3}-\frac{4}{3}\right) & x>y \end{cases} \\
&= \begin{cases} \frac{1}{4}\left(4-3y-y^3\right)\left(x+\frac{x^3}{3}\right) & x<y \\ \frac{1}{4}\left(4-3x-x^3\right)\left(y+\frac{y^3}{3}\right) & x>y \end{cases}
\end{aligned}
$$

we now see the symmetry above as expected.

2.8.6.3 Part (c)

Now we check the solution of part (a) for $f(x)=1$ using the superposition formula and noting that $f\left(y\right)=1$ we obtain

$$
\begin{aligned}
u(x) &= \overbrace{\int_0^x G\left(x,y\right) f\left(y\right) dy}^{y<x} + \overbrace{\int_x^1 G\left(x,y\right) f\left(y\right) dy}^{y>x} \\
&= \int_0^x \frac{1}{4}\left(4-3x-x^3\right)\left(y+\frac{y^3}{3}\right) dy + \int_x^1 \frac{1}{4}\left(4-3y-y^3\right)\left(x+\frac{x^3}{3}\right) dy
\end{aligned}
$$

Hence

$$
\begin{aligned}
u(x) &= \frac{1}{4}\left(4-3x-x^3\right)\int_0^x\left(y+\frac{y^3}{3}\right)dy + \frac{1}{4}\left(x+\frac{x^3}{3}\right)\int_x^1\left(4-3y-y^3\right)dy \\
&= \frac{1}{4}\left(4-3x-x^3\right)\left(\frac{y^2}{2}+\frac{y^4}{12}\right)_0^x + \frac{1}{4}\left(x+\frac{x^3}{3}\right)\left(4y-\frac{3y^2}{2}-\frac{y^4}{4}\right)_x^1 \\
&= \frac{1}{4}\left(4-3x-x^3\right)\left(\frac{x^2}{2}+\frac{x^4}{12}\right) + \frac{1}{4}\left(x+\frac{x^3}{3}\right)\left(4-\frac{3}{2}-\frac{1}{4}-\left(4x-\frac{3x^2}{2}-\frac{x^4}{4}\right)\right) \\
&= \frac{1}{16}x\left(-4x^3+3x^2-8x+9\right)
\end{aligned}
$$

Which agree with solution obtain in part (a)

2.8.6.4 Part (d)

From the solution above $u(x)=\frac{1}{16}\left(-4x^4+3x^3-8x^2+9x\right)$. Hence

$$
\frac{du}{dx} = \frac{1}{16}\left(-16x^3+9x^2-16x+9\right)
$$

Solving for $\frac{du}{dx}=0$ gives

$$
\begin{aligned}
\frac{1}{16}\left(-16x^3+9x^2-16x+9\right) &= 0 \\
-\frac{1}{16}(16x-9)\left(1+x^2\right) &= 0
\end{aligned}
$$

$\left(1+x^2\right)=0$ does not give real solutions. Hence $-\frac{1}{16}(16x-9)=0$ or $16x-9=0$ or

$$x=\frac{9}{16}$$

At this x is the largest displacement which is found by evaluating the solution at this x

$$\begin{aligned}
u\left(\frac{9}{16}\right) &= \frac{1}{16}\left(-4\left(\frac{9}{16}\right)^4+3\left(\frac{9}{16}\right)^3-8\left(\frac{9}{16}\right)^2+9\left(\frac{9}{16}\right)\right)\\
&= \frac{43659}{262144}\\
&= 0.167
\end{aligned}$$

2.8.7 Problem 6.2.7

For n a positive integer, set $f_n(x)=\begin{cases}\frac{1}{2}n & |x-\xi|<\frac{1}{n}\\ 0 & \text{otherwise}\end{cases}$ (a) Find the solution $u_n(x)$ to the boundary value problem $-u''=f_n(x), u(0)=0, u(1)=0$, assuming $0<\xi-\frac{1}{n}<\xi+\frac{1}{n}<1$. (b) Prove that $\lim_{n\to\infty}u_n(x)=G(x;\xi)$ converges to the Green's function (6.51) given by solution to $-cu''=f(x)$ with same BC as

$$G(x;\xi)=\frac{(1-\xi)x-\rho(x-\xi)}{c}=\begin{cases}(1-\xi)\frac{x}{c} & x\le\xi\\ (1-x)\frac{\xi}{c} & x\ge\xi\end{cases}$$

But here $c=1$, so the above becomes

$$G(x;\xi)=(1-\xi)x-\rho(x-\xi)=\begin{cases}(1-\xi)x & x\le\xi\\ (1-x)\xi & x\ge\xi\end{cases}$$

Where ρ is the ramp function. Why should this be the case? (c) Reconfirm the result in part (b) by graphing $u_5(x), u_{15}(x), u_{25}(x)$, along with $G(x;\xi)$ when $\xi=0.3$.

Solution

2.8.7.1 Part a

When $x\neq\xi$, then Green function satisfies $\frac{d^2G(x,y)}{dx^2}=0$. This means that

$$G\left(x,y\right)=A_1x+A_2$$

Hence Green function is

$$G\left(x,y\right)=\begin{cases}A_1x+A_2 & x\le\xi\\ B_1x+B_2 & x\ge\xi\end{cases}$$

At $x=0$, $G\left(0,y\right)=0=A_2$ and at $x=1$, $G\left(1,y\right)=0=B_1+B_2$. Hence $B_2=-B_1$. The above becomes

$$
\begin{aligned}
G\left(x,y\right) &= \begin{cases} A_1 x & x\leq\xi \\ B_1 x - B_1 & x\geq\xi \end{cases} \\
&= \begin{cases} A_1 x & x\leq\xi \\ B_1\left(x-1\right) & x\geq\xi \end{cases} \qquad \text{(A)}
\end{aligned}
$$

Where A_1, B_1 are constants to be found. These are found from the continuity condition and the jump discontinuity condition on $\frac{dG}{dx}$ both at $x=\xi$. The continuity condition at $x=\xi$ gives the first equation as

$$
A\xi = B\left(\xi-1\right) \tag{1}
$$

And $\frac{dG}{dx}$ at $x=\xi$ gives

$$
\lim_{x\to\xi}\frac{dG}{dx} = \begin{cases} A_1 & x\leq\xi \\ B_1 & x\geq\xi \end{cases}
$$

Hence the jump discontinuity condition gives the second equation we want which is

$$
A_1 - B_1 = 1 \tag{2}
$$

Where 1 is used in RHS above since $c=1$. From (1,2) we solve for A_1, B_1. Which gives

$$
\begin{aligned}
B_1 &= -\xi \\
A_1 &= 1-\xi
\end{aligned}
$$

Substituting the above back into Eq (A) gives the Green function

$$
G\left(x;\xi\right) = \begin{cases} \left(1-\xi\right)x & x\leq\xi \\ \left(1-x\right)\xi & x\geq\xi \end{cases} \tag{3}
$$

The solution is now found using superposition formula

$$
\begin{aligned}
u_n\left(x\right) &= \overbrace{\int_0^x G\left(x,\xi\right)f_n\left(\xi\right)d\xi}^{\xi<x} + \overbrace{\int_x^1 G\left(x,\xi\right)f_n\left(\xi\right)d\xi}^{\xi>x} \\
&= \int_0^x \left(1-x\right)\xi f_n\left(\xi\right)d\xi + \int_x^1 \left(1-\xi\right)x f_n\left(\xi\right)d\xi \\
&= \left(1-x\right)\int_0^x \xi f_n\left(\xi\right)d\xi + x\int_x^1 \left(1-\xi\right)f_n\left(\xi\right)d\xi \qquad (4)
\end{aligned}
$$

But $f_n\left(x\right) = \begin{cases} \frac{1}{2}n & \left|x-\xi\right|<\frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$. We are told that $0<\xi-\frac{1}{n}<\xi+\frac{1}{n}<1$. Hence (4) becomes

$$
\begin{aligned}
u_n(x) &= (1-x)\int_{x-\frac{1}{n}}^{x} \xi \frac{n}{2} d\xi + x\int_{x}^{x+\frac{1}{n}} (1-\xi)\frac{n}{2} d\xi \\
&= \frac{(1-x)}{2} n \int_{x-\frac{1}{n}}^{x} \xi d\xi + \frac{x}{2} n \int_{x}^{x+\frac{1}{n}} (1-\xi)\, d\xi \\
&= \frac{(1-x)}{2} n \left(\frac{\xi^2}{2}\right)_{x-\frac{1}{n}}^{x} + \frac{x}{2} n \left(\xi - \frac{\xi^2}{2}\right)_{x}^{x+\frac{1}{n}} \\
&= \frac{(1-x)}{2} n \left(\frac{x^2}{2} - \frac{\left(x-\frac{1}{n}\right)^2}{2}\right) + \frac{x}{2} n \left(\left(\left(x+\frac{1}{n}\right) - \frac{\left(x+\frac{1}{n}\right)^2}{2}\right) - \left(x - \frac{x^2}{2}\right)\right) \\
&= \left(\frac{1}{2}x + \frac{1}{4n}x - \frac{1}{4n} - \frac{1}{2}x^2\right) - \left(\frac{1}{4n}x\,(2nx - 2n + 1)\right) \\
&= -\frac{1}{4n}\left(4nx^2 - 4nx + 1\right) \\
&= x - x^2 - \frac{1}{4n}
\end{aligned}
$$

2.8.7.2 Part b

$$
\begin{aligned}
\lim_{n\to\infty} u_n(x) &= \lim_{n\to\infty} x - x^2 - \frac{1}{4n} \\
&= x(1-x)
\end{aligned}
$$

2.8.7.3 Part c

This is plot of Green function $G(x;\xi) = \begin{cases} (1-\xi)\,x & x \le \xi \\ \xi\,(1-x) & x \ge \xi \end{cases}$ for $\xi = 0.3$

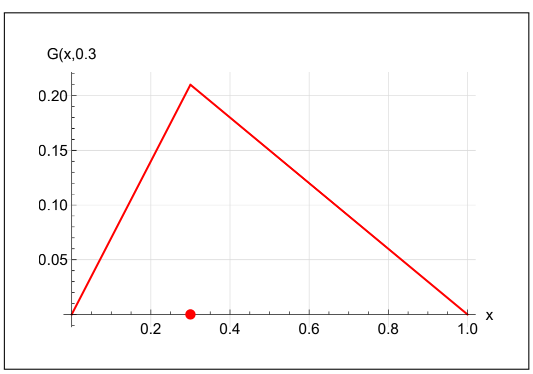


Figure 2.60: Green function

```
green[x_, z_] := Piecewise[{{(1 - z) x, x < z}, {(1 - x) z, x > z}}]
p = Plot[green[x, 0.3], {x, 0, 1}, PlotStyle → Red,
        GridLines → Automatic, GridLinesStyle → LightGray,
        AxesLabel → {"x", "G(x,0.3"}, BaseStyle → 12,
        Epilog → {Red, {PointSize[.025], Point[{0.3, 0}]}}];
```

Figure 2.61: Code for the above plot

These are plots of $u_n(x) = x - x^2 - \frac{1}{4n}$ for different n values.

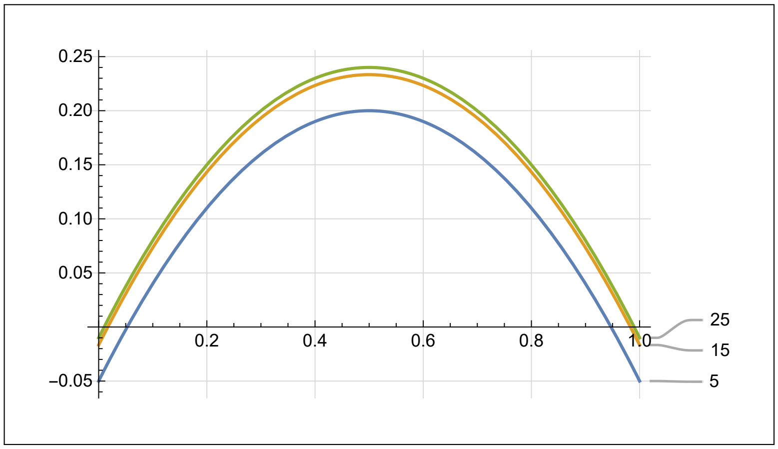


Figure 2.62: Plot of $u_n(x)$ for different n values

```
u[x_, n_] := x - x^2 - 1 / (4 n)
p = Plot[Evaluate[Table[Callout[u[x, n], n], {n, {5, 15, 25}}]], {x, 0, 1},
      AxesOrigin → {0, 0}, GridLines → Automatic,
      GridLinesStyle → LightGray];
```

Figure 2.63: Code for the above plot

Please note that the plots above do not seem to converge well with what is expected which is the Green function plot earlier. I am not able to find out so far where the problem is.

2.8.8 Problem 6.2.11

Let $\omega > 0$. (a) Find the Green's function for the mixed boundary value problem

$$-u'' + \omega^2 u = f(x), \qquad u(0) = 0, u'(1) = 0$$

(b) Use your Green's function to find the solution when $f(x)=\begin{cases} 1 & 0<x<\frac{1}{2} \\ -1 & \frac{1}{2}<x\leq 1\end{cases}$

Solution

2.8.8.1 Part a

When $x\neq\xi$, then Green function satisfies $-\frac{d^2G(x,y)}{dx^2}+\omega^2G(x,y)=0$. This means that $\frac{d^2G(x,y)}{dx^2}-\omega^2G(x,y)=0$ which has solution

$$G(x,y)=A_1\cosh(\omega x)+A_2\sinh(\omega x)$$

Hence Green function is

$$G(x,y)=\begin{cases} A_1\cosh(\omega x)+A_2\sinh(\omega x) & 0<x<y \\ B_1\cosh(\omega x)+B_2\sinh(\omega x) & y<x<1\end{cases} \tag{1A}$$

At $x=0$, $G(0,y)=0=A_1$. And to find conditions at $x=1$, then $G'(x,y)=\omega B_1\sinh(\omega x)+\omega B_2\cosh(\omega x)$. Hence at $x=1$ this gives

$$\begin{aligned} G'(1,y)&=0\\ &=\omega B_1\sinh\omega+\omega B_2\cosh\omega\end{aligned}$$

Therefore $B_1\sinh\omega+B_2\cosh\omega=0$. Or $B_2=-B_1\tanh\omega$. Hence (1A) becomes

$$\begin{aligned} G(x,y)&=\begin{cases} A_2\sinh(\omega x) & 0<x<y \\ B_1\cosh(\omega x)-B_1\tanh\omega\sinh(\omega x) & y<x<1\end{cases}\\ &=\begin{cases} A_2\sinh(\omega x) & 0<x<y \\ B_1(\cosh(\omega x)-\tanh\omega\sinh(\omega x)) & y<x<1\end{cases}\end{aligned}$$

But $\cosh(\omega x)-\tanh\omega\sinh(\omega x)=\frac{\cosh(\omega-\omega x)}{\cosh\omega}$. The above becomes

$$G(x,y)=\begin{cases} A_2\sinh(\omega x) & 0<x<y \\ B_1\frac{\cosh(\omega-\omega x)}{\cosh\omega} & y<x<1\end{cases} \tag{1}$$

We now need to determine A_2,B_1. From continuity condition of $G(x,y)$ at $x=y$ we obtain the first equation

$$A_2\sinh(\omega y)=B_1\frac{\cosh(\omega-\omega y)}{\cosh\omega} \tag{2}$$

And

$$\frac{dG(x,y)}{dx}=\begin{cases} A_2\omega\cosh(\omega x) & x<y \\ B_1\left(\frac{-\omega\sinh(\omega-\omega x)}{\cosh\omega}\right) & x>y\end{cases}$$

Evaluated at $x=y$

$$\frac{dG\left(x,y\right)}{dx}=\begin{cases} A_2\omega\cosh\left(\omega y\right) & x<y\\ B_1\left(\frac{-\omega\sinh\left(\omega-\omega y\right)}{\cosh\omega}\right) & x>y\end{cases}$$

There is a jump discontinuity in $\frac{dG\left(x,y\right)}{dx}$ of value 1 at $x=y$. Therefore this condition gives the second equation we need

$$A_2\omega\cosh\left(\omega y\right)+B_1\frac{\omega\sinh\left(\omega-\omega y\right)}{\cosh\omega}=1 \tag{3}$$

Solving (2,3) for A_2,B_1 gives

$$A_2=\frac{\cosh\left(\omega\left(1-y\right)\right)}{\omega\cosh\left(\omega\right)}$$

$$B_1=\frac{\sinh\left(\omega y\right)}{\omega}$$

Substituting the above into (1) gives the Green function

$$G\left(x,y\right)=\begin{cases}\frac{\cosh\left(\omega\left(1-y\right)\right)}{\omega\cosh\left(\omega\right)}\sinh\left(\omega x\right) & 0<x<y\\ \frac{\cosh\left(\omega\left(1-x\right)\right)}{\omega\cosh\omega}\sinh\left(\omega y\right) & y<x<1\end{cases} \tag{4}$$

2.8.8.2 Part b

Using the superposition formula

$$\begin{aligned} u\left(x\right)&=\overbrace{\int_0^x G\left(x,y\right)f\left(y\right)dy}^{y<x}+\overbrace{\int_x^1 G\left(x,y\right)f\left(y\right)dy}^{y>x}\\ &=\int_0^x\frac{\cosh\left(\omega\left(1-x\right)\right)}{\omega\cosh\left(\omega\right)}\sinh\left(\omega y\right)f\left(y\right)dy+\int_x^1\frac{\cosh\left(\omega\left(1-y\right)\right)}{\omega\cosh\left(\omega\right)}\sinh\left(\omega x\right)f\left(y\right)dy\end{aligned}$$

But $f\left(x\right)=\begin{cases}1 & 0<x<\frac{1}{2}\\ -1 & \frac{1}{2}<x\leq 1\end{cases}$, hence the above reduces to

<u>case $x<\frac{1}{2}$</u>

$$\begin{aligned} u\left(x\right)&=\int_0^x\frac{\cosh\left(\omega\left(1-x\right)\right)}{\omega\cosh\left(\omega\right)}\sinh\left(\omega y\right)dy+\int_x^{\frac{1}{2}}\frac{\cosh\left(\omega\left(1-y\right)\right)}{\omega\cosh\left(\omega\right)}\sinh\left(\omega x\right)dy-\int_{\frac{1}{2}}^1\frac{\cosh\left(\omega\left(1-y\right)\right)}{\omega\cosh\left(\omega\right)}\sinh\left(\omega x\right)dy\\ &=\frac{1}{\omega^2}-\frac{\left(e^{\frac{\omega}{2}}-e^{-\frac{\omega}{2}}+e^{-\omega}\right)e^{\omega x}+\left(e^{\omega}-e^{\frac{\omega}{2}}+e^{-\frac{\omega}{2}}\right)e^{-\omega x}}{\omega^2\left(e^{\omega}+e^{-\omega}\right)}\end{aligned}$$

case $x > \frac{1}{2}$

$$
\begin{aligned}
u(x) &= \int_0^{\frac{1}{2}} \frac{\cosh(\omega(1-x))}{\omega\cosh(\omega)} \sinh\left(\omega y\right) dy - \int_{\frac{1}{2}}^{x} \frac{\cosh\left(\omega\left(1-y\right)\right)}{\omega\cosh(\omega)} \sinh(\omega x)\, dy - \int_x^1 \frac{\cosh\left(\omega\left(1-y\right)\right)}{\omega\cosh(\omega)} \sinh(\omega x)\, dy \\
&= -\frac{1}{\omega^2} - \frac{\left(e^{\frac{-\omega}{2}} - e^{-\omega} + e^{-\frac{3}{2}\omega}\right)e^{\omega x} + \left(e^{\frac{3}{2}\omega} - e^{\omega} + e^{\frac{\omega}{2}}\right)e^{-\omega x}}{\omega^2\left(e^{\omega}+e^{-\omega}\right)}
\end{aligned}
$$

2.8.9 Problem 6.2.12

Suppose $\omega > 0$. Does the Neumann boundary value problem $-u'' + \omega^2 u = f(x), u'(0) = u'(1) = 0$ admit a Green's function? If not, explain why not. If so, find it, and then write down an integral formula for the solution of the boundary value problem.

Solution

To find out if it admits a Green function, we will see if we can solve for the constants that show up in the formulation of Green function. If not able to find a solution, then no Green function.

When $x \neq \xi$, then Green function satisfies $-\frac{d^2G(x,y)}{dx^2} + \omega^2 G\left(x,y\right) = 0$. This means that

$$G\left(x,y\right) = A_1\cosh(\omega x) + A_2\sinh(\omega x)$$

Hence Green function is

$$G\left(x,y\right) = \begin{cases} A_1\cosh(\omega x) + A_2\sinh(\omega x) & 0<x<y \\ B_1\cosh(\omega x) + B_2\sinh(\omega x) & y<x<1 \end{cases} \tag{1A}$$

On the left end, $\frac{d}{dx}G\left(x,y\right) = \omega A_1\sinh(\omega x) + \omega A_2\cosh(\omega x)$. Hence At $x=0$, $G'\left(0,y\right) = 0 = \omega A_2$. Therefore $A_2 = 0$. On the right side $\frac{d}{dx}G\left(x,y\right) = \omega B_1\sinh(\omega x) + \omega B_2\cosh(\omega x)$. At $x=1$, then $G'\left(x,y\right) = \omega B_1\sinh(\omega) + \omega B_2\cosh(\omega) = 0$. Therefore $B_1\sinh\omega + B_2\cosh\omega = 0$. Or $B_2 = -B_1\tanh\omega$. Hence (1A) becomes

$$
\begin{aligned}
G\left(x,y\right) &= \begin{cases} A_1\cosh(\omega x) & 0<x<y \\ B_1\cosh(\omega x) - B_1\tanh\omega\sinh(\omega x) & y<x<1 \end{cases} \\
&= \begin{cases} A_1\cosh(\omega x) & 0<x<y \\ B_1\left(\cosh(\omega x) - \tanh\omega\sinh(\omega x)\right) & y<x<1 \end{cases}
\end{aligned}
$$

But $\cosh(\omega x) - \tanh\omega\sinh(\omega x) = \frac{\cosh(\omega(1-x))}{\cosh\omega}$. The above becomes

$$G\left(x,y\right) = \begin{cases} A_1\cosh(\omega x) & 0<x<y \\ B_1\frac{\cosh(\omega(1-x))}{\cosh\omega} & y<x<1 \end{cases}$$

Now we will try to see if we can determine A_1, B_1. Continuity condition at $x=y$ gives the first equation

$$A_1 \cosh(\omega y) = \frac{B_1}{\cosh \omega} \cosh(\omega(1-y)) \tag{1}$$

And

$$\frac{dG(x,y)}{dx} = \begin{cases} A_1 \omega \sinh(\omega x) & 0 < x < y \\ -\frac{B_1}{\cosh \omega} \omega \sinh(\omega(1-x)) & y < x < 1 \end{cases}$$

Hence at $x = y$ to satisfy the jump discontinuity in $\frac{dG(x,y)}{dx}$ the second equation is

$$A\omega \sinh(\omega y) + \frac{B_1}{\cosh \omega} \omega \sinh(\omega(1-y)) = 1 \tag{2}$$

Solving (1,2) for A, B gives

$$A_1 = \frac{\cosh(\omega(1-y))}{\omega \sinh(\omega)}$$

$$B_1 = \frac{\cosh(\omega y)}{\omega \sinh(\omega)} \cosh(\omega)$$

Hence <u>Green function exist</u>. Substituting the above in Green function above gives

$$G(x,y) = \begin{cases} \frac{\cosh(\omega(1-y))}{\omega \sinh(\omega)} \cosh(\omega x) & 0 < x < y \\ \frac{\cosh(\omega(1-x))}{\omega \sinh(\omega)} \cosh(\omega y) & y < x < 1 \end{cases}$$

Here is a plot of the above when the pulse at $y = 0.25$ with $\omega = 1$

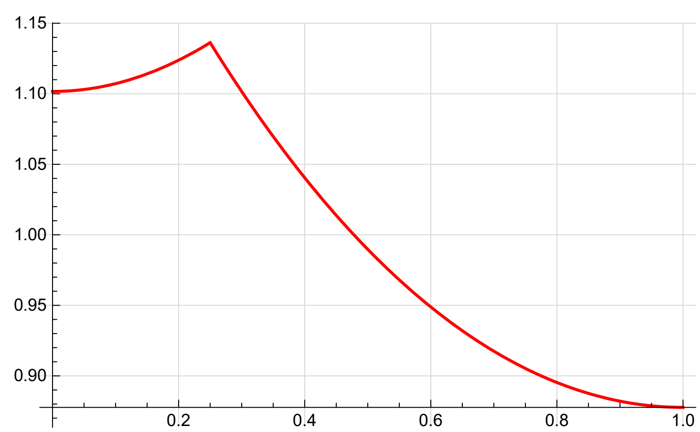


Figure 2.64: Plot of the Green function found

```
p = With[{y = 0.25, w = 1},
        Plot[Cosh[w (1 - y)]/(w Sinh[w]) Cosh[w x] HeavisideTheta[-x + y] +
             Cosh[w (1 - x)]/(w Sinh[w]) Cosh[w y] HeavisideTheta[x - y], {x, 0, 1},
          PlotStyle → Red, GridLines → Automatic, GridLinesStyle → LightGray]
      ];
```

Figure 2.65: Code used for the above plot

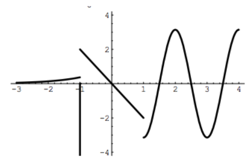
The integral formula is

$$
\begin{aligned}
u(x) &= \int_0^x \frac{\cosh\left(\omega y\right)}{\omega \sinh(\omega)} \cosh(\omega(1-x)) f\left(y\right) dy + \int_x^1 \frac{\cosh\left(\omega\left(1-y\right)\right)}{\omega \sinh(\omega)} \cosh(\omega x) f\left(y\right) dy \\
&= \frac{\cosh(\omega(1-x))}{\omega \sinh(\omega)} \int_0^x \cosh\left(\omega y\right) f\left(y\right) dy + \frac{\cosh(\omega x)}{\omega \sinh(\omega)} \int_x^1 \cosh\left(\omega\left(1-y\right)\right) f\left(y\right) dy
\end{aligned}
$$

2.8.10 Key solution for HW 8

6.1.4c

★ (c) $h'(x) = -e^{-1}\,\delta(x+1) + \begin{cases} \pi\cos\pi x, & x > 1, \\ -2x, & -1 < x < 1, \\ e^x, & x < -1. \end{cases}$

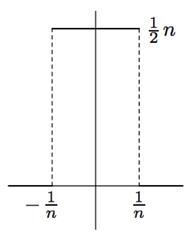


6.1.5b

$$(b)\quad k'(x) = 2\,\delta(x+2) - 2\,\delta(x-2) + \begin{cases} -1, & -2 < x < 0, \\ 1, & 0 < x < 2, \\ 0, & \text{otherwise}, \end{cases}$$
$$= 2\,\delta(x+2) - 2\,\delta(x-2) - \sigma(x+2) + 2\,\sigma(x) - \sigma(x-2),$$
$$k''(x) = 2\,\delta'(x+2) - 2\,\delta'(x-2) - \delta(x+2) + 2\,\delta(x) - \delta(x-2).$$

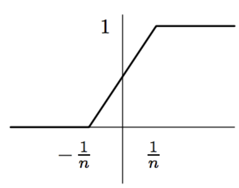
6.1.9

(a)



(b) First, $\lim_{n\to\infty} g_n(x) = 0$ for any $x \neq 0$ since $g_n(x) = 0$ whenever $n > 1/|\,x\,|$. Moreover, $\int_{-\infty}^{\infty} g_n(x)\,dx = 1$, and hence the sequence satisfies (6.11–12), proving
$$\lim_{n\to\infty} g_n(x) = \delta(x).$$

(c) $f_n(x) = \int_{-\infty}^{x} g_n(y)\,dy = \begin{cases} 0, & x < -\frac{1}{n}, \\ \frac{1}{2}nx + \frac{1}{2}, & |\,x\,| < \frac{1}{n}, \\ 1, & x > \ n. \end{cases}$



Since $\dfrac{1}{n} \to 0$ as $n \to \infty$, the limiting function is $\lim_{n\to\infty} f_n(x) = \sigma(x) = \begin{cases} 0 & x < 0, \\ \frac{1}{2} & x = 0, \\ 1 & x > 0. \end{cases}$

(d) $h_n(x) = \frac{1}{2}n\,\delta\big(x + \frac{1}{n}\big) - \frac{1}{2}n\,\delta\big(x - \frac{1}{n}\big)$.

6.1.30

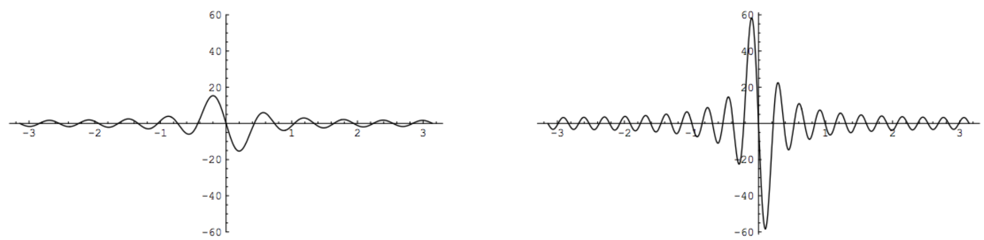
(a) $\delta'(x) \sim \dfrac{\mathrm{i}}{2\pi} \displaystyle\sum_{k=-\infty}^{\infty} k\,e^{\,\mathrm{i}\,k\,x}$.

(b) This is evident since the derivative of $e^{\,\mathrm{i}\,k\,x}$ is $\mathrm{i}\,k\,e^{\,\mathrm{i}\,k\,x}$.

(c) Differentiate formula (6.39) to obtain

$$\frac{\mathrm{i}}{2\pi} \sum_{k=-n}^{n} k\,e^{\,\mathrm{i}\,k\,x} = \frac{\left(n+\frac{1}{2}\right)\cos\left(n+\frac{1}{2}\right)x\;\sin\frac{1}{2}x - \frac{1}{2}\sin\left(n+\frac{1}{2}\right)x\;\cos\frac{1}{2}x}{2\pi\sin^2\frac{1}{2}x}.$$

(d) The graphs of the partial sums $s_{10}(x)$ and $s_{20}(x)$ are:



They indicate weak convergence of the Fourier series, with increasingly rapid oscillations between an envelope, namely $\left(n+\frac{1}{2}\right)/\left(2\pi\sin\frac{1}{2}x\right)$, that has ever-increasing height.

6.1.36

False. Integrating both sides of

$$\delta(x) - \frac{1}{2\pi} \sim \frac{1}{\pi}\left(\cos x + \cos 2x + \cos 3x + \cdots\right),$$

and using (3.72) to find the constant term, yields

$$\sigma(x) - \frac{x}{2\pi} \sim \frac{1}{2} + \frac{1}{\pi}\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right),$$

which agrees with the appropriate combination of (3.49) and (3.73).

6.2.4

(a) $u(x) = \frac{9}{16}x - \frac{1}{2}x^2 + \frac{3}{16}x^3 - \frac{1}{4}x^4;$

(b) $G(x;\xi) = \begin{cases} \left(1 - \frac{3}{4}\xi - \frac{1}{4}\xi^3\right)\left(x + \frac{1}{3}x^3\right), & x \le \xi, \\ \left(1 - \frac{3}{4}x - \frac{1}{4}x^3\right)\left(\xi + \frac{1}{3}\xi^3\right), & x \ge \xi. \end{cases}$

(c) $$\begin{aligned} u(x) &= \int_0^1 G(x;\xi)\,d\xi \\ &= \int_0^x \left(1 - \tfrac{3}{4}x - \tfrac{1}{4}x^3\right)\left(\xi + \tfrac{1}{3}\xi^3\right)d\xi + \int_x^1 \left(1 - \tfrac{3}{4}\xi - \tfrac{1}{4}\xi^3\right)\left(x + \tfrac{1}{3}x^3\right)d\xi \\ &= \tfrac{9}{16}x - \tfrac{1}{2}x^2 + \tfrac{3}{16}x^3 - \tfrac{1}{4}x^4. \end{aligned}$$

(d) Under an impulse force at $x = \xi$, the maximal displacement is at the forcing point, namely $g(x) = G(x,x) = x - \frac{3}{4}x^2 + \frac{1}{3}x^3 - \frac{1}{2}x^4 - \frac{1}{12}x^6$. The maximum value of $g(x^\star) = \frac{1}{3}$ occurs at the solution $x^\star = (1+\sqrt{2})^{1/3} - (1+\sqrt{2})^{-1/3} \approx .596072$ to the equation $g'(x) = 1 - \frac{3}{2}x + x^2 - 2x^3 - \frac{1}{2}x^5 = 0.$ $0 \le x \le \xi - \frac{1}{n}$

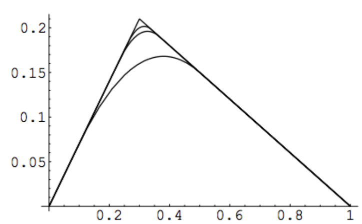
6.2.7

equation $g'(x) = 1 - \frac{3}{2}x + x^2 - 2x^3 - \frac{1}{2}x^5 = 0.$

6.2.7. (a) $$u_n(x) = \begin{cases} x(1-\xi), & 0 \le x \le \xi - \frac{1}{n}, \\ -\frac{1}{4}nx^2 + \left(\frac{1}{2}n - 1\right)x\xi - \frac{1}{4}n\xi^2 + \frac{1}{2}x + \frac{1}{2}\xi - \frac{1}{4n}, & |x - \xi| \le \frac{1}{n}, \\ \xi(1-x), & \xi + \frac{1}{n} \le x \le 1. \end{cases}$$

(b) Since $u_n(x) = G(x;\xi)$ for all $|x - \xi| \ge \frac{1}{n}$, we have $\lim_{n\to\infty} u_n(x) = G(x;\xi)$ for all $x \ne \xi$, while $\lim_{n\to\infty} u_n(\xi) = \lim_{n\to\infty}\left(\xi - \xi^2 - \frac{1}{4n}\right) = \xi - \xi^2 = G(\xi,\xi)$. (Or one can appeal to continuity to infer this.) This limit reflects the fact that the external forces converge to the delta function: $\lim_{n\to\infty} f_n(x) = \delta(x - \xi)$.

(c)



6.2.11

(a) $G(x;\xi) = \begin{cases} \dfrac{\sinh\omega x\,\cosh\omega(1-\xi)}{\omega\cosh\omega}, & x \le \xi, \\ \dfrac{\cosh\omega(1-x)\,\sinh\omega\xi}{\omega\cosh\omega}, & x \ge \xi. \end{cases}$ (b) If $x \le \frac{1}{2}$, then

$$u(x) = \int_0^x \frac{\cosh\omega(1-x)\,\sinh\omega\xi}{\omega\cosh\omega}\,d\xi + \int_x^{1/2} \frac{\sinh\omega x\,\cosh\omega(1-\xi)}{\omega\cosh\omega}\,d\xi - \int_{1/2}^1 \frac{\sinh\omega x\,\cosh\omega(1-\xi)}{\omega\cosh\omega}\,d\xi$$
$$= \frac{1}{\omega^2} - \frac{\left(e^{\omega/2} - e^{-\omega/2} + e^{-\omega}\right)e^{\omega x} + \left(e^{\omega} - e^{\omega/2} + e^{-\omega/2}\right)e^{-\omega x}}{\omega^2(e^{\omega}+e^{-\omega})},$$

while if $x \ge \frac{1}{2}$, then

$$u(x) = \int_0^{1/2} \frac{\cosh\omega(1-x)\,\sinh\omega\xi}{\omega\cosh\omega}\,d\xi - \int_{1/2}^x \frac{\cosh\omega(1-x)\,\sinh\omega\xi}{\omega\cosh\omega}\,d\xi - \int_x^1 \frac{\sinh\omega x\,\cosh\omega(1-\xi)}{\omega\cosh\omega}\,d\xi$$
$$= -\frac{1}{\omega^2} + \frac{\left(e^{-\omega/2} - e^{-\omega} + e^{-3\omega/2}\right)e^{\omega x} + \left(e^{3\omega/2} - e^{\omega} + e^{\omega/2}\right)e^{-\omega x}}{\omega^2(e^{\omega}+e^{-\omega})}.$$

6.2.12

6.2.12. Suppose $\omega > 0$. Does the Neumann boundary value problem $-u'' + \omega^2 u = f(x)$, $u'(0) = u'(1) = 0$ admit a Green's function? If not, explain why not. If so, find it, and then write down an integral formula for the solution of the boundary value problem.

Solution: Yes, it does:

$$G(x;\xi) = \begin{cases} \dfrac{\cosh\omega x\,\cosh\omega(1-\xi)}{\omega\sinh\omega}, & x \le \xi, \\ \dfrac{\cosh\omega(1-x)\,\cosh\omega\xi}{\omega\sinh\omega}, & x \ge \xi. \end{cases}$$

The solution to the boundary value problem is

$$u(x) = \int_0^1 G(x;\xi)f(\xi)\,d\xi$$
$$= \int_0^x \frac{\cosh\omega(1-x)\,\cosh\omega\xi}{\omega\sinh\omega}f(\xi)\,d\xi + \int_x^1 \frac{\cosh\omega x\,\cosh\omega(1-\xi)}{\omega\sinh\omega}f(\xi)\,d\xi.$$

2.9 HW 9

Local contents

2.9.1 Problem 1

Find the eigenvalues and the eigenfunctions for the Dirichlet and Neumann problems for the Laplacian on a rectangle $(0,a)\times(0,b)$

Solution

2.9.1.1 Dirichlet case

$$\begin{aligned}
\nabla^2 u &= -\lambda u\\
u(x,0) &= 0\\
u(x,b) &= 0\\
u\left(0,y\right) &= 0\\
u\left(a,y\right) &= 0
\end{aligned}$$

Let $u\left(x,y\right) = X(x)\,Y(x)$. Substituting this into the PDE $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = -\lambda u$ gives

$$X''Y + Y''X = -\lambda XY$$

Dividing by $XY\neq 0$ gives

$$\begin{aligned}
\frac{X''}{X}+\frac{Y''}{Y} &= -\lambda\\
\frac{X''}{X} &= -\frac{Y''}{Y}-\lambda
\end{aligned}$$

Since the LHS depends on x only and the RHS depends on y only and they are equal, they must be both constant. Say $-\mu$. The above becomes

$$\frac{X''}{X} = -\frac{Y''}{Y}-\lambda = -\mu$$

Two ODE's are therefore obtained from the above. They are

$$\begin{aligned} X'' + \mu X &= 0 \\ X(0) &= 0 \\ X(a) &= 0 \end{aligned} \tag{1}$$

And

$$\begin{aligned} \frac{Y''}{Y} + \lambda &= \mu \\ \frac{Y''}{Y} + (\lambda - \mu) &= 0 \end{aligned}$$

Let $(\lambda - \mu) = \gamma$ constant. Hence the above gives the second ODE in y as

$$\begin{aligned} Y'' + \gamma Y &= 0 \\ Y(0) &= 0 \\ Y(b) &= 0 \end{aligned} \tag{2}$$

Now the eigenvalues μ, γ and eigenfunctions for each ODE is found and from that result the eigenvalue λ is found using

$$\lambda = \gamma + \mu \tag{3}$$

Starting with ODE (1) $X'' + \mu X = 0$

Case $\mu < 0$

The solution to (1) is

$$X = A\cosh\left(\sqrt{|\mu|}x\right) + B\sinh\left(\sqrt{|\mu|}x\right)$$

At $x = 0$, the above gives $0 = A$. Hence $X = B\sinh\left(\sqrt{|\mu|}x\right)$. At $x = a$ this gives $0 = B\sinh\left(\sqrt{|\mu|}a\right)$. But $\sinh\left(\sqrt{|\mu|}a\right) = 0$ only at 0 and $\sqrt{|\mu|}a \neq 0$, therefore $B = 0$ and this leads to trivial solution. Hence $\mu < 0$ is not an eigenvalue.

Case $\mu = 0$

$$X = Ax + B$$

Hence at $x = 0$ this gives $0 = B$ and the solution becomes $X = B$. At $x = a$, $B = 0$. Hence the trivial solution. $\mu = 0$ is not an eigenvalue.

Case $\mu > 0$

Solution is

$$X = A\cos\left(\sqrt{\mu}x\right) + B\sin\left(\sqrt{\mu}x\right)$$

At $x = 0$ this gives $0 = A$ and the solution becomes $X = B\sin\left(\sqrt{\mu}x\right)$. At $x = a$

$$0 = B\sin\left(\sqrt{\mu}a\right)$$

For non-trivial solution we want $\sin\left(\sqrt{\mu}a\right) = 0$ or $\sqrt{\mu}a = k\pi$ where $k = 1, 2, 3, \cdots$, therefore

$$\mu_k = \left(\frac{k\pi}{a}\right)^2 \qquad k = 1, 2, 3, \cdots \tag{4}$$

The corresponding Eigenfunctions are

$$X_k(x) = \sin\left(\frac{k\pi}{a}x\right) \qquad k = 1, 2, 3, \cdots \tag{5}$$

Solving ODE (2) $Y'' + \gamma Y = 0$

The same steps are repeated as above. The only difference is that now we obtain eigenvalues

$$\gamma_m = \left(\frac{m\pi}{b}\right)^2 \qquad m = 1, 2, 3, \cdots \tag{6}$$

And the corresponding eigenfunctions

$$Y_m\left(y\right) = \sin\left(\frac{m\pi}{b}y\right) \qquad m = 1, 2, 3, \cdots \tag{7}$$

From (4,6) we see that the eigenvalues for $\nabla^2 u = -\lambda u$ are, using (3)

$$\begin{aligned}\lambda_{k,m} &= \mu_k + \gamma_m \\ &= \left(\frac{k\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \qquad k = 1, 2, 3, \cdots, m = 1, 2, 3, \cdots\end{aligned}$$

And the eigenfunctions are from (5,7) are

$$\Phi_{k,m}\left(x, y\right) = \sin\left(\frac{k\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right) \qquad k = 1, 2, 3, \cdots, m = 1, 2, 3, \cdots$$

2.9.1.2 Neumann case

$$\begin{aligned}\nabla^2 u &= -\lambda u \\ \frac{\partial}{\partial y}u(x, 0) &= 0 \\ \frac{\partial}{\partial y}u(x, b) &= 0 \\ \frac{\partial}{\partial x}u\left(0, y\right) &= 0 \\ \frac{\partial}{\partial x}u\left(a, y\right) &= 0\end{aligned}$$

Let $u\left(x, y\right) = X(x)\,Y(x)$. Substituting this into the PDE $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\lambda u$ gives

$$X''Y + Y''X = -\lambda XY$$

Dividing by $XY \neq 0$ gives

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$
$$\frac{X''}{X} = -\frac{Y''}{Y} - \lambda$$

Since the LHS depends on x only and the RHS depends on y only and they are equal, they must be both constant. Say $-\mu$. The above becomes

$$\frac{X''}{X} = -\frac{Y''}{Y} - \lambda = -\mu$$

Two ODE's are therefore obtained from the above. They are

$$\begin{aligned} X'' + \mu X &= 0 \qquad (1) \\ X'(0) &= 0 \\ X'(a) &= 0 \end{aligned}$$

And

$$\frac{Y''}{Y} + \lambda = \mu$$
$$\frac{Y''}{Y} + (\lambda - \mu) = 0$$

Let $(\lambda - \mu) = \gamma$ constant. Hence the above gives the second ODE in y as

$$\begin{aligned} Y'' + \gamma Y &= 0 \qquad (2) \\ Y'(0) &= 0 \\ Y'(b) &= 0 \end{aligned}$$

Now we find the eigenvalues μ, γ and eigenfunctions for each ODE and from this result find

$$\lambda = \gamma + \mu \qquad (3)$$

Starting with ODE (1) $X'' + \mu X = 0$

Case $\mu < 0$

The solution to (1) is

$$X(x) = A\cosh\left(\sqrt{|\mu|}x\right) + B\sinh\left(\sqrt{|\mu|}x\right)$$
$$X'(x) = A\sqrt{|\mu|}\sinh\left(\sqrt{|\mu|}x\right) + B\sqrt{|\mu|}\cosh\left(\sqrt{|\mu|}x\right)$$

At $x = 0$, the above gives $0 = B$. Hence $X(x) = A\cosh\left(\sqrt{|\mu|}x\right)$ and $X'(x) = A\sqrt{|\mu|}\sinh\left(\sqrt{|\mu|}x\right)$ At $x = a$ this gives $0 = A\sqrt{|\mu|}\sinh\left(\sqrt{|\mu|}a\right)$. But $\sinh\left(\sqrt{|\mu|}a\right) = 0$ only at 0 and $\sqrt{|\mu|}a \neq 0$, therefore $A = 0$ and this leads to trivial solution. Hence $\mu < 0$ is not an eigenvalue.

Case $\mu = 0$

$$
\begin{aligned}
X &= Ax + B \\
X' &= A
\end{aligned}
$$

At $x = 0$ this gives $0 = A$ and the solution becomes $X = B$, therefore $X' = 0$. At $x = a$, $0 = 0$. Hence any constant B will work. Let this constant be C_0. Therefore $\mu = 0$ is an eigenvalue with corresponding eigenfunction $X_0(x) = C_0$, a constant.

Case $\mu > 0$

Solution is

$$
\begin{aligned}
X(x) &= A\cos\left(\sqrt{\mu}x\right) + B\sin\left(\sqrt{\mu}x\right) \\
X'(x) &= -A\sqrt{\mu}\sin\left(\sqrt{\mu}x\right) + B\sqrt{\mu}\cos\left(\sqrt{\mu}x\right)
\end{aligned}
$$

At $x = 0$ this gives $0 = B$ and the solution becomes $X(x) = A\cos\left(\sqrt{\mu}x\right)$. Hence $X'(x) = -A\sqrt{\mu}\sin\left(\sqrt{\mu}x\right)$. At $x = a$ this gives

$$
0 = -A\sqrt{\mu}\sin\left(\sqrt{\mu}a\right)
$$

For non-trivial solution we want $\sin\left(\sqrt{\mu}a\right) = 0$ or $\sqrt{\mu}a = k\pi$ where $k = 1, 2, 3, \cdots$, therefore

$$
\mu_k = \left(\frac{k\pi}{a}\right)^2 \qquad k = 1, 2, 3, \cdots \tag{4}
$$

The corresponding Eigenfunctions are

$$
X_k(x) = \cos\left(\frac{k\pi}{a}x\right) \qquad k = 1, 2, 3, \cdots \tag{5}
$$

Solving ODE (2) $Y'' + \gamma Y = 0$

The same steps are repeated as above. The only difference is that now we obtain eigenvalues $\gamma = 0$ also and corresponding eigenfunction constant, say D_0 and also obtain

$$
\gamma_m = \left(\frac{m\pi}{b}\right)^2 \qquad m = 1, 2, 3, \cdots \tag{6}
$$

and corresponding eigenfunctions

$$
Y_m(y) = \cos\left(\frac{m\pi}{b}y\right) \qquad m = 1, 2, 3, \cdots \tag{7}
$$

From (4,6) we see that the eigenvalues for $\nabla^2 u = -\lambda u$ are

$$
\begin{aligned}
\lambda_{k,m} &= \begin{cases} 0 & k = 0, m = 0 \\ \mu_k + \gamma_m & k = 1, 2, 3, \cdots, m = 1, 2, 3, \cdots \end{cases} \\
&= \begin{cases} 0 & k = 0, m = 0 \\ \left(\frac{k\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 & k = 1, 2, 3, \cdots, m = 1, 2, 3, \cdots \end{cases}
\end{aligned}
$$

And the eigenfunctions are from (5,7) are

$$\Phi_n\left(x,y\right)=\begin{cases}1 & k=0,m=0\\ \cos\left(\frac{k\pi}{a}x\right)\cos\left(\frac{m\pi}{b}y\right) & k=1,2,3,\cdots,m=1,2,3,\cdots\end{cases}$$

Where in the above the constant eigenfunction that corresponds to the zero eigenvalue is taken as 1.

2.9.2 Problem 2

Prove that the wave equation $u_{tt}\left(x,t\right)=c^2\nabla^2u$, $t>0$, $x\in\Omega\in\Re^d$ with the Dirichlet boundary conditions $u\left(x,t\right)=0$ for $x\in\partial\Omega,t>0$ has solution

$$u\left(x,t\right)=\sum_{n=1}^{\infty}\left(A_n\cos\left(\sqrt{\lambda_n}ct\right)+B_n\sin\left(\sqrt{\lambda_n}ct\right)\right)v_n\left(x\right) \tag{1}$$

Where λ_n,v_n are respectively, eigenvalues and eigenfunctions of the Dirichlet problem for the Laplacian in Ω. Write in an analogous form the solution to the heat equation $u_t\left(x,t\right)=c\nabla^2u$, $t>0$, $x\in\Omega\in\Re^d$ with Dirichlet boundary conditions $u\left(x,t\right)=0$ for $x\in\partial\Omega,t>0$.

For the wave PDE

We will show the solution given solves the PDE by substituting it into the PDE and see if it gives an identity.

$$u_t\left(x,t\right)=\frac{\partial}{\partial t}\sum_{n=1}^{\infty}\left(A_n\cos\left(\sqrt{\lambda_n}ct\right)+B_n\sin\left(\sqrt{\lambda_n}ct\right)\right)v_n\left(x\right)$$

Assuming continuouseigenfunctions, term by term differential is allowed, and the above becomes

$$\begin{aligned}u_t\left(x,t\right)&=\sum_{n=1}^{\infty}\frac{\partial}{\partial t}\left(A_n\cos\left(\sqrt{\lambda_n}ct\right)+B_n\sin\left(\sqrt{\lambda_n}ct\right)\right)v_n\left(x\right)\\&=\sum_{n=1}^{\infty}\left(-A_n\sqrt{\lambda_n}c\sin\left(\sqrt{\lambda_n}ct\right)+B_n\sqrt{\lambda_n}c\cos\left(\sqrt{\lambda_n}ct\right)\right)v_n\left(x\right)\end{aligned}$$

Taking one more time derivatives gives

$$u_{tt}\left(x,t\right)=\sum_{n=1}^{\infty}\left(-A_n\lambda_nc^2\cos\left(\sqrt{\lambda_n}ct\right)-B_n\lambda_nc^2\sin\left(\sqrt{\lambda_n}ct\right)\right)v_n\left(x\right) \tag{2}$$

Similarly for the spatial coordinate

$$\begin{aligned}u_x\left(x,t\right)&=\frac{\partial}{\partial x}\sum_{n=1}^{\infty}\left(A_n\cos\left(\sqrt{\lambda_n}ct\right)+B_n\sin\left(\sqrt{\lambda_n}ct\right)\right)v_n\left(x\right)\\&=\sum_{n=1}^{\infty}\left(A_n\cos\left(\sqrt{\lambda_n}ct\right)+B_n\sin\left(\sqrt{\lambda_n}ct\right)\right)v_n'\left(x\right)\end{aligned}$$

Taking one more space derivatives gives

$$\nabla^2u=\sum_{n=1}^{\infty}\left(A_n\cos\left(\sqrt{\lambda_n}ct\right)+B_n\sin\left(\sqrt{\lambda_n}ct\right)\right)v_n''\left(x\right)$$

But since $v_n(x)$ is an eigenfunction, then $-v_n''(x) = \lambda_n v_n$ and the above simplifies to

$$\nabla^2 u = -\sum_{n=1}^{\infty}\left(A_n \cos\left(\sqrt{\lambda_n}ct\right) + B_n \sin\left(\sqrt{\lambda_n}ct\right)\right)\lambda_n v_n(x) \tag{3}$$

Substituting (2,3) into $u_{tt}(x,t) = c^2\nabla^2 u$ gives

$$\sum_{n=1}^{\infty}\left(-A_n\lambda_n c^2 \cos\left(\sqrt{\lambda_n}ct\right) - B_n\lambda_n c^2 \sin\left(\sqrt{\lambda_n}ct\right)\right)v_n(x) = c^2\left(-\sum_{n=1}^{\infty}\left(A_n \cos\left(\sqrt{\lambda_n}ct\right) + B_n \sin\left(\sqrt{\lambda_n}ct\right)\right)\lambda_n v_n(x)\right)$$

$$c^2\sum_{n=1}^{\infty}\left(-A_n \cos\left(\sqrt{\lambda_n}ct\right) - B_n \sin\left(\sqrt{\lambda_n}ct\right)\right)\lambda_n v_n(x) = -c^2\sum_{n=1}^{\infty}\left(A_n \cos\left(\sqrt{\lambda_n}ct\right) + B_n \sin\left(\sqrt{\lambda_n}ct\right)\right)\lambda_n v_n(x)$$

$$-c^2\sum_{n=1}^{\infty}\left(A_n \cos\left(\sqrt{\lambda_n}ct\right) + B_n \sin\left(\sqrt{\lambda_n}ct\right)\right)\lambda_n v_n(x) = -c^2\sum_{n=1}^{\infty}\left(A_n \cos\left(\sqrt{\lambda_n}ct\right) + B_n \sin\left(\sqrt{\lambda_n}ct\right)\right)\lambda_n v_n(x)$$

The LHS is the same as the RHS. Hence the solution given satisfies the wave PDE.

<u>For the heat PDE</u>

For the heat PDE, we want to show that the following solution

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n ct} v_n(x) \tag{4}$$

Satisfies $u_t(x,t) = c\nabla^2 u$.

$$u_t(x,t) = \frac{\partial}{\partial t}\sum_{n=1}^{\infty} A_n e^{-\lambda_n ct} v_n(x)$$

Assuming term by term differential is allowed the above becomes

$$\begin{aligned} u_t(x,t) &= \sum_{n=1}^{\infty}\frac{\partial}{\partial t} A_n e^{-\lambda_n ct} v_n(x) \\ &= \sum_{n=1}^{\infty} -A_n\lambda_n c e^{-\lambda_n ct} v_n(x) \end{aligned} \tag{5}$$

Similarly for the spatial coordinate

$$\begin{aligned} u_x(x,t) &= \frac{\partial}{\partial x}\sum_{n=1}^{\infty} A_n e^{-\lambda_n ct} v_n(x) \\ &= \sum_{n=1}^{\infty} A_n e^{-\lambda_n ct} v_n'(x) \end{aligned}$$

Taking one more space derivatives gives

$$\nabla^2 u = \sum_{n=1}^{\infty} A_n e^{-\lambda_n ct} v_n''(x)$$

But since $v_n(x)$ is an eigenfunction, then $-v_n''(x) = \lambda_n v_n$. The above becomes

$$\nabla^2 u = -\sum_{n=1}^{\infty} A_n e^{-\lambda_n ct}\lambda_n v_n(x) \tag{6}$$

Substituting (5,6) into $u_t(x,t) = c\nabla^2 u$ gives

$$\sum_{n=1}^{\infty} -A_n\lambda_n c e^{-\lambda_n ct} v_n(x) = c\left(-\sum_{n=1}^{\infty} A_n e^{-\lambda_n ct}\lambda_n v_n(x)\right)$$

$$-c\sum_{n=1}^{\infty} A_n\lambda_n e^{-\lambda_n^2 ct} v_n(x) = -c\sum_{n=1}^{\infty} A_n e^{-\lambda_n ct}\lambda_n v_n(x)$$

The LHS is the same as the RHS. Hence the solution (4) satisfies the heat PDE.

2.9.3 Problem 6.3.9

Suppose $f(x,y) = \begin{cases} 1 & 3x-2y>1 \\ 0 & 3x-2y<1 \end{cases}$ Compute its partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ the sense of generalized functions.

Solution

The following is a plot of the above function in 3D

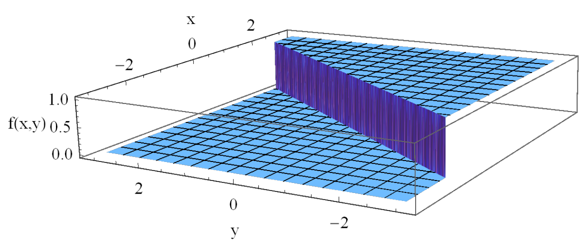


Figure 2.66: Plot of $f(x,y)$

```
f[x_, y_] := Piecewise[{{1, 3 x - 2 y > 1}, {0, 3 x - 2 y > 1}}]
p = ParametricPlot3D[{x, y, f[x, y]}, {x, -3, 3}, {y, -3, 3},
        AxesLabel → {"x", "y", "f(x,y)"}, ImageSize → 400,
        BaseStyle → 12, Exclusions → True,
        ExclusionsStyle → LightGray, PlotTheme -> "Classic", PlotPoints → 50];
```

Figure 2.67: Code used for the above plot

Similar to what we did in 1D, when taking a derivative and there is a jump discontinuity, an impulse $\delta(x)$ is generated at the location where the jump discontinuity is located. The location of the jump here is on the line $3x-2y-1=0$. This is a step function but in 3D. Hence by chain rule

$$\begin{aligned}
\frac{\partial f\left(x,y\right)}{\partial x} &= \frac{\partial}{\partial x}\left(3x-2y-1\right)\delta\left(3x-2y-1\right)\\
&= 3\delta\left(3x-2y-1\right)\\
&= \delta\left(x-\frac{2}{3}y-\frac{1}{3}\right)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial f\left(x,y\right)}{\partial y} &= \frac{\partial}{\partial y}\left(3x-2y-1\right)\delta\left(3x-2y-1\right)\\
&= -2\delta\left(3x-2y-1\right)\\
&= \delta\left(-\frac{3}{2}x+y+\frac{1}{2}\right)
\end{aligned}$$

2.9.4 Problem 6.3.10

Find a series solution to the rectangular boundary value problem 4.91-92 which is

$$\begin{aligned}
\nabla^2 u &= 0 \qquad \text{on a rectangle} \qquad R=\left\{0<x<a,0<y<b\right\}\\
u\left(x,0\right) &= f\left(x\right)\\
u\left(x,b\right) &= 0\\
u\left(0,y\right) &= 0\\
u\left(a,y\right) &= 0
\end{aligned}$$

when the boundary data $f(x) = \delta(x-\xi)$ is a delta function at a point $0 < \xi < a$. Is your solution infinitely differentiable inside the rectangle?

Solution

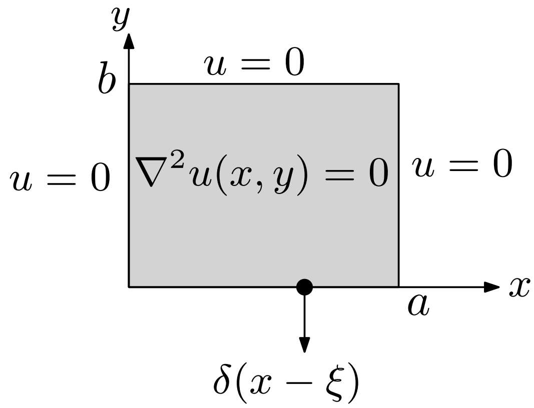


Figure 2.68: The problem to solve. Laplace PDE in rectangle

Let $u\left(x,y\right) = X(x)\,Y(x)$. Substituting this into the PDE $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and simplifying gives

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

Each side depends on different independent variable and they are equal, therefore they must be equal to same constant.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \pm\lambda$$

Since the boundary conditions along the x direction are the homogeneous ones, $-\lambda$ is selected in the above.

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

Two ODE's are obtained

$$X'' + \lambda X = 0 \tag{1}$$

With the boundary conditions

$$X(0) = 0$$
$$X(a) = 0$$

And

$$Y'' - \lambda Y = 0 \tag{2}$$

With the boundary conditions

$$Y(0) = f(x)$$
$$Y(b) = 0$$

In all these cases λ will turn out to be positive. This is shown below.

<u>Case $\lambda < 0$</u>

The solution to (1) is

$$X = A\cosh\left(\sqrt{|\lambda|}x\right) + B\sinh\left(\sqrt{|\lambda|}x\right)$$

At $x = 0$, the above gives $0 = A$. Hence $X = B\sinh\left(\sqrt{|\lambda|}x\right)$. At $x = a$ this gives $X = B\sinh\left(\sqrt{|\lambda|}a\right)$. But $\sinh\left(\sqrt{|\lambda|}a\right) = 0$ only at 0 and $\sqrt{|\lambda|}a \neq 0$, therefore $B = 0$ and this leads to trivial solution. Hence $\lambda < 0$ is not an eigenvalue.

<u>Case $\lambda = 0$</u>

$$X = Ax + B$$

Hence at $x = 0$ this gives $0 = B$ and the solution becomes $X = B$. At $x = a$, $B = 0$. Hence the trivial solution. $\lambda = 0$ is not an eigenvalue.

<u>Case $\lambda > 0$</u>

Solution is

$$X = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

At $x = 0$ this gives $0 = A$ and the solution becomes $X = B\sin\left(\sqrt{\lambda}x\right)$. At $x = a$

$$0 = B\sin\left(\sqrt{\lambda}a\right)$$

For non-trivial solution $\sin\left(\sqrt{\lambda}a\right) = 0$ or $\sqrt{\lambda}a = n\pi$ where $n = 1, 2, 3, \cdots$, therefore

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2 \qquad n = 1, 2, 3, \cdots$$

Eigenfunctions are

$$X_n(x) = B_n \sin\left(\frac{n\pi}{a}x\right) \qquad n = 1, 2, 3, \cdots \tag{3}$$

For the Y ODE, the solution is

$$Y_n = C_n \cosh\left(\frac{n\pi}{a}y\right) + D_n \sinh\left(\frac{n\pi}{a}y\right) \tag{4}$$

Applying B.C. at $y = b$ gives

$$\begin{aligned}
0 &= C_n \cosh\left(\frac{n\pi}{a}b\right) + D_n \sinh\left(\frac{n\pi}{a}b\right)\\
C_n &= -D_n \frac{\sinh\left(\frac{n\pi}{a}b\right)}{\cosh\left(\frac{n\pi}{a}b\right)}\\
&= -D_n \tanh\left(\frac{n\pi}{a}b\right)
\end{aligned}$$

Hence (4) becomes

$$\begin{aligned}Y_n &= -D_n \tanh\left(\frac{n\pi}{a}b\right)\cosh\left(\frac{n\pi}{a}y\right) + D_n \sinh\left(\frac{n\pi}{a}y\right)\\ &= D_n\left(\sinh\left(\frac{n\pi}{a}y\right) - \tanh\left(\frac{n\pi}{a}b\right)\cosh\left(\frac{n\pi}{a}y\right)\right)\end{aligned}$$

Now the complete solution is produced

$$\begin{aligned}u_n\left(x,y\right) &= Y_n X_n\\ &= D_n\left(\sinh\left(\frac{n\pi}{a}y\right) - \tanh\left(\frac{n\pi}{a}b\right)\cosh\left(\frac{n\pi}{a}y\right)\right) B_n \sin\left(\frac{n\pi}{a}x\right)\end{aligned}$$

Let $D_n B_n = B_n$ since a constant. (no need to make up a new symbol).

$$u_n\left(x,y\right) = B_n\left(\sinh\left(\frac{n\pi}{a}y\right) - \tanh\left(\frac{n\pi}{a}b\right)\cosh\left(\frac{n\pi}{a}y\right)\right)\sin\left(\frac{n\pi}{a}x\right)$$

Sum of eigenfunctions is the solution, hence

$$u\left(x,y\right) = \sum_{n=1}^{\infty} B_n\left(\sinh\left(\frac{n\pi}{a}y\right) - \tanh\left(\frac{n\pi}{a}b\right)\cosh\left(\frac{n\pi}{a}y\right)\right)\sin\left(\frac{n\pi}{a}x\right) \tag{5}$$

The nonhomogeneous boundary condition is now resolved. At $y = 0$

$$u\left(x,0\right) = f\left(x\right) = \delta\left(x-\xi\right)$$

Therefore (5) becomes

$$\delta\left(x-\xi\right) = \sum_{n=1}^{\infty} -B_n \tanh\left(\frac{n\pi}{a}b\right)\sin\left(\frac{n\pi}{a}x\right)$$

Multiplying both sides by $\sin\left(\frac{m\pi}{a}x\right)$ and integrating gives

$$\begin{aligned}\int_0^a \delta\left(x-\xi\right)\sin\left(\frac{m\pi}{a}x\right)dx &= -\int_0^a \sin\left(\frac{m\pi}{a}x\right)\sum_{n=1}^{\infty} B_n \tanh\left(\frac{n\pi}{a}b\right)\sin\left(\frac{n\pi}{a}x\right)dx\\ &= -\sum_{n=1}^{\infty} B_n \tanh\left(\frac{n\pi}{a}b\right)\int_0^a \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)dx\\ &= -B_n \tanh\left(\frac{m\pi}{a}b\right)\left(\frac{a}{2}\right)\end{aligned}$$

Hence

$$B_n = -\frac{2}{a}\frac{\int_0^a \delta\left(x-\xi\right)\sin\left(\frac{n\pi}{a}x\right)dx}{\tanh\left(\frac{n\pi}{a}b\right)}$$

But $\int_0^a \delta\left(x-\xi\right)\sin\left(\frac{m\pi}{L}x\right)dx = \sin\left(\frac{m\pi}{L}\xi\right)$ by the property delta function. Therefore

$$B_n = -\frac{2}{a}\frac{\sin\left(\frac{n\pi}{a}\xi\right)}{\tanh\left(\frac{n\pi}{a}b\right)}$$

This completes the solution. (4) becomes

$$
\begin{aligned}
u\left(x,y\right) &= -\frac{2}{a}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{a}\xi\right)}{\tanh\left(\frac{n\pi}{a}b\right)}\left(\sinh\left(\frac{n\pi}{a}y\right)-\tanh\left(\frac{n\pi}{a}b\right)\cosh\left(\frac{n\pi}{a}y\right)\right)\sin\left(\frac{n\pi}{a}x\right)\\
&= -\frac{2}{a}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{a}\xi\right)\sin\left(\frac{n\pi}{a}x\right)\left(\frac{\sinh\left(\frac{n\pi}{a}y\right)}{\tanh\left(\frac{n\pi}{a}b\right)}-\cosh\left(\frac{n\pi}{a}y\right)\right)
\end{aligned}
$$

Looking at the solution above, it is composed of functions that are all differentiable. Hence the solution is infinitely differentiable inside the rectangle.

Here is a plot of the above solution using $a=\pi, b=\frac{1}{2}, \xi=1$.

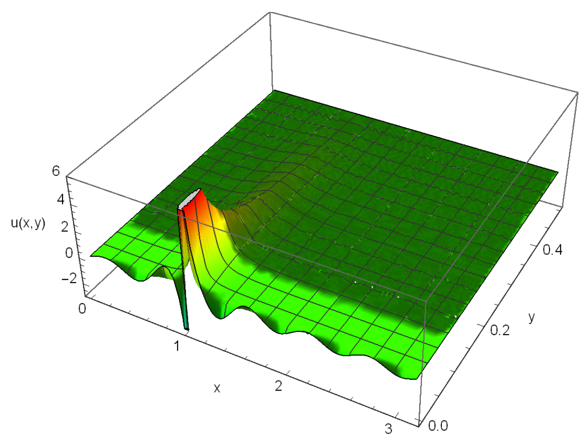


Figure 2.69: Plot of $u(x,y)$

```
u[x_, y_, ξ_] := -2/a Sum[Sin[n π/a ξ] Sin[n π/a x] (Sinh[n π/a y]/Tanh[n π/a b] - Cosh[n π/a y]), {n, 1, 300}];
a = Pi; b = 1/2; ξ = 1;
p = Plot3D[u[x, y, ξ], {x, 0, a}, {y, 0, b}, PlotRange → {Automatic, Automatic, {-3, 7}},
        PlotPoints → 40, AxesLabel → {"x", "y", "u(x,y)"},
        ColorFunction → Function[{x, y, z}, Hue[.45 (1 - z)]]];
```

Figure 2.70: Code used for the above plot

2.9.5 Problem 6.3.18

(a) Use the Method of Images to construct the Green's function for a half-plane $\{y>0\}$ that is subject to homogeneous Dirichlet boundary conditions. Hint : The image point is obtained by reflection. (b) Use your Green's function to solve the boundary value problem

With $y > 0, u(x,0) = 0$

$$-\Delta u = \frac{1}{1+y}$$

Solution

2.9.5.1 Part (a)

The first step is to find Green function in the half-plane $G\left(x,y;x_0,y_0\right)$. To do this we will use Green function in the whole plane, called $\Gamma\left(x,y;x_0,y_0\right)$. There $\left(x,y\right)$ is an arbitrary point in upper half plane and $\left(x_0,y_0\right)$ is fixed point where the impulse is located. We set an impulse at the point $\left(x_0,y_0\right)$ and a negative impulse at $\left(x_0,-y_0\right)$. This way the end effect is that at the boundary which is $x=0$ the half plane Green function is zero which satisfies the boundary conditions of the given PDE. The following diagram helps illustrate this setup

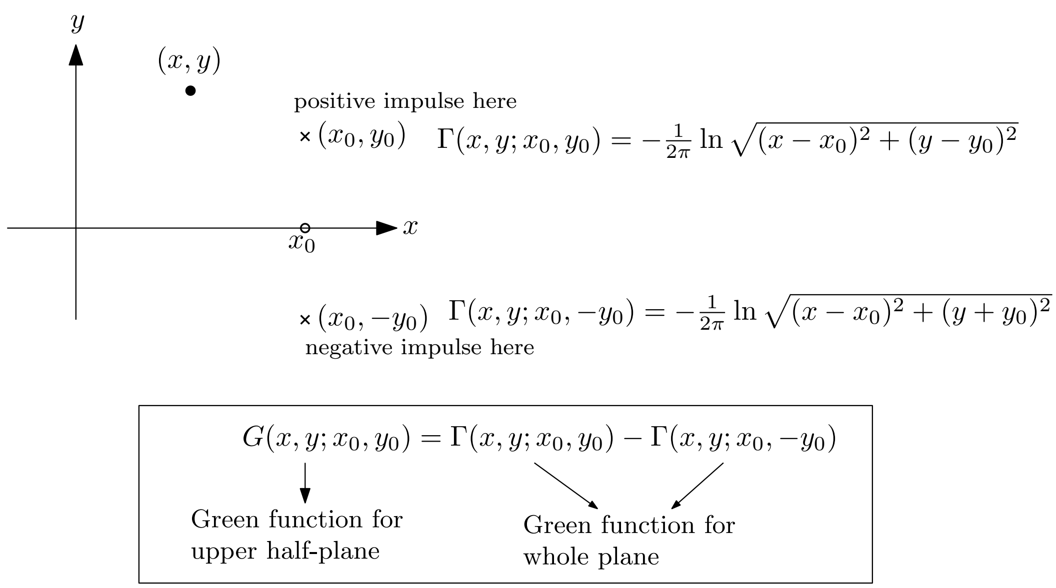


Figure 2.71: Using method of images

Hence

$$\begin{aligned}
G\left(x,y;x_0,y_0\right) &= -\frac{1}{2\pi}\ln\left(\sqrt{(x-x_0)^2+\left(y-y_0\right)^2}\right)+\frac{1}{2\pi}\ln\left(\sqrt{(x-x_0)^2+\left(y+y_0\right)^2}\right)\\
&= -\frac{1}{4\pi}\ln\left((x-x_0)^2+\left(y-y_0\right)^2\right)+\frac{1}{4\pi}\ln\left((x-x_0)^2+\left(y+y_0\right)^2\right)\\
&= \frac{1}{4\pi}\ln\frac{(x-x_0)^2+\left(y+y_0\right)^2}{(x-x_0)^2+\left(y-y_0\right)^2}
\end{aligned}$$

2.9.5.2 Part b

Now that the Green function is known, the solution is

$$
\begin{aligned}
u\left(x,y\right) &= \int_{-\infty}^{x}\int_{0}^{y} G\left(x,y;x_0,y_0\right) f\left(x_0,y_0\right) dx_0 dy_0 \\
&= \int_{-\infty}^{x}\int_{0}^{y} \frac{1}{4\pi}\ln\left(\frac{(x-x_0)^2+\left(y+y_0\right)^2}{(x-x_0)^2+\left(y-y_0\right)^2}\right)\left(\frac{1}{1+y_0}\right) dx_0 dy_0 \\
&= \frac{1}{4\pi}\int_{0}^{y}\left(\frac{1}{1+y_0}\right)\left(\int_{-\infty}^{x}\ln\left(\frac{(x-x_0)^2+\left(y+y_0\right)^2}{(x-x_0)^2+\left(y-y_0\right)^2}\right)dx_0\right)dy_0 \qquad (1)
\end{aligned}
$$

But

$$
\begin{aligned}
\int_{-\infty}^{x}\ln\left(\frac{(x-x_0)^2+\left(y+y_0\right)^2}{(x-x_0)^2+\left(y-y_0\right)^2}\right)dx_0 &= 2y_0\pi - x\ln\left(\left(y+y_0\right)^2\right) + x\ln\left(\frac{\left(y+y_0\right)^2}{\left(y-y_0\right)^2}\right) + x\ln\left(\left(y-y_0\right)^2\right) \\
&= 2y_0\pi - 2x\ln\left(y+y_0\right) + 2x\ln\left(\frac{y+y_0}{y-y_0}\right) + 2x\ln\left(y-y_0\right) \\
&= 2y_0\pi + 2x\ln\left(\frac{y-y_0}{y+y_0}\right) + 2x\ln\frac{y+y_0}{y-y_0} \\
&= 2y_0\pi + 2x\left(\ln\frac{y-y_0}{y+y_0} + \ln\frac{y+y_0}{y-y_0}\right) \\
&= 2y_0\pi + 2x\ln\left(\frac{y-y_0}{y+y_0}\frac{y+y_0}{y-y_0}\right) \\
&= 2y_0\pi
\end{aligned}
$$

Hence (1) becomes

$$
\begin{aligned}
u\left(x,y\right) &= \frac{1}{2}\int_{0}^{y}\frac{y_0}{1+y_0}dy_0 \\
&= \frac{1}{2}\left(y_0 - \ln\left(y_0+1\right)\right)_0^y \\
&= \frac{1}{2}\left(y - \ln\left(y+1\right)\right)
\end{aligned}
$$

Checking: When $y=0$ then $u\left(x,y\right) = -\frac{1}{2}\ln(1) = 0$. Ok. Solution does not depend on x but only on y.

2.9.6 Problem 6.3.21

Provide the details for the following alternative method for solving the homogeneous Dirichlet boundary value problem for the Poisson equation on the unit square:

$$
\begin{aligned}
u_{xx} + u_{yy} &= -f\left(x,y\right) \qquad 0 < x,y < 1 \\
u\left(x,0\right) &= 0 \\
u\left(x,1\right) &= 0 \\
u\left(0,y\right) &= 0 \\
u\left(1,y\right) &= 0
\end{aligned}
$$

(a) Write both $u(x,y)$ and $f(x,y)$ as Fourier sine series in y whose coefficients depend on x. (b) Substitute these series into the differential equation, and equate Fourier coefficients to obtain an infinite system of ordinary boundary value problems for the x-dependent Fourier coefficients of u. (c) Use the Green's functions for each boundary value problem to write out the solution and hence a series for the solution to the original boundary value problem. (d) Implement this method for the following forcing functions (i) $f\left(x,y\right) = \sin\left(\pi y\right)$, (ii) $f\left(x,y\right) = \sin\left(\pi x\right)\sin\left(2\pi y\right)$, (iii) $f\left(x,y\right) = 1$.

Solution

2.9.6.1 Part a

Let

$$
\begin{aligned}
u\left(x,y\right) &= \sum_{n=1}^{\infty} A_n\left(x\right)\sin\left(\sqrt{\lambda_n}y\right) \\
f\left(x,y\right) &= \sum_{n=1}^{\infty} B_n\left(x\right)\sin\left(\sqrt{\lambda_n}y\right)
\end{aligned}
$$

The eigenvalues are known to be $\lambda_n = n^2\pi^2$ for $n = 1,2,\cdots$ for these boundary conditions on $x = 0$ to $x = 1$. Hence the above becomes

$$
u\left(x,y\right) = \sum_{n=1}^{\infty} A_n\left(x\right)\sin\left(n\pi y\right) \tag{1}
$$

$$
f\left(x,y\right) = \sum_{n=1}^{\infty} B_n\left(x\right)\sin\left(n\pi y\right) \tag{2}
$$

2.9.6.2 Part b

From (1)

$$
\begin{aligned}
u_x &= \sum_{n=1}^{\infty} A_n'(x)\sin\left(n\pi y\right)\\
u_{xx} &= \sum_{n=1}^{\infty} A_n''(x)\sin\left(n\pi y\right)\\
u_y &= \sum_{n=1}^{\infty} n\pi A_n(x)\cos\left(n\pi y\right)\\
u_{yy} &= -\sum_{n=1}^{\infty} n^2\pi^2 A_n(x)\sin\left(n\pi y\right)
\end{aligned}
$$

Substituting the above back into the original $u_{xx}+u_{yy}=-f\left(x,y\right)$ gives

$$
\begin{aligned}
\sum_{n=1}^{\infty} A_n''(x)\sin\left(n\pi y\right)-\sum_{n=1}^{\infty} n^2\pi^2 A_n(x)\sin\left(n\pi y\right) &= -\sum_{n=1}^{\infty} B_n(x)\sin\left(n\pi y\right)\\
\sum_{n=1}^{\infty}\left(A_n''(x)-n^2\pi^2 A_n(x)\right)\sin\left(n\pi y\right) &= -\sum_{n=1}^{\infty} B_n(x)\sin\left(n\pi y\right)
\end{aligned}
$$

Equating coefficients in the above gives

$$
A_n''(x)-n^2\pi^2 A_n(x) = -B_n(x)
$$

For all $n=1,2,\cdots$. This is an infinite system of ordinary boundary value problems in $A(x)$ where $B_n(x)$ acts as the external input.

2.9.6.3 Part c

We now want to find Green function for $A_n''(x)-n^2\pi^2A_n(x)=0$ with $A_n(0)=0, A_n(1)=0$. The solution is

$$
A_n(x) = A\cosh(n\pi x)+B\sinh(n\pi x)
$$

Hence the Green function is

$$
G(x;x_0)=\begin{cases} A_1\cosh(n\pi x)+B_1\sinh(n\pi x) & x<x_0\\ A_2\cosh(n\pi x)+B_2\sinh(n\pi x) & x>x_0\end{cases}
$$

At $x=0$, the top branch gives $0=A_1$ and at $x=1$ the lower branch gives $A_2\cosh(n\pi)+B_2\sinh(n\pi)=0$ or $A_2=-B_2\tanh(n\pi)$. Using these in the above gives

$$
\begin{aligned}
G(x;x_0) &= \begin{cases} B_1\sinh(n\pi x) & x<x_0\\ -B_2\tanh(n\pi)\cosh(n\pi x)+B_2\sinh(n\pi x) & x>x_0\end{cases}\\
&= \begin{cases} B_1\sinh(n\pi x) & x<x_0\\ B_2\left(\sinh(n\pi x)-\tanh(n\pi)\cosh(n\pi x)\right) & x>x_0\end{cases} \qquad \text{(1A)}
\end{aligned}
$$

There are two unknowns B_1,B_2 to solve for. Hence we need two equations. The first equation is found by equating the above Green function at $x=x_0$. This gives

$$
B_1\sinh(n\pi x_0)=B_2\left(\sinh(n\pi x_0)-\tanh(n\pi)\cosh(n\pi x_0)\right) \qquad (1)
$$

Taking derivatives of $G(x;x_0)$ gives

$$\frac{d}{dx}G(x;x_0)=\begin{cases} n\pi B_1\cosh(n\pi x) & x<x_0\\ B_2\left(n\pi\cosh(n\pi x)-n\pi\tanh(n\pi)\sinh(n\pi x)\right) & x>x_0\end{cases}$$

The second equation is found by the condition of the jump discontinutiy on the above derivative at $x=x_0$. Hence

$$n\pi B_1\cosh(n\pi x_0)-B_2\left(n\pi\cosh(n\pi x_0)-n\pi\tanh(n\pi)\sinh(n\pi x_0)\right)=1 \tag{2}$$

Solving (1,2) for B_1, B_2 gives

$$\begin{aligned}B_1&=\frac{\cosh(n\pi x_0)-\coth(n\pi)\sinh(n\pi x_0)}{n\pi}=\frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x_0-1))\\ B_2&=\frac{\coth(n\pi)\sinh(n\pi x_0)}{n\pi}=\frac{\sinh(n\pi x_0)}{n\pi\tanh(n\pi)}\end{aligned}$$

Substituting these back in (1A) gives the final Green function

$$\begin{aligned}G(x;x_0)&=\begin{cases}\begin{cases}\frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x_0-1))\sinh(n\pi x) & x<x_0\\ \frac{\sinh(n\pi x_0)}{n\pi\tanh(n\pi)}\left(\sinh(n\pi x)-\tanh(n\pi)\cosh(n\pi x)\right) & x>x_0\end{cases} & x<x_0\\ & x>x_0\end{cases}\\
&=\begin{cases}\frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x_0-1))\sinh(n\pi x) & x<x_0\\ \sinh(n\pi x)\frac{\sinh(n\pi x_0)}{n\pi\tanh(n\pi)}-\frac{\sinh(n\pi x_0)}{n\pi}\cosh(n\pi x) & x>x_0\end{cases}\\
&=\begin{cases}\frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x_0-1))\sinh(n\pi x) & x<x_0\\ \frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x-1))\sinh(n\pi x_0) & x>x_0\end{cases}\end{aligned}$$

Now that the Green function is found, the solution to $A_n''(x)-n^2\pi^2A_n(x)=B_n(x)$ is given by

$$\begin{aligned}A_n(x)&=\int_0^x\frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x-1))\sinh(n\pi x_0)B_n(x_0)\,dx_0\\ &\quad+\int_x^1\frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x_0-1))\sinh(n\pi x)B_n(x_0)\,dx_0\end{aligned}$$

Or

$$\begin{aligned}A_n(x)&=\frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x-1))\int_0^x\sinh(n\pi x_0)B_n(x_0)\,dx_0\\ &\quad+\frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi x)\int_x^1\sinh(n\pi(x_0-1))B_n(x_0)\,dx_0\end{aligned} \tag{3}$$

Now that $A_n(x)$ is found, the solution to the PDE is found from

$$u(x,y)=\sum_{n=1}^{\infty}A_n(x)\sin(n\pi y)$$

Where $A_n(x)$ is given by (3). $B_n(x)$ is the Fourier series coefficient of $f(x,y)$ which needs to be found depending on $f(x,y)$. This is done below.

2.9.6.4 Part d

(i) $f\left(x,y\right)=\sin\left(\pi y\right)$

We first need to find the Fourier coefficients $B_n(x)$. Since $f\left(x,y\right)=\sum_{n=1}^{\infty}B_n(x)\sin\left(n\pi y\right)$, then multiplying both sides by $\sin\left(m\pi y\right)$ and integrating gives

$$\begin{aligned}\int_0^1\sin\left(\pi y\right)\sin\left(m\pi y\right)dy&=\sum_{n=1}^{\infty}B_n(x)\int_0^1\sin\left(m\pi y\right)\sin\left(n\pi y\right)dy\\&=\frac{1}{2}B_m(x)\end{aligned}$$

Therefore

$$B_n(x)=2\int_0^1\sin\left(\pi y\right)\sin\left(n\pi y\right)dy$$

For $n=1$ the above becomes

$$B_1(x)=2\int_0^1\sin^2\left(\pi y\right)dy=1$$

And for all other terms $B_n=0$ due to orthogonality of $\sin$ functions. Therefore now that $B_n(x)$ is found, then from (3) $A_n(x)$ can be found. Only $n=1$ term is needed.

$$\begin{aligned}A_1(x)&=\frac{1}{\pi\sinh(\pi)}\sinh(\pi(x-1))\int_0^x\sinh(\pi x_0)\,dx_0+\frac{1}{\pi\sinh(\pi)}\sinh(\pi x)\int_x^1\sinh(\pi(x_0-1))\,dx_0\\&=\frac{1}{\pi\sinh(\pi)}\sinh(\pi(x-1))\left[\frac{\cosh(\pi x_0)}{\pi}\right]_0^x+\frac{1}{\pi\sinh(\pi)}\sinh(\pi x)\left[\frac{\cosh(\pi(x_0-1))}{\pi}\right]_x^1\\&=\frac{1}{\pi^2\sinh(\pi)}\sinh(\pi(x-1))(\cosh(\pi x)-1)+\frac{1}{\pi^2\sinh(\pi)}\sinh(\pi x)(1-\cosh(\pi(x-1)))\end{aligned}$$

Hence the solution to the PDE is

$$\begin{aligned}u\left(x,y\right)&=\sum_{n=1}^{\infty}A_n(x)\sin\left(n\pi y\right)\\&=A_1(x)\sin\left(\pi y\right)\\&=\left(\frac{1}{\pi^2\sinh(\pi)}\sinh(\pi(x-1))(\cosh(\pi x)-1)+\frac{1}{\pi^2\sinh(\pi)}\sinh(\pi x)(1-\cosh(\pi(x-1)))\right)\sin\left(\pi y\right)\\&=\frac{1}{\pi^2\sinh\pi}(\sinh(\pi(x-1))-\sinh(\pi x)+\sinh\pi)\sin\left(\pi y\right)\end{aligned}$$

The following is a plot of the above solution

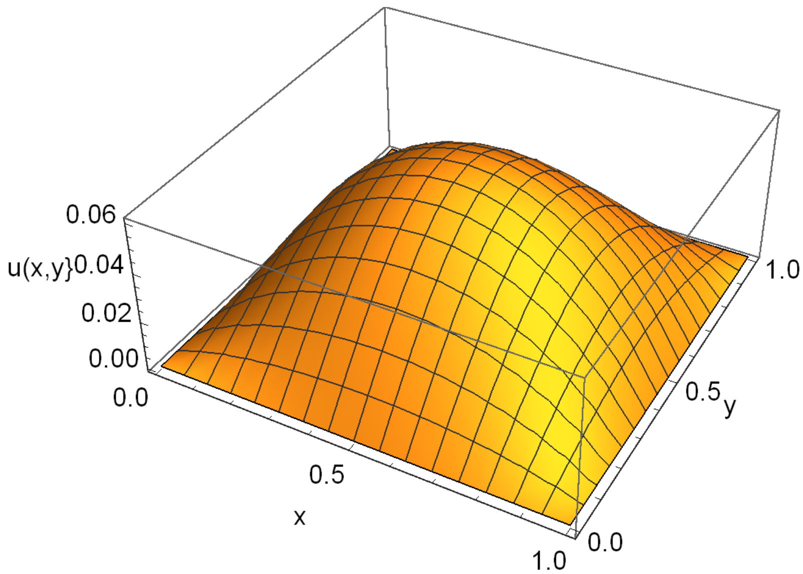


Figure 2.72: Plot of above solution

```
u[x_, y_] := 1/(π^2 Sinh[π]) (Sinh[π (x - 1)] - Sinh[π x] + Sinh[π]) Sin[π y]
p = Plot3D[u[x, y], {x, 0, 1}, {y, 0, 1},
        AxesLabel → {"x", "y", "u(x,y}"},
        BaseStyle → 12];
```

Figure 2.73: Code for the above plot

(ii) $f\left(x,y\right)=\sin\left(\pi x\right)\sin\left(2\pi y\right)$

We first need to find the Fourier coefficients $B_n(x)$. Since $f\left(x,y\right)=\sum_{n=1}^{\infty}B_n(x)\sin\left(n\pi y\right)$, then multiplying both sides by $\sin\left(m\pi y\right)$ and integrating gives

$$\int_0^1 \sin(\pi x)\sin\left(2\pi y\right)\sin\left(m\pi y\right)dy=\sum_{n=1}^{\infty}B_n(x)\int_0^1\sin\left(m\pi y\right)\sin\left(n\pi y\right)dy$$

$$\sin(\pi x)\int_0^1\sin\left(2\pi y\right)\sin\left(m\pi y\right)dy=\frac{1}{2}B_m(x)$$

Therefore

$$B_n(x)=2\sin(\pi x)\int_0^1\sin\left(2\pi y\right)\sin\left(n\pi y\right)dy$$

For $n=2$ the above gives

$$\begin{aligned}B_2(x) &= 2\sin(\pi x)\int_0^1 \sin^2\left(2\pi y\right)dy\\ &= \sin(\pi x)\end{aligned}$$

And for all other terms $B_n = 0$ due to orthogonality. Hence from (3) when $n=2$

$$\begin{aligned}A_2(x) &= \frac{1}{2\pi\sinh(2\pi)}\sinh(2\pi(x-1))\int_0^x \sinh(2\pi x_0)\sin(\pi x_0)\,dx_0\\ &\quad+\frac{1}{2\pi\sinh(2\pi)}\sinh(2\pi x)\int_x^1 \sinh(2\pi(x_0-1))\sin(\pi x_0)\,dx_0\end{aligned}$$

But

$$\int_0^x \sinh(2\pi x_0)\sin(\pi x_0)\,dx_0 = \frac{1}{5\pi}\left(2\cosh(2\pi x)\sin(\pi x)-\cos(\pi x)\sinh(2\pi x)\right)$$

And

$$\int_x^1 \sinh(2\pi(x_0-1))\sin(\pi x_0)\,dx_0 = \frac{-1}{5\pi}\left(2\cosh(2\pi(x-1))\sin(\pi x)+\cos(\pi x)\sinh(2\pi(1-x))\right)$$

Hence

$$\begin{aligned}A_2(x) &= \frac{1}{2\pi\sinh(2\pi)}\sinh(2\pi(x-1))\left(\frac{1}{5\pi}\left(2\cosh(2\pi x)\sin(\pi x)-\cos(\pi x)\sinh(2\pi x)\right)\right)\\ &\quad+\frac{1}{2\pi\sinh(2\pi)}\sinh(2\pi x)\left(\frac{-1}{5\pi}\left(2\cosh(2\pi(x-1))\sin(\pi x)+\cos(\pi x)\sinh(2\pi(1-x))\right)\right)\end{aligned}$$

Or

$$A_2(x) = -\frac{1}{5\pi^2}\sin(\pi x)$$

Hence the PDE solution is

$$\begin{aligned}u\left(x,y\right) &= \sum_{n=1}^{\infty} A_n(x)\sin\left(n\pi y\right)\\ &= A_2(x)\sin\left(2\pi y\right)\\ &= \frac{-1}{5\pi^2}\sin(\pi x)\sin\left(2\pi y\right)\end{aligned}$$

The following is a plot of the above solution

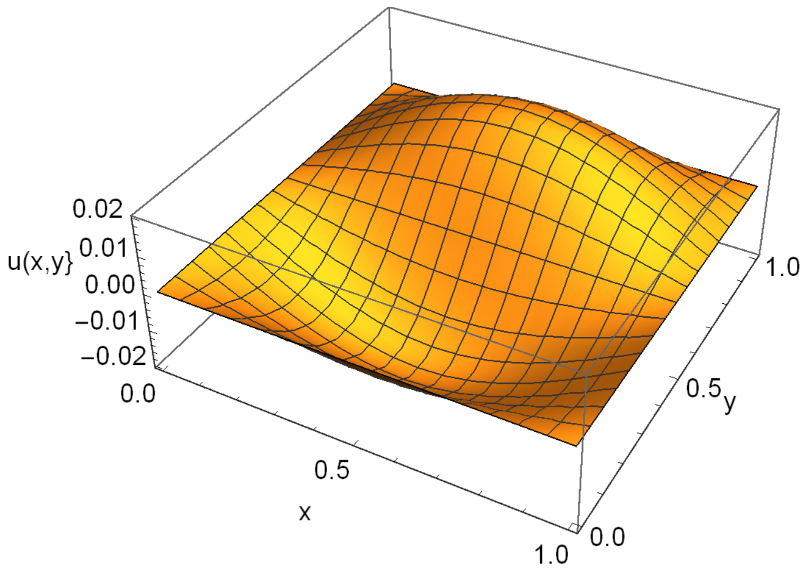


Figure 2.74: Plot of above solution

```
u[x_, y_] := -Sin[Pi x] Sin[2 Pi y] / (5 π^2)
p = Plot3D[u[x, y], {x, 0, 1}, {y, 0, 1},
         AxesLabel → {"x", "y", "u(x,y}"},
         BaseStyle → 12];
```

Figure 2.75: Code for the above plot

(iii) $f\left(x,y\right)=1$

We first need to find the Fourier coefficients $B_n\left(x\right)$. Since $f\left(x,y\right)=\sum_{n=1}^{\infty}B_n\left(x\right)\sin\left(n\pi y\right)$, then multiplying both sides by $\sin\left(m\pi y\right)$ and integrating gives

$$\begin{aligned}
-\int_0^1 \sin\left(m\pi y\right)dy &= \sum_{n=1}^{\infty} B_n\left(x\right)\int_0^1 \sin\left(m\pi y\right)\sin\left(n\pi y\right)dy\\
-\int_0^1 \sin\left(n\pi y\right)dy &= B_n\left(x\right)\frac{1}{2}\\
B_n\left(x\right) &= \frac{2}{n\pi}\left(\cos\left(n\pi y\right)\right)_0^1\\
&= \frac{2}{n\pi}\left(\cos\left(n\pi\right)-1\right)\\
&= \frac{2}{n\pi}\left(\left(-1\right)^n-1\right)
\end{aligned}$$

Hence from (3)

$$\begin{aligned}
A_n\left(x\right) = &\frac{2}{n\pi}\left(\left(-1\right)^n-1\right)\frac{1}{n\pi\sinh\left(n\pi\right)}\sinh\left(n\pi\left(x-1\right)\right)\int_0^x \sinh\left(n\pi x_0\right)dx_0\\
&+\frac{2}{n\pi}\left(\left(-1\right)^n-1\right)\frac{1}{n\pi\sinh\left(n\pi\right)}\sinh\left(n\pi x\right)\int_x^1 \sinh\left(n\pi\left(x_0-1\right)\right)dx_0
\end{aligned}$$

Or

$$\begin{aligned}
A_n\left(x\right) = &\frac{2}{n\pi}\left(\left(-1\right)^n-1\right)\frac{1}{n\pi\sinh\left(n\pi\right)}\sinh\left(n\pi\left(x-1\right)\right)\left[\frac{\cosh\left(n\pi x_0\right)}{n\pi}\right]_0^x\\
&+\frac{2}{n\pi}\left(\left(-1\right)^n-1\right)\frac{1}{n\pi\sinh\left(n\pi\right)}\sinh\left(n\pi x\right)\left[\frac{\cosh\left(n\pi\left(x_0-1\right)\right)}{n\pi}\right]_x^1
\end{aligned}$$

Or

$$\begin{aligned}
A_n\left(x\right) = &\frac{2}{n\pi}\left(\left(-1\right)^n-1\right)\frac{1}{n^2\pi^2\sinh\left(n\pi\right)}\sinh\left(n\pi\left(x-1\right)\right)\left(\cosh\left(n\pi x\right)-1\right)\\
&+\frac{2}{n\pi}\left(\left(-1\right)^n-1\right)\frac{1}{n^2\pi^2\sinh\left(n\pi\right)}\sinh\left(n\pi x\right)\left(1-\cosh\left(n\pi\left(x-1\right)\right)\right)
\end{aligned}$$

Or

$$\begin{aligned}
A_n\left(x\right) &= \frac{2\left(\left(-1\right)^n-1\right)}{n^3\pi^3\sinh\left(n\pi\right)}\left(\sinh\left(n\pi\left(x-1\right)\right)\left(\cosh\left(n\pi x\right)-1\right)+\sinh\left(n\pi x\right)\left(1-\cosh\left(n\pi\left(x-1\right)\right)\right)\right)\\
&= \frac{2\left(\left(-1\right)^n-1\right)}{n^3\pi^3\sinh\left(n\pi\right)}\left(\sinh\left(\pi nx\right)-\sinh\left(\pi n\right)-\sinh\left(\pi nx-\pi n\right)\right)
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
u\left(x,y\right) &= \sum_{n=1}^{\infty} A_n\left(x\right)\sin\left(n\pi y\right)\\
&= \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{\left(\left(-1\right)^n-1\right)}{n^3\sinh\left(n\pi\right)}\left[\sinh\left(\pi nx\right)-\sinh\left(\pi n\right)-\sinh\left(\pi n\left(x-1\right)\right)\right]\sin\left(n\pi y\right)
\end{aligned}$$

The following is a plot of the above solution

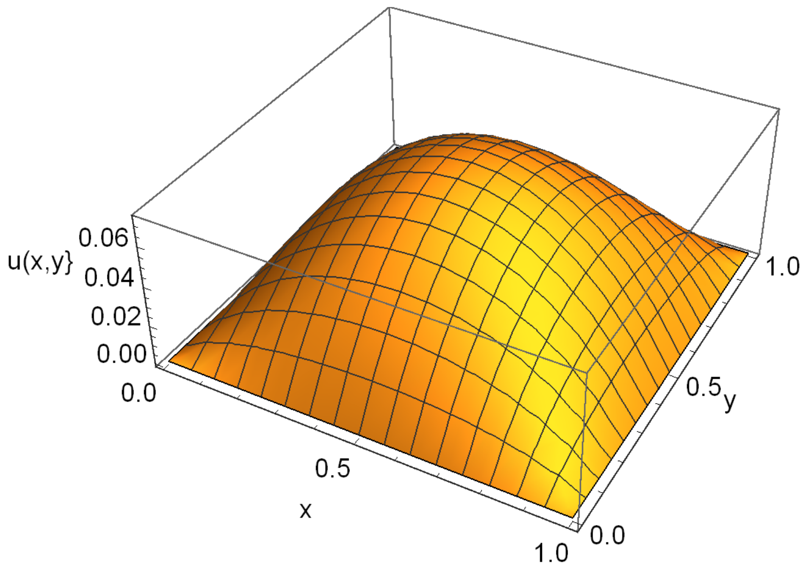


Figure 2.76: Plot of above solution

```
u[x_, y_] :=
     2/π^3 Sum[((-1)^n - 1)/(n^3 Sinh[n π]) (Sinh[n π x] - Sinh[n π] - Sinh[n π (x - 1)]) Sin[n π y], {n, 1, 30}];
p = Plot3D[u[x, y], {x, 0, 1}, {y, 0, 1},
        AxesLabel → {"x", "y", "u(x,y}"},
        BaseStyle → 12];
```

Figure 2.77: Code for the above plot

2.9.7 Problem 6.3.23

Write out the details of how to derive (6.134) from (6.133).

$$
\begin{aligned}
G(x;\xi) &= -\frac{1}{2\pi}\log\|x-\xi\| + \frac{1}{2\pi}\log\frac{\left\|\|\xi\|^2 x - \xi\right\|}{\|\xi\|} \\
&= \frac{1}{2\pi}\log\frac{\left\|\|\xi\|^2 x - \xi\right\|}{\|\xi\|\,\|x-\xi\|} \qquad (6.133)
\end{aligned}
$$

$$
G(r,\theta;\rho,\phi) = \frac{1}{4\pi}\log\left(\frac{1+r^2\rho^2-2r\rho\cos(\theta-\phi)}{r^2+\rho^2-2r\rho\cos(\theta-\phi)}\right) \qquad (6.134)
$$

Solution

Since $x = (r\cos\theta, r\sin\theta)$ and $\boldsymbol{\xi} = \left(\rho\cos\phi, \rho\sin\phi\right)$, then

$$\begin{aligned}\|\xi\|^2 &= \rho^2\cos^2\phi + \rho^2\sin^2\phi \\ &= \rho^2\end{aligned}$$

Hence

$$\begin{aligned}\|\xi\|^2 \boldsymbol{x} &= \rho^2 (r\cos\theta, r\sin\theta) \\ &= \left(r\rho^2\cos\theta, r\rho^2\sin\theta\right)\end{aligned}$$

And therefore

$$\begin{aligned}\|\xi\|^2 \boldsymbol{x} - \boldsymbol{\xi} &= \left(r\rho^2\cos\theta, r\rho^2\sin\theta\right) - \left(\rho\cos\phi, \rho\sin\phi\right) \\ &= \left(r\rho^2\cos\theta - \rho\cos\phi, r\rho^2\sin\theta - \rho\sin\phi\right)\end{aligned}$$

Hence

$$\begin{aligned}\left\|\|\xi\|^2 \boldsymbol{x} - \boldsymbol{\xi}\right\| &= \sqrt{\left(r\rho^2\cos\theta - \rho\cos\phi\right)^2 + \left(r\rho^2\sin\theta - \rho\sin\phi\right)^2} \\ &= \sqrt{\left(r^2\rho^4\cos^2\theta + \rho^2\cos^2\phi - 2r\rho^3\cos\theta\cos\phi\right) + \left(r^2\rho^4\sin^2\theta + \rho^2\sin^2\phi - 2r\rho^3\sin\theta\sin\phi\right)} \\ &= \sqrt{r^2\rho^4\left(\cos^2\theta + \sin^2\theta\right) + \rho^2\left(\cos^2\phi + \sin^2\phi\right) - 2r\rho^3\left(\cos\theta\cos\phi + \sin\theta\sin\phi\right)} \\ &= \sqrt{r^2\rho^4 + \rho^2 - 2r\rho^3\left(\cos\theta\cos\phi + \sin\theta\sin\phi\right)}\end{aligned}$$

But $\cos\theta\cos\phi + \sin\theta\sin\phi = \cos\left(\theta - \phi\right)$. The above becomes

$$\begin{aligned}\left\|\|\xi\|^2 \boldsymbol{x} - \boldsymbol{\xi}\right\| &= \sqrt{r^2\rho^4 + \rho^2 - 2r\rho^3\cos\left(\theta - \phi\right)} \\ &= \rho\sqrt{r^2\rho^2 + 1 - 2r\rho\cos\left(\theta - \phi\right)} \qquad (1)\end{aligned}$$

The above is the numerator of 6.133. Now we find the denominator $\|\xi\|\,\|\boldsymbol{x} - \boldsymbol{\xi}\|$.

$$\begin{aligned}\|\xi\| &= \sqrt{\rho^2\cos^2\phi + \rho^2\sin^2\phi} \\ &= \rho\end{aligned}$$

And

$$\begin{aligned}\|\boldsymbol{x} - \boldsymbol{\xi}\| &= \left\|(r\cos\theta, r\sin\theta) - \left(\rho\cos\phi, \rho\sin\phi\right)\right\| \\ &= \sqrt{\left(r\cos\theta - \rho\cos\phi\right)^2 + \left(r\sin\theta - \rho\sin\phi\right)^2} \\ &= \sqrt{\left(r^2\cos^2\theta + \rho^2\cos^2\phi - 2r\rho\cos\theta\cos\phi\right) + \left(r^2\sin^2\theta + \rho^2\sin^2\phi - 2r\rho\sin\theta\sin\phi\right)} \\ &= \sqrt{r^2\left(\cos^2\theta + \sin^2\theta\right) + \rho^2\left(\cos^2\phi + \sin^2\phi\right) - 2r\rho\left(\cos\theta\cos\phi + \sin\theta\sin\phi\right)} \\ &= \sqrt{r^2 + \rho^2 - 2r\rho\cos\left(\theta - \phi\right)}\end{aligned}$$

Hence

$$\|\xi\|\,\|\boldsymbol{x} - \boldsymbol{\xi}\| = \rho\sqrt{r^2 + \rho^2 - 2r\rho\cos\left(\theta - \phi\right)} \qquad (2)$$

From (1,2)

$$\frac{1}{2\pi}\log\frac{\left\|\|\xi\|^2 x-\xi\right\|}{\|\xi\|\,\|x-\xi\|}=\frac{1}{2\pi}\log\frac{\rho\sqrt{r^2\rho^2+1-2r\rho\cos\left(\theta-\phi\right)}}{\rho\sqrt{r^2+\rho^2-2r\rho\cos\left(\theta-\phi\right)}}$$
$$=\frac{1}{4\pi}\frac{1+r^2\rho^2-2r\rho\cos\left(\theta-\phi\right)}{r^2+\rho^2-2r\rho\cos\left(\theta-\phi\right)}$$

Which is what required to show.

2.9.8 Problem 6.3.27

Consider the wave equation $u_{tt}=c^2u_{xx}$ on the line $-\infty<x<\infty$. Use the d'Alembert formula (2.82) to solve the initial value problem $u(x,0)=\delta(x-a), u_t(x,0)=0$. Can you realize your solution as the limit of classical solutions?

$$u(x,t)=\frac{1}{2}\left(f(x-ct)+f(x+ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)\,ds \tag{2.82}$$

Solution

In (2.82), the function f is the initial conditions and the function g is the initial velocity. Hence the above becomes

$$u(x,t)=\frac{1}{2}\left(\delta\left((x-a)-ct\right)+\delta\left((x-a)+ct\right)\right)$$

But $\delta((x-a)-ct)=\delta(x-a-ct)=\delta(x-(a+ct))$ and $\delta((x-a)+ct)=\delta(x-a+ct)=\delta(x-(a-ct))$. Hence the above becomes

$$u(x,t)=\frac{1}{2}\delta(x-(a+ct))+\frac{1}{2}\delta(x-(a-ct)) \tag{1}$$

The above is two half strength delta pulses, one traveling to the left and one traveling to the right from the starting position. Using the limiting definition of delta function, the solution is the limit of sequence of classical solutions $\lim_{n\to\infty}u_n(x,t)\to u(x,t)$ which has initial position that converges to the delta function and initial velocity which converges to zero as given in this problem. Hence

$$\lim_{n\to\infty}u_n(x,0)=\delta(x-a)$$
$$\lim_{n\to\infty}\frac{\partial}{\partial t}u_n(x,0)=0$$

Using one such definition of limiting function given in 6.10, page 218

$$u_n(x)=\frac{n}{\pi\left(1+n^2x^2\right)}$$

Then

$$u_n(x-a)=\frac{n}{\pi\left(1+n^2(x-a)^2\right)}$$

Hence

$$u_n(x-(a+ct)) = \frac{n}{\pi\left(1+n^2(x-(a+ct))^2\right)}$$

$$u_n(x-(a-ct)) = \frac{n}{\pi\left(1+n^2(x-(a-ct))^2\right)}$$

Using the classical solution $u(x,t) = \frac{1}{2}(u_n(x-(a+ct)) + u_n(x-(a-ct)))$ becomes

$$u(x,t) = \frac{1}{2}\frac{n}{\pi\left(1+n^2(x-(a+ct))^2\right)} + \frac{1}{2}\frac{n}{\pi\left(1+n^2(x-(a-ct))^2\right)}$$

Which converges to (1) $u(x,t) = \frac{1}{2}\delta(x-(a+ct)) + \frac{1}{2}\delta(x-(a-ct))$ as $n \to \infty$.

2.9.9 Problem 6.3.31

(a) Write down a Fourier series for the solution to the initial-boundary value problem

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(-1,t) &= 0 \\ u(1,t) &= 0 \\ u(x,0) &= \delta(x) \\ \frac{\partial u(x,0)}{\partial t} &= 0 \end{aligned}$$

(b) Write down an analytic formula for the solution, i.e., sum your series. (c) In what sense does the series solution in part (a) converge to the true solution? Do the partial sums provide a good approximation to the actual solution?

Solution

2.9.9.1 Part (a)

Since the boundary conditions are at $x=-1$ and at $x=1$, it is a little easier to solve this by first shifting the boundaries to $x=0$ and $x=2$. This is done by transformation. Let

$$z = x+1$$

When $x=-1$ then $z=0$ and when $x=1$ then $z=2$. The PDE in terms of z remains the same but the B.C. are shifted. Hence we want to solve for $v(z,x)$ in

$$\begin{aligned} v_{tt} &= v_{zz} \\ v(0,t) &= 0 \\ v(2,t) &= 0 \end{aligned}$$

No need to worry about initial conditions now, since we will transform back to x before applying initial conditions and therefore will use the original initial conditions. This PDE is now solved by separation. Let $v = Z(z)T(t)$. Substituting into the PDE gives

$$T''Z = Z''T$$
$$\frac{T''}{T} = \frac{Z''}{Z} = -\lambda$$

This gives the boundary value ODE

$$\begin{aligned} Z'' + \lambda Z &= 0 \\ Z(0) &= 0 \\ Z(2) &= 0 \end{aligned} \tag{1}$$

And the time ODE

$$T'' + \lambda T = 0 \tag{2}$$

Solving (1). From the boundary conditions we know only $\lambda > 0$ is an eigenvalue. Hence for $\lambda > 0$ the solution is

$$Z(z) = A\cos\left(\sqrt{\lambda}z\right) + B\sin\left(\sqrt{\lambda}z\right)$$

At $z = 0$ this gives $A = 0$. Hence the solution now becomes $Z(z) = B\sin\left(\sqrt{\lambda}z\right)$. At $z = 2$ the above gives $0 = B\sin\left(2\sqrt{\lambda}\right)$. For non-trivial solution we want $\sin\left(2\sqrt{\lambda}\right) = 0$ which implies $2\sqrt{\lambda} = n\pi$ or

$$\lambda_n = \left(\frac{n\pi}{2}\right)^2 \qquad n = 1, 2, 3, \cdots$$

And the corresponding eigenfunctions

$$Z_n(z) = \sin\left(\frac{n\pi}{2}z\right) \qquad n = 1, 2, 3, \cdots$$

The time ODE (2) now becomes

$$T'' + \left(\frac{n\pi}{2}\right)^2 T = 0$$

Which has solution

$$T_n(t) = A_n\cos\left(\frac{n\pi}{2}t\right) + B_n\sin\left(\frac{n\pi}{2}t\right)$$

Hence the complete solution is

$$v(z,t) = \sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{n\pi}{2}t\right) + B_n\sin\left(\frac{n\pi}{2}t\right)\right)\sin\left(\frac{n\pi}{2}z\right)$$

We are now ready to switch back from z to x. Since $z = x + 1$ then the above becomes

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{n\pi}{2}t\right) + B_n\sin\left(\frac{n\pi}{2}t\right)\right)\sin\left(\frac{n\pi}{2}(x+1)\right) \tag{3}$$

Now we apply initial conditions to find A_n, B_n. At $t = 0, u(x,0) = \delta(x)$. Hence the above gives

$$\delta(x) = \sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi}{2}(x+1)\right)$$

Multiplying both sides by $\sin\left(\frac{m\pi}{2}(x+1)\right)$ and Integrating gives

$$\int_{-1}^{1}\delta(x)\sin\left(\frac{m\pi}{2}(x+1)\right)dx=\sum_{n=1}^{\infty}A_n\int_{-1}^{1}\sin\left(\frac{n\pi}{2}(x+1)\right)\sin\left(\frac{m\pi}{2}(x+1)\right)dx$$

By orthogonality of sin functions only term survives and the above simplifies to

$$\begin{aligned}\int_{-1}^{1}\delta(x)\sin\left(\frac{m\pi}{2}(x+1)\right)dx&=A_m\overbrace{\int_{-1}^{1}\sin^2\left(\frac{m\pi}{2}(x+1)\right)dx}^{1}\\&=A_m\end{aligned}$$

But $\int_{-1}^{1}\delta(x)\sin\left(\frac{m\pi}{2}(x+1)\right)dx=\sin\left(\frac{m\pi}{2}\right)$ since that is where $x=0$. The above reduces to

$$A_n=\sin\left(\frac{n\pi}{2}\right)\qquad n=1,2,3,\cdots$$

The solution (1) becomes

$$u(x,t)=\sum_{n=1}^{\infty}\left(\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}t\right)+B_n\sin\left(\frac{n\pi}{2}t\right)\right)\sin\left(\frac{n\pi}{2}(x+1)\right)\tag{4}$$

Taking time derivatives

$$\frac{\partial}{\partial t}u(x,t)=\sum_{n=1}^{\infty}\left(-\frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{2}t\right)+\frac{n\pi}{2}B_n\cos\left(\frac{n\pi}{2}t\right)\right)\sin\left(\frac{n\pi}{2}(x+1)\right)$$

At $t=0$ the above becomes

$$0=\sum_{n=1}^{\infty}\frac{n\pi}{2}B_n\sin\left(\frac{n\pi}{2}(x+1)\right)$$

Therefore $B_n=0$. Hence the solution (4) becomes

$$u(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}t\right)\sin\left(\frac{n\pi}{2}(x+1)\right)\tag{5}$$

Notice that $\sin\left(\frac{n\pi}{2}\right)$ is zero when n is even.

2.9.9.2 Part b

$$\sin\left(\frac{n\pi}{2}(x+1)\right)=\sin\left(\frac{n\pi}{2}x+\frac{n\pi}{2}\right)$$

Using $\sin(A+B)=\cos A\sin B+\sin A\cos B$, the above becomes, where $A=\frac{n\pi}{2}x$ and $B=\frac{n\pi}{2}$

$$\sin\left(\frac{n\pi}{2}(x+1)\right)=\cos\left(\frac{n\pi}{2}x\right)\sin\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}x\right)\cos\left(\frac{n\pi}{2}\right)$$

Hence (5) becomes

$$\begin{aligned}u(x,t)&=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}t\right)\left(\cos\left(\frac{n\pi}{2}x\right)\sin\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}x\right)\cos\left(\frac{n\pi}{2}\right)\right)\\&=\sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{2}t\right)\left(\cos\left(\frac{n\pi}{2}x\right)\sin^2\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}x\right)\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}\right)\right)\end{aligned}\tag{6}$$

But $\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}\right)=0$, since using $\sin A\cos B=\frac{1}{2}\left(\sin\left(A+B\right)+\sin\left(A-B\right)\right)$ gives

$$\begin{aligned}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}\right)&=\frac{1}{2}\left(\sin\left(n\pi\right)+\sin\left(0\right)\right)\\&=0\end{aligned}$$

Therefore (6) simplifies to

$$u\left(x,t\right)=\sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{2}t\right)\cos\left(\frac{n\pi}{2}x\right)\sin^{2}\left(\frac{n\pi}{2}\right)$$

But $\sin^{2}\left(\frac{n\pi}{2}\right)=0$ when n is even and 1 when n is odd. Hence the above becomes

$$\begin{aligned}u\left(x,t\right)&=\sum_{n=1,3,5,\cdots}^{\infty}\cos\left(\frac{n\pi}{2}t\right)\cos\left(\frac{n\pi}{2}x\right)\\&=\sum_{n=0}^{\infty}\cos\left(\frac{\left(2n+1\right)\pi}{2}t\right)\cos\left(\frac{\left(2n+1\right)\pi}{2}x\right)\end{aligned}$$

Using $\cos A\cos B=\frac{1}{2}\left(\cos\left(A+B\right)+\cos\left(A-B\right)\right)$, then using $A=\frac{(2n+1)\pi}{2}t,B=\frac{(2n+1)\pi}{2}x$ the above becomes

$$\begin{aligned}u\left(x,t\right)&=\sum_{n=0}^{\infty}\frac{1}{2}\left(\cos\left(\frac{\left(2n+1\right)\pi}{2}t+\frac{\left(2n+1\right)\pi}{2}x\right)+\cos\left(\frac{\left(2n+1\right)\pi}{2}t-\frac{\left(2n+1\right)\pi}{2}x\right)\right)\\&=\frac{1}{2}\sum_{n=0}^{\infty}\cos\left(\frac{\left(2n+1\right)\pi}{2}\left(t+x\right)\right)+\frac{1}{2}\sum_{n=0}^{\infty}\cos\left(\frac{\left(2n+1\right)\pi}{2}\left(t-x\right)\right)\qquad(7)\end{aligned}$$

But with help of the computer, found that the sums give

$$\begin{aligned}\sum_{n=0}^{\infty}\cos\left(\frac{\left(2n+1\right)\pi}{2}\left(t+x\right)\right)&=0\\\sum_{n=0}^{\infty}\cos\left(\frac{\left(2n+1\right)\pi}{2}\left(t-x\right)\right)&=0\end{aligned}$$

Hence (7) becomes

$$u\left(x,t\right)=0$$

2.9.9.3 Part c

The solution given by the part b converges to the true solution in the mean sense. Since with wave PDE, there will two pulses, each of half strength moving back and forth on the string each wave with very small width but large amplitude. Solution in part b is giving an averaging value for the solution as zero.

2.9.10 Key solution for HW 9

Problem 1

a)

$\lambda_n = l^2\pi^2/a^2 + m^2\pi^2/b^2$ with eigenfunctions $\sin(l\pi x/a)\sin(m\pi y/b)$, $l, m > 0$

b)

$\lambda_n = l^2\pi^2/a^2 + m^2\pi^2/b^2 \quad l, m \geq 0$

Problem 2

We would like to solve the wave and diffusion equations

$$u_{tt} = c^2\,\Delta u \qquad \textit{and} \qquad u_t = k\,\Delta u$$

in any bounded domain D with one of the classical homogeneous conditions on bdy D and with the standard initial condition. We denote

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \text{or} \qquad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

in two or three dimensions, respectively. For brevity, we continue to use the vector notation $\mathbf{x} = (x, y)$ or (x, y, z). The general discussion that follows works in either dimension, but for definiteness, let's say that we're in three dimensions. Then D is a solid domain and bdy D is a surface.

The first step is to separate the time variable only,

$$u(x, y, z, t) = T(t)v(x, y, z). \qquad (1)$$

Then

$$-\lambda = \frac{T''}{c^2 T} = \frac{\Delta v}{v} \quad \text{or} \quad -\lambda = \frac{T'}{kT} = \frac{\Delta v}{v}, \tag{2}$$

depending on whether we are doing waves or diffusions. In *either* case we get the eigenvalue problem

$$\boxed{\begin{aligned} -\Delta v &= \lambda v \quad \text{in } D \\ v \text{ satisfies } (D), (N), (R) &\quad \text{on bdy } D. \end{aligned}} \tag{3}$$

Therefore, if this problem has eigenvalues λ_n (all positive, say) and eigenfunctions $v_n(x, y, z) = v_n(\mathbf{x})$, then the solutions of the wave equation are

$$\boxed{u(\mathbf{x}, t) = \sum_n [A_n \cos(\sqrt{\lambda_n}\, ct) + B_n \sin(\sqrt{\lambda_n}\, ct)]\, v_n(\mathbf{x})} \tag{4}$$

and the solutions of the diffusion equation are

$$\boxed{u(\mathbf{x}, t) = \sum_n A_n\, e^{-\lambda_n k t}\, v_n(\mathbf{x}).} \tag{5}$$

As usual, the coefficients will be determined by the initial conditions. However, to carry this out, we'll need to know that the eigenfunctions are orthogonal. This is our next goal. One point of notation in (4) and (5): in three dimensions the index n in the sums (4) and (5) will be a *triple index* [(l, m, n), say] and the various series will be *triple series,* one sum for each coordinate.

6.3.9

We rewrite $f(x,y) = \sigma(3x - 2y - 1)$ in terms of the step function. Thus, by the chain rule,
$\dfrac{\partial f}{\partial x} = 3\,\delta(3x - 2y - 1) = \delta\big(x - \tfrac{2}{3}y - \tfrac{1}{3}\big)$, $\quad \dfrac{\partial f}{\partial y} = -2\,\delta(3x - 2y - 1) = -\,\delta\big(y - \tfrac{3}{2}y + \tfrac{1}{2}\big)$.

6.3.10

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2\sin\dfrac{n\pi x}{a}\,\sin\dfrac{n\pi\xi}{a}\,\sinh\dfrac{n\pi(b-y)}{a}}{a\sinh\dfrac{n\pi b}{a}}.$$

Referring back to (4.98), since $|b_n| = \left|\dfrac{2}{a}\sin\dfrac{n\pi\xi}{a}\right| \le \dfrac{2}{a}$, the uniform bound (4.99) holds, and thus the ensuing argument establishes infinite differentiability.

6.3.18

(a) Using the image point $(\xi,-\eta)$, we find $G(x,y;\xi,\eta) = \dfrac{1}{4\pi}\log\dfrac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}$.

(b) $u(x,y) = \dfrac{1}{4\pi}\displaystyle\int_0^\infty\int_{-\infty}^\infty \frac{1}{1+\eta}\log\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}\,d\xi\,d\eta.$

6.3.21

Solution:

(a) We set

$$u(x,y) = \sum_{n=1}^{\infty} b_n(x)\sin n\pi y, \qquad f(x,y) = \sum_{n=1}^{\infty} g_n(x)\sin n\pi y,$$

where

$$b_n(x) = 2\int_0^1 u(x,y)\sin n\pi y\,dy, \qquad g_n(x) = 2\int_0^1 f(x,y)\sin n\pi y\,dy.$$

(b) Substituting into the Poisson equation and the boundary conditions, the resulting boundary value problems for the individual coefficients $b_n(x)$ are

$$-b_n'' + n^2\pi^2 b_n = g_n(x), \qquad b_n(0) = b_n(1) = 0.$$

(c) Using the Green's function solution to the boundary value problem given in (6.65),

$$b_n(x) = \int_0^x \frac{\sinh n\pi(1-x)\sinh n\pi\xi}{n\pi\sinh n\pi}\,g_n(y)\,d\xi + \int_x^1 \frac{\sinh n\pi x\,\sinh n\pi(1-\xi)}{n\pi\sinh n\pi}\,g_n(\xi)\,d\xi.$$

(d) (i) $u(x,y) = \dfrac{1}{\pi^2}\left(1-\dfrac{e^{\pi x}+e^{\pi(1-x)}}{e^\pi+1}\right)\sin\pi y = \dfrac{1}{\pi^2}\left(1-\dfrac{\cosh\pi\left(\frac12-x\right)}{\cosh\frac12\pi}\right)\sin\pi y;$

(ii) $u(x,y) = \dfrac{\sin\pi x\,\sin 2\pi y}{5\pi^2};$

(iii) $u(x,y) = \displaystyle\sum_{k=0}^{\infty}\frac{4}{(2k+1)^3\pi^3}\left(1-\frac{\cosh\left(k+\frac12\right)\pi(1-2x)}{\cosh\left(k+\frac12\right)\pi}\right)\sin(2k+1)\pi y.$

6.3.23

Set
$$\mathbf{x} = (r\cos\theta, r\sin\theta), \qquad \boldsymbol{\xi} = (\rho\cos\varphi, \rho\sin\varphi).$$
Applying the Law of Cosines to the triangle with vertices $\mathbf{0}, \mathbf{x}, \boldsymbol{\xi}$ in Figure 6.13 shows
$$\|\mathbf{x} - \boldsymbol{\xi}\|^2 = \|\mathbf{x}\|^2 + \|\boldsymbol{\xi}\|^2 - 2\|\mathbf{x}\|\,\|\boldsymbol{\xi}\|\cos(\theta - \varphi) = r^2 + \rho^2 - 2r\rho\cos(\theta-\varphi).$$
Applying the Law of Cosines to the triangle with vertices $\mathbf{0}, \mathbf{x}, \boldsymbol{\eta}$ in Figure 6.13 shows
$$\|\mathbf{x} - \boldsymbol{\eta}\|^2 = \|\mathbf{x}\|^2 + \|\boldsymbol{\eta}\|^2 - 2\|\mathbf{x}\|\,\|\boldsymbol{\eta}\|\cos(\theta - \varphi) = r^2 + \frac{1}{\rho^2} - 2\,\frac{r}{\rho}\,\cos(\theta-\varphi),$$
and so
$$\frac{\|\,\|\boldsymbol{\xi}\|^2\,\mathbf{x} - \boldsymbol{\xi}\|^2}{\|\boldsymbol{\xi}\|^2} = \|\boldsymbol{\xi}\|^2\,\|\mathbf{x} - \boldsymbol{\eta}\|^2 = 1 + r^2\rho^2 - 2r\rho\cos(\theta-\varphi).$$
Thus,
$$G(\mathbf{x};\boldsymbol{\xi}) = \frac{1}{4\pi}\log\frac{\|\,\|\boldsymbol{\xi}\|^2\,\mathbf{x} - \boldsymbol{\xi}\|^2}{\|\boldsymbol{\xi}\|^2\,\|\mathbf{x} - \boldsymbol{\xi}\|^2} = \frac{1}{4\pi}\,\log\left(\frac{1 + r^2\rho^2 - 2r\rho\cos(\theta-\varphi)}{r^2 + \rho^2 - 2r\rho\cos(\theta-\varphi)}\right).$$

6.3.27

Solution:
$$u(t,x) = \tfrac{1}{2}\,\delta(x - ct - a) + \tfrac{1}{2}\,\delta(x + ct - a) \qquad (*)$$
consisting of two half-strength delta spikes traveling away from the starting position concentrated on the two characteristic lines. This solution is the limit of a sequence of classical solutions $u^{(n)}(t,x) \longrightarrow u(t,x)$ as $n \longrightarrow \infty$ which have initial conditions that converge to the delta function: $u^{(n)}(0,x) \longrightarrow \delta(x-a)$, $u_t^{(n)}(0,x) = 0$. For example, using (6.10), the initial conditions
$$u^{(n)}(0,x) = \frac{n}{\pi(1 + n^2(x-a)^2)}$$
lead to the classical solutions
$$u^{(n)}(t,x) = \frac{n}{2\pi(1 + n^2(x - ct - a)^2)} + \frac{n}{2\pi(1 + n^2(x + ct - a)^2)}$$
that converge to the delta function solution $(*)$ as $n \longrightarrow \infty$.

6.3.31

(*a*) $u(t,x) = \sum\limits_{k=0}^{\infty} \cos\left(k + \frac{1}{2}\right)\pi t\ \cos\left(k + \frac{1}{2}\right)\pi x;$

(*b*) For any integer k, and $-1 \le x \le 1$,
$$u(t,x) = \begin{cases} \frac{1}{2}\,\delta(x - t + 4k) + \frac{1}{2}\,\delta(x + t - 4k), & 4k - 1 < t < 4k + 1, \\ -\frac{1}{2}\,\delta(x - t + 4k + 2) - \frac{1}{2}\,\delta(x + t - 4k - 2), & 4k + 1 < t < 4k + 3, \\ 0, & t = 2k + 1. \end{cases}$$

(*c*) Because the Fourier series only converges weakly, it cannot be used to approximate the solution; see Figure 6.7 for the one-dimensional version.

2.10 HW 10

Local contents

2.10.1 Problem 1

Show that (assuming sufficient smoothness of the domain and the data) u is a solution to the Dirichlet boundary value problem

$$-\Delta u = f$$

In Ω with B.C. $u = g$ on $\partial\Omega$ iff u is a minimizer of the energy functional, that is

$$E(u) = \min\left\{E(v) : v \in C^2\left(\bar{\Omega}\right)\right\} \text{ such that } u = g \text{ on } \partial\Omega$$

Here

$$E(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - fu\right) dA$$

(note, I will be using dA in the above integral assuming we are in $\mathbb{R}^2$. But the above can also be dV for $\mathbb{R}^3$ just as well and nothing will change in the derivation below. This is easier that writing dx and saying that x is a vector).

Solution

Since the proof is an iff, then we need to show both direction.

Forward direction Given that u solves

$$-\Delta u = f \tag{1}$$

with $u|_{\partial\Omega} = g$. Then we need to show that $E(v) \geq E(u)$ for all $v \in C^2\left(\bar{\Omega}\right)$ that also satisfy same B.C.

Multiplying both sides of (1) by $u - v$ and integrating over the domain gives

$$-\int_\Omega (\Delta u)(u - v)\, dA = \int_\Omega (u - v) f dA \tag{2}$$

For the left integral $\int_\Omega (\Delta u)(u - v)\, dA$, we will do integration by parts. Let $\Delta u \equiv dV, u - v = U$, then $\int_\Omega U dV = \int_{\partial\Omega} UV - \int_\Omega V dU$. Therefore $dU = \nabla(u - v)$ and $V = \nabla u$. After applying integration by parts the (2) now becomes

$$-\left(\int_{\partial\Omega}(u-v)\frac{\partial u}{\partial \boldsymbol{n}}\,dL-\int_{\Omega}\nabla u\cdot\nabla(u-v)\,dA\right)=\int_{\Omega}(u-v)\,f dA$$

But $\int_{\partial\Omega}(u-v)\frac{\partial u}{\partial \boldsymbol{n}}\,dL=0$ because $u=v$ on the boundary $\partial\Omega$ as both are g. The above now simplifies to

$$\int_{\Omega}\nabla u\cdot\nabla(u-v)\;dA=\int_{\Omega}\left(uf-vf\right)\;dA$$
$$\int_{\Omega}\nabla u\cdot(\nabla u-\nabla v)\;dA=\int_{\Omega}\left(uf-vf\right)\;dA$$
$$\int_{\Omega}|\nabla u|^2-\nabla u\cdot\nabla v\;dA=\int_{\Omega}\left(uf-vf\right)\;dA$$
$$\int_{\Omega}|\nabla u|^2-\int_{\Omega}fu\;dA=\int_{\Omega}(\nabla u\cdot\nabla v)-vf\;dA$$

Now we use Schwarz triangle inequality and write $\nabla u\cdot\nabla v\leq\frac{1}{2}\left(|\nabla u|^2+|\nabla v|^2\right)$. This comes from using $ab\leq\frac{1}{2}\left(a^2+b^2\right)$. Using this in the RHS of the above gives

$$\int_{\Omega}|\nabla u|^2\;dA-\int_{\Omega}fu\;dA\leq\int_{\Omega}\frac{1}{2}\left(|\nabla u|^2+|\nabla v|^2\right)-fv\;dA$$
$$\int_{\Omega}|\nabla u|^2\;dA-\int_{\Omega}fu\;dA\leq\int_{\Omega}\frac{1}{2}|\nabla u|^2dA+\left(\frac{1}{2}\int_{\Omega}|\nabla v|^2-fv\;dA\right)$$
$$\int_{\Omega}\frac{1}{2}|\nabla u|^2dA-\int_{\Omega}fu\;dA\leq\frac{1}{2}\int_{\Omega}|\nabla v|^2-fv\;dA$$
$$\int_{\Omega}\frac{1}{2}|\nabla u|^2-fu\;dA\leq\frac{1}{2}\int_{\Omega}|\nabla v|^2-fv\;dA$$

But by definition $\int_{\Omega}\frac{1}{2}|\nabla u|^2-fu\;dA=E(u)$ and $\frac{1}{2}\int_{\Omega}|\nabla v|^2-fv\;dA=E(v)$, therefore the above becomes

$$E(u)\leq E(v)$$

Which is what we wanted to show. Now we will do the other direction.

<u>Reverse direction</u> Given that u minimizes energy among all test functions, i.e. given that $E(u)=\min E(w)$, then need to show that $-\Delta u=f$.

Consider $w=u+\varepsilon v$ where v is any test function $v\in C^2\left(\bar{\Omega}\right)$ and $v=g$ at $\partial\Omega$. Hence

$$\min(E(w))=\min(E(u+\varepsilon v))$$

Therefore $\min(E(u+\varepsilon v))$ is achieved when $\varepsilon=0$, since this then gives $E(u)$ which by assumption is the minimum. Therefore

$$\frac{d}{d\varepsilon}E(u+\varepsilon v)=0$$

At $\varepsilon = 0$. But the above can be written as the following, using the definition of energy

$$\frac{d}{d\varepsilon}\left(\int_{\Omega}\frac{1}{2}\left|\nabla\left(u+\varepsilon v\right)\right|^{2}-f\left(u+\varepsilon v\right)\ dA\right)=0$$

$$\frac{d}{d\varepsilon}\left(\int_{\Omega}\frac{1}{2}\left(\nabla\left(u+\varepsilon v\right)\cdot\nabla\left(u+\varepsilon v\right)\right)-f\left(u+\varepsilon v\right)\ dA\right)=0 \tag{3}$$

Expanding $\nabla\left(u+\varepsilon v\right)\cdot\nabla\left(u+\varepsilon v\right)$ gives

$$\begin{aligned}\nabla\left(u+\varepsilon v\right)\cdot\nabla\left(u+\varepsilon v\right)&=\left(\nabla u+\varepsilon\nabla v\right)\cdot\left(\nabla u+\varepsilon\nabla v\right)\\&=\left|\nabla u\right|^{2}+2\varepsilon\nabla u\cdot\nabla v+\varepsilon^{2}\left|\nabla v\right|^{2}\end{aligned} \tag{4}$$

Substituting (4) into (3) gives

$$\frac{d}{d\varepsilon}\left(\int_{\Omega}v\left(\left|\nabla u\right|^{2}+2\varepsilon\nabla u\cdot\nabla v+\varepsilon^{2}\left|\nabla v\right|^{2}\right)-fu-\varepsilon fv\ dA\right)=0$$

Now we move the derivative inside the take derivative w.r.t. ε giving

$$\left(\int_{\Omega}\frac{1}{2}\left(2\nabla u\cdot\nabla v+2\varepsilon\left|\nabla v\right|^{2}\right)-fv\ dA\right)=0$$

Evaluate at $\varepsilon = 0$ the above becomes

$$\int_{\Omega}\left(\nabla u\cdot\nabla v\right)dA-\int_{\Omega}fv\ dA=0$$

Integration by parts for the first integral. Let $\nabla u = U, dV = \nabla v$, then $\int_{\Omega} UdV = \int_{\partial\Omega} UV - \int_{\Omega} VdU$. Hence the above becomes

$$\left(\int_{\partial\Omega}v\frac{\partial u}{\partial\boldsymbol{n}}\ dL-\int_{\Omega}v\Delta u\ dA\right)-\int_{\Omega}fv\ dA=0$$

But $v = 0$ at boundary $\partial\Omega$. The above simplifies to

$$\begin{aligned}-\int_{\Omega}v\Delta u\ dA-\int_{\Omega}fv\ dA&=0\\\int_{\Omega}v\left(-\Delta u\ -f\ \right)dA&=0\end{aligned}$$

Since the above is true for all v test function then this implies that $-\Delta u\ - f\ = 0$ or

$$-\Delta u\ =f$$

Which is what we wanted to show.

2.10.2 Problem 7.1.1 f

Find the Fourier transform of (f) $f(x) = \begin{cases} e^{-x}\sin x & x>0\\ 0 & x\leq 0\end{cases}$

Solution

$$
\begin{aligned}
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x} \sin x e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \sin x e^{-ikx-x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \sin x e^{-x(1+ik)} dx \qquad (1)
\end{aligned}
$$

Integration by parts. $\int u dv = uv - \int v du$. Let $dv = e^{-x(1+ik)}, v = \frac{e^{-x(1+ik)}}{-(1+ik)}, u = \sin x, du = \cos x$. Hence

$$
\begin{aligned}
I &= \int_{0}^{\infty} \sin x e^{-x(1+ik)} dx \\
&= \left[\sin x \frac{e^{-x(1+ik)}}{-(1+ik)} \right]_0^\infty - \int_0^\infty \cos x \frac{e^{-x(1+ik)}}{-(1+ik)} dx \\
&= \frac{-1}{1+ik} \left[\sin x e^{-x(1+ik)} \right]_0^\infty + \frac{1}{1+ik} \int_0^\infty \cos x e^{-x(1+ik)} dx
\end{aligned}
$$

But $e^{-x(1+ik)} = e^{-x} e^{-ikx}$ and this goes to zero as $x \to \infty$ and since $\sin x = 0$ at $x = 0$ then the first term above is zero. The above reduces to

$$
I = \frac{1}{1+ik} \int_0^\infty \cos x e^{-x(1+ik)} dx
$$

Integration by parts. $\int u dv = uv - \int v du$. Let $dv = e^{-x(1+ik)}, v = \frac{e^{-x(1+ik)}}{-(1+ik)}, u = \cos x, du = -\sin x$. The above becomes

$$
\begin{aligned}
I &= \frac{1}{1+ik} \left(\left[\cos x \frac{e^{-x(1+ik)}}{-(1+ik)} \right]_0^\infty - \int_0^\infty (-\sin x) \frac{e^{-x(1+ik)}}{-(1+ik)} dx \right) \\
&= \frac{1}{1+ik} \left(\left[\cos x \frac{e^{-x(1+ik)}}{-(1+ik)} \right]_0^\infty - \frac{1}{1+ik} \int_0^\infty \sin x e^{-x(1+ik)} dx \right)
\end{aligned}
$$

But $\int_0^\infty \sin x e^{-x(1+ik)} dx = I$. The above becomes

$$
\begin{aligned}
I &= \frac{1}{1+ik} \left(\left[\cos x \frac{e^{-x(1+ik)}}{-(1+ik)} \right]_0^\infty - \frac{1}{1+ik} I \right) \\
&= \frac{1}{1+ik} \left[\cos x \frac{e^{-x(1+ik)}}{-(1+ik)} \right]_0^\infty - \left(\frac{1}{1+ik} \right)^2 I \\
I + \left(\frac{1}{1+ik} \right)^2 I &= \frac{-1}{(1+ik)^2} \left[\cos x e^{-x(1+ik)} \right]_0^\infty
\end{aligned}
$$

Now $\left[\cos x e^{-x(1+ik)}\right]_0^\infty = 0 - 1 = -1$. Hence the above reduces to

$$
\begin{aligned}
I\left(1+\left(\frac{1}{1+ik}\right)^2\right) &= \frac{1}{(1+ik)^2}\\
I &= \frac{\frac{1}{(1+ik)^2}}{1+\left(\frac{1}{1+ik}\right)^2}\\
&= \frac{1}{1+(1+ik)^2}\\
&= \frac{1}{2-k^2+2ik}
\end{aligned}
$$

Therefore

$$
\int_0^\infty \sin x e^{-x(1+ik)} dx = \frac{1}{2-k^2+2ik}
$$

Using (1) the Fourier transform becomes

$$
\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{2-k^2+2ik}
$$

This can be written as real and imaginary parts

$$
\begin{aligned}
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \frac{\left(2-k^2\right)-2ik}{\left(\left(2-k^2\right)+2ik\right)\left(\left(2-k^2\right)-2ik\right)}\\
&= \frac{1}{\sqrt{2\pi}} \frac{\left(2-k^2\right)-2ik}{\left(2-k^2\right)^2+4k^2}\\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{2-k^2}{k^4+4} - i\frac{2k}{k^4+4}\right)
\end{aligned}
$$

2.10.3 Problem 7.1.3 (a,b)

Find the inverse Fourier transform of the function $\frac{1}{k+c}$ when (a) $c=a$ is real (b) $c=ib$ is pure imaginary.

Solution

2.10.3.1 Part a

Using shifting property where $\mathscr{F}\left[f(x)\right] = \hat{f}(k)$ and let $\hat{f}(k) = \frac{1}{k}$ then by shifting property $\mathscr{F}\left[e^{iax} f(x)\right] = \hat{f}(k-a)$, (Theorem 7.4) therefore

$$
\begin{aligned}
\mathscr{F}\left[e^{-iax} f(x)\right] &= \hat{f}(k+a)\\
&= \frac{1}{k+a} \qquad (1)
\end{aligned}
$$

We now just need to find $f(x)$. From table of Fourier transforms on page 272, we see that $\mathscr{F}[\operatorname{sgn}(x)] = \frac{1}{i}\sqrt{\frac{2}{\pi}}\frac{1}{k}$. Hence

$$\mathscr{F}\left[i\sqrt{\frac{\pi}{2}}\operatorname{sgn}(x)\right] = \frac{1}{k}$$

Therefore $f(x) = i\sqrt{\frac{\pi}{2}}\operatorname{sgn}(x)$. Substituting this back into (1) gives

$$\mathscr{F}\left[ie^{-iax}\sqrt{\frac{\pi}{2}}\operatorname{sgn}(x)\right] = \frac{1}{k+a}$$

Or

$$\mathscr{F}^{-1}\left[\frac{1}{k+a}\right] = ie^{-iax}\sqrt{\frac{\pi}{2}}\operatorname{sgn}(x)$$

2.10.3.2 Part b

Using shifting property, given that $\mathscr{F}\big(f(x)\big) = \hat{f}(k)$, let $\hat{f}(k) = \frac{1}{k}$ then by shifting property (Theorem 7.4) $\mathscr{F}\left[e^{i(ib)x}f(x)\right] = \hat{f}(k-ib)$, then

$$\begin{aligned}\mathscr{F}\left[e^{bx}f(x)\right] &= \hat{f}(k+ib)\\ &= \frac{1}{k+ib}\end{aligned}\tag{1}$$

We now just need to find $f(x)$. From table of Fourier transforms on page 272, we see that $\mathscr{F}[\operatorname{sgn}(x)] = \frac{1}{i}\sqrt{\frac{2}{\pi}}\frac{1}{k}$. Hence

$$\mathscr{F}\left[i\sqrt{\frac{\pi}{2}}\operatorname{sgn}(x)\right] = \frac{1}{k}$$

Therefore $f(x) = i\sqrt{\frac{\pi}{2}}\operatorname{sgn}(x)$. Substituting this back into (1) gives

$$\mathscr{F}\left[ie^{bx}\sqrt{\frac{\pi}{2}}\operatorname{sgn}(x)\right] = \frac{1}{k+ib}$$

Or

$$\mathscr{F}^{-1}\left[\frac{1}{k+ib}\right] = ie^{bx}\sqrt{\frac{\pi}{2}}\operatorname{sgn}(x)$$

2.10.4 Problem 7.1.13

Prove the Shift Theorem 7.4 which is

Theorem 7.4: if $f(x)$ has Fourier transform $\hat{f}(k)$, then the Fourier transform of the shifted function $f(x-\xi)$ is $e^{-ik\xi}\hat{f}(k)$. Similarly the transform of the product function $e^{i\alpha x}f(x)$ for real α is the shifted transform $\hat{f}(k-\alpha)$ (note: using α in place of the strange second k that the book uses)

2.10.4.1 Part a

Showing if $f(x)$ has Fourier transform $\hat{f}(k)$, then Fourier transform of the shifted function $f(x-\xi)$ is $e^{-ik\xi}\hat{f}(k)$. From definition, the Fourier transform of $f(x-\xi)$ is given by

$$\mathscr{F}\left[f(x-\xi)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x-\xi)\,e^{-ikx}dx$$

Let $x-\xi=u$. Then $\frac{du}{dx}=1$. The above becomes (limits do not change)

$$\begin{aligned}\mathscr{F}\left[f(x-\xi)\right]&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\,e^{-ik(u+\xi)}du\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\,e^{-iku}e^{-ik\xi}du\\&=e^{-ik\xi}\overbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\,e^{-iku}du}^{\hat{f}(k)}\end{aligned}$$

Therefore

$$\mathscr{F}\left[f(x-\xi)\right]=e^{-ik\xi}\hat{f}(k)$$

Which is what asked to show.

2.10.4.2 Part b

Showing that the Fourier transform of $e^{i\alpha x}f(x)$ is $\hat{f}(k-\alpha)$. From definition, the Fourier transform of $e^{i\alpha x}f(x)$ is

$$\begin{aligned}\mathscr{F}\left[e^{i\alpha x}f(x)\right]&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\alpha x}f(x)\,e^{-ikx}dx\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{-ix(k-\alpha)}dx\end{aligned}$$

But $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{-ix(k-\alpha)}dx$ is $\hat{f}(k-\alpha)$ by replacing k with $k-\alpha$ in the definition of Fourier transform. Hence

$$\mathscr{F}\left[e^{i\alpha x}f(x)\right]=\hat{f}(k-\alpha)$$

Which is what asked to show.

2.10.5 Problem 7.1.20 (a)

The two-dimensional Fourier transform of a function $f(x,y)$ defined for $(x,y)\in\mathbb{R}^2$ is

$$\begin{aligned}\mathscr{F}\left[f(x,y)\right]&=\hat{f}(k,l)\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,e^{-i(kx+ly)}dxdy\end{aligned}$$

(a) compute the Fourier transform of the following functions (i) $e^{-|x|-|y|}$, (iii) The delta function $\delta(x-\xi)\,\delta(y-\eta)$

(b) Show that if $f\left(x,y\right) = g\left(x\right) h\left(y\right)$ then $\hat{f}\left(k,l\right) = \hat{g}\left(k\right) \hat{h}\left(l\right)$

Solution

2.10.5.1 Part a

(i) The Fourier transform of $e^{-|x|-|y|}$ is

$$\begin{aligned}
\hat{f}\left(k,l\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-|x|-|y|}e^{-i\left(kx+ly\right)}dxdy\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-|x|}e^{-|y|}e^{-ikx}e^{-ily}dxdy\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|y|}e^{-ily}\left(\int_{-\infty}^{\infty} e^{-|x|}e^{-ikx}dx\right)dy \qquad (1)
\end{aligned}$$

But $\int_{-\infty}^{\infty} e^{-|x|}e^{-ikx}dx$ is the Fourier transform of $f\left(x\right) = e^{-|x|}$ with $\sqrt{2\pi}$ factor. In other words

$$\int_{-\infty}^{\infty} e^{-|x|}e^{-ikx}dx = \sqrt{2\pi}\hat{g}\left(k\right)$$

Where $\hat{g}\left(k\right)$ is used to indicate the Fourier transform of $e^{-|x|}$. Hence (1) becomes

$$\hat{f}\left(k,l\right) = \frac{\sqrt{2\pi}}{2\pi}\hat{f}_1\left(k\right)\int_{-\infty}^{\infty} e^{-|y|}e^{-ily}dy$$

But $\int_{-\infty}^{\infty} e^{-|y|}e^{-ily}dy = \sqrt{2\pi}\hat{h}\left(l\right)$ Where $\hat{h}\left(l\right)$ is used to indicate the Fourier transform of $e^{-|y|}$. The above becomes

$$\begin{aligned}
\hat{f}\left(k,l\right) &= \frac{\sqrt{2\pi}}{2\pi}\hat{g}\left(k\right)\sqrt{2\pi}\hat{h}\left(l\right)\\
&= \hat{g}\left(k\right)\hat{h}\left(l\right) \qquad (2)
\end{aligned}$$

So now we need to determine $\hat{g}(k)$ and $\hat{h}(l)$ and multiply the result.

$$\begin{aligned}
\hat{g}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{x} e^{-ikx} dx + \int_{0}^{\infty} e^{-x} e^{-ikx} dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{-ikx+x} dx + \int_{0}^{\infty} e^{-ikx-x} dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\left[\frac{e^{-ikx+x}}{1-ik} \right]_{-\infty}^{0} + \left[\frac{e^{-ikx-x}}{-1-ik} \right]_{0}^{\infty} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-ik} \left[e^{-ikx} e^{x} \right]_{-\infty}^{0} - \frac{1}{1+ik} \left[e^{-ikx} e^{-x} \right]_{0}^{\infty} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-ik} (1-0) - \frac{1}{1+ik} (0-1) \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-ik} + \frac{1}{1+ik} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{(1+ik)+(1-ik)}{(1-ik)(1+ik)} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{1+k^2} \right) \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{1+k^2}
\end{aligned}$$

Similarly

$$\begin{aligned}
\hat{h}(l) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|y|} e^{-ily} dy \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{1+l^2}
\end{aligned}$$

Hence from (2) the Fourier transform of $e^{-|x|-|y|}$ is

$$\begin{aligned}
\hat{f}(k,l) &= \hat{g}(k)\, \hat{h}(l) \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{1+k^2} \sqrt{\frac{2}{\pi}} \frac{1}{1+l^2} \\
&= \frac{2}{\pi} \frac{1}{\left(1+k^2\right)\left(1+l^2\right)}
\end{aligned}$$

(ii) The Fourier transform of $\delta(x-\xi)\delta\left(y-\eta\right)$. First we find the Fourier transform of $\delta(x-\xi)$ and then the Fourier transform of $\delta\left(y-\eta\right)$

$$\begin{aligned}\hat{g}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x-\xi)\,e^{-ikx}dx\\ &= \frac{1}{\sqrt{2\pi}}e^{-ik\xi}\end{aligned}$$

And

$$\begin{aligned}\hat{h}(l) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta\left(y-\eta\right)e^{-ily}dy\\ &= \frac{1}{\sqrt{2\pi}}e^{-il\eta}\end{aligned}$$

Hence the Fourier transform of the product $\delta(x-\xi)\delta\left(y-\eta\right)$ is (Using the product rule, which will be proofed in part b also).

$$\begin{aligned}\hat{f}(k,l) &= \hat{g}(k)\hat{h}(l)\\ &= \frac{1}{2\pi}e^{-ik\xi}e^{-il\eta}\end{aligned}$$

The above could be rewritten in terms of trig functions using Euler relation if needed.

2.10.5.2 Part b

By definition, the Fourier transform of $f\left(x,y\right)$ is

$$\hat{f}(k,l) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f\left(x,y\right)e^{-i\left(kx+ly\right)}dxdy$$

But $f\left(x,y\right)=g(x)h\left(y\right)$. Hence the above becomes

$$\begin{aligned}\hat{f}(k,l) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x)h\left(y\right)e^{-i\left(kx+ly\right)}dxdy\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x)h\left(y\right)e^{-ikx}e^{-ily}dxdy\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}h\left(y\right)e^{-ily}\left(\int_{-\infty}^{\infty}g(x)e^{-ikx}dx\right)dy\end{aligned}$$

But $\int_{-\infty}^{\infty}g(x)e^{-ikx}dx=\sqrt{2\pi}\hat{g}(k)$. The above reduces to

$$\hat{f}(k,l) = \frac{1}{2\pi}\sqrt{2\pi}\hat{g}(k)\int_{-\infty}^{\infty}h\left(y\right)e^{-ily}dy$$

But $\int_{-\infty}^{\infty}h\left(y\right)e^{-ily}dy=\sqrt{2\pi}\hat{h}(l)$. Hence the above becomes

$$\begin{aligned}\hat{f}(k,l) &= \frac{1}{2\pi}\sqrt{2\pi}\hat{g}(k)\sqrt{2\pi}\hat{h}(l)\\ &= \hat{g}(k)\hat{h}(l)\end{aligned}$$

Which is what asked to show.

2.10.6 Problem 7.2.2 (a)

Find the Fourier transform of (a) the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2} dz$

Solution

2.10.6.1 Part a

Using

$$1 + \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_{-\infty}^{x} e^{-z^2} dz \tag{1}$$

Taking Fourier transform of both sides, and using the known relation from tables which says

$$\mathscr{F}\left[\int_{-\infty}^{x} f(u)\,du\right] = \frac{1}{ik}\hat{f}(k) + \pi\hat{f}(0)\,\delta(k)$$

And using that Fourier transform of 1 is $\sqrt{2\pi}\delta(k)$ then (1) becomes

$$\sqrt{2\pi}\delta(k) + \mathscr{F}[\operatorname{erf}(x)] = \frac{2}{\sqrt{\pi}}\left(\frac{1}{ik}\hat{f}(k) + \pi\hat{f}(0)\,\delta(k)\right)$$

Where $\hat{f}(k)$ is the Fourier transform of e^{-u^2} (Gaussian) we derived in class as $e^{-u^2} \Leftrightarrow \frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}}$. The above becomes

$$\begin{aligned}
\sqrt{2\pi}\delta(k) + \mathscr{F}[\operatorname{erf}(x)] &= \frac{2}{\sqrt{\pi}}\left(\frac{1}{ik}\frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}} + \pi\left[\frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}}\right]_{k=0}\delta(k)\right)\\
&= \frac{2}{\sqrt{\pi}}\left(\frac{1}{ik}\frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}} + \frac{\pi}{\sqrt{2}}\delta(k)\right)\\
&= \frac{2}{\sqrt{\pi}}\frac{1}{ik}\frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}} + \sqrt{2\pi}\delta(k)
\end{aligned}$$

Therefore the above simplifies to

$$\begin{aligned}
\mathscr{F}[\operatorname{erf}(x)] &= \frac{2}{\sqrt{\pi}}\frac{1}{ik}\frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}}\\
&= \sqrt{\frac{2}{\pi}}\frac{1}{ik}e^{\frac{-k^2}{4}}\\
&= -i\sqrt{\frac{2}{\pi}}\frac{1}{k}e^{\frac{-k^2}{4}}
\end{aligned}$$

2.10.7 Problem 7.2.3 (d)

Find the inverse Fourier transform of the following functions (d) $\frac{k^2}{k-i}$

Solution

Using property that

$$\begin{aligned}\mathscr{F}\left[f'(x)\right] &= ik\hat{f}(k)\\ \mathscr{F}\left[f''(x)\right] &= -k^2\hat{f}(k)\end{aligned} \tag{1}$$

Where in the above $\mathscr{F}\left[f(x)\right] = \hat{f}(k)$. Comparing the above with $\frac{k^2}{k-i}$, we see that

$$\hat{f}(k) = \frac{1}{k-i}$$

Hence we need to find inverse Fourier transform of $\frac{-1}{k-i}$ first in order to find $f(x)$, and then take second derivative of the result. Writing

$$\begin{aligned}\frac{1}{k-i} &= \frac{1}{i\left(\frac{k}{i}-1\right)}\\ &= \frac{1}{i(-ik-1)}\\ &= \frac{-1}{i(ik+1)}\\ &= i\frac{1}{(1+ik)}\end{aligned}$$

From table (page 272 in textbook) we see that

$$\mathscr{F}^{-1}\left[\frac{1}{(ik+1)}\right] = \sqrt{2\pi}e^{-x}\sigma(x)$$

Using $a=1$ in the table entry. Where $\sigma(x)$ is the step function. Hence

$$i\mathscr{F}^{-1}\left[\frac{1}{(ik+1)}\right] = i\sqrt{2\pi}e^{-x}\sigma(x)$$

Therefore

$$f(x) = i\sqrt{2\pi}e^{-x}\sigma(x)$$

Now we take derivative of the above (using product rule)

$$f'(x) = -i\sqrt{2\pi}e^{-x}\sigma(x) + i\sqrt{2\pi}e^{-x}\delta(x)$$

Where $\delta(x)$ is added since derivative of $\sigma(x)$ has jump discontinuity at $x=0$. Taking one more derivative gives

$$\begin{aligned}f''(x) &= i\sqrt{2\pi}e^{-x}\sigma(x) - i\sqrt{2\pi}e^{-x}\delta(x) - i\sqrt{2\pi}e^{-x}\delta(x) + i\sqrt{2\pi}e^{-x}\delta'(x)\\ &= i\sqrt{2\pi}e^{-x}\sigma(x) - 2i\sqrt{2\pi}e^{-x}\delta(x) + i\sqrt{2\pi}e^{-x}\delta'(x)\end{aligned}$$

Therefore

$$\mathscr{F}^{-1}\left[\frac{k^2}{k-i}\right] = i\sqrt{2\pi}e^{-x}\sigma(x) - 2i\sqrt{2\pi}e^{-x}\delta(x) + i\sqrt{2\pi}e^{-x}\delta'(x)$$

2.10.8 Problem 7.2.12

(a) Explain why the Fourier transform of a 2π periodic function $f(x)$ is a linear combinations of delta functions $\hat{f}(k) = \sum_{n=-\infty}^{\infty} c_n \delta(k-n)$ where c_n are the complex Fourier series coefficients (3.65) of $f(x)$ on $[-\pi, \pi]$

$$c_n = \left\langle f, e^{inx} \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \tag{3.65}$$

(b) Find the Fourier transform of the following periodic functions (i) $\sin 2x$ (ii) $\cos^3 x$ (iii) The 2π periodic extension of $f(x) = x$ (iv) The sawtooth function $h(x) = x \bmod 1$. i.e. the fractional part of x

Solution

2.10.8.1 Part a

Since $f(x)$ is periodic, then its can be expressed as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\left(\frac{2\pi}{T}\right)x}$$

But the period $T = 2\pi$ and the above simplifies to

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{1}$$

Taking the Fourier transform of the above gives

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{2}$$

Substituting (1) into (2) gives

$$\begin{aligned}
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} c_n e^{inx} \right) e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} c_n e^{-ix(k-n)} \right) dx
\end{aligned}$$

Changing the order of summation and integration

$$\begin{aligned}
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} c_n e^{-ix(k-n)} dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n \left(\int_{-\infty}^{\infty} e^{-ix(k-n)} dx \right) \qquad (3)
\end{aligned}$$

But from tables we know that $\mathscr{F}(1) = \sqrt{2\pi}\delta(k)$. Which means that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixk} dx = \sqrt{2\pi}\delta(k)$$

Therefore, replacing k by $k-n$ in the above gives

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix(k-n)}dx = \sqrt{2\pi}\delta\left(k-n\right)$$

$$\int_{-\infty}^{\infty} e^{-ix(k-n)}dx = \left(2\pi\right)\delta\left(k-n\right) \tag{4}$$

Substituting (4) into (3) gives

$$\begin{aligned}\hat{f}\left(k\right) &= \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} c_n\left(2\pi\right)\delta\left(k-n\right)\\ &= \sqrt{2\pi}\sum_{n=-\infty}^{\infty} c_n\delta\left(k-n\right)\end{aligned}$$

Note: The books seems to have a typo. It gives the above without the factor $\sqrt{2\pi}$ at the front.

2.10.8.2 Part b

(i) $\sin 2x$. Since this is periodic, then $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin\left(2x\right)e^{-inx}dx$. For $n=2$ this gives $c_2 = -\frac{i}{2}$ and for $n=-2$ it gives $c_{-2} = \frac{i}{2}$ and it is zero for all other n values due to orthogonality of $\sin$ functions. Using the above result obtained in part (a)

$$\begin{aligned}\hat{f}\left(k\right) &= \sqrt{2\pi}\sum_{n=-\infty}^{\infty} c_n\delta\left(k-n\right)\\ &= \sqrt{2\pi}c_{-2}\delta\left(k+2\right) + \sqrt{2\pi}c_2\delta\left(k-2\right)\\ &= \sqrt{2\pi}\frac{i}{2}\delta\left(k+2\right) - \sqrt{2\pi}\frac{i}{2}\delta\left(k-2\right)\\ &= i\sqrt{\frac{\pi}{2}}\delta\left(k+2\right) - i\sqrt{\frac{\pi}{2}}\delta\left(k-2\right)\end{aligned}$$

(ii) $\cos^3 x$. Since this is periodic, then $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^3\left(x\right)e^{-inx}dx$. But $\cos^3\left(x\right) = \frac{1}{4}\cos\left(3x\right) + \frac{3}{4}\cos\left(x\right)$. Hence only $n=\pm 1, n=\pm 3$ will have coefficients and the rest are zero.

$$\begin{aligned}c_{-1} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{3}{4}\cos\left(x\right)e^{ix}dx = \frac{3}{8}\\ c_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{3}{4}\cos\left(x\right)e^{-ix}dx = \frac{3}{8}\\ c_{-3} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{4}\cos\left(3x\right)e^{-3ix}dx = \frac{1}{8}\\ c_3 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{4}\cos\left(3x\right)e^{-3ix}dx = \frac{1}{8}\end{aligned}$$

Therefore, using result from part (a)

$$\begin{aligned}
\hat{f}(k) &= \sqrt{2\pi}\sum_{n=-\infty}^{\infty} c_n \delta(k-n)\\
&= \sqrt{2\pi}\left(\frac{1}{8}\delta(k+3)+\frac{3}{8}\delta(k+1)+\frac{3}{8}\delta(k-1)+\frac{1}{8}\delta(k-3)\right)\\
&= \frac{1}{4}\sqrt{\frac{\pi}{2}}\left(\delta(k+3)+3\delta(k+1)+3\delta(k-1)+\delta(k-3)\right)
\end{aligned}$$

(iii) The 2π periodic extension of $f(x)=x$

Since this is periodic, then

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-inx}dx\\
&= \frac{2i}{n^2}\left(n\pi\cos(n\pi)-\sin(n\pi)\right)\\
&= \frac{2i}{n^2}\left(n\pi(-1)^n\right)\\
&= \frac{2i}{n}\pi(-1)^n
\end{aligned}$$

Therefore, using result from part (a)

$$\begin{aligned}
\hat{f}(k) &= \sqrt{2\pi}\sum_{n=-\infty}^{\infty} c_n\delta(k-n)\\
&= \sqrt{2\pi}\sum_{n=-\infty}^{\infty}\frac{2i}{n}\pi(-1)^n\delta(k-n)\\
&= 2i\pi\sqrt{2\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n}\delta(k-n) \qquad n\neq 0
\end{aligned}$$

(iv) The sawtooth function

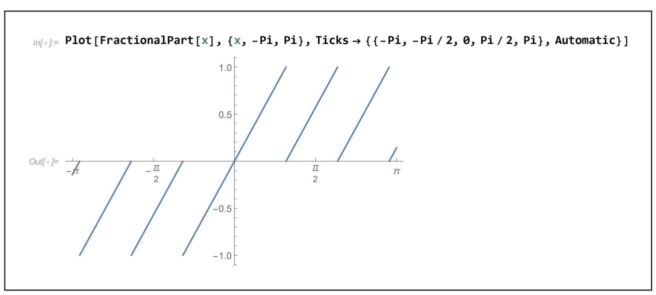


Figure 2.78: Plot of $f(x)$ (Fractional part of x)

2.10.9 Problem 7.3.4

Find a solution to the differential equation $-\frac{d^2u}{dx^2}+4u=\delta(x)$ by using the Fourier transform

Solution

Taking Fourier transform of both sides gives

$$
\begin{aligned}
-(ik)^2 \hat{u}(k) + 4\hat{u}(k) &= \mathscr{F}[\delta(x)] \\
k^2 \hat{u}(k) + 4\hat{u}(k) &= \frac{1}{\sqrt{2\pi}}
\end{aligned}
$$

Solving for $\hat{u}(k)$

$$
\begin{aligned}
\hat{u}(k)\left(k^2+4\right) &= \frac{1}{\sqrt{2\pi}} \\
\hat{u}(k) &= \frac{1}{\sqrt{2\pi}} \frac{1}{k^2+4}
\end{aligned}
$$

Finding inverse Fourier transform. From tables we see that $\mathscr{F}\left(e^{-a|x|}\right) = \sqrt{\frac{2}{\pi}} \frac{a}{k^2+a^2}$. Using $a = 2$

$$
\begin{aligned}
\mathscr{F}\left[e^{-2|x|}\right] &= \sqrt{\frac{2}{\pi}} \frac{2}{k^2+4} \\
\sqrt{\frac{\pi}{2}} \frac{1}{2} \mathscr{F}\left[e^{-2|x|}\right] &= \frac{1}{k^2+4} \\
\sqrt{\frac{\pi}{2}} \mathscr{F}\left[\frac{1}{2} e^{-2|x|}\right] &= \frac{1}{k^2+4} \\
\frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} \mathscr{F}\left[\frac{1}{2} e^{-2|x|}\right] &= \frac{1}{\sqrt{2\pi}} \frac{1}{k^2+4} \\
\frac{1}{2} \mathscr{F}\left[\frac{1}{2} e^{-2|x|}\right] &= \frac{1}{\sqrt{2\pi}} \frac{1}{k^2+4} \\
\mathscr{F}\left[\frac{1}{4} e^{-2|x|}\right] &= \frac{1}{\sqrt{2\pi}} \frac{1}{k^2+4}
\end{aligned}
$$

Therefore

$$
u(x) = \frac{1}{4} e^{-2|x|}
$$

2.10.10 Key solution for HW 10

Problem 1

Proof. 1. Choose $w \in \mathcal{A}$. Then (46) implies

$$0 = \int_U (-\Delta u - f)(u - w)\, dx.$$

An integration by parts yields

$$0 = \int_U Du \cdot D(u-w) - f(u-w)\, dx,$$

and there is no boundary term since $u - w = g - g \equiv 0$ on ∂U. Hence

$$\begin{aligned}\int_U |Du|^2 - uf\, dx &= \int_U Du \cdot Dw - wf\, dx \\ &\leq \int_U \frac{1}{2}|Du|^2\, dx + \int_U \frac{1}{2}|Dw|^2 - wf\, dx,\end{aligned}$$

where we employed the estimates

$$|Du \cdot Dw| \leq |Du||Dw| \leq \frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2,$$

following from the Cauchy–Schwarz and Cauchy inequalities (§B.2). Rearranging, we conclude

$$I[u] \leq I[w] \quad (w \in \mathcal{A}). \tag{48}$$

Since $u \in \mathcal{A}$, (47) follows from (48).

2. Now, conversely, suppose (47) holds. Fix any $v \in C_c^\infty(U)$ and write

$$i(\tau) := I[u + \tau v] \quad (\tau \in \mathbb{R}).$$

Since $u + \tau v \in \mathcal{A}$ for each τ, the scalar function $i(\cdot)$ has a minimum at zero, and thus

$$i'(0) = 0 \quad \left(' = \frac{d}{d\tau}\right),$$

provided this derivative exists. But

$$\begin{aligned} i(\tau) &= \int_U \frac{1}{2}|Du + \tau Dv|^2 - (u + \tau v)f\, dx \\ &= \int_U \frac{1}{2}|Du|^2 + \tau Du \cdot Dv + \frac{\tau^2}{2}|Dv|^2 - (u+\tau v)f\, dx.\end{aligned}$$

Consequently

$$0 = i'(0) = \int_U Du \cdot Dv - vf\, dx = \int_U (-\Delta u - f)v\, dx.$$

This identity is valid for each function $v \in C_c^\infty(U)$ and so $-\Delta u = f$ in U. □

Dirichlet's principle is an instance of the *calculus of variations* applied to Laplace's equation. See Chapter 8 for more.

We have already employed the maximum principle in §2.2.3 to show uniqueness, but now set forth a simple alternative proof. Assume U is open, bounded, and ∂U is C^1.

THEOREM 16 (Uniqueness). *There exists at most one solution $u \in C^2(\bar{U})$ of* (46).

Proof. Assume $\tilde{u}$ is another solution and set $w := u - \tilde{u}$. Then $\Delta w = 0$ in U, and so an integration by parts shows

$$0 = -\int_U w \Delta w \, dx = \int_U |Dw|^2 dx.$$

Thus $Dw \equiv 0$ in U, and, since $w = 0$ on ∂U, we deduce $w = u - \tilde{u} \equiv 0$ in U. □

b. Dirichlet's principle.

Next let us demonstrate that a solution of the boundary-value problem (46) for Poisson's equation can be characterized as the minimizer of an appropriate functional. For this, we define the *energy* functional

$$I[w] := \int_U \frac{1}{2}|Dw|^2 - wf \, dx,$$

w belonging to the *admissible set*

$$\mathcal{A} := \{w \in C^2(\bar{U}) \mid w = g \text{ on } \partial U\}.$$

THEOREM 17 (Dirichlet's principle). *Assume $u \in C^2(\bar{U})$ solves* (46). *Then*

$$I[u] = \min_{w \in \mathcal{A}} I[w]. \tag{47}$$

Conversely, if $u \in \mathcal{A}$ satisfies (47), *then u solves the boundary-value problem* (46).

In other words if $u \in \mathcal{A}$, the PDE $-\Delta u = f$ is equivalent to the statement that u minimizes the energy $I[\cdot]$.

7.1.1f

$$(f)\quad \frac{1}{\sqrt{2\pi}\,(-k^2+2\,\mathrm{i}\,k+2)} = \frac{-k^2-2\,\mathrm{i}\,k+2}{\sqrt{2\pi}\,(k^4+4)}.$$

7.1.3a,b

(a) By the Shift Theorem 7.4, $f(x) = \mathrm{i}\sqrt{\frac{\pi}{2}}\, e^{-\mathrm{i}\,a\,x}\,\operatorname{sign} x$.

(b) Using the Table, if $b > 0$, then $f(x) = \mathrm{i}\sqrt{2\pi}\, e^{b\,x}\,(\sigma(x)-1)$, while if $b < 0$, then $f(x) = \mathrm{i}\sqrt{2\pi}\, e^{b\,x}\,\sigma(x)$. For $b = 0$, use part (a).

7.1.13

Use the change of variables $\widehat{x} = x - \xi$ in the integral:

$$\begin{aligned}\mathcal{F}[\,f(x-\xi)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)\,e^{-\mathrm{i}\,k\,x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\widehat{x})\,e^{-\mathrm{i}\,k\,(\widehat{x}+\xi)}\,d\widehat{x} \\ &= \frac{e^{-\mathrm{i}\,k\,\xi}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\widehat{x})\,e^{-\mathrm{i}\,k\,\widehat{x}}\,d\widehat{x} = e^{-\mathrm{i}\,k\,\xi}\,\widehat{f}(k).\end{aligned}$$

To prove the second statement,

$$\mathcal{F}[e^{\mathrm{i}\,\kappa\,x}\,f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-\mathrm{i}\,(k-\kappa)\,x}\,dx = \widehat{f}(k-\kappa).$$

7.1.20 a): (i), (iii), and b)

$$.\ (a)\quad (i)\quad \frac{2}{\pi(k^2+1)(l^2+1)}, \qquad \star\ (iii)\quad \frac{e^{-\mathrm{i}\,(\xi\,k+\eta\,l)}}{2\pi},$$

7.2.2a

$$(a)\quad -\frac{\mathrm{i}}{k}\sqrt{\frac{2}{\pi}}\;e^{-k^2/4} + \sqrt{2\pi}\,\delta(k).$$

7.2.3d

$$\star\ (d)\quad -\frac{d^2}{dx^2}\left[\sqrt{2\pi}\,e^{-x}\sigma(x)\right] = \sqrt{2\pi}\left[-e^{-x}\sigma(x) + \delta(x) - \delta'(x)\right].$$

7.2.12

(a) Indeed, applying the inverse Fourier transform:

$$f(x) \sim \int_{-\infty}^{\infty} \widehat{f}(k)\,e^{\,i\,k\,x}\,dk = \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty} \delta(k-n)\,e^{\,i\,k\,x}\,dk = \sum_{n=-\infty}^{\infty} c_n e^{\,i\,k\,x}$$

recovers the complex Fourier series for $f(x)$, proving the result.

(b) (i) $\frac{1}{2}\,i\,\delta(x+2) - \frac{1}{2}\,i\,\delta(x-2)$, (iii) $i \displaystyle\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n}\,\delta(k-n)$.

7.3.4

. The Fourier transformed equation is $(k^2+4)\,\widehat{u}(k) = 1/\sqrt{2\pi}$, and hence a solution is $u(x) = \frac{1}{4}\,e^{-2\,|\,x\,|}$.

Chapter 3

Exams

Local contents

3.1 Exam 1, Oct 8, 2019

Local contents

3.1.1 Questions

MATH 5587 (FALL2019): MIDTERM 1

PROFESSOR: SVITLANA MAYBORODA

Name (legibly!): --

Problem 1 (20 points)

(a) Write down an explicit formula for the solution to the initial value problem:

$$\partial_t^2 u - 4\,\partial_x^2 u = 0, \quad u(0,x) = \sin x, \quad \partial_t u(0,x) = \cos x, \quad x \in \mathbb{R}, \quad t \geq 0.$$

(b) True or false: the solution is a periodic function of t. What is the period?

(c) Now solve the forced initial value problem

$$\partial_t^2 u - 4\,\partial_x^2 u = \cos t, \quad u(0,x) = \sin x, \quad \partial_t u(0,x) = \cos x, \quad x \in \mathbb{R}, \quad t \geq 0.$$

(d) True or false: the forced equation exhibits resonance. Explain.

Date: October 8, 2019.

1

Problem 2 (30 points)

(a) Find the (complex or real) Fourier series for the function $f(x) = x$

(b) Does the series converge to the same function, that is, $f(x) = x$, on $\mathbb{R}$? Whether the answer is yes or no, draw the function to which it converges below.

(c) Does the series converge (to the function you identified in (b)):

c1) pointwise? on which interval? Explain. Independently of your answer, write a definition of pointwise convergence.

c2) uniformly? on which interval? Explain. Independently of your answer, write a definition of uniform convergence.

(d) Use the results above to find the (complex or real) Fourier series of $f(x) = x^2$

Problem 3 (15 points)

(a) Write down an explicit formula for the solution to the initial value problem. Show your work.

$$\partial_t u + \frac{1}{x^2+4}\,\partial_x u = 0, \quad u(0,x) = e^{x^3+12x}$$

(b) graph either some of the characteristic curves or the solution to the initial value problem for several values of t. You do not have to do both, just indicate which one you are graphing.

3.2 Exam 2, Nov 7, 2019

Local contents

3.2.1 Questions

MATH 5587 (FALL2019): MIDTERM 2

PROFESSOR: SVITLANA MAYBORODA

Name (legibly!): --

Problem 1 (40 points)

A metal bar, of length $l = 1$ and thermal diffusivity $\gamma = 1$, is taken out of a 100° oven and fully insulated except for a left end which is fixed to a large ice cube, and hence, kept at the constant temperature 0° and the right end which is kept at temperature 50°.

(a) (5 pts) Write down an initial boundary value problem that describes the the temperature of the bar $u(t, x)$.

Equation: ------------------- for $x \in (0, 1)$, $t > 0$

Boundary data: ----------------------

Initial data: -------------------

(b) Use separation of variables to write a series formula for solution $u(t, x)$.

b1) (5 pts) Write $u(t, x) = v(t)w(x)$ and find the equations that v and w satisfies

b2) (10 pts) Solve the equations for v and w and use boundary data to identify possible solutions v, w

Date: November 7, 2019.

b3) (5 pts) Write the global solution $u(t, x)$ as a series and use initial data to write **integral formulas** for coefficients of the series.

b4) (5 pts) Evaluate explicitly coefficients of the series and write the final formula for solution $u(t, x)$

c) (5 pts) What is the equilibrium temperature (that is, the limit as $t \to \infty$) and how fast does the solution go to equilibrium?

Problem 2 (10 points) The solution to the Laplace's equation on a unit disc subject to Dirichlet boundary conditions $u(1,\theta) = h(\theta)$ is given by

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\,\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}\,d\phi.$$

Show that if u achieves its maximum at the center of the disc then u is constant on the entire disc.

Problem 3 (15 points) The solution to the Laplace's equation $\Delta u = 0$ on a square $0 < x < a$, $0 < y < b$, with boundary data

$$u(x,0) = f(x),\ u(x,b) = 0,\ u(0,y) = 0,\ u(a,y) = 0$$

is given by

$$u(x,y) = \sum_{n=1}^{\infty} b_n \frac{\sin\frac{n\pi x}{a}\,\sinh\frac{n\pi(b-y)}{a}}{\sinh\frac{n\pi b}{a}}$$

where $b_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$.

Show that whenever $\int_0^a |f(x)|\,dx$ is finite, the coefficients of the series in the formula for u,

$$b_n \frac{\sinh\frac{n\pi(b-y)}{a}}{\sinh\frac{n\pi b}{a}} \rightarrow 0$$

go to zero exponentially fast as $n \to \infty$.

Now assume that $a = b = 1$ and $f(x) = x$ when $x < 1/2$ and $f(x) = 1 - x$ when $x > 1/2$. What can you say about the smoothness of f? What can you say about the smoothness of the solution and why?

3.3 Final exam, Dec 10, 2019

Local contents

3.3.1 Questions

MATH 5587 (FALL 2019): FINAL

PROFESSOR: SVITLANA MAYBORODA

Name (legibly!): ----------------------------------

Problem 1 (35 points)

Write down the solution to the following initial-boundary value problem in the form of the Fourier series.

$u=X(x)T(t)$

$$u_{tt} = u_{xx} \quad u(t,0) = u(t,\pi) = 0, \quad u(0,x) = 0, \quad u_t(0,x) = 1$$

1.1) (5 pts) Write $u(t,x) = v(t)w(x)$ and find the equations that v and w satisfies

1.2) (5 pts) Solve the equations for v and w and use boundary data $u(t,0)$, $u(t,\pi)$, as well as $u(0,x)$ to identify possible solutions v, w

1.3) (5 pts) Write the global solution $u(t,x)$ as a series and use $u_t(0,x)$ to write integral formulas for coefficients of the series.

Date: December 10, 2019.

1

1.4) (5 pts) Evaluate explicitly coefficients of the series and write the final formula for solution $u(t, x)$

1.5) (15 pts) Now solve the same equation on an infinite interval (be careful to evaluate and write the solution legibly using cases depending on the values of x and t):

$$u_{tt} = u_{xx} \quad u(0, x) = 0 \text{ for all } x\,, \quad u_t(0, x) = 1 \text{ for } x < 0\,, \quad u_t(0, x) = 0 \text{ for } x > 0$$

Problem 2 (10 points) Find the Fourier transform of the Gaussian $h(x) = e^{-x^2}$ (you have to show your work, do not use a table value).

Problem 3 (20 points) (a) (10 pts) Find the Green function and write the solution using the Green function representation for the equation $-\frac{d^2u}{dx^2} + 4u = h(x)$.

(b) (10 pts) Now find the Green function and write the solution using the Green function representation for a similar equation on a finite interval:

$$-u''(x) = f(x) \quad \text{on } (0,1), \quad u(0) = 0, \quad u(1) = 2u'(1).$$

Concise Table of Fourier Transforms

$f(x)$	$\widehat{f}(k)$
1	$\sqrt{2\pi}\,\delta(k)$
$\delta(x)$	$\dfrac{1}{\sqrt{2\pi}}$
$\sigma(x)$	$\sqrt{\dfrac{\pi}{2}}\,\delta(k) - \dfrac{\mathrm{i}}{\sqrt{2\pi}\,k}$
$\operatorname{sign} x$	$-\mathrm{i}\sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{k}$
$\sigma(x+a) - \sigma(x-a)$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin a k}{k}$
$e^{-a x}\,\sigma(x)$	$\dfrac{1}{\sqrt{2\pi}\,(a + \mathrm{i}\,k)}$
$e^{a x}\,(1 - \sigma(x))$	$\dfrac{1}{\sqrt{2\pi}\,(a - \mathrm{i}\,k)}$
$e^{-a\,\lvert x \rvert}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{a}{k^2 + a^2}$
$e^{-a x^2}$	$\dfrac{e^{-k^2/(4a)}}{\sqrt{2a}}$
$\tan^{-1} x$	$-\mathrm{i}\sqrt{\dfrac{\pi}{2}}\,\dfrac{e^{-\lvert k \rvert}}{k}$
$f(cx + d)$	$\dfrac{e^{\mathrm{i}\,k d/c}}{\lvert c \rvert}\,\widehat{f}\left(\dfrac{k}{c}\right)$
$\overline{f(x)}$	$\overline{\widehat{f}(-k)}$
$\widehat{f}(x)$	$f(-k)$
$f'(x)$	$\mathrm{i}\,k\,\widehat{f}(k)$
$x\,f(x)$	$\mathrm{i}\,\widehat{f}'(k)$
$f * g(x)$	$\sqrt{2\pi}\,\widehat{f}(k)\,\widehat{g}(k)$

Note: The parameters a, c, d are real, with $a > 0$ and $c \neq 0$.

Chapter 4

Study and misc. items

Local contents

4.1 Few pages from Strauss W. PDE book to study

By Section 11.3, the eigenfunctions for all three problems (in D, in D_1, and in D_2) are complete. Among the eigenfunctions of $-\Delta$ in the rectangle D are the products $v_n w_m$. Suppose now that there were an eigenfunction $u(x, y)$ in the rectangle, other than these products. Then, for some λ, $-\Delta u = \lambda u$ in D and u would satisfy the boundary conditions. If λ were different from every one of the sums $\alpha_n + \beta_m$, then we would know (from Section 10.1) that u is orthogonal to all the products $v_n w_m$. Hence

$$0 = (u, v_n w_m) = \int \left[\int u(x, y) v_n(x)\, dx \right] w_m(y)\, dy. \tag{13}$$

So, by the completeness of the w_m,

$$\int u(x, y) v_n(x)\, dx = 0 \qquad \text{for all } y. \tag{14}$$

By the completeness of the v_n, (14) would imply that $u(x, y) = 0$ for all x, y. So $u(x, y)$ wasn't an eigenfunction after all.

One possibility remains, namely, that $\lambda = \alpha_n + \beta_m$ for certain n and m. This could be true for one pair m, n or several such pairs. If λ were such a sum, we would consider the difference

$$\psi(x, y) = u(x, y) - \sum c_{nm} v_n(x) w_m(y), \tag{15}$$

where the sum is over all the n, m pairs for which $\lambda = \alpha_n + \beta_m$ and where $c_{nm} = (u, v_n w_m)/\|v_n w_m\|^2$. The function ψ defined by (15) is constructed so as to be orthogonal to *all* the products $v_n w_m$, for both $\alpha_n + \beta_m = \lambda$ and $\alpha_n + \beta_m \neq \lambda$. It follows by the same reasoning as above that $\psi(x, y) \equiv 0$. Hence $u(x, y) = \Sigma\, c_{nm} v_n(x) w_m(y)$, summed over $\alpha_n + \beta_m = \lambda$. That is, u was *not* a new eigenfunction at all, but was just a linear combination of those old products $v_n w_m$ which have the same eigenvalue λ. This completes the proof of Theorem 2. □

EXERCISES

1. Verify that all the functions (7) are solutions of (1) if a is an eigenvalue λ_N and if $\int f v_N\, dx = 0$. Why does the series in (7) converge?
2. Use the completeness to show that the solutions of the wave equation in any domain with a standard set of BC satisfy the usual expansion $u(x, t) = \Sigma_{n=1}^{\infty} [A_n \cos(\sqrt{\lambda_n}\, ct) + B_n \sin(\sqrt{\lambda_n}\, ct)] v_n(x)$. In particular, show that the series converges in the L^2 sense.
3. Provide the details of the proof that $\psi(x, y)$, defined by (15), is identically zero.

11.6 ASYMPTOTICS OF THE EIGENVALUES

The main purpose of this section is to show that $\lambda_n \to +\infty$. In fact, we'll show exactly *how fast* the eigenvalues go to infinity. For the case of the Dirichlet boundary condition, the precise result is as follows.

Theorem 1. For a *two*-dimensional problem $-\Delta u = \lambda u$ in any plane domain D with $u = 0$ on bdy D, the eigenvalues satisfy the limit relation

$$\boxed{\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{A},} \tag{1}$$

where A is the *area* of D.

For a *three*-dimensional problem in any solid domain, the relation is

$$\boxed{\lim_{n\to\infty} \frac{\lambda_n^{3/2}}{n} = \frac{6\pi^2}{V},} \tag{2}$$

where V is the *volume* of D.

Example 1. The Interval

Let's compare Theorem 1 with the *one*-dimensional case where $\lambda_n = n^2\pi^2/l^2$. In that case,

$$\lim_{n\to\infty} \frac{\lambda_n^{1/2}}{n} = \frac{\pi}{l}, \tag{3}$$

where l is the length of the interval! The same result (3) was also derived for the one-dimensional Neumann condition in Section 4.2 and the Robin conditions in Section 4.3. □

Example 2. The Rectangle

Here the domain is $D = \{0 < x < a,\ 0 < y < b\}$ in the plane. We showed explicitly in Section 10.1 that

$$\lambda_n = \frac{l^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} \tag{4}$$

with the eigenfunction $\sin(l\pi x/a) \cdot \sin(m\pi y/b)$. Since the eigenvalues are naturally numbered using a pair of integer indices, it is difficult to see the relationship between (4) and (1). For this purpose it is convenient to introduce the *enumeration function*

$$N(\lambda) \equiv \textit{the number of eigenvalues that do not exceed } \lambda. \tag{5}$$

If the eigenvalues are written in increasing order as in (11.1.2), then $N(\lambda_n) = n$. Now we can express $N(\lambda)$ another way using (4). Namely, $N(\lambda)$ is the number of integer lattice points (l, m) which are contained within the quarter-ellipse

$$\frac{l^2}{a^2} + \frac{m^2}{b^2} \le \frac{\lambda}{\pi^2} \qquad (l > 0,\ m > 0) \tag{6}$$

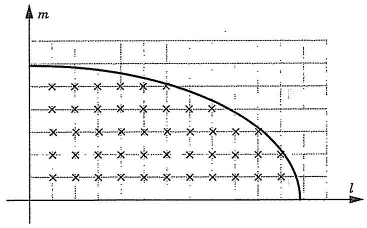


Figure 1

in the (l, m) plane (see Figure 1). Each such lattice point is the upper right corner of a square lying within the quarter ellipse. Therefore, $N(\lambda)$ is at most the area of this quarter ellipse:

$$N(\lambda) \le \frac{\lambda ab}{4\pi}. \tag{7}$$

For large λ, $N(\lambda)$ and this area may differ by approximately the length of the perimeter, which is of the order $\sqrt{\lambda}$. Precisely,

$$\frac{\lambda ab}{4\pi} - C\sqrt{\lambda} \le N(\lambda) \le \frac{\lambda ab}{4\pi} \tag{8}$$

for some constant C. Substituting $\lambda = \lambda_n$ and $N(\lambda) = n$, (8) takes the form

$$\frac{\lambda_n ab}{4\pi} - C\sqrt{\lambda_n} \le n \le \frac{\lambda_n ab}{4\pi}, \tag{9}$$

where the constant C does not depend on n. Therefore, upon dividing by n, we deduce that

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{ab}, \tag{10}$$

which is Theorem 1 for a rectangle. □

For the Neumann condition, the only difference is that l and m are allowed to be zero, but the result is exactly the same:

$$\lim_{n\to\infty} \frac{\tilde{\lambda}_n}{n} = \frac{4\pi}{ab}. \tag{11}$$

To prove Theorem 1, we will need the maximin principle. It is like the minimum principle of Section 11.1 but with more general constraints. The idea is that *any* orthogonality constraints *other* than those in Section 11.1 will lead to smaller minimum values of the Rayleigh quotient.

Theorem 2. **Maximin Principle** Fix a positive integer $n \ge 2$. Fix $n-1$ arbitrary trial functions $y_1(\mathbf{x}), \ldots, y_{n-1}(\mathbf{x})$. Let

$$\boxed{\lambda_{n*} = \min \frac{\|\nabla w\|^2}{\|w\|^2}} \tag{12}$$

among all trial functions w that are orthogonal to $y_1, \ldots, y_{n-1}$. Then

$$\boxed{\lambda_n = \max \lambda_{n*}} \tag{13}$$

over *all* choices of the $n-1$ trial functions $y_1, \ldots, y_{n-1}$.

Proof. Fix an arbitrary choice of $y_1, \ldots, y_{n-1}$. Let $w(\mathbf{x}) = \Sigma_{j=1}^n c_j v_j(\mathbf{x})$ be a linear combination of the first n eigenfunctions which is chosen to be orthogonal to $y_1, \ldots, y_{n-1}$. That is, the constants $c_1, \ldots, c_n$ are chosen to satisfy the linear system

$$0 = \left(\sum_{j=1}^n c_j v_j, y_k\right) = \sum_{j=1}^n (v_j, y_k) c_j \qquad (\text{for } k = 1, \ldots, n-1).$$

Being a system of only $n-1$ equations in n unknowns, it has a solution $c_1, \ldots, c_n$, not all of which constants are zero. Then, by definition (12) of λ_{n*},

$$\begin{aligned} \lambda_{n*} &\le \frac{\|\nabla w\|^2}{\|w\|^2} = \frac{\Sigma_{j,k}\, c_j c_k(-\Delta v_j, v_k)}{\Sigma_{j,k}\, c_j c_k(v_j, v_k)} \\ &= \frac{\Sigma_{j=1}^n \lambda_j c_j^2}{\Sigma_{j=1}^n c_j^2} \le \frac{\Sigma_{j=1}^n \lambda_n c_j^2}{\Sigma_{j=1}^n c_j^2} = \lambda_n, \end{aligned} \tag{14}$$

where we've again taken $\|v_j\| = 1$. This inequality (14) is true for every choice of $y_1, \ldots, y_{n-1}$. Hence, $\max \lambda_{n*} \le \lambda_n$. This proves half of (13).

To demonstrate the equality in (13), we need only exhibit a special choice of $y_1, \ldots, y_{n-1}$ for which $\lambda_{n*} = \lambda_n$. Our special choice is the first $n-1$ eigenfunctions: $y_1 = v_1, \ldots, y_{n-1} = v_{n-1}$. By the minimum principle for the nth eigenvalue in Section 11.1, we know that

$$\lambda_{n*} = \lambda_n \qquad \text{for this choice.} \tag{15}$$

The maximin principle (13) follows directly from (14) and (15). □

The same maximin principle is also valid for the *Neumann boundary condition* if we use the "free" trial functions that don't satisfy any boundary condition. Let's denote the Neumann eigenvalues by $\tilde{\lambda}_j$. Now we shall simultaneously consider the Neumann and Dirichlet cases.

Theorem 3. $\tilde{\lambda}_j \le \lambda_j$ for all $j = 1, 2, \ldots$.

Proof. Let's begin with the first eigenvalues. By Theorems 11.1.1 and 11.3.1, both $\tilde{\lambda}_1$ and λ_1 are expressed as the same minimum of the Rayleigh

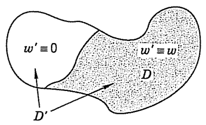


Figure 2

quotient except that the test functions for λ_1 satisfy one extra constraint (namely, that $w = 0$ on bdy D). Having one less constraint, $\tilde{\lambda}_1$ has a greater chance of being small. Thus $\tilde{\lambda}_1 \le \lambda_1$.

Now let $n \ge 2$. For the same reason of having one extra constraint, we have

$$\tilde{\lambda}_{n*} \le \lambda_{n*}. \tag{16}$$

We take the maximum of both sides of (16) over all choices of trial functions $y_1, \ldots, y_{n-1}$. By the maximin principle of this section (Theorem 2 and its Neumann analog), we have

$$\tilde{\lambda}_n = \max \tilde{\lambda}_{n*} \le \max \lambda_{n*} = \lambda_n. \qquad \square$$

Example 3.

For the interval $(0, l)$ in one dimension, the eigenvalues are $\lambda_n = n^2\pi^2/l^2$ and $\tilde{\lambda}_n = (n-1)^2\pi^2/l^2$ (using our present notation with n running from 1 to ∞). It is obvious that $\tilde{\lambda}_n < \lambda_n$. □

The general principle which is illustrated by Theorem 3 is that

any additional constraint will increase the value of the maximin. (17)

In particular, we can use this principle as follows to prove the monotonicity of the eigenvalues with respect to the domain.

Theorem 4. If the domain is enlarged, each eigenvalue is decreased.

That is, if one domain D is contained in another domain D', then $\lambda_n \ge \lambda'_n$ and $\tilde{\lambda}_n \ge \tilde{\lambda}'_n$, where we use primes on eigenvalues to refer to the larger domain D' (see Figure 2).

Proof. In the Dirichlet case, consider the maximin expression (13) for D. If $w(\mathbf{x})$ is any trial function in D, we define $w(\mathbf{x})$ in all of D' by setting it equal to zero outside D; that is,

$$w'(\mathbf{x}) = \begin{cases} w(\mathbf{x}) & \text{for } \mathbf{x} \text{ in } D \\ 0 & \text{for } \mathbf{x} \text{ in } D' \text{ but } \mathbf{x} \text{ not in } D. \end{cases} \tag{18}$$

Thus every trial function in D corresponds to a trial function in D' (but not conversely). So, compared to the trial functions for D', the trial functions for D have the extra constraint of vanishing in the rest of D'. By the general principle (17), the maximin for D is larger than the maximin for D'. It follows that $\lambda_n \ge \lambda'_n$, as we wanted to prove. But we should beware that we are avoiding the difficulty that by extending the function to be zero, it is most likely no longer a C^2 function and therefore not a trial function. The good thing about the extended function $w'(\mathbf{x})$ is that it still is continuous. For a rigorous justification of this point, see [CH] or [Ga].

The same kind of reasoning is valid in the Neumann case. Indeed, the maximin principle for the Neumann boundary condition states that

$$\tilde{\lambda}_n = \max \tilde{\lambda}_{n*} \qquad \text{where } \tilde{\lambda}_{n*} = \min \frac{\|\nabla w\|^2}{\|w\|^2} \tag{19}$$

and the competing trial functions $w(\mathbf{x})$ do not satisfy any boundary condition at all. As above, these test functions on D may be extended to the larger domain D' by setting them equal to zero outside D. In this case, the new trial functions $w'(\mathbf{x})$ may be *discontinuous* at the part of the boundary of D which is internal to D' (see Figure 1.) But in any case there are again more trial functions for D' than for D. That is, the maximin for D has more constraints, so that $\tilde{\lambda}_n \ge \tilde{\lambda}'_n$. Again see [CH] for a complete proof. □

SUBDOMAINS

Our next step in establishing Theorem 1 is to *divide the general domain D into a finite number of subdomains* $D_1, \ldots, D_m$ by introducing inside D a system of smooth surfaces $S_1, S_2, \ldots$ (see Figure 3). Let D have Dirichlet eigenvalues $\lambda_1 \le \lambda_2 \le \cdots$ and Neumann eigenvalues $\tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \cdots$. Each of the subdomains $D_1, \ldots, D_m$ has its own collection of eigenvalues. We combine *all* of the Dirichlet eigenvalues of *all* of the subdomains $D_1, \ldots, D_m$ into a single increasing sequence $\mu_1 \le \mu_2 \le \cdots$. We combine *all* of their Neumann eigenvalues into another single increasing sequence $\tilde{\mu}_1 \le \tilde{\mu}_2 \le \cdots$.

By the maximin principle, each of these numbers can be obtained as the maximum over trial functions $y_1, \ldots, y_{n-1}$ of the minimum over trial functions w orthogonal to $y_1, \ldots, y_{n-1}$. As discussed above, although each μ_n is a Dirichlet eigenvalue of a single one of the subdomains, the trial functions can be defined in all of D simply by making them vanish in the other subdomains. Thus each of the competing trial functions for μ_n has the extra restriction, compared with the trial functions for λ_n for D, of vanishing on the internal boundaries. It follows from the general principle (17) that

$$\lambda_n \le \mu_n \qquad \text{for each } n = 1, 2, \ldots. \tag{20}$$

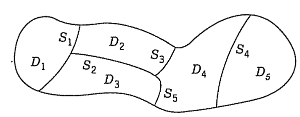


Figure 3

On the other hand, the trial functions defining $\tilde{\lambda}_n$ for the Neumann problem in D are arbitrary C^2 functions. As above, we can characterize $\tilde{\mu}_n$ as

$$\tilde{\mu}_n = \max \tilde{\mu}_{n*} \qquad \tilde{\mu}_{n*} = \min \frac{\|\nabla w\|^2}{\|w\|^2}, \tag{21}$$

where the competing trial functions are arbitrary on each subdomain and orthogonal to $y_1, \ldots, y_{n-1}$. But these trial functions *are allowed to be discontinuous on the internal boundaries,* so they comprise a significantly more extensive class than the trial functions for $\tilde{\lambda}_n$, which required to be continuous in D. Therefore, by (17) we have $\tilde{\mu}_n \le \tilde{\lambda}_n$ for each n. Combining this with Theorem 3 and (20), we have proved the following inequalities.

Theorem 5.

$$\boxed{\tilde{\mu}_n \le \tilde{\lambda}_n \le \lambda_n \le \mu_n.}$$

Example 4.

Let D be the union of a finite number of rectangles $D = D_1 \cup D_2 \cup \cdots$ in the plane as in Figure 4. Each particular μ_n corresponds to one of these rectangles, say D_p (where p depends on n). Let $A(D_p)$ denote the area of D_p. Let $M(\lambda)$ be the enumeration function for the sequence $\mu_1, \mu_2, \ldots$ defined above:

$$M(\lambda) \equiv \text{the number of } \mu_1, \mu_2, \ldots \text{ that do not exceed } \lambda. \tag{22}$$

Then, adding up the integer lattice points which are located within D, we get

$$\lim_{\lambda\to\infty} \frac{M(\lambda)}{\lambda} = \sum_p \frac{A(D_p)}{4\pi} = \frac{A(D)}{4\pi}, \tag{23}$$

as for the case of a single rectangle. Since $M(\mu_n) = n$, the reciprocal of (23) takes the form

$$\lim_{n\to\infty} \frac{\mu_n}{n} = \frac{4\pi}{A(D)}. \tag{24}$$

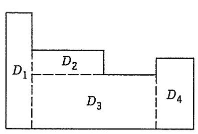


Figure 4

Similarly,

$$\lim_{n\to\infty} \frac{\tilde{\mu}_n}{n} = \frac{4\pi}{A(D)}. \tag{25}$$

By Theorem 5 it follows that all the limits are equal: $\lim \lambda_n/n = \lim \tilde{\lambda}_n/n = 4\pi/A(D)$. This proves Theorem 1 for unions of rectangles. □

Now an arbitrary plane domain D can be approximated by unions of rectangles just as in the construction of a double integral (and as in Section 8.4). With the help of Theorem 5, it is possible to prove Theorem 1. The details are omitted but the proof may be found in [CH].

THREE DIMENSIONS

The *three-dimensional case* works the same way. We limit ourselves, however, to the basic example.

Example 5. The Rectangular Box

Let $D = \{0 < x < a, 0 < y < b, 0 < z < c\}$. As in Example 2, the enumeration function $N(\lambda)$ is approximately the volume of the ellipsoid

$$\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{k^2}{c^2} \le \frac{\lambda}{\pi^2}$$

in the first octant. Thus for large λ

$$\begin{aligned} N(\lambda) &\sim \frac{1}{8}\,\frac{4\pi}{3}\,\frac{a\lambda^{1/2}}{\pi}\,\frac{b\lambda^{1/2}}{\pi}\,\frac{c\lambda^{1/2}}{\pi} \\ &= \lambda^{3/2}\frac{abc}{6\pi^2} \end{aligned} \tag{26}$$

and the same for the Neumann case. Substituting $\lambda = \lambda_n$ and $N(\lambda) = n$, we deduce that

$$\lim_{n\to\infty} \frac{\lambda_n^{3/2}}{n} = \frac{6\pi^2}{abc} = \lim_{n\to\infty} \frac{\tilde{\lambda}_n^{3/2}}{n}. \tag{27}$$

For the union of a finite number of boxes of volume V(D), we deduce that

$$\lim_{n\to\infty} \frac{\lambda_n^{3/2}}{n} = \frac{6\pi^2}{V(D)} = \lim_{n\to\infty} \frac{\tilde{\lambda}_n^{3/2}}{n}.$$

Then a general domain is approximated by unions of boxes. □

For the very general case of a *symmetric differential operator* as (11.4.1), the statement of the theorem is modified (in three dimensions, say) to read

$$\lim_{n\to\infty} \frac{\lambda_n^{3/2}}{n} = \lim_{n\to\infty} \frac{\tilde{\lambda}_n^{3/2}}{n} = \frac{6\pi^2}{\iiint_D [m(\mathbf{x})/p(\mathbf{x})]^{3/2}\, d\mathbf{x}}. \tag{28}$$

EXERCISES

1. Prove that (9) implies (10).
2. (a) For a circular drumhead (D = disk), verify Theorem 1 directly from Section 10.2 and the properties of Bessel functions.
   (b) Do the same in the Neumann case.
3. (a) For a spherical ball, verify Theorem 1 directly from Section 10.3 and the properties of Bessel functions.
   (b) Do the same in the Neumann case.
4. Explain how it is possible that λ_2 is *both* a maximin *and* a minimax.
5. For $-\Delta$ in the *ellipsoid* $D = \{x^2 + y^2/4 < 1\}$ with Dirichlet BCs use the monotonicity of the eigenvalues with respect to the domain to find estimates for the first two eigenvalues. Inscribe or circumscribe rectangles or circles, for which we already know the exact values.
   (a) Find upper bounds.
   (b) Find lower bounds.
6. In the proof of Theorem 1 for an arbitrary domain D, one must approximate D by unions of rectangles. This is a delicate limiting procedure. Outline the main steps required to carry out the proof.
7. Use the surface area of an ellipsoid to write the inequalities that make (26) a more precise statement.
8. For a symmetric differential operator in three dimensions as in (11.4.1), explain why Theorem 1 should be modified to be (28).
9. Consider the Dirichlet BCs in a domain D. Show that the first eigenfunction $v_1(\mathbf{x})$ *vanishes at no point* of D by the following method.
   (a) Suppose on the contrary that $v_1(\mathbf{x}) = 0$ at some point in D. Show that both $D^+ = \{\mathbf{x} \in D : v_1(\mathbf{x}) > 0\}$ and $D^- = \{\mathbf{x} \in D : v_1(\mathbf{x}) < 0\}$ are nonempty. (*Hint:* Use the maximum principle in Exercise 7.4.26.)
   (b) Let $v^+(\mathbf{x}) = v_1(\mathbf{x})$ for $\mathbf{x} \in D^+$ and $v^+(\mathbf{x}) = 0$ for $\mathbf{x} \in D^-$. Let $v^- = v_1 - v^+$. Notice that $|v_1| = v^+ - v^-$. Noting that $v_1 = 0$ on bdy D, we may deduce that $\nabla v^+ = \nabla v_1$ in D, and $\nabla v^+ = 0$ outside D. Similarly for ∇v^-. Show that the Rayleigh quotient Q for the function $|v_1|$ is equal to λ_1. Therefore, both v_1 and $|v_1|$ are eigenfunctions with the eigenvalue λ_1.
   (c) Use the maximum principle on $|v_1|$ to show that $v_1 > 0$ in all of D or $v_1 < 0$ in all of D.
   (d) Deduce that λ_1 is a simple eigenvalue (*Hint:* If $u(x)$ were another eigenfunction with eigenvalue λ_1, let w be the component of u orthogonal to v_1. Applying part (c) to w, we know that $w > 0$ or $w < 0$ or $w \equiv 0$ in D. Conclude that $w \equiv 0$ in D.)

10. Show that the nodes of the nth eigenfunction $v_n(\mathbf{x})$ divide the domain D into *at most n* pieces, assuming (for simplicity) that the eigenvalues are distinct, by the following method. Assume Dirichlet BCs.
    (a) Suppose on the contrary that $\{\mathbf{x} \in D : v_n(\mathbf{x}) \neq 0\}$ has at least $n+1$ disconnected components $D_1 \cup D_2 \cup \cdots \cup D_{n+1}$. Let $w_j(\mathbf{x}) = v_n(\mathbf{x})$ for $\mathbf{x} \in D_j$, and $w_j(\mathbf{x}) = 0$ elsewhere. You may assume that $\nabla w_j(\mathbf{x}) = \nabla v_n(\mathbf{x})$ for $\mathbf{x} \in D_j$, and $\nabla w_j(\mathbf{x}) = 0$ elsewhere. Show that the Rayleigh quotient for w_j equals λ_n.
    (b) Show that the Rayleigh quotient for any linear combination $w = c_1 w_1 + \cdots + c_{n+1} w_{n+1}$ also equals λ_n.
    (c) Let $y_1, \ldots, y_n$ be any trial functions. Choose the $n+1$ coefficients c_j so that w is orthogonal to each of $y_1, \ldots, y_n$. Use the maximin principle to deduce that $\lambda_{n+1} \leq \|\nabla w\|^2/\|w\|^2 = \lambda_n$. Hence deduce that $\lambda_{n+1} = \lambda_n$, which contradicts our assumption.

4.2 Study notes, cheat sheet

4.2.1 Linear and Nonlinear Waves (Chapter 2)

stationary waves such as $u_t + 3u = 0$

Transport and Traveling Waves such as $u_t + cu_x = u$. Uniform transport. Speed c is constant. Characteristics are $\frac{dx}{dt} = c$. When speed is not constant, we get Nonuniform Transport. Characteristics is $\frac{dx}{dt} = c(x)$. Nonlinear Transport: $u_t + u\, u_x = 0$ where wave speed depends not on position x but on u itself.

d'Almbert

$$u(x,t) = \frac{1}{2}\Big(f(x-ct) + f(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

With extranl force $u_{tt} = c^2 u_{xx} + F(x,t)$ we add the term $\frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} F(s,y)\,dy\right) ds$. The limits are the same as above, but replace t by $t-s$. remeber ds goes with t and dy goes with x.

4.2.2 Fourier series (Chapter 3)

Just need to know the F.S. definition. Either complex one or standard.

4.2.3 Seperation of variables (Chapter 4)

Theorem 4.2. If $u(t,x)$ is a solution to the heat equation with piecewise continuous initial data $f(x) = u(t0,x)$, or, more generally, initial data satisfying (4.27), then, for any $t > t_0$, the solution $u(t,x)$ is an infinitely differentiable function of x. (page 128) .

"In other words, the heat equation instantaneously smoothes out any discontinuities and corners in the initial temperature profile by fast damping of the high-frequency modes"

Heat PDE in 1D.

Inhomogeneous Boundary Conditions convert to homogeneous by using reference function.

Wave PDE in 1D. Fixed ends. d'Alembert Formula for Bounded Intervals: For Dirichlet do odd extension of initial position. For Neumann (free) boundary conditions, do even extension.

Laplace PDE on disk and on recrangle. in polar Laplace becomes $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$. When doing seperations, rememebr to use the angular ODE for finding the eigenvalues first. The radial ODE becomes Euler ODE. Solve using assuming $R(r) = r^k$. For disk, the solution is $u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n \left(A_n \cos(n\theta + B_n \sin n\theta)\right)$

Laplace PDE maximum principle. Lots of theorem here.

Characteristics and the Cauchy Problem see HW 7, Problem 4.4.16. This is for second order pde. Write pde as $Au_{xx} + Bu_{xy} + Cu_{yy} = G$ and then Characteristics is $A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0$. This gives ode $\frac{dy}{dx}$ which is the Characteristics.

Laplacian in 3D with no angle dependencty is $u_{rr} + \frac{2}{r}u_r = 0$

4.2.4 Generalized functions and Green function (Chapter 6)

$\delta(x-\xi)$: "In general, a unit impulse at position $a < \xi < b$ will be described by something called the delta function".

Two ways to define $\delta(x-\xi)$. one based on limit of function as $n \to \infty$ and one based on how it acts inside integral. For limit, use this one

$$g_n(x) = \frac{n}{\pi\left(1 + n^2x^2\right)}$$

Then $\lim_{n\to\infty} g_n(x) = \delta_0(x)$. And the above also meets the integral relation $\int_{-\infty}^{\infty} g_n(x)\,dx = \frac{1}{\pi}\left[\arctan(nx)\right]_{-\infty}^{\infty} = 1$.

For calculus, remember this: When taking derivative of a function with jump discontitty, we get an impulse at location of the jump with magnitude of the jump. Direction is negative if the jump is down and positive if the jump is up, this is when moving from left to right. For example derivative of unit step is $\delta(x)$. And the integral of $\delta(x)$ is unit step (or 1). Hence if $f(x) = g(x) + \sigma(x)$ where $\sigma(x)$ is unit step and $g(x)$ is continuous everywhere, then $f'(x) = g'(x) + \delta(x)$

Fourier series of $\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi}(\cos x + \cos 2x + \cos 3x + \cdots)$

Green function for 1D boundary value problems.

Remember when satisfying the jump discontinuity, it is $A + \frac{1}{p} = B$ where p is one which matches when the ODE is written as $py'' + q(x)y' + ry = f(x)$ in the original ODE. And A is the top term and B is the bottom term, as is

$$\left[\frac{d}{dx}G(x;\xi)\right]_{x=\xi} = \begin{cases} A & x < \xi \\ B & x > \xi \end{cases}$$

So the second equation is

$$A + \frac{1}{p} = B$$

That is really the only tricky part in finding Green function. Getting the sign right here. So if the ODE is $-cy'' = f(x)$ then here $p = -c$ (notice, sign is negative, i.e. $p = -c$ including the sign) and the jump is $\frac{1}{p} = \frac{1}{-c} = -\frac{1}{c}$ and hence the equation becomes

$$A + \frac{1}{p} = B$$
$$A - \frac{1}{c} = B$$

And if the ODE is given as $cy'' = f(x)$ then $p = c$ and the equation becomes

$$A + \frac{1}{p} = B$$
$$A + \frac{1}{c} = B$$

"Thus, the Neumann boundary value problem admits a solution if and only if there is no net force on the bar." (page 239). This means $-u'' = f(x)$ with $u'(0) = 0 = u'(1)$ has Green function and solution if $\int_0^1 f(x)\,dx = 0$. If this holds, the $-u'' = f(x)$ has solution (but the solution is not unique) and any constant value is a solution.

Green function for Laplace $-\Delta u = f(x,y)$

Some relations: $\nabla \cdot \nabla u = \Delta u = u_{xx} + u_{yy}$. i.e. divergence of the gradient of u is Laplacian of u. Green function in full space for Laplacian in 2D is

$$G(x,y;\xi,\eta) = \frac{-1}{2\pi}\ln r$$

where r is distance from (x,y) to where the pulse is (ξ,η), i.e. $\sqrt{(x-\xi)^2 + (y-\eta)^2}$. In 3D, it is $\frac{1}{4\pi r}$.

Method of images To find $G(x,y;\xi,\eta)$ in say upper half, put a negative pulse at $(\xi,-\eta)$ and then use $G_{upper}(x,y;\xi,\eta) = G_{full}(x,y;\xi,\eta) - G_{full}(x,y;\xi,-\eta)$

For disk

$$G(\boldsymbol{x};\boldsymbol{\xi}) = \frac{1}{2\pi}\ln\left(\frac{\left\|\|\boldsymbol{\xi}\|^2\boldsymbol{x} - \boldsymbol{\xi}\right\|}{\|\boldsymbol{\xi}\|\,\|\boldsymbol{x} - \boldsymbol{\xi}\|}\right)$$

In polar it becomes

$$G(r,\theta;\rho,\phi) = \frac{1}{4\pi}\ln\left(\frac{1 + r^2\rho^2 - \beta}{r^2 + \rho^2 - \beta}\right)$$

Where $\beta = 2r\rho\cos(\theta - \phi)$ where (r,θ) is variable point and pulse fixed at (ρ,ϕ), all using polar coordinates.

4.2.5 Fourier transform (chapter 7)

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx} dx$$
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx} dk$$

Table of Fourier transforms on page 272 will be given in exam also. Remember the shift property

$$\hat{f}(k-a) \Leftrightarrow e^{iax} f(x)$$
$$f(x-a) \Leftrightarrow e^{-iax} \hat{f}(k)$$

Gaussian integrals, for any b is

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$
$$\int_{-\infty}^{\infty} e^{-(x+b)^2} dx = \sqrt{\pi}$$
$$\int_{-\infty}^{\infty} e^{-(x-b)^2} dx = \sqrt{\pi}$$
$$\int_{-\infty}^{\infty} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}} \qquad a > 0$$
$$\int_{-\infty}^{\infty} e^{-a(x-b)^2} dx = \sqrt{\frac{\pi}{a}} \qquad a > 0$$

Derivative and integrals

$$f(x) \Leftrightarrow \hat{f}(k)$$
$$f'(x) \Leftrightarrow (ik)\, \hat{f}(k)$$
$$f''(x) \Leftrightarrow (ik)^2\, \hat{f}(k) = -k^2 \hat{f}(k)$$

Remember this also $x f(x) \Leftrightarrow i\frac{d\hat{f}(k)}{dk}$. On smoothness of $f(x)$ and relation to decay of $\hat{f}(k)$. see book page 276 "the smoothness of the function $f(x)$ is manifested in the rate of decay of its Fourier transform $f(k)$." and "Thus, the smoother $f(x)$, the more rapid the decay of its Fourier transform" and "This result can be viewed as the Fourier transform version of the Riemann–Lebesgue Lemma 3.46.)"

Integration

$$\int_{-\infty}^{x} f(x)\, dx \Leftrightarrow \frac{1}{ik} \hat{f}(k) + \pi \hat{f}(0)\, \delta(k)$$

Easy to remember when comparing it to $f'(x) \Leftrightarrow (ik)\, \hat{f}(k)$. Just change (ik) from numerator to denominator and add $\pi \hat{f}(0)\, \delta(k)$.

In context of generalized functions, we write

$$\int_{-\infty}^{\infty} f(x)\, dx = \sqrt{2\pi} \hat{f}(0)$$

So if we know the F.T. of $f(x)$ we do the above integration by using the above relation directly by evaluating $\hat{f}(k)$ at $k=0$. This can be handy. For example let us apply this to the Gaussian,. $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{2\pi}\hat{f}(0)$ where $\hat{f}(k) = \mathscr{F}\left(e^{-x^2}\right) = \frac{1}{\sqrt{2}} e^{-\frac{k^2}{4}}$. Hence $\hat{f}(0) = \frac{1}{\sqrt{2}}$ and therefore $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{2\pi}\frac{1}{\sqrt{2}} = \sqrt{\pi}$

Green function

Using F.T, to find Green function. Used only for infinite space. Put a $\delta_y(x)$ in RHS, solve for $\hat{G}\left(y,t\right)$ then find the inverse Fourier transform to get $G(x,t)$. For example for heat pde.

Weyl's law for eigenvalues convergence for large n. For 2D

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{A}$$

Where here $\lambda_n = \frac{l^2\pi^2}{a^2} + \frac{k^2\pi^2}{b^2}, l = 1,2,3,\cdots, k = 1,2,3,\cdots$. So λ_n are sorted in order. This is for reactangle with width a and high b.