my study notes

Maths 4512
Differential Equations with Applications

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Contents

1 Population models (Section 1.5 in book)

■ The most basic model is call Malthusian model given by $\frac{dp}{dt} = ap(t)$ which says that rate of change of population is proportional to current population size. a is constant. The solution is $p(t) = p_0 e^{a(t-t_0)}$. Where $p(t)$ is population at time t and p_0 is initial population at time t_0. This model is OK when population is small. A better model is called logistic model given by

$$\frac{dp}{dt} = ap(t) - bp^2(t)$$
$$p(t_0) = p_0$$

Where b is the competition factor. Also constant and positive. It is much smaller than a. The solution to the above is

$$p(t) = \frac{ap_0}{bp_0 + \left(a - bp_0\right)e^{-a(t-t_0)}} \tag{1}$$

In this model, we are normally given p_0 and given $(t - t_0)$ and given what is called the limiting value $\frac{a}{b}$ which is $\lim_{t\to\infty} p(t)$. Then asked to find population $p(t)$ after sometime. This will be $(t - t_0)$. We need to find a. Once we find a, then we find b from the limiting value. The trick is to find a. To do this, we first use (1) from the information given. The problem will always say that the population doubles every so many years, or the population increases at rate of some percentage per year and so on. Use this to find a from (1). Now we know b. Then use (1) again now to find the population at some future time as the problem says. See HW1, last problem for an example.

■ If a problem says substance decays exponentially, this means $M(t) = M_0 e^{-Ct}$. where $C > 0$. Need to find C from other problem information. Typically problem gives half life to do this. For example, see problem section 1.8, problem 3. It says:

substance x decays exponentially, and only half of the given quantity remains after 2 years. How long it takes for 5 lb decay to 1 lb? Solution is

$$M = M_0 e^{-Ct}$$

After 2 years, $M = \frac{M_0}{2}$, hence $\frac{M_0}{2} = M_0 e^{-2C}$. Hence $\frac{1}{2} = e^{-2C}$ or $\ln\left(\frac{1}{2}\right) = -2C$, hence $C = -\frac{1}{2}\ln\left(\frac{1}{2}\right) = \frac{1}{2}\ln(2)$. Now we know C, we can finish the solution.

$$\begin{aligned}
M &= M_0 e^{-\frac{1}{2}\ln(2)t} \\
1 &= 5e^{-\frac{1}{2}\ln(2)t} \\
\frac{1}{5} &= e^{-\frac{1}{2}\ln(2)t} \\
\ln\left(\frac{1}{5}\right) &= -\frac{1}{2}\ln\left(\frac{1}{5}\right)t \\
t &= -2\frac{\ln\left(\frac{1}{5}\right)}{\ln(2)} \\
&= 2\frac{\ln 5}{\ln 2} \\
&= 4.643 \text{ years}
\end{aligned}$$

If it says it grows exponentially, then $M = M_0 e^{Ct}$ instead.

2 Mixing problems (Section 1.8(b) in book)

The main idea is to set an ODE using $\frac{dS(t)}{dt} = R_{in} - R_{out}$ where R_{in} is rate of mass of salt coming into the tank and R_{out} is rate of mass of salt leaving tank. This gives an ODE to solve for $S(t)$ using initial conditions which is given. At end, divide by volume of tank to get concentration at time t. See book example at page 54.

3 Example 1, page 369

Book solution for example 1 is wrong. So I typed corrected solution.

Solve $\dot{\boldsymbol{x}} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} e^t \\ e^t \end{pmatrix}$ with $\boldsymbol{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

<u>Solution</u>

$$
\begin{aligned}
(sI - A)\,\boldsymbol{X}(s) &= \boldsymbol{F}(s) + \boldsymbol{x}(0) \\
\left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\right] \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} &= \begin{pmatrix} \frac{1}{s-1} \\ \frac{1}{s-1} \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
\begin{pmatrix} s-1 & -4 \\ -1 & s-1 \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} &= \begin{pmatrix} \frac{1}{s-1} + 2 \\ \frac{1}{s-1} + 1 \end{pmatrix}
\end{aligned}
$$

Multiplying the second row by $(s-1)$ and adding the result to the first row to obtain Gaussian elimination. First multiplying second row by $(s-1)$ gives

$$
\begin{aligned}
\begin{pmatrix} s-1 & -4 \\ -(s-1) & (s-1)^2 \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} &= \begin{pmatrix} \frac{1}{s-1} + 2 \\ 1 + (s-1) \end{pmatrix} \\
\begin{pmatrix} s-1 & -4 \\ -(s-1) & (s-1)^2 \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} &= \begin{pmatrix} \frac{1}{s-1} + 2 \\ s \end{pmatrix}
\end{aligned}
$$

Now replacing row 2 by row 2 plus row 1 gives

$$
\begin{aligned}
\begin{pmatrix} s-1 & -4 \\ 0 & (s-1)^2 - 4 \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} &= \begin{pmatrix} \frac{1}{s-1} + 2 \\ s + \left(\frac{1}{s-1} + 2\right) \end{pmatrix} \\
\begin{pmatrix} s-1 & -4 \\ 0 & s^2 - 2s - 3 \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} &= \begin{pmatrix} \frac{1}{s-1} + 2 \\ \frac{1}{s-1}\left(s^2 + s - 1\right) \end{pmatrix} \qquad (1)
\end{aligned}
$$

Hence

$$
\begin{aligned}
x_2(s) &= \frac{1}{s-1}\left(\frac{s^2+s-1}{s^2-2s-3}\right) \\
&= \frac{s^2+s-1}{(s-1)(s-3)(s+1)} \qquad (2)
\end{aligned}
$$

Partial fractions:

$$
\begin{aligned}
\frac{s^2+s-1}{(s-1)(s-3)(s+1)} &= \frac{A}{(s-1)} + \frac{B}{(s-3)} + \frac{C}{(s+1)} \\
A = \left(\frac{s^2+s-1}{(s-3)(s+1)}\right)_{s=1} &= \frac{1+1-1}{(1-3)(1+1)} = -\frac{1}{4}
\end{aligned}
$$

And

$$
B = \left(\frac{s^2+s-1}{(s-1)(s+1)}\right)_{s=3} = \frac{9+3-1}{(3-1)(3+1)} = \frac{11}{8}
$$

And

$$
C = \left(\frac{s^2+s-1}{(s-1)(s-3)}\right)_{s=-1} = \frac{1-1-1}{(-1-1)(-1-3)} = -\frac{1}{8}
$$

Hence

$$
x_2(s) = -\frac{1}{4}\frac{1}{(s-1)} + \frac{11}{8}\frac{1}{(s-3)} - \frac{1}{8}\frac{1}{(s+1)} \qquad (3)
$$

Therefore

$$
x_2(t) = -\frac{1}{4}e^{t} + \frac{11}{8}e^{3t} - \frac{1}{8}e^{-t}
$$

Now we go back to (1) and use the first row to find $x_1(s)$ since we know $x_2(s)$ which is given in (2). This results in

$$
\begin{aligned}
(s-1)x_1(s) - 4x_2(s) &= \frac{1}{s-1} + 2 \\
(s-1)x_1(s) &= \frac{1}{s-1} + 2 + 4x_2 \\
x_1(s) &= \frac{1}{(s-1)^2} + \frac{2}{s-1} + \frac{4}{(s-1)}x_2 \\
&= \frac{1}{(s-1)^2} + \frac{2}{s-1} + \frac{4}{(s-1)}\left(\frac{s^2+s-1}{(s-1)(s-3)(s+1)}\right) \\
&= \frac{1}{(s-1)^2} + \frac{2}{s-1} + \frac{4}{(s-1)}\left(-\frac{1}{4}\frac{1}{(s-1)} + \frac{11}{8}\frac{1}{(s-3)} - \frac{1}{8}\frac{1}{(s+1)}\right) \\
&= \frac{1}{(s-1)^2} + \frac{2}{s-1} - \frac{1}{(s-1)^2} + \frac{44}{8}\frac{1}{(s-1)(s-3)} - \frac{4}{8}\frac{1}{(s-1)(s+1)} \\
&= \frac{2}{s-1} + \frac{44}{8}\frac{1}{(s-1)(s-3)} - \frac{4}{8}\frac{1}{(s-1)(s+1)} \\
&= \frac{2}{s-1} + \frac{11}{2}\frac{1}{(s-1)(s-3)} - \frac{1}{2}\frac{1}{(s-1)(s+1)} \\
&= \frac{2}{s-1} + \frac{11}{2}\left(\frac{1}{2}\frac{1}{s-3} - \frac{1}{2}\frac{1}{s-1}\right) - \frac{1}{2}\left(\frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}\right) \\
&= \frac{2}{s-1} + \frac{11}{4}\frac{1}{s-3} - \frac{11}{4}\frac{1}{s-1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{4}\frac{1}{s+1} \\
&= \frac{-1}{s-1} + \frac{11}{4}\frac{1}{s-3} + \frac{1}{4}\frac{1}{s+1}
\end{aligned}
$$

Therefore

$$x_1(t) = -e^t + \frac{11}{4}e^{3t} + \frac{1}{4}e^{-t}$$

We see that book solution is wrong. It gives $x_1(t) = 2e^{3t} + \frac{1}{2}e^t - \frac{1}{2}e^{-t}$.

Solving the same problem, but using the variation of parameters method:

Since $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$ then

$$
\begin{aligned}
\det(A - \lambda I) &= 0 \\
\det\begin{pmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix} &= 0 \\
(1-\lambda)^2 - 4 &= 0
\end{aligned}
$$

Hence roots are $\lambda = -1, \lambda = 3$

$\lambda = -1$

$$
\begin{aligned}
\begin{pmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

From first row, $2v_1 + 4v_2 = 0$ or $v_1 = -2v_2$. Hence $\boldsymbol{v}^1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\boldsymbol{x}^1(t) = e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\lambda = 3$

$$
\begin{aligned}
\begin{pmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

From first row, $-2v_1+4v_2=0$ or $v_1=2v_2$. Hence $\boldsymbol{v}^1=\begin{pmatrix}2\\1\end{pmatrix}$ and $\boldsymbol{x}^2(t)=e^{3t}\begin{pmatrix}2\\1\end{pmatrix}$. Therefore

$$X(t)=\begin{pmatrix}-2e^{-t} & 2e^{3t}\\ e^{-t} & e^{3t}\end{pmatrix}$$

$$X(0)=\begin{pmatrix}-2 & 2\\ 1 & 1\end{pmatrix}$$

Therefore $X^{-1}(0)=\frac{adj(X(0))}{\det(X(0))}=\frac{\begin{pmatrix}1 & -1\\ -2 & -2\end{pmatrix}^T}{-4}=-\frac{1}{4}\begin{pmatrix}1 & -2\\ -1 & -2\end{pmatrix}=\begin{pmatrix}-\frac{1}{4} & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{2}\end{pmatrix}$. Hence

$$\begin{aligned}e^{At}&=X(t)X^{-1}(0)\\&=\begin{pmatrix}-2e^{-t} & 2e^{3t}\\ e^{-t} & e^{3t}\end{pmatrix}\begin{pmatrix}-\frac{1}{4} & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{2}\end{pmatrix}\\&=\begin{pmatrix}\frac{1}{2}e^{-t}+\frac{1}{2}e^{3t} & -e^{-t}+e^{3t}\\ -\frac{1}{4}e^{-t}+\frac{1}{4}e^{3t} & \frac{1}{2}e^{-t}+\frac{1}{2}e^{3t}\end{pmatrix}\end{aligned}$$

Using (since $t_0=0$)

$$\boldsymbol{x}(t)=e^{At}\boldsymbol{x}(0)+e^{At}\int_0^t e^{-As}f(s)\,ds$$

But

$$\begin{aligned}e^{-As}f(s)&=\begin{pmatrix}\frac{1}{2}e^{s}+\frac{1}{2}e^{-3s} & -e^{s}+e^{-3s}\\ -\frac{1}{4}e^{s}+\frac{1}{4}e^{-3s} & \frac{1}{2}e^{s}+\frac{1}{2}e^{-3s}\end{pmatrix}\begin{pmatrix}e^s\\ e^s\end{pmatrix}\\&=\begin{pmatrix}\frac{1}{2}e^{2s}+\frac{1}{2}e^{-2s}-e^{2s}+e^{-2s}\\ -\frac{1}{4}e^{2s}+\frac{1}{4}e^{-2s}+\frac{1}{2}e^{2s}+\frac{1}{2}e^{-2s}\end{pmatrix}\\&=\begin{pmatrix}-\frac{1}{2}e^{2s}+\frac{3}{2}e^{-2s}\\ \frac{1}{4}e^{2s}+\frac{3}{4}e^{-2s}\end{pmatrix}\end{aligned}$$

Integrating

$$\begin{aligned}\int_0^t e^{-As}f(s)\,ds&=\begin{pmatrix}\int_0^t -\frac{1}{2}e^{2s}+\frac{3}{2}e^{-2s}ds\\ \int_0^t \frac{1}{4}e^{2s}+\frac{3}{4}e^{-2s}ds\end{pmatrix}\\&=\begin{pmatrix}-\frac{1}{4}\left(e^{2s}\right)_0^t-\frac{3}{4}\left(e^{-2s}\right)_0^t\\ \frac{1}{8}\left(e^{2s}\right)_0^t-\frac{3}{8}\left(e^{-2s}\right)_0^t\end{pmatrix}\\&=\begin{pmatrix}-\frac{1}{4}\left(e^{2t}-1\right)-\frac{3}{4}\left(e^{-2t}-1\right)\\ \frac{1}{8}\left(e^{2t}-1\right)-\frac{3}{8}\left(e^{-2t}-1\right)\end{pmatrix}\\&=\begin{pmatrix}-\frac{1}{4}e^{2t}+\frac{1}{4}-\frac{3}{4}e^{-2t}+\frac{3}{4}\\ \frac{1}{8}e^{2t}-\frac{1}{8}-\frac{3}{8}e^{-2t}+\frac{3}{8}\end{pmatrix}\\&=\begin{pmatrix}-\frac{1}{4}e^{2t}-\frac{3}{4}e^{-2t}+1\\ \frac{1}{8}e^{2t}-\frac{3}{8}e^{-2t}+\frac{1}{4}\end{pmatrix}\end{aligned}$$

Hence

$$\begin{aligned}e^{At}\int_0^t e^{-As}f(s)\,ds&=\begin{pmatrix}\frac{1}{2}e^{-t}+\frac{1}{2}e^{3t} & -e^{-t}+e^{3t}\\ -\frac{1}{8}e^{-t}+\frac{1}{4}e^{3t} & \frac{1}{2}e^{-t}+\frac{1}{2}e^{3t}\end{pmatrix}\begin{pmatrix}-\frac{1}{4}e^{2t}-\frac{3}{4}e^{-2t}+1\\ \frac{1}{8}e^{2t}-\frac{3}{8}e^{-2t}+\frac{1}{4}\end{pmatrix}\\&=\begin{pmatrix}\frac{1}{4}e^{-t}-e^{t}+\frac{3}{4}e^{3t}\\ -\frac{1}{8}e^{-t}-\frac{1}{4}e^{2t}+\frac{3}{8}e^{4t}\end{pmatrix}\end{aligned}$$

And

$$e^{At}\boldsymbol{x}(0) = \begin{pmatrix} \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t} & -e^{-t} + e^{3t} \\ -\frac{1}{8}e^{-t} + \frac{1}{4}e^{3t} & \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix}$$

Hence

$$\boldsymbol{x}(t) = \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix} + \begin{pmatrix} \frac{1}{4}e^{-t} - e^{t} + \frac{3}{4}e^{3t} \\ -\frac{1}{8}e^{-t} - \frac{1}{4}e^{2t} + \frac{3}{8}e^{4t} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{4}e^{-t} - e^{t} + \frac{11}{4}e^{3t} \\ -\frac{1}{8}e^{-t} - \frac{1}{4}e^{2t} + \frac{11}{8}e^{4t} \end{pmatrix}$$

Therefore

$$x_1(t) = \frac{1}{4}e^{-t} - e^{t} + \frac{11}{4}e^{3t}$$
$$x_2(t) = -\frac{1}{8}e^{-t} - \frac{1}{4}e^{2t} + \frac{11}{8}e^{4t}$$

Which agrees with result using Laplace transform method.

4 Orthogonal projections

Given $F(x, y, c)$ we need to find the orthogonal projections. The first step is to find the slope of the orthogonal projection, which is given by (it is orthogonal to the given curve slope)

$$\frac{dy}{dx} = \frac{F_y}{F_x} \tag{1}$$

Next step, check if c still shows up in the above (i.e. did not cancel out), then solve for c from $F(x, y, c) = 0$ and replace it in (1). Now (1) will not have c in it any more. Next, solve (1) for y. This gives the curve for the orthogonal projection. This solution will have new c in it (since we need to integrate to find y). See HW 2 for example problem.

5 Existence-uniqueness for 1D ODE

Given by theorem 2 for existence and uniqueness: given $\frac{dy}{dx} = f(x, y)$, with initial value $y(x_0) = y_0$. Let f and $\frac{\partial f}{\partial x}$be continuous in the rectangle $R : t \leq t \leq t_0 + a, |y - y_0| \leq b$. Compute $M = \max_{(x,y)} |f(x, y)|$ and set $\alpha = \min\left(a, \frac{b}{M}\right)$ then ODE has at least one solution in interval $t \leq t \leq t_0 + \alpha$ and this solution is unique. (I do not know why book split this into theorem 2 and 2').

Notice in the above, if f or $\frac{\partial f}{\partial x}$ not continuous in the range (the range must include the initial point) then not unique solution exist. For example $y' = \sin(2t)\, y^{\frac{1}{3}}$ with $y(0) = 0$. Here f' is not continuous at $y = 0$.

How to use the above The first step is to find M. This is done by finding maximum in R. This is normally done by inspection from looking at $f(x, y)$. Next, let $g(y) = \frac{b}{M}$. Find where this one is maximum. set its value in $\alpha = \min\left(a, \frac{b}{M}\right)$ and this finds α. Done. Example. Show $y(t)$ solution to $\frac{dy}{dt} = t^2 + e^{-y^2}, y(0) = 0$ exists in $0 \leq t \leq \frac{1}{2}$ and $|y(t)| \leq 1$. Here it is clear $M = \frac{5}{4}$ and hence $\alpha = \min\left(\frac{1}{2}, \frac{b}{M}\right)$ but $b = 1$, hence $\alpha = \min\left(\frac{1}{2}, \frac{1}{\frac{5}{4}}\right) = \alpha = \min\left(\frac{1}{2}, \frac{4}{5}\right) = \frac{1}{2}$. Therefore solution exist for $0 \leq t \leq 0 + \alpha$ or $0 \leq t \leq \frac{1}{2}$.

5.1 practice problems

5.1.1 Problem 5, section 1.10

Show that the solution $y(t)$ exists on $y(0) = 0; 0 \leq t \leq \frac{1}{3}$

$$y' = 1 + y + y^2 \cos t$$

solution

Here $a = \frac{1}{3}$.

$$\begin{aligned} M &= \max\left(f\left(t, y\right)\right) \\ &= 1 + b + b^2 \end{aligned}$$

Hence

$$\begin{aligned} \alpha &= \min\left(\frac{1}{3}, \frac{b}{M}\right) \\ &= \min\left(\frac{1}{3}, \frac{b}{1 + b + b^2}\right) \end{aligned}$$

Let $g(b) = \frac{b}{1+b+b^2}$ then $\frac{dg}{dp} = \frac{\left(1+b+b^2\right) - b(1+2b)}{\left(1+b+b^2\right)^2}$. Setting this to zero and solving for b

$$\begin{aligned} \left(1 + b + b^2\right) - b(1 + 2b) &= 0 \\ 1 + b + b^2 - b - 2b^2 &= 0 \\ 1 - b^2 &= 0 \end{aligned}$$

Hence $b = 1$. At $b = 1$, then $g(b) = \frac{1}{1+1+1} = \frac{1}{3}$. Therefore

$$\begin{aligned} \alpha &= \min\left(\frac{1}{3}, \frac{1}{3}\right) \\ &= \frac{1}{3} \end{aligned}$$

Therefore $y(t)$ solution exist for $0 \leq t \leq 0 + \alpha$ or $0 \leq t \leq \frac{1}{3}$.

5.1.2 Problem 16, section 1.10

Consider $y' = t^2 + y^2, y(0) = 0$ and let R be rectangle $0 \leq t \leq a, -b \leq y \leq b$. (a) Show the solution exist for $0 \leq t \leq \min\left(a, \frac{b}{a^2+b^2}\right)$ (b) Show the maximum value of $\frac{b}{a^2+b^2}$, for a fixed is $\frac{1}{2a}$. (c) Show that $\alpha = \min\left(a, \frac{1}{2}a\right)$ is largest when $a = \frac{1}{\sqrt{2}}$

$$y' = 1 + y + y^2 \cos t$$

solution

(a)

$$\begin{aligned} M &= \max\left(f\left(t, y\right)\right) \\ &= a^2 + b^2 \end{aligned}$$

Hence

$$\begin{aligned} \alpha &= \min\left(a, \frac{b}{M}\right) \\ &= \min\left(\frac{1}{3}, \frac{b}{a^2 + b^2}\right) \end{aligned}$$

Hence solution exist for $0 \leq t \leq \min\left(a, \frac{b}{a^2+b^2}\right)$.

(b) Let $g(b) = \frac{b}{a^2+b^2}$ then $\frac{dg}{dp} = \frac{(a^2+b^2)-b(2b)}{(1+b+b^2)^2}$. Setting this to zero and solving for b

$$\begin{aligned}(a^2+b^2) - b(2b) &= 0\\ a^2+b^2-2b^2 &= 0\\ a^2 &= b^2\end{aligned}$$

Hence $b = \pm a$. At $b = a$, then $g(b) = \frac{a}{a^2+a^2} = \frac{a}{2a^2} = \frac{1}{2a}$.

(c)

$$\begin{aligned}\alpha &= \min\left(a, g_{\max}(b)\right)\\ &= \min\left(a, \frac{1}{2a}\right)\end{aligned}$$

Solving $a = \frac{1}{2a}$ or $a^2 = \frac{1}{2}$. Hence $a = \frac{1}{\sqrt{2}}$ gives largest value.

5.1.3 Problem 17, section 1.10

Prove that $y(t) = -1$ is only solution for $y' = t(1+y), y(0) = -1$

solution

Since $f = t(1+y)$ is continuous for all t, y and $f_y = t$ is continuous for all y, then if we find a solution, it will be unique solution by theorem 2'. But $y(t) = -1$ is a solution since we can show easily it satisfies the ODE. Hence it is the only solution over all t by theorem 2'

5.1.4 Problem 19, section 1.10

Find solution of $y' = t\sqrt{1-y^2}, y(0) = 1$ other than $y(t) = 1$. Does this violate theorem 2'?

solution

$$\begin{aligned}\frac{dy}{dt} &= t\sqrt{1-y^2}\\ \int \frac{dy}{\sqrt{1-y^2}} &= \int t\,dt\\ \arcsin(y) &= \frac{t^2}{2} + C\end{aligned}$$

At $t = 0$

$$\arcsin(1) = C$$

Hence solution is $\arcsin(y) = \frac{t^2}{2} + \arcsin(1)$ or

$$\begin{aligned}y(t) &= \sin\left(\frac{t^2}{2} + \arcsin(1)\right)\\ &= \sin\left(\frac{t^2}{2} + \frac{\pi}{2}\right)\\ &= \sin\left(\frac{1}{2}\left(t^2+\pi\right)\right)\end{aligned}$$

This does not violate theorem 2' because $f(t,y) = t\sqrt{1-y^2}$, hence $f_y = \frac{-ty}{\sqrt{1-y^2}}$ which is not continuous at $y = \pm 1$. But $y = -1$ is the initial conditions. Hence theorem 2' do not apply. Theorem 2' applies in the region where both f, f_y are continuous.

6 Stability of system

Algorithm 1 Determining stability of system $\dot{x} = Ax + g(x)$

```
if system is linear, i.e. ẋ = Ax then
    determine eigenvalues λ_i of A by solving |A − λI| = 0
    if all eigenvalues have real part smaller than zero then
        return stable
    else
        if at least one eigenvalue have positive real part then
            return not stable
        else                                ▷ we get here if at least one λ has zero real part
            for all λ_i with zero real part do
                M = multiplicity of λ_i
                N = number of linearly independent eigenvectors that λ_i can generate
                if N < M then
                    return not stable
                end if
            end for
            return stable
        end if
    end if
else                                        ▷ system not linear
    will only consider case when origin is equilibrium point
    determine the Jacobian matrix J
    evaluate J at origin x = 0
    determine eigenvalues λ_i of J by solving |J − λI| = 0
    if all eigenvalues have real part smaller than zero then
        return stable
    else
        if at least one eigenvalue have positive real part then
            return not stable
        else                                ▷ we get here if at least one λ has zero real part
            return unable to decide
        end if
    end if
end if
```

7 Laplace

If $Y(s)$ has form $\frac{s}{s^2+as+b}$ where roots of quadratic are complex, then complete the square. Write $s^2 + as + b = (s + A)^2 + B$ and find A, B. Then

$$\begin{aligned}
\frac{s}{s^2 + as + b} &= \frac{s}{(s + A)^2 + B} \\
&= \frac{s + A - A}{(s + A)^2 + B} \\
&= \frac{s + A}{(s + A)^2 + B} - A\frac{1}{(s + A)^2 + B} \\
&= \frac{\tilde{s}}{\tilde{s}^2 + B} - A\frac{1}{\tilde{s}^2 + B} \\
&= \frac{\tilde{s}}{\tilde{s}^2 + B} - \frac{A}{\sqrt{B}}\frac{\sqrt{B}}{\tilde{s}^2 + B}
\end{aligned}$$

And now use tables. Due to shifting, multiply result by e^{-Bt}. So inverse Laplace of the above is $e^{-Bt}\left(\cos\sqrt{B}t - \frac{A}{\sqrt{B}}\sin\sqrt{B}t\right)$