University Course

Math 4512
Differential Equations with Applications

University of Minnesota, Twin Cities
Fall 2019

My Class Notes

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Fall 2019

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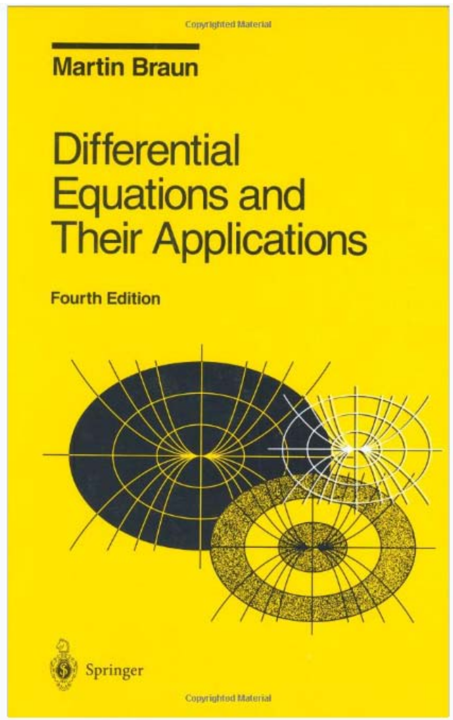
Chapter 1

Introduction

1.1 Links

1. class web page (needs login) `https://canvas.umn.edu/courses/135946`

1.2 Text book



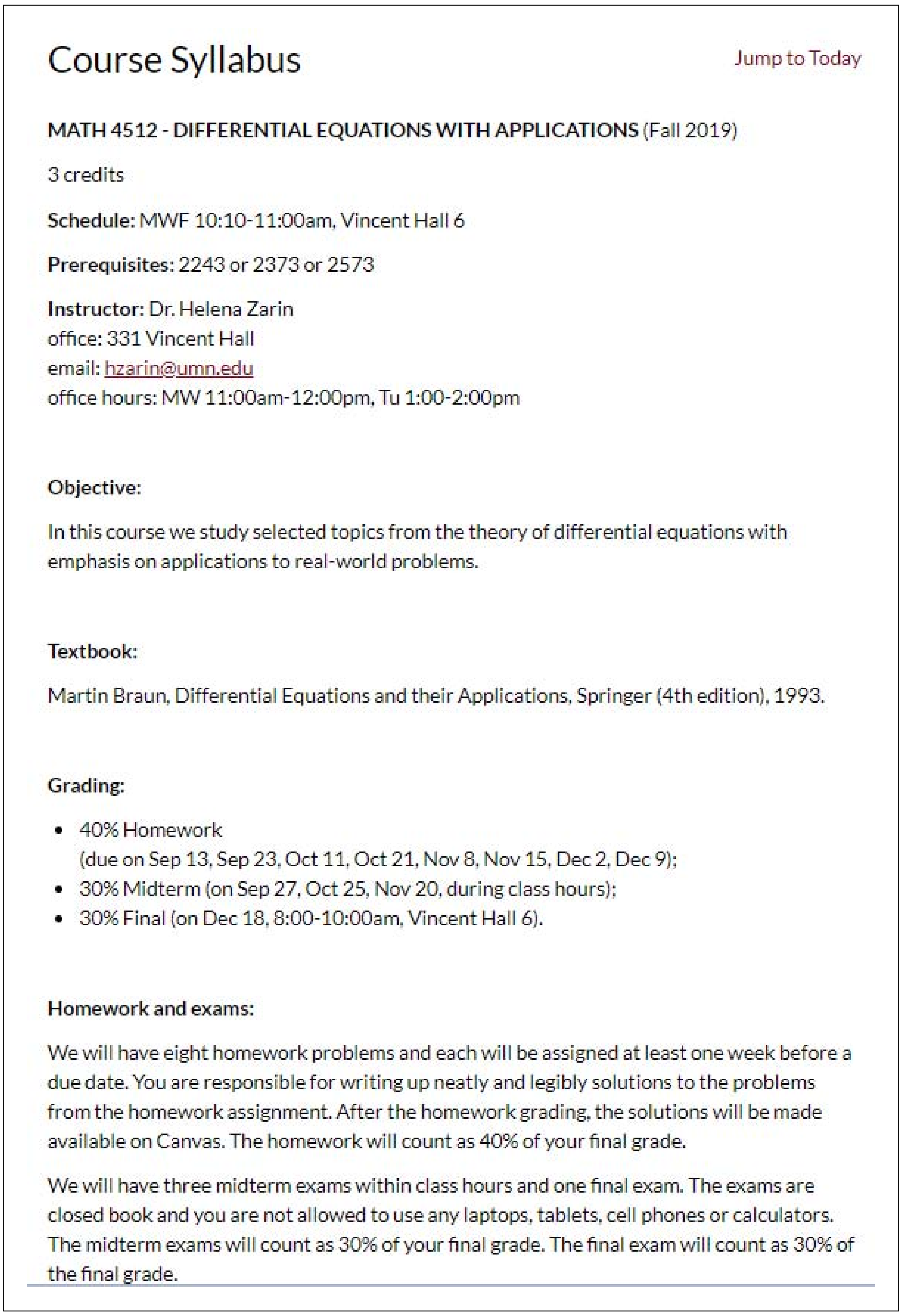
Martin Braun

Differential Equations and Their Applications

Fourth Edition

Springer

1.3 syllabus



Course Syllabus

MATH 4512 - DIFFERENTIAL EQUATIONS WITH APPLICATIONS (Fall 2019)

3 credits

Schedule: MWF 10:10-11:00am, Vincent Hall 6

Prerequisites: 2243 or 2373 or 2573

Instructor: Dr. Helena Zarin
office: 331 Vincent Hall
email: hzarin@umn.edu
office hours: MW 11:00am-12:00pm, Tu 1:00-2:00pm

Objective:

In this course we study selected topics from the theory of differential equations with emphasis on applications to real-world problems.

Textbook:

Martin Braun, Differential Equations and their Applications, Springer (4th edition), 1993.

Grading:

- 40% Homework
  (due on Sep 13, Sep 23, Oct 11, Oct 21, Nov 8, Nov 15, Dec 2, Dec 9);
- 30% Midterm (on Sep 27, Oct 25, Nov 20, during class hours);
- 30% Final (on Dec 18, 8:00-10:00am, Vincent Hall 6).

Homework and exams:

We will have eight homework problems and each will be assigned at least one week before a due date. You are responsible for writing up neatly and legibly solutions to the problems from the homework assignment. After the homework grading, the solutions will be made available on Canvas. The homework will count as 40% of your final grade.

We will have three midterm exams within class hours and one final exam. The exams are closed book and you are not allowed to use any laptops, tablets, cell phones or calculators. The midterm exams will count as 30% of your final grade. The final exam will count as 30% of the final grade.

Chapter 2

HWs

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2.1 HW table lookup

Table 2.1: HW table

#	grade	HW subject
1	93/100	Section 1.2,1.4,1.5. Solve separable ODE. Solve ODE by $u = \frac{y}{t}$ substitution. population model problem. Asking what the population will be after some time
2	98/100	section 1.8,1.10,1.13. Tank mixing. Finding Orthogonal projection. Find where solution exist. Show that some given solution for initial value ODE is unique. Euler numerical solution problem.
3	100/100	section 2.1,2.2,2.4,2.5. Show that 2 functions are linearly independent. Finding Wronskian. Solving second order ODE with constant coefficients. Using Variation of parameters to find particular solution. Using Guessing (undetermined coefficients) method to find particular solution (RHS is $1 + t^2 + e^{-2t}$).
4	100/100	section 2.6,2.9,2.10. Vibration problem. Using Laplace method to solve second order initial value problem. Finding inverse Laplace of expression.
5	100/100	section 3.1-3.5. Converting pair of first order ODE's to system. Determine if set of vectors form vector space. (check if closed under addition or scalar multiplication). Find basis in 3D given 2 basis (i.e. need to find third base vector). Given 3 solutions, determine if they are linearly independent. (solve $c_i x^i = 0$ for c_i and show all c are zero. Find determinant of 4 by 4 matrix. Finding inverse of Matrix.
6	100/100	section 3.8-3.9,3.10. Solving system $x' = Ax$ using the eigenvalue/eigenvector method, eigenvalues all different and real. Same as above, but 2 of eigenvalues are complex. When one eigenvalue is complex, just find the eigenvector for it, and find the real and imaginary parts of $x(t) = e^{\lambda t} v(t)$ which will give the two solutions associated with both complex eigenvalues. i.e. only need to find one eigenvector with there are two complex eigenvalues (since they are conjugates). Same as above, but one eigenvalue of multiplicity 3.
7	94/100	section 4.1,4.2,4.3. Find all equilibrium points. Determine the stability of all solutions to system (find the eigenvalues). Given non-linear system, determine if origin is equilibrium point and check if stable or not (Use the Jacobian). If non-linear system, and real part is zero, then unable to decide.
8	99/100	section 4.4,4.7. Finding orbit equation for 2 by 2 system using $\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$. Drawing phase diagrams.

2.2 HW 1

Local contents

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2.2.1 Problem 8, section 1.2

Solve$\frac{dy}{dt} + \sqrt{1+t^2}y = 0, y(0) = \sqrt{5}$

Solution

This is separable first order ODE. Therefore

$$\int \frac{dy}{y} = -\int \sqrt{1+t^2}dt \tag{1}$$

The LHS becomes

$$\int \frac{dy}{y} = \ln|y| \tag{2}$$

For the RHS of (1), the integral $\int \sqrt{1+t^2}dt$ can be evaluated as follows. Let $t = \sinh(\theta)$. Hence $\frac{dt}{d\theta} = \cosh(\theta)$. Therefore

$$\begin{aligned}\int \sqrt{1+t^2}dt &= \int \sqrt{1+\sinh^2(\theta)}\cosh(\theta)\,d\theta\\ &= \int \cosh^2(\theta)\,d\theta\\ &= \int \frac{1}{2}(1+\cosh(2\theta))\,d\theta\\ &= \frac{1}{2}\left(\int d\theta + \int \cosh(2\theta)\,d\theta\right)\\ &= \frac{1}{2}\left(\theta + \frac{\sinh(2\theta)}{2}\right)\\ &= \frac{1}{2}\theta + \frac{\sinh(2\theta)}{4}\end{aligned}$$

Since $\sinh(2\theta) = 2\sinh\theta\cosh\theta$, the above becomes

$$\int \sqrt{1+t^2}dt = \frac{1}{2}\theta + \frac{\sinh\theta\cosh\theta}{2}$$

Since $\cosh^2(\theta) - \sinh^2(\theta) = 1$ then $\cosh^2\theta = 1 + \sinh^2(\theta)$ and the above becomes

$$\int \sqrt{1+t^2}dt = \frac{1}{2}\left(\theta + \sinh\theta\sqrt{1+\sinh^2(\theta)}\right)$$

But $t = \sinh(\theta)$ and $\theta = \operatorname{arcsinh}(t)$. Therefore the above becomes

$$\int \sqrt{1+t^2}dt = \frac{1}{2}\left(\operatorname{arcsinh}(t) + t\sqrt{1+t^2}\right) \tag{3}$$

Using (2,3) in (1) gives

$$\ln|y| = -\frac{1}{2}\left(\operatorname{arcsinh}(t) + t\sqrt{1+t^2}\right) + C \tag{4}$$

Where C is arbitrary constant of integration. Writing $\operatorname{arcsinh}(t)$ using known identity as $\ln\left|t+\sqrt{1+t^2}\right|$. And since $\sqrt{1+t^2}$ is always larger than t, then the absolute sign is not needed. Eq. (4) becomes

$$\begin{aligned}\ln|y| &= -\frac{1}{2}\left(\ln\left(t+\sqrt{1+t^2}\right) + t\sqrt{1+t^2}\right) + C\\ |y| &= e^{-\frac{1}{2}\left(\ln\left(t+\sqrt{1+t^2}\right)+t\sqrt{1+t^2}\right)}e^C\\ y &= C_1 e^{-\frac{1}{2}\left(\ln\left(t+\sqrt{1+t^2}\right)+t\sqrt{1+t^2}\right)}\\ &= C_1 e^{-\frac{1}{2}\ln\left(t+\sqrt{1+t^2}\right)}e^{t\sqrt{1+t^2}}\end{aligned}$$

Therefore the general solution is

$$y(t) = C_1\frac{e^{t\sqrt{1+t^2}}}{\left(t+\sqrt{1+t^2}\right)^{\frac{1}{2}}}$$

Now initial conditions are used to determine C_1. From $y(0) = \sqrt{5}$ then the above gives

$$\sqrt{5} = C_1$$

Therefore the particular solution is

$$y(t)=\sqrt{5}\frac{e^{t\sqrt{1+t^2}}}{\left(t+\sqrt{1+t^2}\right)^{\frac{1}{2}}}$$

2.2.2 Problem 17, section 1.2

Find a continuous solution of the IVP $y+y'=g(t),y(0)=0$ where

$$g(t)=\begin{cases}2 & 0\leq t\leq 1\\ 0 & t>1\end{cases}$$

Solution

This is linear first order ODE. The integrating factor is $\mu=e^{\int dt}=e^t$. Hence the ODE becomes

$$\begin{aligned}\frac{d}{dt}\left(y\mu\right)&=\mu g(t)\\ \frac{d}{dt}\left(ye^t\right)&=e^t g(t)\end{aligned}$$

Integrating gives

$$ye^t=\int e^t g(t)\,dt+C \tag{1}$$

Breaking the problem into two phases, and solving the above for $0\leq t\leq 1$ gives

$$\begin{aligned}ye^t&=\int 2e^t dt+C\\ &=2e^t+C\\ y(t)&=2+Ce^{-t}\end{aligned}$$

Applying initial conditions gives $0=2+C$, or $C=-2$ and the above becomes

$$y(t)=2-2e^{-t}\qquad 0\leq t\leq 1 \tag{2}$$

The above solution is valid for $0\leq t\leq 1$.

To solve for $t>1$, initial conditions are first found for $t=1$. At $t=1$ the above gives

$$y(1)=2-\frac{2}{e}$$

Hence for $t>1$, initial conditions are $y(1)=2-\frac{2}{e}$. Now the second phase is solved. From (1)

$$ye^t=\int e^t g(t)\,dt+C$$

But now $g(t)=0$. The above simplifies to

$$\begin{aligned}ye^t&=C \qquad (3)\\ y&=Ce^{-t}\end{aligned}$$

But at $t=1,y=2-\frac{2}{e}$. Therefore

$$\begin{aligned}2-\frac{2}{e}&=Ce^{-1}\\ C&=2e-2\\ &=2(e-1)\end{aligned}$$

Substituting the above C into (3) gives

$$y=2(e-1)e^{-t}\qquad t>1 \tag{4}$$

Using (2,4) the final solution is therefore

$$y(t)=\begin{cases}2-2e^{-t} & 0\leq t\leq 1\\ 2(e-1)e^{-t} & t>1\end{cases}$$

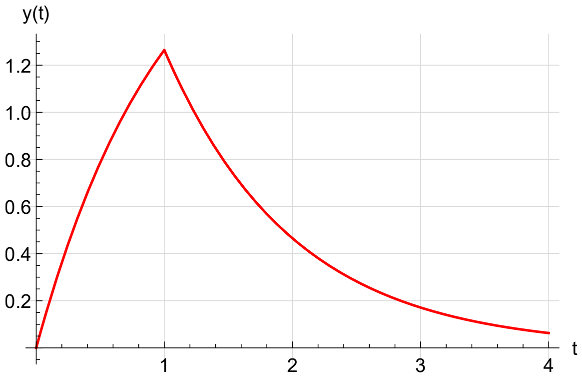


Figure 2.1: Plot of the solution $y(t)$

2.2.3 Problem 10, section 1.4

Solve $\cos y \frac{dy}{dt} = \frac{-t\sin y}{1+t^2}, y(1) = \frac{\pi}{2}$

Solution

This is separable first order ODE

$$\int \frac{\cos y}{\sin y} dy = -\int \frac{t}{1+t^2} dt$$

But $\int \frac{\cos y}{\sin y} dy = \int \frac{\frac{d}{dy}\sin y}{\sin y} dy = \ln\left|\sin(y)\right|$ and $\int \frac{t}{1+t^2} dt = \frac{1}{2}\ln\left|1+t^2\right| = \frac{1}{2}\ln\left(1+t^2\right)$ since $1+t^2$ is positive. Hence the above becomes

$$\ln\left|\sin(y)\right| = -\frac{1}{2}\ln\left(1+t^2\right) + C$$

Where C is the integration constant. Hence

$$\begin{aligned}\left|\sin(y)\right| &= e^{-\frac{1}{2}\ln\left(1+t^2\right)+C}\\ &= e^{-\frac{1}{2}\ln\left(1+t^2\right)}e^{C}\end{aligned}$$

Therefore

$$\begin{aligned}\sin(y) &= C_1 e^{-\frac{1}{2}\ln\left(1+t^2\right)}\\ &= C_1\frac{1}{\sqrt{1+t^2}} \qquad (1)\end{aligned}$$

From initial conditions $y(1) = \frac{\pi}{2}$ the above becomes

$$\begin{aligned}\sin\left(\frac{\pi}{2}\right) &= C_1\frac{1}{\sqrt{2}}\\ C_1 &= \sqrt{2}\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}\sin(y) &= \sqrt{2}\frac{1}{\sqrt{1+t^2}}\\ y(t) &= \arcsin\left(\frac{\sqrt{2}}{\sqrt{1+t^2}}\right)\end{aligned}$$

2.2.4 Problem 18, section 1.4

Solve $\frac{dy}{dt} = \frac{t+y}{t-y}$

Solution

Let $u = \frac{y}{t}$ or $y = ut$. Hence $\frac{dy}{dt} = u + t\frac{du}{dt}$. Therefore the ODE becomes

$$\begin{aligned} u + t\frac{du}{dt} &= \frac{t + ut}{t - ut} \\ u + t\frac{du}{dt} &= \frac{t(1+u)}{t(1-u)} \\ t\frac{du}{dt} &= \frac{(1+u)}{(1-u)} - u \\ &= -\frac{u^2+1}{u-1} \\ &= \frac{1+u^2}{1-u} \end{aligned}$$

This is now separable ODE. Therefore

$$\begin{aligned} \frac{1-u}{1+u^2}\frac{du}{dt} &= \frac{1}{t} \\ \int \frac{1-u}{1+u^2}du &= \int \frac{1}{t}dt \qquad (1) \end{aligned}$$

But

$$\begin{aligned} \int \frac{1-u}{1+u^2}du &= \int \frac{1}{1+u^2}du - \int \frac{u}{1+u^2}du \\ &= \arctan(u) - \frac{1}{2}\ln\left|1+u^2\right| \end{aligned}$$

but $1 + u^2$ is positive. Hence

$$\int \frac{1-u}{1+u^2}du = \arctan(u) - \frac{1}{2}\ln\left(1+u^2\right)$$

And $\int \frac{1}{t}dt = \ln|t|$. Hence (1) becomes

$$\arctan(u) - \frac{1}{2}\ln\left(1+u^2\right) = \ln|t| + C$$

But $\frac{y}{t}$, and the above becomes

$$\arctan\left(\frac{y}{t}\right) - \frac{1}{2}\ln\left(1 + \left(\frac{y}{t}\right)^2\right) = \ln|t| + C$$

The above solution is implicit in $y(t)$.

2.2.5 Problem 4, section 1.5

Suppose that a population doubles its original size in 100 years, and triples it in 200 years. Show that this population cannot satisfy the Malthusian law of population growth.

Solution

In Malthusian law of population growth, the rate at which population changes is fixed in the model. It is given by a below

$$\frac{dp}{dt} = ap(t)$$

Where a is constant. But the problem says the population is doubled in first 100 years. So if p_0 was initial population, then after 100 years the population now has become $2p_0$. There one will expect that after another 100 years the population will double again to become $4p_0$.

But the problem says that the population triples in 200 years, becoming $3p_0$ and not $4p_0$. This shows that the rate of growth is not constant. Hence this do not satisfy Malthusian law of population growth.

2.2.6 Problem 6(a), section 1.5

A population grows according to the logistic law, with a limiting population of 5×10^8 individuals. When the population is low it doubles every 40 minutes. What will the population be after two hours if initially it was (a) 10^8 ?

Solution

In the logistic law, the population model is given by

$$\frac{dp}{dt} = ap - bp^2$$

Where $p(t)$ is population at time t and a is the growth rate (constant) and b is the competition rate (also constant). In this model

$$\lim_{t\to\infty} p(t) = \frac{a}{b}$$

Therefore

$$\frac{a}{b} = 5\times 10^8 \tag{1}$$

The problem says that $a = 100\%$ (per 40 minute) or $a = 1$ (per 40 minute). Therefore $a = \frac{1}{40}$ per minute. And $p_0 = 10^8$. Using the solution of this model, given in the textbook at page 30 as

$$p(t) = \frac{ap_0}{bp_0 + \left(a - bp_0\right)e^{-a(t-t_0)}} \tag{3}$$

And using $t_0 = 0$, then the population size at t is now be calculated. From (1), $b = \frac{\frac{1}{40}}{5\times 10^8} = \frac{1}{2}\times 10^{-10} = 5\times 10^{-11}$. Eq. (3) now becomes

$$\begin{aligned} p(t) &= \frac{\frac{1}{40}\left(10^8\right)}{\left(5\times 10^{-11}\right)\left(10^8\right) + \left(\frac{1}{40} - \left(5\times 10^{-11}\right)\left(10^8\right)\right)e^{-\frac{1}{40}t}} \\ &= \frac{\frac{1}{40}\left(10^8\right)}{\left(5\times\frac{1}{1000}\right) + \left(\frac{1}{40} - 5\times\frac{1}{1000}\right)e^{-\frac{1}{40}t}} \end{aligned}$$

For $t = 120$ (minutes) the above becomes

$$\begin{aligned} p(120) &= \frac{\frac{1}{40}\left(10^8\right)}{\left(5\times\frac{1}{1000}\right) + \left(\frac{1}{40} - 5\times\frac{1}{1000}\right)e^{-\frac{1}{40}120}} \\ &= \frac{\frac{1}{40}\left(10^8\right)}{\left(5\times\frac{1}{1000}\right) + \left(\frac{1}{40} - 5\times\frac{1}{1000}\right)e^{-3}} \\ &= 4.1696\times 10^8 \end{aligned}$$

Hence

$$\boxed{p(120) = 4.1696\times 10^8}$$

The inflection point is

$$\begin{aligned} \frac{a}{2b} &= \frac{\frac{1}{40}}{(2)\left(5\times 10^{-11}\right)} \\ &= 2.5\times 10^8 \end{aligned}$$

The following plot was generated to compare the population $p(t)$ between case (a) and case (b). It shows that when starting with initial population of $p_0 = 10^8$ which is case (a) and when starting with $p_0 = 10^9$ which is case (b), both populations will eventually reach the limiting population of 5×10^8. The S curve shows up only when starting with population below the limiting population.

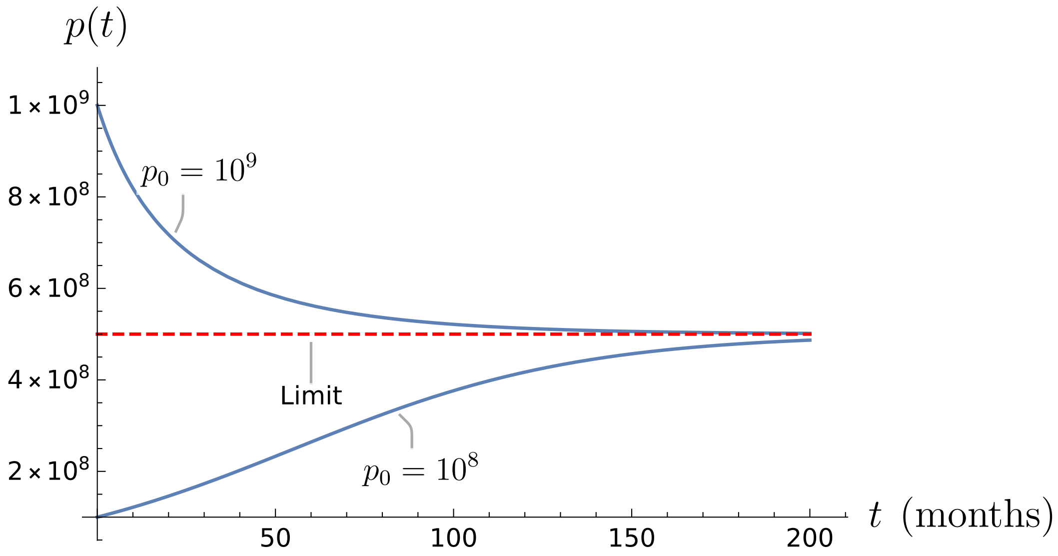
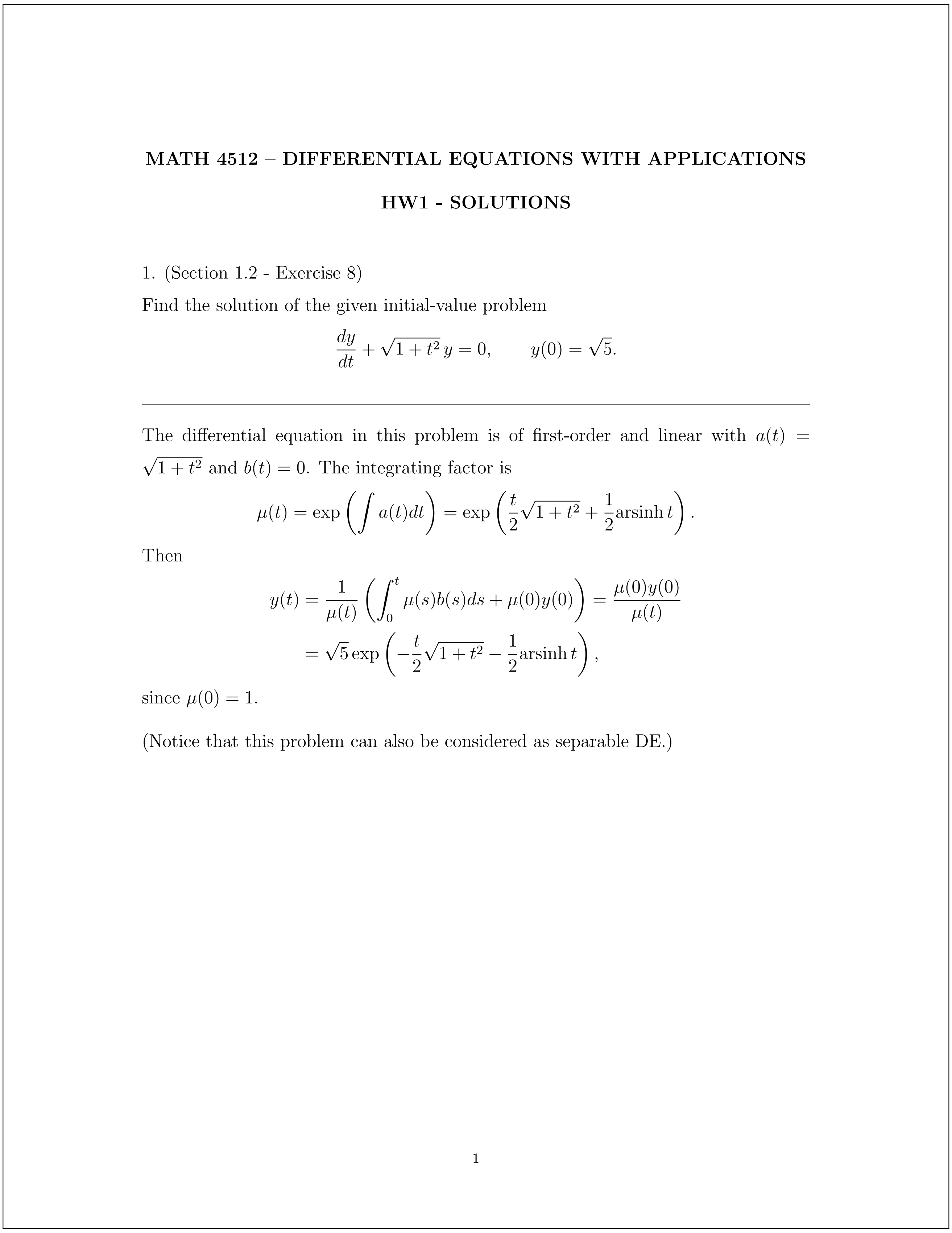


Figure 2.2: Population $p(t)$ change depends on p_0

2.2.7 Key solution for HW 1



MATH 4512 – DIFFERENTIAL EQUATIONS WITH APPLICATIONS

HW1 - SOLUTIONS

1. (Section 1.2 - Exercise 8)

Find the solution of the given initial-value problem

$$\frac{dy}{dt} + \sqrt{1+t^2}\, y = 0, \qquad y(0) = \sqrt{5}.$$

The differential equation in this problem is of first-order and linear with $a(t) = \sqrt{1+t^2}$ and $b(t) = 0$. The integrating factor is

$$\mu(t) = \exp\left(\int a(t)dt\right) = \exp\left(\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\operatorname{arsinh} t\right).$$

Then

$$\begin{aligned} y(t) &= \frac{1}{\mu(t)}\left(\int_0^t \mu(s)b(s)ds + \mu(0)y(0)\right) = \frac{\mu(0)y(0)}{\mu(t)} \\ &= \sqrt{5}\exp\left(-\frac{t}{2}\sqrt{1+t^2} - \frac{1}{2}\operatorname{arsinh} t\right), \end{aligned}$$

since $\mu(0) = 1$.

(Notice that this problem can also be considered as separable DE.)

2. (Section 1.2 - Exercise 17)

Find a continuous solution of the initial-value problem

$$\frac{dy}{dt} + y = g(t), \qquad y(0) = 0,$$

where

$$g(t) = \begin{cases} 2, & 0 \leq t \leq 1, \\ 0, & t > 1. \end{cases}$$

The differential equation is linear with $a(t) = 1$. The integrating factor is

$$\mu(t) = \mathrm{e}^{\int dt} = \mathrm{e}^{\mathrm{t}}$$

and the solution is

$$y(t) = \frac{1}{\mu(t)} \left(\int_0^t \mu(s) g(s) ds + \mu(0) y(0) \right) = \mathrm{e}^{-t} \int_0^t \mathrm{e}^s g(s) ds.$$

If $t \in [0, 1]$, then

$$y(t) = \mathrm{e}^{-t} \int_0^t 2 \cdot \mathrm{e}^s ds = 2\mathrm{e}^{-t} \mathrm{e}^s \Big|_0^t = 2\mathrm{e}^{-t}(\mathrm{e}^t - 1) = 2(1 - \mathrm{e}^{-t}).$$

If $t > 1$, then

$$y(t) = \mathrm{e}^{-t} \left(\int_0^1 2 \cdot \mathrm{e}^s ds + \int_1^t 0 \cdot \mathrm{e}^s ds \right) = 2\mathrm{e}^{-t} \mathrm{e}^s \Big|_0^1 = 2\mathrm{e}^{-t}(\mathrm{e} - 1).$$

Finally we get

$$y(t) = \begin{cases} 2(1 - \mathrm{e}^{-t}), & 0 \leq t \leq 1, \\ 2\mathrm{e}^{-t}(\mathrm{e} - 1), & t > 1. \end{cases}$$

The function y is continuous since

$$\lim_{t \to 1+} y(t) = \lim_{t \to 1+} 2\mathrm{e}^{-t}(\mathrm{e} - 1) = 2\mathrm{e}^{-1}(\mathrm{e} - 1) = 2(1 - \mathrm{e}^{-1}) = y(1).$$

3. (Section 1.4 - Exercise 10)

Solve initial-value problem

$$\cos y \, \frac{dy}{dt} = \frac{-t \sin y}{1+t^2}, \qquad y(1) = \frac{\pi}{2}$$

and determine the interval of existence of its solution.

First we will find a general solution to the separable DE:

$$\begin{aligned}
\frac{\cos y}{\sin y} \, \frac{dy}{dt} &= -\frac{t}{1+t^2} \\
\int \cot y dy &= -\int \frac{t}{1+t^2} dt \\
\ln|\sin y| &= -\frac{1}{2} \ln|1+t^2| + c_1 \\
|\sin y| &= c_2 (1+t^2)^{-1/2}, \qquad c_2 = \mathrm{e}_1^c \\
\sin y &= \frac{c}{\sqrt{1+t^2}} \\
y(t) &= \arcsin\left(\frac{c}{\sqrt{1+t^2}} \right).
\end{aligned}$$

The initial condition

$$y(1) = \arcsin\left(\frac{c}{\sqrt{2}} \right) = \frac{\pi}{2}$$

implies $c/\sqrt{2} = 1$, i.e. $c = \sqrt{2}$. The final solution is

$$y(t) = \arcsin\left(\sqrt{\frac{2}{1+t^2}} \right)$$

and it is well defined if

$$\sqrt{\frac{2}{1+t^2}} \in [-1, 1].$$

Notice that for $t \geq 1$ or $t \leq -1$ we have that

$$0 < \frac{2}{1+t^2} \leq 1,$$

because

$$1 - \frac{2}{1+t^2} = \frac{t^2-1}{t^2+1} \geq 0.$$

Since we are looking for a continuous solution that contains $t_0 = 1$, we conclude that the interval of existence for the solution is $[1, \infty)$.

4. (Section 1.4 - Exercise 18)

Find the general solution for

$$\frac{dy}{dt} = \frac{t+y}{t-y}.$$

Following the hint, we will use the substitution

$$y(t) = tv(t), \qquad \frac{dy}{dt} = v(t) + t\frac{dv}{dt}.$$

Then the differential equation transforms into

$$\begin{aligned} v + t\frac{dv}{dt} &= \frac{t+tv}{t-tv} = \frac{1+v}{1-v} \\ t\frac{dv}{dt} &= \frac{1+v}{1-v} - v = \frac{1+v^2}{1-v} \\ \frac{1-v}{1+v^2}\frac{dv}{dt} &= \frac{1}{t}. \end{aligned}$$

From

$$\int \frac{1-v}{1+v^2}dv = \int \frac{1}{1+v^2}dv - \int \frac{v}{1+v^2}dv = \arctan v - \frac{1}{2}\ln(1+v^2)$$

it follows

$$\begin{aligned} \int \frac{1-v}{1+v^2}dv &= \int \frac{1}{t}dt \\ \arctan v - \frac{1}{2}\ln(1+v^2) &= \ln|t| + c \\ \arctan\frac{y}{t} - \frac{1}{2}\ln\left(1+\frac{y^2}{t^2}\right) &= \ln|t| + c. \end{aligned}$$

In the last step we again applied substitution in order to get the implicit form of the general solution.

5. (Section 1.5 - Exercise 4)

Suppose that a population doubles its original size in 100 years, and triples it in 200 years. Show that this population cannot satisfy the Malthusian law of population growth.

The Malthusian law of population growth

$$\frac{dp}{dt} = a\,p(t), \qquad p(t_0) = p_0, \qquad a = const,$$

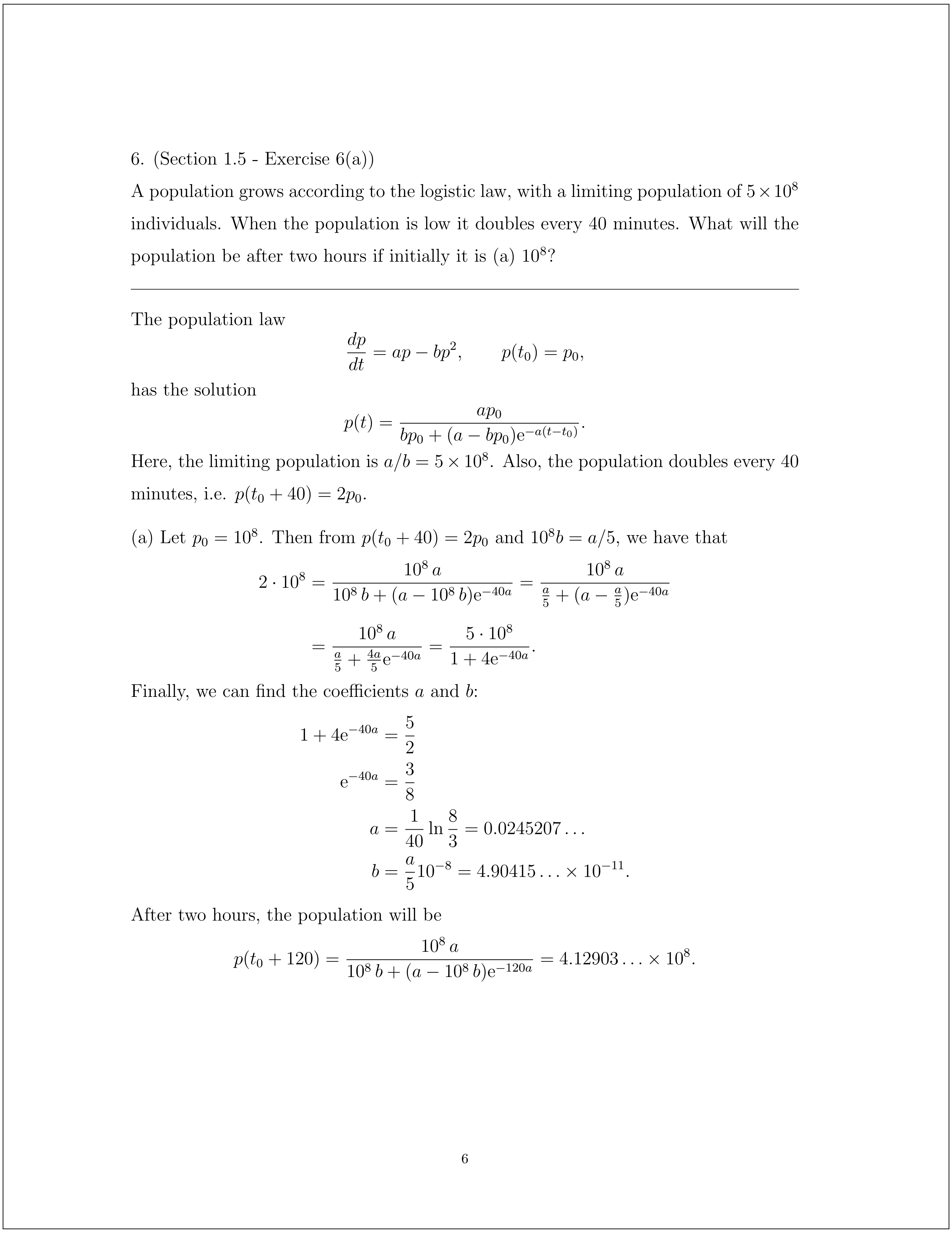
describes an exponential growth of a population. The solution to this problem is

$$p(t) = p_0\,\mathrm{e}^{a(t-t_0)}.$$

Suppose a population doubles its original size in 100 years. Then $p(t_0 + 100) = 2p_0$. Similarly, if this population triples its size in 200 years, then we should also have $p(t_0 + 200) = 3p_0$. From these two conditions we get

$$\begin{aligned} 2p_0 &= p(t_0+100) = p_0\,\mathrm{e}^{a(t_0+100-t_0)} = p_0\,\mathrm{e}^{100a} &&\longrightarrow && a = \frac{\ln 2}{100} = 0.00693147\ldots \\ 3p_0 &= p(t_0+200) = p_0\,\mathrm{e}^{a(t_0+200-t_0)} = p_0\,\mathrm{e}^{200a} &&\longrightarrow && a = \frac{\ln 3}{200} = 0.00549306\ldots, \end{aligned}$$

which shows that the Malthusain law cannot be satisfied.



6. (Section 1.5 - Exercise 6(a))

A population grows according to the logistic law, with a limiting population of 5×10^8 individuals. When the population is low it doubles every 40 minutes. What will the population be after two hours if initially it is (a) 10^8?

The population law

$$\frac{dp}{dt} = ap - bp^2, \qquad p(t_0) = p_0,$$

has the solution

$$p(t) = \frac{ap_0}{bp_0 + (a - bp_0)\mathrm{e}^{-a(t-t_0)}}.$$

Here, the limiting population is $a/b = 5 \times 10^8$. Also, the population doubles every 40 minutes, i.e. $p(t_0 + 40) = 2p_0$.

(a) Let $p_0 = 10^8$. Then from $p(t_0 + 40) = 2p_0$ and $10^8 b = a/5$, we have that

$$\begin{aligned} 2 \cdot 10^8 &= \frac{10^8\, a}{10^8\, b + (a - 10^8\, b)\mathrm{e}^{-40a}} = \frac{10^8\, a}{\frac{a}{5} + (a - \frac{a}{5})\mathrm{e}^{-40a}} \\ &= \frac{10^8\, a}{\frac{a}{5} + \frac{4a}{5}\mathrm{e}^{-40a}} = \frac{5 \cdot 10^8}{1 + 4\mathrm{e}^{-40a}}. \end{aligned}$$

Finally, we can find the coefficients a and b:

$$\begin{aligned} 1 + 4\mathrm{e}^{-40a} &= \frac{5}{2} \\ \mathrm{e}^{-40a} &= \frac{3}{8} \\ a &= \frac{1}{40} \ln \frac{8}{3} = 0.0245207\ldots \\ b &= \frac{a}{5} 10^{-8} = 4.90415\ldots \times 10^{-11}. \end{aligned}$$

After two hours, the population will be

$$p(t_0 + 120) = \frac{10^8\, a}{10^8\, b + (a - 10^8\, b)\mathrm{e}^{-120a}} = 4.12903\ldots \times 10^8.$$

2.3 HW 2

Local contents

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2.3.1 Section 1.8, problem 8

A tank contains 300 gallons of water and 100 gallons of pollutant. Fresh water is pumped into the tank at rate 2 gal/min, and the well stirred mixture leaves at the same rate. How long does it take for the concentration of pollutants in the tank to decrease to $\frac{1}{10}$ of its original value?

Solution

Let $V(t)$ be the volume in gallons of the pollutant at time t. Hence

$$\frac{dV(t)}{dt} = R_{in} - R_{out} \tag{1}$$

Where R_{in} is the rate in gallons per min that the pollutant is entering the tank and R_{out} is the rate in gallons per min that the pollutant is leaving the tank. In this problem

$$R_{in} = 0 \tag{1A}$$

Since no pollutant enters the tank. And $R_{out} = 2$ gal/min. But each gallon that leaves contains the ratio $\frac{V(t)}{400}$ of pollutant at any moment of time. This is because the volume of the tank is fixed at 400 gallons since same volume enters as it leaves. Hence

$$R_{out} = 2\frac{V(t)}{400} \qquad \text{gal/min} \tag{1B}$$

Using (1A,1B) in (1) gives

$$\begin{aligned} \frac{dV(t)}{dt} &= -\frac{2}{400}V(t) \\ \frac{dV(t)}{dt} + \frac{1}{200}V(t) &= 0 \end{aligned}$$

This is a linear ODE. The integration factor is $I = e^{\int \frac{1}{200}dt} = e^{\frac{t}{200}}$. Therefore the above can be written as

$$\begin{aligned} \frac{d}{dt}(V(t)I) &= 0 \\ \frac{d}{dt}\left(Ve^{\frac{t}{200}}\right) &= 0 \end{aligned}$$

Integrating gives the general solution as

$$Ve^{\frac{t}{200}} = C \tag{1}$$

Using initial conditions, at $t = 0$, $V = 100$ gallons. Substituting these in the above to solve for C gives

$$100 = C$$

Hence the solution (1) becomes

$$V(t) = 100e^{\frac{-t}{200}} \tag{2}$$

To find the time t when $V(t) = 10$ gallons (this is $\frac{1}{10}$ of the original volume of pollutant, which is 100 gallons), then the above becomes

$$10 = 100e^{\frac{-1}{200}t_0}$$

Solving for t_0 gives

$$\begin{aligned}
\frac{1}{10} &= e^{\frac{-1}{200}t_0}\\
\ln\left(\frac{1}{10}\right) &= \frac{-1}{200}t_0\\
t_0 &= -200\ln\left(\frac{1}{10}\right)
\end{aligned}$$

Hence

$$\boxed{t_0 = 460.517 \text{ minutes}}$$

This is the time it takes for the pollutant volume to decrease to $\frac{1}{10}$ of its original value in the tank.

2.3.2 Section 1.8, problem 14

Find the orthogonal trajectory of the curve $y = c\sin x$

Solution

Let

$$F\left(x,y,c\right) = c\sin x - y \tag{1}$$

Then $F_x = c\cos x$ and $F_y = -1$. Hence the slope of the orhogonal projection is given by

$$\begin{aligned}
\frac{dy}{dx} &= \frac{F_y}{F_x}\\
&= \frac{-1}{c\cos x}
\end{aligned}$$

From (1), we need to solve for c from $F\left(x,y,c\right) = 0$ which gives $c\sin x - y = 0$ or $c = \frac{y}{\sin x}$. Substituting this back into the above result gives

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-1}{\left(\frac{y}{\sin x}\right)\cos x}\\
&= \frac{-\sin x}{y\cos x}\\
&= -\frac{1}{y}\tan x
\end{aligned}$$

The above gives the ODE to sovle for the orthogonal trajectory curves. This is separable. Integrating gives

$$\int y dy = -\int \tan x dx$$

But $\int \tan x dx = -\ln|\cos(x)|$. Hence the above becomes

$$\begin{aligned}
\frac{y^2}{2} &= \ln\left(|\cos(x)|\right) + C_1\\
y^2 &= 2\ln\left(|\cos x|\right) + C
\end{aligned}$$

Where $C = 2C_1$. Solving for y gives two solutions

$$y(x) = \pm\sqrt{2\ln\left(|\cos x|\right) + C}$$

For illustration, the above was plotted for $C = 1,2,3,4,5$ in the following (shown in red color) against the function $\sin(x)$ (in blue color). It shows the projection curves all cross $\sin(x)$ at 90^0 everywhere as expected.

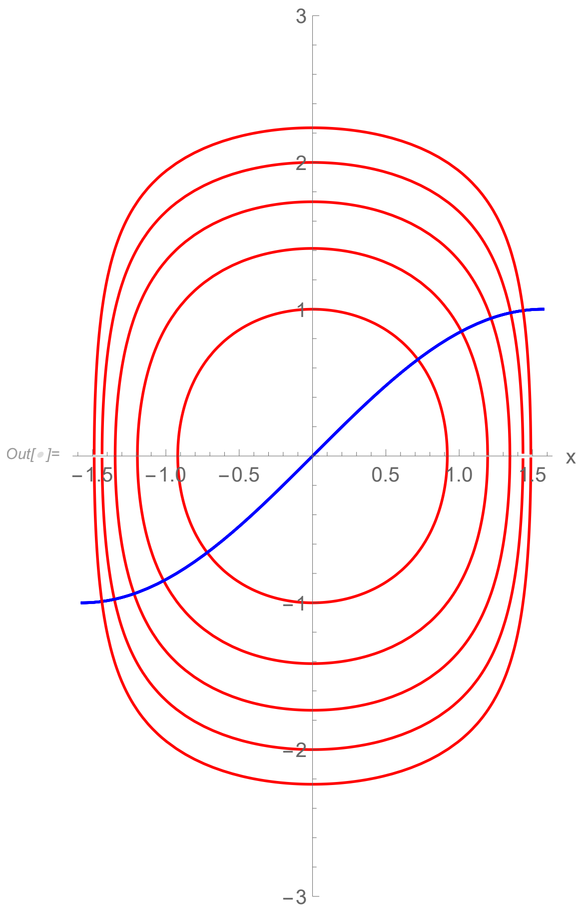
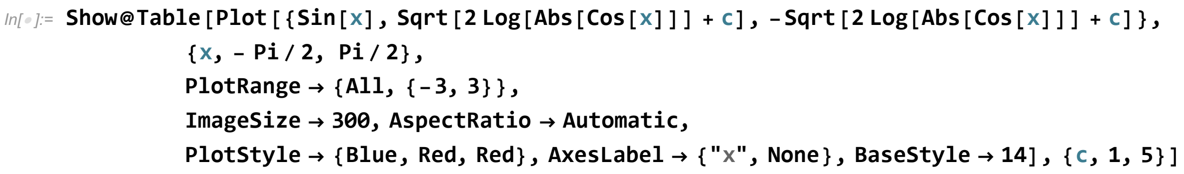


Figure 2.3: Orthogonal projections for different C values



```
In[ ]:= Show@Table[Plot[{Sin[x], Sqrt[2 Log[Abs[Cos[x]]] + c], -Sqrt[2 Log[Abs[Cos[x]]] + c]},
           {x, - Pi / 2, Pi / 2},
           PlotRange → {All, {-3, 3}},
           ImageSize → 300, AspectRatio → Automatic,
           PlotStyle → {Blue, Red, Red}, AxesLabel → {"x", None}, BaseStyle → 14], {c, 1, 5}]
```

Figure 2.4: code used for the above

The following plot is over a larger x range, from -2π to 2π

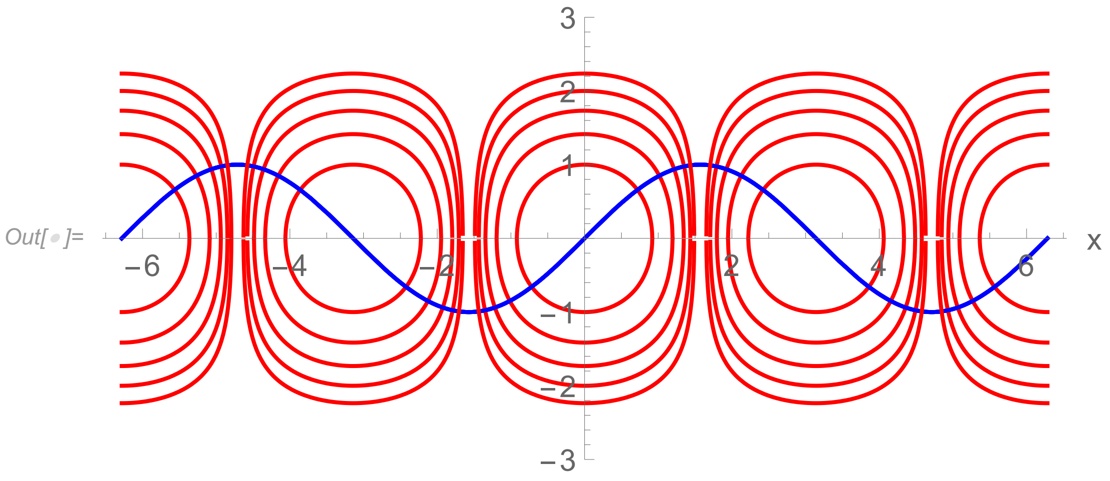


Figure 2.5: Orthogonal projections for different C values

2.3.3 section 1.10, problem 4

Show that the solution $y(t)$ of the given initial value problem exists on the specified interval.

$$y' = y^2 + \cos\left(t^2\right) \qquad y(0) = 0; \qquad 0 \le t \le \frac{1}{2}$$

Solution

Writing the ODE as

$$\begin{aligned} y' &= f\left(t,y\right) \\ &= y^2 + \cos\left(t^2\right) \end{aligned}$$

Let R be rectangle $0 \leq t \leq \frac{1}{2}, y_0 - b \leq y \leq y_0 + b$. But $y_0 = 0$ as given. Therefore

$$R = \left[0, \frac{1}{2}\right] \times [-b, b]$$

Now

$$\begin{aligned} M &= \max_{\left(t,y\right)\in R} \left|f\left(t,y\right)\right| \\ &= \max_{\left(t,y\right)\in R} \left|y^2 + \cos\left(t^2\right)\right| \\ &= b^2 + 1 \end{aligned}$$

Hence

$$\alpha = \min\left(a, \frac{b}{M}\right)$$

But $a = \frac{1}{2}, M = b^2 + 1$, therefore the above becomes

$$\alpha = \min\left(\frac{1}{2}, \frac{b}{b^2+1}\right)$$

The largest value α can obtain is when $g\left(b\right) = \frac{b}{b^2+1}$ is maximum.

$$\begin{aligned} g'\left(b\right) &= \frac{\left(b^2+1\right) - b\left(2b\right)}{\left(b^2+1\right)^2} \\ &= \frac{b^2 + 1 - 2b^2}{\left(b^2+1\right)^2} \\ &= \frac{1 - b^2}{\left(b^2+1\right)^2} \end{aligned}$$

Hence $g'\left(b\right) = 0$ gives $1 - b^2 = 0$ or $b = \pm 1$. Taking $b = 1$ gives $g_{\max}\left(b\right) = \frac{1}{1^2+1} = \frac{1}{2}$. Therefore

$$\begin{aligned} \alpha &= \min\left(\frac{1}{2}, \frac{1}{2}\right) \\ &= \frac{1}{2} \end{aligned}$$

This shows that the solution $y\left(t\right)$ exists on

$$t_0 \leq t \leq t_0 + \alpha$$

But $t_0 = 0, \alpha = \frac{1}{2}$, therefore

$$0 \leq t \leq \frac{1}{2}$$

Hence a unique solution exist inside rectangle

$$R = \left[0, \frac{1}{2}\right] \times [-1, 1]$$

2.3.4 Section 1.10, problem 17

Prove that $y\left(t\right) = -1$ is the only solution of the initial value problem

$$y' = t\left(1 + y\right) \qquad y\left(0\right) = -1$$

Solution

The solution is found first to show it is $y\left(t\right) = -1$, then using the uniqueness theory, one can show it is unique. The above ODE is separable. Hence

$$
\begin{aligned}
\int \frac{dy}{1+y} &= \int t dt \\
\ln\left(\left|1+y\right|\right) &= \frac{t^2}{2} + C \\
\left|1+y\right| &= e^{\frac{t^2}{2}+C} \\
1+y &= C_1 e^{\frac{t^2}{2}} \qquad (1)
\end{aligned}
$$

Applying initial conditions gives

$$
\begin{aligned}
1-1 &= C_1 \\
C_1 &= 0
\end{aligned}
$$

Hence the solution (1) becomes

$$
\begin{aligned}
1+y &= 0 \\
y(t) &= -1
\end{aligned}
$$

To show the above is the only solution we need to show the uniqueness theorem applies to this ODE over all of $\Re$. Let

$$
\begin{aligned}
y' &= f\left(t,y\right) \\
&= t\left(1+y\right)
\end{aligned}
$$

The above shows that $f\left(t,y\right)$ is continuous in t over $-\infty < t < \infty$ and continuous in y over $-\infty < y < \infty$. Now

$$
\frac{\partial f}{\partial y} = t
$$

Hence $\frac{\partial f}{\partial y}$ is also continuous in y over $-\infty < y < \infty$. Therefore a solution exist and is unique in any region that includes the initial conditions. Hence the solution $y(t) = -1$ found above is the only solution.

2.3.5 Section 1.13, problem 2

Using Euler's method with step size $h = 0.1$, determine an approximate value of the solution at $t = 1$ for

$$
y' = 2ty \qquad y(0) = 2
$$

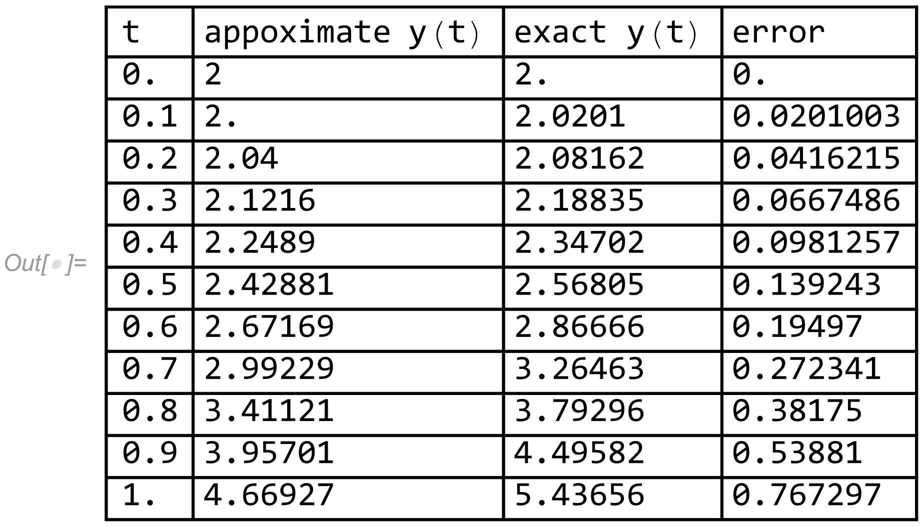
Which has analytical solution $y(t) = 2e^{t^2}$. Compute approximate value at $t = 1$ using just $h = 0.1$, and compare with $y(1)$.

Solution

Euler method is given by

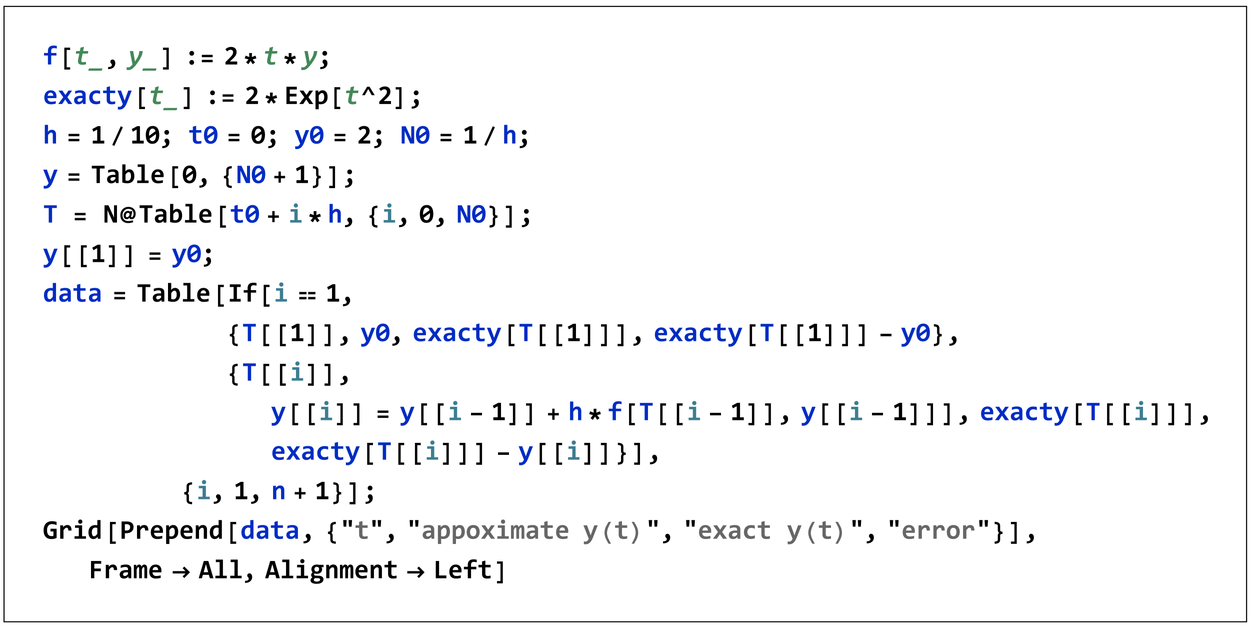
$$
\begin{aligned}
y_1 &= y_0 + hf\left(t_0, y_0\right) \\
y_2 &= y_1 + hf\left(t_1, y_1\right) \\
&\vdots \\
y_{k+1} &= y_k + hf\left(t_k, y_k\right)
\end{aligned}
$$

Where $y_0 = 2$ in this problem, and $t_1 = t_0 + h, t_2 = t_1 + h$ and so on. Where $h = 0.1$. The following table shows the numerical value of $y(t)$ found at each t starting from $0, 0.1, 0.2, \cdots, 1.0$ and comparing it to the exact $y(t)$ and the error at each step using a small Mathematica program which implements the above method.



t	appoximate y(t)	exact y(t)	error
0.	2	2.	0.
0.1	2.	2.0201	0.0201003
0.2	2.04	2.08162	0.0416215
0.3	2.1216	2.18835	0.0667486
0.4	2.2489	2.34702	0.0981257
0.5	2.42881	2.56805	0.139243
0.6	2.67169	2.86666	0.19497
0.7	2.99229	3.26463	0.272341
0.8	3.41121	3.79296	0.38175
0.9	3.95701	4.49582	0.53881
1.	4.66927	5.43656	0.767297

Figure 2.6: Table to compare Euler method with exact



```
f[t_, y_] := 2*t*y;
exacty[t_] := 2*Exp[t^2];
h = 1/10; t0 = 0; y0 = 2; N0 = 1/h;
y = Table[0, {N0 + 1}];
T  = N@Table[t0 + i*h, {i, 0, N0}];
y[[1]] = y0;
data = Table[If[i == 1,
            {T[[1]], y0, exacty[T[[1]]], exacty[T[[1]]] - y0},
            {T[[i]],
               y[[i]] = y[[i - 1]] + h*f[T[[i - 1]], y[[i - 1]]], exacty[T[[i]]],
               exacty[T[[i]]] - y[[i]]}],
         {i, 1, n + 1}];
Grid[Prepend[data, {"t", "appoximate y(t)", "exact y(t)", "error"}],
   Frame → All, Alignment → Left]
```

Figure 2.7: Code for Euler method to generate the above table

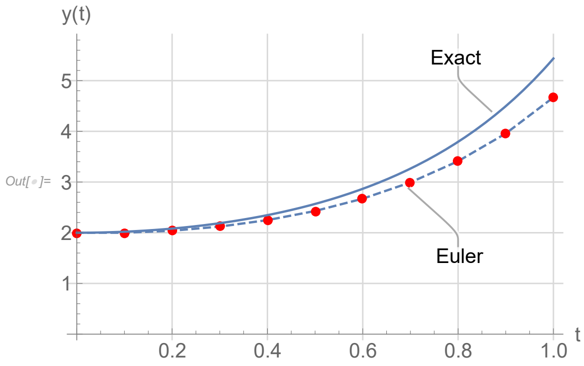
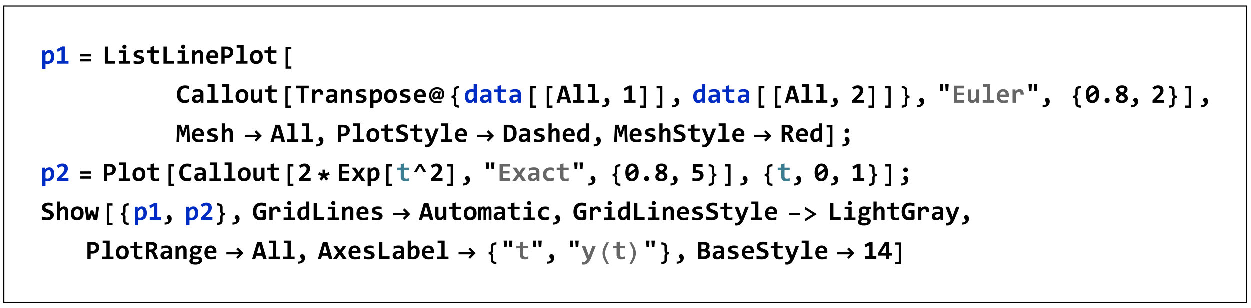


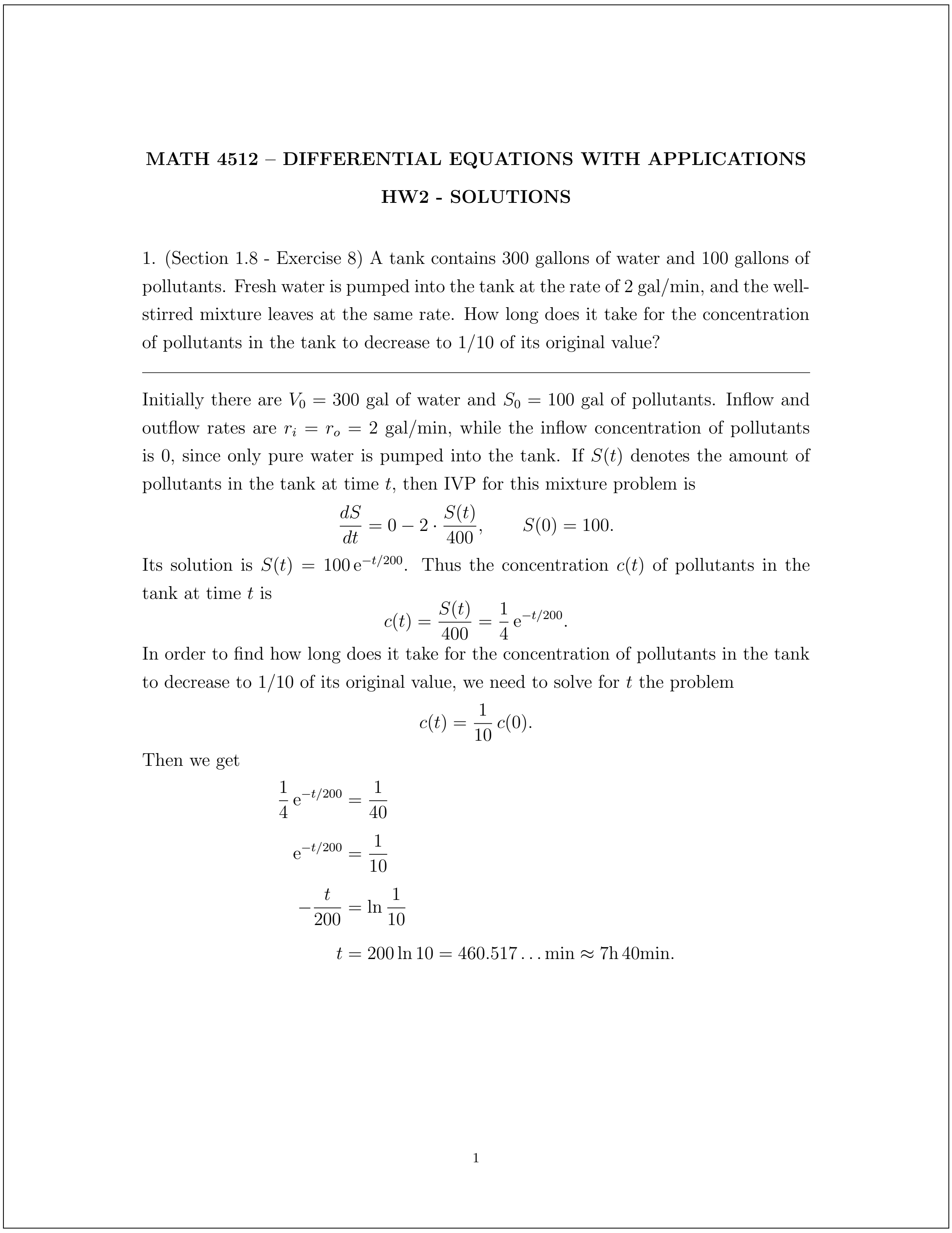
Figure 2.8: Plot of exact vs. Euler



```
p1 = ListLinePlot[
          Callout[Transpose@{data[[All, 1]], data[[All, 2]]}, "Euler", {0.8, 2}],
          Mesh → All, PlotStyle → Dashed, MeshStyle → Red];
p2 = Plot[Callout[2*Exp[t^2], "Exact", {0.8, 5}], {t, 0, 1}];
Show[{p1, p2}, GridLines → Automatic, GridLinesStyle -> LightGray,
   PlotRange → All, AxesLabel → {"t", "y(t)"}, BaseStyle → 14]
```

Figure 2.9: Code to make plot

2.3.6 Key solution for HW 2



MATH 4512 – DIFFERENTIAL EQUATIONS WITH APPLICATIONS

HW2 - SOLUTIONS

1. (Section 1.8 - Exercise 8) A tank contains 300 gallons of water and 100 gallons of pollutants. Fresh water is pumped into the tank at the rate of 2 gal/min, and the well-stirred mixture leaves at the same rate. How long does it take for the concentration of pollutants in the tank to decrease to 1/10 of its original value?

Initially there are $V_0 = 300$ gal of water and $S_0 = 100$ gal of pollutants. Inflow and outflow rates are $r_i = r_o = 2$ gal/min, while the inflow concentration of pollutants is 0, since only pure water is pumped into the tank. If $S(t)$ denotes the amount of pollutants in the tank at time t, then IVP for this mixture problem is

$$\frac{dS}{dt} = 0 - 2 \cdot \frac{S(t)}{400}, \qquad S(0) = 100.$$

Its solution is $S(t) = 100\,\mathrm{e}^{-t/200}$. Thus the concentration $c(t)$ of pollutants in the tank at time t is

$$c(t) = \frac{S(t)}{400} = \frac{1}{4}\,\mathrm{e}^{-t/200}.$$

In order to find how long does it take for the concentration of pollutants in the tank to decrease to 1/10 of its original value, we need to solve for t the problem

$$c(t) = \frac{1}{10}\,c(0).$$

Then we get

$$\begin{aligned} \frac{1}{4}\,\mathrm{e}^{-t/200} &= \frac{1}{40} \\ \mathrm{e}^{-t/200} &= \frac{1}{10} \\ -\frac{t}{200} &= \ln\frac{1}{10} \\ t &= 200\ln 10 = 460.517\ldots \mathrm{min} \approx 7\mathrm{h}\,40\mathrm{min}. \end{aligned}$$

2. (Section 1.8 - Exercise 14)

Find the orthogonal trajectories of the given family of curves

$$y = c\,\sin x.$$

Here we can take $F(x, y, c) = y - c\,\sin x$. Then from

$$F_x = -c\,\cos x, \qquad F_y = 1, \qquad c = \frac{y}{\sin x},$$

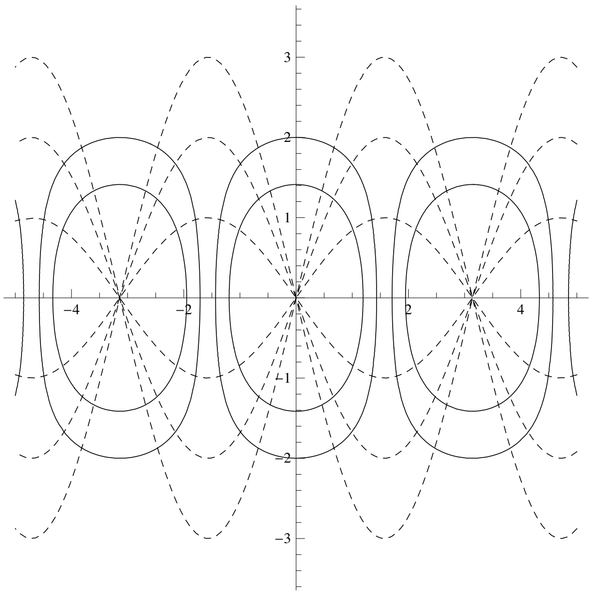
the orthogonal trajectories of the given family are the solution curves of the equation

$$\frac{dy}{dx} = \frac{F_y}{F_x} = -\frac{\tan x}{y}.$$

This is a separable differential equation and we solve it as follows:

$$\int y\,dy = -\int \tan x dx$$

$$\frac{y^2}{2} = \ln|\cos x| + c.$$



Curves $y = c\,\sin x$ (dashed) and $\frac{y^2}{2} = \ln|\cos x| + c$ (solid).

3. (Section 1.10 - Exercise 4)
Show that the solution y of the initial-value problem

$$\frac{dy}{dt} = y^2 + \cos t^2, \qquad y(0) = 0,$$

exists on the interval $0 \le t \le \frac{1}{2}$.

Let $f(t, y) = y^2 + \cos t^2$. The functions f and $f_y = 2y$ are continuous on a rectangle

$$R = [t_0, t_0 + a] \times [y_0 - b, y_0 + b] = [0, a] \times [-b, b],$$

for arbitrary constants $a > 0$ and $b > 0$. Then there exists a unique solution of the IVP on the interval $[0, \alpha]$, with

$$\alpha = \min\{a, \frac{b}{M}\}, \qquad M = \max_{(t,y)\in R} |y^2 + \cos t^2|.$$

Since

$$M = \max_{(t,y)\in R} |y^2 + \cos t^2| = b^2 + 1,$$

then

$$\alpha = \min\{a, \frac{b}{M}\} = \min\{a, \frac{b}{b^2 + 1}\}.$$

Let

$$g(b) = \frac{b}{b^2 + 1}.$$

For $b > 0$, the function g is positive and

$$g'(b) = \frac{1 - b^2}{(b^2 + 1)^2} = \frac{(1 - b)(1 + b)}{(b^2 + 1)^2}.$$

The point $b = 1$ is the local maximum of g on $(0, \infty)$ since

$$b \in (0, 1) \Rightarrow g'(b) > 0 \Rightarrow g \text{ is increasing},$$

$$b \in (1, \infty) \Rightarrow g'(b) < 0 \Rightarrow g \text{ is decreasing}.$$

Therefore

$$\frac{b}{b^2 + 1} = g(b) \le g(1) = \frac{1}{2}.$$

Consequently, the largest possible value for α is $1/2$ (obtained for $b = 1$ and any $a \ge 1/2$), that concludes the proof.

4. (Section 1.10 - Exercise 17)
Prove that $y(t) = -1$ is the only solution of the initial-value problem

$$\frac{dy}{dt} = t(1+y), \qquad y(0) = -1.$$

First notice that the constant function $y(t) = -1$ is the solution of the given IVP (its derivative is zero, $1 + y = 0$ and $y(0) = -1$). In order to prove that this is the only solution, we need to analyze the function $f(t, y) = t(1 + y)$ and its partial derivative $f_y = t$. On a rectangle

$$R = [0, a] \times [-1 - b, -1 + b],$$

both f and f_y are continuous functions, for arbitrary positive constants a, b. Let

$$M = \max_{(t,y)\in R} |f(t, y)| = \max_{(t,y)\in R} |t(1 + y)| = a\,b,$$

and

$$\alpha = \min\{a, \frac{b}{M}\} = \min\{a, \frac{1}{a}\} = 1.$$

(Remark: Conclusion $\alpha = 1$ can be deduced from assuming first that $\min\{a, 1/a\} = a$. Then $a \leq 1/a$ and

$$a - \frac{1}{a} \leq 0$$

$$\frac{a^2 - 1}{a} \leq 0$$

$$\frac{(a-1)(a+1)}{a} \leq 0 \quad \longrightarrow \quad a \leq 1 \quad \longrightarrow \quad \min\{a, \frac{1}{a}\} = a \leq 1.$$

Similarly, assuming $\min\{a, 1/a\} = 1/a$ we obtain $1/a \leq a$ and

$$a - \frac{1}{a} \geq 0$$

$$\frac{(a-1)(a+1)}{a} \geq 0 \quad \longrightarrow \quad a \geq 1 \quad \longrightarrow \quad \min\{a, \frac{1}{a}\} = \frac{1}{a} \leq 1.)$$

From the existence-uniqueness theorem, we conclude that the solution $y(t) = -1$ of the IVP is unique in the interval $t_0 \leq t \leq t_0 + \alpha$, i.e. when $0 \leq t \leq 1$.

5. (Section 1.13 - Exercise 2 with $h = 0.1$)
Using Euler's method with step size $h = 0.1$, determine an approximate value of the solution at $t = 1$ for the initial-vale problem

$$\frac{dy}{dt} = 2ty, \qquad y(0) = 2,$$

and compare the results with the exact solution $y(t) = 2\mathrm{e}^{t^2}$.

Let $t_0 = 0$, $y_0 = 2$ and $f(t, y) = 2ty$. Using equidistant points

$$t_{k+1} = t_k + h, \qquad k = 0, 1, \ldots, 9, \qquad h = 0.1,$$

Euler's method

$$y_{k+1} = y_k + h\, f(t_k, y_k), \qquad k = 0, 1, \ldots, 9, \qquad y_0 = y(t_0),$$

will generate the following data

k	t_k	y_k
0	0	2
1	0.1	2
2	0.2	2.04
3	0.3	2.1216
4	0.4	2.2489
5	0.5	2.42881
6	0.6	2.67169
7	0.7	2.99229
8	0.8	3.41121
9	0.9	3.95701
10	1	4.66927

From this table we read $y_{10} = 4.66927$ is the approximation to $y(1) = 2\,\mathrm{e} = 5.43656$. Absolute error is

$$|y(1) - y_{10}| = 0.767297.$$

2.4 HW 3

Local contents

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2.4.1 Section 2.1, problem 11

Let $y_1(t) = t^2$ and $y_2(t) = t|t|$

1. Show that y_1, y_2 are linearly dependent (L.D.) on the interval $0 \leq t \leq 1$
2. Show that y_1, y_2 are linearly independent (L.I.) on the interval $-1 \leq t \leq 1$
3. Show that $W\left[y_1, y_2\right](t)$ is identically zero.
4. Show that y_1, y_2 can never be two solutions of (3) which is $y'' + p(t)y' + q(t)y = 0$, on the interval $-1 < t < 1$ if both p, q are continuous in this interval.

Solution

2.4.1.1 Part a

On the interval $0 \leq t \leq 1$, then $|t| = t$ since t is positive. Hence $y_2(t) = t^2$, which is the same as $y_1(t) = t^2$. Therefore they are linearly dependent (same solution). In other words, $y_1(t) = c_1 y_2(t)$ where $c_1 = 1$.

2.4.1.2 Part b

When $t \leq 0$ now $y_2(t) = -t^2$. Hence we have $y_1 = y_2$ for $0 \leq t \leq 1$ and $y_1 = -y_2$ for $-1 \leq t < 0$. Therefore it is not possible to find the same constant c such that $y_1 = cy_2$ which will work for all t regions. This implies that $y_1(t)$ and $y_2(t)$ are linearly independent on $-1 \leq t \leq 1$.

2.4.1.3 Part c

$$W\left[y_1, y_2\right](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

If $W(t) = 0$ is in some region or at some point, then it must be zero anywhere. Therefore let us pick the interval $0 \leq t \leq 1$ to calculate $W(t)$. This way we avoid having to deal with the $|t|$ when taking derivatives since on this interval, $y_1 = t^2$ and also $y_2 = t^2$. Now $W(t)$ becomes

$$\begin{aligned} W(t) &= t^2(2t) - t^2(2t) \\ &= 0 \end{aligned}$$

Therefore $W(t) = 0$ everywhere.

2.4.1.4 Part d

Since p, q are continuous on $-1 < t < 1$, then by uniqueness theorem, we know there are two fundamental solutions y_1, y_2, which must be linearly independent that their linear combination give the general solution $y(t) = c_1 y_1(t) + c_2 y_2(t)$.

But from part(b) above we found that the given functions y_1, y_2 are not linearly independent on $-1 < t < 1$, hence these can never be the fundamental solutions to $y'' + p(t)y' + q(t)y = 0$.

2.4.2 Section 2.2.1, problem 6 (page 144, complex roots)

Solve $y'' + 2y' + 5y = 0$ with $y(0) = 0, y'(0) = 2$

Solution

Let $y = e^{\lambda t}$. Substituting in the above ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + 5e^{\lambda t} = 0$$
$$e^{\lambda t}\left(\lambda^2 + 2\lambda + 5\right) = 0$$

Since $e^{\lambda t} \neq 0$, the above simplifies to $\lambda^2 + 2\lambda + 5 = 0$. The roots are $\lambda = \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} = \frac{-2}{2} \pm \frac{1}{2}\sqrt{4 - 4(5)}$ or $\lambda = -1 \pm \frac{1}{2}\sqrt{-16}$. Hence

$$\lambda = -1 \pm 2i$$

Therefore the general solution is linear combination of

$$\begin{aligned} y(t) &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ &= c_1 e^{(-1+2i)t} + c_2 e^{(-1-2i)t} \\ &= e^{-t}\left(c_1 e^{2it} + c_2 e^{-2it}\right) \end{aligned}$$

But $c_1 e^{2it} + c_2 e^{-2it}$ can be rewritten, using Euler relation, as $C_1 \cos 2t + C_2 \sin 2t$. The above solution becomes

$$y(t) = e^{-t}\left(C_1 \cos 2t + C_2 \sin 2t\right) \tag{1}$$

C_1, C_2 are now found from initial conditions. At $t = 0$

$$0 = C_1$$

The solution (1) simplifies to

$$y(t) = C_2 e^{-t} \sin 2t \tag{2}$$

Taking time derivative gives

$$y'(t) = C_2\left(-e^{-t} \sin 2t + 2e^{-t} \cos 2t\right)$$

At $t = 0$ the above becomes

$$\begin{aligned} 2 &= 2C_2 \\ C_2 &= 1 \end{aligned}$$

Substituting the above in (2) gives the final general solution

$$y(t) = e^{-t} \sin 2t$$

2.4.3 section 2.2.2, problem 6 (page 149, equal roots)

Solve the following initial-value problems $y'' + 2y' + y = 0$ with $y(2) = 1, y'(2) = -1$

Solution

Let $y = e^{\lambda t}$. Substituting in the above ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + e^{\lambda t} = 0$$
$$e^{\lambda t}\left(\lambda^2 + 2\lambda + 1\right) = 0$$

Since $e^{\lambda t} \neq 0$, the above simplifies to $\lambda^2 + 2\lambda + 1 = 0$ or $(\lambda + 1)^2 = 0$. Hence there is a double root $\lambda = -1$. One fundamental solution is

$$y_1 = e^{-t}$$

To find the second solution, reduction of order is used. Let the second solution be

$$\begin{aligned} y_2(t) &= y_1(t)\, u(t) \\ &= e^{-t} u \end{aligned} \tag{1}$$

Hence

$$y_2' = -e^{-t} u + e^{-t} u' \tag{2}$$
$$y_2'' = e^{-t} u - e^{-t} u' - e^{-t} u' + e^{-t} u'' \tag{3}$$

Substituting (1,2,3) into the ODE gives (since y_2 is assumed to be a solution)

$$\begin{aligned}\left(e^{-t}u - e^{-t}u' - e^{-t}u' + e^{-t}u''\right) + 2\left(-e^{-t}u + e^{-t}u'\right) + \left(e^{-t}u\right) &= 0\\ \left(u - u' - u' + u''\right) + 2\left(-u + u'\right) + u &= 0\\ u'' - 2u' + u - 2u + 2u' + u &= 0\\ u'' &= 0\end{aligned}$$

Hence the solution is $u = C_1 t + C_2$. Therefore from (1) the second solution is

$$\begin{aligned}y_2(t) &= y_1(t)\,u(t)\\ &= e^{-t}\left(C_1 t + C_2\right)\end{aligned}$$

Therefore the general solution is

$$\begin{aligned}y(t) &= C_3 y_1 + C_4 y_2\\ &= C_3 e^{-t} + C_4 e^{-t}\left(C_1 t + C_2\right)\end{aligned}$$

Combining constants gives

$$\begin{aligned}y(t) &= C_3 e^{-t} + e^{-t}\left(C_1 t + C_2\right)\\ &= \left(C_3 + C_2\right)e^{-t} + C_1 t e^{-t}\end{aligned}$$

Let $A = \left(C_3 + C_2\right), B = C_1$, then the final solution is

$$y(t) = Ae^{-t} + Bte^{-t} \tag{4}$$

Now A, B are found from initial conditions $y(2) = 1, y'(2) = -1$. First initial condition gives from (4)

$$1 = Ae^{-2} + 2Be^{-2} \tag{5}$$

Taking derivative of (4) gives

$$y'(t) = -Ae^{-t} + B\left(e^{-t} - te^{-t}\right)$$

Applying second initial condition on the above gives

$$\begin{aligned}-1 &= -Ae^{-2} + B\left(e^{-2} - 2e^{-2}\right)\\ &= -Ae^{-2} - Be^{-2} \qquad (6)\end{aligned}$$

Now we need to solve (5,6) for (A, B). Adding (5,6) gives

$$0 = Be^{-2}$$

Hence $B = 0$. Therefore from (5) we can now solve for A

$$\begin{aligned}1 &= Ae^{-2}\\ A &= e^2\end{aligned}$$

Hence (4) now becomes

$$\begin{aligned}y(t) &= e^2 e^{-t}\\ &= e^{2-t}\end{aligned}$$

2.4.4 Section 2.4, problem 6 (page 156, Variation of parameters)

Solve the following initial-value problems $y'' + 4y' + 4y = t^{\frac{5}{2}}e^{-2t}$ with $y(0) = 0, y'(0) = 0$

Solution

The first step is to solve the homogenous ODE $y'' + 4y' + 4y = 0$. The characteristic equation is

$$\begin{aligned}\lambda^2 + 4\lambda + 4 &= 0\\ (\lambda + 2)(\lambda + 2) &= 0\end{aligned}$$

Hence a double root at $\lambda = -2$. The first solution is $y_1 = e^{=2t}$. Therefore the second solution is $y_2 = te^{-2t}$ (obtained using reduction of order as was done in the above problem with equal roots). Therefore the homogenous $y_h(t)$ is

$$y_h(t) = C_1 e^{-2t} + C_2 te^{-2t}$$

To find the particular solution $y_p(t)$, Variation of parameters will be used. Assuming the particular solution is

$$y_p(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$$

Where

$$u_1(t) = -\int \frac{y_2(t)\,f(t)}{W(t)}dt \tag{1}$$

And

$$u_2(t) = \int \frac{y_1(t)\,f(t)}{W(t)}dt \tag{2}$$

Where in the above $f(t) = t^{\frac{5}{2}}e^{-2t}$ and $y_1 = e^{=2t}, y_2 = te^{-2t}$. We now need to find $W(t)$

$$\begin{aligned}
W(t) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= y_1y_2' - y_2y_1' \\
&= e^{=2t}\left(e^{-2t} - 2te^{-2t}\right) + 2te^{-2t}e^{-2t} \\
&= e^{-4t} - 2te^{-4t} + 2te^{-4t} \\
&= e^{-4t}
\end{aligned}$$

Therefore (1) becomes

$$\begin{aligned}
u_1(t) &= -\int \frac{te^{-2t}t^{\frac{5}{2}}e^{-2t}}{e^{-4t}}dt \\
&= -\int t^{\frac{5}{2}+1}dt \\
&= -\int t^{\frac{7}{2}}dt \\
&= -\frac{t^{\frac{9}{2}}}{\frac{9}{2}} \\
&= \frac{-2}{9}t^{\frac{9}{2}}
\end{aligned}$$

And (2) becomes

$$\begin{aligned}
u_2(t) &= \int \frac{e^{-2t}t^{\frac{5}{2}}e^{-2t}}{e^{-4t}}dt \\
&= \int t^{\frac{5}{2}}dt \\
&= \frac{t^{\frac{7}{2}}}{\frac{7}{2}} \\
&= \frac{2}{7}t^{\frac{7}{2}}
\end{aligned}$$

Since $y_p(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, then using the above results we obtain the particular solution

$$\begin{aligned}
y_p(t) &= \left(\frac{-2}{9}t^{\frac{9}{2}}\right)e^{-2t} + \left(\frac{2}{7}t^{\frac{7}{2}}\right)te^{-2t} \\
&= e^{-2t}\left(\frac{-2}{9}t^{\frac{9}{2}} + \frac{2}{7}t^{\frac{9}{2}}\right) \\
&= \frac{4}{63}e^{-2t}t^{\frac{9}{2}}
\end{aligned}$$

Since $y(t) = y_h(t) + y_p(t)$ then the final solution is

$$\begin{aligned}
y(t) &= \left(C_1e^{-2t} + C_2te^{-2t}\right) + \frac{4}{63}e^{-2t}t^{\frac{9}{2}} \\
&= e^{-2t}\left(C_1 + C_2t + \frac{4}{63}t^{\frac{9}{2}}\right) \qquad (3)
\end{aligned}$$

Now initial conditions are applied to find C_1, C_2. From $y(0) = 0$, then (3) becomes

$$0 = C_1$$

Hence the solution (3) simplifies to

$$y(t)=e^{-2t}\left(C_2t+\frac{4}{63}t^{\frac{9}{2}}\right) \tag{4}$$

Taking derivatives

$$\begin{aligned}y'(t)&=-2e^{-2t}\left(C_2t+\frac{4}{63}t^{\frac{9}{2}}\right)+e^{-2t}\left(C_2+\left(\frac{4}{63}\right)\left(\frac{7}{2}\right)t^{\frac{7}{2}}\right)\\&=-2e^{-2t}\left(C_2t+\frac{4}{63}t^{\frac{9}{2}}\right)+e^{-2t}\left(C_2+\frac{2}{9}t^{\frac{7}{2}}\right)\end{aligned}$$

Applying the second BC $y'(0)=0$ to the above gives

$$0=C_2$$

The solution (4) now reduces to

$$y(t)=\frac{4}{63}t^{\frac{9}{2}}e^{-2t}$$

Which is just the particular solution. This makes sense, since both initial conditions are zero, then the homogenous solution will be zero.

2.4.5 Section 2.5, problem 14 (page 164, Guessing method)

Find the particular solution for $y''+2y'=1+t^2+e^{-2t}$

Solution

The first step is to solve the homogeneous solution $y_h(t)$ of the ODE $y''+2y'=0$. Let $u=y'$. Then the ODE becomes

$$u'+2u=0$$

The integrating factor is $I=e^{\int 2dt}=e^{2t}$. The above becomes

$$\begin{aligned}\frac{d}{dt}\left(ue^{2t}\right)&=0\\ue^{2t}&=C_1\\u&=C_1e^{-2t}\end{aligned}$$

But $y'=u$. Integrating gives

$$\begin{aligned}y_h(t)&=\int C_1e^{-2t}dt+C_2\\&=\frac{-1}{2}C_1e^{-2t}+C_2\\&=C_3e^{-2t}+C_2\end{aligned}$$

Hence the fundamental solutions are

$$\begin{aligned}y_1&=e^{-2t}\\y_2&=1\end{aligned}$$

We now go back to the original ODE and find the particular solution y_p. Since the RHS is $p(t)+e^{-2t}$ where $p(t)=1+t^2$, we can use linearity and find particular solution $y_{p_1}(t)$ associated with $p(t)$ only and then find $y_{p_2}(t)$ associated with e^{-2t} only and then add them together to obtain $y_p(t)$. In other words

$$y_p(t)=y_{p_1}(t)+y_{p_2}(t)$$

To find $y_{p_1}(t)$ associated with $1+t^2$ we guess $y_{p_1}(t)=C_0+C_1t+C_2t^2$. But because the ODE is missing the y term in it, then we have to multiply this guess by an extra t. Therefore it becomes

$$y_{p_1}(t)=t\left(C_0+C_1t+C_2t^2\right)$$

To find $y_{p_2}(t)$ associated with e^{-2t} we guess $y_{p_2}=Ae^{-2t}$. But because e^{-2t} is also a fundamental solution of the homogenous solution found above, we have to again adjust this and multiply the guess by t. Hence it becomes

$$y_{p_2}(t)=Ate^{-2t}$$

Therefore the full guess for particular solution becomes

$$\begin{aligned} y_p(t) &= y_{p_1}(t) + y_{p_2}(t) \\ &= t\left(C_0 + C_1 t + C_2 t^2\right) + Ate^{-2t} \\ &= tC_0 + C_1 t^2 + C_2 t^3 + Ate^{-2t} \qquad \text{(1A)} \end{aligned}$$

Now

$$y_p'(t) = C_0 + 2C_1 t + 3C_2 t^2 + Ae^{-2t} - 2Ate^{-2t} \qquad (1)$$

And

$$y_p''(t) = 2C_1 + 6C_2 t - 2Ae^{-2t} - 2Ae^{-2t} + 4Ate^{-2t} \qquad (2)$$

Substituting (1,2) into LHS of $y'' + 2y' = 1 + t^2 + e^{-2t}$ gives

$$\begin{aligned} \left(2C_1 + 6C_2 t - 2Ae^{-2t} - 2Ae^{-2t} + 4Ate^{-2t}\right) + 2\left(C_0 + 2C_1 t + 3C_2 t^2 + Ae^{-2t} - 2Ate^{-2t}\right) &= 1 + t^2 + e^{-2t} \\ 2C_0 + 2C_1 + 4tC_1 + 6tC_2 - 2Ae^{-2t} + 6t^2 C_2 &= 1 + t^2 + e^{-2t} \\ e^{-2t}(-2A) + t(4C_1 + 6C_2) + t^2(6C_2) + (2C_0 + 2C_1) &= 1 + t^2 + e^{-2t} \end{aligned}$$

Comparing coefficients gives

$$\begin{aligned} -2A &= 1 \\ 4C_1 + 6C_2 &= 0 \\ 6C_2 &= 1 \\ 2C_0 + 2C_1 &= 1 \end{aligned}$$

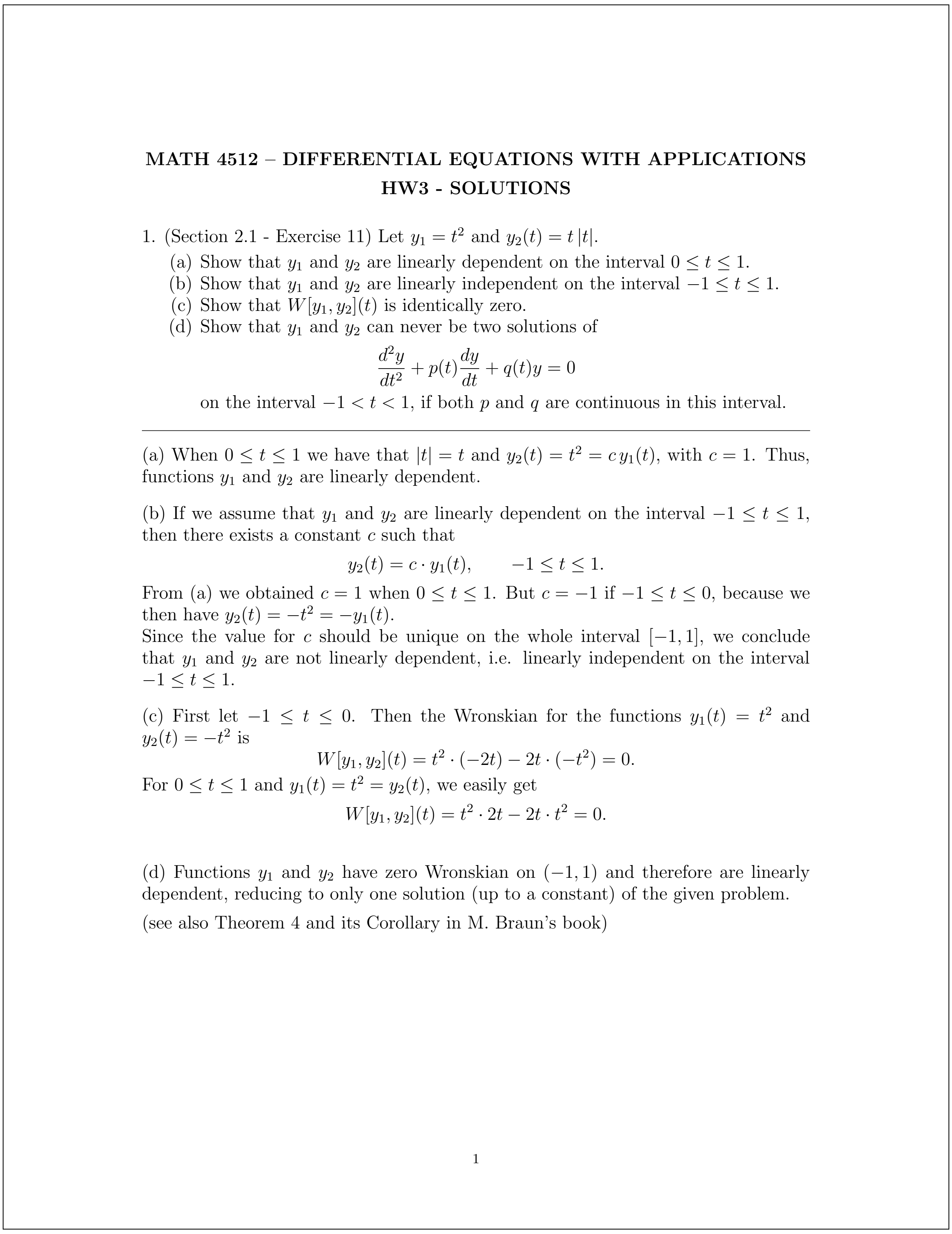
Solving gives $A = -\frac{1}{2}, C_0 = \frac{3}{4}, C_1 = -\frac{1}{4}, C_2 = \frac{1}{6}$. Substituting the above in (1A) gives the particular solution as

$$\begin{aligned} y_p(t) &= t\left(\frac{3}{4} - \frac{1}{4}t + \frac{1}{6}t^2\right) - \frac{1}{2}te^{-2t} \\ &= \frac{3}{4}t - \frac{1}{4}t^2 + \frac{1}{6}t^3 - \frac{1}{2}te^{-2t} \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= C_3 e^{-2t} + C_2 + \left(\frac{3}{4}t - \frac{1}{4}t^2 + \frac{1}{6}t^3 - \frac{1}{2}te^{-2t}\right) \end{aligned}$$

2.4.6 Key solution for HW 3



MATH 4512 – DIFFERENTIAL EQUATIONS WITH APPLICATIONS
HW3 - SOLUTIONS

1. (Section 2.1 - Exercise 11) Let $y_1 = t^2$ and $y_2(t) = t\,|t|$.
   - (a) Show that y_1 and y_2 are linearly dependent on the interval $0 \le t \le 1$.
   - (b) Show that y_1 and y_2 are linearly independent on the interval $-1 \le t \le 1$.
   - (c) Show that $W[y_1, y_2](t)$ is identically zero.
   - (d) Show that y_1 and y_2 can never be two solutions of

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

on the interval $-1 < t < 1$, if both p and q are continuous in this interval.

(a) When $0 \le t \le 1$ we have that $|t| = t$ and $y_2(t) = t^2 = c\,y_1(t)$, with $c = 1$. Thus, functions y_1 and y_2 are linearly dependent.

(b) If we assume that y_1 and y_2 are linearly dependent on the interval $-1 \le t \le 1$, then there exists a constant c such that

$$y_2(t) = c \cdot y_1(t), \qquad -1 \le t \le 1.$$

From (a) we obtained $c = 1$ when $0 \le t \le 1$. But $c = -1$ if $-1 \le t \le 0$, because we then have $y_2(t) = -t^2 = -y_1(t)$.
Since the value for c should be unique on the whole interval $[-1, 1]$, we conclude that y_1 and y_2 are not linearly dependent, i.e. linearly independent on the interval $-1 \le t \le 1$.

(c) First let $-1 \le t \le 0$. Then the Wronskian for the functions $y_1(t) = t^2$ and $y_2(t) = -t^2$ is

$$W[y_1, y_2](t) = t^2 \cdot (-2t) - 2t \cdot (-t^2) = 0.$$

For $0 \le t \le 1$ and $y_1(t) = t^2 = y_2(t)$, we easily get

$$W[y_1, y_2](t) = t^2 \cdot 2t - 2t \cdot t^2 = 0.$$

(d) Functions y_1 and y_2 have zero Wronskian on $(-1, 1)$ and therefore are linearly dependent, reducing to only one solution (up to a constant) of the given problem.

(see also Theorem 4 and its Corollary in M. Braun's book)

1

2. (Subsection 2.2.1 - Exercise 6) Solve the initial-value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0, \qquad y(0) = 0, \qquad y'(0) = 2.$$

The characteristic equation

$$r^2 + 2r + 5 = 0$$

has two complex roots $r_1 = -1 + 2i$ and $r_2 = -1 - 2i$. The general solution has the form

$$y(t) = c_1 \mathrm{e}^{-t} \cos(2t) + c_2 \mathrm{e}^{-t} \sin(2t),$$

with

$$y'(t) = (-c_1 + 2c_2)\mathrm{e}^{-t} \cos(2t) + (-2c_1 - c_2)\mathrm{e}^{-t} \sin(2t).$$

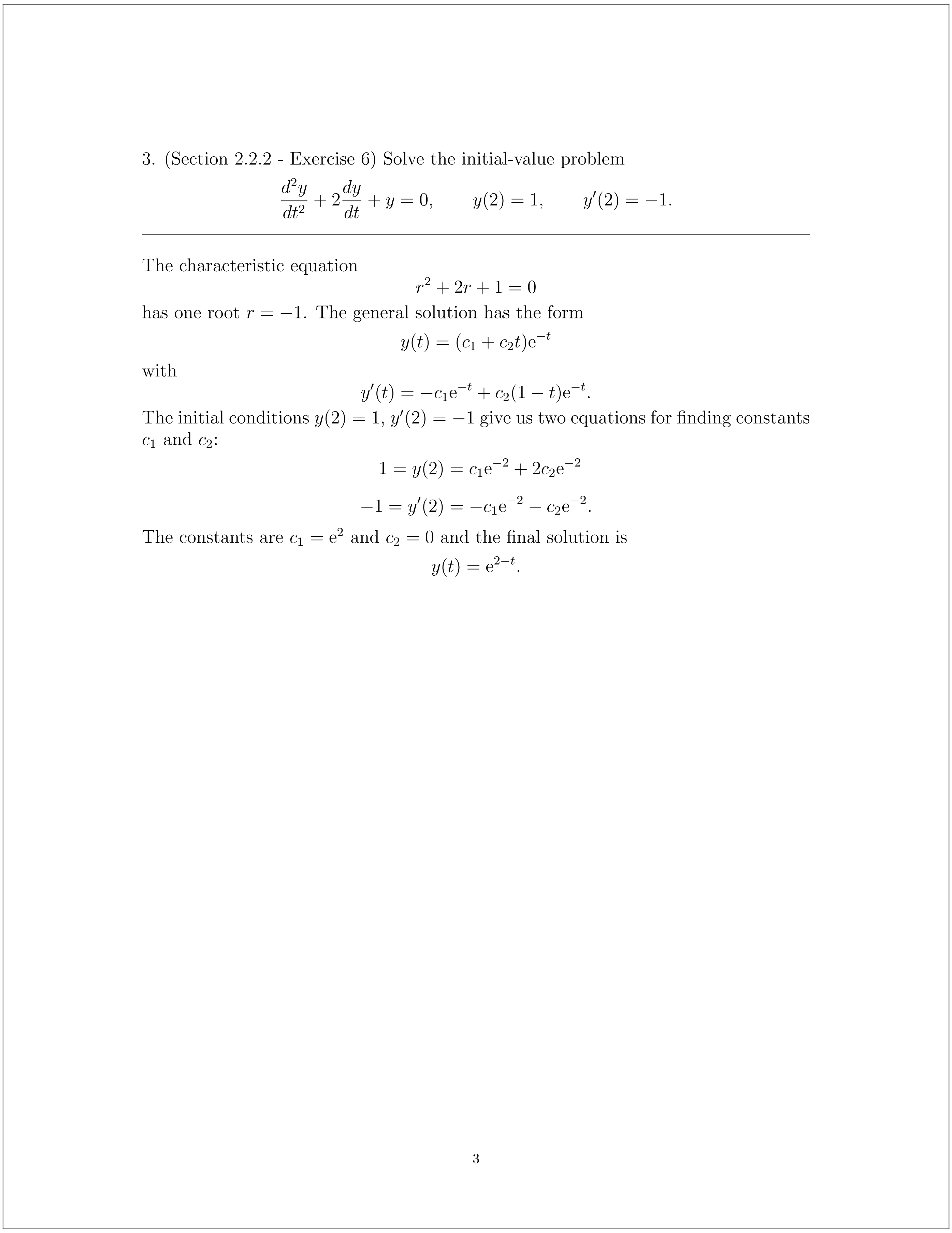
The initial conditions $y(0) = 0$, $y'(0) = 2$ give us two equations for finding constants c_1 and c_2:

$$0 = y(0) = c_1$$

$$2 = y'(0) = -c_1 + 2c_2.$$

The constants are $c_1 = 0$ and $c_2 = 1$ and the final solution is

$$y(t) = \mathrm{e}^{-t} \sin(2t).$$



3. (Section 2.2.2 - Exercise 6) Solve the initial-value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0, \qquad y(2) = 1, \qquad y'(2) = -1.$$

The characteristic equation

$$r^2 + 2r + 1 = 0$$

has one root $r = -1$. The general solution has the form

$$y(t) = (c_1 + c_2 t)\mathrm{e}^{-t}$$

with

$$y'(t) = -c_1\mathrm{e}^{-t} + c_2(1-t)\mathrm{e}^{-t}.$$

The initial conditions $y(2) = 1$, $y'(2) = -1$ give us two equations for finding constants c_1 and c_2:

$$1 = y(2) = c_1\mathrm{e}^{-2} + 2c_2\mathrm{e}^{-2}$$

$$-1 = y'(2) = -c_1\mathrm{e}^{-2} - c_2\mathrm{e}^{-2}.$$

The constants are $c_1 = \mathrm{e}^2$ and $c_2 = 0$ and the final solution is

$$y(t) = \mathrm{e}^{2-t}.$$

4. (Section 2.4 - Exercise 6) Solve the initial-value problem

$$y'' + 4y' + 4y = t^{5/2}\mathrm{e}^{-2t}, \qquad y(0) = 0, \qquad y'(0) = 0.$$

First we solve the homogeneous problem. The characteristic equation

$$r^2 + 4r + 4 = 0$$

has one root $r = -2$. The functions

$$y_1(t) = \mathrm{e}^{-2t}, \qquad y_2(t) = t\,\mathrm{e}^{-2t},$$

form the fundamental set of solutions. The Wronskian is

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = \mathrm{e}^{-4t}.$$

Now we find the particular solution ψ in the form

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

If $g(t) = t^{5/2}\mathrm{e}^{-2t}$, then

$$u_1(t) = -\int \frac{g(t)y_2(t)}{W[y_1, y_2](t)}dt = -\int t^{7/2}dt = -\frac{2}{9}t^{9/2}$$

$$u_2(t) = \int \frac{g(t)y_1(t)}{W[y_1, y_2](t)}dt = \int t^{5/2}dt = \frac{2}{7}t^{7/2}.$$

The particular solution is

$$\psi(t) = -\frac{2}{9}t^{9/2}\mathrm{e}^{-2t} + \frac{2}{7}t^{7/2}t\,\mathrm{e}^{-2t} = \frac{4}{63}t^{9/2}\mathrm{e}^{-2t}.$$

The general solution of the starting problem has the form

$$y(t) = (c_1 + c_2 t)\mathrm{e}^{-2t} + \psi(t)$$

with

$$y'(t) = -2c_1\mathrm{e}^{-2t} + c_2(1 - 2t)\mathrm{e}^{-2t} + \frac{4}{63}\left(\frac{9}{2}t^{7/2} - 2t^{9/2}\right)\mathrm{e}^{-2t}.$$

The initial conditions $y(0) = y'(0) = 0$ give us two equations for finding constants c_1 and c_2:

$$0 = y(0) = c_1$$

$$0 = y'(0) = -2c_1 + c_2.$$

The constants are $c_1 = c_2 = 0$ and the final solution is

$$y(t) = \psi(t) = \frac{4}{63}t^{9/2}\mathrm{e}^{-2t}.$$

5. (Section 2.5 - Exercise 14) Find a particular solution of

$$y'' + 2y' = 1 + t^2 + \mathrm{e}^{-2t}.$$

We will find the function ψ by splitting our problem into two parts:
First we will find a particular solution ψ_1 of the problem

$$y'' + 2y' = 1 + t^2.$$

Then we will find a particular solution ψ_2 of the problem

$$y'' + 2y' = \mathrm{e}^{-2t}.$$

Finally, $\psi = \psi_1 + \psi_2$. In both of those cases, we will use the guessing technique.

(1) We propose

$$\psi_1(t) = t(At^2 + Bt + C),$$

with unknown constants A, B, C. Using

$$\psi_1'(t) = 3At^2 + 2Bt + C$$
$$\psi_1''(t) = 6At + 2B$$

the differential equation $\psi_1'' + 2\psi_1' = 1 + t^2$ becomes

$$6At^2 + (6A + 4B)t + 2B + 2C = 1 + t^2.$$

This further implies $A = 1/6$, $B = -1/4$, $C = 3/4$, and

$$\psi_1(t) = \frac{t^3}{6} - \frac{t^2}{4} + \frac{3t}{4}.$$

(2) Now we propose

$$\psi_2(t) = Dt\mathrm{e}^{-2t}.$$

with an unknown constant D. Using

$$\psi_2'(t) = D\mathrm{e}^{-2t} - 2Dt\mathrm{e}^{-2t}$$
$$\psi_2''(t) = -4D\mathrm{e}^{-2t} + 4Dt\mathrm{e}^{-2t}$$

the differential equation $\psi_2'' + 2\psi_2' = \mathrm{e}^{-2t}$ becomes

$$-4D\mathrm{e}^{-2t} + 4Dt\mathrm{e}^{-2t} + 2(D\mathrm{e}^{-2t} - 2Dt\mathrm{e}^{-2t}) = \mathrm{e}^{-2t}$$
$$-2D\mathrm{e}^{-2t} = \mathrm{e}^{-2t}.$$

This further implies $D = -1/2$, and

$$\psi_2(t) = -\frac{t}{2}\,\mathrm{e}^{-2t}.$$

The particular solution for the starting problems is

$$\psi(t) = \psi_1(t) + \psi_2(t) = \frac{t^3}{6} - \frac{t^2}{4} + \frac{3t}{4} - \frac{t}{2}\,\mathrm{e}^{-2t}.$$

2.5 HW 4

Local contents

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2.5.1 Section 2.6, problem 4

A small object of mass 1 kg is attached to a spring with spring constant $k = 2$ N/m. This spring-mass system is immersed in a viscous medium with damping constant $c = 3$ N s/m. At time $t = 0$, the mass is lowered $\frac{1}{2}$ m below its equilibrium position, and released. Show that the mass will creep back to its equilibrium position as t approaches infinity.

Solution

The ODE is

$$my''(t) + cy' + ky = 0$$

Where $m = 1, c = 3, k = 2$. The above becomes

$$y''(t) + 3y' + 2y = 0$$

And initial conditions, using equilibrium position as $y = 0$ and hence below the equilibrium position y is taken as negative. Therefore

$$\begin{aligned} y(0) &= -\frac{1}{2} \\ y'(0) &= 0 \end{aligned}$$

The characteristic equation is

$$\begin{aligned} r^2 + 3r + 2 &= 0 \\ r &= \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} \\ &= \frac{-3}{2} \pm \frac{1}{2}\sqrt{9 - 4(2)} \\ &= \frac{-3}{2} \pm \frac{1}{2}\sqrt{9 - 8} \\ &= \frac{-3}{2} \pm \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} r_1 &= -1 \\ r_2 &= -2 \end{aligned}$$

Therefore the solution to the ODE is

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} \tag{1}$$

At $t = 0$ the above becomes

$$-\frac{1}{2} = c_1 + c_2 \tag{2}$$

Taking derivative of (1) gives

$$y'(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$$

At $t = 0$ the above becomes

$$0 = -c_1 - 2c_2 \tag{3}$$

From (3) $c_1 = -2c_2$. Substituting into (2) gives

$$\begin{aligned} -\frac{1}{2} &= -2c_2 + c_2 \\ &= -c_2 \end{aligned}$$

Hence

$$c_2 = \frac{1}{2}$$

Therefore from (3)

$$\begin{aligned} 0 &= -c_1 - 2\left(\frac{1}{2}\right) \\ 0 &= -c_1 - 1 \\ c_1 &= -1 \end{aligned}$$

Hence the solution (1) becomes

$$y(t) = -e^{-t} + \frac{1}{2}e^{-2t}$$

We see now that as $t \rightarrow \infty$ the terms e^{-t}, e^{-2t} both go to zero. Therefore

$$\lim_{t\rightarrow\infty} y(t) = 0$$

Hence the mass will go back to equilibrium position $y = 0$ after long time.

The following is a plot of the solution above

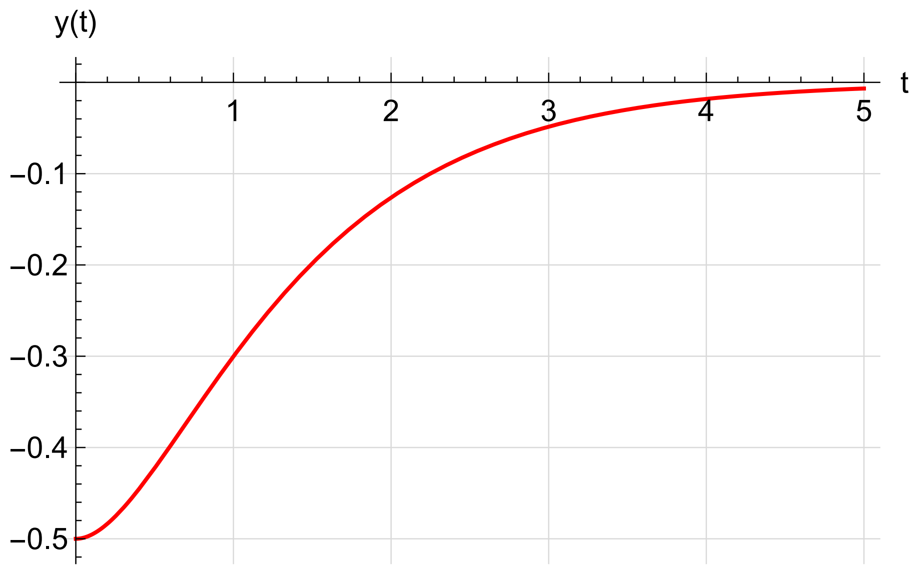
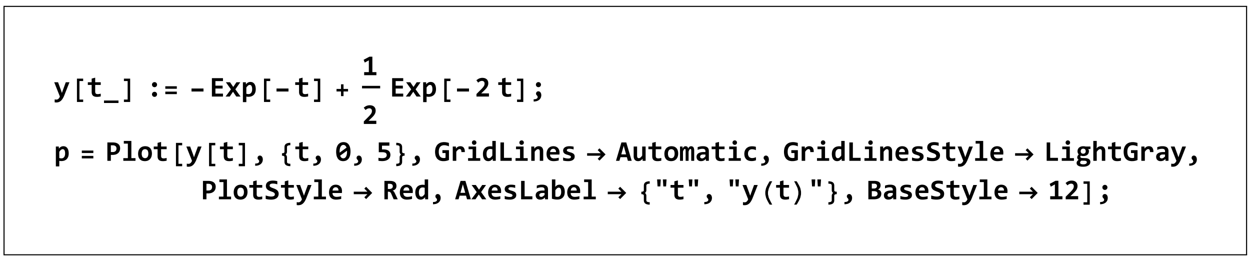


Figure 2.10: Plot showing solution in time



```
y[t_] := -Exp[-t] + 1/2 Exp[-2 t];
p = Plot[y[t], {t, 0, 5}, GridLines → Automatic, GridLinesStyle → LightGray,
      PlotStyle → Red, AxesLabel → {"t", "y(t)"}, BaseStyle → 12];
```

Figure 2.11: Code used for the above plot

2.5.2 Section 2.9, problem 18

Find the Laplace transform of the solution of the following initial value problem.

$$\begin{aligned} y'' + y &= t^2 \sin t \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

<u>Solution</u>

First we find the solution to the ODE then find its Laplace transform. The solution is given by

$$y(t) = y_h(t) + y_p(t)$$

Where $y_h(t)$ is the homogeneous solution to $y'' + y = 0$ and $y_p(t)$ is the particular solution to $y'' + y = t^2 \sin t$.

The characteristic equation is $r^2 + 1 = 0$. Hence $r^2 = -1$ or

$$r = \pm i$$

Therefore

$$y_h(t) = c_1 e^{it} + c_2 e^{-it} \tag{1}$$

To find the particular solution, we find the particular solution for $y'' + y = t^2 e^{it}$ instead, and then take the imaginary part. For this ODE, the RHS is $t^2 e^{it}$, therefore we start by guessing the particular solution to be

$$y_p = \left(At^2 + Bt + C\right) e^{it}$$

But from (1) we see that e^{it} is a fundamental solution to the homogeneous ode. Hence we adjust the above by multiplying by an extra t giving

$$y_p = \left(At^3 + Bt^2 + Ct\right) e^{it}$$

We now substitute the above back into $y'' + y = t^2 e^{it}$ in order to find A, B, C.

$$y_p' = \left(3At^2 + 2Bt + C\right) e^{it} + i\left(At^3 + Bt^2 + Ct\right) e^{it}$$

And

$$\begin{aligned} y_p'' &= (6At + 2B) e^{it} + i\left(3At^2 + 2Bt + C\right) e^{it} + i\left(3At^2 + 2Bt + C\right) e^{it} + i^2\left(At^3 + Bt^2 + Ct\right) e^{it} \\ &= (6At + 2B) e^{it} + i\left(3At^2 + 2Bt + C\right) e^{it} + i\left(3At^2 + 2Bt + C\right) e^{it} - \left(At^3 + Bt^2 + Ct\right) e^{it} \end{aligned}$$

Substituting the above in $y_p'' + y_p = t^2 e^{it}$ gives

$$\begin{aligned} (6At + 2B) e^{it} + i\left(3At^2 + 2Bt + C\right) e^{it} + i\left(3At^2 + 2Bt + C\right) e^{it} \\ - \left(At^3 + Bt^2 + Ct\right) e^{it} + \left(At^3 + Bt^2 + Ct\right) e^{it} = t^2 e^{it} \end{aligned}$$

Canceling e^{it}

$$\begin{aligned} (6At + 2B) + i\left(3At^2 + 2Bt + C\right) + i\left(3At^2 + 2Bt + C\right) - \left(At^3 + Bt^2 + Ct\right) + \left(At^3 + Bt^2 + Ct\right) &= t^2 \\ (6At + 2B) + i\left(3At^2 + 2Bt + C\right) + i\left(3At^2 + 2Bt + C\right) &= t^2 \\ (6At + 2B) + 2i\left(3At^2 + 2Bt + C\right) &= t^2 \\ (2B + 2iC) + t(6A + 4iB) + t^2(6iA) &= t^2 \end{aligned}$$

Comparing coefficients gives

$$\begin{aligned} 2B + 2iC &= 0 \\ 6A + 4iB &= 0 \\ 6Ai &= 1 \end{aligned}$$

Hence $A = \frac{1}{6i} = -\frac{i}{6}$. From the second equation

$$\begin{aligned} 6\left(-\frac{i}{6}\right) + 4iB &= 0 \\ -i + 4iB &= 0 \\ B &= \frac{i}{4i} = \frac{1}{4} \end{aligned}$$

From the first equation

$$\begin{aligned} \frac{1}{2} + 2iC &= 0 \\ 2C &= -\frac{1}{2i} \\ C &= \frac{i}{4} \end{aligned}$$

Substituting the above back into $y_p = \left(At^3 + Bt^2 + Ct\right)e^{it}$ gives

$$\begin{aligned}
y_p &= \left(-\frac{i}{6}t^3 + \frac{1}{4}t^2 + \frac{i}{4}t\right)e^{it}\\
&= \left(-\frac{i}{6}t^3 + \frac{1}{4}t^2 + \frac{i}{4}t\right)(\cos t + i\sin t)\\
&= -\frac{i}{6}t^3\cos t + \frac{1}{4}t^2\cos t + \frac{i}{4}t\cos t - \frac{i}{6}t^3(i\sin t) + \frac{1}{4}t^2(i\sin t) + \frac{i}{4}t(i\sin t)\\
&= -\frac{i}{6}t^3\cos t + \frac{1}{4}t^2\cos t + \frac{i}{4}t\cos t + \frac{1}{6}t^3\sin t + \frac{1}{4}it^2\sin t - \frac{1}{4}t\sin t\\
&= \left(\frac{1}{4}t^2\cos t + \frac{1}{6}t^3\sin t - \frac{1}{4}t\sin t\right) + i\left(-\frac{1}{6}t^3\cos t + \frac{1}{4}t\cos t + \frac{1}{4}t^2\sin t\right)
\end{aligned}$$

The particular solution of the original ODE $y'' + y = t^2\sin t$ is the $\underline{\text{imaginary part}}$ of the above which is

$$y_p = -\frac{1}{6}t^3\cos t + \frac{1}{4}t\cos t + \frac{1}{4}t^2\sin t$$

The homogeneous solution from (1) is $y_h(t) = c_1e^{it} + c_2e^{-it}$ which can be written using Euler relation as $y_h(t) = C_1\cos t + C_2\sin t$, therefore the general solution is

$$\begin{aligned}
y(t) &= y_h(t) + y_p(t)\\
&= C_1\cos t + C_2\sin t - \frac{1}{6}t^3\cos t + \frac{1}{4}t\cos t + \frac{1}{4}t^2\sin t \qquad (2)
\end{aligned}$$

What is left is to find C_1, C_2 from initial conditions. At $t = 0$ the above becomes

$$0 = C_1$$

Hence (2) becomes

$$y(t) = C_2\sin t - \frac{1}{6}t^3\cos t + \frac{1}{4}t\cos t + \frac{1}{4}t^2\sin t \qquad (3)$$

Taking derivative gives

$$y'(t) = C_2\cos t - \frac{3}{6}t^2\cos t + \frac{1}{6}t^3\sin t + \frac{1}{4}\cos t - \frac{1}{4}t\sin t + \frac{2}{4}t\sin t + \frac{1}{4}t^2\cos t$$

At $t = 0$ the above becomes

$$\begin{aligned}
0 &= C_2 + \frac{1}{4}\\
C_2 &= -\frac{1}{4}
\end{aligned}$$

Hence (3) becomes the final solution

$$y(t) = -\frac{1}{4}\sin t - \frac{1}{6}t^3\cos t + \frac{1}{4}t\cos t + \frac{1}{4}t^2\sin t \qquad (3A)$$

The following is a plot of the above solution. The solution blows up in time due to resonance.

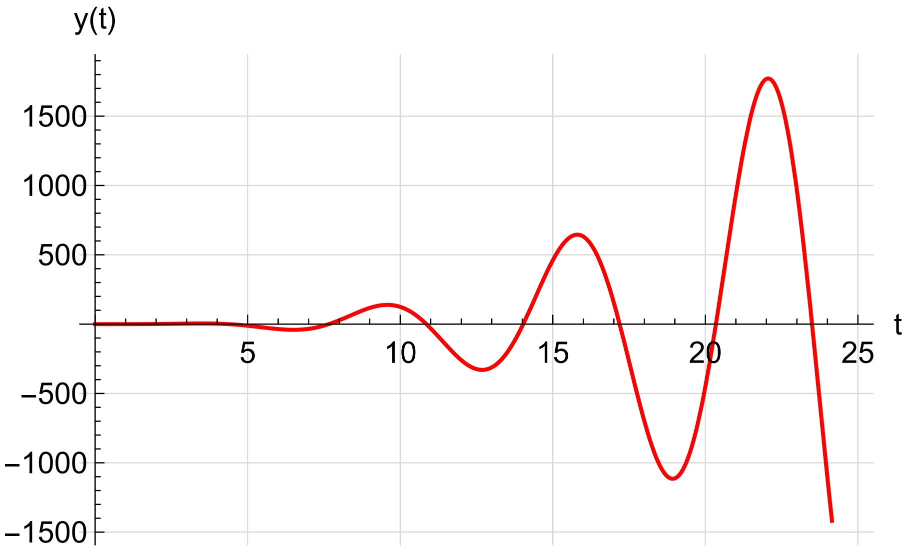
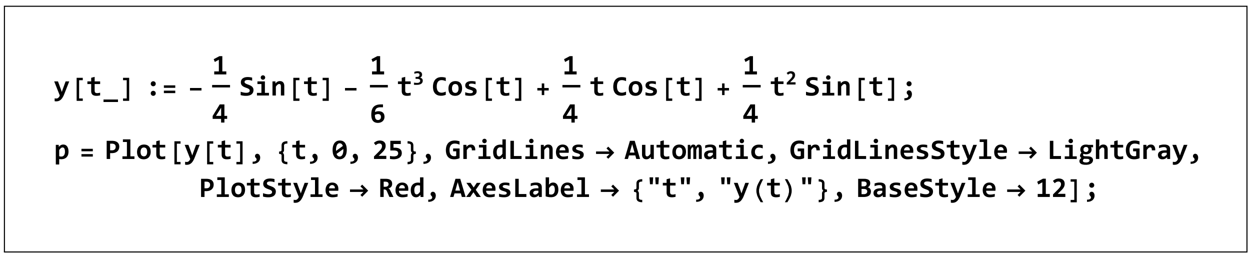


Figure 2.12: Plot showing solution in time



```
y[t_] := -1/4 Sin[t] - 1/6 t^3 Cos[t] + 1/4 t Cos[t] + 1/4 t^2 Sin[t];
p = Plot[y[t], {t, 0, 25}, GridLines → Automatic, GridLinesStyle → LightGray,
        PlotStyle → Red, AxesLabel → {"t", "y(t)"}, BaseStyle → 12];
```

Figure 2.13: Code used for the above plot

The problem now asks to find the Laplace transform of the above. To obtain the Laplace Transform of the above, the following relations will be used (In the following, the notation ⇔ means the Laplace transform from left to right and the inverse Laplace transform from right to left).

$$\begin{aligned}\sin(at) &\Leftrightarrow \frac{a}{a^2+s^2}\\ \cos(at) &\Leftrightarrow \frac{s}{a^2+s^2}\\ t^n f(t) &\Leftrightarrow (-1)^n \frac{d^n}{ds^n}F(s)\end{aligned}$$

Hence

$$\sin(t) \Leftrightarrow \frac{1}{1+s^2} \tag{4}$$

$$\cos(t) \Leftrightarrow \frac{s}{1+s^2} \tag{5}$$

And

$$t\sin(t) \Leftrightarrow (-1)\frac{d}{ds}\mathscr{L}(\sin(t))$$

But

$$\begin{aligned}\frac{d}{ds}\mathscr{L}(\sin(t)) &= \frac{d}{ds}\left(\frac{1}{1+s^2}\right)\\ &= \frac{-2s}{\left(1+s^2\right)^2}\end{aligned}$$

Therefore

$$\begin{aligned}t\sin(t) &\Leftrightarrow (-1)\frac{-2s}{\left(1+s^2\right)^2}\\ &\Leftrightarrow \frac{2s}{\left(1+s^2\right)^2}\end{aligned} \tag{6}$$

And

$$\mathscr{L}\left(t^2 \sin(t)\right) = (-1)^2 \frac{d^2}{ds^2} \mathscr{L}(\sin(t))$$

But

$$\begin{aligned}
\frac{d^2}{ds^2} \mathscr{L}(\sin(t)) &= \frac{d}{ds} \left(\frac{-2s}{\left(1+s^2\right)^2} \right) \\
&= \frac{-2\left(1+s^2\right)^2 + 2s(2)\left(1+s^2\right)(2s)}{\left(1+s^2\right)^2} \\
&= \frac{-2\left(1+s^2\right)^2 + 8s^2\left(1+s^2\right)}{\left(1+s^2\right)^4} \\
&= \frac{-2\left(1+s^2\right) + 8s^2}{\left(1+s^2\right)^3} \\
&= \frac{-2+6s^2}{\left(1+s^2\right)^3}
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathscr{L}\left(t^2 \sin(t)\right) &= (-1)^2 \frac{-2+6s^2}{\left(1+s^2\right)^3} \\
&= \frac{-2+6s^2}{\left(1+s^2\right)^3} \qquad (7)
\end{aligned}$$

And

$$\mathscr{L}(t \cos(t)) = (-1) \frac{d}{ds} \mathscr{L}(\cos(t))$$

But

$$\begin{aligned}
\frac{d}{ds} \mathscr{L}(\cos(t)) &= \frac{d}{ds} \left(\frac{s}{1+s^2} \right) \\
&= \frac{\left(1+s^2\right) - s(2s)}{\left(1+s^2\right)^2} \\
&= \frac{1-s^2}{\left(1+s^2\right)^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathscr{L}(t \cos(t)) &= (-1) \frac{1-s^2}{\left(1+s^2\right)^2} \\
&= \frac{s^2-1}{\left(1+s^2\right)^2} \qquad (8)
\end{aligned}$$

And

$$\mathscr{L}\left(t^2 \cos(t)\right) = (-1)^2 \frac{d^2}{ds^2} \mathscr{L}(\cos(t))$$

But

$$\begin{aligned}
\frac{d^2}{ds^2}\mathscr{L}(\cos(t)) &= \frac{d}{ds}\left(\frac{1-s^2}{\left(1+s^2\right)^2}\right)\\
&= \frac{-2s\left(1+s^2\right)^2-\left(1-s^2\right)(2)\left(1+s^2\right)(2s)}{\left(1+s^2\right)^4}\\
&= \frac{-2s\left(1+s^2\right)-\left(1-s^2\right)(2)(2s)}{\left(1+s^2\right)^3}\\
&= \frac{-2s\left(1+s^2\right)-4s\left(1-s^2\right)}{\left(1+s^2\right)^3}\\
&= \frac{-2s-2s^3-4s+4s^3}{\left(1+s^2\right)^3}\\
&= \frac{-6s+2s^3}{\left(1+s^2\right)^3}
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathscr{L}\left(t^2\cos(t)\right) &= (-1)^2\left(\frac{-6s+2s^3}{\left(1+s^2\right)^3}\right)\\
&= \frac{-6s+2s^3}{\left(1+s^2\right)^3} \qquad (9)
\end{aligned}$$

And finally

$$\mathscr{L}\left(t^3\cos(t)\right) = (-1)^3\frac{d^3}{ds^3}\mathscr{L}(\cos(t))$$

But

$$\begin{aligned}
\frac{d^3}{ds^3}\mathscr{L}(\cos(t)) &= \frac{d}{ds}\left(\frac{-6s+2s^3}{\left(1+s^2\right)^3}\right)\\
&= \frac{\left(-6+6s^2\right)\left(1+s^2\right)^3-\left(-6s+2s^3\right)3\left(1+s^2\right)^2(2s)}{\left(1+s^2\right)^6}\\
&= \frac{\left(-6+6s^2\right)\left(1+s^2\right)-\left(-6s+2s^3\right)3(2s)}{\left(1+s^2\right)^4}\\
&= \frac{-6s^4+36s^2-6}{\left(1+s^2\right)^4}
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathscr{L}\left(t^3\cos(t)\right) &= (-1)^3\left(\frac{-6s^4+36s^2-6}{\left(1+s^2\right)^4}\right)\\
&= \frac{6s^4-36s^2+6}{\left(1+s^2\right)^4} \qquad (10)
\end{aligned}$$

Using (4,5,6,7,8,9,10) in (3A) gives

$$\begin{aligned}
\mathscr{L}\left(y(t)\right) &= -\frac{1}{4}\mathscr{L}(\sin t) - \frac{1}{6}\mathscr{L}\left(t^3\cos t\right) + \frac{1}{4}\mathscr{L}(t\cos t) + \frac{1}{4}\mathscr{L}\left(t^2\sin t\right)\\
&= -\frac{1}{4}\frac{1}{1+s^2} - \frac{1}{6}\frac{6s^4-36s^2+6}{\left(1+s^2\right)^4} + \frac{1}{4}\frac{s^2-1}{\left(1+s^2\right)^2} + \frac{1}{4}\frac{-2+6s^2}{\left(1+s^2\right)^3}\\
&= -\frac{1}{4}\frac{\left(1+s^2\right)^3}{\left(1+s^2\right)^4} - \frac{1}{6}\frac{6s^4-36s^2+6}{\left(1+s^2\right)^4} + \frac{1}{4}\frac{\left(s^2-1\right)\left(1+s^2\right)^2}{\left(1+s^2\right)^4} + \frac{1}{4}\frac{\left(-2+6s^2\right)\left(1+s^2\right)}{\left(1+s^2\right)^4}\\
&= \frac{-\frac{1}{4}\left(1+s^2\right)^3 - \frac{1}{6}\left(6s^4-36s^2+6\right) + \frac{1}{4}\left(s^2-1\right)\left(1+s^2\right)^2 + \frac{1}{4}\left(-2+6s^2\right)\left(1+s^2\right)}{\left(1+s^2\right)^4}\\
&= \frac{-\frac{1}{4}\left(1+s^2\right)^3 - \frac{1}{6}\left(6s^4-36s^2+6\right) + \frac{1}{4}\left(s^2-1\right)\left(1+s^2\right)^2 + \frac{1}{4}\left(-2+6s^2\right)\left(1+s^2\right)}{\left(1+s^2\right)^4}\\
&= \frac{-\frac{1}{4}\left(1+s^2\right)^3 - \frac{1}{6}\left(6s^4-36s^2+6\right) + \frac{1}{4}\left(s^2-1\right)\left(1+s^2\right)^2 + \frac{1}{4}\left(-2+6s^2\right)\left(1+s^2\right)}{\left(1+s^2\right)^4}
\end{aligned}$$

Which can be simplified to

$$\mathscr{L}\left(y(t)\right) = \frac{6s^2-2}{\left(1+s^2\right)^4}$$

2.5.3 section 2.10, problem 14

Find the inverse Laplace transform of each of the following functions

$$\frac{1}{s(s+4)^2}$$

<u>Solution</u>

Let

$$F(s) = \frac{1}{s(s+4)^2}$$

Using partial fractions

$$\frac{1}{s(s+4)^2} = \frac{A}{s} + \frac{B}{(s+4)} + \frac{C}{(s+4)^2}$$

Hence

$$\begin{aligned}
\frac{1}{s(s+4)^2} &= \frac{A(s+4)^2 + Bs(s+4) + Cs}{s(s+4)^2}\\
&= \frac{A\left(s^2+16+8s\right) + Bs^2 + 4Bs + Cs}{s(s+4)^2}\\
&= \frac{s^2(A+B) + s(8A+4B+C) + 16A}{s(s+4)^2}
\end{aligned}$$

Comparing coefficients gives

$$\begin{aligned}
16A &= 1\\
8A + 4B + C &= 0\\
A + B &= 0
\end{aligned}$$

From first equation $A = \frac{1}{16}$. From the third equation $B = -\frac{1}{16}$. From the second equation $8\left(\frac{1}{16}\right) + 4\left(-\frac{1}{16}\right) + C = 0$, hence $C = -\frac{1}{4}$, Therefore

$$\frac{1}{s(s+4)^2} = \frac{1}{16}\frac{1}{s} - \frac{1}{16}\frac{1}{(s+4)} - \frac{1}{4}\frac{1}{(s+4)^2} \tag{1}$$

Now we use the relation

$$\mathscr{L}^{-1}\left(\frac{1}{s}\right)=H_0(t) \tag{2}$$

For $\frac{1}{(s+4)}$ we will use the relation that

$$\mathscr{L}\left(e^{at}f(t)\right)=F(s-a)$$

Where here $\mathscr{L}\left(f(t)\right)=F(s)$. Therefore if we take $F(s)=\frac{1}{s}$ then we see that $\mathscr{L}\left(e^{-4t}f(t)\right)=\frac{1}{s+4}$. Therefore

$$\mathscr{L}^{-1}\left(\frac{1}{s+4}\right)=e^{-4t} \tag{3}$$

For $\frac{1}{(s+4)^2}$ we will use the relation that

$$\mathscr{L}\left(t^n f(t)\right)=(-1)^n F^{(n)}(s)$$

If we put $n=1$ and $f(t)=e^{-4t}$ then

$$\begin{aligned}\mathscr{L}\left(te^{-4t}\right)&=(-1)\frac{d}{ds}\left(\frac{1}{s+4}\right)\\&=(-1)\left(\frac{-1}{(s+4)^2}\right)\\&=\frac{1}{(s+4)^2}\end{aligned}$$

Therefore we see that

$$\mathscr{L}^{-1}\left(\frac{1}{(s+4)^2}\right)=te^{-4t} \tag{4}$$

Substituting (2,3,4) back into (1) gives

$$\begin{aligned}\mathscr{L}^{-1}\left(\frac{1}{s(s+4)^2}\right)&=\frac{1}{16}\mathscr{L}^{-1}\left(\frac{1}{s}\right)-\frac{1}{16}\mathscr{L}^{-1}\left(\frac{1}{(s+4)}\right)-\frac{1}{4}\mathscr{L}^{-1}\left(\frac{1}{(s+4)^2}\right)\\&=\frac{1}{16}H_0(t)-\frac{1}{16}e^{-4t}-\frac{1}{4}te^{-4t}\end{aligned}$$

Or, taking $t\geq 0$, then $H_0(t)$ can be replaced by 1 and above can be simplifies to

$$\mathscr{L}^{-1}\left(\frac{1}{s(s+4)^2}\right)=\frac{1}{16}-\frac{1}{16}e^{-4t}-\frac{1}{4}te^{-4t}$$

2.5.4 Section 2.10, problem 20

Solve the following initial-value problems by the method of Laplace transforms

$$\begin{aligned}y''+y&=t\sin t\\y(0)&=1\\y'(0)&=2\end{aligned}$$

Solution

Taking the Laplace transform of the ODE gives

$$\begin{aligned}\mathscr{L}\left(y''+y\right)&=\mathscr{L}(t\sin t)\\\mathscr{L}y''+\mathscr{L}y&=\mathscr{L}(t\sin t)\end{aligned} \tag{1}$$

But from the above problem Section 2.9, problem 18 we have already found that

$$\mathscr{L}(t\sin t)=\frac{2s}{\left(1+s^2\right)^2}$$

And using

$$\mathscr{L}y''=s^2Y(s)-sy(0)-y'(0)$$

where $Y(s)=\mathscr{L}\left(y(t)\right)$, then (1) becomes

$$s^2Y(s)-sy(0)-y'(0)+Y(s)=\frac{2s}{\left(1+s^2\right)^2}$$

Substituting the initial conditions into the above gives

$$\begin{aligned} s^2Y(s) - s - 2 + Y(s) &= \frac{2s}{\left(1+s^2\right)^2} \\ Y(s)\left(s^2+1\right) - s - 2 &= \frac{2s}{\left(1+s^2\right)^2} \\ Y(s)\left(s^2+1\right) &= \frac{2s}{\left(1+s^2\right)^2} + s + 2 \\ Y(s) &= 2\frac{s}{\left(1+s^2\right)^3} + \frac{s}{\left(s^2+1\right)} + 2\frac{1}{\left(s^2+1\right)} \qquad (1A) \end{aligned}$$

Now we ready to apply inverse Laplace transform using the relations

$$\mathscr{L}\cos t = \frac{s}{s^2+1} \qquad (2A)$$

$$\mathscr{L}\sin t = \frac{1}{s^2+1} \qquad (2B)$$

The only term left is $\frac{s}{\left(1+s^2\right)^3}$. But this is the same as $\frac{s}{\left(1+s^2\right)^2}\frac{1}{(1+s)}$ and we already found that $\frac{2s}{\left(1+s^2\right)^2} \Leftrightarrow t\sin t$ from above solving section 2.9, problem 18, and $\frac{1}{(1+s)} \Leftrightarrow \sin t$. Therefore we can use convolution as follows

$$\left(\frac{2s}{\left(1+s^2\right)^2}\right)\left(\frac{1}{(1+s)}\right) \Leftrightarrow \int_0^t f(\tau)g(t-\tau)\,d\tau$$

Where we assume that $\frac{2s}{\left(1+s^2\right)^2} \Leftrightarrow f(t) = t\sin t$ and $\frac{1}{(1+s)} \Leftrightarrow g(t) = \sin t$. Hence the above becomes

$$\frac{2s}{\left(1+s^2\right)^3} \Leftrightarrow \int_0^t \tau\sin(\tau)\sin(t-\tau)\,d\tau \qquad (2)$$

Let $A = \tau, B = t - \tau$ and using $\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$, then

$$\begin{aligned} \sin(\tau)\sin(t-\tau) &= \frac{1}{2}(\cos(\tau-(t-\tau)) - \cos(\tau+(t-\tau))) \\ &= \frac{1}{2}(\cos(2\tau-t) - \cos(t)) \end{aligned}$$

Substituting the above in (2) gives

$$\begin{aligned} \frac{s}{\left(1+s^2\right)^3} &\Leftrightarrow \int_0^t \tau\left(\frac{1}{2}(\cos(2\tau-t) - \cos(t))\right)d\tau \\ &\Leftrightarrow \frac{1}{2}\int_0^t \tau\cos(2\tau-t)\,d\tau - \frac{1}{2}\int_0^t \tau\cos(t)\,d\tau \\ &\Leftrightarrow \frac{1}{2}\int_0^t \tau\cos(2\tau-t)\,d\tau - \frac{1}{2}\cos(t)\int_0^t \tau d\tau \qquad (3) \end{aligned}$$

Using integration by parts on the first integral. Let $u = \tau, dv = \cos(2\tau - t), du = 1, v = \frac{\sin(2\tau-t)}{2}$, hence

$$\begin{aligned} \int_0^t \tau\cos(2\tau-t)\,d\tau &= \frac{1}{2}\left[\tau\sin(2\tau-t)\right]_0^t - \int_0^t \frac{\sin(2\tau-t)}{2}d\tau \\ &= \frac{1}{2}\left[t\sin(t)\right] - \frac{1}{2}\int_0^t \sin(2\tau-t)\,d\tau \\ &= \frac{1}{2}t\sin(t) + \frac{1}{4}\left[\cos(2\tau-t)\right]_0^t \\ &= \frac{1}{2}t\sin(t) + \frac{1}{4}\left[\cos(t) - \cos(-t)\right] \\ &= \frac{1}{2}t\sin(t) + \frac{1}{4}\left[\cos(t) - \cos(t)\right] \\ &= \frac{1}{2}t\sin(t) \end{aligned}$$

Substituting the above in (3) gives

$$\begin{aligned}\frac{s}{\left(1+s^2\right)^3} &\Leftrightarrow \frac{1}{2}\left(\frac{1}{2}t\sin(t)\right)-\frac{1}{4}t^2\cos(t)\\ &\Leftrightarrow \frac{1}{4}t\sin(t)-\frac{1}{4}t^2\cos(t) \qquad \text{(2C)}\end{aligned}$$

We have found the inverse Laplace transform for all the terms. Substituting (2A,2B,2C) into (1A) gives

$$\begin{aligned}\mathscr{L}^{-1}Y(s) &= \mathscr{L}^{-1}\frac{2s}{\left(1+s^2\right)^3}+\mathscr{L}^{-1}\frac{s}{\left(s^2+1\right)}+2\mathscr{L}^{-1}\frac{1}{\left(s^2+1\right)}\\ y(t) &= \left(\frac{1}{4}t\sin(t)-\frac{1}{4}t^2\cos(t)\right)+\cos t+2\sin t\\ &= -\frac{1}{4}t^2\cos t+\frac{1}{4}t\sin t+\cos t+2\sin t\end{aligned}$$

The following is a plot of the above solution. The solution blows up in time due to resonance.

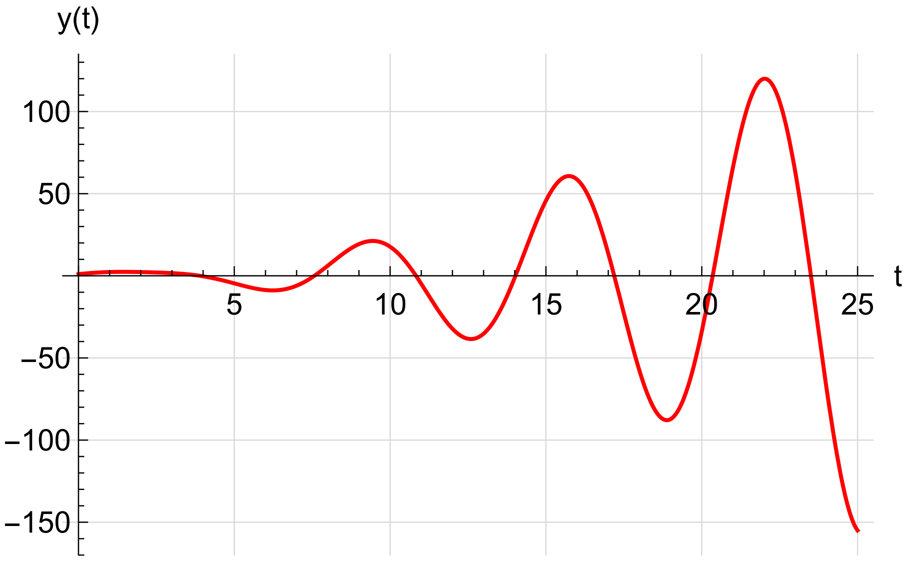
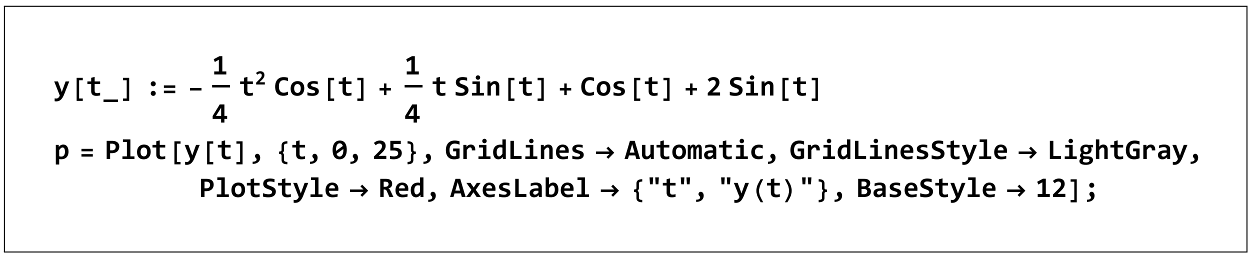


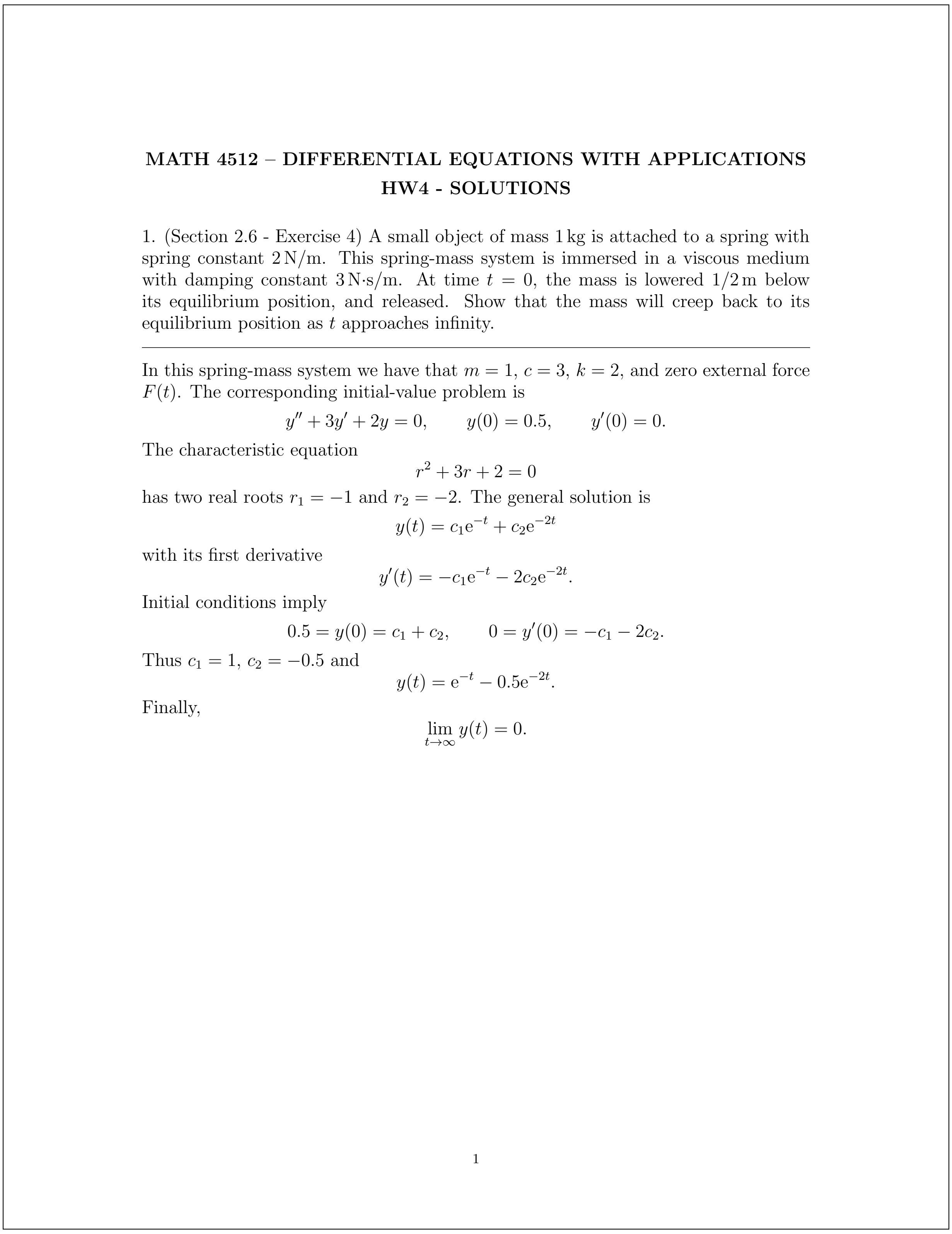
Figure 2.14: Plot showing solution in time



```
y[t_] := - 1/4 t^2 Cos[t] + 1/4 t Sin[t] + Cos[t] + 2 Sin[t]
p = Plot[y[t], {t, 0, 25}, GridLines → Automatic, GridLinesStyle → LightGray,
          PlotStyle → Red, AxesLabel → {"t", "y(t)"}, BaseStyle → 12];
```

Figure 2.15: Code used for the above plot

2.5.5 Key solution for HW 4



MATH 4512 – DIFFERENTIAL EQUATIONS WITH APPLICATIONS
HW4 - SOLUTIONS

1. (Section 2.6 - Exercise 4) A small object of mass 1 kg is attached to a spring with spring constant 2 N/m. This spring-mass system is immersed in a viscous medium with damping constant 3 N·s/m. At time $t = 0$, the mass is lowered 1/2 m below its equilibrium position, and released. Show that the mass will creep back to its equilibrium position as t approaches infinity.

In this spring-mass system we have that $m = 1$, $c = 3$, $k = 2$, and zero external force $F(t)$. The corresponding initial-value problem is

$$y'' + 3y' + 2y = 0, \qquad y(0) = 0.5, \qquad y'(0) = 0.$$

The characteristic equation

$$r^2 + 3r + 2 = 0$$

has two real roots $r_1 = -1$ and $r_2 = -2$. The general solution is

$$y(t) = c_1 e^{-t} + c_2 e^{-2t}$$

with its first derivative

$$y'(t) = -c_1 e^{-t} - 2c_2 e^{-2t}.$$

Initial conditions imply

$$0.5 = y(0) = c_1 + c_2, \qquad 0 = y'(0) = -c_1 - 2c_2.$$

Thus $c_1 = 1$, $c_2 = -0.5$ and

$$y(t) = e^{-t} - 0.5e^{-2t}.$$

Finally,

$$\lim_{t\to\infty} y(t) = 0.$$

2. (Section 2.9 - Exercise 18) Find the Laplace transform of the solution of the following initial value problem

$$y'' + y = t^2 \sin t, \qquad y(0) = y'(0) = 0.$$

The Laplace transform $Y(s)$ of the solution $y(t)$ can be obtained from the formula

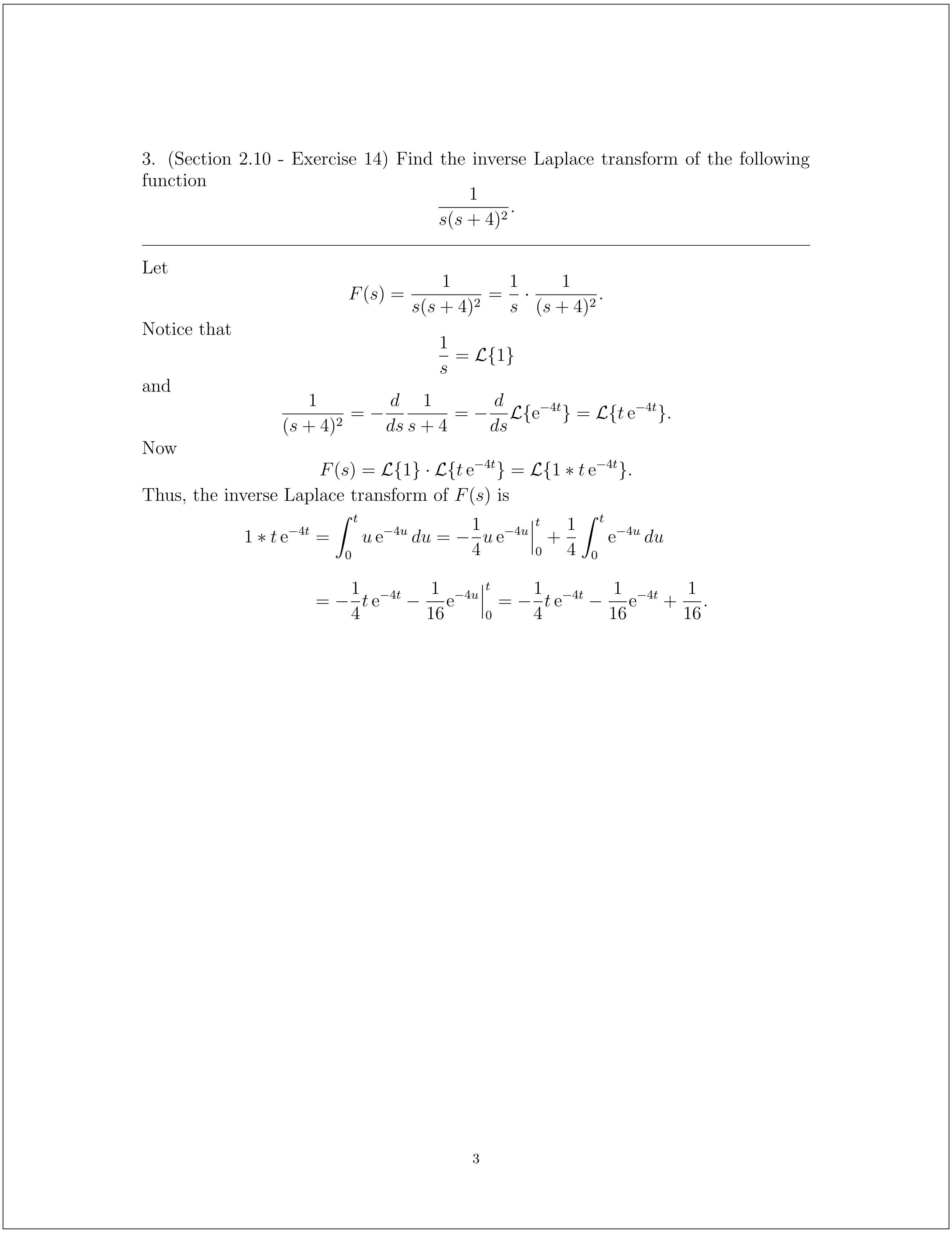
$$Y(s) = \frac{1}{s^2+1} F(s),$$

where $F(s) = \mathcal{L}\{t^2 \sin t\}$. Next we find $F(s)$:

$$\begin{aligned} F(s) &= -\mathcal{L}\{-t \cdot t \sin t\} = -\frac{d}{ds}\mathcal{L}\{t \sin t\} \\ &= \frac{d}{ds}\mathcal{L}\{-t \sin t\} = \frac{d}{ds}\left(\frac{d}{ds}\mathcal{L}\{\sin t\}\right) = \frac{d^2}{ds^2}\mathcal{L}\{\sin t\} \\ &= \frac{d^2}{ds^2}\frac{1}{s^2+1} = \frac{2(3s^2-1)}{(s^2+1)^3}. \end{aligned}$$

Finally, the Laplace transform of the solution of the initial value problem is

$$Y(s) = \frac{2(3s^2-1)}{(s^2+1)^4}.$$



3. (Section 2.10 - Exercise 14) Find the inverse Laplace transform of the following function

$$\frac{1}{s(s+4)^2}.$$

Let

$$F(s) = \frac{1}{s(s+4)^2} = \frac{1}{s} \cdot \frac{1}{(s+4)^2}.$$

Notice that

$$\frac{1}{s} = \mathcal{L}\{1\}$$

and

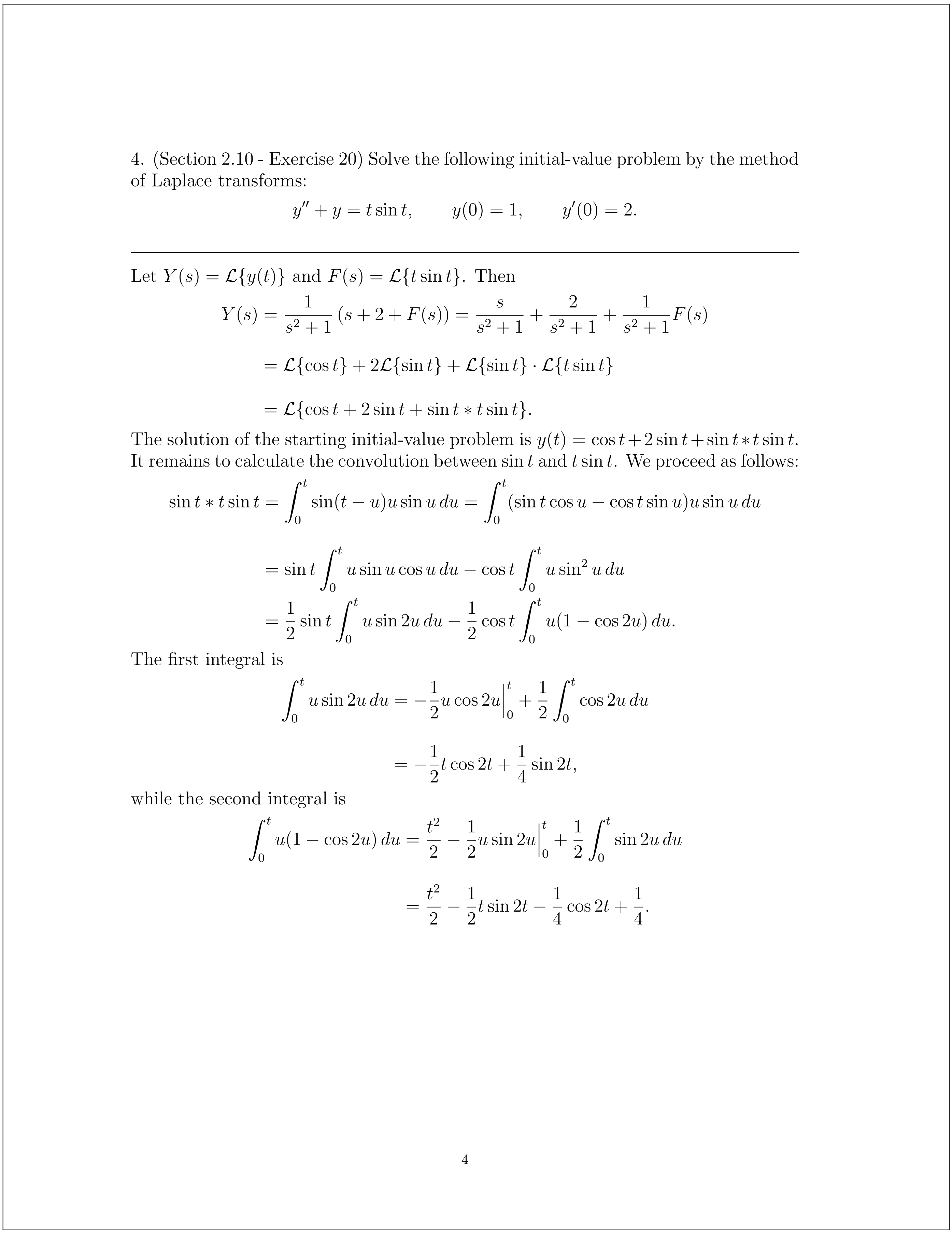
$$\frac{1}{(s+4)^2} = -\frac{d}{ds}\frac{1}{s+4} = -\frac{d}{ds}\mathcal{L}\{\mathrm{e}^{-4t}\} = \mathcal{L}\{t\,\mathrm{e}^{-4t}\}.$$

Now

$$F(s) = \mathcal{L}\{1\} \cdot \mathcal{L}\{t\,\mathrm{e}^{-4t}\} = \mathcal{L}\{1 * t\,\mathrm{e}^{-4t}\}.$$

Thus, the inverse Laplace transform of $F(s)$ is

$$\begin{aligned} 1 * t\,\mathrm{e}^{-4t} &= \int_0^t u\,\mathrm{e}^{-4u}\,du = -\frac{1}{4}u\,\mathrm{e}^{-4u}\Big|_0^t + \frac{1}{4}\int_0^t \mathrm{e}^{-4u}\,du \\ &= -\frac{1}{4}t\,\mathrm{e}^{-4t} - \frac{1}{16}\mathrm{e}^{-4u}\Big|_0^t = -\frac{1}{4}t\,\mathrm{e}^{-4t} - \frac{1}{16}\mathrm{e}^{-4t} + \frac{1}{16}. \end{aligned}$$



4. (Section 2.10 - Exercise 20) Solve the following initial-value problem by the method of Laplace transforms:

$$y'' + y = t\sin t, \qquad y(0) = 1, \qquad y'(0) = 2.$$

Let $Y(s) = \mathcal{L}\{y(t)\}$ and $F(s) = \mathcal{L}\{t\sin t\}$. Then

$$\begin{aligned} Y(s) &= \frac{1}{s^2+1}\left(s + 2 + F(s)\right) = \frac{s}{s^2+1} + \frac{2}{s^2+1} + \frac{1}{s^2+1}F(s) \\ &= \mathcal{L}\{\cos t\} + 2\mathcal{L}\{\sin t\} + \mathcal{L}\{\sin t\} \cdot \mathcal{L}\{t\sin t\} \\ &= \mathcal{L}\{\cos t + 2\sin t + \sin t * t\sin t\}. \end{aligned}$$

The solution of the starting initial-value problem is $y(t) = \cos t + 2\sin t + \sin t * t\sin t$. It remains to calculate the convolution between $\sin t$ and $t\sin t$. We proceed as follows:

$$\begin{aligned} \sin t * t\sin t &= \int_0^t \sin(t-u)u\sin u\,du = \int_0^t (\sin t\cos u - \cos t\sin u)u\sin u\,du \\ &= \sin t\int_0^t u\sin u\cos u\,du - \cos t\int_0^t u\sin^2 u\,du \\ &= \frac{1}{2}\sin t\int_0^t u\sin 2u\,du - \frac{1}{2}\cos t\int_0^t u(1-\cos 2u)\,du. \end{aligned}$$

The first integral is

$$\begin{aligned} \int_0^t u\sin 2u\,du &= -\frac{1}{2}u\cos 2u\Big|_0^t + \frac{1}{2}\int_0^t \cos 2u\,du \\ &= -\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t, \end{aligned}$$

while the second integral is

$$\begin{aligned} \int_0^t u(1-\cos 2u)\,du &= \frac{t^2}{2} - \frac{1}{2}u\sin 2u\Big|_0^t + \frac{1}{2}\int_0^t \sin 2u\,du \\ &= \frac{t^2}{2} - \frac{1}{2}t\sin 2t - \frac{1}{4}\cos 2t + \frac{1}{4}. \end{aligned}$$

Then

$$\begin{aligned}\sin t * t\sin t &= \frac{1}{2}\sin t\left(-\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t\right)\\ &\quad - \frac{1}{2}\cos t\left(\frac{t^2}{2} - \frac{1}{2}t\sin 2t - \frac{1}{4}\cos 2t + \frac{1}{4}\right)\\ &= -\frac{t^2}{4}\cos t + \frac{t}{4}\left(\sin 2t\cos t - \cos 2t\sin t\right)\\ &\quad + \frac{1}{8}\left(\cos 2t\cos t + \sin 2t\sin t\right) - \frac{1}{8}\cos t\\ &= -\frac{t^2}{4}\cos t + \frac{t}{4}\sin t + \frac{1}{8}\cos t - \frac{1}{8}\cos t = -\frac{t^2}{4}\cos t + \frac{t}{4}\sin t.\end{aligned}$$

Finally

$$y(t) = \cos t + 2\sin t - \frac{t^2}{4}\cos t + \frac{t}{4}\sin t = \left(1 - \frac{t^2}{4}\right)\cos t + \left(2 + \frac{t}{4}\right)\sin t.$$

2.6 HW 5

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2.6.1 Section 3.1, problem 4

Convert the pair of second-order equations

$$\begin{aligned} y''(t) + 3z'(t) + 2y(t) &= 0 \\ z''(t) + 3y'(t) + 2z(t) &= 0 \end{aligned}$$

into a system of 4 first-order equations for the variables $x_1 = y, x_2 = y', x_3 = z, x_4 = z'$

Solution

$$x_1 = y, x_2 = y', x_3 = z, x_4 = z'$$

Taking derivative gives

$$\begin{aligned} \dot{x}_1 &= y', \dot{x}_2 = y'', \dot{x}_3 = z', \dot{x}_4 = z'' \\ \dot{x}_1 &= x_2, \dot{x}_2 = -\left(3z' + 2y\right), \dot{x}_3 = x_4, \dot{x}_4 = -\left(3y' + 2z\right) \end{aligned}$$

Hence

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -3x_4 - 2x_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -3x_2 - 2x_3 \end{aligned}$$

Or In Matrix form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & -3 & -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$

2.6.2 Section 3.2, problem 4

Determine whether the given set of elements $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ where $x_1 + x_2 + x_3 = 1$ form a vector space under the properties of vector addition and scalar multiplication defined in Section 3.1.

Solution

We need to check that using vector addition + and scalar multiplication c the following is true. For any $\boldsymbol{x}, \boldsymbol{y}$ in V then $\left(\boldsymbol{x} + \boldsymbol{y}\right)$ is still in V. And for $\boldsymbol{x}$ in V then $c\boldsymbol{x}$ is still in V.

Checking for addition This fails. Here is an example. Let $\boldsymbol{x} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \boldsymbol{y} = \begin{pmatrix} \frac{2}{3} \\ 0 \\ \frac{1}{3} \end{pmatrix}$. Both $\boldsymbol{x}, \boldsymbol{y}$ are in

V, but vector addition gives

$$\begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{pmatrix}+\begin{pmatrix}\frac{2}{3}\\0\\\frac{1}{3}\end{pmatrix}=\begin{pmatrix}1\\\frac{1}{3}\\\frac{2}{3}\end{pmatrix}$$

We see that the resulting vector is not in V, because sum of its elements $1+\frac{1}{3}+\frac{2}{3}=2\neq 1$. Hence not in V.

So it does not form a vector space.

2.6.3 Section 3.3, problem 16

Find basis for $\Re^3$ which includes the vectors $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\3\\4\end{pmatrix}$

Solution

We need to find 3rd vector $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ such that it is linearly independent to above two vectors. If we take the cross product of the above two vectors, then we get a vector that is perpendicular to the plane that the two given vectors span. This will give us the third vector we need

$$\begin{aligned}\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}1\\3\\4\end{pmatrix}&=\begin{vmatrix}i&j&k\\1&1&0\\1&3&4\end{vmatrix}\\&=i(4)-j(4)+k(3-1)\\&=4i-4j+2k\end{aligned}$$

Hence the third vector is $\begin{pmatrix}4\\-4\\2\end{pmatrix}$.

To verify this result, we now check that $c_1\begin{pmatrix}1\\1\\0\end{pmatrix}+c_2\begin{pmatrix}1\\3\\4\end{pmatrix}+c_3\begin{pmatrix}4\\-4\\2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ implies $c_1=0,c_2=0,c_3=0$ as only solution. Writing the above as

$$\begin{pmatrix}1&1&4\\1&3&-4\\0&4&2\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Hence augmented matrix is

$$\begin{pmatrix}1&1&4&0\\1&3&-4&0\\0&4&2&0\end{pmatrix}$$

Replacing row 2 by row 2 minus row 1

$$\begin{pmatrix}1&1&4&0\\0&2&-8&0\\0&4&2&0\end{pmatrix}$$

Replacing row 3 by row 3 minus twice row 2

$$\begin{pmatrix}1&1&4&0\\0&2&-8&0\\0&0&16&0\end{pmatrix}$$

This implies that Gaussian elimination gives

$$\begin{pmatrix}1 & 1 & 4\\0 & 2 & -8\\0 & 0 & 16\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Back substituting gives $c_3 = 0$. From second row we obtain $2c_2 - 8c_3 = 0$, hence $c_2 = 0$ and from first row $c_1 + c_2 + 4c_3 = 0$ hence $c_1 = 0$. This shows that

$$\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\3\\4\end{pmatrix},\begin{pmatrix}4\\-4\\2\end{pmatrix}$$

Are linearly independent. Hence they span $\Re^3$ and form a basis.

2.6.4 Section 3.4, problem 6

Determine whether the given solutions are a basis for the set of all solutions

$$\dot{x}=\begin{pmatrix}4 & -2 & 2\\-1 & 3 & 1\\1 & -1 & 5\end{pmatrix}x$$

$$x^1(t)=\begin{pmatrix}e^{2t}\\e^{2t}\\0\end{pmatrix},x^2(t)=\begin{pmatrix}0\\e^{4t}\\e^{4t}\end{pmatrix},x^3(t)=\begin{pmatrix}e^{6t}\\0\\e^{6t}\end{pmatrix}$$

Solution

We pick $t = 0$ to check linear independence (we can choose any t value, but $t = 0$ is the simplest). At $t = 0$ the given solutions become

$$x^1=\begin{pmatrix}1\\1\\0\end{pmatrix},x^2=\begin{pmatrix}0\\1\\1\end{pmatrix},x^3=\begin{pmatrix}1\\0\\1\end{pmatrix}$$

we now check that $c_1\begin{pmatrix}1\\1\\0\end{pmatrix}+c_2\begin{pmatrix}0\\1\\1\end{pmatrix}+c_3\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ implies $c_1 = 0, c_2 = 0, c_3 = 0$ as only solution. Writing the above as

$$\begin{pmatrix}1 & 0 & 1\\1 & 1 & 0\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Hence augmented matrix is

$$\begin{pmatrix}1 & 0 & 1 & 0\\1 & 1 & 0 & 0\\0 & 1 & 1 & 0\end{pmatrix}$$

Replacing row 2 by row 2 minus row 1 gives

$$\begin{pmatrix}1 & 0 & 1 & 0\\0 & 1 & -1 & 0\\0 & 1 & 1 & 0\end{pmatrix}$$

Replacing row 3 by row 3 minus row 2 gives

$$\begin{pmatrix}1 & 0 & 1 & 0\\0 & 1 & -1 & 0\\0 & 0 & 2 & 0\end{pmatrix}$$

This implies that Gaussian elimination gives

$$\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

Back substituting gives $2c_3 = 0$ or $c_3 = 0$. From second row $c_2 - c_3 = 0$. Hence $c_2 = 0$ and from first row $c_1 + c_3 = 0$, hence $c_1 = 0$. This shows that

$$x^1(t)=\begin{pmatrix}e^{2t}\\ e^{2t}\\ 0\end{pmatrix}, x^2(t)=\begin{pmatrix}0\\ e^{4t}\\ e^{4t}\end{pmatrix}, x^3(t)=\begin{pmatrix}e^{6t}\\ 0\\ e^{6t}\end{pmatrix}$$

Are linearly independent. Hence they form basis for the set of all solutions for the system given above.

2.6.5 Section 3.5, problem 6

Compute the determinant of

$$\begin{pmatrix}2 & -1 & 6 & 3\\ 1 & 0 & 1 & -1\\ 1 & 3 & 0 & 2\\ 1 & -1 & 1 & 0\end{pmatrix}$$

Solution

$$\begin{vmatrix}2 & -1 & 6 & 3\\ 1 & 0 & 1 & -1\\ 1 & 3 & 0 & 2\\ 1 & -1 & 1 & 0\end{vmatrix}=2\begin{vmatrix}0 & 1 & -1\\ 3 & 0 & 2\\ -1 & 1 & 0\end{vmatrix}+\begin{vmatrix}1 & 1 & -1\\ 1 & 0 & 2\\ 1 & 1 & 0\end{vmatrix}+6\begin{vmatrix}1 & 0 & -1\\ 1 & 3 & 2\\ 1 & -1 & 0\end{vmatrix}-3\begin{vmatrix}1 & 0 & 1\\ 1 & 3 & 0\\ 1 & -1 & 1\end{vmatrix} \tag{1}$$

But

$$\begin{aligned}\begin{vmatrix}0 & 1 & -1\\ 3 & 0 & 2\\ -1 & 1 & 0\end{vmatrix}&=0-\begin{vmatrix}3 & 2\\ -1 & 0\end{vmatrix}-\begin{vmatrix}3 & 0\\ -1 & 1\end{vmatrix}\\ &=-2-3\\ &=-5\end{aligned} \tag{2}$$

And

$$\begin{aligned}\begin{vmatrix}1 & 1 & -1\\ 1 & 0 & 2\\ 1 & 1 & 0\end{vmatrix}&=\begin{vmatrix}0 & 2\\ 1 & 0\end{vmatrix}-\begin{vmatrix}1 & 2\\ 1 & 0\end{vmatrix}-\begin{vmatrix}1 & 0\\ 1 & 1\end{vmatrix}\\ &=-2+2-1\\ &=-1\end{aligned} \tag{3}$$

And

$$\begin{aligned}\begin{vmatrix}1 & 0 & -1\\ 1 & 3 & 2\\ 1 & -1 & 0\end{vmatrix}&=\begin{vmatrix}2 & 2\\ -1 & 0\end{vmatrix}+0-\begin{vmatrix}1 & 3\\ 1 & -1\end{vmatrix}\\ &=2-(-1-3)\\ &=6\end{aligned} \tag{4}$$

And

$$\begin{aligned}\begin{vmatrix}1 & 0 & 1\\ 1 & 3 & 0\\ 1 & -1 & 1\end{vmatrix}&=\begin{vmatrix}3 & 0\\ -1 & 1\end{vmatrix}+0+\begin{vmatrix}1 & 3\\ 1 & -1\end{vmatrix}\\ &=3+(-1-3)\\ &=-1\end{aligned} \tag{5}$$

Substituting (2,3,4,5) into (1) gives

$$\begin{aligned}\begin{vmatrix}2 & -1 & 6 & 3\\ 1 & 0 & 1 & -1\\ 1 & 3 & 0 & 2\\ 1 & -1 & 1 & 0\end{vmatrix} &= 2(-5) + (-1) + 6(6) - 3(-1)\\ &= 28\end{aligned}$$

2.6.6 Section 3.6, problem 10

Find the inverse if it exist of

$$A = \begin{pmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{pmatrix}$$

Solution

$$A^{-1} = \frac{1}{|A|} adj(A)^T \tag{1}$$

But

$$\begin{aligned}|A| &= \cos\theta\begin{vmatrix}1 & 0\\ 0 & \cos\theta\end{vmatrix} - 0 - \sin\theta\begin{vmatrix}0 & 1\\ \sin\theta & 0\end{vmatrix}\\ &= \cos^2\theta + \sin^2\theta\\ &= 1 \end{aligned}\tag{2}$$

And

$$\begin{aligned}adj(A) &= \begin{pmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & \cos^2\theta + \sin^2\theta & 0\\ \sin\theta & 0 & \cos\theta\end{pmatrix}\\ &= \begin{pmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{pmatrix}\end{aligned}$$

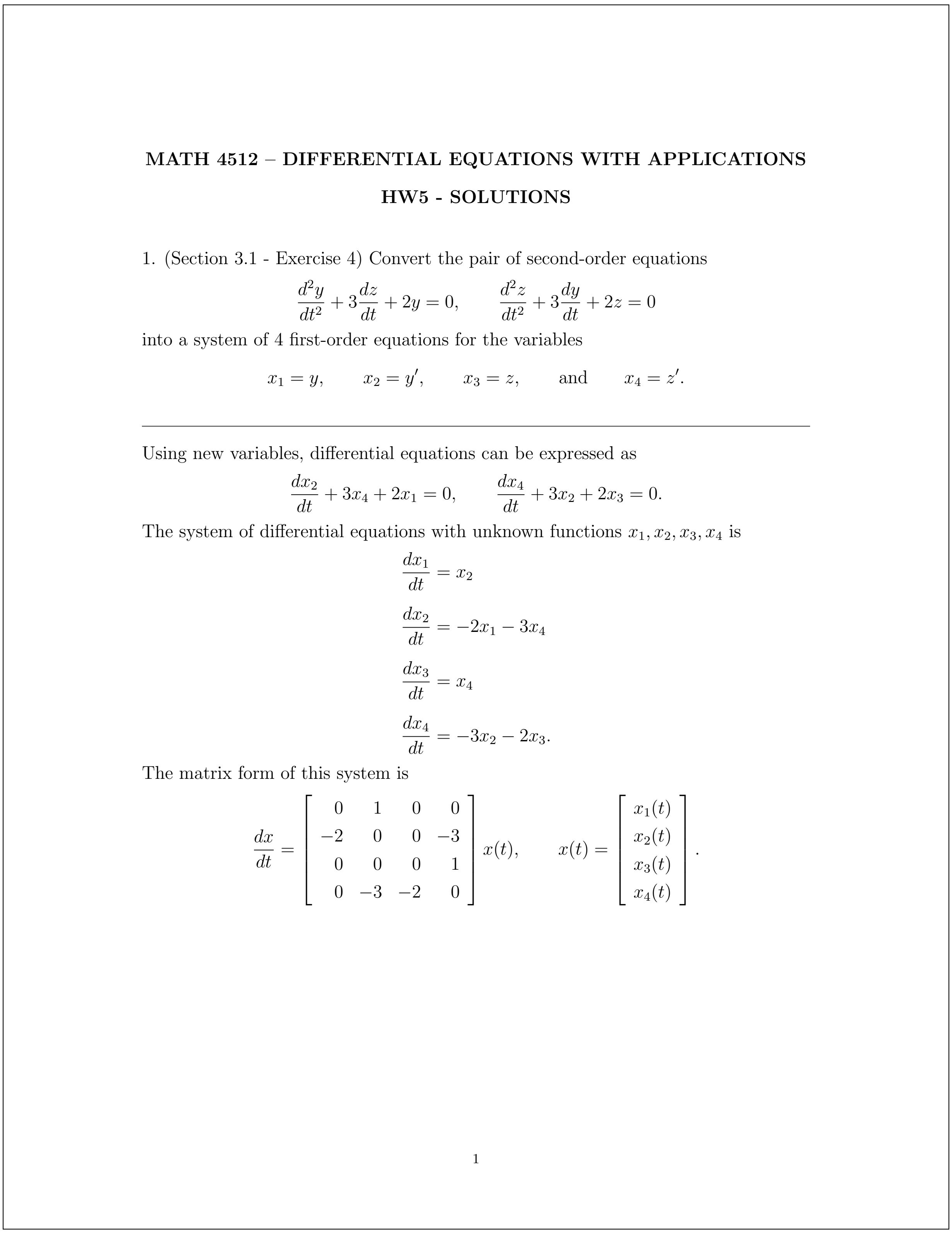
Hence

$$adj(A)^T = \begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix} \tag{3}$$

Substituting (2,3) into (1) gives

$$A^{-1} = \begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}$$

2.6.7 Key solution for HW 5



MATH 4512 – DIFFERENTIAL EQUATIONS WITH APPLICATIONS

HW5 - SOLUTIONS

1. (Section 3.1 - Exercise 4) Convert the pair of second-order equations

$$\frac{d^2y}{dt^2} + 3\frac{dz}{dt} + 2y = 0, \qquad \frac{d^2z}{dt^2} + 3\frac{dy}{dt} + 2z = 0$$

into a system of 4 first-order equations for the variables

$$x_1 = y, \qquad x_2 = y', \qquad x_3 = z, \qquad \text{and} \qquad x_4 = z'.$$

Using new variables, differential equations can be expressed as

$$\frac{dx_2}{dt} + 3x_4 + 2x_1 = 0, \qquad \frac{dx_4}{dt} + 3x_2 + 2x_3 = 0.$$

The system of differential equations with unknown functions x_1, x_2, x_3, x_4 is

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -2x_1 - 3x_4 \\ \frac{dx_3}{dt} &= x_4 \\ \frac{dx_4}{dt} &= -3x_2 - 2x_3. \end{aligned}$$

The matrix form of this system is

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & -3 & -2 & 0 \end{bmatrix} x(t), \qquad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}.$$

2. (Section 3.2 - Exercise 4) Determine whether the set of all elements $x = [x_1, x_2, x_3]^\top$ where $x_1 + x_2 + x_3 = 1$ forms a vector space under the properties of vector addition and scalar multiplication.

Let V denote the given set of vectors, i.e.

$$V = \{x = [x_1, x_2, x_3]^\top \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 1\}.$$

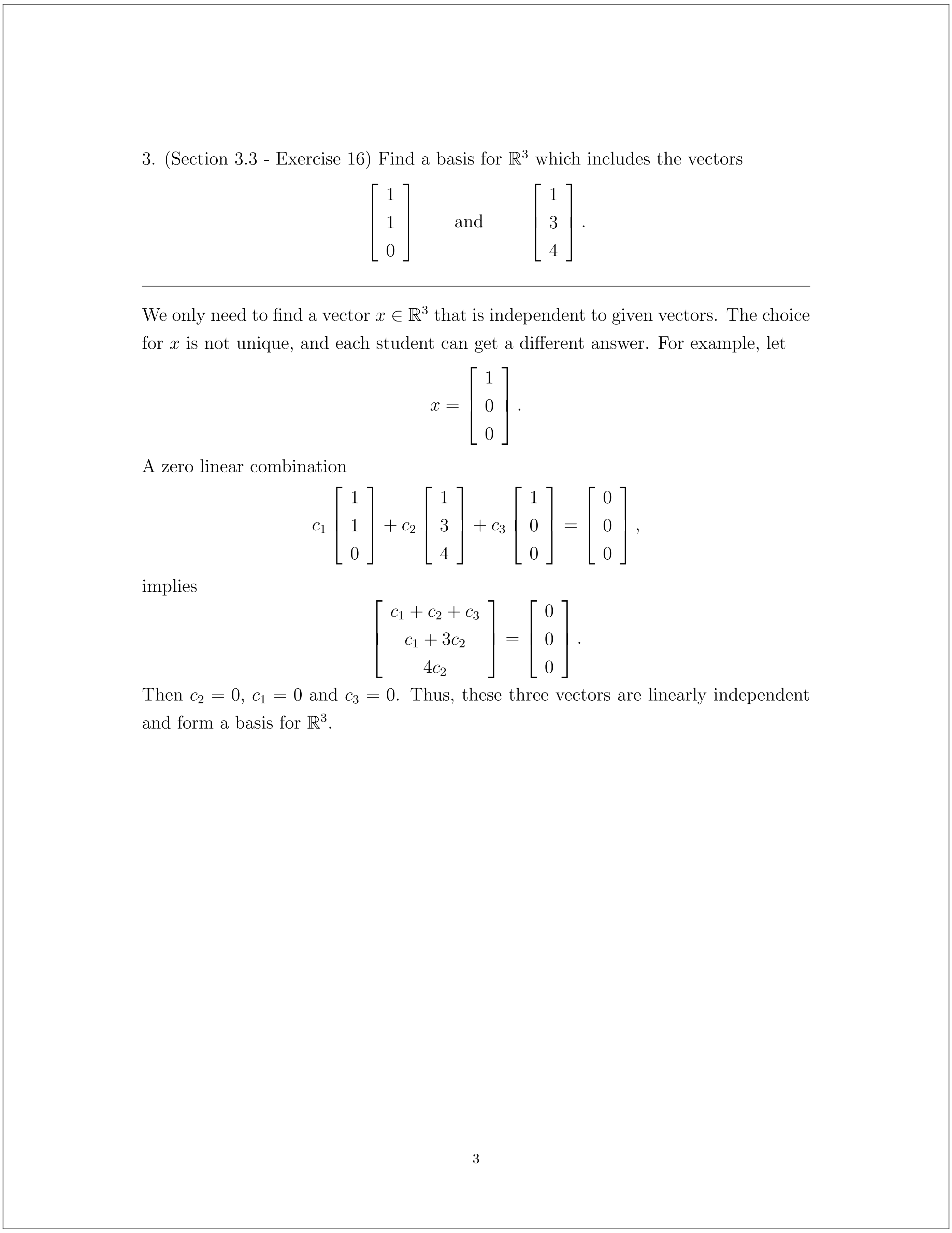
Consider vectors $x, y \in V$ where

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Since

$$x + y = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \notin V,$$

we conclude that V is not a vector space.



3. (Section 3.3 - Exercise 16) Find a basis for $\mathbb{R}^3$ which includes the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

We only need to find a vector $x \in \mathbb{R}^3$ that is independent to given vectors. The choice for x is not unique, and each student can get a different answer. For example, let

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

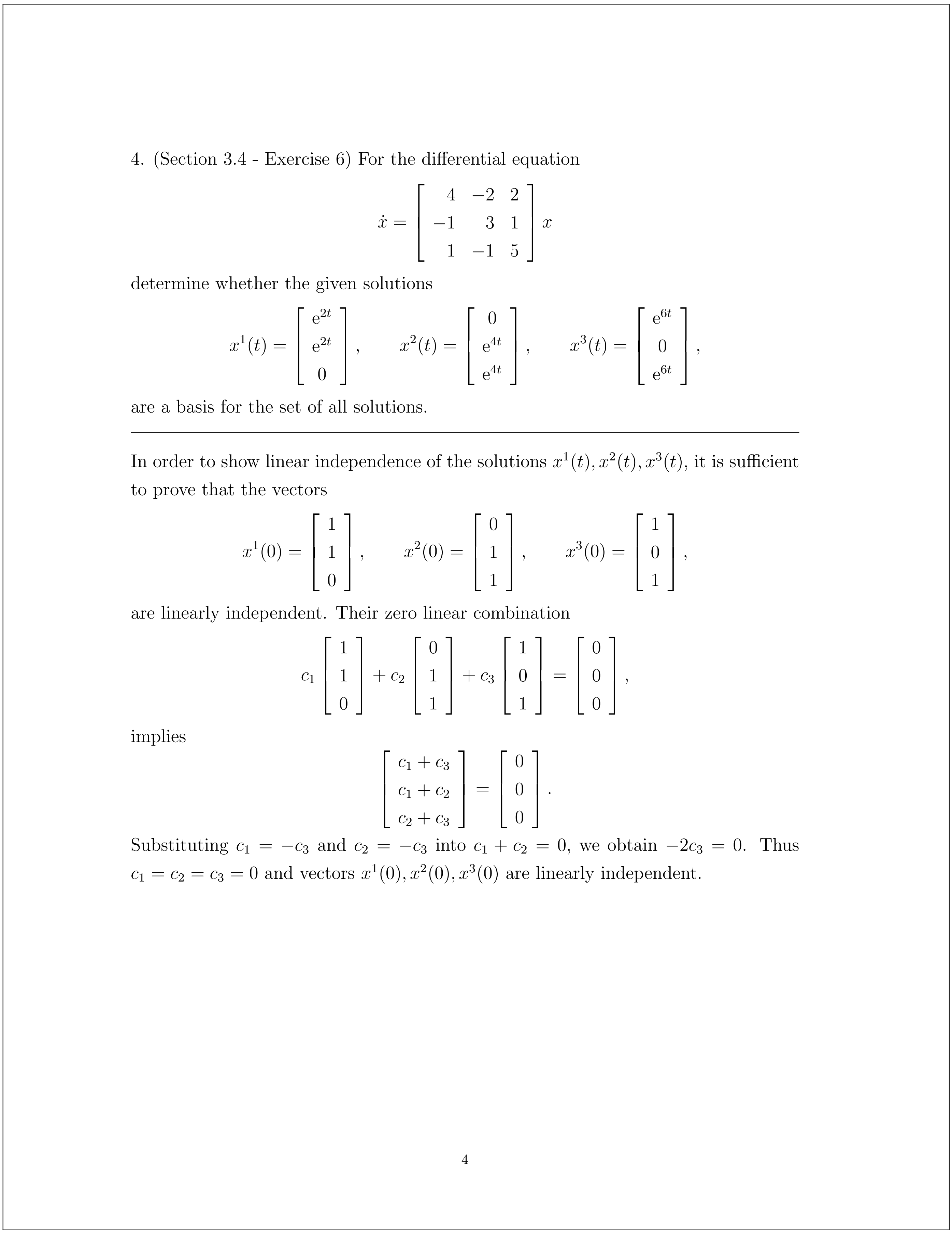
A zero linear combination

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

implies

$$\begin{bmatrix} c_1 + c_2 + c_3 \\ c_1 + 3c_2 \\ 4c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then $c_2 = 0$, $c_1 = 0$ and $c_3 = 0$. Thus, these three vectors are linearly independent and form a basis for $\mathbb{R}^3$.



4. (Section 3.4 - Exercise 6) For the differential equation

$$\dot{x} = \begin{bmatrix} 4 & -2 & 2 \\ -1 & 3 & 1 \\ 1 & -1 & 5 \end{bmatrix} x$$

determine whether the given solutions

$$x^1(t) = \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ 0 \end{bmatrix}, \qquad x^2(t) = \begin{bmatrix} 0 \\ \mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix}, \qquad x^3(t) = \begin{bmatrix} \mathrm{e}^{6t} \\ 0 \\ \mathrm{e}^{6t} \end{bmatrix},$$

are a basis for the set of all solutions.

In order to show linear independence of the solutions $x^1(t), x^2(t), x^3(t)$, it is sufficient to prove that the vectors

$$x^1(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad x^2(0) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad x^3(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

are linearly independent. Their zero linear combination

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

implies

$$\begin{bmatrix} c_1 + c_3 \\ c_1 + c_2 \\ c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Substituting $c_1 = -c_3$ and $c_2 = -c_3$ into $c_1 + c_2 = 0$, we obtain $-2c_3 = 0$. Thus $c_1 = c_2 = c_3 = 0$ and vectors $x^1(0), x^2(0), x^3(0)$ are linearly independent.

5. (Section 3.5 - Exercise 6) Compute the determinant of the matrix

$$\begin{bmatrix} 2 & -1 & 6 & 3 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & 0 & 2 \\ 1 & -1 & 1 & 0 \end{bmatrix}.$$

One of the ways to find the determinant of the given matrix is:

$$\begin{vmatrix} 2 & -1 & 6 & 3 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & 0 & 2 \\ 1 & -1 & 1 & 0 \end{vmatrix} \overset{-R_2+R_4}{=\!=\!=} \begin{vmatrix} 2 & -1 & 6 & 3 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & 0 & 2 \\ 0 & -1 & 0 & 1 \end{vmatrix}$$

$$\overset{4^{th}\ row\ exp.}{=\!=\!=\!=} (-1)(-1)^{4+2} \begin{vmatrix} 2 & 6 & 3 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix} + 1(-1)^{4+4} \begin{vmatrix} 2 & -1 & 6 \\ 1 & 0 & 1 \\ 1 & 3 & 0 \end{vmatrix}$$

$$= -(4-6-3-12) + (-1+18-6) = 17 + 11 = 28.$$

6. (Section 3.6 - Exercise 10) Find the inverse, it is exists, of the given matrix

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

Then

$$\det A = \begin{vmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \neq 0,$$

and A^{-1} exists. The cofactor matrix C for A is

$$C = \begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ 0 & \cos\theta \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ \sin\theta & \cos\theta \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ \sin\theta & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & -\sin\theta \\ 0 & \cos\theta \end{vmatrix} & +\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} & -\begin{vmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{vmatrix} \\ +\begin{vmatrix} 0 & -\sin\theta \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} \cos\theta & -\sin\theta \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = A.$$

Thus $\mathrm{adj}A = C^\top = A^\top$ and

$$A^{-1} = \frac{1}{\det A}\,\mathrm{adj}A = A^\top = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

2.7 HW 6

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2.7.1 Section 3.8, problem 12

Solve

$$\dot{x} = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix}$$

Solution

The first step is to find the eigenvalues. For this we need to solve

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \begin{vmatrix} 3-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 4 & 1 & -3-\lambda \end{vmatrix} &= 0 \\ (3-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ 4 & -3-\lambda \end{vmatrix} - 2\begin{vmatrix} -1 & 2-\lambda \\ 4 & 1 \end{vmatrix} &= 0 \\ (3-\lambda)((2-\lambda)(-3-\lambda)-1) - ((3+\lambda)-4) - 2(-1-4(2-\lambda)) &= 0 \\ \lambda^3 - 2\lambda^2 - \lambda + 2 &= 0 \end{aligned}$$

Guessing a root at $\lambda = 1$ is verified to be correct since $1-2-1+2=0$. Now that we know one root, we can do long division $\frac{(\lambda^3-2\lambda^2-\lambda+2)}{(\lambda-1)} = \lambda^2 - \lambda - 2$. Therefore the charaterstic polynominal factors to

$$\begin{aligned} \lambda^3 - 2\lambda^2 - \lambda + 2 &= (\lambda-1)\left(\lambda^2-\lambda-2\right) \\ &= (\lambda-1)(\lambda-2)(\lambda+1) \end{aligned}$$

Hence the eigenvalues are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1$.

For $\lambda_1 = 1$

$$\begin{aligned} (A - \lambda_1 I)\,\boldsymbol{v}_1 &= 0 \\ \begin{pmatrix} 3-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 4 & 1 & -3-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 3-1 & 1 & -2 \\ -1 & 2-1 & 1 \\ 4 & 1 & -3-1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 2 & 1 & -2 \\ -1 & 1 & 1 \\ 4 & 1 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Let $v_1 = 1$. First equation gives $2 + v_2 - 2v_3 = 0$ and the second equation gives $-1 + v_2 + v_3 = 0$. Subtracting gives $3 - 3v_3 = 0$, giving $v_3 = 1$. Therefore $v_2 = 0$. Hence

$$\boldsymbol{v}^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

<u>For $\lambda_2 = 2$</u>

$$(A - \lambda_2 I)\,\boldsymbol{v}^2 = 0$$

$$\begin{pmatrix} 3-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 4 & 1 & -3-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3-2 & 1 & -2 \\ -1 & 2-2 & 1 \\ 4 & 1 & -3-2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -2 \\ -1 & 0 & 1 \\ 4 & 1 & -5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Let $v_1 = 1$. Hence first equation gives $1 + v_2 - 2v_3 = 0$ and second equation gives $-1 + v_3 = 0$. Therefore $v_3 = 1$ and $v_2 = 2v_3 - 1 = 1$. Hence

$$\boldsymbol{v}^2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

<u>For $\lambda_3 = -1$</u>

$$(A - \lambda_3 I)\,\boldsymbol{v}^3 = 0$$

$$\begin{pmatrix} 3-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 4 & 1 & -3-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3+1 & 1 & -2 \\ -1 & 2+1 & 1 \\ 4 & 1 & -3+1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 & -2 \\ -1 & 3 & 1 \\ 4 & 1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Let $v_1 = 1$. Hence first equation gives $4 + v_2 - 2v_3 = 0$ and second equation gives $-1 + 3v_2 + v_3 = 0$. Multiplying $4 + v_2 - 2v_3 = 0$ by -3 and adding it to $-1 + 3v_2 + v_3 = 0$ gives $(-12 - 3v_2 + 6v_3 + (-1 + 3v_2 + v_3)) = 0$ or $-13+7v_3 = 0$ Hence $v_3 = \frac{13}{7}$. Therefore $v_2 = 2v_3 - 4 = 2\left(\frac{13}{7}\right) - 4 = -\frac{2}{7}$. Hence

$$\boldsymbol{v}^3 = \begin{pmatrix} 1 \\ -\frac{2}{7} \\ \frac{13}{7} \end{pmatrix}$$

Therefore

$$\boldsymbol{x}^1(t) = e^{\lambda_1 t}\boldsymbol{v}^1 = e^t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{x}^2(t) = e^{\lambda_2 t}\boldsymbol{v}^2 = e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\boldsymbol{x}^3(t) = e^{\lambda_3 t}\boldsymbol{v}^3 = e^{-t} \begin{pmatrix} 1 \\ -\frac{2}{7} \\ \frac{13}{7} \end{pmatrix}$$

Hence the general solution is

$$\begin{aligned}\boldsymbol{x}(t) &= c_1\boldsymbol{x}^1(t) + c_2\boldsymbol{x}^2(t) + c_3\boldsymbol{x}^3(t)\\ &= e^t\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2e^{2t}\begin{pmatrix}1\\1\\1\end{pmatrix} + c_3e^{-t}\begin{pmatrix}1\\-\frac{2}{7}\\\frac{13}{7}\end{pmatrix}\end{aligned}$$

Or

$$\boldsymbol{x}(t) = \begin{pmatrix}c_1e^t + c_2e^{2t} + c_3e^{-t}\\ c_2e^{2t} - \frac{2}{7}c_3e^{-t}\\ c_1e^t + c_2e^{2t} + \frac{13}{7}c_3e^{-t}\end{pmatrix} \tag{A}$$

Initial conditions are now used to find c_1, c_2, c_3. At $t = 0$ the above reduces to

$$\boldsymbol{x}(0) = \begin{pmatrix}1\\4\\-7\end{pmatrix}$$

$$\begin{pmatrix}c_1 + c_2 + c_3\\ c_2 - \frac{2}{7}c_3\\ c_1 + c_2 + \frac{13}{7}c_3\end{pmatrix} = \begin{pmatrix}1\\4\\-7\end{pmatrix}$$

$$\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & -\frac{2}{7}\\ 1 & 1 & \frac{13}{7}\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}1\\4\\-7\end{pmatrix} \tag{1}$$

Gaussian elimination on $\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & -\frac{2}{7}\\ 1 & 1 & \frac{13}{7}\end{pmatrix}\begin{pmatrix}1\\4\\-7\end{pmatrix}$. Replacing row 3 by row 3 - row 1 gives

$$\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & -\frac{2}{7}\\ 0 & 0 & \frac{13}{7}-1\end{pmatrix}\begin{pmatrix}1\\4\\-7-1\end{pmatrix} \rightarrow \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & -\frac{2}{7}\\ 0 & 0 & \frac{6}{7}\end{pmatrix}\begin{pmatrix}1\\4\\-8\end{pmatrix}$$

Hence (1) becomes

$$\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & -\frac{2}{7}\\ 0 & 0 & \frac{6}{7}\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}1\\4\\-8\end{pmatrix}$$

Back substitution gives $\frac{6}{7}c_3 = -8$, or $c_3 = -\frac{28}{3}$. From second row

$$\begin{aligned}c_2 - \frac{2}{7}c_3 &= 4\\ c_2 &= 4 + \frac{2}{7}c_3\\ &= 4 + \frac{2}{7}\left(-\frac{28}{3}\right)\\ &= \frac{4}{3}\end{aligned}$$

From first row

$$\begin{aligned}c_1 + c_2 + c_3 &= 1\\ c_1 &= 1 - c_2 - c_3\\ &= 1 - \frac{4}{3} + \frac{28}{3}\\ &= 9\end{aligned}$$

Using the above values of c_1, c_2, c_3, Eq (A) becomes

$$\begin{aligned}
x(t) &= \begin{pmatrix} c_1 e^t + c_2 e^{2t} + c_3 e^{-t} \\ c_2 e^{2t} - \frac{2}{7} c_3 e^{-t} \\ c_1 e^t + c_2 e^{2t} + \frac{13}{7} c_3 e^{-t} \end{pmatrix} \\
&= \begin{pmatrix} 9e^t + \frac{4}{3} e^{2t} - \frac{28}{3} e^{-t} \\ \frac{4}{3} e^{2t} - \frac{2}{7}\left(-\frac{28}{3}\right) e^{-t} \\ 9e^t + \frac{4}{3} e^{2t} + \frac{13}{7}\left(-\frac{28}{3}\right) e^{-t} \end{pmatrix} \\
&= \begin{pmatrix} 9e^t + \frac{4}{3} e^{2t} - \frac{28}{3} e^{-t} \\ \frac{4}{3} e^{2t} + \frac{8}{3} e^{-t} \\ 9e^t + \frac{4}{3} e^{2t} - \frac{52}{3} e^{-t} \end{pmatrix} \qquad (2)
\end{aligned}$$

Therefore

$$\begin{aligned}
x_1(t) &= 9e^t + \frac{4}{3} e^{2t} - \frac{28}{3} e^{-t} \\
x_2(t) &= \frac{4}{3} e^{2t} + \frac{8}{3} e^{-t} \\
x_3(t) &= 9e^t + \frac{4}{3} e^{2t} - \frac{52}{3} e^{-t}
\end{aligned}$$

2.7.2 Section 3.9, problem 2 (complex roots)

Find general solution of

$$\dot{x} = \begin{pmatrix} 1 & -5 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} x$$

Solution

The first step is to find the eigenvalues. For this we need to solve

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\begin{vmatrix} 1-\lambda & -5 & 0 \\ 1 & -3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} &= 0 \\
(1-\lambda)\begin{vmatrix} -3-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ 0 & 1-\lambda \end{vmatrix} &= 0 \\
(1-\lambda)((-3-\lambda)(1-\lambda)) + 5(1-\lambda) &= 0
\end{aligned}$$

Factoring $(1-\lambda)$ gives

$$\begin{aligned}
(1-\lambda)((-3-\lambda)(1-\lambda) + 5) &= 0 \\
(1-\lambda)\left(\lambda^2 + 2\lambda - 3 + 5\right) &= 0 \\
(1-\lambda)\left(\lambda^2 + 2\lambda + 2\right) &= 0
\end{aligned}$$

Hence one root is $\lambda_1 = 1$. Now we find roots of $\left(\lambda^2 + 2\lambda + 2\right)$. $\lambda = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} = -1 \pm \frac{1}{2}\sqrt{4 - 4(2)} = -1 \pm \frac{1}{2}\sqrt{-4}$. Hence

$$\lambda = -1 \pm i$$

Therefore the roots are

$$\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -1 + i \\
\lambda_3 &= -1 - i
\end{aligned}$$

For $\lambda_1 = 1$

$$
\begin{aligned}
(A-\lambda_1 I)\,\boldsymbol{v}^1 &= 0\\
\begin{pmatrix}1-\lambda_1 & -5 & 0\\ 1 & -3-\lambda_1 & 0\\ 0 & 0 & 1-\lambda_1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} &= \begin{pmatrix}0\\0\\0\end{pmatrix}\\
\begin{pmatrix}1-1 & -5 & 0\\ 1 & -3-1 & 0\\ 0 & 0 & 1-1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} &= \begin{pmatrix}0\\0\\0\end{pmatrix}\\
\begin{pmatrix}0 & -5 & 0\\ 1 & -4 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} &= \begin{pmatrix}0\\0\\0\end{pmatrix}
\end{aligned}
$$

Hence v_3 is arbitrary, say $v_3 = 1$. And $v_2 = 0$ from first equation. And from second equation $v_1 = 0$. Therefore

$$\boldsymbol{v}^1 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Hence

$$
\begin{aligned}
\boldsymbol{x}^1(t) &= e^{\lambda_1 t}\boldsymbol{v}^1\\
&= e^t\begin{pmatrix}0\\0\\1\end{pmatrix}
\end{aligned}
$$

For $\lambda_2 = -1 + i$

$$
\begin{aligned}
(A-\lambda_2 I)\,\boldsymbol{v}^2 &= 0\\
\begin{pmatrix}1-\lambda_2 & -5 & 0\\ 1 & -3-\lambda_2 & 0\\ 0 & 0 & 1-\lambda_2\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} &= \begin{pmatrix}0\\0\\0\end{pmatrix}\\
\begin{pmatrix}1-(-1+i) & -5 & 0\\ 1 & -3-(-1+i) & 0\\ 0 & 0 & 1-(-1+i)\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} &= \begin{pmatrix}0\\0\\0\end{pmatrix}\\
\begin{pmatrix}2-i & -5 & 0\\ 1 & -2-i & 0\\ 0 & 0 & 2-i\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} &= \begin{pmatrix}0\\0\\0\end{pmatrix}
\end{aligned}
$$

From last equation $v_3 = 0$. from second equation $v_1 = (2+i)\,v_2$. Hence

$$\boldsymbol{v}^2 = \begin{pmatrix}(2+i)\,v_2\\ v_2\\ 0\end{pmatrix} = v_2\begin{pmatrix}2+i\\ 1\\ 0\end{pmatrix}$$

Choosing $v_2 = 1$ the above becomes

$$\boldsymbol{v}^2 = \begin{pmatrix}2+i\\ 1\\ 0\end{pmatrix}$$

Hence

$$\boldsymbol{x}^2_{\lambda_2}(t) = e^{\lambda_2 t}\boldsymbol{v}^2 = e^{(-1+i)t}\begin{pmatrix}2+i\\ 1\\ 0\end{pmatrix}$$

Since this is complex root, we will now find the real and imaginary parts of the above, and use these to generate $x^2(t), x^3(t)$ from the above.

$$
\begin{aligned}
e^{(-1+i)t}\begin{pmatrix}2+i\\1\\0\end{pmatrix} &= e^{-t}e^{it}\begin{pmatrix}2+i\\1\\0\end{pmatrix}\\
&= e^{-t}(\cos t + i\sin t)\begin{pmatrix}2+i\\1\\0\end{pmatrix}\\
&= \left(e^{-t}\cos t + ie^{-t}\sin t\right)\begin{pmatrix}2+i\\1\\0\end{pmatrix}\\
&= \begin{pmatrix}\left(e^{-t}\cos t + ie^{-t}\sin t\right)(2+i)\\ \left(e^{-t}\cos t + ie^{-t}\sin t\right)\\ 0\end{pmatrix}\\
&= \begin{pmatrix}2e^{-t}\cos t + ie^{-t}\cos t + 2ie^{-t}\sin t - e^{-t}\sin t\\ e^{-t}\cos t + ie^{-t}\sin t\\ 0\end{pmatrix}\\
&= \begin{pmatrix}\left(2e^{-t}\cos t - e^{-t}\sin t\right) + i\left(e^{-t}\cos t + 2e^{-t}\sin t\right)\\ e^{-t}\cos t + ie^{-t}\sin t\\ 0\end{pmatrix}
\end{aligned}
$$

The real of the above is

$$
x^2(t) = \begin{pmatrix}2e^{-t}\cos t - e^{-t}\sin t\\ e^{-t}\cos t\\ 0\end{pmatrix}
$$

And imaginary part is

$$
x^3(t) = \begin{pmatrix}e^{-t}\cos t + 2e^{-t}\sin t\\ e^{-t}\sin t\\ 0\end{pmatrix}
$$

We have now obtain the three eigenvectors we want. Hence the general solution is

$$
\begin{aligned}
x(t) &= c_1x^1(t) + c_2x^2(t) + c_3x^3(t)\\
&= c_1e^t\begin{pmatrix}0\\0\\1\end{pmatrix} + c_2e^{-t}\begin{pmatrix}2\cos t - \sin t\\ \cos t\\ 0\end{pmatrix} + c_3e^{-t}\begin{pmatrix}\cos t + 2\sin t\\ \sin t\\ 0\end{pmatrix}
\end{aligned}
$$

2.7.3 Section 3.10, problem 6 (Equal roots)

Solve

$$
\dot{x} = \begin{pmatrix}-4 & -4 & 0\\ 10 & 9 & 1\\ -4 & -3 & 1\end{pmatrix}x, \qquad x(0) = \begin{pmatrix}2\\1\\-1\end{pmatrix}
$$

Solution

The first step is to find the eigenvalues. For this we need to solve

$$\begin{aligned}
\det(A-\lambda I) &= 0\\
\begin{vmatrix} -4-\lambda & -4 & 0\\ 10 & 9-\lambda & 1\\ -4 & -3 & 1-\lambda \end{vmatrix} &= 0\\
(-4-\lambda)\begin{vmatrix} 9-\lambda & 1\\ -3 & 1-\lambda \end{vmatrix} + 4\begin{vmatrix} 10 & 1\\ -4 & 1-\lambda \end{vmatrix} &= 0\\
(-4-\lambda)\left((9-\lambda)(1-\lambda)+3\right)+4\left((10)(1-\lambda)+4\right) &= 0\\
(\lambda-2)^3 &= 0
\end{aligned}$$

Hence root is $\lambda = 2$ of multiplicity 3.

To eigenvectors we start as before, using $\lambda = 2$.

$$\begin{aligned}
(A-\lambda I)\,\boldsymbol{v}^1 &= 0\\
\begin{pmatrix} -4-\lambda & -4 & 0\\ 10 & 9-\lambda & 1\\ -4 & -3 & 1-\lambda \end{pmatrix}\begin{pmatrix} v_1\\ v_2\\ v_3 \end{pmatrix} &= \begin{pmatrix} 0\\ 0\\ 0 \end{pmatrix}\\
\begin{pmatrix} -4-2 & -4 & 0\\ 10 & 9-2 & 1\\ -4 & -3 & 1-2 \end{pmatrix}\begin{pmatrix} v_1\\ v_2\\ v_3 \end{pmatrix} &= \begin{pmatrix} 0\\ 0\\ 0 \end{pmatrix}\\
\begin{pmatrix} -6 & -4 & 0\\ 10 & 7 & 1\\ -4 & -3 & -1 \end{pmatrix}\begin{pmatrix} v_1\\ v_2\\ v_3 \end{pmatrix} &= \begin{pmatrix} 0\\ 0\\ 0 \end{pmatrix}
\end{aligned}$$

Now we check if the eigenvalue is complete or defective. Using the first 2 rows we obtain

$$\begin{aligned}
-6v_1 - 4v_2 &= 0\\
10v_1 + 7v_2 + v_3 &= 0
\end{aligned}$$

Solving gives $v_1 = 2v_3, v_2 = -3v_3$. Hence

$$\begin{pmatrix} v_1\\ v_2\\ v_3 \end{pmatrix} = \begin{pmatrix} 2v_3\\ -3v_3\\ v_3 \end{pmatrix} = v_3\begin{pmatrix} 2\\ -3\\ 1 \end{pmatrix}$$

Choosing $v_3 = 1$ gives

$$\boldsymbol{v}^1 = \begin{pmatrix} 2\\ -3\\ 1 \end{pmatrix}$$

Lets see now if we can obtain another linearly independent eigenvector. Using the first row and the third row

$$\begin{aligned}
-6v_1 - 4v_2 &= 0\\
-4v_1 - 3v_2 - v_3 &= 0
\end{aligned}$$

Solving gives $v_1 = 2v_3, v_2 = -3v_3$. Which is the same as the one found above. Finally using the second and third row

$$\begin{aligned}
10v_1 + 7v_2 + v_3 &= 0\\
-4v_1 - 3v_2 - v_3 &= 0
\end{aligned}$$

Solving gives $v_1 = 2v_3, v_2 = -3v_3$ which is the same as above. So the eigenvalue 2 is defective.

$$\boldsymbol{x}^1(t) = e^{2t}\begin{pmatrix} 2\\ -3\\ 1 \end{pmatrix}$$

Since the eigenvalue is defective, to find the second and third eigenvectors we do the following. To find $\boldsymbol{v}^2$. We need to solve

$$(A-\lambda I)^2 \boldsymbol{v}^2 = 0 \tag{1}$$

But $A-\lambda I = \begin{pmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{pmatrix}$ from earlier. Hence

$$\begin{aligned}(A-\lambda I)^2 &= \begin{pmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{pmatrix}\begin{pmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -4 & -4 & -4 \\ 6 & 6 & 6 \\ -2 & -2 & -2 \end{pmatrix}\end{aligned}$$

Therefore (1) becomes

$$\begin{pmatrix} -4 & -4 & -4 \\ 6 & 6 & 6 \\ -2 & -2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Using the first equation $-4v_1 - 4v_2 - 4v_3 = 0$ or equivalently $v_1 + v_2 + v_3 = 0$. Therefore $v_1 = -v_2 - v_3$. Hence

$$\begin{aligned}\boldsymbol{v}^2 &= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\ &= \begin{pmatrix} -v_2 - v_3 \\ v_2 \\ v_3 \end{pmatrix} \\ &= v_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + v_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\end{aligned}$$

Taking $v_2 = 1, v_3 = 0$ gives

$$\boldsymbol{v}^2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Let us check the above choice is valid: $(A-\lambda I)\boldsymbol{v}^2 = \begin{pmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ which is not zero. Good, so we can use it. Therefore

$$\begin{aligned}\boldsymbol{x}^2(t) &= e^{\lambda t}\left(\boldsymbol{v}^2 + t(A-\lambda I)\boldsymbol{v}^2\right) \\ &= e^{2t}\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right) \\ &= e^{2t}\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}\right) \\ &= e^{2t}\begin{pmatrix} -1+2t \\ 1-3t \\ t \end{pmatrix}\end{aligned}$$

Now we find the third eigenvector $\boldsymbol{v}^3$. We need to solve

$$(A-\lambda I)^3 \boldsymbol{v}^3 = 0 \tag{1}$$

But $(A-\lambda I)^2 = \begin{pmatrix} -4 & -4 & -4 \\ 6 & 6 & 6 \\ -2 & -2 & -2 \end{pmatrix}$ from earlier. Hence

$$\begin{aligned} (A-\lambda I)^3 &= \begin{pmatrix} -4 & -4 & -4 \\ 6 & 6 & 6 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Therefore v_3 is arbitrary as long as $(A-\lambda I)^2 v_3 \neq 0$. Let us pick $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Checking this choice is valid: $\begin{pmatrix} -4 & -4 & -4 \\ 6 & 6 & 6 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ -2 \end{pmatrix}$. Not zero. Good, so we can use it. Therefore

$$\begin{aligned} \boldsymbol{x}^3(t) &= e^{\lambda t}\left(\boldsymbol{v}^3 + t(A-\lambda I)\boldsymbol{v}^3 + \frac{t^2}{2}(A-\lambda I)^2 \boldsymbol{v}^3\right) \\ &= e^{2t}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} -4 & -4 & -4 \\ 6 & 6 & 6 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) \\ &= e^{2t}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -6 \\ 10 \\ -4 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} -4 \\ 6 \\ -2 \end{pmatrix}\right) \\ &= e^{2t} \begin{pmatrix} 1-6t-2t^2 \\ 10t+3t^2 \\ -4t-t^2 \end{pmatrix} \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} \boldsymbol{x}(t) &= c_1 \boldsymbol{x}^1(t) + c_2 \boldsymbol{x}^2(t) + c_3 \boldsymbol{x}^3(t) \\ &= c_1 e^{2t} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -1+2t \\ 1-3t \\ t \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1-6t-2t^2 \\ 10t+3t^2 \\ -4t-t^2 \end{pmatrix} \\ &= \begin{pmatrix} e^{2t}\left(2c_1 + c_2(-1+2t) + c_3\left(1-6t-2t^2\right)\right) \\ e^{2t}\left(-3c_1 + c_2(1-3t) + c_3\left(10t+3t^2\right)\right) \\ e^{2t}\left(c_1 + tc_2 + c_3\left(-4t-t^2\right)\right) \end{pmatrix} \end{aligned}$$

Now we find c_i from initial conditions. At $t=0$

$$\begin{aligned} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} &= c_1 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2c_1 - c_2 + c_3 \\ -3c_1 + c_2 \\ c_1 \end{pmatrix} \end{aligned}$$

Or

$$\begin{pmatrix} 2 & -1 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \tag{2}$$

From last row, $c_1 = -1$. From second row $-3c_1 + c_2 = 1$, hence $c_2 = 1 - 3 = -2$. From first row

$2c_1 - c_2 + c_3 = 2$, hence $c_3 = 2 - 2 + 2 = 2$. Therefore the general solution becomes

$$\begin{aligned}
\boldsymbol{x}(t) &= c_1\boldsymbol{x}^1(t) + c_2\boldsymbol{x}^2(t) + c_3\boldsymbol{x}^3(t)\\
&= -e^{2t}\begin{pmatrix}2\\-3\\1\end{pmatrix} - 2e^{2t}\begin{pmatrix}-1+2t\\1-3t\\t\end{pmatrix} + 2e^{2t}\begin{pmatrix}1-6t-2t^2\\10t+3t^2\\-4t-t^2\end{pmatrix}\\
&= e^{2t}\begin{pmatrix}-2-2(-1+2t)+2\left(1-6t-2t^2\right)\\3-2(1-3t)+2\left(10t+3t^2\right)\\-1-2t+2\left(-4t-t^2\right)\end{pmatrix}\\
&= e^{2t}\begin{pmatrix}-4t^2-16t+2\\6t^2+26t+1\\-2t^2-10t-1\end{pmatrix}
\end{aligned}$$

Or

$$\begin{aligned}
x_1(t) &= e^{2t}\left(-4t^2-16t+2\right)\\
x_2(t) &= e^{2t}\left(6t^2+26t+1\right)\\
x_3(t) &= e^{2t}\left(-2t^2-10t-1\right)
\end{aligned}$$

This is a plot of the solutions. The solutions all blow up in time due to positive exponential terms.

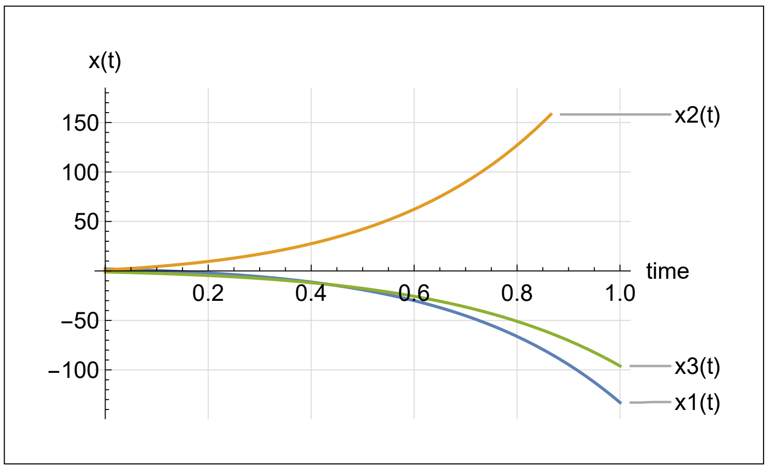
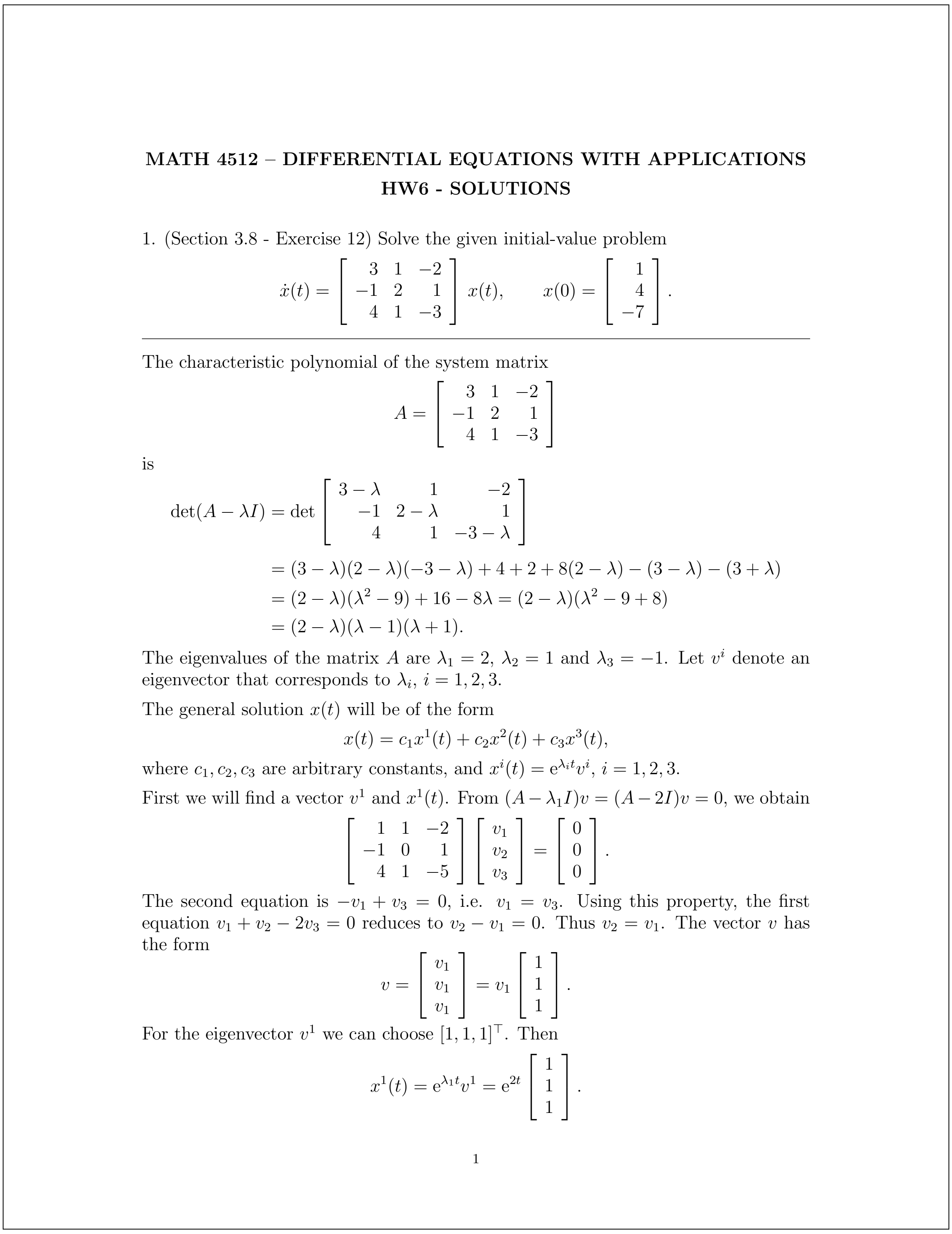


Figure 2.16: Plot of the solutions above

2.7.4 Key solution for HW 6



MATH 4512 – DIFFERENTIAL EQUATIONS WITH APPLICATIONS
HW6 - SOLUTIONS

1. (Section 3.8 - Exercise 12) Solve the given initial-value problem

$$\dot{x}(t) = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{bmatrix} x(t), \qquad x(0) = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}.$$

The characteristic polynomial of the system matrix

$$A = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{bmatrix}$$

is

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 4 & 1 & -3-\lambda \end{bmatrix} \\ &= (3-\lambda)(2-\lambda)(-3-\lambda) + 4 + 2 + 8(2-\lambda) - (3-\lambda) - (3+\lambda) \\ &= (2-\lambda)(\lambda^2 - 9) + 16 - 8\lambda = (2-\lambda)(\lambda^2 - 9 + 8) \\ &= (2-\lambda)(\lambda - 1)(\lambda + 1). \end{aligned}$$

The eigenvalues of the matrix A are $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = -1$. Let v^i denote an eigenvector that corresponds to λ_i, $i = 1, 2, 3$.

The general solution $x(t)$ will be of the form

$$x(t) = c_1 x^1(t) + c_2 x^2(t) + c_3 x^3(t),$$

where c_1, c_2, c_3 are arbitrary constants, and $x^i(t) = \mathrm{e}^{\lambda_i t} v^i$, $i = 1, 2, 3$.

First we will find a vector v^1 and $x^1(t)$. From $(A - \lambda_1 I)v = (A - 2I)v = 0$, we obtain

$$\begin{bmatrix} 1 & 1 & -2 \\ -1 & 0 & 1 \\ 4 & 1 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The second equation is $-v_1 + v_3 = 0$, i.e. $v_1 = v_3$. Using this property, the first equation $v_1 + v_2 - 2v_3 = 0$ reduces to $v_2 - v_1 = 0$. Thus $v_2 = v_1$. The vector v has the form

$$v = \begin{bmatrix} v_1 \\ v_1 \\ v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For the eigenvector v^1 we can choose $[1, 1, 1]^\top$. Then

$$x^1(t) = \mathrm{e}^{\lambda_1 t} v^1 = \mathrm{e}^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

1

Now we proceed with finding a vector v^2 and $x^2(t)$. From $(A-\lambda_2 I)v = (A-I)v = 0$, we obtain

$$\begin{bmatrix} 2 & 1 & -2 \\ -1 & 1 & 1 \\ 4 & 1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Multiplying the first equation $2v_1 + v_2 - 2v_3 = 0$ by -1 and adding to the second equation $-v_1 + v_2 + v_3 = 0$ leads to $-3v_1 + 3v_3 = 0$. Then $v_1 = v_3$. This relation implies $v_2 = 0$. Now the vector v has the form

$$v = \begin{bmatrix} v_1 \\ 0 \\ v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

For the eigenvector v^2 we can choose $[1, 0, 1]^\top$. Then

$$x^2(t) = \mathrm{e}^{\lambda_2 t} v^2 = \mathrm{e}^{t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

At the end we will find a vector v^3 and $x^3(t)$. From $(A - \lambda_3 I)v = (A + I)v = 0$, we obtain

$$\begin{bmatrix} 4 & 1 & -2 \\ -1 & 3 & 1 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Multiplying the second equation $-v_1 + 3v_2 + v_3 = 0$ by 2 and adding to the first equation $4v_1 + v_2 - 2v_3 = 0$ leads to $2v_1 + 7v_2 = 0$. Then $v_1 = -7v_2/2$. Using this relation in the second equation, we find $v_3 = v_1 - 3v_2 = -13v_2/2$. The vector v has the form

$$v = \begin{bmatrix} -\frac{7}{2}v_2 \\ v_2 \\ -\frac{13}{2}v_2 \end{bmatrix} = v_2 \begin{bmatrix} -\frac{7}{2} \\ 1 \\ -\frac{13}{2} \end{bmatrix}.$$

For the eigenvector v^3 we can choose

$$v = -2 \begin{bmatrix} -\frac{7}{2} \\ 1 \\ -\frac{13}{2} \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 13 \end{bmatrix}$$

Then

$$x^3(t) = \mathrm{e}^{\lambda_3 t} v^3 = \mathrm{e}^{-t} \begin{bmatrix} 7 \\ -2 \\ 13 \end{bmatrix}.$$

The general solution is

$$x(t) = c_1 \mathrm{e}^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{-t} \begin{bmatrix} 7 \\ -2 \\ 13 \end{bmatrix}$$

Initial condition implies

$$\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} = x(0) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ -2 \\ 13 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + 7c_3 \\ c_1 - 2c_3 \\ c_1 + c_2 + 13c_3 \end{bmatrix}.$$

Subtracting first and third equation gives $6c_3 = -8$, and consequently $c_3 = -4/3$. From the second equation it follows $c_1 = 2c_3 + 4 = 4/3$. Finally, $c_2 = 1 - c_1 - 7c_3 = 9$. The solution of the initial-value problem is

$$x(t) = \frac{4}{3}\mathrm{e}^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 9\mathrm{e}^{t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{3}\mathrm{e}^{-t} \begin{bmatrix} 7 \\ -2 \\ 13 \end{bmatrix}.$$

2. (Section 3.9 - Exercise 2) Find the general solution of the given system of differential equations

$$\dot{x}(t) = \begin{bmatrix} 1 & -5 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t).$$

The characteristic polynomial of the system matrix

$$A = \begin{bmatrix} 1 & -5 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & -5 & 0 \\ 1 & -3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda)^2(-3-\lambda) + 5(1-\lambda) = (1-\lambda)(-3+2\lambda+\lambda^2+5) \\ &= (1-\lambda)(\lambda^2+2\lambda+2). \end{aligned}$$

The eigenvalues of the matrix A are $\lambda_1 = 1$, $\lambda_2 = -1+i$ and $\lambda_3 = -1-i$.

In order to find $x^1(t)$, we proceed as in the previous exercise. We start from solving $(A - \lambda_1 I)v = (A - I)v = 0$, i.e.

$$\begin{bmatrix} 0 & -5 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the first equation we obtain $v_2 = 0$, while from the second $v_1 - 4v_2 = 0$ it follows $v_1 = 0$. Thus the vector v has the form

$$v = \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

For the eigenvector v^1 we can choose $[0, 0, 1]^\top$. Then

$$x^1(t) = \mathrm{e}^{\lambda_1 t} v^1 = \mathrm{e}^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Other eigenvalues of the matrix A are complex. In order to obtain two remaining linearly independent real solutions $x^2(t)$ and $x^3(t)$, it is sufficient to consider $\lambda_2 = -1 + i$. We first find a complex vector v that solves $(A - \lambda_2 I)v = 0$, i.e.

$$\begin{bmatrix} 2-i & -5 & 0 \\ 1 & -2-i & 0 \\ 0 & 0 & 2-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the last equation we obtain $v_3 = 0$, while from the second it follows $v_1 = (2+i)v_2$. A complex eigenvector v that corresponds to $\lambda_2 = i$ has the form

$$v = \begin{bmatrix} (2+i)v_2 \\ v_2 \\ 0 \end{bmatrix} = v_2 \begin{bmatrix} 2+i \\ 1 \\ 0 \end{bmatrix}.$$

A complex-valued solution is

$$\phi(t) = \mathrm{e}^{(-1+i)t} \begin{bmatrix} 2+i \\ 1 \\ 0 \end{bmatrix} = \mathrm{e}^{-t}(\cos t + i \sin t) \begin{bmatrix} 2+i \\ 1 \\ 0 \end{bmatrix}$$

$$= \mathrm{e}^{-t} \begin{bmatrix} (2\cos t - \sin t) + i(2\sin t + \cos t) \\ \cos t + i \sin t \\ 0 \end{bmatrix}$$

$$= \mathrm{e}^{-t} \begin{bmatrix} 2\cos t - \sin t \\ \cos t \\ 0 \end{bmatrix} + i\mathrm{e}^{-t} \begin{bmatrix} 2\sin t + \cos t \\ \sin t \\ 0 \end{bmatrix}.$$

Now,

$$x^2(t) = \mathrm{e}^{-t} \begin{bmatrix} 2\cos t - \sin t \\ \cos t \\ 0 \end{bmatrix}, \qquad x^3(t) = \mathrm{e}^{-t} \begin{bmatrix} 2\sin t + \cos t \\ \sin t \\ 0 \end{bmatrix},$$

and the general solution has the form

$$x(t) = c_1 x^1(t) + c_2 x^2(t) + c_3 x^3(t)$$

$$= c_1 \mathrm{e}^{t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-t} \begin{bmatrix} 2\cos t - \sin t \\ \cos t \\ 0 \end{bmatrix} + c_3 \mathrm{e}^{-t} \begin{bmatrix} 2\sin t + \cos t \\ \sin t \\ 0 \end{bmatrix}.$$

3. (Section 3.10 - Exercise 6) Solve the initial-value problem

$$\dot{x}(t) = \begin{bmatrix} -4 & -4 & 0 \\ 10 & 9 & 1 \\ -4 & -3 & 1 \end{bmatrix} x(t), \qquad x(0) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

The characteristic polynomial of the system matrix

$$A = \begin{bmatrix} -4 & -4 & 0 \\ 10 & 9 & 1 \\ -4 & -3 & 1 \end{bmatrix}$$

is

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} -4-\lambda & -4 & 0 \\ 10 & 9-\lambda & 1 \\ -4 & -3 & 1-\lambda \end{bmatrix} \\ &= -(4+\lambda)(9-\lambda)(1-\lambda) + 16 - 3(4+\lambda) + 40(1-\lambda) = (2-\lambda)^3. \end{aligned}$$

The eigenvalue of the matrix A is $\lambda = 2$.

In order to find $x^1(t)$, we start from solving $(A - \lambda I)v = (A - 2I)v = 0$, i.e.

$$\begin{bmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the first equation $-6v_1 - 4v_2 = 0$ we obtain $v_2 = -3v_1/2$, which together with the last equation $-4v_1 - 3v_2 - v_3 = 0$ implies $v_3 = -4v_1 + 9v_1/2 = v_1/2$. The vector v has the form

$$v = \begin{bmatrix} v_1 \\ -3v_1/2 \\ v_1/2 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ -3/2 \\ 1/2 \end{bmatrix}.$$

For the eigenvector we can choose $2[1, -3/2, 1/2]^\top = [2, -3, 1]^\top$. Then

$$x^1(t) = \mathrm{e}^{2t} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

Now we will find $x^2(t)$. First we need to find a vector v such that

$$(A - 2I)^2 v = 0 \qquad \text{and} \qquad (A - 2I)v \neq 0.$$

This implies that v needs to satisfy

$$\begin{bmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{bmatrix}^2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -4 & -4 & -4 \\ 6 & 6 & 6 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which is equivalent to $v_1 + v_2 + v_3 = 0$. The condition $(A - 2I)v \neq 0$ is

$$\begin{bmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -6v_1 - 4v_2 \\ 10v_1 + 7v_2 + v_3 \\ -4v_1 - 3v_2 - v_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can choose $v = [1, 0, -1]^\top$. Then

$$x^2(t) = \mathrm{e}^{2t}(v + t(A - 2I)v) = \mathrm{e}^{2t} \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -6 \\ 9 \\ -3 \end{bmatrix} \right) = \mathrm{e}^{2t} \begin{bmatrix} -6t + 1 \\ 9t \\ -3t - 1 \end{bmatrix}.$$

It remains to determine $x^3(t)$. First we need to find a vector v such that

$$(A - 2I)^3 v = 0 \qquad \text{and} \qquad (A - 2I)^2 v \neq 0.$$

Calculation shows the matrix $(A - 2I)^3$ is a zero matrix. Then any vector v with the property $v_1 + v_2 + v_3 \neq 0$ can be used to generate $x^3(t)$. Let $v = [1, 0, 0]^\top$. Then $(A - 2I)v = [-6, 10, -4]^\top$, $(A - 2I)^2 v = [-4, 6, -2]^\top$ and

$$\begin{aligned} x^3(t) &= \mathrm{e}^{2t} \left(v + t(A - 2I)v + \frac{t^2}{2}(A - 2I)^2 v \right) \\ &= \mathrm{e}^{2t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ 10 \\ -4 \end{bmatrix} + t^2 \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} \right) = \mathrm{e}^{2t} \begin{bmatrix} -2t^2 - 6t + 1 \\ 3t^2 + 10t \\ -t^2 - 4t \end{bmatrix}. \end{aligned}$$

The general solution is

$$x(t) = c_1 \mathrm{e}^{2t} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{2t} \begin{bmatrix} -6t + 1 \\ 9t \\ -3t - 1 \end{bmatrix} + c_3 \mathrm{e}^{2t} \begin{bmatrix} -2t^2 - 6t + 1 \\ 3t^2 + 10t \\ -t^2 - 4t \end{bmatrix}.$$

From the initial condition $x(0) = [2, 1, -1]^\top$ we get

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then equating second components from both sides we obtain $-3c_1 = 1$, $c_1 = -1/3$. The last equation $-1 = c_1 - c_2$ results in $c_2 = c_1 + 1 = 2/3$. Finally, $c_3 = 2 - 2c_1 - c_2 = 2$. The final solution is

$$\begin{aligned} x(t) &= -\frac{1}{3}\mathrm{e}^{2t} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \frac{2}{3}\mathrm{e}^{2t} \begin{bmatrix} -6t + 1 \\ 9t \\ -3t - 1 \end{bmatrix} + 2\mathrm{e}^{2t} \begin{bmatrix} -2t^2 - 6t + 1 \\ 3t^2 + 10t \\ -t^2 - 4t \end{bmatrix} \\ &= \mathrm{e}^{2t} \begin{bmatrix} -4t^2 - 16t + 2 \\ 6t^2 + 26t + 1 \\ -2t^2 - 10t - 1 \end{bmatrix}. \end{aligned}$$

2.8 HW 7

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2.8.1 Section 4.1, problem 6 (page 377)

find all equilibrium values of the given system of differential equations.

$$\frac{dx}{dt} = \cos y$$
$$\frac{dy}{dt} = \sin x - 1$$

solution

The system can be written as $\dot{x}(t) = f(x(t))$, where $x(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $f = \begin{pmatrix} \cos y \\ \sin x - 1 \end{pmatrix}$. Equilibrium points are solution to $\begin{pmatrix} \cos y \\ \sin x - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This gives two equations to solve

$$\cos y = 0$$
$$\sin x - 1 = 0$$

The first equation has solution $y = \frac{\pi}{2} \pm 2n\pi$ for all integer n values. And the second equation is $\sin x = 1$ which has solution $x = \frac{\pi}{2} + 2n\pi$ for all integer n values. Since we want both components of f to be zero for equilibrium, then the common values that makes both zero at the same values is given by

$$\{x, y\} = \frac{\pi}{2} + 2n\pi$$

For all integer n. Here is partial list of values $\{x, y\} = \left\{\cdots, -\frac{7}{2}\pi, -\frac{3}{2}\pi, \frac{\pi}{2}, \frac{3}{2}\pi, \frac{5}{2}\pi, \cdots\right\}$. At any one of such values $f = \begin{pmatrix} \cos y \\ \sin x - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

2.8.2 Section 4.1, problem 8

find all equilibrium values of the given system of differential equations.

$$\frac{dx}{dt} = x - y^2$$
$$\frac{dy}{dt} = x^2 - y$$
$$\frac{dz}{dt} = e^z - x$$

solution

We need to find values of x, y, z which solves

$$x - y^2 = 0$$
$$x^2 - y = 0$$
$$e^z - x = 0$$

From the first equation $x = y^2$. From the third equation $e^z = x$ or $z = \ln x$. A solution that satisfies all these is

$$
\begin{aligned}
x &= 1 \\
y &= 1 \\
z &= 0
\end{aligned}
$$

At the above values the system is in equilibrium. No other real solutions exist.

2.8.3 Section 4.2, problem 9

Determine the stability or instability of all solutions of the following systems of differential equations

$$
\dot{x} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix} x
$$

solution

The stability is determined from the eigenvalues. Therefore we need to find the eigenvalues of A first.

$$
\begin{aligned}
|A - \lambda I| &= 0 \\
\begin{vmatrix} -\lambda & 2 & 0 & 0 \\ -2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 2 \\ 0 & 0 & -2 & -\lambda \end{vmatrix} &= 0 \\
-\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & -2 & -\lambda \end{vmatrix} - 2 \begin{vmatrix} -2 & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & -2 & -\lambda \end{vmatrix} &= 0 \\
-\lambda \left(-\lambda \left(\lambda^2 + 4 \right) \right) - 2 \left(-2 \left(\lambda^2 + 4 \right) \right) &= 0 \\
\lambda^2 \left(\lambda^2 + 4 \right) + 4 \left(\lambda^2 + 4 \right) &= 0 \\
\left(\lambda^2 + 4 \right) \left(\lambda^2 + 4 \right) &= 0
\end{aligned}
$$

Hence roots are

$$
\begin{aligned}
\lambda_{1,2} &= 2i \qquad \text{multiplicity 2} \\
\lambda_{3,4} &= -2i \qquad \text{multiplicity 2}
\end{aligned}
$$

The real part is zero for all the above 4 eigenvalues. Since the real part is zero, then to check if it is stable, we need to check if the eigenvalue $2i$ and $-2i$ are defective or not. A defective eigenvalue is one which generates n linearly independent vectors where n is less than the multiplicity of the eigenvalue. So basically we need to find the eigenvectors associated with $\lambda = 2i$ and see if we obtain 2 linearly independent eigenvectors or not. If we obtain only one eigenvector, then the system is not stable. Same for $\lambda = -2i$.

Case $\lambda = 2i$

$$
\begin{pmatrix} -\lambda & 2 & 0 & 0 \\ -2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 2 \\ 0 & 0 & -2 & -\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
\begin{pmatrix} -2i & 2 & 0 & 0 \\ -2 & -2i & 0 & 0 \\ 0 & 0 & -2i & 2 \\ 0 & 0 & -2 & -2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

The first two rows give same information, which is $-2v_1 - 2iv_2 = 0$. Or $v_1 = -iv_2$. Row 3 and

4 also give same information, which is $-2v_3 - 2iv_4 = 0$ or $v_3 = -iv_4$. Hence the eigenvector is

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} -iv_2 \\ v_2 \\ -iv_4 \\ v_4 \end{pmatrix} = v_2 \begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix} + v_4 \begin{pmatrix} 0 \\ 0 \\ -i \\ 1 \end{pmatrix}$$

Hence we found two linearly independent eigenvector associated with $\lambda = 2i$ which is the same number as the multiplicity which is 2. Hence this eigenvalue is not defective. Therefore stable eigenvalue. Now we check for the other eigenvalue $\lambda = -2i$ using same method.

Case $\lambda = -2i$

$$\begin{pmatrix} -\lambda & 2 & 0 & 0 \\ -2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 2 \\ 0 & 0 & -2 & -\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2i & 2 & 0 & 0 \\ -2 & 2i & 0 & 0 \\ 0 & 0 & 2i & 2 \\ 0 & 0 & -2 & 2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The first two rows give same information, which is $-2v_1 + 2iv_2 = 0$. Or $v_1 = iv_2$. Row 3 and 4 also give same information, which is $-2v_3 + 2iv_4 = 0$ or $v_3 = iv_4$. Hence the eigenvector is

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} iv_2 \\ v_2 \\ iv_4 \\ v_4 \end{pmatrix} = v_2 \begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix} + v_4 \begin{pmatrix} 0 \\ 0 \\ i \\ 1 \end{pmatrix}$$

Hence we found two linearly independent eigenvector associated with $\lambda = -2i$ which is the same number as the multiplicity which is 2. Hence this eigenvalue is not defective. Therefore stable eigenvalue.

In summary, the eigenvalues are $\{2i, 2i, -2i, -2i\}$ and the associated eigenvectors are

$$\overbrace{\begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -i \\ 1 \end{pmatrix}}^{\lambda=2i}, \overbrace{\begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ i \\ 1 \end{pmatrix}}^{\lambda=-2i}$$

Therefore the system is stable.

2.8.4 Section 4.2, problem 10

Determine the stability or instability of all solutions of the following systems of differential equations

$$\dot{x} = \begin{pmatrix} 0 & 2 & 1 & 0 \\ -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix} x$$

solution

The stability is determined from the eigenvalues. Therefore we need to find the eigenvalues of A first.

$$|A-\lambda I|=0$$

$$\begin{vmatrix}-\lambda&2&1&0\\-2&-\lambda&0&1\\0&0&-\lambda&2\\0&0&-2&-\lambda\end{vmatrix}=0$$

$$-\lambda\begin{vmatrix}-\lambda&0&1\\0&-\lambda&2\\0&-2&-\lambda\end{vmatrix}-2\begin{vmatrix}-2&0&1\\0&-\lambda&2\\0&-2&-\lambda\end{vmatrix}+\begin{vmatrix}-2&-\lambda&1\\0&0&2\\0&0&-\lambda\end{vmatrix}=0$$

$$-\lambda\left(-\lambda\begin{vmatrix}-\lambda&2\\-2&-\lambda\end{vmatrix}+\begin{vmatrix}0&-\lambda\\0&-2\end{vmatrix}\right)-2\left(-2\begin{vmatrix}-\lambda&2\\-2&-\lambda\end{vmatrix}+\begin{vmatrix}0&-\lambda\\0&-2\end{vmatrix}\right)-2(0)=0$$

$$-\lambda\left(-\lambda\left(\lambda^2+4\right)+0\right)-2\left(-2\left(\lambda^2+4\right)+0\right)=0$$

$$\lambda^2\left(\lambda^2+4\right)+4\left(\lambda^2+4\right)=0$$

$$\left(\lambda^2+4\right)\left(\lambda^2+4\right)=0$$

Hence the eigenvalues are the same as last problem.

$$\lambda_{1,2}=2i\qquad\text{multiplicity 2}$$

$$\lambda_{3,4}=-2i\qquad\text{multiplicity 2}$$

The real part is zero for all the above 4 eigenvalues. Since the real part is zero, then to check if it is stable, we need to check if the eigenvalue $2i$ and $-2i$ are defective or not.

A defective eigenvalue is one which generates n linearly independent vectors where n is less than the multiplicity of the eigenvalue. So basically we need to find the eigenvectors associated with $\lambda=2i$ and see if we obtain 2 linearly independent eigenvectors or not. If we obtain only one eigenvector, then the system is not stable. Same for $\lambda=-2i$.

Case $\lambda=2i$

$$\begin{pmatrix}-\lambda&2&1&0\\-2&-\lambda&0&1\\0&0&-\lambda&2\\0&0&-2&-\lambda\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}-2i&2&1&0\\-2&-2i&0&1\\0&0&-2i&2\\0&0&-2&-2i\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Hence

$$\begin{aligned}-2iv_1+2v_1+v_3&=0\\-2v_1-2iv_2+v_4&=0\\-2iv_3+2v_4&=0\\-2v_3-2iv_4&=0\end{aligned}$$

Third and fourth equations gives same information which is $-2v_3=2iv_4$ or $v_3=-iv_4$. Substituting this into first two equations gives

$$\begin{aligned}-2iv_1+2v_1-iv_4&=0\\-2v_1-2iv_2+v_4&=0\end{aligned}$$

Multiplying second equation by $-i$ and adding the two equations gives $-2iv_4=0$. Hence $v_4=0$, which implies $v_3=0$. Therefore the above reduces to

$$\begin{aligned}-2iv_1+2v_1&=0\\-2v_1-2iv_2&=0\end{aligned}$$

These two equations give the same information which is $-2v_1=2iv_2$ or $v_1=-iv_2$. Therefore the eigenvector is

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} -iv_2 \\ v_2 \\ 0 \\ 0 \end{pmatrix} = v_2 \begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

So only <u>one eigenvector</u> was found. But the multiplicity of the eigenvalue is two. Hence this eigenvalue is defective. Therefore the system is <u>unstable</u>. No need to check the second eigenvalue because if one eigenvalue with zero real part is defective then that is enough to make the system unstable.

2.8.5 Section 4.3, problem 8

Verify that the origin is an equilibrium point of each of the following systems of equations and determine, if possible, whether it is stable or unstable.

$$\begin{aligned} \dot{x} &= y + \cos y - 1 \\ \dot{y} &= -\sin x + x^3 \end{aligned}$$

<u>solution</u>

At $x = 0, y = 0$ the above becomes

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= 0 \end{aligned}$$

Hence origin $(0,0)$ is equilibrium point. To check if it is stable equilibrium or not, we find the Jacobian matrix, evaluate it at the origin and check the eigenvalues that results. The Jacobian is

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 - \sin y \\ -\cos x + 2x^2 & 0 \end{pmatrix}$$

At $x = 0, y = 0$ the above becomes

$$J_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Hence

$$p(\lambda) = |J - \lambda I| = \det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0$$

Therefore $\lambda = \pm i$. The <u>real part is zero</u>. Since this is a nonlinear system, then we are <u>not able to determine</u> the stability of equilibrium at the origin.

2.8.6 Section 4.3, problem 10

Verify that the origin is an equilibrium point of each of the following systems of equations and determine, if possible, whether it is stable or unstable.

$$\begin{aligned} \dot{x} &= \ln\left(1 + x + y^2\right) \\ \dot{y} &= -y + x^3 \end{aligned}$$

<u>Solution</u>

At $x = 0, y = 0$ the above becomes

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= 0 \end{aligned}$$

Hence origin $(0,0)$ is equilibrium point. To check if it is stable equilibrium or not, we find the Jacobian matrix, evaluate it at the origin and check the eigenvalues that results. The Jacobian is

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+x+y^2} & \frac{2y}{1+x+y^2} \\ 3x^2 & -1 \end{pmatrix}$$

At $x = 0, y = 0$ the above becomes

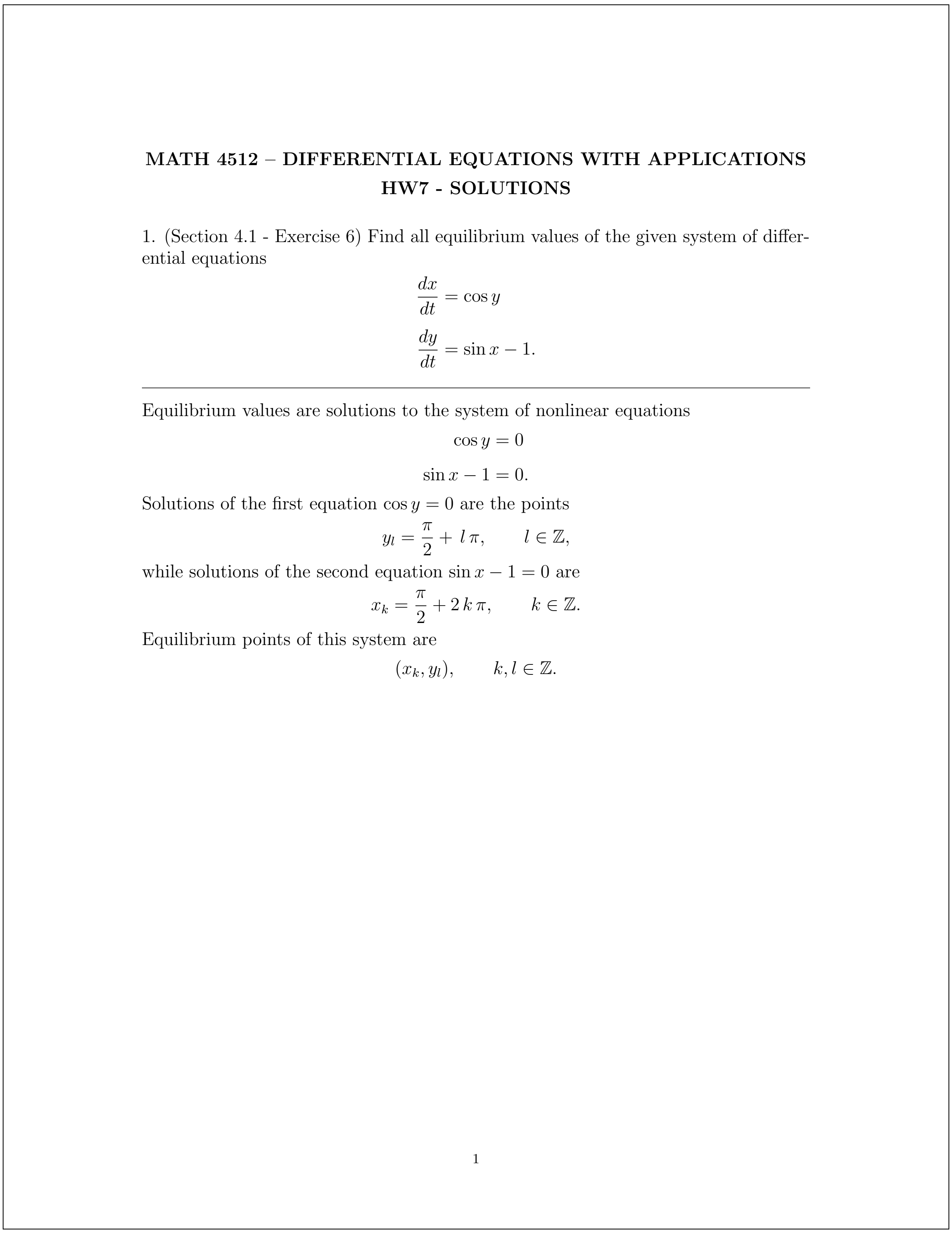
$$J_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence

$$\begin{aligned} p(\lambda) &= |J - \lambda I| \\ &= \det \begin{pmatrix} 1-\lambda & 1 \\ 0 & -1-\lambda \end{pmatrix} \\ &= (1-\lambda)(-1-\lambda) \\ &= -1 - \lambda + \lambda + \lambda^2 \\ &= \lambda^2 - 1 \end{aligned}$$

Hence eigenvalues are $\lambda = \pm 1$. Since one of the eigenvalues is positive, the origin is <u>not a stable</u> equilibrium point.

2.8.7 Key solution for HW 7



MATH 4512 – DIFFERENTIAL EQUATIONS WITH APPLICATIONS
HW7 - SOLUTIONS

1. (Section 4.1 - Exercise 6) Find all equilibrium values of the given system of differential equations

$$\frac{dx}{dt} = \cos y$$
$$\frac{dy}{dt} = \sin x - 1.$$

Equilibrium values are solutions to the system of nonlinear equations

$$\cos y = 0$$
$$\sin x - 1 = 0.$$

Solutions of the first equation $\cos y = 0$ are the points

$$y_l = \frac{\pi}{2} + l\,\pi, \qquad l \in \mathbb{Z},$$

while solutions of the second equation $\sin x - 1 = 0$ are

$$x_k = \frac{\pi}{2} + 2\,k\,\pi, \qquad k \in \mathbb{Z}.$$

Equilibrium points of this system are

$$(x_k, y_l), \qquad k, l \in \mathbb{Z}.$$

1

2. (Section 4.1 - Exercise 8) Find all equilibrium values of the given system of differential equations

$$\frac{dx}{dt} = x - y^2$$
$$\frac{dy}{dt} = x^2 - y$$
$$\frac{dz}{dt} = e^z - x.$$

Equilibrium values are solutions to the system of nonlinear equations

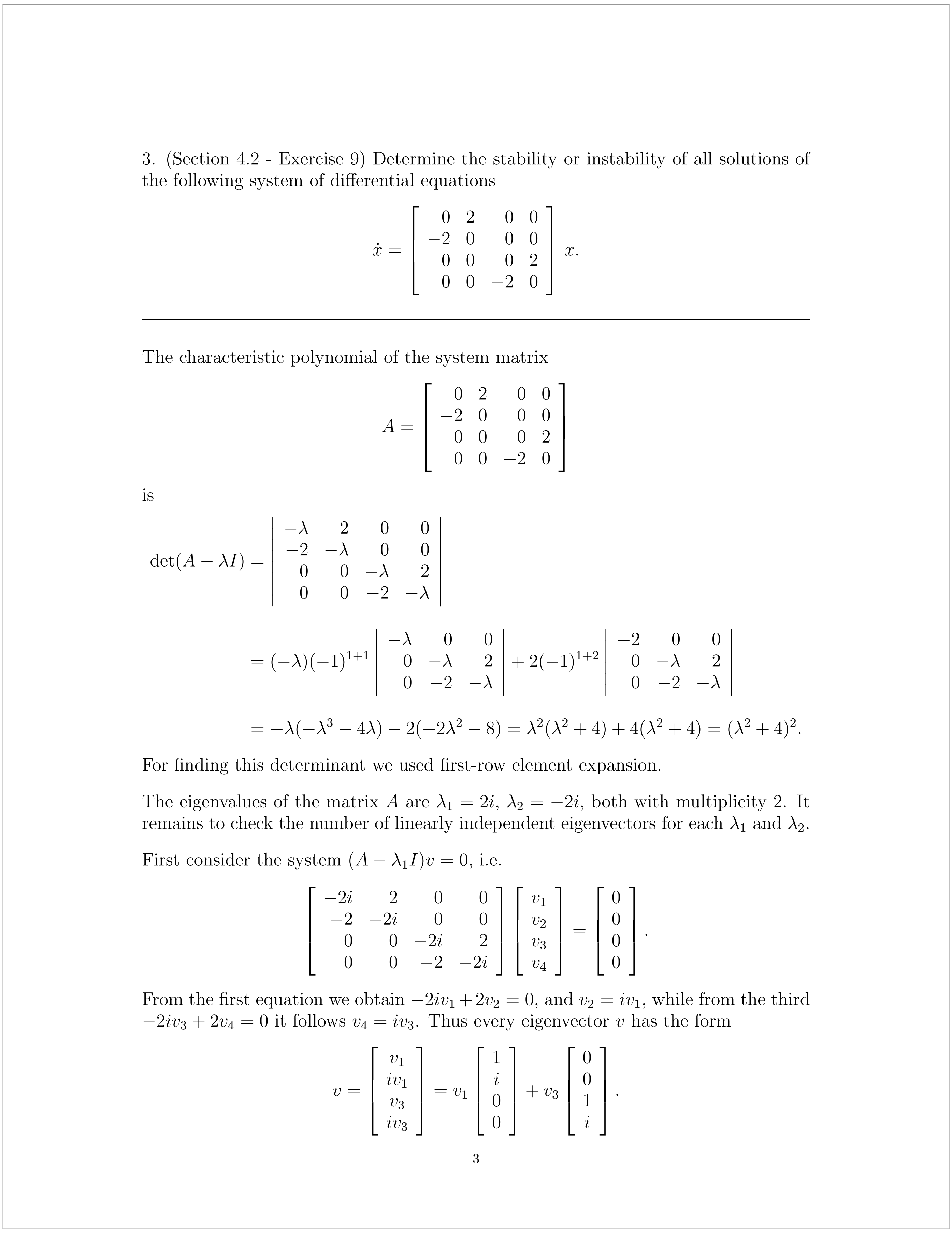
$$x - y^2 = 0$$
$$x^2 - y = 0$$
$$e^z - x = 0.$$

From $x = y^2$ we obtain $y^4 - y = 0$. Real solutions of this equation are

$$y_1 = 0, \qquad y_2 = 1.$$

Then $x_1 = y_1^2 = 0$ and $x_2 = y_2^2 = 1$. Notice that there is no value z_1 such that $e^{z_1} = x_1 = 0$, while $z_2 = 0$, where $e^{z_2} = x_2 = 1$. Therefore, the only equilibrium point is

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$



3. (Section 4.2 - Exercise 9) Determine the stability or instability of all solutions of the following system of differential equations

$$\dot{x} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix} x.$$

The characteristic polynomial of the system matrix

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 2 & 0 & 0 \\ -2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 2 \\ 0 & 0 & -2 & -\lambda \end{vmatrix}$$

$$= (-\lambda)(-1)^{1+1} \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & -2 & -\lambda \end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix} -2 & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & -2 & -\lambda \end{vmatrix}$$

$$= -\lambda(-\lambda^3 - 4\lambda) - 2(-2\lambda^2 - 8) = \lambda^2(\lambda^2 + 4) + 4(\lambda^2 + 4) = (\lambda^2 + 4)^2.$$

For finding this determinant we used first-row element expansion.

The eigenvalues of the matrix A are $\lambda_1 = 2i$, $\lambda_2 = -2i$, both with multiplicity 2. It remains to check the number of linearly independent eigenvectors for each λ_1 and λ_2.

First consider the system $(A - \lambda_1 I)v = 0$, i.e.

$$\begin{bmatrix} -2i & 2 & 0 & 0 \\ -2 & -2i & 0 & 0 \\ 0 & 0 & -2i & 2 \\ 0 & 0 & -2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the first equation we obtain $-2iv_1 + 2v_2 = 0$, and $v_2 = iv_1$, while from the third $-2iv_3 + 2v_4 = 0$ it follows $v_4 = iv_3$. Thus every eigenvector v has the form

$$v = \begin{bmatrix} v_1 \\ iv_1 \\ v_3 \\ iv_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}.$$

Notice that

$$\begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}$$

are linearly independent eigenvectors for $\lambda_1 = 2i$ that generate all other eigenvectors. Since the multiplicity of λ_1 is the same as the number of linearly independent eigenvectors, we proceed with analysis of the second eigenvalue.
Consider the system $(A - \lambda_2 I)v = 0$, i.e.

$$\begin{bmatrix} 2i & 2 & 0 & 0 \\ -2 & 2i & 0 & 0 \\ 0 & 0 & 2i & 2 \\ 0 & 0 & -2 & 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the second equation $-2v_1 + 2iv_2 = 0$ we obtain $v_1 = iv_2$, while from the last equation $-2v_3 + 2iv_4 = 0$ it follows $v_3 = iv_4$. Thus every eigenvector v has the form

$$v = \begin{bmatrix} iv_2 \\ v_2 \\ iv_4 \\ v_4 \end{bmatrix} = v_2 \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix} + v_4 \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}.$$

Similarly to previous case, vectors

$$\begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}$$

are linearly independent eigenvectors for $\lambda_2 = -2i$ that generate all other eigenvectors.

Since the multiplicity of each λ_1 and λ_2 is the same as the number of corresponding linearly independent eigenvectors, we conclude that every solution of the starting system of DEs is stable.

4. (Section 4.2 - Exercise 10) Determine the stability or instability of all solutions of the following system of differential equations

$$\dot{x} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix} x.$$

The characteristic polynomial of the system matrix

$$A = \begin{bmatrix} 0 & 2 & 1 & 0 \\ -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 2 & 1 & 0 \\ -2 & -\lambda & 0 & 1 \\ 0 & 0 & -\lambda & 2 \\ 0 & 0 & -2 & -\lambda \end{vmatrix}$$

$$= (-\lambda)(-1)^{1+1} \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 2 \\ 0 & -2 & -\lambda \end{vmatrix} + (-2)(-1)^{2+1} \begin{vmatrix} 2 & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & -2 & -\lambda \end{vmatrix}$$

$$= -\lambda(-\lambda^3 - 4\lambda) + 2(2\lambda^2 + 8) = \lambda^2(\lambda^2 + 4) + 4(\lambda^2 + 4) = (\lambda^2 + 4)^2.$$

For finding this determinant we used first-column element expansion.

Eigenvectors for $\lambda_1 = 2i$ solve $(A - \lambda_1 I)v = 0$, i.e.

$$\begin{bmatrix} -2i & 2 & 1 & 0 \\ -2 & -2i & 0 & 1 \\ 0 & 0 & -2i & 2 \\ 0 & 0 & -2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The third equation $-2iv_3 + 2v_4 = 0$ implies $v_4 = iv_3$. The second equation can be written as

$$0 = -2v_1 - 2iv_2 + v_4 = -2v_1 - 2iv_2 + iv_3 = -i(-2iv_1 + 2v_2 - v_3).$$

Combining the last relation with the first equation $-2iv_1 + 2v_2 + v_3 = 0$, we obtain $v_3 = 0$. Consequently $v_4 = 0$ and $v_2 = iv_1$. Every eigenvector v corresponding to $\lambda_1 = 2i$ can be represented as

$$v = \begin{bmatrix} v_1 \\ iv_1 \\ 0 \\ 0 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix}.$$

Since the number of linearly independent eigenvectors is smaller than the multiplicity 2 of λ_1, we conclude that every solution of the starting system of DEs is unstable.

5. (Section 4.3 - Exercise 8) Verify that the origin is an equilibrium point of the following system of equations

$$\dot{x} = y + \cos y - 1$$
$$\dot{y} = -\sin x + x^3$$

and determine (if possible) whether it is stable or unstable.

Vector $[0,0]^\top$ is obviously an equilibrium point of this system.

First approach.
From the expansions

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots$$

we can write down $\dot{x} = y + \cos y - 1$, and $\dot{y} = x^3 - \sin x$, as

$$\dot{x} = y - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots = y + g_1(y)$$
$$\dot{y} = x^3 - x + \frac{x^3}{3!} - \frac{x^5}{5!} - \cdots = -x + g_2(x).$$

Then

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} g_1(y) \\ g_2(x) \end{bmatrix}.$$

The characteristic polynomial of the previous matrix is

$$p(\lambda) = \det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1.$$

Since its roots $\lambda_1 = i$, $\lambda_2 = -i$, both have zero real part, we cannot determine whether the vector $[0,0]^\top$ is stable or not.
(At this point of the course, we can only apply the theory from Sections 4.1-4.3).

Second approach.
Let $f_1(x,y) = y + \cos y - 1$ and $f_2(x,y) = -\sin x + x^3$. The Jacobian matrix for nonlinear vector-valued function

$$f(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$$

evaluated at the equilibrium point $[0,0]^\top$ is

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x}(0,0) & \frac{\partial f_1}{\partial y}(0,0) \\ \frac{\partial f_2}{\partial x}(0,0) & \frac{\partial f_2}{\partial y}(0,0) \end{bmatrix} = \begin{bmatrix} 0 & 1 - \sin y \\ -\cos x + 3x^2 & 0 \end{bmatrix}_{x=0,y=0} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We obtained the same matrix and we can proceed as in the first approach.

6. (Section 4.3 - Exercise 10) Verify that the origin is an equilibrium point of the following system of equations

$$\dot{x} = \ln(1 + x + y^2)$$
$$\dot{y} = -y + x^3$$

and determine (if possible) whether it is stable or unstable.

Again the vector $[0, 0]^\top$ is obviously an equilibrium point of this system.

First approach.
Here we will use expansion

$$\ln(1 + x + y^2) = x + y^2 - \frac{(x + y^2)^2}{2} + \frac{(x + y^2)^3}{3} - \cdots .$$

Then

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} g(x, y) \\ x^3 \end{bmatrix},$$

where

$$g(x, y) = y^2 - \frac{(x + y^2)^2}{2} + \frac{(x + y^2)^3}{3} - \cdots .$$

The characteristic polynomial of the previous matrix is

$$p(\lambda) = \det \begin{bmatrix} 1 - \lambda & 0 \\ 0 & -1 - \lambda \end{bmatrix} = -(1 - \lambda)(1 + \lambda).$$

Since one eigenvalue of A has positive real part, the equilibrium value $[0, 0]^\top$ for this system is unstable.

Second approach.
Let $f_1(x, y) = \ln(1+x+y^2)$ and $f_2(x, y) = -y+x^3$. The Jacobian matrix for nonlinear vector-valued function

$$f(x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix}$$

evaluated at the equilibrium point $[0, 0]^\top$ is

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x}(0, 0) & \frac{\partial f_1}{\partial y}(0, 0) \\ \frac{\partial f_2}{\partial x}(0, 0) & \frac{\partial f_2}{\partial y}(0, 0) \end{bmatrix} = \begin{bmatrix} \frac{1}{1 + x + y^2} & \frac{2y}{1 + x + y^2} \\ 3x^2 & -1 \end{bmatrix}_{x=0,y=0}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We obtained the same matrix and we can proceed as in the first approach.

2.9 HW 8

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2.9.1 Section 4.4, problem 1

In each of Problems l-3, verify that $x(t), y(t)$ is a solution of the given system of equations, and find its orbit.

$$\begin{aligned}\dot{x} &= 1\\ \dot{y} &= 2(1-x)\sin\left((1-x)^2\right)\\ x(t) &= 1+t\\ y(t) &= \cos\left(t^2\right)\end{aligned}$$

solution

Since $x(t)=1+t$ then $\dot{x}=1$. Verified OK. And since $y(t)=\cos\left(t^2\right)$ then $\dot{y}=-2t\sin\left(t^2\right)$. But $t=x-1$, hence $\dot{y}=-2(x-1)\sin\left((1-x)^2\right)$ or

$$\dot{y}=2(1-x)\sin\left((1-x)^2\right)$$

Verified OK. Both solutions verified. Now we need to find system orbit. The Orbit is given by the equation

$$\frac{dy}{dx}=\frac{g\left(x,y\right)}{f\left(x,y\right)}$$

When we write the given system in the following form

$$\begin{aligned}\dot{x} &= f\left(x,y\right)\\ \dot{y} &= g\left(x,y\right)\end{aligned}$$

We see now that $f\left(x,y\right)=1$ and $g\left(x,y\right)=2(1-x)\sin\left((1-x)^2\right)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{2(1-x)\sin\left((1-x)^2\right)}{1}\\ &= 2(1-x)\sin\left((1-x)^2\right)\end{aligned}$$

This is first order ODE. Since separable, we can solve it by integration

$$y(x)=\int 2(1-x)\sin\left((1-x)^2\right)dx$$

Let $u=(1-x)^2$, then $\frac{du}{dx}=2(1-x)(-1)=-2\sqrt{u}$. Substituting in the above gives

$$\begin{aligned}y(x) &= \int 2\sqrt{u}\sin(u)\frac{du}{-2\sqrt{u}}\\ &= -\int\sin(u)\,du\\ &= -(-\cos(u))+C\\ &= \cos(u)+C\\ &= \cos\left((1-x)^2\right)+C\end{aligned}$$

Therefore the equation of the orbit is

$$y(x)=\cos\left((1-x)^2\right)+C$$

For different values of C, different orbit results.

2.9.2 Section 4.4, problem 2

In each of Problems l-3, verify that $x(t), y(t)$ is a solution of the given system of equations, and find its orbit.

$$\begin{aligned}\dot{x} &= e^{-x}\\ \dot{y} &= e^{e^{x}-1}\\ x(t) &= \ln(1+t)\\ y(t) &= e^{t}\end{aligned}$$

Solution

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}\ln(1+t)\\ \dot{x} &= \frac{1}{1+t}\end{aligned}$$

But $e^{-x} = e^{-\ln(1+t)} = \frac{1}{1+t}$. Verified OK. And

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}e^{t}\\ \dot{y} &= e^{t}\end{aligned}$$

But $x-1 = \ln(1+t)-1$. Hence $\ln(1+t) = x$. Therefore $1+t = e^{x}$ or $t = e^{x}-1$. Therefore $\dot{y} = e^{t} = e^{e^{x}-1}$. Verified OK.

Now we need to find system orbit. The Orbit is given by the equation

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

When we write the given system in the following form

$$\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}$$

We see now that $f(x,y) = e^{-x}$ and $g(x,y) = e^{e^{x}-1}$. Therefore

$$\frac{dy}{dx} = \frac{e^{e^{x}-1}}{e^{-x}}$$

Integrating

$$\int dy = \int \frac{e^{e^{x}-1}}{e^{-x}}dx$$

Let $e^{x} = u, du = e^{x}dx$. Hence the RHS $\int \frac{e^{e^{x}-1}}{e^{-x}}dx = \int \frac{e^{u-1}}{\frac{1}{u}}\frac{du}{u} = \int e^{u-1}du = e^{u-1} = e^{e^{x}-1}$. The above becomes

$$y = e^{e^{x}-1} + C$$

The orbits are given by the above equation for different C

2.9.3 Section 4.4, problem 3

In each of Problems l-3, verify that $x(t), y(t)$ is a solution of the given system of equations, and find its orbit.

$$\begin{aligned}\dot{x} &= 1+x^{2}\\ \dot{y} &= \left(1+x^{2}\right)\sec^{2}x\\ x(t) &= \tan t\\ y(t) &= \tan(\tan t)\end{aligned}$$

solution

Orbits given by

$$\begin{aligned}\frac{dy}{dx} &= \frac{\left(1+x^2\right)\sec^2 x}{1+x^2}\\ &= \sec^2 x\end{aligned}$$

Hence

$$\int dy = \int \sec^2 x dx$$

But $\sec^2 x = \frac{1}{\cos^2 x}$. And $\frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$. Hence $\int \sec^2 x dx = \tan x$. Therefore the above gives

$$y = \tan x + C$$

The orbits are given by the above equation for different C. (do not know why book gives only $y = \tan x$)

2.9.4 Section 4.4, problem 8

Find the orbits of each of the following systems

$$\begin{aligned}\dot{x} &= y + x^2 y\\ \dot{y} &= 3x + xy^2\end{aligned}$$

<u>Solution</u>

The Orbit is given by the equation

$$\frac{dy}{dx} = \frac{g\left(x,y\right)}{f\left(x,y\right)}$$

When we write the given system in the following form

$$\begin{aligned}\dot{x} &= f\left(x,y\right)\\ \dot{y} &= g\left(x,y\right)\end{aligned}$$

We see now that $f\left(x,y\right) = y + x^2 y$ and $g\left(x,y\right) = 3x + xy^2$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{3x+xy^2}{y+x^2y}\\ &= \frac{x\left(3+y^2\right)}{y\left(1+x^2\right)}\\ &= \frac{x}{\left(1+x^2\right)}\frac{\left(3+y^2\right)}{y}\end{aligned}$$

Hence it is separable.

$$\begin{aligned}\int \frac{y}{3+y^2}dy &= \int \frac{x}{1+x^2}dx\\ \frac{1}{2}\ln\left(3+y^2\right) &= \frac{1}{2}\ln\left(1+x^2\right) + C_2\\ \ln\left(3+y^2\right) &= \ln\left(1+x^2\right) + C_1\end{aligned}$$

Therefore

$$\begin{aligned}3+y^2 &= e^{\ln\left(1+x^2\right)+C_1}\\ &= e^{C_1}e^{\ln\left(1+x^2\right)}\\ &= C\left(1+x^2\right)\end{aligned}$$

Hence

$$\begin{aligned}y^2 &= C\left(1+x^2\right) - 3\\ y\left(x\right) &= \pm\sqrt{C\left(1+x^2\right)-3}\end{aligned}$$

The above gives the equations for the orbit. For each C value, there is a different orbit curve. Now we need to find equilibrium points, since these are orbits also. We need to solve

$$\begin{aligned} 0 &= y + x^2 y \\ 0 &= 3x + xy^2 \end{aligned}$$

Or

$$\begin{aligned} 0 &= y\left(1 + x^2\right) \\ 0 &= x\left(3 + y^2\right) \end{aligned}$$

First equation gives $y = 0$ as only real solution. When $y = 0$ then second equation gives $x = 0$. Hence $(0,0)$ is also an orbit. So the orbits are

$$\begin{aligned} y^2 &= C\left(1 + x^2\right) - 3 \qquad C \neq 3 \\ \left(x,y\right) &= (0,0) \end{aligned}$$

And when $C = 3$ we obtain orbits $y^2 = 3\left(1 + x^2\right) - 3 = 3x^2$, with additional orbits (notice that we have to exclude $x = 0$ from each one below, since $x = 0$ is allready included in $\left(x,y\right) = (0,0)$)

$$\begin{aligned} y &= \sqrt{3}x \qquad x > 0 \\ y &= \sqrt{3}x \qquad x < 0 \\ y &= -\sqrt{3}x \qquad x > 0 \\ y &= -\sqrt{3}x \qquad x < 0 \end{aligned}$$

Hence there are 6 possible orbits in total.

2.9.5 Section 4.7, problem 3

Draw the phase portraits of each of the following systems of differential equations

$$\dot{x} = \begin{pmatrix} 4 & -1 \\ -2 & 5 \end{pmatrix} x$$

<u>solution</u>

$$\begin{aligned} \det\left(A - \lambda I\right) &= 0 \\ \begin{vmatrix} 4-\lambda & -1 \\ -2 & 5-\lambda \end{vmatrix} &= 0 \\ \left(4-\lambda\right)\left(5-\lambda\right) - 2 &= 0 \\ \lambda^2 - 9\lambda + 18 &= 0 \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= 6 \\ \lambda_2 &= 3 \end{aligned}$$

<u>Case $\lambda_1 = 6$</u>

$$\begin{aligned} \begin{pmatrix} 4-\lambda & -1 \\ -2 & 5-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 4-6 & -1 \\ -2 & 5-6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -2 & -1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

From first row $-2v_1 - v_2 = 0$. Hence $v_2 = -2v_1$. Therefore the first eigenvector is $v^1 = \begin{pmatrix} v_1 \\ -2v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ by setting $v_1 = 1$

<u>Case $\lambda_1 = 3$</u>

$$\begin{pmatrix} 4-\lambda & -1 \\ -2 & 5-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 4-3 & -1 \\ -2 & 5-3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first row $v_1 - v_2 = 0$. Hence $v_2 = v_1$. Therefore the second eigenvector is $v^2 = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by setting $v_1 = 1$

Since eigenvalues are both real and both are positive, then $(0,0)$ is unstable node. Here is a the Phase portrait. The lines marked red and blue are the two eigenvectors found above. The arrows are all leaving $(0,0)$ which means this is unstable equilibrium point.

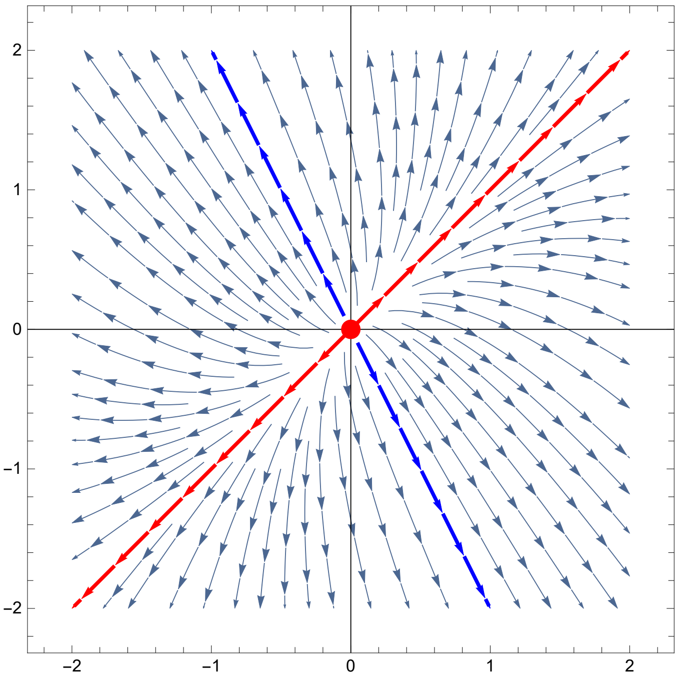
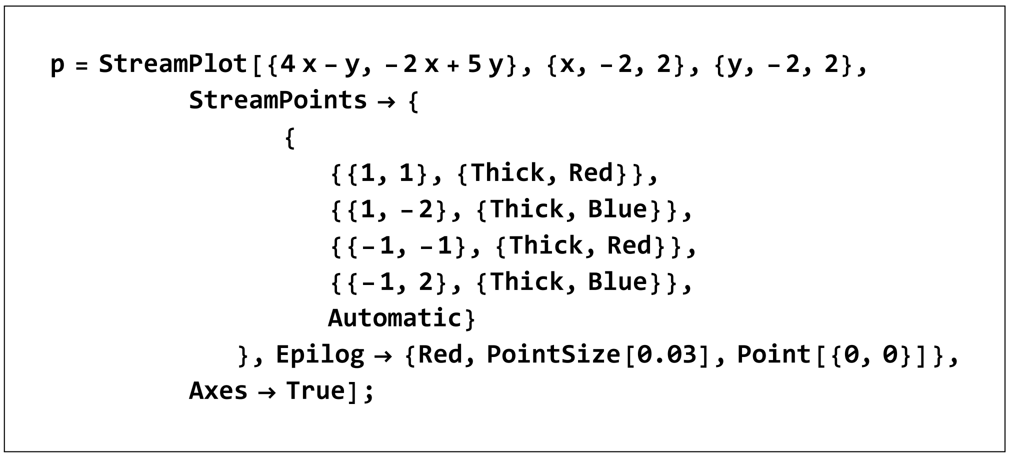


Figure 2.17: Phase portrait



```
p = StreamPlot[{4 x - y, -2 x + 5 y}, {x, -2, 2}, {y, -2, 2},
        StreamPoints → {
              {
                 {{1, 1}, {Thick, Red}},
                 {{1, -2}, {Thick, Blue}},
                 {{-1, -1}, {Thick, Red}},
                 {{-1, 2}, {Thick, Blue}},
                 Automatic}
           }, Epilog → {Red, PointSize[0.03], Point[{0, 0}]},
        Axes → True];
```

Figure 2.18: Code used

2.9.6 Section 4.7, problem 6

Draw the phase portraits of each of the following systems of differential equations

$$\dot{x}=\begin{pmatrix}3 & -1\\ 5 & -3\end{pmatrix}x$$

Solution

$$\begin{aligned}\det\left(A-\lambda I\right)&=0\\ \begin{vmatrix}3-\lambda & -1\\ 5 & -3-\lambda\end{vmatrix}&=0\\ \left(3-\lambda\right)\left(-3-\lambda\right)+5&=0\\ \lambda^{2}-4&=0\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=2\\ \lambda_2&=-2\end{aligned}$$

We see that one eigenvalue is stable and one is not stable.

Case $\lambda_1=2$

$$\begin{aligned}\begin{pmatrix}3-\lambda & -1\\ 5 & -3-\lambda\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}&=\begin{pmatrix}0\\ 0\end{pmatrix}\\ \begin{pmatrix}3-2 & -1\\ 5 & -3-2\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}&=\begin{pmatrix}0\\ 0\end{pmatrix}\\ \begin{pmatrix}1 & -1\\ 5 & -5\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}&=\begin{pmatrix}0\\ 0\end{pmatrix}\end{aligned}$$

From first row $v_1-v_2=0$. Hence $v_2=v_1$. Therefore the first eigenvector is $\boldsymbol{v}^1=\begin{pmatrix}v_1\\ v_1\end{pmatrix}=v_1\begin{pmatrix}1\\ 1\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}$ by setting $v_1=1$

Case $\lambda_1=-2$

$$\begin{aligned}\begin{pmatrix}3-\lambda & -1\\ 5 & -3-\lambda\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}&=\begin{pmatrix}0\\ 0\end{pmatrix}\\ \begin{pmatrix}3+2 & -1\\ 5 & -3+2\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}&=\begin{pmatrix}0\\ 0\end{pmatrix}\\ \begin{pmatrix}5 & -1\\ 5 & -1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}&=\begin{pmatrix}0\\ 0\end{pmatrix}\end{aligned}$$

From first row $5v_1-v_2=0$. Hence $v_2=5v_1$. Therefore the first eigenvector is $\boldsymbol{v}^1=\begin{pmatrix}v_1\\ 5v_1\end{pmatrix}=v_1\begin{pmatrix}1\\ 5\end{pmatrix}=\begin{pmatrix}1\\ 5\end{pmatrix}$ by setting $v_1=1$.

Since one eigenvalue is stable and one is not, then $(0,0)$ is unstable saddle point. Here is a the Phase portrait. The lines marked red and blue are the two eigenvectors found above.

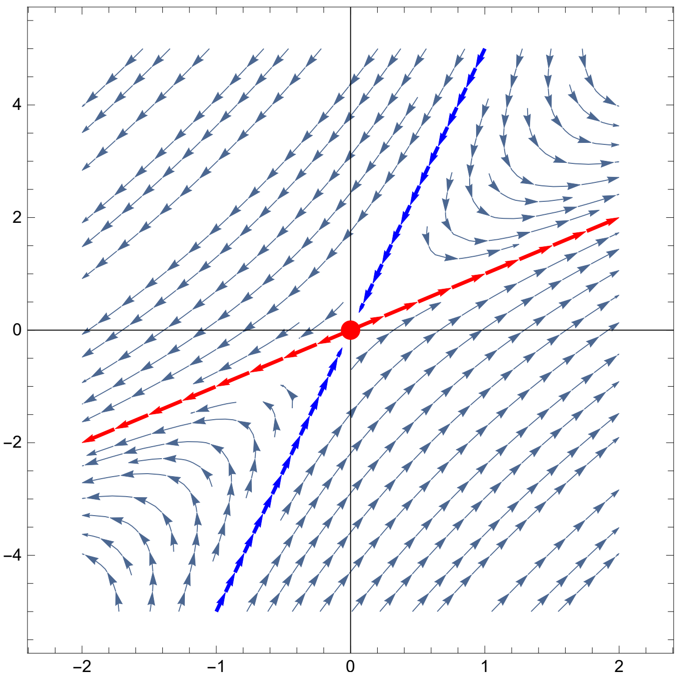
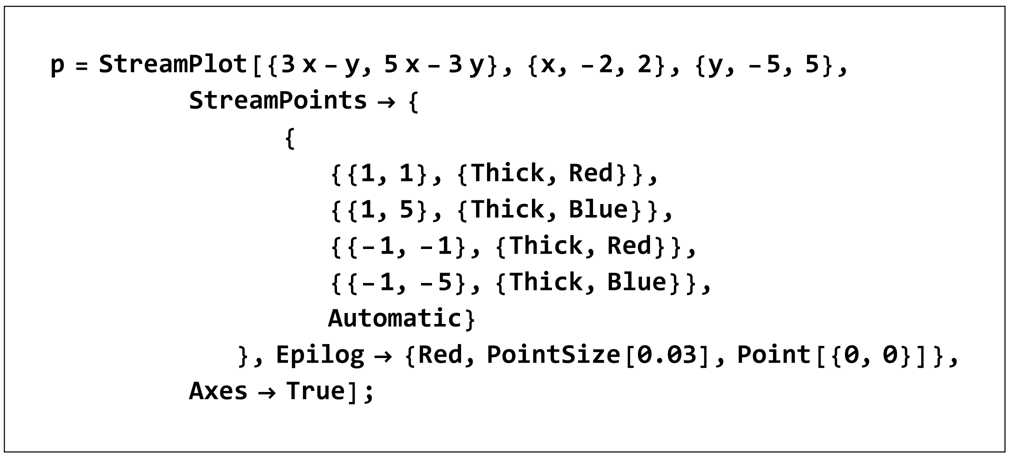


Figure 2.19: Phase portrait



```
p = StreamPlot[{3 x - y, 5 x - 3 y}, {x, -2, 2}, {y, -5, 5},
        StreamPoints → {
              {
                 {{1, 1}, {Thick, Red}},
                 {{1, 5}, {Thick, Blue}},
                 {{-1, -1}, {Thick, Red}},
                 {{-1, -5}, {Thick, Blue}},
                 Automatic}
           }, Epilog → {Red, PointSize[0.03], Point[{0, 0}]},
        Axes → True];
```

Figure 2.20: Code used

2.9.7 Section 4.7, problem 9

Draw the phase portraits of each of the following systems of differential equations

$$\dot{x} = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix} x$$

solution

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & -1 \\ -5 & -2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(-2-\lambda) + 5 = 0$$

$$\lambda^2 + 1 = 0$$

Hence

$$\lambda_1 = i$$

$$\lambda_2 = -i$$

The real part is zero. Hence $(0,0)$ equilibrium point is called CENTER. it is stable, but not asymptotically stable.

Case $\lambda_1 = i$

$$\begin{pmatrix} 2-\lambda & 1 \\ -5 & -2-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-i & 1 \\ -5 & -2-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From second row $-5v_1 - (2+i)\,v_2 = 0$. Hence $v_2 = -\frac{5}{(2+i)}v_1$. Therefore the first eigenvector is $v^1 = \begin{pmatrix} v_1 \\ -\frac{5}{(2+i)}v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ -\frac{5}{(2+i)} \end{pmatrix} = \begin{pmatrix} -(2+i) \\ 5 \end{pmatrix}$ by setting $v_1 = 1$

Case $\lambda_1 = -i$

$$\begin{pmatrix} 2-\lambda & 1 \\ -5 & -2-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2+i & 1 \\ -5 & -2+i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From second row $-5v_1 + (-2+i)\,v_2 = 0$. Hence $v_2 = -\frac{5}{(-2+i)}v_1$. Therefore the first eigenvector is $v^1 = \begin{pmatrix} v_1 \\ -\frac{5}{(-2+i)}v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ -\frac{5}{(-2+i)} \end{pmatrix} = \begin{pmatrix} -2+i \\ 5 \end{pmatrix}$ by setting $v_1 = 1$

$(0,0)$ equilibrium point is called CENTER with curves making closed circles around $(0,0)$ as shown below

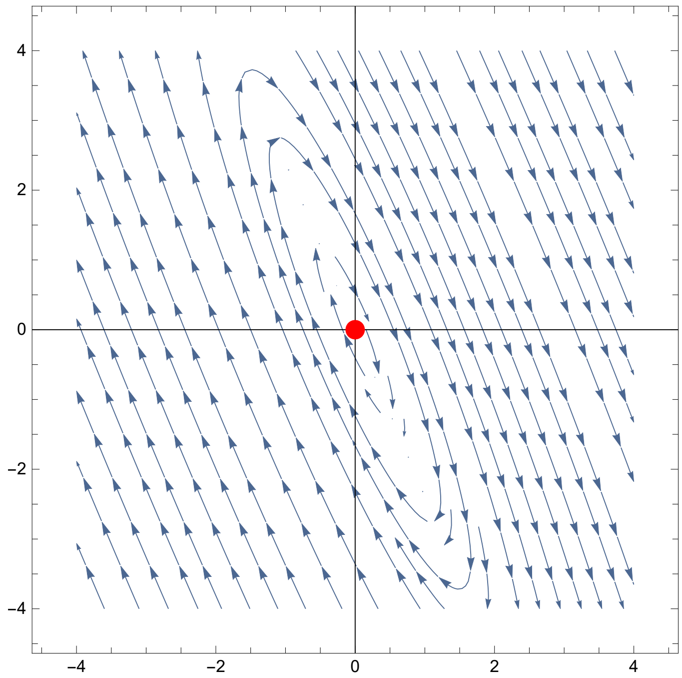
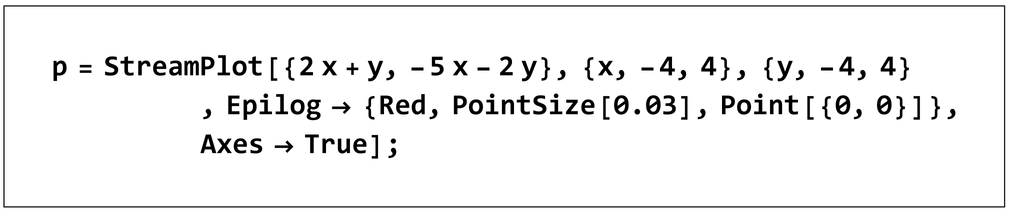


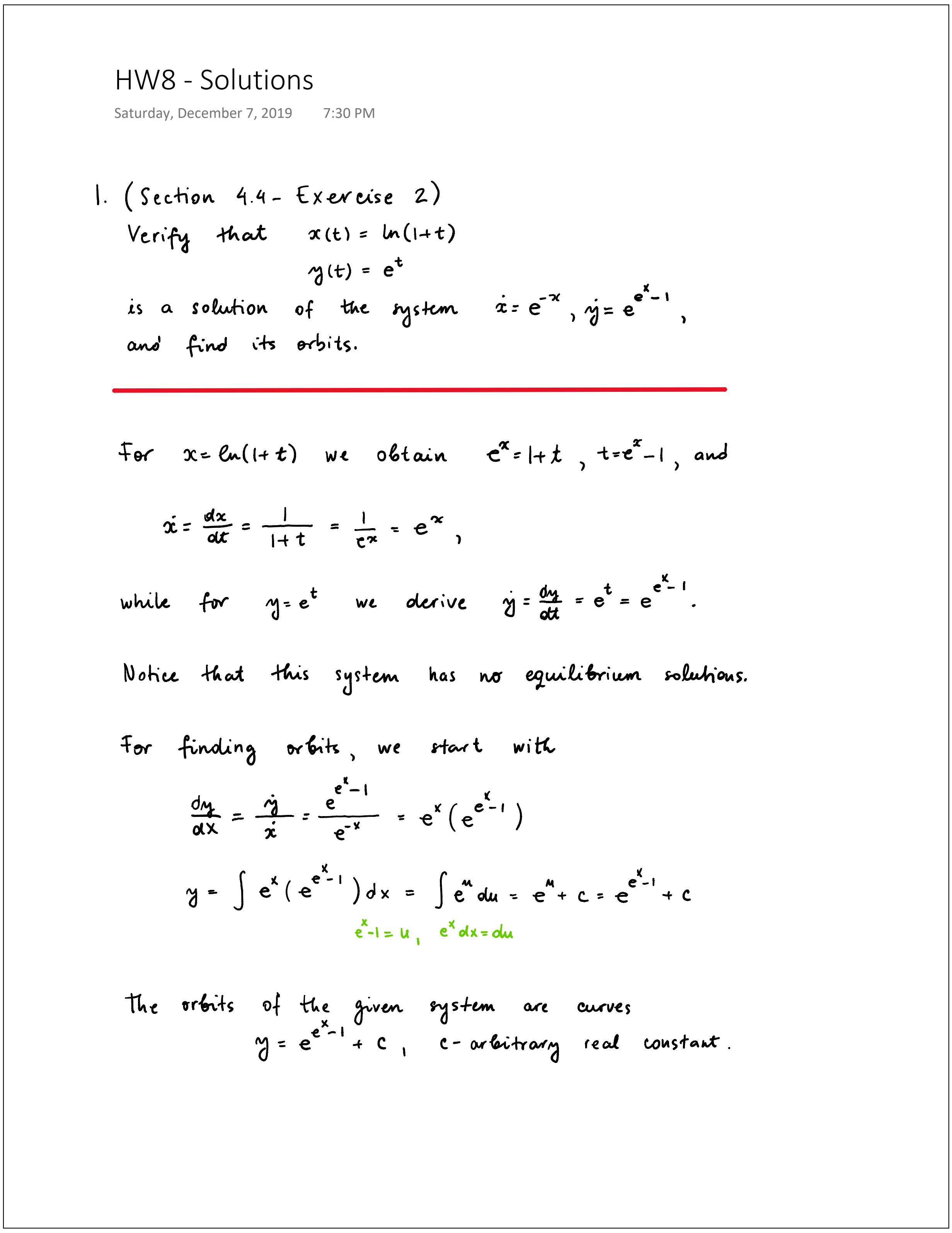
Figure 2.21: Phase portrait



```
p = StreamPlot[{2 x + y, -5 x - 2 y}, {x, -4, 4}, {y, -4, 4}
        , Epilog → {Red, PointSize[0.03], Point[{0, 0}]},
        Axes → True];
```

Figure 2.22: Code used

2.9.8 Key solution for HW 8



HW8 - Solutions

Saturday, December 7, 2019 7:30 PM

1. (Section 4.4 - Exercise 2)

Verify that $x(t) = \ln(1+t)$

$y(t) = e^{t}$

is a solution of the system $\dot{x} = e^{-x}$, $\dot{y} = e^{e^{x}-1}$,

and find its orbits.

For $x = \ln(1+t)$ we obtain $e^{x} = 1+t$, $t = e^{x}-1$, and

$$\dot{x} = \frac{dx}{dt} = \frac{1}{1+t} = \frac{1}{e^{x}} = e^{x},$$

while for $y = e^{t}$ we derive $\dot{y} = \frac{dy}{dt} = e^{t} = e^{e^{x}-1}$.

Notice that this system has no equilibrium solutions.

For finding orbits, we start with

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{e^{e^{x}-1}}{e^{-x}} = e^{x}\left(e^{e^{x}-1}\right)$$

$$y = \int e^{x}\left(e^{e^{x}-1}\right)dx = \int e^{u}\,du = e^{u} + c = e^{e^{x}-1} + c$$

$e^{x}-1 = u$, $e^{x}dx = du$

The orbits of the given system are curves

$y = e^{e^{x}-1} + c$, c - arbitrary real constant.

2. (Section 4.4 - Exercise 8)

Find orbits of the system $\dot{x} = y + x^2 y$
$\dot{y} = 3x + xy^2$.

Solving equations
$$y + x^2 y = y(1+x^2) = 0$$
$$3x + xy^2 = x(3+y^2) = 0$$
we find the only equilibrium point of the system is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
(notice that $1+x^2 > 0$, for all x, and $3+y^2 > 0$, for all y)

For finding orbits, consider the differential equation
$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{3x+xy^2}{y+x^2y} = \frac{x(3+y^2)}{y(1+x^2)} .$$

It is separable and we can solve it in the following way:
$$\int \frac{y}{3+y^2}\, dy = \int \frac{x}{1+x^2}\, dx$$

$r = 3+y^2$, $dr = 2y dy$
$s = 1+x^2$, $ds = 2x dx$

$$\frac{1}{2}\int \frac{dr}{r} = \frac{1}{2}\int \frac{ds}{s}$$

$$\ln|3+y^2| = \ln|1+x^2| + c_1$$

$$|3+y^2| = c_2\,|1+x^2|, \quad c_2 = e^{c_1}$$
$$3+y^2 = c(1+x^2)$$
$$y^2 = c(1+x^2) - 3$$

Orbits of the given sistem are :

1) equilibrium point $(0,0)$
2) the curves $y^2 = c(1+x^2)-3$, $c \neq 3$
3) the half-lines
   $y = \sqrt{3}\,x$, $x>0$
   $y = \sqrt{3}\,x$, $x<0$
   $y = -\sqrt{3}\,x$, $x>0$, and
   $y = -\sqrt{3}\,x$, $x<0$.

(remark : if in the solution curves $y^2 = c(1+x^2)-3$ we formally take $c=3$, we obtain $y^2 = 3(1+x^2)-3 = 3x^2$, and $y = \pm\sqrt{3}\cdot x$ – here we need to exclude $(0,0)$, thus orbits are 4 half-lines)

3. (Section 4.7 - Exercise 3)

Draw the phase portrait of the system

$$\dot{x} = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix} x .$$

For system matrix $A = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}$ we find

$$\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & -1 \\ -2 & 5-\lambda \end{vmatrix} = (4-\lambda)(5-\lambda) - 2 = 20 - 4\lambda - 5\lambda + \lambda^2 - 2$$

$$= \lambda^2 - 9\lambda + 18 = 0 \quad , \quad \lambda_{1,2} = \frac{9 \pm \sqrt{81-72}}{2} = \frac{9 \pm 3}{2}$$

Eigenvalues of A are $\lambda_1 = 3$, $\lambda_2 = 6$.

Equilibrium solution $(0,0)$ is nodal source.

$\lambda_1 = 3:$ $(A - \lambda_1 I)v = 0$

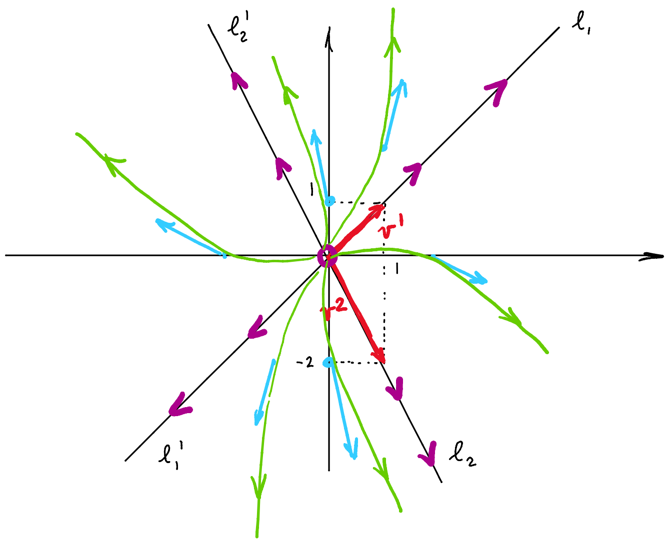
$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = v_2$$

$\Rightarrow$ eigenvector for $\lambda_1 = 3$ is $v^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 6:$ $(A - \lambda_2 I) = 0$

$$\begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -2v_1 - v_2 = 0, \quad v_2 = -2v_1$$

$\Rightarrow$ eigenvector for $\lambda_2 = 6$ is $v^2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$



	$\dot{x}$	$\dot{y}$
(0,1)	-1	5
(0,-2)	2	-10
(1,2)	2	8
(-2,0)	-8	4
(2,0)	8	-4
(-1,-2)	-2	-8

4. (Section 4.7 - Exercise 6)

Draw the phase portrait of the system

$$\dot{x} = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} x.$$

For the system matrix $A = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}$ we find

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda) + 5 = -9 - 3\lambda + 3\lambda + \lambda^2 + 5$$

$$= \lambda^2 - 4 = 0$$

Eigenvalues of A are $\lambda_1 = -2$, $\lambda_2 = 2$, and equilibrium point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is saddle point.

$\lambda_1 = -2$: $(A - \lambda_1 I)v = 0$

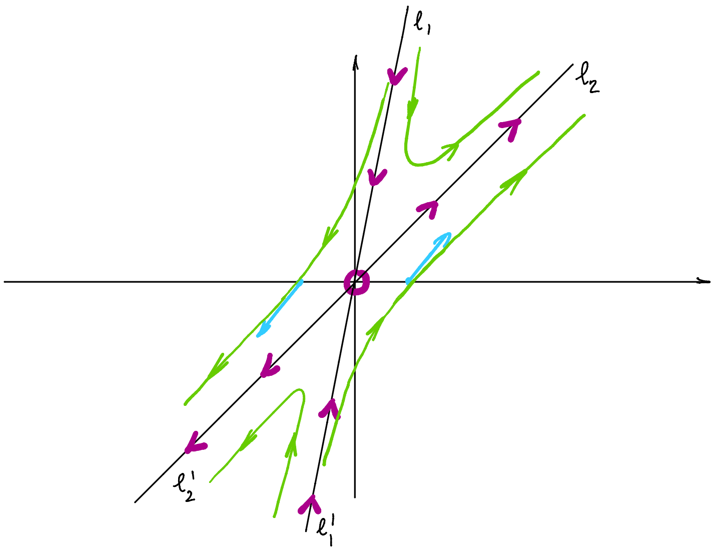
$$\begin{bmatrix} 5 & -1 \\ 5 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 5v_1 - v_2 = 0, \quad v_2 = 5v_1$$

$\Rightarrow$ eigenvector for $\lambda_1 = -2$ is $v^1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$\lambda_2 = 2$: $(A - \lambda_2 I)v = 0$

$$\begin{bmatrix} 1 & -1 \\ 5 & -5 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = v_2$$

$\Rightarrow$ eigenvector for $\lambda_2 = 2$ is $v^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



	$\dot{x}$	$\dot{y}$
(-1,0)	-3	-5
(1,0)	3	5

5. (Section 4.7 - Exercise 9)

Draw the phase portrait of the system

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ -5 & -2 \end{bmatrix} x.$$

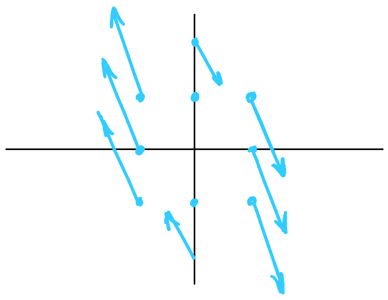
For the system matrix $A = \begin{bmatrix} 2 & 1 \\ -5 & -2 \end{bmatrix}$ we find

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -5 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 5 = -4 - 2\lambda + 2\lambda + \lambda^2 + 5$$

$$= \lambda^2 + 1 = 0 \quad , \quad \lambda_1 = i \; , \quad \lambda_2 = \bar{\lambda}_1 = -i$$

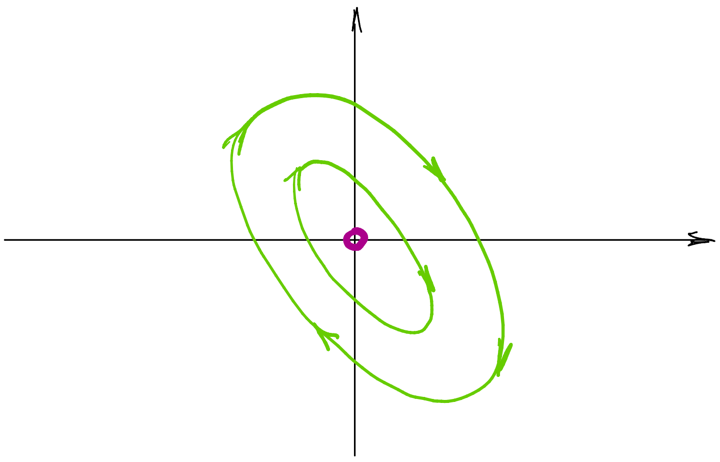
Eigenvalues of A are $\lambda_1 = i$, $\lambda_2 = -i$. They are complex, with zero real part, and consequently equilibrium point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is center.

Let's just first pick few points and determine $\dot{x}, \dot{y}$.



	$\dot{x}$	$\dot{y}$
(1, 1)	3	-7
(-1, -1)	-3	7
(1, 0)	2	-5
(-1, 0)	-2	5
(1, -1)	1	-3
(0, 2)	2	-4

Direction of arrows points to clockwise orientation of orbits.



Chapter 3

Exams

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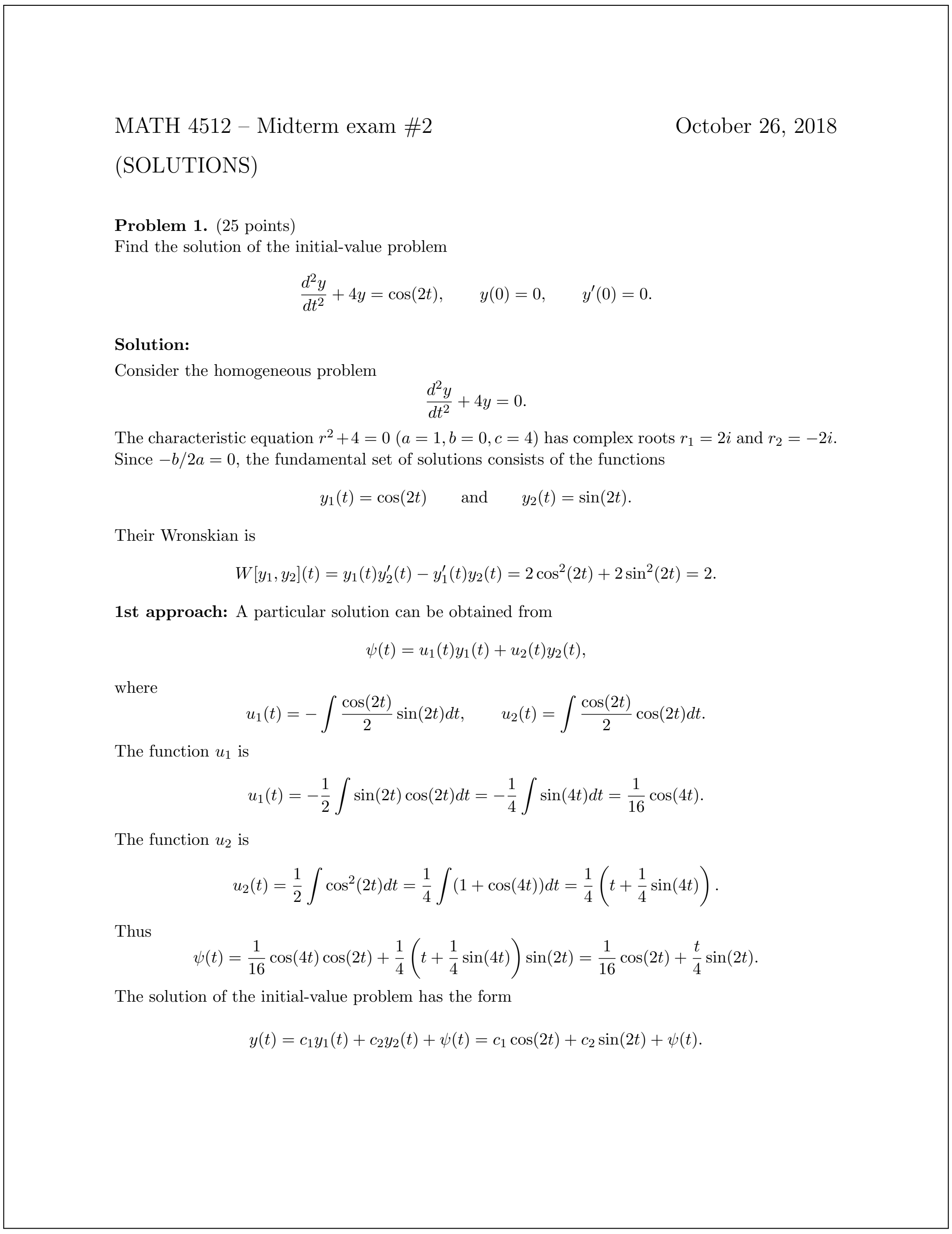
3.1 practice exams

3.1.1 second midterm 2018



MATH 4512 – Midterm exam #2 October 26, 2018

(SOLUTIONS)

Problem 1. (25 points)
Find the solution of the initial-value problem

$$\frac{d^2y}{dt^2} + 4y = \cos(2t), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution:
Consider the homogeneous problem

$$\frac{d^2y}{dt^2} + 4y = 0.$$

The characteristic equation $r^2 + 4 = 0$ ($a = 1, b = 0, c = 4$) has complex roots $r_1 = 2i$ and $r_2 = -2i$. Since $-b/2a = 0$, the fundamental set of solutions consists of the functions

$$y_1(t) = \cos(2t) \qquad \text{and} \qquad y_2(t) = \sin(2t).$$

Their Wronskian is

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = 2\cos^2(2t) + 2\sin^2(2t) = 2.$$

1st approach: A particular solution can be obtained from

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

where

$$u_1(t) = -\int \frac{\cos(2t)}{2}\sin(2t)dt, \qquad u_2(t) = \int \frac{\cos(2t)}{2}\cos(2t)dt.$$

The function u_1 is

$$u_1(t) = -\frac{1}{2}\int \sin(2t)\cos(2t)dt = -\frac{1}{4}\int \sin(4t)dt = \frac{1}{16}\cos(4t).$$

The function u_2 is

$$u_2(t) = \frac{1}{2}\int \cos^2(2t)dt = \frac{1}{4}\int (1 + \cos(4t))dt = \frac{1}{4}\left(t + \frac{1}{4}\sin(4t)\right).$$

Thus

$$\psi(t) = \frac{1}{16}\cos(4t)\cos(2t) + \frac{1}{4}\left(t + \frac{1}{4}\sin(4t)\right)\sin(2t) = \frac{1}{16}\cos(2t) + \frac{t}{4}\sin(2t).$$

The solution of the initial-value problem has the form

$$y(t) = c_1y_1(t) + c_2y_2(t) + \psi(t) = c_1\cos(2t) + c_2\sin(2t) + \psi(t).$$

From the condition $y(0) = 0$, we get

$$0 = y(0) = c_1 y_1(0) + c_2 y_2(0) + \psi(0) = c_1 + \frac{1}{16} \qquad \text{and} \qquad c_1 = -\frac{1}{16}.$$

Since

$$\begin{aligned} y'(t) &= -2c_1 \sin(2t) + 2c_2 \cos(2t) + \psi'(t) \\ \psi'(t) &= -\frac{1}{8}\sin(2t) + \frac{1}{4}\sin(2t) + \frac{t}{2}\cos(2t) = \frac{1}{8}\sin(2t) + \frac{t}{2}\cos(2t), \end{aligned}$$

the second initial condition $y'(0) = 0$ further implies

$$0 = y'(0) = 2c_2 + \psi'(0) = 2c_2 \qquad \text{and} \qquad c_2 = 0.$$

The solution of the starting problem is now

$$y(t) = -\frac{1}{16}y_1(t) + \psi(t) = -\frac{1}{16}\cos(2t) + \frac{1}{16}\cos(2t) + \frac{t}{4}\sin(2t) = \frac{t}{4}\sin(2t).$$

2nd approach (guessing): Consider the complex-valued problem $\frac{d^2y}{dt^2} + 4y = \mathrm{e}^{2it}$ and guess its particular solution $\phi(t) = At\mathrm{e}^{2it}$. From

$$\phi'(t) = A\mathrm{e}^{2it} + 2iAt\mathrm{e}^{2it}, \qquad \phi''(t) = 4iA\mathrm{e}^{2it} - 4At\mathrm{e}^{2it},$$

we get

$$\mathrm{e}^{2it} = \phi''(t) + 4\phi(t) = 4iA\mathrm{e}^{2it} - 4At\mathrm{e}^{2it} - 4At\mathrm{e}^{2it} = 4iA\mathrm{e}^{2it}.$$

Thus $A = 1/(4i) = -i/4$ and

$$\phi(t) = -\frac{i}{4}t\mathrm{e}^{2it} = -\frac{i}{4}t(\cos(2t) + i\sin(2t)) = \frac{1}{4}t\sin(2t) - \frac{i}{4}t\cos(2t).$$

The particular solution of the starting problem is $\psi(t) = \mathrm{Re}(\phi(t)) = \frac{t}{4}\sin(2t)$. In the first approach we derived the general form of the solution

$$y(t) = c_1\cos(2t) + c_2\sin(2t) + \psi(t).$$

Applying the initial condition, we obtain $0 = y(0) = c_1$. From

$$y'(t) = -2c_1\sin(2t) + 2c_2\cos(2t) + \frac{1}{4}\sin(2t) + \frac{t}{2}\cos(2t),$$

we now get $0 = y'(0) = 2c_2$, i.e. $c_2 = 0$. The general solution of the IVP is $y(t) = \psi(t) = \frac{t}{4}\sin(2t)$.

3rd approach (Laplace transforms): Let $Y(s) = \mathcal{L}\{y(t)\}$. Then

$$Y(s) = \frac{1}{s^2+4}\mathcal{L}\{\cos(2t)\} = \frac{1}{s^2+4}\frac{s}{s^2+4} = \frac{s}{(s^2+4)^2}.$$

We can write now

$$Y(s) = \frac{1}{2}\frac{2}{s^2+4}\frac{s}{s^2+4} = \frac{1}{2}\mathcal{L}\{\sin(2t)\}\mathcal{L}\{\cos(2t)\} = \frac{1}{2}\mathcal{L}\{\sin(2t) * \cos(2t)\},$$

which will give us $y(t) = \frac{1}{2}\sin(2t) * \cos(2t)$. One can calculate this convolution and get $y(t) = \frac{t}{4}\sin(2t)$. Instead, notice the following

$$\frac{d}{ds}\left(\frac{1}{s^2+4}\right) = -\frac{2s}{(s^2+4)^2}.$$

Therefore

$$Y(s) = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+4}\right) = -\frac{1}{4}\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = -\frac{1}{4}\frac{d}{ds}\mathcal{L}\{\sin(2t)\} = -\frac{1}{4}\mathcal{L}\{-t\,\sin(2t)\},$$

and consequently $y(t) = \frac{t}{4}\sin(2t)$.

Problem 2. (25 points)
Find a function $g(t)$, $t \geq 0$, such that

$$\mathcal{L}\{g(t)\} = \frac{s^2}{(s^2+9)^2}, \qquad s > 0.$$

Solution:
First we have that

$$\frac{s^2}{(s^2+9)^2} = s \cdot \frac{s}{(s^2+9)^2} = -\frac{s}{6}\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = -\frac{s}{6}\frac{d}{ds}\mathcal{L}\{\sin(3t)\}$$
$$= -\frac{s}{6}\mathcal{L}\{-t\sin(3t)\} = s\,\mathcal{L}\{\frac{1}{6}t\sin(3t)\}.$$

If we introduce $H(s) = \mathcal{L}\{h(t)\}$ with $h(t) = \frac{1}{6}t\sin(3t)$, then

$$\frac{s^2}{(s^2+9)^2} = s\,H(s) = \mathcal{L}\{h'(t)\} + h(0) = \mathcal{L}\{h'(t)\}.$$

This gives us

$$g(t) = h'(t) = \frac{1}{6}\sin(3t) + \frac{t}{2}\cos(3t).$$

We could have also start from

$$\frac{s^2}{(s^2+9)^2} = \frac{s}{s^2+9} \cdot \frac{s}{s^2+9} = \mathcal{L}\{\cos(3t)\}\mathcal{L}\{\cos(3t)\} = \mathcal{L}\{\cos(3t) * \cos(3t)\}.$$

The convolution $g(t) = \cos(3t) * \cos(3t)$ is

$$\cos(3t) * \cos(3t) = \int_0^t \cos(3t - 3u)\cos(3u)du = \int_0^t (\cos(3t)\cos(3u) + \sin(3t)\sin(3u))\cos(3u)du$$
$$= \cos(3t)\int_0^t \cos^2(3u)du + \sin(3t)\int_0^t \sin(3u)\cos(3u)du$$
$$= \frac{1}{2}\cos(3t)\int_0^t (1+\cos(6u))du + \frac{1}{2}\sin(3t)\int_0^t \sin(6u)du$$
$$= \frac{1}{2}\cos(3t)\left(t + \frac{1}{6}\sin(6t)\right) + \frac{1}{2}\sin(3t)\frac{1}{6}(-\cos(6t)+1)$$
$$= \frac{t}{2}\cos(3t) + \frac{1}{12}\sin(6t)\cos(3t) - \frac{1}{12}\sin(3t)\cos(6t) + \frac{1}{12}\sin(3t)$$
$$= \frac{t}{2}\cos(3t) + \frac{1}{12}\sin(3t) + \frac{1}{12}\sin(3t) = \frac{t}{2}\cos(3t) + \frac{1}{6}\sin(3t).$$

Problem 3. (50 points)
Consider the following initial-value problem

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t\,\mathrm{e}^t, \qquad y(0) = y'(0) = 0. \tag{P}$$

(i) Find fundamental set of solutions for the homogeneous differential equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0.$$

(ii) Find a particular solution of the initial-value problem (P).

(iii) Using the results from (i) and (ii), find the solution of (P) that satisfies the given initial conditions.

(iv) Solve the problem (P) using Laplace transforms.

Solution:
(i) The characteristic equation $r^2 - 2r + 1 = 0$ has one real root $r = 1$. The fundamental set of solutions is

$$y_1(t) = \mathrm{e}^t, \qquad y_2(t) = t\,\mathrm{e}^t,$$

with the Wronskian

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = \mathrm{e}^t(\mathrm{e}^t + t\,\mathrm{e}^t) - t\,\mathrm{e}^{2t} = \mathrm{e}^{2t}.$$

(ii) The particular solution can be obtained from

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

where

$$u_1(t) = -\int \frac{t\,\mathrm{e}^t}{\mathrm{e}^{2t}} t\,\mathrm{e}^t dt = -\int t^2 dt = -\frac{t^3}{3}$$
$$u_2(t) = \int \frac{t\,\mathrm{e}^t}{\mathrm{e}^{2t}} \mathrm{e}^t dt = \int t dt = \frac{t^2}{2}.$$

Thus

$$\psi(t) = -\frac{t^3}{3}\mathrm{e}^t + \frac{t^3}{2}\mathrm{e}^t = \frac{t^3}{6}\mathrm{e}^t.$$

We can also guess particular solution as $\psi(t) = t^2(A_1 t + A_0)\mathrm{e}^t$ and obtain the same function ($A_0 = 0$, $A_1 = 1/6$).
(iii) The solution of the problem (P) has the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \psi(t) = c_1\mathrm{e}^t + c_2 t\,\mathrm{e}^t + \frac{t^3}{6}\mathrm{e}^t.$$

From the condition $y(0) = 0$, we immediately get $c_1 = 0$. Since

$$y'(t) = c_1 \mathrm{e}^t + c_2(1+t)\mathrm{e}^t + \frac{3t^2 + t^3}{6}\mathrm{e}^t,$$

the second initial condition $y'(0) = 0$ further implies $c_2 = 0$. The solution of (P) is now

$$y(t) = \psi(t) = \frac{t^3}{6}\mathrm{e}^t.$$

(iv) Let $Y(s) = \mathcal{L}\{y(t)\}$. Applying $y(0) = y'(0) = 0$, we have that

$$Y(s) = \frac{1}{s^2 - 2s + 1}\mathcal{L}\{t\,\mathrm{e}^t\} = \frac{1}{(s-1)^2}\left(-\frac{d}{ds}F(s)\right)$$

where $F(s) = \mathcal{L}\{\mathrm{e}^t\} = (s-1)^{-1}$, $s > 1$. From

$$\frac{d}{ds}F(s) = -\frac{1}{(s-1)^2},$$

we conclude

$$Y(s) = \frac{1}{(s-1)^4}.$$

On lectures we showed

$$\mathcal{L}\{t^n \mathrm{e}^{at}\} = \frac{n!}{(s-a)^{n+1}}, \qquad n \in \mathbb{N}.$$

With $a = 1$, $n = 3$, this implies

$$Y(s) = \frac{1}{(s-1)^4} = \frac{1}{3!}\mathcal{L}\{t^3\mathrm{e}^t\} = \mathcal{L}\{\frac{1}{6}t^3\mathrm{e}^t\},$$

and finally

$$y(t) = \frac{1}{6}t^3\mathrm{e}^t.$$

3.2 Exam 1, Sept 27, 2019

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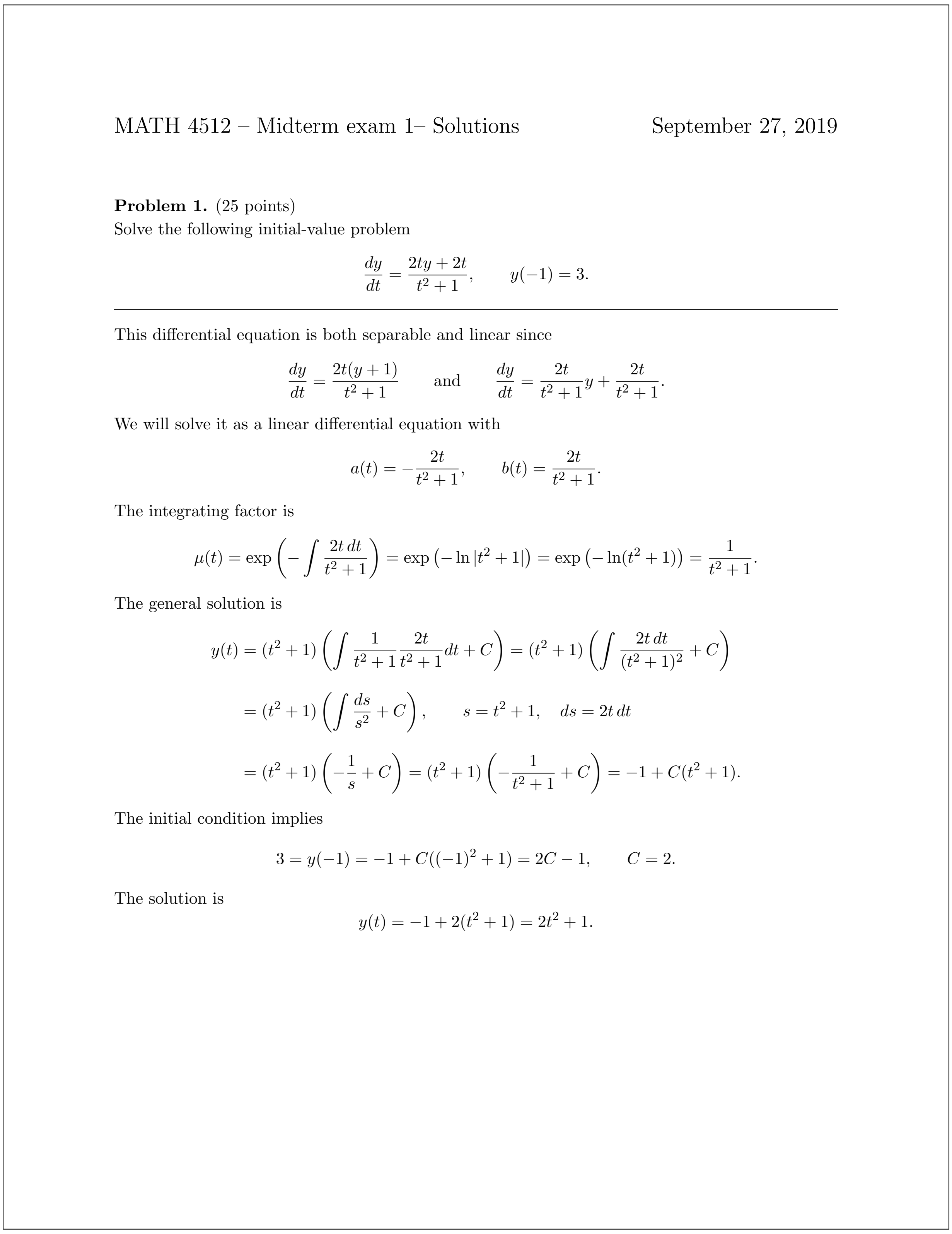
3.2.1 Key solution 118

3.2.1 Key solution



MATH 4512 – Midterm exam 1– Solutions September 27, 2019

Problem 1. (25 points)
Solve the following initial-value problem

$$\frac{dy}{dt} = \frac{2ty + 2t}{t^2+1}, \qquad y(-1) = 3.$$

This differential equation is both separable and linear since

$$\frac{dy}{dt} = \frac{2t(y+1)}{t^2+1} \qquad \text{and} \qquad \frac{dy}{dt} = \frac{2t}{t^2+1}y + \frac{2t}{t^2+1}.$$

We will solve it as a linear differential equation with

$$a(t) = -\frac{2t}{t^2+1}, \qquad b(t) = \frac{2t}{t^2+1}.$$

The integrating factor is

$$\mu(t) = \exp\left(-\int \frac{2t\,dt}{t^2+1}\right) = \exp\left(-\ln|t^2+1|\right) = \exp\left(-\ln(t^2+1)\right) = \frac{1}{t^2+1}.$$

The general solution is

$$y(t) = (t^2+1)\left(\int \frac{1}{t^2+1}\frac{2t}{t^2+1}dt + C\right) = (t^2+1)\left(\int \frac{2t\,dt}{(t^2+1)^2} + C\right)$$

$$= (t^2+1)\left(\int \frac{ds}{s^2} + C\right), \qquad s = t^2+1, \quad ds = 2t\,dt$$

$$= (t^2+1)\left(-\frac{1}{s} + C\right) = (t^2+1)\left(-\frac{1}{t^2+1} + C\right) = -1 + C(t^2+1).$$

The initial condition implies

$$3 = y(-1) = -1 + C((-1)^2+1) = 2C - 1, \qquad C = 2.$$

The solution is

$$y(t) = -1 + 2(t^2+1) = 2t^2 + 1.$$

Problem 2. (25 points)
A tank initially contains 60 gal of pure water. Brine containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the well-stirred solution leaves the tank at the same rate.

(a) If $S(t)$ is the amount of salt in the tank at time t, write the differential equation for the time rate of change of $S(t)$ and solve it.

(b) Find the concentration $c(t)$ of the salt in the tank at time t.

(c) What would be the limiting concentration of salt as $t \to \infty$?

The inflow rate is $r_i = 2\,\text{gal/min}$, the outflow rate is $r_o = 2\,\text{gal/min}$, the inflow salt concentration is $c_i = 1\,\text{lb/gal}$, $V_0 = 60$ gal is the initial volume and $S(0) = 0$ is the initial condition (only pure water is initially in the tank).
(a) The time rate of change of $S(t)$ is given by

$$\frac{dS}{dt} = r_i c_i - r_o \frac{S}{60} = 2 - \frac{S}{30}.$$

This differential equation is linear

$$\frac{dS}{dt} + \frac{S}{30} = 2$$

and the integrating factor is

$$\mu(t) = \exp\left(\int \frac{dt}{30}\right) = \mathrm{e}^{t/30}.$$

The solution is

$$S(t) = \mathrm{e}^{-t/30}\left(\int 2\,\mathrm{e}^{t/30} dt + C\right) = \mathrm{e}^{-t/30}\left(60\,\mathrm{e}^{t/30} + C\right) = 60 + C\,\mathrm{e}^{-t/30}.$$

From the initial condition it follows

$$0 = S(0) = 60 + C, \qquad C = -60.$$

Finally

$$S(t) = 60 - 60\,\mathrm{e}^{-t/30}.$$

(b) Concentration $c(t)$ of the salt in the tank at time t is

$$c(t) = \frac{S(t)}{60} = 1 - \mathrm{e}^{-t/30}.$$

(c) The limiting concentration is

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty} (1 - \mathrm{e}^{-t/30}) = 1.$$

Problem 3. (30 points)
A population of butterflies grows according to the logistic law

$$\frac{dp}{dt} = 0.002p(100-p) = 0.2p - 0.002p^2, \qquad t \geq 0.$$

(a) Find the population $p(t)$ as the function of time t, if the initial population is 60.

(b) Find $\lim_{t\to\infty} p(t)$ and determine limiting population in this model.

(a) We start from

$$\frac{dp}{dt} = 0.002p(100-p).$$

Using partial fractions

$$\frac{1}{p(100-p)} = \frac{1}{100}\left(\frac{1}{p} + \frac{1}{100-p}\right),$$

we derive

$$\int \frac{dp}{p(100-p)} = \int 0.002\,dt$$

$$\frac{1}{100}\int\left(\frac{1}{p} + \frac{1}{100-p}\right)dp = 0.002\int dt$$

$$\int\left(\frac{1}{p} + \frac{1}{100-p}\right)dp = 0.2\int dt$$

$$\ln|p| - \ln|100-p| = 0.2t + C_1$$

$$\ln\left|\frac{p}{100-p}\right| = 0.2t + C_1$$

$$\frac{p}{100-p} = C\mathrm{e}^{0.2t}.$$

From the given initial condition $p(0) = 60$, we can find the constant C:

$$\frac{p(0)}{100-p(0)} = C\mathrm{e}^{0.2\cdot 0}, \qquad \frac{60}{100-60} = C, \qquad C = \frac{3}{2}.$$

The next step is to find the function $p(t)$:

$$\frac{p}{100-p} = \frac{3}{2}\mathrm{e}^{0.2t}$$

$$p = \frac{3}{2}\mathrm{e}^{0.2t}(100-p) = 150\,\mathrm{e}^{0.2t} - \frac{3}{2}\mathrm{e}^{0.2t}p$$

$$p\left(1 + \frac{3}{2}\mathrm{e}^{0.2t}\right) = 150\,\mathrm{e}^{0.2t}$$

$$p(t) = \frac{150\,\mathrm{e}^{0.2t}}{1 + \frac{3}{2}\mathrm{e}^{0.2t}} = \frac{150}{\mathrm{e}^{-0.2t} + \frac{3}{2}}.$$

The solution can be obtained directly from the formula

$$p(t) = \frac{ap_0}{bp_0 + (a - bp_0)\mathrm{e}^{-a(t-t_0)}}.$$

Here $a = 0.2$, $b = 0.002$, $t_0 = 0$, and $p_0 = 60$. Then

$$p(t) = \frac{12}{0.12 + (0.2 - 0.12)\mathrm{e}^{-0.2\,t}} = \frac{12}{0.12 + 0.08\mathrm{e}^{-0.2\,t}} = \frac{150}{1.5 + \mathrm{e}^{-0.2\,t}}.$$

(b) The limiting population in this model is

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{150}{1.5 + \mathrm{e}^{-0.2\,t}} = \frac{150}{1.5} = 100 = \frac{a}{b}.$$

Problem 4. (20 points)
Find the orthogonal trajectories of the given family of curves

$$y = c\,\mathrm{e}^{x}.$$

Here we can take $F(x, y, c) = y - c\,\mathrm{e}^{x}$. Then from

$$F_x = -c\,\mathrm{e}^{x}, \qquad F_y = 1, \qquad c = \frac{y}{\mathrm{e}^{x}},$$

the orthogonal trajectories of the given family are the solution curves of the equation

$$\frac{dy}{dx} = \frac{F_y}{F_x} = -\frac{1}{y}.$$

This is a separable differential equation and we solve it as follows:

$$\int y\,dy = -\int dx$$

$$\frac{y^2}{2} = -x + c.$$

3.3 Exam 2, Oct 25, 2019

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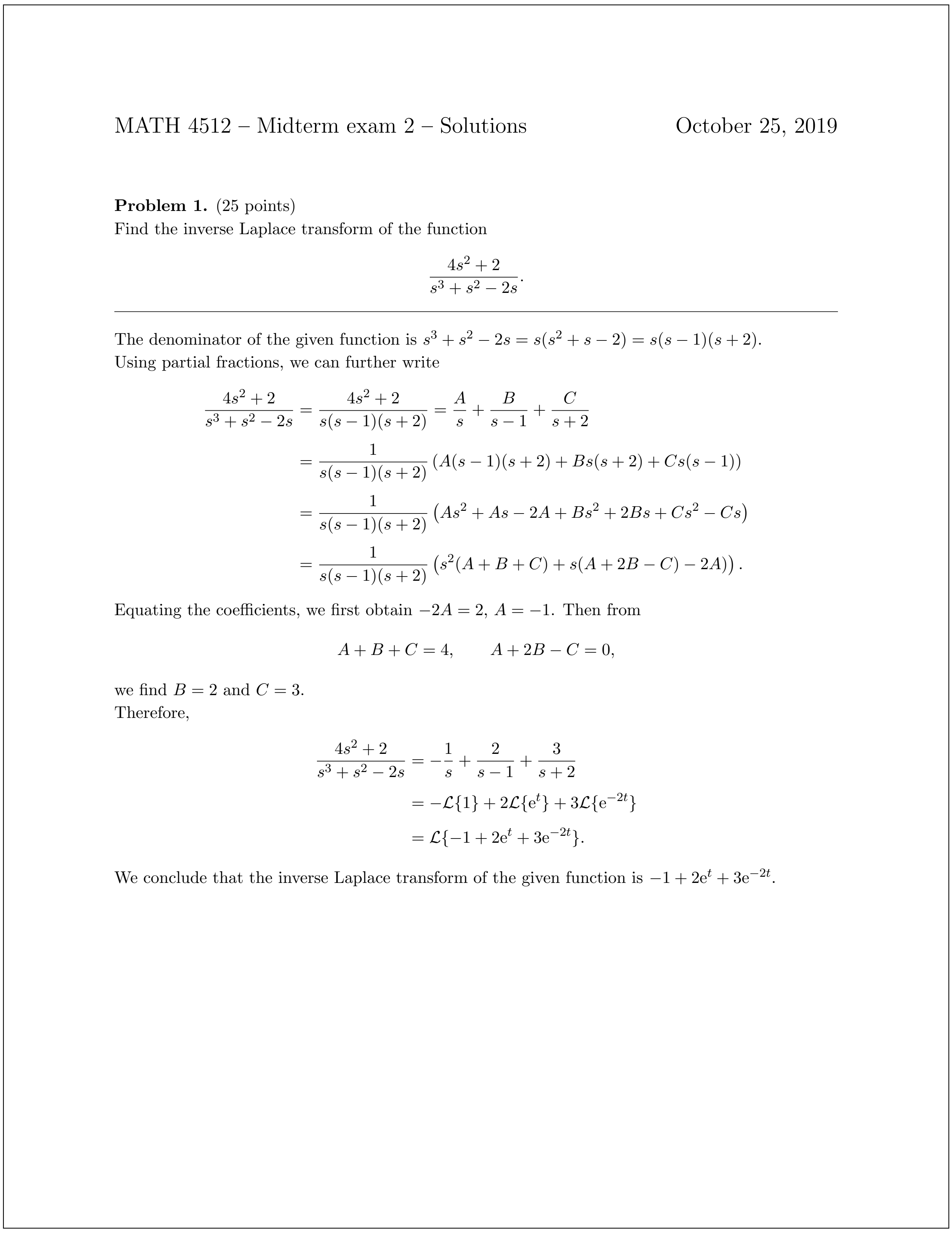
3.3.1 Key solution 123

3.3.1 Key solution



MATH 4512 – Midterm exam 2 – Solutions October 25, 2019

Problem 1. (25 points)
Find the inverse Laplace transform of the function

$$\frac{4s^2+2}{s^3+s^2-2s}.$$

The denominator of the given function is $s^3+s^2-2s = s(s^2+s-2) = s(s-1)(s+2)$.
Using partial fractions, we can further write

$$\begin{aligned}
\frac{4s^2+2}{s^3+s^2-2s} = \frac{4s^2+2}{s(s-1)(s+2)} &= \frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+2}\\
&= \frac{1}{s(s-1)(s+2)}\left(A(s-1)(s+2)+Bs(s+2)+Cs(s-1)\right)\\
&= \frac{1}{s(s-1)(s+2)}\left(As^2+As-2A+Bs^2+2Bs+Cs^2-Cs\right)\\
&= \frac{1}{s(s-1)(s+2)}\left(s^2(A+B+C)+s(A+2B-C)-2A\right).
\end{aligned}$$

Equating the coefficients, we first obtain $-2A=2$, $A=-1$. Then from

$$A+B+C=4, \qquad A+2B-C=0,$$

we find $B=2$ and $C=3$.
Therefore,

$$\begin{aligned}
\frac{4s^2+2}{s^3+s^2-2s} &= -\frac{1}{s}+\frac{2}{s-1}+\frac{3}{s+2}\\
&= -\mathcal{L}\{1\}+2\mathcal{L}\{\mathrm{e}^t\}+3\mathcal{L}\{\mathrm{e}^{-2t}\}\\
&= \mathcal{L}\{-1+2\mathrm{e}^t+3\mathrm{e}^{-2t}\}.
\end{aligned}$$

We conclude that the inverse Laplace transform of the given function is $-1+2\mathrm{e}^t+3\mathrm{e}^{-2t}$.

Problem 2. (25 points)
Solve the following initial-value problem

$$y'' - 3y' - 4y = \mathrm{e}^{2t}, \qquad y(0) = 0, \qquad y'(0) = -\frac{1}{2}.$$

For finding the particular solution, use **the method of variation of parameters**.

First we solve the homogeneous problem. The characteristic equation

$$r^2 - 3r - 4 = 0$$

has two real roots $r_1 = 4$ and $r_2 = -1$. The functions

$$y_1(t) = \mathrm{e}^{4t}, \qquad y_2(t) = \mathrm{e}^{-t},$$

form a fundamental set of solutions.
The particular solution $\psi(t)$ we seek in the form

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

The Wronskian for y_1, y_2 is

$$W(t) = W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = -\mathrm{e}^{4t}\mathrm{e}^{-t} - 4\mathrm{e}^{4t}\mathrm{e}^{-t} = -5\mathrm{e}^{3t}.$$

Functions u_1, u_2 we find as

$$u_1(t) = -\int \frac{\mathrm{e}^{2t}\,\mathrm{e}^{-t}}{-5\mathrm{e}^{3t}}dt = \frac{1}{5}\int \mathrm{e}^{-2t}dt = -\frac{1}{10}\mathrm{e}^{-2t}$$
$$u_2(t) = \int \frac{\mathrm{e}^{2t}\,\mathrm{e}^{4t}}{-5\mathrm{e}^{3t}}dt = -\frac{1}{5}\int \mathrm{e}^{3t}dt = -\frac{1}{15}\mathrm{e}^{3t}.$$

The particular solution is

$$\psi(t) = -\frac{1}{10}\mathrm{e}^{-2t}\,\mathrm{e}^{4t} - \frac{1}{15}\mathrm{e}^{3t}\,\mathrm{e}^{-t} = -\frac{1}{6}\mathrm{e}^{2t}.$$

The general solution is

$$y(t) = c_1y_1(t) + c_2y_2(t) + \psi(t) = c_1\mathrm{e}^{4t} + c_2\mathrm{e}^{-t} - \frac{1}{6}\mathrm{e}^{2t}.$$

From $y(0) = 0$, we get the first condition $c_1 + c_2 - 1/6 = 0$ for the constants c_1, c_2. The first derivative of $y(t)$ is

$$y'(t) = 4c_1\mathrm{e}^{4t} - c_2\mathrm{e}^{-t} - \frac{1}{3}\mathrm{e}^{2t}.$$

Now, from $y'(0) = -1/2$ we get $4c_1 - c_2 - 1/3 = -1/2$. The constants are $c_1 = 0$, $c_2 = 1/6$, and the final solution is

$$y(t) = \frac{1}{6}\mathrm{e}^{-t} - \frac{1}{6}\mathrm{e}^{2t}.$$

Problem 3. (20 points)
Which of the following functions

(a) $e^{-t/3}(A_3t^3 + A_0)t^2$

(b) $e^{-t/3}(A_3t^3 + A_2t^2 + A_1t + A_0)t$

(c) $e^{-t/3}(A_3t^3 + A_2t^2 + A_1t + A_0)t^2$

(d) $e^{-t/3}(A_3t^3 + A_2t^2 + A_1t + A_0)$

should be chosen as a guessing for the particular solution of $9y'' + 6y' + y = e^{-t/3}(t^3 - 1)$?

The characteristic equation

$$9r^2 + 6r + 1 = 0$$

has one double root $r = -1/3$ that coincides with the exponent $\alpha = -1/3$ in the right-hand side $g(t) = e^{\alpha t}(t^3 - 1)$. Thus the correct guessing for the particular solution is the answer (c), i.e.

$$\psi(t) = e^{-t/3}\, t^2\, (A_3t^3 + A_2t^2 + A_1t + A_0).$$

Problem 4. (30 points)
A spring-mass-dashpot system with $m = 1$, $k = 2$ and $c = 2$ (in their respective units) hangs in equilibrium. At time $t = 0$, an external force $F(t) = t - \pi$ N starts acting on the hanging object. Find the position $y(t)$ of the object at anytime $t > 0$. Over time, what do you expect to occur within this system?

The initial-value problem describing this system is

$$y'' + 2y' + 2y = t - \pi, \qquad y(0) = y'(0) = 0.$$

The characteristic equation $r^2 + 2r + 2 = 0$ has complex roots $r_1 = -1 + i$, $r_2 = -1 - i$. Therefore, the functions

$$y_1(t) = \mathrm{e}^{-t}\cos t, \qquad y_2(t) = \mathrm{e}^{-t}\sin t,$$

form a fundamental set of solutions.
The particular solution can be found using the guessing method. Then

$$\psi(t) = At + B,$$

with $\psi'(t) = A$ and $\psi''(t) = 0$. The differential equation with the function ψ reduces to

$$\psi'' + 2\psi' + 2\psi = 2A + 2At + 2B = 2At + 2(A + B) = t - \pi.$$

Equating coefficients we obtain $A = 1/2$, $B = -\pi/2 - 1/2$. Hence, the particular solution is

$$\psi(t) = \frac{t}{2} - \frac{\pi}{2} - \frac{1}{2}.$$

The general solution

$$y(t) = c_1\mathrm{e}^{-t}\cos t + c_2\mathrm{e}^{-t}\sin t + \frac{t}{2} - \frac{\pi}{2} - \frac{1}{2}$$

has its first derivative

$$y'(t) = -c_1\mathrm{e}^{-t}\cos t - c_1\mathrm{e}^{-t}\sin t - c_2\mathrm{e}^{-t}\sin t + c_2\mathrm{e}^{-t}\cos t + \frac{1}{2}.$$

Initial conditions $y(0) = 0$ and $y'(0) = 0$ further imply

$$0 = c_1 - \frac{\pi}{2} - \frac{1}{2}, \qquad 0 = -c_1 + c_2 + \frac{1}{2}.$$

Then

$$c_1 = \frac{\pi}{2} + \frac{1}{2}, \qquad c_2 = \frac{\pi}{2},$$

and the function $y(t)$ that describes the position of the object at anytime $t > 0$ is

$$y(t) = \left(\frac{\pi}{2} + \frac{1}{2}\right)\mathrm{e}^{-t}\cos t + \frac{\pi}{2}\mathrm{e}^{-t}\sin t + \frac{t}{2} - \frac{\pi}{2} - \frac{1}{2}.$$

Over time we expect the spring to break since

$$\lim_{t\to\infty} y(t) = \infty$$

(the positive direction of the position $y(t)$ is downwards).

3.4 Exam 3, Nov 20, 2019

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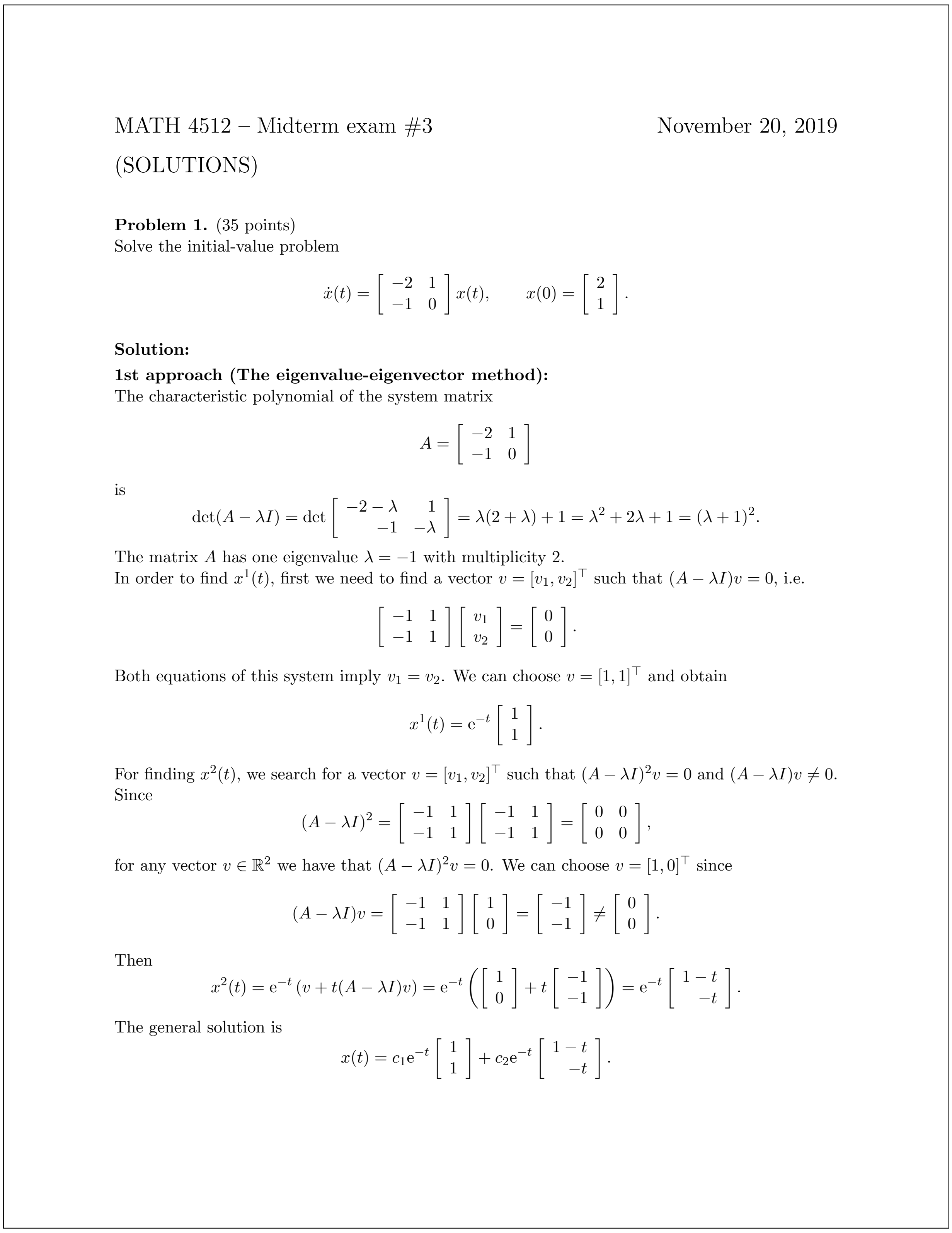
3.4.1 Key solution



MATH 4512 – Midterm exam #3 November 20, 2019

(SOLUTIONS)

Problem 1. (35 points)
Solve the initial-value problem

$$\dot{x}(t) = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} x(t), \qquad x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Solution:
1st approach (The eigenvalue-eigenvector method):
The characteristic polynomial of the system matrix

$$A = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$$

is

$$\det(A - \lambda I) = \det \begin{bmatrix} -2-\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda(2+\lambda) + 1 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2.$$

The matrix A has one eigenvalue $\lambda = -1$ with multiplicity 2.
In order to find $x^1(t)$, first we need to find a vector $v = [v_1, v_2]^\top$ such that $(A - \lambda I)v = 0$, i.e.

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Both equations of this system imply $v_1 = v_2$. We can choose $v = [1,1]^\top$ and obtain

$$x^1(t) = \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For finding $x^2(t)$, we search for a vector $v = [v_1, v_2]^\top$ such that $(A - \lambda I)^2 v = 0$ and $(A - \lambda I)v \neq 0$.
Since

$$(A - \lambda I)^2 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

for any vector $v \in \mathbb{R}^2$ we have that $(A - \lambda I)^2 v = 0$. We can choose $v = [1,0]^\top$ since

$$(A - \lambda I)v = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then

$$x^2(t) = \mathrm{e}^{-t}\left(v + t(A - \lambda I)v\right) = \mathrm{e}^{-t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = \mathrm{e}^{-t} \begin{bmatrix} 1-t \\ -t \end{bmatrix}.$$

The general solution is

$$x(t) = c_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-t} \begin{bmatrix} 1-t \\ -t \end{bmatrix}.$$

From the initial condition $x(0) = [2, 1]^\top$ we obtain

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = x(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 \end{bmatrix}.$$

Then $c_1 = c_2 = 1$ and the final solution is

$$x(t) = \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \mathrm{e}^{-t} \begin{bmatrix} 1 - t \\ -t \end{bmatrix} = \mathrm{e}^{-t} \begin{bmatrix} 2 - t \\ 1 - t \end{bmatrix}.$$

2nd approach (Laplace transforms):
We will determine $X(s) = \mathcal{L}(x(t))$ from the condition $(sI - A)X(s) = x(0)$, i.e.

$$\begin{bmatrix} s+2 & -1 \\ 1 & s \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

From the first equation we have $(s+2)X_1(s) - X_2(s) = 2$ and $X_2(s) = (s+2)X_1(s) - 2$. The second equation is now

$$1 = X_1(s) + sX_2(s) = X_1(s) + s(s+2)X_1(s) - 2s = X_1(s)(s^2 + 2s + 1) - 2s.$$

Then

$$X_1(s) = \frac{2s+1}{(s+1)^2} = \frac{2}{s+1} - \frac{1}{(s+1)^2}.$$

Using

$$\begin{aligned} \frac{1}{s+1} &= \mathcal{L}\{\mathrm{e}^{-t}\} \\ \frac{1}{(s+1)^2} &= -\frac{d}{ds}\left(\frac{1}{s+1}\right) = -\frac{d}{ds}\mathcal{L}\{\mathrm{e}^{-t}\} = \mathcal{L}\{t\mathrm{e}^{-t}\}, \end{aligned}$$

we further derive

$$X_1(s) = 2\mathcal{L}\{\mathrm{e}^{-t}\} - \mathcal{L}\{t\mathrm{e}^{-t}\} = \mathcal{L}\{2\mathrm{e}^{-t} - t\mathrm{e}^{-t}\}.$$

Therefore

$$x_1(t) = (2-t)\mathrm{e}^{-t}.$$

Now

$$\begin{aligned} X_2(s) &= (s+2)X_1(s) - 2 = (s+2)\frac{2s+1}{(s+1)^2} - 2 = \frac{s}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2} \\ &= \mathcal{L}\{\mathrm{e}^{-t}\} - \mathcal{L}\{t\mathrm{e}^{-t}\} = \mathcal{L}\{(1-t)\mathrm{e}^{-t}\}, \end{aligned}$$

and $x_2(t) = (1-t)\mathrm{e}^{-t}$. The final solution is

$$x(t) = \mathrm{e}^{-t} \begin{bmatrix} 2 - t \\ 1 - t \end{bmatrix}.$$

Problem 2. (35 points)
Transforming the second-order differential equation

$$y''(t) - 4y'(t) + 5y(t) = 0$$

into a system of first-order differential equations, find its solution that satisfies

$$y(\pi) = 0, \qquad y'(\pi) = -1.$$

Solution:
Introducing

$$x_1(t) = y(t) \qquad \text{and} \qquad x_2(t) = y'(t),$$

the differential equation becomes

$$x_2'(t) = -5x_1(t) + 4x_2(t).$$

Therefore we obtain the following initial-value problem

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad \begin{bmatrix} x_1(\pi) \\ x_2(\pi) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

The system matrix

$$A = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}$$

has the characteristic polynomial

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -5 & 4-\lambda \end{bmatrix} = -\lambda(4-\lambda) + 5 = \lambda^2 - 4\lambda + 5,$$

with the roots $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$ as eigenvalues of A. For determining $x^1(t)$ and $x^2(t)$, it is sufficient to consider just $\lambda_1 = 2 + i$.
A complex eigenvector $v = [v_1, v_2]^\top$ that corresponds to λ_1 satisfies $(A - \lambda_1 I)v = 0$, i.e.

$$\begin{bmatrix} -2-i & 1 \\ -5 & 2-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From the first equation we obtain $v_2 = (2+i)v_1$. Thus, the vector v has the form

$$v = \begin{bmatrix} v_1 \\ (2+i)v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 2+i \end{bmatrix}$$

and we can choose $v_1 = 1$. Then a complex-valued solution of the system is

$$\phi(t) = \mathrm{e}^{(2+i)t} \begin{bmatrix} 1 \\ 2+i \end{bmatrix} = \mathrm{e}^{2t}(\cos t + i \sin t) \begin{bmatrix} 1 \\ 2+i \end{bmatrix}$$

$$= \mathrm{e}^{2t} \begin{bmatrix} \cos t + i \sin t \\ 2\cos t - \sin t + i(2\sin t + \cos t) \end{bmatrix}.$$

Taking $x^1(t) = \mathrm{Re}(\phi(t))$ and $x^2(t) = \mathrm{Im}(\phi(t))$, we obtain a general solution of the form

$$x(t) = c_1 \mathrm{e}^{2t} \begin{bmatrix} \cos t \\ 2\cos t - \sin t \end{bmatrix} + c_2 \mathrm{e}^{2t} \begin{bmatrix} \sin t \\ 2\sin t + \cos t \end{bmatrix}.$$

The initial condition $x(\pi) = [0, -1]^\top$ implies

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} = x(\pi) = c_1 \mathrm{e}^{2\pi} \begin{bmatrix} -1 \\ -2 \end{bmatrix} + c_2 \mathrm{e}^{2\pi} \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Hence $c_1 = 0$ and $c_2 = \mathrm{e}^{-2\pi}$. The solution of the initial value problem is

$$x(t) = \mathrm{e}^{2(t-\pi)} \begin{bmatrix} \sin t \\ 2\sin t + \cos t \end{bmatrix},$$

while the solution of the second-order differential equation with $y(\pi) = 0$, $y'(\pi) = -1$, is

$$y(t) = x_1(t) = \mathrm{e}^{2(t-\pi)} \sin t.$$

Problem 3. (30 points)
For the matrix

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

determine e^{At}.

Solution:
We will determine e^{At} from the relation

$$\mathrm{e}^{At} = X(t)X(0)^{-1},$$

where $X(t)$ is a fundamental matrix solution of the system $\dot{x}(t) = Ax(t)$. The characteristic polynomial of the system matrix A is

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 \\ -1 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda).$$

The matrix A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$.
In order to find $x^1(t)$, first we need to find a vector $v = [v_1, v_2]^\top$ such that $(A - \lambda_1 I)v = 0$, i.e.

$$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The second equation implies $v_1 = v_2$. We can choose $v = [1, 1]^\top$ and obtain

$$x^1(t) = \mathrm{e}^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For finding $x^2(t)$, we search for a vector $v = [v_1, v_2]^\top$ such that $(A - \lambda_2 I)v = 0$, i.e.

$$\begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From both equations we get $v_1 = 0$. Choosing $v_2 = 1$, we obtain

$$x^2(t) = \mathrm{e}^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The fundamental matrix solution of the system $\dot{x}(t) = Ax(t)$ is

$$X(t) = \begin{bmatrix} \mathrm{e}^t & 0 \\ \mathrm{e}^t & \mathrm{e}^{2t} \end{bmatrix}.$$

The inverse matrix of

$$X(0) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

is

$$X(0)^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Therefore

$$\mathrm{e}^{At} = \begin{bmatrix} \mathrm{e}^t & 0 \\ \mathrm{e}^t & \mathrm{e}^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^t & 0 \\ \mathrm{e}^t - \mathrm{e}^{2t} & \mathrm{e}^{2t} \end{bmatrix}.$$

3.4.2 practice problems for third exam

1

Exercise 6 / Section 3.12

$$\dot{x} = \begin{bmatrix} -1 & -1 & -2 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} , \qquad x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution:

$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & -1 & -2 \\ 1 & 1-\lambda & 1 \\ 2 & 1 & 3-\lambda \end{vmatrix} \xrightarrow{R_3+R_1} \begin{vmatrix} 1-\lambda & 0 & 1-\lambda \\ 1 & 1-\lambda & 1 \\ 2 & 1 & 3-\lambda \end{vmatrix} .$$

$$= (1-\lambda)^2(3-\lambda) + \cancel{(1-\lambda)} - 2(1-\lambda)^2 - \cancel{(1-\lambda)} = (1-\lambda)^2(3-\lambda-2) = (1-\lambda)^3 = 0$$

System matrix has one eigenvalue $\lambda = 1$ of multiplicity 3.

1) finding $x^1(t)$:

$$(A-\lambda I)v = \begin{bmatrix} -2 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \left.\begin{aligned} v_1 + v_3 &= 0 \\ 2v_1 + v_2 + 2v_3 &= 0 \end{aligned}\right\} \rightarrow \begin{aligned} v_3 &= -v_1 \\ v_2 &= 0 \end{aligned}$$

choose $v^1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, then $x^1(t) = e^{\lambda t} v^1 = e^{t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

2) finding $x^2(t)$:

$$(A-\lambda I)^2 v = \begin{bmatrix} -2 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} v = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2

$v_1 + v_3 = 0 \quad \rightarrow \quad v_1 = -v_3$

v_2 - arbitrary , choose $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$(A - \lambda I)\, v = \begin{bmatrix} -2 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x^2(t) = e^{\lambda t}\left(v + t(A - \lambda I)v\right) = e^{t}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right) = e^{t} \begin{bmatrix} -t \\ 1 \\ t \end{bmatrix}$$

3) finding $x^3(t)$:

$$(A - \lambda I)^3 v = (A - \lambda I)^2 (A - \lambda I) v = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} v$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ v is arbitrary vector such that $(A - \lambda I)^2 v \neq 0$

choose $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $(A - \lambda I)^2 v = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

we need also $(A - \lambda I) v = \begin{bmatrix} -2 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

$$x^3(t) = e^{\lambda t}\left(v + t(A - \lambda I)v + \frac{t^2}{2}(A - \lambda I)^2 v\right)$$

$$x^3(t) = e^t\left(\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}-2\\1\\2\end{bmatrix} + \frac{t^2}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix}\right) = e^t\begin{bmatrix}-\frac{t^2}{2}-2t+1\\ t\\ \frac{t^2}{2}+2t\end{bmatrix}$$

4) fund. matrix solution

$$X(t) = e^t\begin{bmatrix}1 & -t & -\frac{t^2}{2}-2t+1\\ 0 & 1 & t\\ -1 & t & \frac{t^2}{2}+2t\end{bmatrix}$$

5) matrix $e^{At} = X(t)\,X(0)^{-1}$

$$X(0) = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ -1 & 0 & 0 & 0 & 0 & 1\end{array}\right] \xrightarrow{R_1+R_3} \left[\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 1 & 0 & 1\end{array}\right] \xrightarrow{-R_3+R_1} \left[\begin{array}{ccc|ccc}\multicolumn{3}{c|}{\overbrace{\hphantom{1\ 0\ 0}}^{I}} & \multicolumn{3}{c}{\overbrace{\hphantom{0\ 0\ -1}}^{X(0)^{-1}}}\\ 1 & 0 & 0 & 0 & 0 & -1\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 1 & 0 & 1\end{array}\right]$$

$$e^{At} = X(t)\,X(0)^{-1} = e^t\begin{bmatrix}1 & -t & -\frac{t^2}{2}-2t+1\\ 0 & 1 & t\\ -1 & t & \frac{t^2}{2}+2t\end{bmatrix}\cdot\begin{bmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$$

$$= e^t\begin{bmatrix}-\frac{t^2}{2}-2t+1 & -t & -\frac{t^2}{2}-2t\\ t & 1 & t\\ \frac{t^2}{2}+2t & t & \frac{t^2}{2}+2t+1\end{bmatrix}$$

b) Since $t_0 = 0$ and $x^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, the solution of the IVP is

$$x(t) = e^{At} \int_0^t e^{-As} f(s)\, ds \quad , \quad \text{with} \quad f(s) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{s}.$$

$$e^{-As} f(s) = e^{-s} \begin{bmatrix} -\frac{s^2}{2} + 2s + 1 & s & -\frac{s^2}{2} + 2s \\ -s & 1 & -s \\ \frac{s^2}{2} - 2s & -s & \frac{s^2}{2} - 2s + 1 \end{bmatrix} \cdot e^{s} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{s^2}{2} + 2s + 1 \\ -s \\ \frac{s^2}{2} - 2s \end{bmatrix}$$

$$\int_0^t e^{-As} f(s)\, ds = \begin{bmatrix} -\frac{t^3}{6} + t^2 + t \\ -\frac{t^2}{2} \\ \frac{t^3}{6} - t^2 \end{bmatrix} = t \begin{bmatrix} -\frac{t^2}{6} + t + 1 \\ -\frac{t}{2} \\ \frac{t^2}{6} - t \end{bmatrix}$$

$$x(t) = e^{At} \int_0^t e^{-As} f(s)\, ds = t e^{t} \begin{bmatrix} -\frac{t^2}{2} - 2t + 1 & -t & -\frac{t^2}{2} - 2t \\ t & 1 & t \\ \frac{t^2}{2} + 2t & t & \frac{t^2}{2} + 2t + 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{t^2}{6} + t + 1 \\ -\frac{t}{2} \\ \frac{t^2}{6} - t \end{bmatrix}$$

$$= t e^{t} \begin{bmatrix} -\frac{t^2}{6} - t + 1 \\ \frac{t}{2} \\ \frac{t^2}{6} + t \end{bmatrix}$$

Exercise 8 / Section 3.12 Find the solution of the initial-value problem

$$y''' + y' = \sec t \cdot \tan t$$
$$y(0) = y'(0) = y''(0) = 0$$

Solution: First we transform the differential equation into a system. Let

$$x_1(t) = y(t), \quad x_2(t) = y'(t), \quad x_3(t) = y''(t).$$

Then $y''' + y' = x_3' + x_2 = \sec t \cdot \tan t$, and

$$x_1'(t) = x_2(t), \qquad x_1(0) = 0$$
$$x_2'(t) = x_3(t), \qquad x_2(0) = 0$$
$$x_3'(t) = -x_2(t) + \sec t \cdot \tan t, \qquad x_3(0) = 0.$$

The matrix form of this system is

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \sec t \cdot \tan t \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix} = -\lambda^3 - \lambda = -\lambda(\lambda^2 + 1) = 0$$

Eigenvalues of the system matrix are $\lambda_1 = 0$, $\lambda_2 = i$, $\lambda_3 = -i$.

$\lambda_1 = 0$: $(A - \lambda_1 I)v = Av = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$v_2 = 0$
$v_3 = 0$
v_1 - arbitrary

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choose $v^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ – then $x^1(t) = e^{\lambda_1 t} v^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = i$:

$$(A - \lambda_2 I) v = \begin{bmatrix} -i & 1 & 0 \\ 0 & -i & 1 \\ 0 & -1 & -i \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \rightarrow & -iv_1 + v_2 = 0 \\ & v_1 = -iv_2 \\ \searrow & -iv_2 + v_3 = 0 \\ & v_3 = iv_2 \end{matrix}$$

$v = \begin{bmatrix} -iv_2 \\ v_2 \\ iv_2 \end{bmatrix} = v_2 \begin{bmatrix} -i \\ 1 \\ i \end{bmatrix}$, choose $v^2 = \begin{bmatrix} -i \\ 1 \\ i \end{bmatrix}$

complex-valued solution is $\phi(t) = e^{\lambda_2 t} v^2 = e^{it} \begin{bmatrix} -i \\ 1 \\ i \end{bmatrix}$

$$= (\cos t + i \sin t) \begin{bmatrix} -i \\ 1 \\ i \end{bmatrix} = \begin{bmatrix} \sin t - i \cos t \\ \cos t + i \sin t \\ -\sin t + i \cos t \end{bmatrix}$$

$x^2(t) = \operatorname{Re}(\phi(t)) = \begin{bmatrix} \sin t \\ \cos t \\ -\sin t \end{bmatrix}$, $x^3(t) = \operatorname{Im}(\phi(t)) = \begin{bmatrix} -\cos t \\ \sin t \\ \cos t \end{bmatrix}$

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fundamental matrix solution is

$$X(t) = \begin{bmatrix} 1 & \sin t & -\cos t \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}$$

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$$X(0) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{X(0)^{-1}}$$

$$e^{At} = X(t)\, X(0)^{-1} = \begin{bmatrix} 1 & \sin t & -\cos t \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \sin t & 1-\cos t \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}$$

$$x(t) = e^{A(t-t_0)} x^0 + e^{At} \int_{t_0}^{t} e^{-As} f(s)\, ds = e^{At} \int_0^t e^{-As} f(s)\, ds$$

$$e^{-As} f(s) = \begin{bmatrix} 1 & -\sin s & 1-\cos s \\ 0 & \cos s & -\sin s \\ 0 & \sin s & \cos s \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \sec s \cdot \tan s \end{bmatrix} = \begin{bmatrix} 1-\cos s \\ -\sin s \\ \cos s \end{bmatrix} \cdot \sec s \cdot \tan s$$

$$\int_0^t (1-\cos s) \sec s \cdot \tan s\, ds = \int_0^t \sec s \cdot \tan s\, ds - \int_0^t \tan s\, ds$$

$$= \left(\sec s + \ln|\cos s|\right)\Big|_0^t = \sec t - 1 + \ln|\cos t|$$

$$-\int_0^t \sin s \cdot \sec s \cdot \tan s\, ds = \int_0^t (1 - \sec^2 s)\, ds = t - \tan s\Big|_0^t = t - \tan t$$

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$$\int_0^t \cos s \cdot \sec s \cdot \tan s \, ds = \int_0^t \tan s \, ds = -\ln|\cos s| \Big|_0^t = -\ln|\cos t|$$

$$\Rightarrow \int_0^t e^{-As} f(s) ds = \begin{bmatrix} \sec t - 1 + \ln|\cos t| \\ t - \tan t \\ -\ln|\cos t| \end{bmatrix}$$

$$e^{At} \int_0^t e^{-As} f(s) ds = \begin{bmatrix} 1 & \sin t & 1-\cos t \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix} \cdot \begin{bmatrix} \sec t - 1 + \ln|\cos t| \\ t - \tan t \\ -\ln|\cos t| \end{bmatrix}$$

$$= \begin{bmatrix} -1 + t \cdot \sin t + \cos t (1 + \ln|\cos t|) \\ t \cos t - \sin t (1 + \ln|\cos t|) \\ -t \sin t + \sin t \cdot \tan t - \cos t \cdot \ln|\cos t| \end{bmatrix}$$

* Solution of the starting IVP is

$$y(t) = x_1(t) = -1 + t \sin t + \cos t (1 + \ln|\cos t|).$$

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* THE SOLUTION IN THE BOOK SEEMS TO BE INCORRECT *

Exercise 7/Section 3.13 Using Laplace transforms solve

$$\dot{x} = \begin{bmatrix} 4 & 5 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 4e^t \cos t \\ 0 \end{bmatrix}, \qquad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

<u>Solution</u>: Matrix $sI - A$ is $\begin{bmatrix} s-4 & -5 \\ 2 & s+2 \end{bmatrix}$.

The system for finding $X(s) = \mathcal{L}\{x(t)\}$ is

$$(sI - A)\,X(s) = F(s) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{where} \quad F(s) = \begin{bmatrix} \mathcal{L}\{4e^t \cos t\} \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} s-4 & -5 \\ 2 & s+2 \end{bmatrix} \cdot \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} F_1(s) + 1 \\ 1 \end{bmatrix}, \qquad F_1(s) = \mathcal{L}\{4e^t \cos t\} = 4 \cdot \frac{s-1}{(s-1)^2 + 1}$$

$$\begin{aligned} (s-4)\,X_1(s) - 5X_2(s) &= F_1(s) + 1 \\ 2X_1(s) + (s+2)\,X_2(s) &= 1 \end{aligned} \quad \Rightarrow \quad X_1(s) = \frac{1}{2}\big(1 - (s+2)X_2(s)\big)$$

$$\frac{s-4}{2}\big(1 - (s+2)X_2(s)\big) - 5X_2(s) = F_1(s) + 1 \qquad (\text{mult. by } 2)$$

$$-(s-4)(s+2)\,X_2(s) - 10\,X_2(s) = 2\big(F_1(s) + 1\big) - (s-4)$$

$$X_2(s)\big(-s^2 - 2s + 4s + 8 - 10\big) = 2\big(F_1(s) + 1\big) - (s-4)$$

$$X_2(s)\big(-s^2 + 2s - 2\big) = 2\big(F_1(s) + 1\big) - (s-4)$$

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$$X_2(s) = \frac{s-4}{s^2-2s+2} - 2\,\frac{F_1(s)+1}{s^2-2s+2} = \frac{s-6}{s^2-2s+2} - 2\,\frac{F_1(s)}{s^2-2s+2}$$

Notice that s^2-2s+2 has complex roots $1 \pm i$. Therefore we will use $s^2-2s+2 = (s-1)^2+1$.

The first term in $X_2(s)$ is $\frac{s-6}{s^2-2s+2}$. It can be transformed as

$$\frac{s-6}{s^2-2s+2} = \frac{s-1}{(s-1)^2+1} - \frac{5}{(s-1)^2+1} = \mathcal{L}\{e^t\cos t\} - 5\,\mathcal{L}\{e^t \sin t\}.$$

Continue with the second term in $X_2(s)$:

$$-2\,\frac{F_1(s)}{s^2-2s+2} = -2\cdot\frac{1}{(s-1)^2+1}\cdot\frac{4(s-1)}{(s-1)^2+1} = -8\,\frac{s-1}{((s-1)^2+1)^2}$$

From $$\frac{s}{(s^2+1)^2} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\frac{1}{2}\frac{d}{ds}\mathcal{L}\{\sin t\} = -\frac{1}{2}\mathcal{L}\{-t\sin t\}$$
$$= \mathcal{L}\left\{\frac{t}{2}\sin t\right\},$$

we further derive $$-8\,\frac{s-1}{((s-1)^2+1)^2} = -8\,\mathcal{L}\left\{e^t\cdot\frac{t}{2}\sin t\right\}$$
$$= \mathcal{L}\{-4te^t\sin t\}.$$

Then $X_2(s) = \mathcal{L}\{e^t\cos t - 5e^t\sin t - 4te^t\sin t\} = \mathcal{L}\{x_2(t)\}$.

$$X_1(s) = \frac{1}{2}\Big(1-(s+2)X_2(s)\Big) = \frac{1}{2}\left(1 - \frac{(s+2)(s-6)}{s^2-2s+2} + 2(s+2)\frac{F_1(s)}{s^2-2s+2}\right)$$

$$X_1(s) = \frac{1}{2}\left(\frac{s^2-2s+2-(s^2-4s-12)}{s^2-2s+2} + 2(s+2)\frac{4(s-1)}{((s-1)^2+1)^2} \right)$$

$$= \frac{s+7}{(s-1)^2+1} + 4\frac{(s-1)(s+2)}{((s-1)^2+1)^2}$$

The first term in $X_1(s)$ we write as

$$\frac{s+7}{(s-1)^2+1} = \frac{s-1}{(s-1)^2+1} + \frac{8}{(s-1)^2+1} = \mathcal{L}\{e^t\cos t\} + 8\mathcal{L}\{e^t\sin t\}.$$

The second term we transform into

$$4\frac{(s-1)(s+2)}{((s-1)^2+1)^2} = 4\frac{(s-1)(s-1+3)}{((s-1)^2+1)^2} = 4\frac{(s-1)^2}{((s-1)^2+1)^2} + 12\frac{s-1}{((s-1)^2+1)^2}$$

$$= 4\frac{1}{(s-1)^2+1} - 4\frac{1}{((s-1)^2+1)^2} + 12\frac{s-1}{((s-1)^2+1)^2}$$

$$= 4\mathcal{L}\{e^t\sin t\} - 4\mathcal{L}\{\tfrac{1}{2}e^t(\sin t - t\cos t)\}$$

$$+ 12\mathcal{L}\{e^t\cdot\tfrac{1}{2}t\sin t\}$$

Here we have used $\frac{1}{(s^2+1)^2} = \mathcal{L}\{\sin t * \sin t\} = \mathcal{L}\{\frac{1}{2}(\sin t - t\cos t)\}$

and thus $\frac{1}{((s-1)^2+1)^2} = \mathcal{L}\{\frac{1}{2}e^t(\sin t - t\cos t)\}.$

Finally, $X_1(s) = \mathcal{L}\{e^t\cos t + 8e^t\sin t + 4e^t\sin t - 2e^t(\sin t - t\cos t)$

$+ 6e^t t\sin t\} =$

$= \mathcal{L}\{10e^t\sin t + e^t\cos t + 6te^t\sin t + 2te^t\cos t\} = \mathcal{L}\{x_1(t)\}$

3.5 Final exam, Dec 18, 2019

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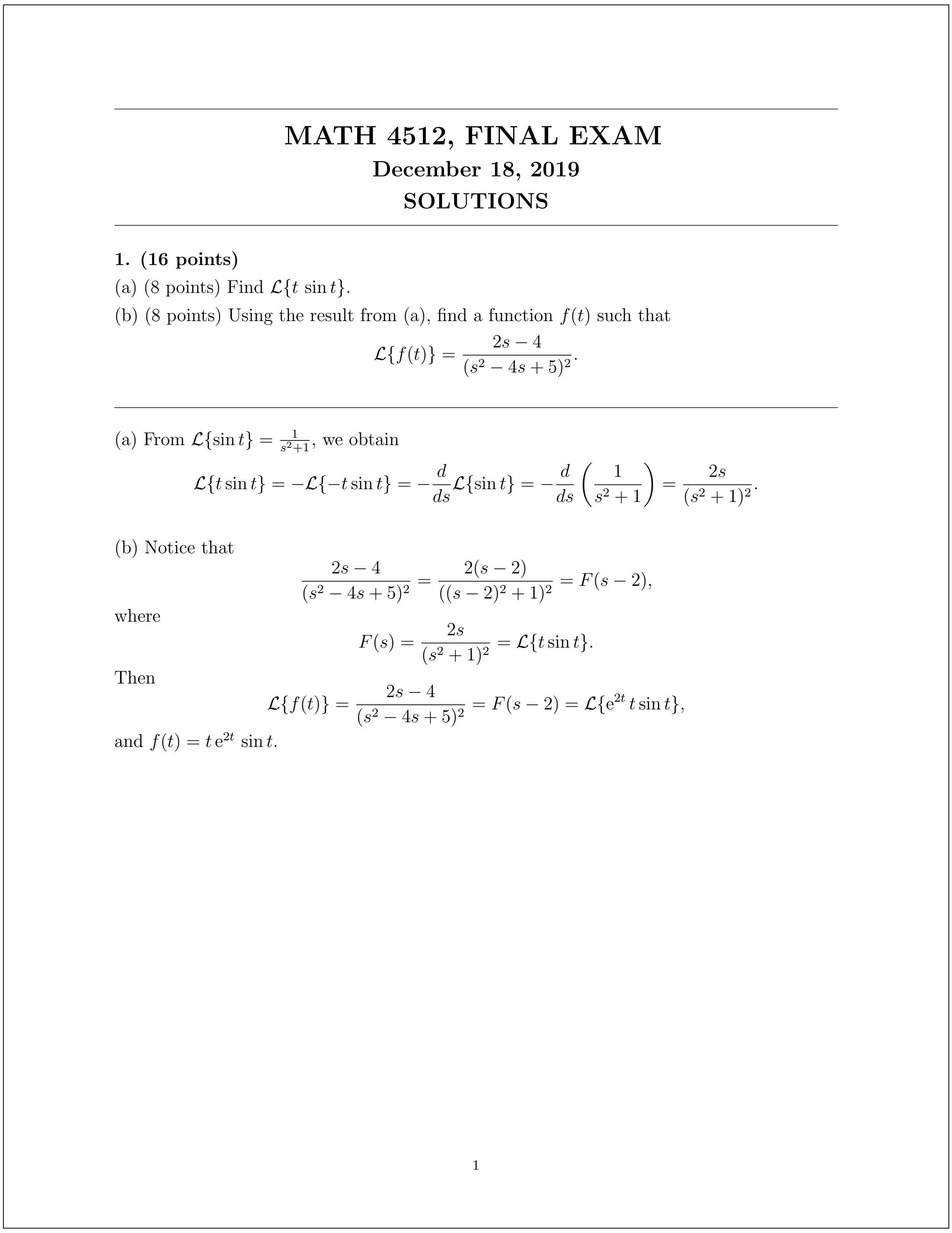
3.5.1 Key solution



MATH 4512, FINAL EXAM

December 18, 2019

SOLUTIONS

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1. (16 points)

(a) (8 points) Find $\mathcal{L}\{t\,\sin t\}$.

(b) (8 points) Using the result from (a), find a function $f(t)$ such that

$$\mathcal{L}\{f(t)\} = \frac{2s-4}{(s^2-4s+5)^2}.$$

(a) From $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$, we obtain

$$\mathcal{L}\{t\sin t\} = -\mathcal{L}\{-t\sin t\} = -\frac{d}{ds}\mathcal{L}\{\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}.$$

(b) Notice that

$$\frac{2s-4}{(s^2-4s+5)^2} = \frac{2(s-2)}{((s-2)^2+1)^2} = F(s-2),$$

where

$$F(s) = \frac{2s}{(s^2+1)^2} = \mathcal{L}\{t\sin t\}.$$

Then

$$\mathcal{L}\{f(t)\} = \frac{2s-4}{(s^2-4s+5)^2} = F(s-2) = \mathcal{L}\{\mathrm{e}^{2t}\,t\sin t\},$$

and $f(t) = t\,\mathrm{e}^{2t}\,\sin t$.

2. (18 points)

(a) (3 points) Write an initial-value problem describing vibrations of a small object of mass 1 kg attached to a spring with spring constant 9 N/m, and immersed in a viscous medium with damping constant 6 N s/m. At time $t = 0$, the mass, which is hanging in rest, is acted upon by an external force $F(t) = \cos t$ N.

(b) (8 points) Find a particular solution $\psi(t)$ of the differential equation from (a).

(c) (7 points) Solve the initial-value problem from (a).

(a) Here $m = 1$, $k = 9$, $c = 6$, and $F(t) = \cos t$. The IVP describing position y of this object in dependence of time t, with initial conditions $y(0) = y'(0) = 0$, is

$$y''(t) + 6y'(t) + 9y(t) = \cos t, \qquad y(0) = y'(0) = 0.$$

(b) The characteristic equation for $y''(t) + 6y'(t) + 9y(t) = 0$ is $r^2 + 6r + 9 = (r + 3)^2 = 0$ has a double root $r = -3$.

We will use guessing for the particular solution $\phi(t)$ of the complex-valued problem

$$y''(t) + 6y'(t) + 9y(t) = \mathrm{e}^{i\,t}.$$

Let $\phi(t) = A\,\mathrm{e}^{i\,t}$. Then $\phi'(t) = A\,i\,\mathrm{e}^{i\,t}$, $\phi''(t) = -A\,\mathrm{e}^{i\,t}$, and

$$\mathrm{e}^{i\,t} = \phi''(t) + 6\phi'(t) + 9\phi(t) = (-A + 6A\,i + 9A)\mathrm{e}^{i\,t} = (8 + 6i)A\,\mathrm{e}^{i\,t}.$$

We obtain

$$A = \frac{1}{8 + 6i} = \frac{8 - 6i}{100} = \frac{4}{50} - \frac{3}{50}i,$$

and

$$\phi(t) = \left(\frac{4}{50} - \frac{3}{50}i\right)\mathrm{e}^{i\,t} = \left(\frac{4}{50} - \frac{3}{50}i\right)(\cos t + i\sin t)$$

$$= \frac{4}{50}\cos t + \frac{3}{50}\sin t + i\left(\frac{4}{50}\sin t - \frac{3}{50}\cos t\right).$$

The particular solution $\psi(t)$ of the differential equation $y''(t) + 6y'(t) + 9y(t) = \cos t$ is

$$\psi(t) = \operatorname{Re}\phi(t) = \frac{4}{50}\cos t + \frac{3}{50}\sin t.$$

(c) The general solution is

$$y(t) = (c_1 + c_2\,t)\mathrm{e}^{-3t} + \frac{4}{50}\cos t + \frac{3}{50}\sin t.$$

From $y(0) = 0$ we obtain

$$0 = y(0) = c_1 + \frac{4}{50}, \qquad c_1 = -\frac{4}{50} = -\frac{2}{25}.$$

Since

$$y'(t) = c_2\, \mathrm{e}^{-3t} - 3(c_1 + c_2\, t)\mathrm{e}^{-3t} - \frac{4}{50}\sin t + \frac{3}{50}\cos t,$$

the initial condition $y'(0) = 0$ implies

$$0 = y'(0) = c_2 - 3c_1 + \frac{3}{50}, \qquad c_2 = 3c_1 - \frac{3}{50} = -\frac{15}{50} = -\frac{3}{10}.$$

The solution of the IVP is

$$y(t) = \left(-\frac{2}{25} - \frac{3}{10}\, t\right)\mathrm{e}^{-3t} + \frac{4}{50}\cos t + \frac{3}{50}\sin t.$$

3. (32 points) Consider the linear system of differential equations

$$\dot{x} = Ax, \qquad A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}.$$

(a) (5 points) Determine stability of all solutions to $\dot{x} = Ax$.

(b) (10 points) Find the general solution to $\dot{x} = Ax$.

(c) (10 points) Find e^{At}.

(d) (7 points) Solve the initial-value problem

$$\dot{x} = Ax, \qquad x(0) = \begin{bmatrix} 4 \\ -5 \\ 0 \end{bmatrix}.$$

(a) The characteristic polynomial of the matrix A is

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = (-1-\lambda) \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix}$$

$$= -(1+\lambda)(1 - 2\lambda + \lambda^2 + 4) = -(1+\lambda)(\lambda^2 - 2\lambda + 5).$$

Eigenvalues of the matrix A are $\lambda_1 = -1$, $\lambda_2 = 1 + 2i$, and $\lambda_3 = 1 - 2i$. Since both λ_2 and λ_3 have positive real part, all solutions of the system $\dot{x} = Ax$ are unstable.

(b) Eigenvector for $\lambda_1 = -1$ satisfies $(A - \lambda_1 I)v = 0$. Then

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & -2 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Adding second equation $2v_1 + 2v_2 - 2v_3 = 0$ and third equation $3v_1 + 2v_2 + 2v_3 = 0$, we obtain

$$5v_1 + 4v_2 = 0, \qquad v_1 = -\frac{4}{5}\, v_2.$$

From second equation it follows

$$v_3 = v_1 + v_2 = -\frac{4}{5}\, v_2 + v_2 = \frac{1}{5}\, v_2.$$

Every eigenvector corresponding to λ_1 has the form

$$v = \begin{bmatrix} -4/5 \\ 1 \\ 1/5 \end{bmatrix} v_2$$

and we can choose

$$v^1 = \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}.$$

For eigenvalue $\lambda_2 = 1 + 2i$, we solve $(A - \lambda_2 I)v = 0$, i.e.

$$\begin{bmatrix} -2-2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The first equation immediately gives $v_1 = 0$, while from the second it follows $v_3 = -iv_2$. Then

$$v = \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} v_2,$$

and we can choose

$$v = \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}.$$

The complex-valued solution $\mathrm{e}^{\lambda_2 t} v$ can be written as

$$\mathrm{e}^{\lambda_2 t} v = \mathrm{e}^{(1+2i)t} \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} = \mathrm{e}^{t}(\cos 2t + i \sin 2t) \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}$$

$$= \mathrm{e}^{t} \begin{bmatrix} 0 \\ \cos 2t \\ \sin 2t \end{bmatrix} + i\,\mathrm{e}^{t} \begin{bmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{bmatrix}.$$

The general solution to $\dot{x} = Ax$ is

$$x(t) = c_1\,\mathrm{e}^{-t} \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix} + c_2\,\mathrm{e}^{t} \begin{bmatrix} 0 \\ \cos 2t \\ \sin 2t \end{bmatrix} + c_3\,\mathrm{e}^{t} \begin{bmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{bmatrix}.$$

(c) The fundamental matrix solutions $X(t)$ for this system is

$$X(t) = \begin{bmatrix} -4\mathrm{e}^{-t} & 0 & 0 \\ 5\mathrm{e}^{-t} & \mathrm{e}^{t}\cos 2t & \mathrm{e}^{t}\sin 2t \\ \mathrm{e}^{-t} & \mathrm{e}^{t}\sin 2t & -\mathrm{e}^{t}\cos 2t \end{bmatrix}.$$

Then

$$X(0) = \begin{bmatrix} -4 & 0 & 0 \\ 5 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

From

$$\left[\begin{array}{ccc|ccc} -4 & 0 & 0 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_3]{-\frac{1}{4}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/4 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\xrightarrow[R_1+R_3]{-5R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/4 & 0 & 0 \\ 0 & 1 & 0 & 5/4 & 1 & 0 \\ 0 & 0 & 1 & -1/4 & 0 & -1 \end{array}\right]$$

we obtain

$$X(0)^{-1} = \begin{bmatrix} -1/4 & 0 & 0 \\ 5/4 & 1 & 0 \\ -1/4 & 0 & -1 \end{bmatrix}.$$

Finally,

$$e^{At} = X(t)X(0)^{-1} = \begin{bmatrix} e^{-t} & 0 & 0 \\ -\frac{5}{4}e^{-t} + \frac{5}{4}e^{t}\cos 2t - \frac{1}{4}e^{t}\sin 2t & e^{t}\cos 2t & -e^{t}\sin 2t \\ -\frac{1}{4}e^{-t} + \frac{5}{4}e^{t}\sin 2t + \frac{1}{4}e^{t}\cos 2t & e^{t}\sin 2t & e^{t}\cos 2t \end{bmatrix}.$$

(d) The initial-value problem

$$\dot{x} = Ax, \qquad x(0) = \begin{bmatrix} 4 \\ -5 \\ 0 \end{bmatrix}.$$

can be solved using the formula $x(t) = e^{At}x(0)$, or from the initial condition

$$\begin{bmatrix} 4 \\ -5 \\ 0 \end{bmatrix} = x(0) = c_1 \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -4c_1 \\ 5c_1 + c_2 \\ c_1 - c_3 \end{bmatrix}.$$

Then $c_1 = -1$, $c_2 = -5 - 5c_1 = 0$, and $c_3 = c_1 = -1$. The solution to the IVP is

$$x(t) = -e^{-t} \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix} - e^{t} \begin{bmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{bmatrix} = \begin{bmatrix} 4e^{-t} \\ -5e^{-t} - e^{t}\sin 2t \\ -e^{-t} + e^{t}\cos 2t \end{bmatrix}.$$

4. (22 points) Consider the autonomous nonlinear system of differential equations

$$\begin{aligned}\dot{x} &= 4y\\ \dot{y} &= 2x + xy^2.\end{aligned}$$

(a) (7 points) Find orbits of the system.

(b) (7 points) Determine stability of equilibrium solutions of the system.

(c) (8 points) Write the nonlinear system as

$$\dot{z} = Az + g(z), \qquad z = \begin{bmatrix} x \\ y \end{bmatrix},$$

and draw the phase portrait of $\dot{z} = Az$.

(a) The differential equation

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2x + xy^2}{4y} = \frac{x(2+y^2)}{4y}$$

is separable, and we can solve it in the following way:

$$\begin{aligned}\int \frac{y}{2+y^2}\,dy &= \frac{1}{4}\int x\,dx\\ \frac{1}{2}\ln(2+y^2) &= \frac{x^2}{8} + c_1\\ \ln(2+y^2) &= \frac{x^2}{4} + c_2\\ y^2 &= c\,\mathrm{e}^{x^2/4} - 2.\end{aligned}$$

The only equilibrium solution is $(0,0)$. Thus, the orbits of the given system are

- equilibrium point $(0,0)$,
- curves $y^2 = c\,\mathrm{e}^{x^2/4} - 2$, $c \neq 2$,
- four curves
  - (1) $y = \sqrt{2\,\mathrm{e}^{x^2/4} - 2}$, $x > 0$,
  - (2) $y = \sqrt{2\,\mathrm{e}^{x^2/4} - 2}$, $x < 0$,
  - (3) $y = -\sqrt{2\,\mathrm{e}^{x^2/4} - 2}$, $x > 0$,
  - (4) $y = -\sqrt{2\,\mathrm{e}^{x^2/4} - 2}$, $x < 0$.

(b) The nonlinear system in the matrix form is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ xy^2 \end{bmatrix}.$$

The characteristic polynomial of the system matrix

$$A = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$$

is

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 4 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 8.$$

The eigenvalues of A are $\lambda_1 = -\sqrt{8}$ and $\lambda_2 = \sqrt{8}$. Since one eigenvalue has positive real part, the equilibrium solution $(0,0)$ is unstable.

(c) In (b) we already derived the matrix form

$$\dot{z} = Az + g(z), \qquad z = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}.$$

Eigenvalues of A are $\lambda_1 = -\sqrt{8}$ and $\lambda_2 = \sqrt{8}$, and the equilibrium solution $(0,0)$ is saddle. In order to draw the phase portrait for $\dot{z} = Az$, we will determine eigenvectors corresponding to λ_1, λ_2.

From

$$(A - \lambda_1 I)v = \begin{bmatrix} \sqrt{8} & 4 \\ 2 & \sqrt{8} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

we obtain

$$\sqrt{8}\, v_1 + 4\, v_2 = 0, \qquad v_1 = -\frac{4}{\sqrt{8}}\, v_2 = -\sqrt{2}\, v_2,$$

and we can choose

$$v^1 = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}.$$

From

$$(A - \lambda_2 I)v = \begin{bmatrix} -\sqrt{8} & 4 \\ 2 & -\sqrt{8} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

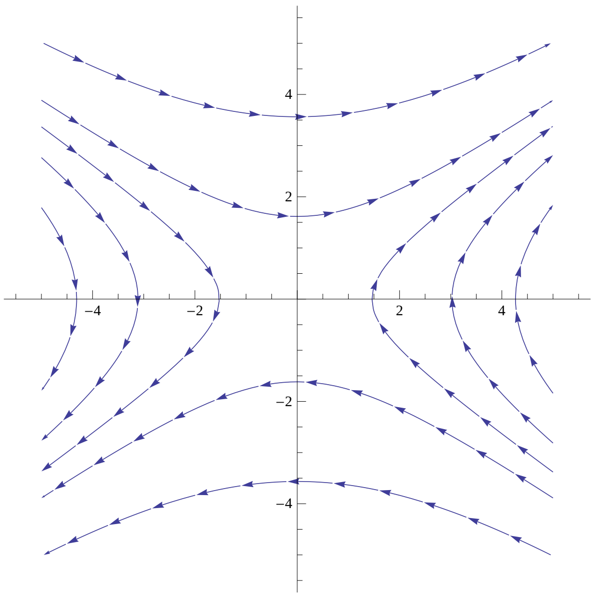
we obtain

$$-\sqrt{8}\, v_1 + 4\, v_2 = 0, \qquad v_1 = \frac{4}{\sqrt{8}}\, v_2 = \sqrt{2}\, v_2,$$

and we can choose

$$v^2 = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}.$$

The phase portrait of $\dot{z} = Az$:



5. (12 points) Determine whether the following statements are true or false. Explain your answer.

(a) (4 points) Initial-value problem

$$\frac{dy}{dt} = \mathrm{e}^t\,(y+1)^{2/3}, \qquad y(0) = -1,$$

has a unique solution $y(t) = -1$. True/False

(b) (4 points) Families of curves

$$y = c\,\tan x, \qquad y^2 + \sin^2 x = c,$$

are orthogonal. True/False

(c) (4 points) Vector-valued functions

$$x(t) = \begin{bmatrix} 3\mathrm{e}^t \\ -\mathrm{e}^t \\ \mathrm{e}^t \end{bmatrix}, \qquad y(t) = \begin{bmatrix} \sin t \\ \cos t \\ -\cos t \end{bmatrix}, \qquad z(t) = \begin{bmatrix} -\mathrm{e}^{-2t} \\ 1 - \mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} - 1 \end{bmatrix},$$

are linearly independent. True/False

(a) False.

Notice that a constant function $y(t) = -1$ is one solution to the IVP.
Let $f(t, y) = \mathrm{e}^t\,(y+1)^{2/3}$ and $y_0 = -1$. Though function f is continuous for all $t \in \mathbb{R}$ and all $y \in \mathbb{R}$, its partial derivative

$$\frac{\partial f}{\partial y} = \mathrm{e}^t\,\frac{2}{3}(y+1)^{-1/3}$$

is not continuous in any neighborhood of y_0. Thus this IVP has more than one solution.

(b) True.

Starting from

$$F(x, y, c) = c\tan x - y, \qquad F_y = -1, \qquad F_x = \frac{c}{\cos^2 x} = \frac{y}{\tan x}\frac{1}{\cos^2 x} = \frac{y}{\sin x \cos x},$$

we can derive

$$\frac{dy}{dx} = \frac{F_y}{F_x} = \frac{-\sin x \cos x}{y}$$

$$\int y\,dy = -\int \sin x \cos x\,dx$$

$$\frac{y^2}{2} = -\frac{\sin^2 x}{2} + c_1$$

$$y^2 + \sin^2 x = c.$$

(c) False.

Choose $t = 0$ and consider a zero linear combination $c_1x(0) + c_2y(0) + c_3z(0) = 0$:

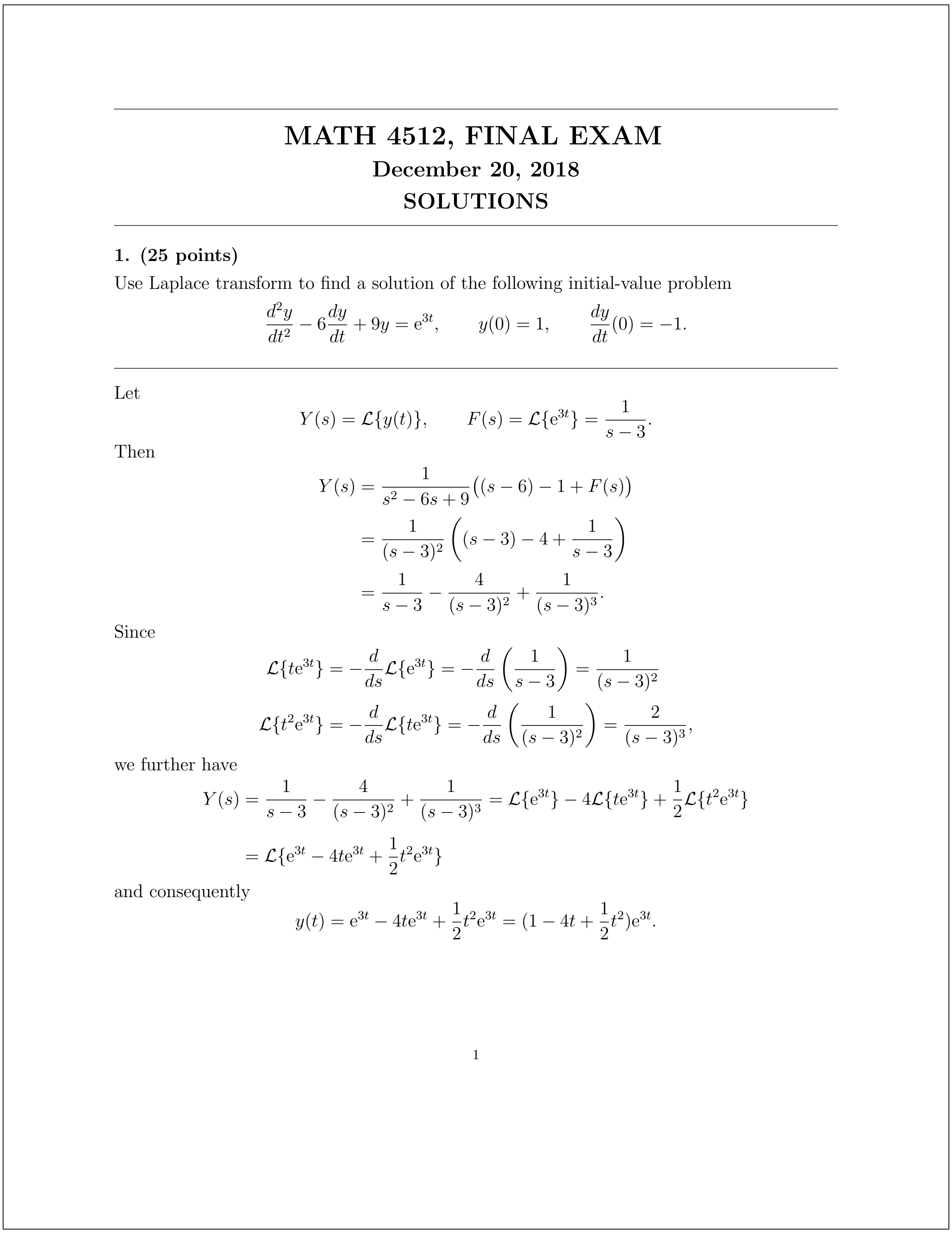
$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3c_1 - c_3 \\ -c_1 + c_2 \\ c_1 - c_2 \end{bmatrix}.$$

We can choose, for example, the following nonzero constants

$$c_1 = 1, \qquad c_2 = 1, \qquad c_3 = 3,$$

and obtain $c_1x(0) + c_2y(0) + c_3z(0) = 0$. This implies that the vector-valued functions $x(t), y(t), z(t)$ are linearly dependent.

3.5.2 practice final exam for 2018



MATH 4512, FINAL EXAM

December 20, 2018

SOLUTIONS

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1. (25 points)

Use Laplace transform to find a solution of the following initial-value problem

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = \mathrm{e}^{3t}, \qquad y(0) = 1, \qquad \frac{dy}{dt}(0) = -1.$$

Let

$$Y(s) = \mathcal{L}\{y(t)\}, \qquad F(s) = \mathcal{L}\{\mathrm{e}^{3t}\} = \frac{1}{s-3}.$$

Then

$$\begin{aligned} Y(s) &= \frac{1}{s^2 - 6s + 9}\big((s-6) - 1 + F(s)\big) \\ &= \frac{1}{(s-3)^2}\left((s-3) - 4 + \frac{1}{s-3}\right) \\ &= \frac{1}{s-3} - \frac{4}{(s-3)^2} + \frac{1}{(s-3)^3}. \end{aligned}$$

Since

$$\begin{aligned} \mathcal{L}\{t\mathrm{e}^{3t}\} &= -\frac{d}{ds}\mathcal{L}\{\mathrm{e}^{3t}\} = -\frac{d}{ds}\left(\frac{1}{s-3}\right) = \frac{1}{(s-3)^2} \\ \mathcal{L}\{t^2\mathrm{e}^{3t}\} &= -\frac{d}{ds}\mathcal{L}\{t\mathrm{e}^{3t}\} = -\frac{d}{ds}\left(\frac{1}{(s-3)^2}\right) = \frac{2}{(s-3)^3}, \end{aligned}$$

we further have

$$\begin{aligned} Y(s) = \frac{1}{s-3} - \frac{4}{(s-3)^2} + \frac{1}{(s-3)^3} &= \mathcal{L}\{\mathrm{e}^{3t}\} - 4\mathcal{L}\{t\mathrm{e}^{3t}\} + \frac{1}{2}\mathcal{L}\{t^2\mathrm{e}^{3t}\} \\ = \mathcal{L}\{\mathrm{e}^{3t} - 4t\mathrm{e}^{3t} + \frac{1}{2}t^2\mathrm{e}^{3t}\} \end{aligned}$$

and consequently

$$y(t) = \mathrm{e}^{3t} - 4t\mathrm{e}^{3t} + \frac{1}{2}t^2\mathrm{e}^{3t} = (1 - 4t + \frac{1}{2}t^2)\mathrm{e}^{3t}.$$

2. (25 points)

Transform the differential equation

$$\frac{d^2u}{dt^2} + \frac{du}{dt} - 2u = 0$$

into a system of differential equations

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \tag{1}$$

(a) Determine stability of all solutions of (1).

(b) Find a general solution of (1).

(c) Find equilibrium points of (1) and examine their stability.

(d) Draw the phase portrait of (1).

Introducing $x(t) = u(t)$, $y(t) = \dot{u}(t)$, we have that

$$\begin{aligned} \dot{x}(t) &= y(t) \\ \dot{y}(t) &= 2x(t) - y(t). \end{aligned}$$

Therefore

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}.$$

(a) The characteristic polynomial of the matrix A is

$$p(\lambda) = \det \begin{bmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{bmatrix} = \lambda(1+\lambda) - 2 = \lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1).$$

The eigenvalues of A are $\lambda_1 = -2$ and $\lambda_2 = 1$. Since one eigenvalue has positive real part, all solutions of (1) are unstable.

(b) First we will determine eigenvectors corresponding to $\lambda_1 = -2$ and $\lambda_2 = 1$.
From

$$(A - \lambda_1 I)v = (A + 2I)v = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

we can choose $v^1 = [1, -2]^\top$, while from

$$(A - \lambda_2 I)v = (A - I)v = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

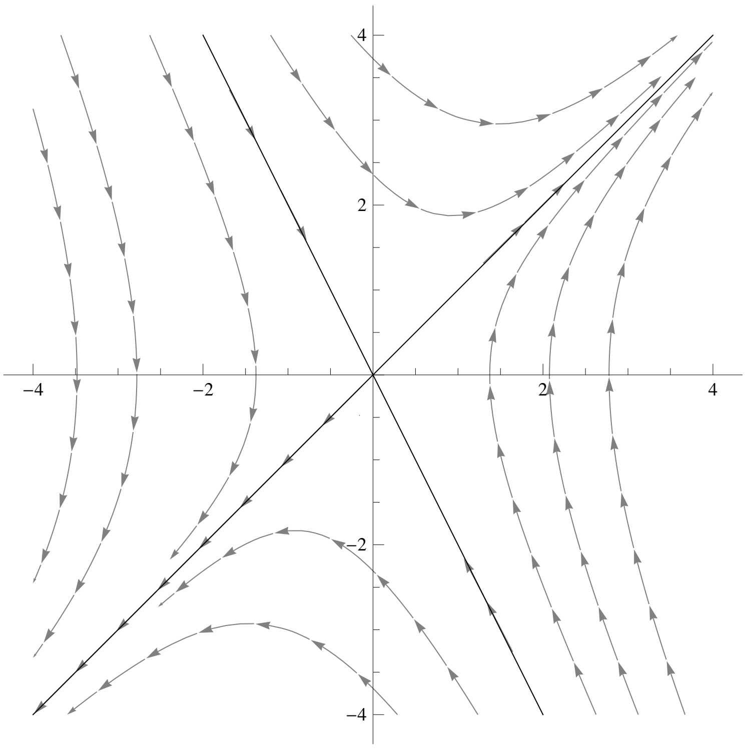
we can choose $v^2 = [1, 1]^\top$. The general solution of (1) has the form

$$c_1 \mathrm{e}^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \mathrm{e}^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(c) The system (1) has only one equilibrium point $[0,0]^\top$. This constant solution is saddle since

$$-2 = \lambda_1 < 0 < \lambda_2 = 1.$$

(d) The phase portrait of (1):



3. (25 points)

Consider the system of nonlinear differential equations

$$\begin{aligned}\dot{x} &= y + 3yx^2\\ \dot{y} &= 4x.\end{aligned}$$

(a) Find orbits of the system.

(b) Find orthogonal trajectories of the family of curves obtained in (a).

(a) Consider the differential equation

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{4x}{y+3yx^2} = \frac{4x}{y(1+3x^2)}.$$

Since it is separable, we can solve it in the following way:

$$\begin{aligned} y\frac{dy}{dx} &= \frac{4x}{1+3x^2}\\ \frac{d}{dx}\frac{y^2}{2} &= \frac{4x}{1+3x^2}\\ \frac{y^2}{2} &= \int \frac{4x}{1+3x^2}dx = \frac{4}{6}\int \frac{ds}{s} = \frac{2}{3}\ln|1+3x^2| + c_1\\ y^2 &= \frac{4}{3}\ln(1+3x^2) + c. \end{aligned}$$

The only equilibrium value is $[0,0]^\top$. Thus, the orbits of the given system are

- the equilibrium point $[0,0]^\top$,
- the curve $y^2 = \frac{4}{3}\ln(1+3x^2)$,
- the curves $y^2 = \frac{4}{3}\ln(1+3x^2) + c$, $c \neq 0$.

(b) Let $F(x,y,c) = \frac{4}{3}\ln(1+3x^2) - y^2 + c$. From

$$F_x = \frac{8x}{1+3x^2}, \qquad F_y = -2y,$$

we obtain that the orthogonal trajectories y need to satisfy

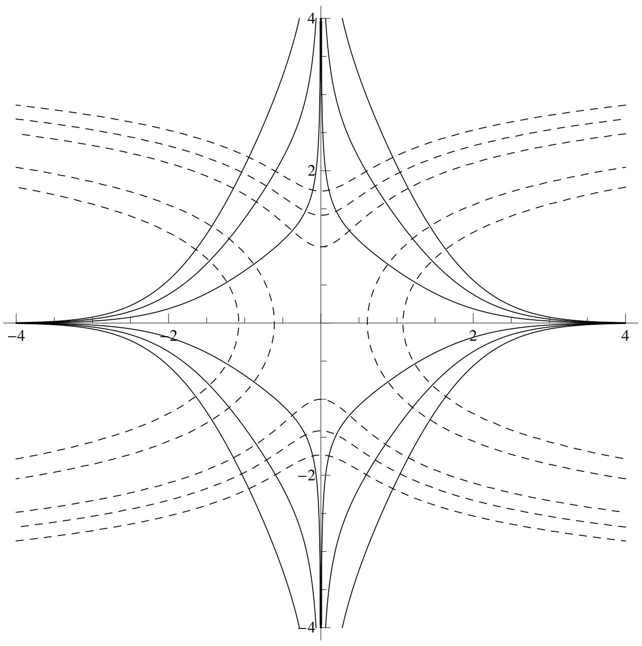
$$\frac{dy}{dx} = \frac{F_y}{F_x} = -\frac{y(1+3x^2)}{4x}.$$

This is a separable problem and we solve it as follows:

$$\begin{aligned}
\frac{1}{y}\frac{dy}{dx} &= -\frac{1+3x^2}{4x} \\
\frac{d}{dx}(\ln|y|) &= -\frac{1+3x^2}{4x} \\
\ln|y| &= -\int \frac{1+3x^2}{4x}dx = -\frac{1}{4}\int\frac{dx}{x} - \frac{3}{4}\int x dx \\
\ln|y| &= -\frac{1}{4}\ln|x| - \frac{3x^2}{8} + c_1 \\
|y| &= c|x|^{-1/4}\exp(-3x^2/8).
\end{aligned}$$

Orthogonal trajectories are the curves that satisfy

$$|y| = c|x|^{-1/4}\exp(-3x^2/8).$$



Orbits $y^2 = \frac{4}{3}\ln(1+3x^2) + c$ (dashed) and
orthogonal trajectories $|y| = c|x|^{-1/4}\exp(-3x^2/8)$ (solid).

4. (25 points)

The charge $Q(t)$ on the capacitor within closed electric circuit satisfies the differential equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t),$$

with an inductance L, a resistance R, a capacitance C, and a voltage source $E(t)$ at time t. If $L = 1\text{H}$, $R = 2\Omega$, $C = 0.2\text{F}$, and $E(t) = 17\cos(2t)\text{V}$, find charge $Q(t)$ that satisfies

$$Q(0) = 0\,\text{C}, \quad \frac{dQ}{dt}(0) = 9\,\text{A}.$$

Useful identities and properties:

$$\begin{aligned} \sin(2\theta) &= 2\sin\theta\cos\theta, & \int e^{at}\sin(bt)dt &= \frac{e^{at}}{a^2+b^2}(a\sin(bt) - b\cos(bt)) + c, \\ \cos(2\theta) &= 2\cos^2\theta - 1, & \int e^{at}\cos(bt)dt &= \frac{e^{at}}{a^2+b^2}(a\cos(bt) + b\sin(bt)) + c. \end{aligned}$$

With the given data, we are solving the following initial-value problem

$$\frac{d^2Q}{dt^2} + 2\frac{dQ}{dt} + 5Q = 17\cos(2t), \qquad Q(0) = 0, \quad \frac{dQ}{dt}(0) = 9.$$

The characteristic equation

$$r^2 + 2r + 5 = 0$$

has complex roots $r_1 = -1 + 2i$, $r_2 = -1 - 2i$. The functions

$$y_1(t) = e^{-t}\cos(2t), \qquad y_2(t) = e^{-t}\sin(2t),$$

form the fundamental set of solutions. The Wronskian is

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = 2e^{-2t}.$$

Now we search a particular solution ψ in the form

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

With $E(t) = 17\cos(2t)$, we obtain

$$\begin{aligned} u_1(t) &= -\int \frac{E(t)y_2(t)}{W[y_1,y_2](t)}dt = -\frac{17}{2}\int e^t\sin(2t)\cos(2t)dt = -\frac{17}{4}\int e^t\sin(4t)dt \\ &= -\frac{1}{4}e^t(\sin(4t) - 4\cos(4t)) \\ u_2(t) &= \int \frac{E(t)y_1(t)}{W[y_1,y_2](t)}dt = \frac{17}{2}\int e^t\cos^2(2t)dt = \frac{17}{4}\int e^t(1 + \cos(4t))dt \\ &= \frac{17}{4}e^t + \frac{1}{4}e^t(\cos(4t) + 4\sin(4t)). \end{aligned}$$

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The particular solution is

$$\begin{aligned}\psi(t) &= u_1(t)y_1(t) + u_2(t)y_2(t)\\ &= -\frac{1}{4}(\sin(4t) - 4\cos(4t))\cos(2t) + \frac{17}{4}\sin(2t) + \frac{1}{4}(\cos(4t) + 4\sin(4t))\sin(2t)\\ &= \cos(2t) + 4\sin(2t).\end{aligned}$$

Here we have used double-angle formulae.

The particular solution can be found using guessing

$$\psi(t) = a\cos(2t) + b\sin(2t), \qquad a, b \text{ are constants.}$$

Then from

$$\begin{aligned}\psi'(t) &= -2a\sin(2t) + 2b\cos(2t)\\ \psi''(t) &= -4a\cos(2t) - 4b\sin(2t)\end{aligned}$$

we get

$$17\cos(2t) = \psi''(t) + 2\psi'(t) + 5\psi(t) = (-4a + b)\sin(2t) + (a + 4b)\cos(2t).$$

Finally, $a = 1$, $b = 4$, and $\psi(t) = \cos(2t) + 4\sin(2t)$.

The general solution of the starting problem has the form

$$Q(t) = c_1y_1(t) + c_2y_2(t) + \psi(t) = c_1\mathrm{e}^{-t}\cos(2t) + c_2\mathrm{e}^{-t}\sin(2t) + \psi(t).$$

The initial condition $Q(0) = 0$, and $y_1(0) = 1$, $y_2(0) = 0$, $\psi(0) = 1$, imply

$$0 = Q(0) = c_1 + 1, \qquad c_1 = -1.$$

Now, from

$$\begin{aligned}y_1'(t) &= -\mathrm{e}^{-t}(\cos(2t) + 2\sin(2t))\\ y_2'(t) &= \mathrm{e}^{-t}(2\cos(2t) - \sin(2t))\\ \psi'(t) &= 8\cos(2t) - 2\sin(2t),\end{aligned}$$

the second condition $Q'(0) = 9$ further implies

$$9 = c_1y_1'(0) + c_2y_2'(0) + \psi'(0) = -c_1 + 2c_2 + 8, \qquad c_2 = 0.$$

The final solution (the charge on the capacitor at time t) is

$$\begin{aligned}Q(t) &= -y_1(t) + \psi(t)\\ &= -\mathrm{e}^{-t}\cos(2t) + \cos(2t) + 4\sin(2t)\\ &= (1 - \mathrm{e}^{-t})\cos(2t) + 4\sin(2t).\end{aligned}$$

This initial-value problem

$$\frac{d^2Q}{dt^2} + 2\frac{dQ}{dt} + 5Q = 17\cos(2t), \qquad Q(0) = 0, \quad \frac{dQ}{dt}(0) = 9,$$

can also be solved using Laplace transforms. Let

$$Y(s) = \mathcal{L}\{Q(t)\}, \qquad F(s) = \mathcal{L}\{17\cos(2t)\} = \frac{17s}{s^2+4}.$$

Then

$$\begin{aligned} Y(s) &= \frac{1}{s^2+2s+5}\big(9 + F(s)\big) = \frac{1}{s^2+2s+5}\left(9 + \frac{17s}{s^2+4}\right) \\ &= \frac{9s^2+17s+36}{(s^2+2s+5)(s^2+4)} = -\frac{s+1}{s^2+2s+5} + \frac{s+8}{s^2+4}, \end{aligned}$$

where in the last step we use partial fractions. Then

$$\begin{aligned} -\frac{s+1}{s^2+2s+5} &= -\frac{s+1}{(s+1)^2+4} = -\mathcal{L}\{\mathrm{e}^{-t}\cos(2t)\} \\ \frac{s+8}{s^2+4} &= \frac{s}{s^2+4} + 4\frac{2}{s^2+4} = \mathcal{L}\{\cos(2t)\} + 4\mathcal{L}\{\sin(2t)\}, \end{aligned}$$

and consequently

$$\begin{aligned} Y(s) &= -\mathcal{L}\{\mathrm{e}^{-t}\cos(2t)\} + \mathcal{L}\{\cos(2t)\} + 4\mathcal{L}\{\sin(2t)\} \\ Q(t) &= -\mathrm{e}^{-t}\cos(2t) + \cos(2t) + 4\sin(2t) \\ &= (1 - \mathrm{e}^{-t})\cos(2t) + 4\sin(2t). \end{aligned}$$

Chapter 4

study notes

4.1 Population models (Section 1.5 in book)

■ The most basic model is call Malthusian model given by $\frac{dp}{dt} = ap(t)$ which says that rate of change of population is proportional to current population size. a is constant. The solution is $p(t) = p_0 e^{a(t-t_0)}$. Where $p(t)$ is population at time t and p_0 is initial population at time t_0. This model is OK when population is small. A better model is called logistic model given by

$$\frac{dp}{dt} = ap(t) - bp^2(t)$$
$$p(t_0) = p_0$$

Where b is the competition factor. Also constant and positive. It is much smaller than a. The solution to the above is

$$p(t) = \frac{ap_0}{bp_0 + \left(a - bp_0\right) e^{-a(t-t_0)}} \tag{1}$$

In this model, we are normally given p_0 and given $(t - t_0)$ and given what is called the limiting value $\frac{a}{b}$ which is $\lim_{t\to\infty} p(t)$. Then asked to find population $p(t)$ after sometime. This will be $(t - t_0)$. We need to find a. Once we find a, then we find b from the limiting value. The trick is to find a. To do this, we first use (1) from the information given. The problem will always say that the population doubles every so many years, or the population increases at rate of some percentage per year and so on. Use this to find a from (1). Now we know b. Then use (1) again now to find the population at some future time as the problem says. See HW1, last problem for an example.

■ If a problem says substance decays exponentially, this means $M(t) = M_0 e^{-Ct}$. where $C > 0$. Need to find C from other problem information. Typically problem gives half life to do this. For example, see problem section 1.8, problem 3. It says:

substance x decays exponentially, and only half of the given quantity remains after 2 years. How long it takes for 5 lb decay to 1 lb? Solution is

$$M = M_0 e^{-Ct}$$

After 2 years, $M = \frac{M_0}{2}$, hence $\frac{M_0}{2} = M_0 e^{-2C}$. Hence $\frac{1}{2} = e^{-2C}$ or $\ln\left(\frac{1}{2}\right) = -2C$, hence

$C = -\frac{1}{2}\ln\left(\frac{1}{2}\right) = \frac{1}{2}\ln(2)$. Now we know C, we can finish the solution.

$$\begin{aligned} M &= M_0 e^{-\frac{1}{2}\ln(2)t} \\ 1 &= 5e^{-\frac{1}{2}\ln(2)t} \\ \frac{1}{5} &= e^{-\frac{1}{2}\ln(2)t} \\ \ln\left(\frac{1}{5}\right) &= -\frac{1}{2}\ln\left(\frac{1}{5}\right)t \\ t &= -2\frac{\ln\left(\frac{1}{5}\right)}{\ln(2)} \\ &= 2\frac{\ln 5}{\ln 2} \\ &= 4.643 \text{ years} \end{aligned}$$

If it says it grows exponentially, then $M = M_0 e^{Ct}$ instead.

4.2 Mixing problems (Section 1.8(b) in book)

The main idea is to set an ODE using $\frac{dS(t)}{dt} = R_{in} - R_{out}$ where R_{in} is rate of mass of salt coming into the tank and R_{out} is rate of mass of salt leaving tank. This gives an ODE to solve for $S(t)$ using initial conditions which is given. At end, divide by volume of tank to get concentration at time t. See book example at page 54.

4.3 Example 1, page 369

Book solution for example 1 is wrong. So I typed corrected solution.

Solve $\dot{x} = \begin{pmatrix}1 & 4\\ 1 & 1\end{pmatrix} \boldsymbol{x} + \begin{pmatrix}e^t\\ e^t\end{pmatrix}$ with $\boldsymbol{x}(0) = \begin{pmatrix}2\\ 1\end{pmatrix}$

Solution

$$(sI - A)\boldsymbol{X}(s) = \boldsymbol{F}(s) + \boldsymbol{x}(0)$$

$$\left[\begin{pmatrix}s & 0\\ 0 & s\end{pmatrix} - \begin{pmatrix}1 & 4\\ 1 & 1\end{pmatrix}\right]\begin{pmatrix}x_1(s)\\ x_2(s)\end{pmatrix} = \begin{pmatrix}\frac{1}{s-1}\\ \frac{1}{s-1}\end{pmatrix} + \begin{pmatrix}2\\ 1\end{pmatrix}$$

$$\begin{pmatrix}s-1 & -4\\ -1 & s-1\end{pmatrix}\begin{pmatrix}x_1(s)\\ x_2(s)\end{pmatrix} = \begin{pmatrix}\frac{1}{s-1}+2\\ \frac{1}{s-1}+1\end{pmatrix}$$

Multiplying the second row by $(s-1)$ and adding the result to the first row to obtain Gaussian elimination. First multiplying second row by $(s-1)$ gives

$$\begin{pmatrix}s-1 & -4\\ -(s-1) & (s-1)^2\end{pmatrix}\begin{pmatrix}x_1(s)\\ x_2(s)\end{pmatrix} = \begin{pmatrix}\frac{1}{s-1}+2\\ 1+(s-1)\end{pmatrix}$$

$$\begin{pmatrix}s-1 & -4\\ -(s-1) & (s-1)^2\end{pmatrix}\begin{pmatrix}x_1(s)\\ x_2(s)\end{pmatrix} = \begin{pmatrix}\frac{1}{s-1}+2\\ s\end{pmatrix}$$

Now replacing row 2 by row 2 plus row 1 gives

$$\begin{pmatrix}s-1 & -4\\ 0 & (s-1)^2-4\end{pmatrix}\begin{pmatrix}x_1(s)\\ x_2(s)\end{pmatrix} = \begin{pmatrix}\frac{1}{s-1}+2\\ s+\left(\frac{1}{s-1}+2\right)\end{pmatrix}$$

$$\begin{pmatrix}s-1 & -4\\ 0 & s^2-2s-3\end{pmatrix}\begin{pmatrix}x_1(s)\\ x_2(s)\end{pmatrix} = \begin{pmatrix}\frac{1}{s-1}+2\\ \frac{1}{s-1}\left(s^2+s-1\right)\end{pmatrix} \tag{1}$$

Hence

$$\begin{aligned}x_2(s) &= \frac{1}{s-1}\left(\frac{s^2+s-1}{s^2-2s-3}\right)\\ &= \frac{s^2+s-1}{(s-1)(s-3)(s+1)}\end{aligned} \tag{2}$$

Partial fractions:

$$\frac{s^2+s-1}{(s-1)(s-3)(s+1)} = \frac{A}{(s-1)} + \frac{B}{(s-3)} + \frac{C}{(s+1)}$$

$$A = \left(\frac{s^2+s-1}{(s-3)(s+1)}\right)_{s=1} = \frac{1+1-1}{(1-3)(1+1)} = -\frac{1}{4}$$

And

$$B = \left(\frac{s^2+s-1}{(s-1)(s+1)}\right)_{s=3} = \frac{9+3-1}{(3-1)(3+1)} = \frac{11}{8}$$

And

$$C = \left(\frac{s^2+s-1}{(s-1)(s-3)}\right)_{s=-1} = \frac{1-1-1}{(-1-1)(-1-3)} = -\frac{1}{8}$$

Hence

$$x_2(s) = -\frac{1}{4}\frac{1}{(s-1)} + \frac{11}{8}\frac{1}{(s-3)} - \frac{1}{8}\frac{1}{(s+1)} \tag{3}$$

Therefore

$$x_2(t) = -\frac{1}{4}e^t + \frac{11}{8}e^{3t} - \frac{1}{8}e^{-t}$$

Now we go back to (1) and use the first row to find $x_1(s)$ since we know $x_2(s)$ which is given in (2). This results in

$$
\begin{aligned}
(s-1)\,x_1(s) - 4x_2(s) &= \frac{1}{s-1} + 2\\
(s-1)\,x_1(s) &= \frac{1}{s-1} + 2 + 4x_2\\
x_1(s) &= \frac{1}{(s-1)^2} + \frac{2}{s-1} + \frac{4}{(s-1)}x_2\\
&= \frac{1}{(s-1)^2} + \frac{2}{s-1} + \frac{4}{(s-1)}\left(\frac{s^2+s-1}{(s-1)(s-3)(s+1)}\right)\\
&= \frac{1}{(s-1)^2} + \frac{2}{s-1} + \frac{4}{(s-1)}\left(-\frac{1}{4}\frac{1}{(s-1)} + \frac{11}{8}\frac{1}{(s-3)} - \frac{1}{8}\frac{1}{(s+1)}\right)\\
&= \frac{1}{(s-1)^2} + \frac{2}{s-1} - \frac{1}{(s-1)^2} + \frac{44}{8}\frac{1}{(s-1)(s-3)} - \frac{4}{8}\frac{1}{(s-1)(s+1)}\\
&= \frac{2}{s-1} + \frac{44}{8}\frac{1}{(s-1)(s-3)} - \frac{4}{8}\frac{1}{(s-1)(s+1)}\\
&= \frac{2}{s-1} + \frac{11}{2}\frac{1}{(s-1)(s-3)} - \frac{1}{2}\frac{1}{(s-1)(s+1)}\\
&= \frac{2}{s-1} + \frac{11}{2}\left(\frac{1}{2}\frac{1}{s-3} - \frac{1}{2}\frac{1}{s-1}\right) - \frac{1}{2}\left(\frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}\right)\\
&= \frac{2}{s-1} + \frac{11}{4}\frac{1}{s-3} - \frac{11}{4}\frac{1}{s-1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{4}\frac{1}{s+1}\\
&= \frac{-1}{s-1} + \frac{11}{4}\frac{1}{s-3} + \frac{1}{4}\frac{1}{s+1}
\end{aligned}
$$

Therefore

$$x_1(t) = -e^t + \frac{11}{4}e^{3t} + \frac{1}{4}e^{-t}$$

We see that book solution is wrong. It gives $x_1(t) = 2e^{3t} + \frac{1}{2}e^t - \frac{1}{2}e^{-t}$.

Solving the same problem, but using the <u>variation of parameters method</u>:

Since $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$ then

$$
\begin{aligned}
\det(A - \lambda I) &= 0\\
\det\begin{pmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix} &= 0\\
(1-\lambda)^2 - 4 &= 0
\end{aligned}
$$

Hence roots are $\lambda = -1, \lambda = 3$

<u>$\lambda = -1$</u>

$$
\begin{aligned}
\begin{pmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}\\
\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

From first row, $2v_1 + 4v_2 = 0$ or $v_1 = -2v_2$. Hence $\boldsymbol{v}^1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\boldsymbol{x}^1(t) = e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

<u>$\lambda = 3$</u>

$$
\begin{aligned}
\begin{pmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}\\
\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

From first row, $-2v_1 + 4v_2 = 0$ or $v_1 = 2v_2$. Hence $v^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{x}^2(t) = e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Therefore

$$X(t) = \begin{pmatrix} -2e^{-t} & 2e^{3t} \\ e^{-t} & e^{3t} \end{pmatrix}$$

$$X(0) = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$$

Therefore $X^{-1}(0) = \frac{adj(X(0))}{\det(X(0))} = \frac{\begin{pmatrix} 1 & -1 \\ -2 & -2 \end{pmatrix}^T}{-4} = -\frac{1}{4}\begin{pmatrix} 1 & -2 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$. Hence

$$\begin{aligned} e^{At} &= X(t)\,X^{-1}(0) \\ &= \begin{pmatrix} -2e^{-t} & 2e^{3t} \\ e^{-t} & e^{3t} \end{pmatrix}\begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t} & -e^{-t} + e^{3t} \\ -\frac{1}{4}e^{-t} + \frac{1}{4}e^{3t} & \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t} \end{pmatrix} \end{aligned}$$

Using (since $t_0 = 0$)

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + e^{At}\int_0^t e^{-As} f(s)\,ds$$

But

$$\begin{aligned} e^{-As} f(s) &= \begin{pmatrix} \frac{1}{2}e^{s} + \frac{1}{2}e^{-3s} & -e^{s} + e^{-3s} \\ -\frac{1}{4}e^{s} + \frac{1}{4}e^{-3s} & \frac{1}{2}e^{s} + \frac{1}{2}e^{-3s} \end{pmatrix}\begin{pmatrix} e^{s} \\ e^{s} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}e^{2s} + \frac{1}{2}e^{-2s} - e^{2s} + e^{-2s} \\ -\frac{1}{4}e^{2s} + \frac{1}{4}e^{-2s} + \frac{1}{2}e^{2s} + \frac{1}{2}e^{-2s} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2}e^{2s} + \frac{3}{2}e^{-2s} \\ \frac{1}{4}e^{2s} + \frac{3}{4}e^{-2s} \end{pmatrix} \end{aligned}$$

Integrating

$$\begin{aligned} \int_0^t e^{-As} f(s)\,ds &= \begin{pmatrix} \int_0^t -\frac{1}{2}e^{2s} + \frac{3}{2}e^{-2s} ds \\ \int_0^t \frac{1}{4}e^{2s} + \frac{3}{4}e^{-2s} ds \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{4}\left(e^{2s}\right)_0^t - \frac{3}{4}\left(e^{-2s}\right)_0^t \\ \frac{1}{8}\left(e^{2s}\right)_0^t - \frac{3}{8}\left(e^{-2s}\right)_0^t \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{4}\left(e^{2t} - 1\right) - \frac{3}{4}\left(e^{-2t} - 1\right) \\ \frac{1}{8}\left(e^{2t} - 1\right) - \frac{3}{8}\left(e^{-2t} - 1\right) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{4}e^{2t} + \frac{1}{4} - \frac{3}{4}e^{-2t} + \frac{3}{4} \\ \frac{1}{8}e^{2t} - \frac{1}{8} - \frac{3}{8}e^{-2t} + \frac{3}{8} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{4}e^{2t} - \frac{3}{4}e^{-2t} + 1 \\ \frac{1}{8}e^{2t} - \frac{3}{8}e^{-2t} + \frac{1}{4} \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned} e^{At}\int_0^t e^{-As} f(s)\,ds &= \begin{pmatrix} \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t} & -e^{-t} + e^{3t} \\ -\frac{1}{8}e^{-t} + \frac{1}{4}e^{3t} & \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t} \end{pmatrix}\begin{pmatrix} -\frac{1}{4}e^{2t} - \frac{3}{4}e^{-2t} + 1 \\ \frac{1}{8}e^{2t} - \frac{3}{8}e^{-2t} + \frac{1}{4} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4}e^{-t} - e^{t} + \frac{3}{4}e^{3t} \\ -\frac{1}{8}e^{-t} - \frac{1}{4}e^{2t} + \frac{3}{8}e^{4t} \end{pmatrix} \end{aligned}$$

And

$$\begin{aligned} e^{At}\boldsymbol{x}(0) &= \begin{pmatrix} \frac{1}{2}e^{-t}+\frac{1}{2}e^{3t} & -e^{-t}+e^{3t} \\ -\frac{1}{8}e^{-t}+\frac{1}{4}e^{3t} & \frac{1}{2}e^{-t}+\frac{1}{2}e^{3t} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \boldsymbol{x}(t) &= \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix} + \begin{pmatrix} \frac{1}{4}e^{-t}-e^{t}+\frac{3}{4}e^{3t} \\ -\frac{1}{8}e^{-t}-\frac{1}{4}e^{2t}+\frac{3}{8}e^{4t} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4}e^{-t}-e^{t}+\frac{11}{4}e^{3t} \\ -\frac{1}{8}e^{-t}-\frac{1}{4}e^{2t}+\frac{11}{8}e^{4t} \end{pmatrix} \end{aligned}$$

Therefore

$$\begin{aligned} x_1(t) &= \frac{1}{4}e^{-t}-e^{t}+\frac{11}{4}e^{3t} \\ x_2(t) &= -\frac{1}{8}e^{-t}-\frac{1}{4}e^{2t}+\frac{11}{8}e^{4t} \end{aligned}$$

Which agrees with result using Laplace transform method.

4.4 Orthogonal projections

Given $F\left(x,y,c\right)$ we need to find the orthogonal projections. The first step is to find the slope of the orthogonal projection, which is given by (it is orthogonal to the given curve slope)

$$\frac{dy}{dx}=\frac{F_y}{F_x} \tag{1}$$

Next step, check if c still shows up in the above (i.e. did not cancel out), then solve for c from $F\left(x,y,c\right)=0$ and replace it in (1). Now (1) will not have c in it any more. Next, solve (1) for y. This gives the curve for the orthogonal projection. This solution will have new c in it (since we need to integrate to find y). See HW 2 for example problem.

4.5 Existence-uniqueness for 1D ODE

Given by theorem 2 for existence and uniqueness: given $\frac{dy}{dx}=f\left(x,y\right)$, with initial value $y\left(x_0\right)=y_0$. Let f and $\frac{\partial f}{\partial x}$be continuous in the rectangle $R: t \leq t \leq t_0+a, \left|y-y_0\right| \leq b$. Compute $M=\max_{(x,y)}\left|f\left(x,y\right)\right|$ and set $\alpha=\min\left(a,\frac{b}{M}\right)$ then ODE has at least one solution in interval $t \leq t \leq t_0+\alpha$ and this solution is unique. (I do not know why book split this into theorem 2 and 2').

Notice in the above, if f or $\frac{\partial f}{\partial x}$ not continuous in the range (the range must include the initial point) then not unique solution exist. For example $y'=\sin\left(2t\right)y^{\frac{1}{3}}$ with $y\left(0\right)=0$. Here f' is not continuous at $y=0$.

How to use the above The first step is to find M. This is done by finding maximum in R. This is normally done by inspection from looking at $f\left(x,y\right)$. Next, let $g\left(y\right)=\frac{b}{M}$. Find where this one is maximum. set its value in $\alpha=\min\left(a,\frac{b}{M}\right)$ and this finds α. Done. Example. Show $y\left(t\right)$ solution to $\frac{dy}{dt}=t^2+e^{-y^2}, y\left(0\right)=0$ exists in $0 \leq t \leq \frac{1}{2}$ and $\left|y\left(t\right)\right| \leq 1$. Here it is clear $M=\frac{5}{4}$ and hence $\alpha=\min\left(\frac{1}{2},\frac{b}{M}\right)$ but $b=1$, hence $\alpha=\min\left(\frac{1}{2},\frac{1}{\frac{5}{4}}\right)=\alpha=\min\left(\frac{1}{2},\frac{4}{5}\right)=\frac{1}{2}$. Therefore solution exist for $0 \leq t \leq 0+\alpha$ or $0 \leq t \leq \frac{1}{2}$.

4.5.1 practice problems

4.5.1.1 Problem 5, section 1.10

Show that the solution $y\left(t\right)$ exists on $y\left(0\right)=0; 0 \leq t \leq \frac{1}{3}$

$$y'=1+y+y^2\cos t$$

solution

Here $a=\frac{1}{3}$.

$$\begin{aligned} M &= \max\left(f\left(t,y\right)\right) \\ &= 1+b+b^2 \end{aligned}$$

Hence

$$\begin{aligned} \alpha &= \min\left(\frac{1}{3},\frac{b}{M}\right) \\ &= \min\left(\frac{1}{3},\frac{b}{1+b+b^2}\right) \end{aligned}$$

Let $g\left(b\right)=\frac{b}{1+b+b^2}$ then $\frac{dg}{dp}=\frac{\left(1+b+b^2\right)-b(1+2b)}{\left(1+b+b^2\right)^2}$. Setting this to zero and solving for b

$$\begin{aligned} \left(1+b+b^2\right)-b\left(1+2b\right) &= 0 \\ 1+b+b^2-b-2b^2 &= 0 \\ 1-b^2 &= 0 \end{aligned}$$

Hence $b=1$. At $b=1$, then $g\left(b\right)=\frac{1}{1+1+1}=\frac{1}{3}$. Therefore

$$\begin{aligned} \alpha &= \min\left(\frac{1}{3},\frac{1}{3}\right) \\ &= \frac{1}{3} \end{aligned}$$

Therefore $y\left(t\right)$ solution exist for $0 \leq t \leq 0+\alpha$ or $0 \leq t \leq \frac{1}{3}$.

4.5.1.2 Problem 16, section 1.10

Consider $y' = t^2 + y^2, y(0) = 0$ and let R be rectangle $0 \leq t \leq a, -b \leq y \leq b$. (a) Show the solution exist for $0 \leq t \leq \min\left(a, \frac{b}{a^2+b^2}\right)$ (b) Show the maximum value of $\frac{b}{a^2+b^2}$, for a fixed is $\frac{1}{2a}$. (c) Show that $\alpha = \min\left(a, \frac{1}{2}a\right)$ is largest when $a = \frac{1}{\sqrt{2}}$

$$y' = 1 + y + y^2 \cos t$$

<u>solution</u>

(a)

$$\begin{aligned} M &= \max\left(f\left(t,y\right)\right) \\ &= a^2 + b^2 \end{aligned}$$

Hence

$$\begin{aligned} \alpha &= \min\left(a, \frac{b}{M}\right) \\ &= \min\left(\frac{1}{3}, \frac{b}{a^2+b^2}\right) \end{aligned}$$

Hence solution exist for $0 \leq t \leq \min\left(a, \frac{b}{a^2+b^2}\right)$.

(b) Let $g(b) = \frac{b}{a^2+b^2}$ then $\frac{dg}{dp} = \frac{\left(a^2+b^2\right)-b(2b)}{\left(1+b+b^2\right)^2}$. Setting this to zero and solving for b

$$\begin{aligned} \left(a^2+b^2\right) - b\left(2b\right) &= 0 \\ a^2 + b^2 - 2b^2 &= 0 \\ a^2 &= b^2 \end{aligned}$$

Hence $b = \pm a$. At $b = a$, then $g(b) = \frac{a}{a^2+a^2} = \frac{a}{2a^2} = \frac{1}{2a}$.

(c)

$$\begin{aligned} \alpha &= \min\left(a, g_{\max}\left(b\right)\right) \\ &= \min\left(a, \frac{1}{2a}\right) \end{aligned}$$

Solving $a = \frac{1}{2a}$ or $a^2 = \frac{1}{2}$. Hence $a = \frac{1}{\sqrt{2}}$ gives largest value.

4.5.1.3 Problem 17, section 1.10

Prove that $y(t) = -1$ is only solution for $y' = t\left(1+y\right), y(0) = -1$

<u>solution</u>

Since $f = t\left(1+y\right)$ is continuous for all t, y and $f_y = t$ is continuous for all y, then if we find a solution, it will be unique solution by theorem 2'. But $y(t) = -1$ is a solution since we can show easily it satisfies the ODE. Hence it is the only solution over all t by theorem 2'

4.5.1.4 Problem 19, section 1.10

Find solution of $y' = t\sqrt{1-y^2}, y(0) = 1$ other than $y(t) = 1$. Does this violate theorem 2'?

<u>solution</u>

$$\begin{aligned}\frac{dy}{dt} &= t\sqrt{1-y^2}\\ \int\frac{dy}{\sqrt{1-y^2}} &= \int tdt\\ \arcsin\left(y\right) &= \frac{t^2}{2}+C\end{aligned}$$

At $t=0$

$$\arcsin\left(1\right)=C$$

Hence solution is $\arcsin\left(y\right)=\frac{t^2}{2}+\arcsin\left(1\right)$ or

$$\begin{aligned}y\left(t\right) &= \sin\left(\frac{t^2}{2}+\arcsin\left(1\right)\right)\\ &= \sin\left(\frac{t^2}{2}+\frac{\pi}{2}\right)\\ &= \sin\left(\frac{1}{2}\left(t^2+\pi\right)\right)\end{aligned}$$

This does not violate theorem 2' because $f\left(t,y\right)=t\sqrt{1-y^2}$, hence $f_y=\frac{-ty}{\sqrt{1-y^2}}$ which is not continuous at $y=\pm1$. But $y=-1$ is the initial conditions. Hence theorem 2' do not apply. Theorem 2' applies in the region where both f,f_y are continuous.

4.6 Stability of system

Algorithm 1 Determining stability of system $\dot{x} = Ax + g(x)$

```
if system is linear, i.e. $\dot{x} = Ax$ then
    determine eigenvalues $\lambda_i$ of $A$ by solving $|A - \lambda I| = 0$
    if all eigenvalues have real part smaller than zero then
        return stable
    else
        if at least one eigenvalue have positive real part then
            return not stable
        else                              ▷ we get here if at least one $\lambda$ has zero real part
            for all $\lambda_i$ with zero real part do
                $M$ = multiplicity of $\lambda_i$
                $N$ = number of linearly independent eigenvectors that $\lambda_i$ can generate
                if $N < M$ then
                    return not stable
                end if
            end for
            return stable
        end if
    end if
else                                                                      ▷ system not linear
    will only consider case when origin is equilibrium point
    determine the Jacobian matrix $J$
    evaluate $J$ at origin $x = 0$
    determine eigenvalues $\lambda_i$ of $J$ by solving $|J - \lambda I| = 0$
    if all eigenvalues have real part smaller than zero then
        return stable
    else
        if at least one eigenvalue have positive real part then
            return not stable
        else                              ▷ we get here if at least one $\lambda$ has zero real part
            return unable to decide
        end if
    end if
end if
```

4.7 Laplace

If $Y(s)$ has form $\frac{s}{s^2+as+b}$ where roots of quadratic are complex, then complete the square. Write $s^2 + as + b = (s + A)^2 + B$ and find A, B. Then

$$\begin{aligned}
\frac{s}{s^2 + as + b} &= \frac{s}{(s + A)^2 + B} \\
&= \frac{s + A - A}{(s + A)^2 + B} \\
&= \frac{s + A}{(s + A)^2 + B} - A\frac{1}{(s + A)^2 + B} \\
&= \frac{\tilde{s}}{\tilde{s}^2 + B} - A\frac{1}{\tilde{s}^2 + B} \\
&= \frac{\tilde{s}}{\tilde{s}^2 + B} - \frac{A}{\sqrt{B}}\frac{\sqrt{B}}{\tilde{s}^2 + B}
\end{aligned}$$

And now use tables. Due to shifting, multiply result by e^{-Bt}. So inverse Laplace of the above is $e^{-Bt}\left(\cos\sqrt{B}t - \frac{A}{\sqrt{B}}\sin\sqrt{B}t\right)$