MATH 4512 – Midterm exam 2 – Solutions October 25, 2019

Problem 1. (25 points)
Find the inverse Laplace transform of the function

$$\frac{4s^2+2}{s^3+s^2-2s}.$$

The denominator of the given function is $s^3+s^2-2s = s(s^2+s-2) = s(s-1)(s+2)$.
Using partial fractions, we can further write

$$\begin{aligned}
\frac{4s^2+2}{s^3+s^2-2s} = \frac{4s^2+2}{s(s-1)(s+2)} &= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2} \\
&= \frac{1}{s(s-1)(s+2)}\left(A(s-1)(s+2) + Bs(s+2) + Cs(s-1)\right) \\
&= \frac{1}{s(s-1)(s+2)}\left(As^2 + As - 2A + Bs^2 + 2Bs + Cs^2 - Cs\right) \\
&= \frac{1}{s(s-1)(s+2)}\left(s^2(A+B+C) + s(A+2B-C) - 2A\right).
\end{aligned}$$

Equating the coefficients, we first obtain $-2A = 2$, $A = -1$. Then from

$$A+B+C = 4, \qquad A+2B-C = 0,$$

we find $B = 2$ and $C = 3$.
Therefore,

$$\begin{aligned}
\frac{4s^2+2}{s^3+s^2-2s} &= -\frac{1}{s} + \frac{2}{s-1} + \frac{3}{s+2} \\
&= -\mathcal{L}\{1\} + 2\mathcal{L}\{\mathrm{e}^{t}\} + 3\mathcal{L}\{\mathrm{e}^{-2t}\} \\
&= \mathcal{L}\{-1 + 2\mathrm{e}^{t} + 3\mathrm{e}^{-2t}\}.
\end{aligned}$$

We conclude that the inverse Laplace transform of the given function is $-1 + 2\mathrm{e}^{t} + 3\mathrm{e}^{-2t}$.

Problem 2. (25 points)
Solve the following initial-value problem

$$y'' - 3y' - 4y = \mathrm{e}^{2t}, \qquad y(0) = 0, \qquad y'(0) = -\frac{1}{2}.$$

For finding the particular solution, use **the method of variation of parameters**.

First we solve the homogeneous problem. The characteristic equation

$$r^2 - 3r - 4 = 0$$

has two real roots $r_1 = 4$ and $r_2 = -1$. The functions

$$y_1(t) = \mathrm{e}^{4t}, \qquad y_2(t) = \mathrm{e}^{-t},$$

form a fundamental set of solutions.
The particular solution $\psi(t)$ we seek in the form

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

The Wronskian for y_1, y_2 is

$$W(t) = W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = -\mathrm{e}^{4t}\mathrm{e}^{-t} - 4\mathrm{e}^{4t}\mathrm{e}^{-t} = -5\mathrm{e}^{3t}.$$

Functions u_1, u_2 we find as

$$u_1(t) = -\int \frac{\mathrm{e}^{2t}\,\mathrm{e}^{-t}}{-5\mathrm{e}^{3t}} dt = \frac{1}{5}\int \mathrm{e}^{-2t} dt = -\frac{1}{10}\mathrm{e}^{-2t}$$
$$u_2(t) = \int \frac{\mathrm{e}^{2t}\,\mathrm{e}^{4t}}{-5\mathrm{e}^{3t}} dt = -\frac{1}{5}\int \mathrm{e}^{3t} dt = -\frac{1}{15}\mathrm{e}^{3t}.$$

The particular solution is

$$\psi(t) = -\frac{1}{10}\mathrm{e}^{-2t}\,\mathrm{e}^{4t} - \frac{1}{15}\mathrm{e}^{3t}\,\mathrm{e}^{-t} = -\frac{1}{6}\mathrm{e}^{2t}.$$

The general solution is

$$y(t) = c_1y_1(t) + c_2y_2(t) + \psi(t) = c_1\mathrm{e}^{4t} + c_2\mathrm{e}^{-t} - \frac{1}{6}\mathrm{e}^{2t}.$$

From $y(0) = 0$, we get the first condition $c_1 + c_2 - 1/6 = 0$ for the constants c_1, c_2. The first derivative of $y(t)$ is

$$y'(t) = 4c_1\mathrm{e}^{4t} - c_2\mathrm{e}^{-t} - \frac{1}{3}\mathrm{e}^{2t}.$$

Now, from $y'(0) = -1/2$ we get $4c_1 - c_2 - 1/3 = -1/2$. The constants are $c_1 = 0$, $c_2 = 1/6$, and the final solution is

$$y(t) = \frac{1}{6}\mathrm{e}^{-t} - \frac{1}{6}\mathrm{e}^{2t}.$$

Problem 3. (20 points)
Which of the following functions

(a) $e^{-t/3}(A_3t^3 + A_0)t^2$

(b) $e^{-t/3}(A_3t^3 + A_2t^2 + A_1t + A_0)t$

(c) $e^{-t/3}(A_3t^3 + A_2t^2 + A_1t + A_0)t^2$

(d) $e^{-t/3}(A_3t^3 + A_2t^2 + A_1t + A_0)$

should be chosen as a guessing for the particular solution of $9y'' + 6y' + y = e^{-t/3}(t^3 - 1)$?

The characteristic equation

$$9r^2 + 6r + 1 = 0$$

has one double root $r = -1/3$ that coincides with the exponent $\alpha = -1/3$ in the right-hand side $g(t) = e^{\alpha t}(t^3 - 1)$. Thus the correct guessing for the particular solution is the answer (c), i.e.

$$\psi(t) = e^{-t/3}\, t^2\, (A_3t^3 + A_2t^2 + A_1t + A_0).$$

Problem 4. (30 points)
A spring-mass-dashpot system with $m = 1$, $k = 2$ and $c = 2$ (in their respective units) hangs in equilibrium. At time $t = 0$, an external force $F(t) = t - \pi$ N starts acting on the hanging object. Find the position $y(t)$ of the object at anytime $t > 0$. Over time, what do you expect to occur within this system?

The initial-value problem describing this system is

$$y'' + 2y' + 2y = t - \pi, \qquad y(0) = y'(0) = 0.$$

The characteristic equation $r^2 + 2r + 2 = 0$ has complex roots $r_1 = -1 + i$, $r_2 = -1 - i$. Therefore, the functions

$$y_1(t) = \mathrm{e}^{-t}\cos t, \qquad y_2(t) = \mathrm{e}^{-t}\sin t,$$

form a fundamental set of solutions.
The particular solution can be found using the guessing method. Then

$$\psi(t) = At + B,$$

with $\psi'(t) = A$ and $\psi''(t) = 0$. The differential equation with the function ψ reduces to

$$\psi'' + 2\psi' + 2\psi = 2A + 2At + 2B = 2At + 2(A + B) = t - \pi.$$

Equating coefficients we obtain $A = 1/2$, $B = -\pi/2 - 1/2$. Hence, the particular solution is

$$\psi(t) = \frac{t}{2} - \frac{\pi}{2} - \frac{1}{2}.$$

The general solution

$$y(t) = c_1\mathrm{e}^{-t}\cos t + c_2\mathrm{e}^{-t}\sin t + \frac{t}{2} - \frac{\pi}{2} - \frac{1}{2}$$

has its first derivative

$$y'(t) = -c_1\mathrm{e}^{-t}\cos t - c_1\mathrm{e}^{-t}\sin t - c_2\mathrm{e}^{-t}\sin t + c_2\mathrm{e}^{-t}\cos t + \frac{1}{2}.$$

Initial conditions $y(0) = 0$ and $y'(0) = 0$ further imply

$$0 = c_1 - \frac{\pi}{2} - \frac{1}{2}, \qquad 0 = -c_1 + c_2 + \frac{1}{2}.$$

Then

$$c_1 = \frac{\pi}{2} + \frac{1}{2}, \qquad c_2 = \frac{\pi}{2},$$

and the function $y(t)$ that describes the position of the object at anytime $t > 0$ is

$$y(t) = \left(\frac{\pi}{2} + \frac{1}{2}\right)\mathrm{e}^{-t}\cos t + \frac{\pi}{2}\mathrm{e}^{-t}\sin t + \frac{t}{2} - \frac{\pi}{2} - \frac{1}{2}.$$

Over time we expect the spring to break since

$$\lim_{t\to\infty} y(t) = \infty$$

(the positive direction of the position $y(t)$ is downwards).