MATH 4512 – Midterm exam 1– Solutions September 27, 2019

Problem 1. (25 points)
Solve the following initial-value problem

$$\frac{dy}{dt} = \frac{2ty + 2t}{t^2 + 1}, \qquad y(-1) = 3.$$

This differential equation is both separable and linear since

$$\frac{dy}{dt} = \frac{2t(y+1)}{t^2+1} \qquad \text{and} \qquad \frac{dy}{dt} = \frac{2t}{t^2+1} y + \frac{2t}{t^2+1}.$$

We will solve it as a linear differential equation with

$$a(t) = -\frac{2t}{t^2+1}, \qquad b(t) = \frac{2t}{t^2+1}.$$

The integrating factor is

$$\mu(t) = \exp\left(-\int \frac{2t\,dt}{t^2+1}\right) = \exp\big(-\ln|t^2+1|\big) = \exp\big(-\ln(t^2+1)\big) = \frac{1}{t^2+1}.$$

The general solution is

$$y(t) = (t^2+1)\left(\int \frac{1}{t^2+1}\frac{2t}{t^2+1} dt + C\right) = (t^2+1)\left(\int \frac{2t\,dt}{(t^2+1)^2} + C\right)$$

$$= (t^2+1)\left(\int \frac{ds}{s^2} + C\right), \qquad s = t^2+1, \quad ds = 2t\,dt$$

$$= (t^2+1)\left(-\frac{1}{s} + C\right) = (t^2+1)\left(-\frac{1}{t^2+1} + C\right) = -1 + C(t^2+1).$$

The initial condition implies

$$3 = y(-1) = -1 + C((-1)^2+1) = 2C - 1, \qquad C = 2.$$

The solution is

$$y(t) = -1 + 2(t^2+1) = 2t^2 + 1.$$

Problem 2. (25 points)
A tank initially contains 60 gal of pure water. Brine containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the well-stirred solution leaves the tank at the same rate.

(a) If $S(t)$ is the amount of salt in the tank at time t, write the differential equation for the time rate of change of $S(t)$ and solve it.

(b) Find the concentration $c(t)$ of the salt in the tank at time t.

(c) What would be the limiting concentration of salt as $t \to \infty$?

The inflow rate is $r_i = 2\,\text{gal/min}$, the outflow rate is $r_o = 2\,\text{gal/min}$, the inflow salt concentration is $c_i = 1\,\text{lb/gal}$, $V_0 = 60$ gal is the initial volume and $S(0) = 0$ is the initial condition (only pure water is initially in the tank).
(a) The time rate of change of $S(t)$ is given by

$$\frac{dS}{dt} = r_i c_i - r_o \frac{S}{60} = 2 - \frac{S}{30}.$$

This differential equation is linear

$$\frac{dS}{dt} + \frac{S}{30} = 2$$

and the integrating factor is

$$\mu(t) = \exp\left(\int \frac{dt}{30}\right) = \mathrm{e}^{t/30}.$$

The solution is

$$S(t) = \mathrm{e}^{-t/30}\left(\int 2\,\mathrm{e}^{t/30} dt + C\right) = \mathrm{e}^{-t/30}\left(60\,\mathrm{e}^{t/30} + C\right) = 60 + C\,\mathrm{e}^{-t/30}.$$

From the initial condition it follows

$$0 = S(0) = 60 + C, \qquad C = -60.$$

Finally

$$S(t) = 60 - 60\,\mathrm{e}^{-t/30}.$$

(b) Concentration $c(t)$ of the salt in the tank at time t is

$$c(t) = \frac{S(t)}{60} = 1 - \mathrm{e}^{-t/30}.$$

(c) The limiting concentration is

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty} (1 - \mathrm{e}^{-t/30}) = 1.$$

Problem 3. (30 points)
A population of butterflies grows according to the logistic law

$$\frac{dp}{dt} = 0.002p(100 - p) = 0.2p - 0.002p^2, \qquad t \geq 0.$$

(a) Find the population $p(t)$ as the function of time t, if the initial population is 60.

(b) Find $\lim_{t\to\infty} p(t)$ and determine limiting population in this model.

(a) We start from

$$\frac{dp}{dt} = 0.002p(100 - p).$$

Using partial fractions

$$\frac{1}{p(100-p)} = \frac{1}{100}\left(\frac{1}{p} + \frac{1}{100-p}\right),$$

we derive

$$\begin{aligned}
\int \frac{dp}{p(100-p)} &= \int 0.002\,dt \\
\frac{1}{100}\int \left(\frac{1}{p} + \frac{1}{100-p}\right) dp &= 0.002 \int dt \\
\int \left(\frac{1}{p} + \frac{1}{100-p}\right) dp &= 0.2 \int dt \\
\ln|p| - \ln|100-p| &= 0.2t + C_1 \\
\ln\left|\frac{p}{100-p}\right| &= 0.2t + C_1 \\
\frac{p}{100-p} &= C\mathrm{e}^{0.2t}.
\end{aligned}$$

From the given initial condition $p(0) = 60$, we can find the constant C:

$$\frac{p(0)}{100-p(0)} = C\mathrm{e}^{0.2\cdot 0}, \qquad \frac{60}{100-60} = C, \qquad C = \frac{3}{2}.$$

The next step is to find the function $p(t)$:

$$\begin{aligned}
\frac{p}{100-p} &= \frac{3}{2}\mathrm{e}^{0.2t} \\
p &= \frac{3}{2}\mathrm{e}^{0.2t}(100-p) = 150\,\mathrm{e}^{0.2t} - \frac{3}{2}\mathrm{e}^{0.2t}p \\
p\left(1 + \frac{3}{2}\mathrm{e}^{0.2t}\right) &= 150\,\mathrm{e}^{0.2t} \\
p(t) &= \frac{150\,\mathrm{e}^{0.2t}}{1 + \frac{3}{2}\mathrm{e}^{0.2t}} = \frac{150}{\mathrm{e}^{-0.2t} + \frac{3}{2}}.
\end{aligned}$$

The solution can be obtained directly from the formula

$$p(t) = \frac{ap_0}{bp_0 + (a - bp_0)\mathrm{e}^{-a(t-t_0)}}.$$

Here $a = 0.2$, $b = 0.002$, $t_0 = 0$, and $p_0 = 60$. Then

$$p(t) = \frac{12}{0.12 + (0.2 - 0.12)\mathrm{e}^{-0.2\,t}} = \frac{12}{0.12 + 0.08\mathrm{e}^{-0.2\,t}} = \frac{150}{1.5 + \mathrm{e}^{-0.2\,t}}.$$

(b) The limiting population in this model is

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{150}{1.5 + \mathrm{e}^{-0.2\,t}} = \frac{150}{1.5} = 100 = \frac{a}{b}.$$

Problem 4. (20 points)
Find the orthogonal trajectories of the given family of curves

$$y = c\,\mathrm{e}^x.$$

Here we can take $F(x, y, c) = y - c\,\mathrm{e}^x$. Then from

$$F_x = -c\,\mathrm{e}^x, \qquad F_y = 1, \qquad c = \frac{y}{\mathrm{e}^x},$$

the orthogonal trajectories of the given family are the solution curves of the equation

$$\frac{dy}{dx} = \frac{F_y}{F_x} = -\frac{1}{y}.$$

This is a separable differential equation and we solve it as follows:

$$\int y\,dy = -\int dx$$

$$\frac{y^2}{2} = -x + c.$$