MATH 4512 – DIFFERENTIAL EQUATIONS WITH APPLICATIONS
HW6 - SOLUTIONS

1. (Section 3.8 - Exercise 12) Solve the given initial-value problem

$$\dot{x}(t) = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{bmatrix} x(t), \qquad x(0) = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}.$$

The characteristic polynomial of the system matrix

$$A = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{bmatrix}$$

is

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 4 & 1 & -3-\lambda \end{bmatrix} \\ &= (3-\lambda)(2-\lambda)(-3-\lambda) + 4 + 2 + 8(2-\lambda) - (3-\lambda) - (3+\lambda) \\ &= (2-\lambda)(\lambda^2 - 9) + 16 - 8\lambda = (2-\lambda)(\lambda^2 - 9 + 8) \\ &= (2-\lambda)(\lambda-1)(\lambda+1). \end{aligned}$$

The eigenvalues of the matrix A are $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = -1$. Let v^i denote an eigenvector that corresponds to λ_i, $i = 1, 2, 3$.

The general solution $x(t)$ will be of the form

$$x(t) = c_1 x^1(t) + c_2 x^2(t) + c_3 x^3(t),$$

where c_1, c_2, c_3 are arbitrary constants, and $x^i(t) = \mathrm{e}^{\lambda_i t} v^i$, $i = 1, 2, 3$.

First we will find a vector v^1 and $x^1(t)$. From $(A - \lambda_1 I)v = (A - 2I)v = 0$, we obtain

$$\begin{bmatrix} 1 & 1 & -2 \\ -1 & 0 & 1 \\ 4 & 1 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The second equation is $-v_1 + v_3 = 0$, i.e. $v_1 = v_3$. Using this property, the first equation $v_1 + v_2 - 2v_3 = 0$ reduces to $v_2 - v_1 = 0$. Thus $v_2 = v_1$. The vector v has the form

$$v = \begin{bmatrix} v_1 \\ v_1 \\ v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For the eigenvector v^1 we can choose $[1, 1, 1]^\top$. Then

$$x^1(t) = \mathrm{e}^{\lambda_1 t} v^1 = \mathrm{e}^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Now we proceed with finding a vector v^2 and $x^2(t)$. From $(A-\lambda_2 I)v = (A-I)v = 0$, we obtain

$$\begin{bmatrix} 2 & 1 & -2 \\ -1 & 1 & 1 \\ 4 & 1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Multiplying the first equation $2v_1 + v_2 - 2v_3 = 0$ by -1 and adding to the second equation $-v_1 + v_2 + v_3 = 0$ leads to $-3v_1 + 3v_3 = 0$. Then $v_1 = v_3$. This relation implies $v_2 = 0$. Now the vector v has the form

$$v = \begin{bmatrix} v_1 \\ 0 \\ v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

For the eigenvector v^2 we can choose $[1, 0, 1]^\top$. Then

$$x^2(t) = \mathrm{e}^{\lambda_2 t} v^2 = \mathrm{e}^{t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

At the end we will find a vector v^3 and $x^3(t)$. From $(A-\lambda_3 I)v = (A+I)v = 0$, we obtain

$$\begin{bmatrix} 4 & 1 & -2 \\ -1 & 3 & 1 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Multiplying the second equation $-v_1 + 3v_2 + v_3 = 0$ by 2 and adding to the first equation $4v_1 + v_2 - 2v_3 = 0$ leads to $2v_1 + 7v_2 = 0$. Then $v_1 = -7v_2/2$. Using this relation in the second equation, we find $v_3 = v_1 - 3v_2 = -13v_2/2$. The vector v has the form

$$v = \begin{bmatrix} -\frac{7}{2}v_2 \\ v_2 \\ -\frac{13}{2}v_2 \end{bmatrix} = v_2 \begin{bmatrix} -\frac{7}{2} \\ 1 \\ -\frac{13}{2} \end{bmatrix}.$$

For the eigenvector v^3 we can choose

$$v = -2 \begin{bmatrix} -\frac{7}{2} \\ 1 \\ -\frac{13}{2} \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 13 \end{bmatrix}$$

Then

$$x^3(t) = \mathrm{e}^{\lambda_3 t} v^3 = \mathrm{e}^{-t} \begin{bmatrix} 7 \\ -2 \\ 13 \end{bmatrix}.$$

The general solution is

$$x(t) = c_1 \mathrm{e}^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{-t} \begin{bmatrix} 7 \\ -2 \\ 13 \end{bmatrix}$$

Initial condition implies

$$\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} = x(0) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ -2 \\ 13 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + 7c_3 \\ c_1 - 2c_3 \\ c_1 + c_2 + 13c_3 \end{bmatrix}.$$

Subtracting first and third equation gives $6c_3 = -8$, and consequently $c_3 = -4/3$. From the second equation it follows $c_1 = 2c_3 + 4 = 4/3$. Finally, $c_2 = 1 - c_1 - 7c_3 = 9$. The solution of the initial-value problem is

$$x(t) = \frac{4}{3}\mathrm{e}^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 9\mathrm{e}^{t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{3}\mathrm{e}^{-t} \begin{bmatrix} 7 \\ -2 \\ 13 \end{bmatrix}.$$

2. (Section 3.9 - Exercise 2) Find the general solution of the given system of differential equations

$$\dot{x}(t) = \begin{bmatrix} 1 & -5 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t).$$

The characteristic polynomial of the system matrix

$$A = \begin{bmatrix} 1 & -5 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & -5 & 0 \\ 1 & -3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda)^2(-3-\lambda) + 5(1-\lambda) = (1-\lambda)(-3+2\lambda+\lambda^2+5) \\ &= (1-\lambda)(\lambda^2+2\lambda+2). \end{aligned}$$

The eigenvalues of the matrix A are $\lambda_1 = 1$, $\lambda_2 = -1+i$ and $\lambda_3 = -1-i$.

In order to find $x^1(t)$, we proceed as in the previous exercise. We start from solving $(A - \lambda_1 I)v = (A - I)v = 0$, i.e.

$$\begin{bmatrix} 0 & -5 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the first equation we obtain $v_2 = 0$, while from the second $v_1 - 4v_2 = 0$ it follows $v_1 = 0$. Thus the vector v has the form

$$v = \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

For the eigenvector v^1 we can choose $[0,0,1]^\top$. Then

$$x^1(t) = \mathrm{e}^{\lambda_1 t} v^1 = \mathrm{e}^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Other eigenvalues of the matrix A are complex. In order to obtain two remaining linearly independent real solutions $x^2(t)$ and $x^3(t)$, it is sufficient to consider $\lambda_2 = -1+i$. We first find a complex vector v that solves $(A - \lambda_2 I)v = 0$, i.e.

$$\begin{bmatrix} 2-i & -5 & 0 \\ 1 & -2-i & 0 \\ 0 & 0 & 2-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the last equation we obtain $v_3 = 0$, while from the second it follows $v_1 = (2+i)v_2$. A complex eigenvector v that corresponds to $\lambda_2 = i$ has the form

$$v = \begin{bmatrix} (2+i)v_2 \\ v_2 \\ 0 \end{bmatrix} = v_2 \begin{bmatrix} 2+i \\ 1 \\ 0 \end{bmatrix}.$$

A complex-valued solution is

$$\begin{aligned}
\phi(t) = \mathrm{e}^{(-1+i)t} \begin{bmatrix} 2+i \\ 1 \\ 0 \end{bmatrix} &= \mathrm{e}^{-t}(\cos t + i \sin t) \begin{bmatrix} 2+i \\ 1 \\ 0 \end{bmatrix} \\
&= \mathrm{e}^{-t} \begin{bmatrix} (2\cos t - \sin t) + i(2\sin t + \cos t) \\ \cos t + i \sin t \\ 0 \end{bmatrix} \\
&= \mathrm{e}^{-t} \begin{bmatrix} 2\cos t - \sin t \\ \cos t \\ 0 \end{bmatrix} + i\mathrm{e}^{-t} \begin{bmatrix} 2\sin t + \cos t \\ \sin t \\ 0 \end{bmatrix}.
\end{aligned}$$

Now,

$$x^2(t) = \mathrm{e}^{-t} \begin{bmatrix} 2\cos t - \sin t \\ \cos t \\ 0 \end{bmatrix}, \qquad x^3(t) = \mathrm{e}^{-t} \begin{bmatrix} 2\sin t + \cos t \\ \sin t \\ 0 \end{bmatrix},$$

and the general solution has the form

$$\begin{aligned}
x(t) &= c_1 x^1(t) + c_2 x^2(t) + c_3 x^3(t) \\
&= c_1 \mathrm{e}^{t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-t} \begin{bmatrix} 2\cos t - \sin t \\ \cos t \\ 0 \end{bmatrix} + c_3 \mathrm{e}^{-t} \begin{bmatrix} 2\sin t + \cos t \\ \sin t \\ 0 \end{bmatrix}.
\end{aligned}$$

3. (Section 3.10 - Exercise 6) Solve the initial-value problem

$$\dot{x}(t) = \begin{bmatrix} -4 & -4 & 0 \\ 10 & 9 & 1 \\ -4 & -3 & 1 \end{bmatrix} x(t), \qquad x(0) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

The characteristic polynomial of the system matrix

$$A = \begin{bmatrix} -4 & -4 & 0 \\ 10 & 9 & 1 \\ -4 & -3 & 1 \end{bmatrix}$$

is

$$\det(A - \lambda I) = \det \begin{bmatrix} -4-\lambda & -4 & 0 \\ 10 & 9-\lambda & 1 \\ -4 & -3 & 1-\lambda \end{bmatrix}$$
$$= -(4+\lambda)(9-\lambda)(1-\lambda) + 16 - 3(4+\lambda) + 40(1-\lambda) = (2-\lambda)^3.$$

The eigenvalue of the matrix A is $\lambda = 2$.

In order to find $x^1(t)$, we start from solving $(A - \lambda I)v = (A - 2I)v = 0$, i.e.

$$\begin{bmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the first equation $-6v_1 - 4v_2 = 0$ we obtain $v_2 = -3v_1/2$, which together with the last equation $-4v_1 - 3v_2 - v_3 = 0$ implies $v_3 = -4v_1 + 9v_1/2 = v_1/2$. The vector v has the form

$$v = \begin{bmatrix} v_1 \\ -3v_1/2 \\ v_1/2 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ -3/2 \\ 1/2 \end{bmatrix}.$$

For the eigenvector we can choose $2[1, -3/2, 1/2]^\top = [2, -3, 1]^\top$. Then

$$x^1(t) = \mathrm{e}^{2t} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

Now we will find $x^2(t)$. First we need to find a vector v such that

$$(A - 2I)^2 v = 0 \qquad \text{and} \qquad (A - 2I)v \neq 0.$$

This implies that v needs to satisfy

$$\begin{bmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{bmatrix}^2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -4 & -4 & -4 \\ 6 & 6 & 6 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which is equivalent to $v_1 + v_2 + v_3 = 0$. The condition $(A - 2I)v \neq 0$ is

$$\begin{bmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -6v_1 - 4v_2 \\ 10v_1 + 7v_2 + v_3 \\ -4v_1 - 3v_2 - v_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can choose $v = [1, 0, -1]^\top$. Then

$$x^2(t) = \mathrm{e}^{2t}(v + t(A - 2I)v) = \mathrm{e}^{2t} \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -6 \\ 9 \\ -3 \end{bmatrix} \right) = \mathrm{e}^{2t} \begin{bmatrix} -6t + 1 \\ 9t \\ -3t - 1 \end{bmatrix}.$$

It remains to determine $x^3(t)$. First we need to find a vector v such that

$$(A - 2I)^3 v = 0 \qquad \text{and} \qquad (A - 2I)^2 v \neq 0.$$

Calculation shows the matrix $(A - 2I)^3$ is a zero matrix. Then any vector v with the property $v_1 + v_2 + v_3 \neq 0$ can be used to generate $x^3(t)$. Let $v = [1, 0, 0]^\top$. Then $(A - 2I)v = [-6, 10, -4]^\top$, $(A - 2I)^2 v = [-4, 6, -2]^\top$ and

$$\begin{aligned} x^3(t) &= \mathrm{e}^{2t} \left(v + t(A - 2I)v + \frac{t^2}{2}(A - 2I)^2 v \right) \\ &= \mathrm{e}^{2t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ 10 \\ -4 \end{bmatrix} + t^2 \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} \right) = \mathrm{e}^{2t} \begin{bmatrix} -2t^2 - 6t + 1 \\ 3t^2 + 10t \\ -t^2 - 4t \end{bmatrix}. \end{aligned}$$

The general solution is

$$x(t) = c_1 \mathrm{e}^{2t} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{2t} \begin{bmatrix} -6t + 1 \\ 9t \\ -3t - 1 \end{bmatrix} + c_3 \mathrm{e}^{2t} \begin{bmatrix} -2t^2 - 6t + 1 \\ 3t^2 + 10t \\ -t^2 - 4t \end{bmatrix}.$$

From the initial condition $x(0) = [2, 1, -1]^\top$ we get

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then equating second components from both sides we obtain $-3c_1 = 1$, $c_1 = -1/3$. The last equation $-1 = c_1 - c_2$ results in $c_2 = c_1 + 1 = 2/3$. Finally, $c_3 = 2 - 2c_1 - c_2 = 2$. The final solution is

$$\begin{aligned} x(t) &= -\frac{1}{3}\mathrm{e}^{2t} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \frac{2}{3}\mathrm{e}^{2t} \begin{bmatrix} -6t + 1 \\ 9t \\ -3t - 1 \end{bmatrix} + 2\mathrm{e}^{2t} \begin{bmatrix} -2t^2 - 6t + 1 \\ 3t^2 + 10t \\ -t^2 - 4t \end{bmatrix} \\ &= \mathrm{e}^{2t} \begin{bmatrix} -4t^2 - 16t + 2 \\ 6t^2 + 26t + 1 \\ -2t^2 - 10t - 1 \end{bmatrix}. \end{aligned}$$