University Course

Math 228A

Numerical Solution of Differential Equations

University of California, Davis

Fall 2010

My Class Notes

Nasser M. Abbasi

Fall 2010

Contents

Chapter 1

Introduction

I took this course in Fall 2010 to learn about numerical solutions of PDE's.

course description from catalog

228A-228B-228C. Numerical Solution of Differential Equations (4-4-4) Lecture 3 hours; term paper or discussion 1 hour. Prerequisite: course 128C. Numerical solutions of initial-value, eigenvalue and boundary-value problems for ordinary differential equations. Numerical solution of parabolic and hyperbolic partial differential equations. Offered in alternate years

official class syllabus

See Professor Guy web page HTML In case the syllabus goes away, here is an archive image

Math 228A
Numerical Methods for PDEs
Fall Quarter 2010

Instructor:	Professor Bob Guy
Office:	MSB 2136
Email:	guy@math.ucdavis.edu
Phone:	754-9201
Office Hours:	Tuesday 3-4
	Thursday 3-4

Textbook:
    1. R. J. LeVeque. Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems. SIAM, 2007.
    2. W. Briggs, V.E. Henson, and S. McCormick. A Multigrid Tutorial, Second Edition. SIAM, 2000.

    Both books are published by SIAM. SIAM members receive a 30% discount on orders. Free membership in SIAM is available by joining the UCD SIAM student chapter. http://siam.math.ucdavis.edu

Webpage:
    http://www.math.ucdavis.edu/~guy/teaching/228a/
    Homework and announcements will be posted here.

Class:
    Tuesday and Thursday 1:40-3:00 in Physics 140

TA:	Hsiao-Chieh (Arcade) Tseng
Office:	MSB 2123
Email:	hctseng@math.ucdavis.edu
Office Hours:	Monday 1:30-2:30
	Friday 1:30-2:30

Homework
You are encouraged to talk with your classmates about homework problems. However, you must do your own write-up and write your own codes. All aspects of your write up must be clearly presented. Your writing should be clear and grammatically correct. Your codes must be thoroughly commented. All tables and figures must be appropriately labeled. You will be graded on the quality of your presentation.

What we will cover
This coarse is part of the sequence 228A-C on numerical methods for partial differential equations. The first quarter (228A) will focus on elliptic problems. The topics we will cover this quarter are listed below.

- Introduction to Poisson equation
- Finite differences
- Convergence and accuracy
- Iterative methods
- Multigrid
- Krylov subspace methods, CG & GMRES
- FFT

Grading
Your grade will be based on your homework assignments. We will likely have 4 or 5 homework assignments during this quarter.

Programing
This class will require writing computer programs. You may use any language. If you do not have a strong preference of language, it is recommended that you use MATLAB, because it is easy to use and very powerful. All codes will be turned in and must be thoroughly commented. You can create a computer account in the math department at http://www.math.ucdavis.edu/comp/class-accts.

Text book
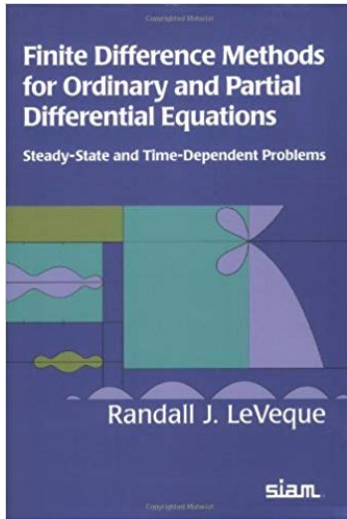
Chapter 2

HWs

Local contents

2.1 Summary of HW's

HW	description	grade
1	Poisson pde, eigenvalue problem, refinment study 1D	5/5
2	Refinment studies, 1D, LTE, convergence	5/5
3	2D, Jacobi, SOR, Gauss-Seidel, direct solver, sparse matrices	5/5
4	Multigrid V cycle algorithm, using to solve Poisson 2D	5/5
5	Conjugate Gradient	5/5

2.2 HW 1

2.2.1 Problem description

Math 228A
Homework 1
Due Tuesday, 10/12/10

1. Let L be the linear operator $Lu = u_{xx}$, $u_x(0) = u_x(1) = 0$.

   (a) Find the eigenfunctions and corresponding eigenvalues of L.

   (b) Show that the eigenfunctions are orthogonal in the $L^2[0,1]$ inner product:

   $$\langle u, v \rangle = \int_0^1 uv\, dx.$$

   (c) It can be shown that the eigenfunctions, $\phi_j(x)$, form a complete set in $L^2[0,1]$. This means that for any $f \in L^2[0,1]$, $f(x) = \sum_j \alpha_j \phi_j(x)$. Express the solution to

   $$u_{xx} = f, \quad u_x(0) = u_x(1) = 0, \tag{1}$$

   as a series solution of the eigenfunctions.

   (d) Note that equation (1) does not have a solution for all f. Express the condition for existence of a solution in terms of the eigenfunctions of L.

2. Define the functional $F : X \to \Re$ by

   $$F(u) = \int_0^1 \frac{1}{2}(u_x)^2 + fu\, dx,$$

   where X is the space of real valued functions on $[0,1]$ that have at least one continuous derivative and are zero at $x = 0$ and $x = 1$. The Frechet derivative of F at a point u is defined to be the linear operator $F'(u)$ for which

   $$F(u + v) = F(u) + F'(u)v + R(v),$$

   where

   $$\lim_{||v|| \to 0} \frac{||R(v)||}{||v||} = 0.$$

   One way to compute the derivative is

   $$F'(u)v = \lim_{\epsilon \to 0} \frac{F(u + \epsilon v) - F(u)}{\epsilon}.$$

   Note that this looks just like a directional derivative.

   (a) Compute the Frechet derivative of F.

   (b) $u \in X$ is a critical point of F if $F'(u)v = 0$ for all $v \in X$. Show that if u is a solution to the Poisson equation

   $$u_{xx} = f, \quad u(0) = u(1) = 0,$$

   then it is a critical point of F.

Finite element methods are based on these "weak formulations" of the problem. The Ritz method is based on minimizing F and the Galerkin method is based on finding the critical points of $F'(u)$.

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Figure 2.1: problem description

2.2.2 Problem 1

L is a second order differential operator defined by $Lu \equiv u_{xx}$ with boundary conditions on u given as $u_x(0) = u_x(1) = 0$

2.2.2.1 part a

Let $\phi(x)$ be an eigenfunction of the operator L associated with an eigenvalue λ. To obtain the eigenfunctions and eigenvalues, we solve an eigenvalue problem $L\phi = \lambda\phi$ where λ is scalar. Hence the problem is to solve the differential equation

$$\phi_{xx} - \lambda\phi = 0 \tag{1}$$

with B.C. given as $\phi'(0) = \phi'(1) = 0$. The characteristic equation is

$$r^2 - \lambda = 0$$

The roots are $r = \pm\sqrt{\lambda}$, therefore the solution to the eigenvalue problem (1) is

$$\phi(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x} \tag{2}$$

Where c_1, c_2 are constants.

$$\phi'(x) = c_1\sqrt{\lambda}e^{\sqrt{\lambda}x} - \sqrt{\lambda}c_2 e^{-\sqrt{\lambda}x} \tag{3}$$

First we determine the allowed values of the eigenvalues λ which satisfies the boundary conditions.

1. Assume $\lambda = 0$ The solution (2) becomes $\phi(x) = c_1 + c_2$. Hence the solution is a constant. In other words, when the eigenvalue is zero, the eigenfunction is a constant. Let us now see if this eigenfunction satisfies the B.C. Since $\phi(x)$ is constant, then $\phi'(x) = 0$, and this does satisfy the B.C. at both $x = 0$ and $x = 1$. Hence $\lambda = 0$ is an eigenvalue with a corresponding eigenfunction being a constant. We can take the constant as 1.

2. Assume $\lambda > 0$ From the first BC we have, from (3), that $\phi'(0) = 0 = c_1\sqrt{\lambda} - \sqrt{\lambda}c_2$ or

$$c_1 = c_2$$

and from the second BC we have that $\phi'(1) = 0 = c_1\sqrt{\lambda}e^{\sqrt{\lambda}} - \sqrt{\lambda}c_2 e^{-\sqrt{\lambda}}$ or

$$c_1 e^{\sqrt{\lambda}} - c_2 e^{-\sqrt{\lambda}} = 0$$

From the above 2 equations, we find that $e^{\sqrt{\lambda}} = e^{-\sqrt{\lambda}}$ which is not possible for positive λ. Hence λ can not be positive.

3. Assume $\lambda < 0$. Let $\lambda = -\beta^2$ form some positive β. Then the solution (2) becomes

$$\phi(x) = c_1 e^{i\beta x} + c_2 e^{-i\beta x}$$

which can be transformed using the Euler relation to obtain

$$\phi(x) = c_1\cos\beta x + c_2\sin\beta x$$
$$\phi'(x) = -c_1\beta\sin\beta x + c_2\beta\cos\beta x \tag{4}$$

Now consider the BC's. Since $\phi'(0) = 0$ we obtain $c_2 = 0$ and from $\phi'(1) = 0$ we obtain $0 = c_1\beta\sin\beta$ and hence for non trivial solution, i.e. for $c_1 \neq 0$, we must have that

$$\sin\beta = 0$$

or

$$\beta = \pm n\pi$$

but since β is positive, we consider only $\beta_n = n\pi$, where n is positive integer $n = 1, 2, 3, \cdots$

Conclusion: The eigenvalues are

$$\lambda_n = -\left(\beta_n\right)^2 = -(n\pi)^2 = \left\{0, -\pi^2, -(2\pi)^2, -(3\pi)^2, \cdots\right\}$$

And the corresponding eigenfunctions are $\phi_n(x) = \cos \beta_n x = \cos n\pi x = \{1, \cos \pi x, \cos 2\pi x, \cos 3\pi x, \cdots\}$ where $n = 0, 1, 2, \cdots$

2.2.2.2 part (b)

Given inner product defined as $\langle u, v \rangle = \displaystyle\int_0^1 uvdx$, then

$$\langle \phi_n, \phi_m \rangle = \int_0^1 \left(\cos \beta_n x\right)\left(\cos \beta_m x\right) dx$$

$$= \int_0^1 (\cos n\pi x)(\cos m\pi x)\, dx$$

$$= \begin{cases} 0 & n \neq m \\ \dfrac{1}{2} & n = m \end{cases}$$

Also, the first eigenfucntion, $\phi_0(x) = 1$ is orthogonal to all other eigenfunctions, since $\displaystyle\int_0^1 (\cos n\pi x)\, dx = \frac{1}{n\pi} [\sin n\pi x]_0^1 = 0$ for any integer $n > 0$.

Hence all the eigenfunctions are orthogonal to each others in $L^2[0,1]$ space.

2.2.2.3 Part (c)

Given

$$u_{xx} = f$$

$u_x(0) = u_x(1) = 0$. This is $Lu = f$. We have found the eigenfunctions $\phi(x)$ of L above. These are basis of the function space of L where f resides in. We can express f as a linear combination of the eigenfunctions of the operator L, hence we write

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$$

where $\phi_n(x)$ is the nth eigenfunction of L and a_n is the corresponding coordinate (scalar).

Therefore the differential equation above can be written as

$$Lu = f(x) = \sum_{n=0} a_n \phi_n(x) \tag{1}$$

But since

$$L\phi_n = \lambda_n \phi_n$$

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Then

$$L^{-1} = \frac{1}{\lambda_n}$$

Therefore, using (1), the solution is

$$u(x) = \sum_n \left(\frac{a_n}{\lambda_n}\right)\phi_n(x) \tag{2}$$

Now to find a_n, using $f(x) = \sum_n a_n\phi_n(x)$, we multiply each side by an eigenfunction, say $\phi_m(x)$ and integrate

$$\int_0^1 \phi_m(x) f(x) \, dx = \int_0^1 \phi_m(x) \sum_n a_n\phi_n(x)(x) \, dx$$

$$= \int_0^1 \sum_n a_n\phi_m(x)\phi_n(x) \, dx$$

$$= \sum_n a_n \int_0^1 \phi_m(x)\phi_n(x) \, dx$$

The RHS is 1/2 when $n = m$ and zero otherwise, hence the above becomes

$$\int_0^1 \phi_n(x) f(x) \, dx = \frac{a_n}{2}$$

Or

$$a_n = 2\int_0^1 \cos(n\pi x) f(x) \, dx \tag{3}$$

Where a_n as given by (3).

If we know $f(x)$ we can determine a_n and hence the solution is now found.

2.2.2.4   part (d)

The solution found above

$$u(x) = \sum_n \left(\frac{a_n}{\lambda_n}\right)\phi_n(x)$$

Is not possible for all f. Only an f which has $a_0 = 0$ is possible. This is because $\lambda_0 = 0$, then a_0 has to be zero to obtain a solution (since L^{-1} does not exist if an eigenvalue is zero).

$a_0 = 0$ implies, by looking at (3) above, that when $n = 0$ we have

$$0 = \int_0^1 f(x) \, dx$$

So only the functions $f(x)$ which satisfy the above can be a solution to $Lu = f$ with the B.C. given.

To review: We found that $\lambda = 0$ to be a valid eigenvalue due to the B.C. being Von Neumann boundary conditions. This in resulted in a_0 having to be zero. This implied that $\int_0^1 f(x)\,dx = 0$.

Having a zero eigenvalue effectively removes one of the space dimensions that $f(x)$ can resides in.

In addition to this restriction, the function $f(x)$ is *assumed* to meet the Dirichlet conditions for Fourier series expansion, and these are

1. $f(x)$ must have a finite number of extrema in any given interval

2. $f(x)$ must have a finite number of discontinuities in any given interval

3. $f(x)$ must be absolutely integrable over a period.

4. $f(x)$ must be bounded

2.2.3 Problem 2

2.2.3.1 Part (a)

Applying the definition given

$$F'(u)\,v = \lim_{\varepsilon \to 0} \frac{F(u + \varepsilon v) - F(u)}{\varepsilon} \tag{1}$$

And using $F(u) = \int_0^1 \frac{1}{2}(u_x)^2 + f\ u\ dx$, then (1) becomes

$$F'(u)\,v = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(\int_0^1 \frac{1}{2}\left[(u + \varepsilon v)_x\right]^2 + f\ (u + \varepsilon v)\ dx - \int_0^1 \frac{1}{2}(u_x)^2 + f\ u\ dx \right)$$

Simplify the above, we obtain

$$F'(u)\,v = \lim_{\varepsilon \to 0}\left(\int_0^1 \frac{\varepsilon}{2}v_x^2 dx + \int_0^1 u_x v_x dx + \int_0^1 f v dx \right)$$

Hence, as $\varepsilon \to 0$ only the first integral above vanishes (since v_x is bounded), and we have

$$F'(u)\,v = \int_0^1 u_x v_x + f v dx \tag{1A}$$

2.2.3.2 Part (b)

The solution to $u_{xx} = f(x)$ with $u(0) = u(1) = 0$ was found in class to be

$$u(x) = \sum_n \left(\frac{a_n}{\lambda_n}\right)\phi_n(x) \tag{2}$$

where

$$\phi_n(x) = \sin(n\pi x)$$

are the eigenfunctions associated with the eigenvalues $\lambda_n = -n^2\pi^2$.

Now we can use this solution in the definition of $F'(u)v$ found in (1A) from part (a). Substitute $u(x)$ from (2) into (1A), and also substitute $f = \sum_n a_n \phi_n(x)$ into (1A), we obtain

$$F'(u)v = \int_0^1 \left(\sum_n \left(\frac{a_n}{\lambda_n} \right) \phi_n(x) \right)' v' + \left(\sum_n a_n \phi_n(x) \right) v \, dx \tag{4}$$

We need to show that the above becomes zero for any $v(x) \in X$.

$$F'(u)v = \int_0^1 \sum_n v' \left(\frac{a_n}{\lambda_n} \right) \phi_n'(x) + \sum_n v a_n \phi_n(x) \, dx$$

$$= \int_0^1 \sum_n \left(v' \left(\frac{a_n}{\lambda_n} \right) \phi_n'(x) + v a_n \phi_n(x) \right) dx$$

$$= \sum_n a_n \left(\int_0^1 \frac{1}{\lambda_n} v' \phi_n'(x) + v \phi_n(x) \, dx \right) \tag{5}$$

Now we pay attention to the integral term above. If we can show this is zero, then we are done.

$$I = \frac{1}{\lambda_n} \int_0^1 v' \phi_n'(x) + \int_0^1 v \phi_n(x) \, dx$$

$$= I_1 + I_2 \tag{6}$$

Integrate by parts I_1

$$I_1 = \frac{1}{\lambda_n} \int_0^1 \phi_n'(x) \overbrace{v' dx}^{udv}$$

$$= \frac{1}{\lambda_n} \left(\left[\phi_n'(x) v \right]_0^1 - \int_0^1 v(x) \phi_n''(x) \, dx \right)$$

$$= \frac{1}{\lambda_n} \left(\overbrace{\left[v(1) \phi_n'(1) - v(0) \phi_n'(0) \right]}^{\text{zero due to boundaries on } v(x) \in X} - \int_0^1 v(x) \phi_n''(x) \, dx \right)$$

$$= -\frac{1}{\lambda_n} \int_0^1 v(x) \phi_n''(x) \, dx$$

But since $\phi_n(x) = \sin n\pi x$, then $\phi_n'(x) = n\pi \cos n\pi x$ and $\phi_n''(x) = -n^2\pi^2 \sin n\pi x = -n^2\pi^2 \phi_n(x)$ then

$$I_1 = \frac{n^2\pi^2}{\lambda_n} \int_0^1 v(x) \phi_n(x) \, dx$$

But also $\lambda_n = -n^2\pi^2$ hence the above becomes

$$I_1 = -\int_0^1 v(x)\,\phi_n(x)\,dx$$

Therefore (6) can be written as

$$I = I_1 + I_2$$
$$= -\int_0^1 v(x)\,\phi_n(x)\,dx + \int_0^1 v(x)\,\phi_n(x)\,dx$$
$$= 0$$

Therefore, from (5), we see that

$$F'(u)\,v = 0$$

Hence we showed that if u is solution to $u_{xx} = f$ with $u(0) = u(1) = 0$, then $F'(u)\,v = 0$.

2.2.4 Problem (3)

2.2.4.1 Part (a)

Notations used: let $\tilde{f}$ to mean the approximate discrete solution at a grid point. Let f to mean the exact solution.

Using the method of undetermined coefficients, let the second derivative approximation be

$$\tilde{f}''(x) = af\left(x - \frac{h}{2}\right) + bf(x) + cf(x+h) \tag{1}$$

Where a, b, c are constants to be found. Now using Taylor expansion, since

$$f(x+\Delta) = f(x) + \Delta f'(x) + \frac{\Delta^2}{2!}f''(x) + \frac{\Delta^3}{3!}f'''(x) + O\left(h^4\right)$$

Hence apply the above to each of the terms in the RHS of (1) and simplify

$$f\left(x - \frac{h}{2}\right) = f(x) - \frac{h}{2}f'(x) + \frac{\left(-\frac{h}{2}\right)^2}{2!}f''(x) + \frac{\left(-\frac{h}{2}\right)^3}{3!}f''(x) + \frac{\left(-\frac{h}{2}\right)^4}{4!}f^{(4)}(x) + O\left(h^5\right)$$

$$f(x) = f(x)$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f''(x) + \frac{h^4}{4!}f^{(4)}(x) + O\left(h^5\right)$$

Substitute the above 3 terms in (1)

$$\tilde{f}''(x) = a\left(f(x) - \frac{h}{2}f'(x) + \frac{h^2}{8}f''(x) - \frac{h^3}{8\times 6}f'''(x) + \frac{h^4}{16\times 24}f^{(4)}(x) + O\left(h^5\right)\right)$$
$$+ bf(x)$$
$$+ c\left(f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x) + O\left(h^5\right)\right)$$

Collect terms

$$\tilde{f}''(x) = (a+b+c)f(x) + f'(x)h\left(-\frac{a}{2}+c\right) + f''(x)h^2\left(\frac{a}{8}+\frac{c}{2}\right) + f'''(x)h^3\left(\frac{-a}{8\times 6}+\frac{c}{6}\right) \quad (2)$$

$$+ f^{(4)}h^4\left(\frac{a}{16\times 24}+\frac{c}{24}\right) + O\left(h^5\right)$$

Hence for $\tilde{f}''(x)$ to best approximate $f''(x)$, we need

$$(a+b+c) = 0$$

$$-\frac{a}{2}+c = 0$$

$$h^2\left(\frac{a}{8}+\frac{c}{2}\right) = 1$$

Solving the above 3 equations we find

$$a = \frac{8}{3h^2}$$

$$b = -\frac{4}{h^2}$$

$$c = \frac{4}{3h^2}$$

Hence (1) becomes

$$\tilde{f}''(x) = af\left(x-\frac{h}{2}\right) + bf(x) + cf(x+h)$$

$$= \frac{8}{3h^2}f\left(x-\frac{h}{2}\right) - \frac{4}{h^2}f(x) + \frac{4}{3h^2}f(x+h)$$

To examine the local truncation error, from (2), and using the solution we just found for a, b, c we find

$$\tilde{f}''(x) = f''(x) + f'''(x)h^3\left(\frac{-\left(\frac{8}{3h^2}\right)}{8\times 6} + \frac{\left(\frac{4}{3h^2}\right)}{6}\right) + f^{(4)}h^4\left(\frac{\left(\frac{8}{3h^2}\right)}{16\times 24} + \frac{\left(\frac{4}{3h^2}\right)}{24}\right) + O\left(h^5\right)$$

$$= f''(x) + f'''(x)h^3\left(\frac{1}{6h^2}\right) + f^{(4)}h^4\left(\frac{1}{16h^2}\right) + O\left(h^5\right)$$

$$= f''(x) + f'''(x)\left(\frac{h}{6}\right) + f^{(4)}h^2\left(\frac{1}{16}\right) + O\left(h^5\right)$$

We can truncate at either $f'''(x)$ or $f^{(4)}$. In the first case, we obtain

$$\tilde{f}''(x) = f''(x) + O(h)$$

Where $O(h) = \frac{f'''(x)}{6}h$, hence $p = 1$ in this case, and with the truncation error $\tau = \frac{f'''(x_j)}{6}h$ at each grid point.

In the second case, we obtain

$$\tilde{f}''(x) = f''(x) + \frac{f'''(x)}{6}h + O\left(h^2\right)$$

Where $O\left(h^2\right) = \frac{f^{(4)}}{16}h^2$ and $p = 2$ in this case, and with the truncation error $\tau = \frac{f^{(4)}(x_j)}{16}h^2$ at each grid point. We see that τ is smaller if we use $p = 2$ than $p = 1$.

The accuracy then depends on where we decide to truncate. For example, at $p = 1$, the error is dominated by $O(h)$, and at $p = 2$, it is $O\left(h^2\right)$.

2.2.4.2   part (b) Refinement study

Given $f(x) = \cos(2\pi x)$, first, let us find the accuracy of this scheme. The finite difference approximation formula found is

$$\tilde{f}''(x) = \frac{8}{3h^2}f\left(x - \frac{h}{2}\right) - \frac{4}{h^2}f(x) + \frac{4}{3h^2}f(x + h) \tag{1}$$

And the exact value is

$$\frac{d^2}{dx^2}\cos(2\pi x) = -4\pi^2\cos 2\pi x \tag{2}$$

To find the local error τ

$$\tau = \tilde{f}''(x) - f''(x)$$

Substitute $f(x) = \cos(2\pi x)$ in the RHS of (1) to find the approximation of the second derivative and subtract the exact result value of the second derivative from it.

Plug $f(x) = \cos(2\pi x)$ in RHS of (1) we obtain

$$\tilde{f}''(x) = \frac{8}{3h^2}\cos\left(2\pi\left(x - \frac{h}{2}\right)\right) - \frac{4}{h^2}\cos(2\pi x) + \frac{4}{3h^2}\cos(2\pi(x + h))$$

$$= \frac{8}{3h^2}\cos(2\pi x - \pi h) - \frac{4}{h^2}\cos(2\pi x) + \frac{4}{3h^2}\cos(2\pi x + 2\pi h)$$

Hence the local error τ is

$$\tau = \tilde{f}''(x) - f''(x)$$

$$= \left[\frac{8}{3h^2}\cos(2\pi x - \pi h) - \frac{4}{h^2}\cos(2\pi x) + \frac{4}{3h^2}\cos(2\pi x + 2\pi h)\right] + 4\pi^2\cos 2\pi x$$

We notice that τ depends on h and x. At $x = 1$,

$$\tau = \left[\frac{8}{3h^2}\cos(2\pi - \pi h) - \frac{4}{h^2} + \frac{4}{3h^2}\cos(2\pi + 2\pi h)\right] + 4\pi^2$$

$$= \frac{4}{3h^2}\left(\cos(2\pi h) + 2\cos(\pi h) + 3h^2\pi^2 - 3\right)$$

In the following we plot local error τ as a function of h in linear scale and log scale. Here is the result.
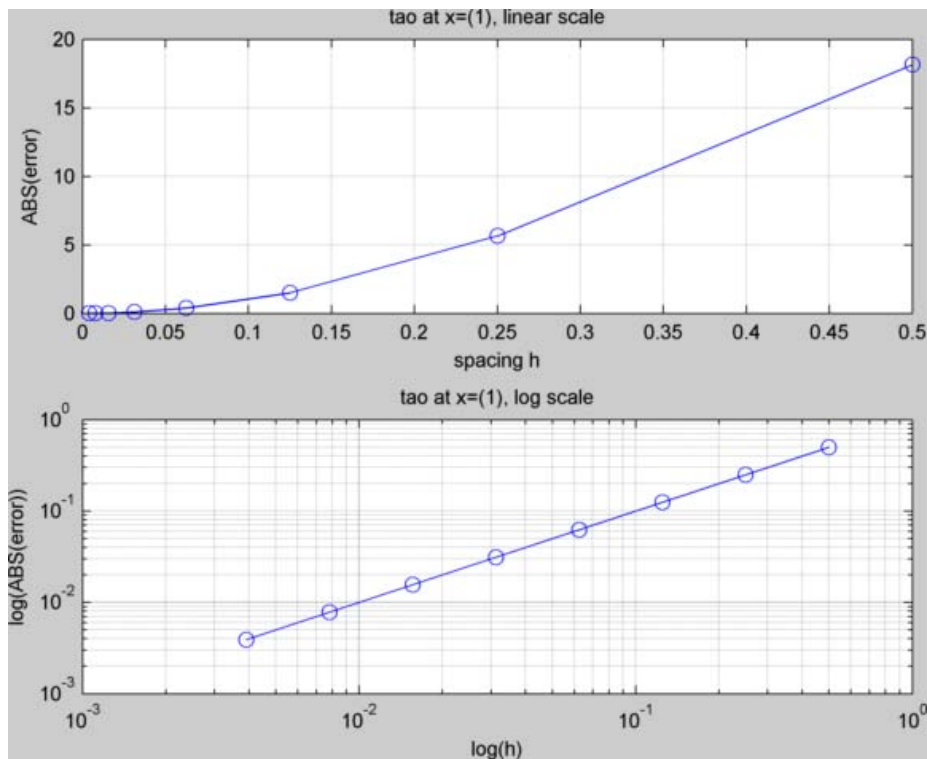
Figure 2.2: matlab HW1 partb

We notice that the log plot shows the slope $p = 2$ and not $p = 1$. This is because the $O(h)$ part turned out to be zero at $x = 1$ this is because $O(h) = \frac{f'''(x)}{6}h = \frac{(8\pi^3 \sin 2\pi x)}{6}h$ and this term is zero at $x = 1$, so the dominant error term became $O\left(h^2\right)$ which is $\frac{f^{(4)}\left(x_j\right)}{16}h^2 = \frac{16\pi^4 \cos 2\pi x}{16}h^2$ or $\pi^4 h^2$ or $O\left(h^2\right)$.

This is why we obtained $p = 2$ and not $p = 1$ at $x = 1$.

The following table show the ratio of the local error between each 2 successive halving of the spacing h. Each time h is halved, and the ratio of the error (absolute local error) is shown. We see for $x = 1$ that the ratio approaches 4. This indicates that $p = 2$.

```
 1  EDU>> nma_HW1_partb()
 2  h                   error              ratio
 3  5.0000E-001         1.8145E+001        0.0000E+000
 4  2.5000E-001         5.6483E+000        3.2125E+000
 5  1.2500E-001         1.4936E+000        3.7816E+000
 6  6.2500E-002         3.7872E-001        3.9439E+000
 7  3.1250E-002         9.5014E-002        3.9859E+000
 8  1.5625E-002         2.3775E-002        3.9965E+000
 9  7.8125E-003         5.9449E-003        3.9991E+000
10  3.9063E-003         1.4863E-003        3.9998E+000
```

.

2.2.4.3   Part(c)

The refinement study in part (b) showed that the local error became smaller as h become smaller, and the error was $O\left(h^2\right)$ since $p = 2$ in the log plot.

But this is not a good test as it was done only for one point $x = 1$. We need to examine the approximation scheme at other points as well. The reason is the local error at an x location is

$$\tau = \left[\frac{8}{3h^2}\cos\left(2\pi x - \pi h\right) - \frac{4}{h^2}\cos\left(2\pi x\right) + \frac{4}{3h^2}\cos\left(2\pi x + 2\pi h\right)\right] + 4\pi^2\cos 2\pi x$$

which can be seen to be a function of x and h. In (b) we found that at $x = 1$, $\tau = O\left(h^2\right)$ and this was because the dominant error term $O(h)$ happened to vanish at $x = 1$.

But if we examine τ at different point, say $x = 0.2$, then we will see that τ is $O\left(h\right)$ and $p = 1$.

Here is a plot of τ at $x = 0.2$ and at $x = 1$. Both showing what happens as h becomes smaller. We see that the at $x = 1$ the approximation was more accurate ($p = 2$) but at $x = 0.2$ the approximation was less accurate ($p = 1$). What we conclude from this, is that a single test is not sufficient for determine the accuracy for all points. More tests are needed at other points to have more confidence. To verify that at $x = 0.2$ we indeed have $p = 1$, we generate the error table as shown above, but for $x = 0.2$ this time.
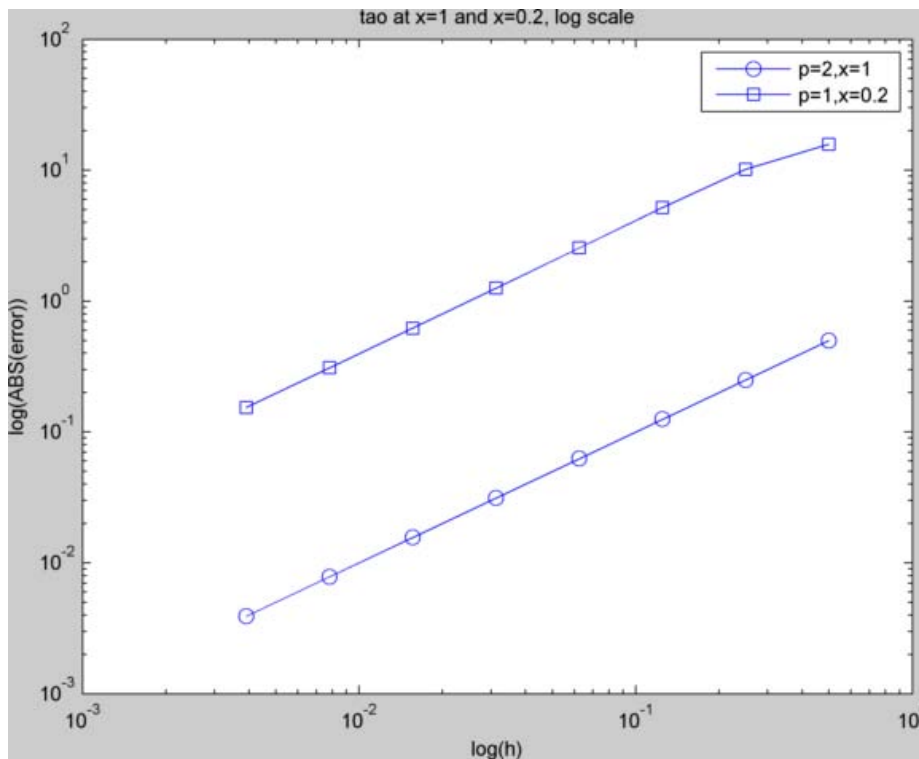
Figure 2.3: matlab HW1 partc

```
 1  EDU>> nma_HW1_partc()
 2     h                    error                  ratio
 3  5.0000E-001          1.5752E+001           0.0000E+000
 4  2.5000E-001          1.0149E+001           1.5520E+000
 5  1.2500E-001          5.1898E+000           1.9557E+000
 6  6.2500E-002          2.5508E+000           2.0345E+000
 7  3.1250E-002          1.2551E+000           2.0324E+000
 8  1.5625E-002          6.2133E-001           2.0200E+000
 9  7.8125E-003          3.0897E-001           2.0110E+000
10  3.9063E-003          1.5404E-001           2.0057E+000
```

.

We see that the ratio becomes 2 this time, not 4 as we half the spacing each time. This mean $p = 1$. This means the accuracy of the formula used can depend on the location.

2.2.4.4 Part (d)

The points that we need to interpolate are $\left[\left[x - \frac{h}{2}, u\left(x - \frac{h}{2}\right)\right], [x, u(x)], [x + h, u(x + h)]\right]$ where $u = \cos(2\pi x)$

Since we require a quadratic polynomial, then we write

$$p(x) = a + bx + cx^2$$

Where $p(x)$ is the interpolant. Evaluate the above at each of the 3 points. Choose $x = 1$, hence the points are

$$\begin{pmatrix} 1 - \frac{h}{2}, u\left(1 - \frac{h}{2}\right) \\ 1, u(x) \\ 1 + h, u(1 + h) \end{pmatrix}$$

Evaluate $p(x)$ at each of these points

$$p\left(1 - \frac{h}{2}\right) = \cos\left(2\pi\left(1 - \frac{h}{2}\right)\right) = a + b\left(1 - \frac{h}{2}\right) + c\left(1 - \frac{h}{2}\right)^2$$

$$p(1) = \cos(2\pi) = a + b + c$$

$$p(1 + h) = \cos(2\pi(1 + h)) = a + b(1 + h) + c(1 + h)^2$$

or

$$\begin{pmatrix} \left(1 - \frac{h}{2}\right)^2 & \left(1 - \frac{h}{2}\right) & 1 \\ 1 & 1 & 1 \\ (1 + h)^2 & (1 + h) & 1 \end{pmatrix}\begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{pmatrix} \cos\left(2\pi\left(1 - \frac{h}{2}\right)\right) \\ \cos(2\pi 1) \\ \cos(2\pi(1 + h)) \end{pmatrix}$$

$$Av = b$$

Solving the above Vandermonde system, we obtain

$$a = \frac{1}{3h^2}(4(1 + h)\cos(h\pi) + (h - 2)(3 + 3h - \cos(2\pi h)))$$

$$b = \frac{-1}{3h^2}4((h - 4)\cos(\pi h) - h - 8)\sin^2\left(\frac{\pi h}{2}\right)$$

$$c = \frac{2}{3h^2}(2\cos(\pi h) - 3 + \cos(2\pi h))$$

Hence

$$p(x) = \left[\frac{1}{3h^2}(4(1 + h)\cos(h\pi) + (h - 2)(3 + 3h - \cos(2\pi h)))\right] \tag{1}$$

$$- \left[\frac{1}{3h^2}4((h - 4)\cos(\pi h) - h - 8)\sin^2\left(\frac{\pi h}{2}\right)\right]x$$

$$+ \left[\frac{2}{3h^2}(2\cos(\pi h) - 3 + \cos(2\pi h))\right]x^2$$

Recall, that we found, for $u = \cos(2\pi x)$, the finite difference formula was

$$\tilde{u}''(x) = \left[\frac{8}{3h^2}\cos(2\pi x - \pi h) - \frac{4}{h^2}\cos(2\pi x) + \frac{4}{3h^2}\cos(2\pi x + 2\pi h)\right] \tag{2}$$

Take the second derivative of $p(x)$ shown in (1) above

$$p''(x) = \frac{4}{3h^2}(2\cos(\pi h) + \cos(2\pi h) - 3) \tag{3}$$

But we notice that $\tilde{u}''(x)$ evaluated at $x = 1$ is

$$\tilde{u}''(1) = \frac{4}{3h^2}(2\cos(\pi h) + \cos(2\pi h) - 3)$$

which is the same as $p''(x)$.

Therefore, $p''(x)$ is the same as as the finite difference approximation evaluated at the central point of the 3 points, used to generate p.

In other words, given 3 points

$$\begin{pmatrix} x_0 - \frac{h}{2}, u(x_0) \\ x_0, u(x_0) \\ x_0 + h, u(x_0 + h) \end{pmatrix}$$

Where $u(x)$ is some function (here it was $\cos(2\pi x)$), and we generate a quadratic interpolant polynomial $p(x)$ using the above 3 points, then $p''(x)$ will given the same value as the finite difference formula evaluated at x_0.

$$p''(x)\big|_{x=x_0} = \tilde{u}(x)\big|_{x=x_0}$$

For this to be valid, $p(x)$ must have been generated with the center point being x_0. If we pick another center point x_1, and therefore have the 3 points $x_1 - h/2, x_1, x_1 + h$, and then generate a polynomial $q(x)$ as above, then we will find

$$q''(x)\big|_{x=x_1} = \tilde{u}(x)\big|_{x=x_1}$$

This is illustrated by the following diagram

$$p''(x)\big|_{x=x_0} = \tilde{u}(x)\big|_{x=x_0} \qquad\qquad q''(x)\big|_{x=x_1} = \tilde{u}(x)\big|_{x=x_1}$$
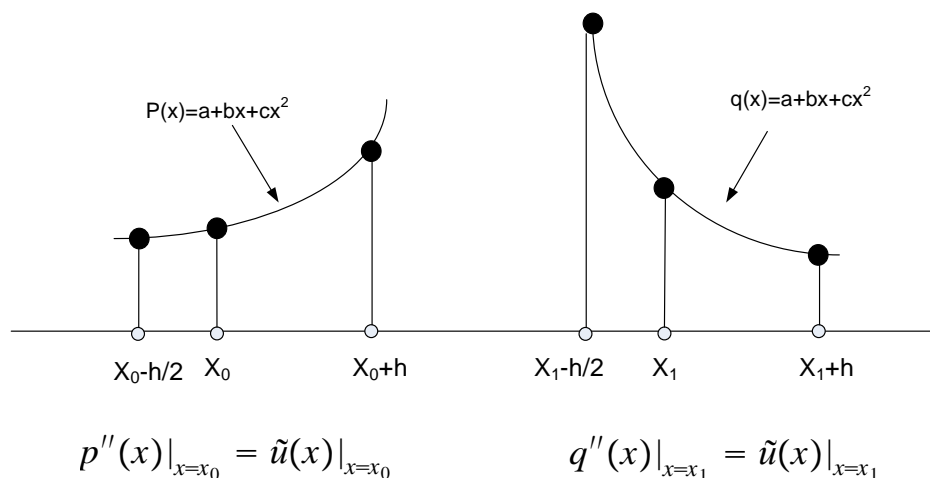
Figure 2.4: prob3 c

2.2.5 Appendix (Source code)

2.2.5.1 Matlab

```matlab
%-- by Nasser M. Abbasi, Math 228A, UC Davis, Fall 2010
%-- implement part b, problem 3
function nma_HW1_partb()
%-- Generate h values to use, and define tao(h) function
N = 8;
pointAt=1;
data  = arrayfun( @(i) [1/(2^i) , local_error(1/(2^i),pointAt)] ,1:N, ...
                  'UniformOutput',false);
data = reshape(cell2mat(data),2,N)';

%-- plot the tao(h) in linear and log scale
set(0,'defaultaxesfontsize',8) ;
set(0,'defaulttextfontsize',8);

subplot(2,1,1);
plot(data(:,1),data(:,2),'-o'); grid on;
title('tao at x=(1), linear scale');
xlabel('spacing h'); ylabel('ABS(error)');

subplot(2,1,2);
loglog(data(:,1),data(:,2),'-o'); grid on;
title('tao at x=(1), log scale');
xlabel('log(h)'); ylabel('log(ABS(error))');
export_fig matlab_HW1_partb.png

%-- now generate the error table, find ratio first
error_ratio = zeros(N,1);
for i=2:N
    error_ratio(i) = data((i-1),2)/data(i,2);
end

%-- print table
fprintf('h\t\t\t\t error\t\t\t\t ratio\n');
for i=1:N
    fprintf('%6.4E\t\t%6.4E\t\t%6.4E\n',data(i,1),data(i,2),error_ratio(i));
end

end

function tao=local_error(h,x)
tao=8/(3*h^2)*cos(2*pi*(x-h/2))-4/h^2*cos(2*pi*x)+4/(3*h^2)*...
    cos(2*pi*(x+h))+(2*pi)^2*cos(2*pi*x);
end
```

```matlab
%-- by Nasser M. Abbasi, Math 228A, UC Davis, Fall 2010
```

```matlab
%-- implement part c, problem 3
function nma_HW1_partc()

%-- Generate h values to use, and define tao(h) function
%-- plot the tao(h) in linear and log scale
set(0,'defaultaxesfontsize',8) ;
set(0,'defaulttextfontsize',8);

%build data, x-axis is spacing h, y-axis is error
N = 8;
pointAt=1.0;
data  = arrayfun( @(i) [1/(2^i) , local_error(1/(2^i),pointAt)] ,1:N, ...
                  'UniformOutput',false);
data = reshape(cell2mat(data),2,N)';

loglog(data(:,1),data(:,1),'-o'); grid off;
title('tao at x=1 and x=0.2, log scale');
xlabel('log(h)'); ylabel('log(ABS(error))');
hold on;

pointAt=0.2;
data  = arrayfun( @(i) [1/(2^i) , local_error(1/(2^i),pointAt)] ,1:N,...
                  'UniformOutput',false);
data = reshape(cell2mat(data),2,N)';
loglog(data(:,1),data(:,2),'-s');
legend('p=2,x=1','p=1,x=0.2');

export_fig matlab_HW1_partc.png

%-- now generate the error table, find ratio first
error_ratio = zeros(N,1);
for i=2:N
    error_ratio(i) = data((i-1),2)/data(i,2);
end

%-- print table
fprintf('h\t\t\t\t error\t\t\t\t ratio\n');
for i=1:N
   fprintf('%6.4E\t\t%6.4E\t\t%6.4E\n',data(i,1),data(i,2),error_ratio(i));
end

end

function tao=local_error(h,x)
tao=8/(3*h^2)*cos(2*pi*(x-h/2))-4/h^2*cos(2*pi*x)+4/(3*h^2)*...
    cos(2*pi*(x+h))+(2*pi)^2*cos(2*pi*x);
end
```

2.2.5.2 Mathematica

HW 1, problem 3, computational part. math 228A UC davis fall 2010
Nasser M. Abbasi

This is the code used to generate the plots and tables used in HW1

define local error function

```
localError[h_, x_] :=
  Module[{}, 8/(3 h^2) Cos[2 π x - π h] - 4/h^2 Cos[2 π x] + 4/(3 h^2) Cos[2 π x + 2 π h]] + (2 π)^2 Cos[2 π x];
```

define a function to make the plots

```
makePlot[x_, s_, title_, xlabel_, ylabel_, f_] := Module[{data, n = 8},
  data = Table[{ 1/(2^i), Abs@localError[1/(2^i), x] }, {i, 1, n}];
  f[data, Joined → True, AxesOrigin → {0, 0},
    GridLines → Automatic, AspectRatio → 1, Frame → True, PlotRange → All,
    FrameLabel → {{ylabel, None}, {xlabel, title}}, PlotStyle → s, ImageSize → Full]
 ]
```

make plot for problem 3, part b

```
title = Style["local error at x=1, log scale", 16];
xlabel = Style["h", 16]; ylabel = Style["local error", 16];
p1 = makePlot[1, {Thick, Dashed}, title, xlabel, ylabel, ListLogLogPlot];
title = Style["local error at x=1, linear scale", 16];
p2 = makePlot[1, {Thick, Dashed}, title, xlabel, ylabel, ListPlot];
Framed[Grid[{{p1, p2}}], ImageSize → {600, 300}]
```
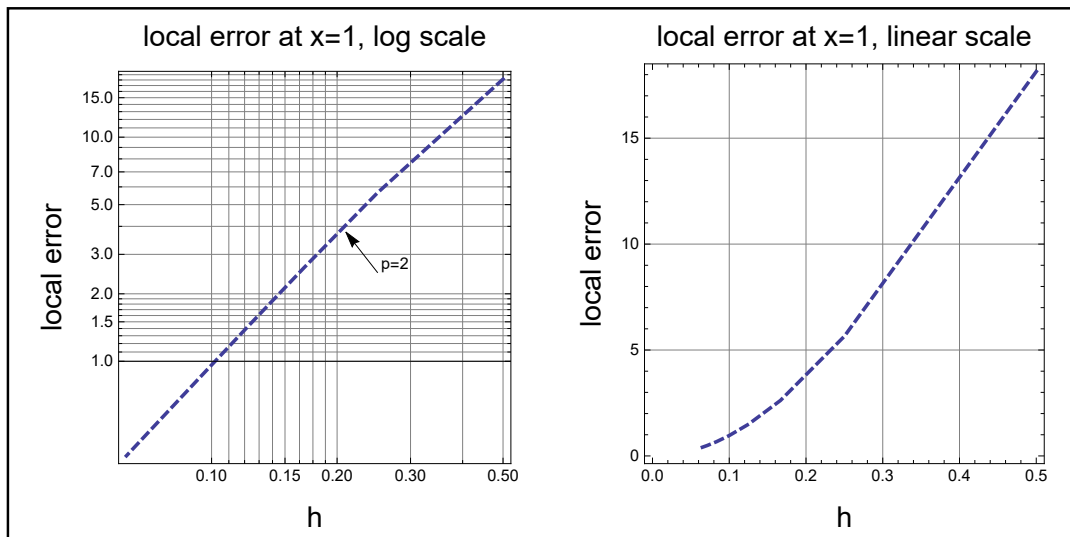
Generate error table, problem 3, part b

```
n = 14;
x = 1;
data = Table[{ 1 / (2^i), Abs@localError[1 / (2^i), x] }, {i, 1, n}];
data = Table[{ data[[i, 1]], data[[i, 2]], If[i == 1, 0, data[[i - 1, 2]] / data[[i, 2]] ] }, {i, 1, n}];
t = TableForm[N[data, $MachinePrecision], TableHeadings →
     {None, {"h", "local error τ", "ratio"}}, TableSpacing → {1, 6}, TableAlignments → Left];
Labeled[Framed@ScientificForm[t, {8, 6}, NumberFormat → (Row[{#1, "e", #3}] &),
    NumberPadding → {"", "0"}], Style["local error as function of h at x=1", 14], Top]
```

local error as function of h at x=1

h	local error τ	ratio
5.000000e-1	1.814508e1	0.000000e
2.500000e-1	5.648307e	3.212482e
1.250000e-1	1.493636e	3.781581e
6.250000e-2	3.787161e-1	3.943947e
3.125000e-2	9.501408e-2	3.985895e
1.562500e-2	2.377451e-2	3.996468e
7.812500e-3	5.944941e-3	3.999117e
3.906250e-3	1.486317e-3	3.999779e
1.953125e-3	3.715845e-4	3.999945e
9.765625e-4	9.289644e-5	3.999986e
4.882812e-4	2.322413e-5	3.999997e
2.441406e-4	5.806034e-6	3.999999e
1.220703e-4	1.451509e-6	4.000000e
6.103516e-5	3.628771e-7	4.000000e

Generate table for problem 3, part (c)

```
n = 14;
x = 0.2;
data = Table[{ 1 / (2^i), Abs@localError[1 / (2^i), x] }, {i, 1, n}];
data = Table[{ data[[i, 1]], data[[i, 2]], If[i == 1, 0, data[[i - 1, 2]] / data[[i, 2]] ] }, {i, 1, n}];
t = TableForm[N[data, $MachinePrecision], TableHeadings →
     {None, {"h", "local error τ", "ratio"}}, TableSpacing → {1, 6}, TableAlignments → Left];
Labeled[Framed@ScientificForm[t, {8, 6}, NumberFormat → (Row[{#1, "e", #3}] &),
    NumberPadding → {"", "0"}], Style["local error as function of h at x=0.2", 14], Top]
```

local error as function of h at x=0.2

h	local error τ	ratio
5.000000e-1	1.575174e1	0.000000e
2.500000e-1	1.014949e1	1.551974e
1.250000e-1	5.189762e	1.955675e
6.250000e-2	2.550829e	2.034539e
3.125000e-2	1.255100e	2.032371e
1.562500e-2	6.213251e-1	2.020037e
7.812500e-3	3.089650e-1	2.010989e
3.906250e-3	1.540406e-1	2.005737e
1.953125e-3	7.690765e-2	2.002930e
9.765625e-4	3.842539e-2	2.001480e
4.882812e-4	1.920555e-2	2.000744e
2.441406e-4	9.600990e-3	2.000372e
1.220703e-4	4.800048e-3	2.000186e
6.103516e-5	2.399851e-3	2.000144e

Generate plot for part (C)

```
title = Style["local error at different x locations, log scale", 16];
xlabel = Style["h", 16]; ylabel = Style["local error", 16];
p1 = makePlot[1, {Thick, Dashed}, title, xlabel, ylabel, ListLogLogPlot];
p2 = makePlot[0.2, {Thick, Black}, title, xlabel, ylabel, ListLogLogPlot];
Show[{p1, p2}, ImageSize → 500]
```

local error at different x locations, log scale
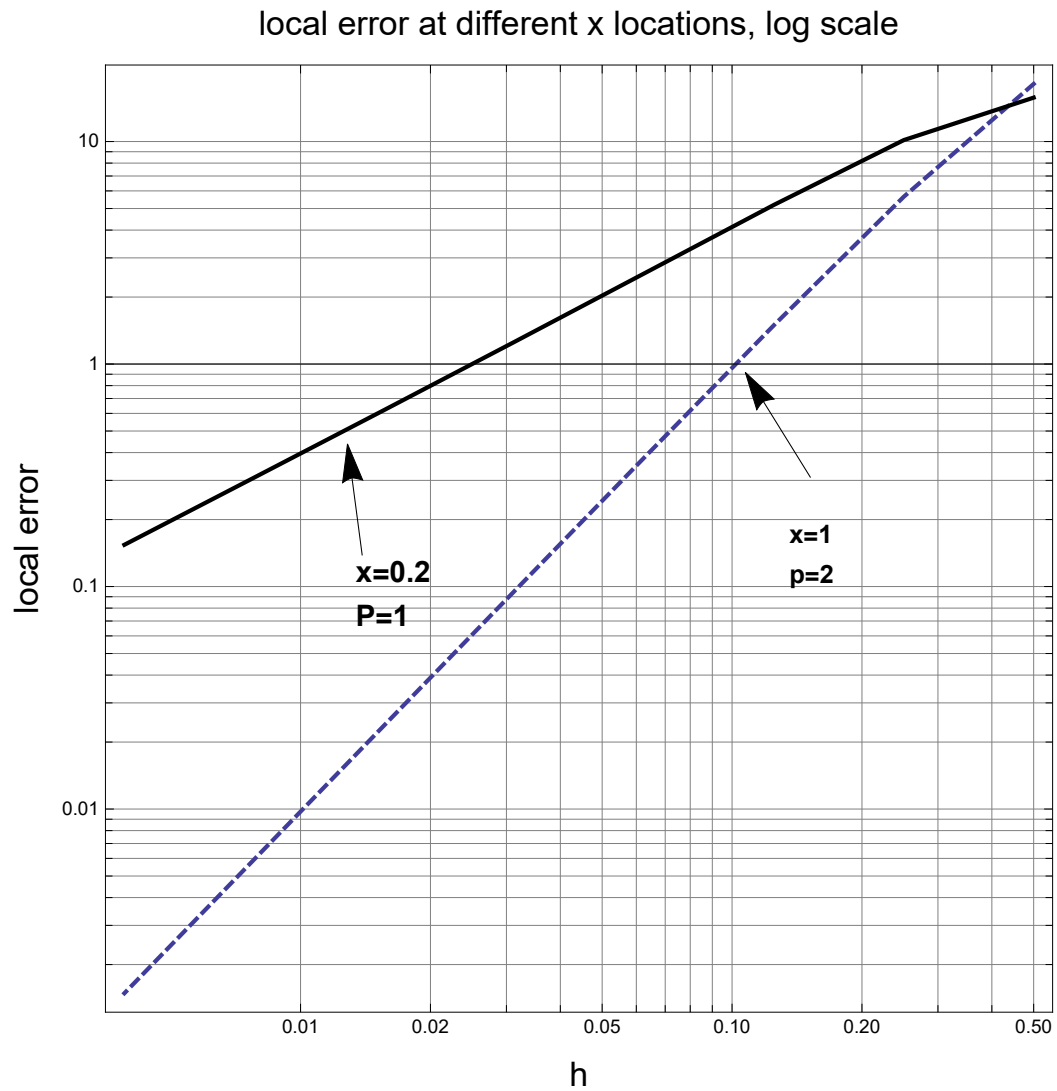
2.3   HW 2

2.3.1   Problem description

Math 228A
Homework 2
Due Friday, 10/22/08, 4:00

1. Use the standard 3-point discretization of the Laplacian on a regular mesh to find a numerical solution to the PDEs below. Perform a refinement study using the exact solution to compute the error that shows the rate of convergence for both the 1-norm and the max norm.

    (a) $u_{xx} = \exp(x), \quad u(0) = 0, \quad u(1) = 1$
    (b) $u_{xx} = 2\cos^2(\pi x), \quad u_x(0) = 0, \quad u_x(1) = 1$

2. Propose a discretization scheme for

$$u_{xx} = f, \quad u_x(0) - \alpha u(0) = g, \quad u(1) = b.$$

What is the form of the matrix and right hand side in your discrete equations? What order of accuracy do you expect?

3. As a general rule, we usually think that an $O(h^p)$ local truncation error (LTE) leads to an $O(h^p)$ error. However, in some cases the LTE can be lower order at some points without lowering the order of the error. Consider the standard second-order discretization of the Poisson equation on $[0, 1]$ with homogeneous boundary conditions. The standard discretization of this problem gives an $O(h^2)$ LTE provided the the solution is at least C^4. The LTE may be lower order because the solution is not C^4 or because we use a lower order discretization at some points.

    (a) Suppose that the LTE is $O(h^p)$ at the first grid point $(x_1 = h)$. What effect does this have on the error? What is the smallest value of p that gives a second order accurate error? Hint: Use equation (2.46) from LeVeque to aid in your argument.
    (b) Suppose that the LTE is $O(h^p)$ at an interior point (i.e. a point that does not limit to the boundary as $h \to 0$). What effect does this have on the error? What is the smallest value of p that gives a second order accurate error?
    (c) Verify the results of your analysis from parts (a) and (b) using numerical tests.

Figure 2.5: problem description

2.3.2   Problem 1

Use standard 3-point discretization of the Laplacian on a regular mesh to find numerical solution to the PDE below and perform refinement study to compute the error that shows the rate of convergence for both the 1-norm and the max-norm.

2.3.2.1   part (a)

The differential equation with its boundary conditions is $u_{xx} = e^x$ with $u(0) = 0, u(1) = 1$.

Finding the analytical solution

Let D be a differential operator $D \equiv \frac{d}{dx}$. Since $(D-1)\,e^x = 0$, then applying $(D-1)$ to both sides of the differential equation results in

$$\left[(D-1)\,D^2\right](u) = 0$$

Thus the characteristic equation is $(r-1)\,r^2 = 0$ with roots $r_1 = 1, r_2 = 0$ and $r_3 = 0$. Therefore the complete solution is

$$u(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + x c_3 e^{r_3 x}$$

It helps to designate in the above which part is the particular solution and which is the homogeneous

$$u(x) = \overbrace{c_1 e^x}^{u_p} + \overbrace{c_2 + c_3 x}^{y_h}$$

c_1 is found by substituting the particular solution in the original differential equation giving $c_1 = 1$. The solution becomes[1]

$$u(x) = e^x + c_2 + x c_3$$

The remaining constants c_2 and c_3 are found by satisfying the boundary conditions on the above solution. Applying $u(0) = 0$ gives $c_2 = -1$ and applying $u(1) = 1$ gives $c_3 = 2 - e$. The solution becomes

$$u(x) = 2x - \exp(1)\,x + \exp(x) - 1$$

Scheme to use for the numerical solution

The numbering used is the one described in the class. The first grid point has an index $j = 0$, and the last grid point has an index $j = N + 1$. U is the unknown to solve for. It is the numerical solution of the differential equation at the grid points. Lower case u is the exact analytical solution found earlier.

Because this problem has Dirichlet boundary conditions, U is known at $j = 0$ and at $j = N+1$, therefore only internal grid points are used to solve for U. These internal points are numbered $1 \cdots N$. The spacing between each grid point is $h = \frac{len}{N+1}$ where the length of the domain len is always taken to have value 1.

The physical x-coordinate of each grid point is given by $x_j = jh$. For example, when $j = 0$, the first point will have a physical x-coordinate of 0, and when $j = N + 1$, the last grid point will have a physical x-coordinate of 1.

The diagram below helps illustrate these relations and the notations used.

---

[1]another way to solve this is to integrate the original differential equation twice to obtain this general solution.
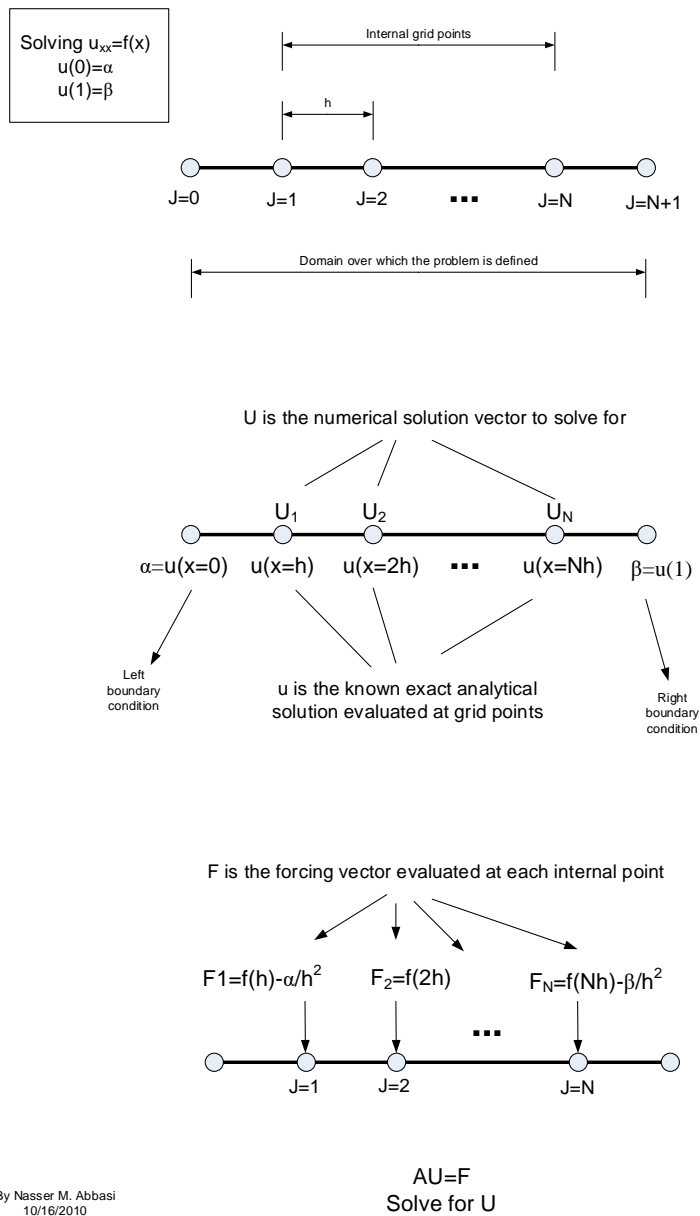
Figure 2.6: problem 1 part a scheme

The standard 3-point central difference formula $\frac{U_{j-1}-2U_j+U_{j+1}}{h^2}$ was used to approximate u_{xx} on each internal grid point. This approximation has local truncation error (LTE) of $O\left(h^2\right)$. Therefore, at each internal grid point j the equation $u_{xx} = f(x)$ is approximated by $\frac{U_{j-1}-2U_j+U_{j+1}}{h^2} = f_{x=jh}$.
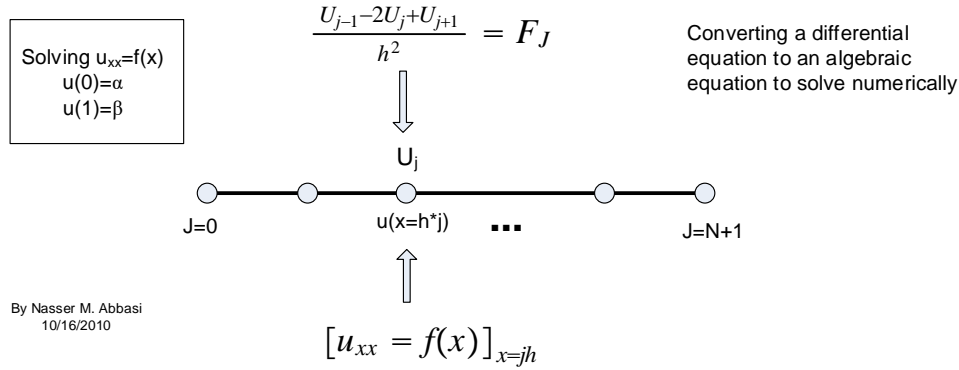
Figure 2.7: problem 1 part a scheme 2

For the special case of $j = 1$ (the first internal grid point on left side), the above formula was modified by replacing U_o by its given value (U_0 is on the boundary and its value is known). Therefore, for $j = 1$ the discrete equation becomes

$$\frac{\alpha - 2U_0 + U_1}{h^2} = F_1$$

$$\frac{-2U_0 + U_1}{h^2} = F_1 - \frac{\alpha}{h^2} \tag{2}$$

The same was done for $j = N$ (the last internal grid point on right side). The standard 3 points formula was modified by replacing U_{N+1} by its given value. Therefore, for $j = N$ the discrete equation becomes

$$\frac{U_{N-1} - 2U_N + \beta}{h^2} = F_N$$

$$\frac{U_{N-1} - 2U_N}{h^2} = F_N - \frac{\beta}{h^2} \tag{3}$$

The above equations are collected and put in the form $Au = f$ resulting in

$$\frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ \cdot \\ \cdot \\ \cdot \\ U_N \end{pmatrix} = \begin{pmatrix} f_h - \frac{\alpha}{h^2} \\ f_{2h} \\ f_{3h} \\ \cdot \\ \cdot \\ f_{(N-1)h} \\ F_{Nh} - \frac{\beta}{h^2} \end{pmatrix}$$

The above system is solved for the unknown vector U. These are the values of the numerical solution at the internal grid points. The A matrix above is symmetric and tridiagonal and of size $N \times N$ where N is the number of internal grid points.

Error norm calculation

The total error at a grid point j is given by

$$e_j = U_j - u\left(x_j\right)$$

Where U_j is the numerical solution at the jth grid point and $u\left(x_j\right)$ is the exact solution sampled at the same grid point location. e is a vector of length N and represents the difference between U and u at each point.

To measure the size of the error vector e, a grid norm is used in place of the standard vector norm. The following are the definitions of the norms used.

1. max-norm $\left\|e^h\right\|_{\text{max}} = \max_j \left|e_j\right|$

2. 1-norm $\left\|e^h\right\|_1 = h\sum_{j=1}^{N}\left|e_j\right|$

3. 2-norm $\left\|e^h\right\|_2 = \sqrt{h\sum_{j=1}^{N}\left|e_j\right|^2}$

Description of method used in Refinement study

The goal of the refinement study is to check that the scheme selected is stable. Since the scheme selected is consistent (it has an LTE of order $O\left(h^2\right)$ therefore $\|\tau\| \to 0$ as $h \to 0$), this implies that the scheme will be stable if it can be shown that the numerical solution converges to the exact solution[2].

For a stable scheme the following relation must hold

$$\|e\| \le \left\|A^{-1}\right\|\|\tau\|$$

therefore stability can be established by verifying that error norm $\|e\|$ is also of order $O\left(h^2\right)$. This implies that $\left\|A^{-1}\right\|$ is $O\left(1\right)$. In addition, showing that $\|e\|$ is of order $O\left(h^2\right)$ implies that $\|e\| \to 0$ as $h \to 0$, which means convergence.

These are the steps carried out in the refinement study

1. The system $AU = f$ is formulated based on the scheme described above. These equations contain h (the grid space) in them as a free parameter. Initially h is given an initial starting value.

2. $AU = f$ is solved for U.

3. The total error vector $e = U - u$ is calculated and the error grid norm $\|e\|$ found using the the above definitions of norms.

4. The spacing h is divided by 2 and the above steps are repeated. The number of iterations was selected so that numerical convergence can be clearly observed. It is found that 5 or 6 iterations was sufficient to show convergence in this problem.

---

[2]Lax-Richtmyer theorem

5. The error table and the log plots are generated. Convergence is verified by showing from the error table results that the total error norm $\|e\|$ has an order of accuracy of $O\left(h^2\right)$. This implies the numerical scheme selected is stable.

Results of the refinement study

This is a plot of $\log(h)$ against the log of the various error norms. The source code is in the appendix.
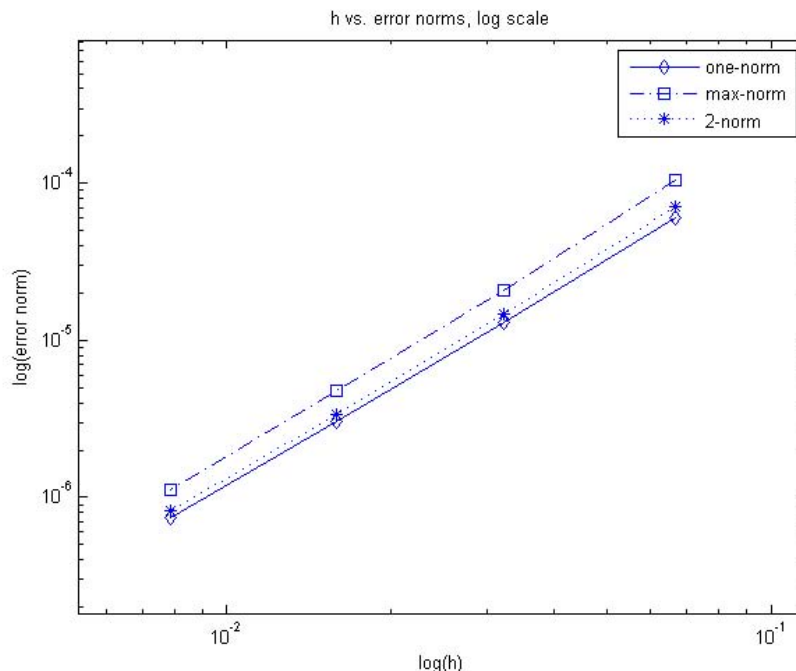
Figure 2.8: matlab HW2 part a logplot

The error table is the following

```
 1  EDU>> nma_HW2_part_a
 2  N         h          emax        ratio         e1         ratio        e2          ratio
 3  16    6.6667e-002  1.0444e-004 0.0000e+000 5.9816e-005 0.0000e+000 7.0865e-005 0.0000e+000
 4  32    3.2258e-002  2.0983e-005 4.9772e+000 1.3040e-005 4.5870e+000 1.4799e-005 4.7886e+000
 5  64    1.5873e-002  4.7448e-006 4.4223e+000 3.0536e-006 4.2706e+000 3.4030e-006 4.3488e+000
 6  128   7.8740e-003  1.1299e-006 4.1992e+000 7.3937e-007 4.1300e+000 8.1708e-007 4.1648e+000
 7  256   3.9216e-003  2.7583e-007 4.0964e+000 1.8194e-007 4.0637e+000 2.0025e-007 4.0802e+000
 8  512   1.9569e-003  6.8147e-008 4.0476e+000 4.5130e-008 4.0316e+000 4.9573e-008 4.0396e+000
 9  1024  9.7752e-004  1.6937e-008 4.0236e+000 1.1238e-008 4.0157e+000 1.2333e-008 4.0196e+000
10  2048  4.8852e-004  4.2220e-009 4.0115e+000 2.8042e-009 4.0076e+000 3.0758e-009 4.0096e+000
```

.

Conclusion

The study was carried out using 3 different norms (max norm,1-norm and 2-norm). It is found that for each norm the total error ratio converged to 4, implying $p = 2$, hence the total

error norm was $O\left(h^2\right)$ the same as LTE. This shows that the numerical scheme is stable and the numerical solution converges to the exact solution.

2.3.2.2 Part (b)

Solve $u_{xx} = 2\cos^2\left(\pi x\right)$ with $u_x\left(0\right) = 0, u_x\left(1\right) = 1$

2.3.2.3 Finding analytical solution

The existence of a solution is first verified. This is done in this case since for Neumann boundary conditions a solution might not even exist if the problem is not well posed. Integrating both sides of the PDE gives

$$\int_0^1 u_{xx}\left(x\right)dx = \int_0^1 2\cos^2\left(\pi x\right)dx$$
$$= u_x\left(1\right) - u_x\left(0\right)$$
$$= 1$$

Substituting the values of u_x on the boundaries, the above becomes zero, hence the problem is well posed. Now the analytical solution is found.

Integrating the differential equation once

$$u_x = x + \frac{\sin\left(2\pi x\right)}{2\pi} + c_1 \tag{1}$$

And Integrating again

$$u\left(x\right) = \frac{x^2}{2} - \frac{\cos\left(2\pi x\right)}{4\pi^2} + c_1 x + c_2 \tag{2}$$

Substituting the boundary condition $u_x\left(0\right) = 0$ in the above[3] gives $c_1 = 0$, therefore the solution is

$$u\left(x\right) = \frac{x^2}{2} - \frac{\cos\left(2\pi x\right)}{4\pi^2} + c_2 \tag{3}$$

This solution is not unique. The constant c_2 is arbitrary and an infinite number of solutions exist. A solution exist which is up to an arbitrary additive constant. To select a constant for the purpose of the numerical analysis in the refinement study, the constant is selected to give the solution zero mean. Hence

$$\int_0^1 u\left(x\right)dx = 0$$
$$\int_0^1 \left(\frac{x^2}{2} - \frac{\cos\left(2\pi x\right)}{4\pi^2} + c_2\right)dx = 0$$
$$\frac{1}{6} + c_2 = 0$$

Therefore $c_2 = -\frac{1}{6}$. Therefore, the exact solution used in the refinement study to compare

---

[3]The other condition $u_x\left(1\right) = 1$ could also have been used, but the result would remain the same

the numerical solution against is

$$u(x) = \frac{x^2}{2} - \frac{\cos(2\pi x)}{4\pi^2} - \frac{1}{6}$$

<u>Scheme to use for the numerical solution</u>

Two schemes formulated, both having a local truncation error $\|\tau\|$ of order $O(h^2)$. Hence both schemes are consistent.

2.3.2.3.1   First scheme   The standard 3 point centered difference formula $\frac{U_{j-1}-2U_j+U_{j+1}}{h^2}$ is used for approximating u_{xx} on internal grid points. However in this problem U_o and U_{N+1} are not known at the left most and the right most grid points (grid points of index 0 and $N+1$), therefore the grid points used includes these 2 points in addition to the standard internal grid points $(1 \cdots N)$ resulting in the A matrix having size $N+2$. (A will have 2 additional rows as compared to part (a)).

Now, two imaginary points are introduced into the domain, one to the left of $j = 0$, and one to the right of $j = N+1$
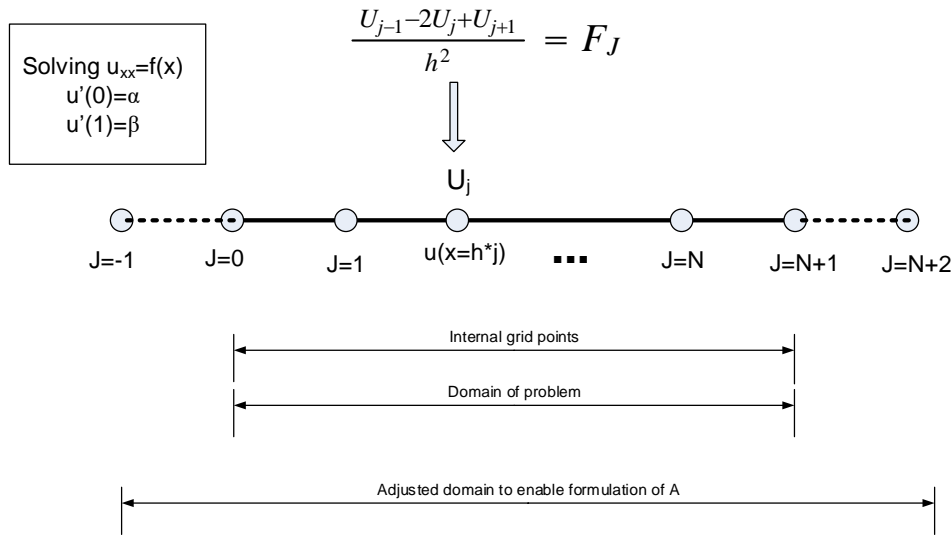
Figure 2.9: problem 1 part b scheme

For $j = 0$ the standard 3 points formulate is used to approximate u_{xx}

$$\frac{U_{-1} - 2U_0 + U_1}{h^2} = f_0 \tag{1}$$

And $u_x(0) = \alpha$ the 2 points central difference scheme is used

$$\frac{U_1 - U_{-1}}{2h} = \alpha \tag{2}$$

Now U_{-1} is eliminated using (1) and (2). From equation (2)

$$U_{-1} = U_1 - 2h\alpha$$

substituting the above into (1)

$$\frac{(U_1 - 2h\alpha) - 2U_0 + U_1}{h^2} = f_0$$

$$\frac{-2U_0 + 2U_1}{h^2} = f_0 + \frac{2\alpha}{h} \tag{3}$$

The above equation (3) is the discrete equation for node $j = 0$ (the first row in the A matrix). Similarly for the right-most node $j = N + 1$, u_{xx} is approximated using

$$\frac{U_N - 2U_{N+1} + U_{N+2}}{h^2} = f_{N+1} \tag{4}$$

And $u_x(1) = \beta$ at node $j = N + 1$ is approximated using 2 points central difference

$$\frac{U_{N+2} - U_N}{2h} = \beta \tag{5}$$

U_{N+2} is eliminated using (4) and (5). From equation (5)

$$U_{N+2} = U_N + 2h\beta$$

substituting the above into (4)

$$\frac{U_N - 2U_{N+1} + \left(U_N + 2h\beta\right)}{h^2} = f_{N+1}$$

$$\frac{2U_N - 2U_{N+1}}{h^2} = f_{N+1} - \frac{2\beta}{h} \tag{6}$$

The above equation (6) is the discrete equation for node $j = N + 1$ (the last row in the A matrix). $Au = f$ is now setup resulting in

$$\frac{1}{h^2} \begin{pmatrix} -2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ \cdot \\ \cdot \\ N_{N-1} \\ U_N \\ U_{N+1} \end{pmatrix} = \begin{pmatrix} f_0 + \frac{2\alpha}{h} \\ f_1 \\ f_2 \\ \cdot \\ \cdot \\ f_N \\ f_{N+1} - \frac{2\beta}{h} \end{pmatrix}$$

The A matrix is tridiagonal, not symmetrical and singular since $null(A) = 1^T$. In some books, this matrix is called the Laplacian matrix.

2.3.2.3.2 Second scheme In this scheme, the first order derivative for the Neumann boundary condition at the left point and at the right point is approximated using

$$\alpha = \frac{1}{h} \left(\frac{3}{2} U_0 - 2U_1 + \frac{1}{2} U_2 \right)$$

$$\beta = \frac{1}{h} \left(\frac{3}{2} U_{N+1} - 2U_N + \frac{1}{2} U_{N-1} \right)$$

respectively.

The derivation of the above formulas is shown in example 1.2 in the textbook. The above approximation has an LTE of $O\left(h^2\right)$. For the other internal grid points, the standard 3 points centred difference $\frac{U_{j-1}-2U_j+U_{j+1}}{h^2}$ is used to approximate u_{xx}. Using the above approximations, the system $AU = f$ becomes

$$\frac{1}{h^2}\begin{pmatrix} \frac{3}{2} & -2 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -2 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ \cdot \\ \cdot \\ U_{N-1} \\ U_N \\ U_{N+1} \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{h} \\ F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_N \\ \frac{\beta}{h} \end{pmatrix}$$

The A matrix is tridiagonal, not symmetrical and is also singular since $null(A) = 1^T$.

2.3.2.3.3   Augmenting the systems before solving   The A matrix for both schemes above is singular. Both have $u = (1,1,1,\cdots,1,1)^T$ as the null vector. Using Matlab to verify, the null command can be used as follows

```
1  EDU>> A=nma_FDM_matrix_laplace_1D_Neumann_scheme_1(6)
2  A =
3  -2      2      0      0      0      0
4  1     -2      1      0      0      0
5  0      1     -2      1      0      0
6  0      0      1     -2      1      0
7  0      0      0      1     -2      1
8  0      0      0      0      2     -2
9  DU>>  null(A,'r')'
10 ans =
11 1      1      1      1      1      1
```

.

For the second scheme

```
1  EDU>> A=nma_FDM_matrix_laplace_1D_Neumann_scheme_2(6)
2  A =
3  1.5000    -2.0000     0.5000          0          0          0
4  1.0000    -2.0000     1.0000          0          0          0
5  0          1.0000    -2.0000     1.0000          0          0
6  0          0          1.0000    -2.0000     1.0000          0
7  0          0          0          1.0000    -2.0000     1.0000
8  0          0          0          0.5000    -2.0000     1.5000
9  EDU>>  null(A,'r')'
10 ans =
```

```
11 | 1      1      1      1      1      1
```

.

Since the matrix A is singular, before solving by Gaussian elimination it is augmented as follows

$$A \rightarrow \begin{pmatrix} A & v \\ u & 0 \end{pmatrix}$$

where v is the the null of A^* (the adjoint of A, or for a real matrix, the same as A^T) and u is the null of A.

v is found for the above 2 schemes. For the first scheme:

```
1 | EDU>> A=nma_FDM_matrix_laplace_1D_Neumann_scheme_1(6);
2 | EDU>> null(A','r')'
3 | ans =
4 | 1      2      2      2      2      1
```

.

And for the second scheme

```
1 | EDU>> A=nma_FDM_matrix_laplace_1D_Neumann_scheme_2(6);
2 | EDU>> null(A','r')'
3 | ans =
4 | 1.0000   -1.5000   -1.0000   -1.0000   -1.5000    1.0000
```

.

Now that v and u are found, the augmented A can be formulated.

The force vector is also augmented as follows

$$f \rightarrow \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{7}$$

The augmented system is now solved for U using Gaussian elimination. In Matlab

$$U = A \backslash f$$

The resulting solution U will have an extra element at the end, which is not used in the solution. This element, called λ was verified to be equal $\frac{v \cdot f}{v \cdot v}$.

Notice that last element of f was set to 0 in (7), this is because it corresponds to having selected an exact solution with a zero mean. When the last element of f is set to 0, this corresponds to having $\sum_{i=0}^{N+1} U_i = 0$, which implies the mean of the numerical solution is zero.

This follows because $null(A)$ was found to be 1^T, meaning that, from looking at $\begin{pmatrix} A & v \\ 1^T & 0 \end{pmatrix} \begin{pmatrix} U \\ \lambda \end{pmatrix} =$

$\begin{pmatrix} f \\ 0 \end{pmatrix}$, the $dot\left(1^T, U\right) = 0$, which is the same as saying that $\sum\limits_{i=0}^{N+1} U_i = 0$ or the mean of the numerical solution is zero.

To summarize: A constant in the analytical solution was earlier selected which resulted in the mean of the exact solution to be zero. This constant was found to be $\frac{-1}{6}$. The last entry in the augment f vector is set to be zero to cause the numerical solution to have zero mean as well.

Now that the numerical solution is found, the error vector $e = U - u$ is found and its norms are calculated to complete the refinement study. Below is the result for both schemes described above.

Refinement study results
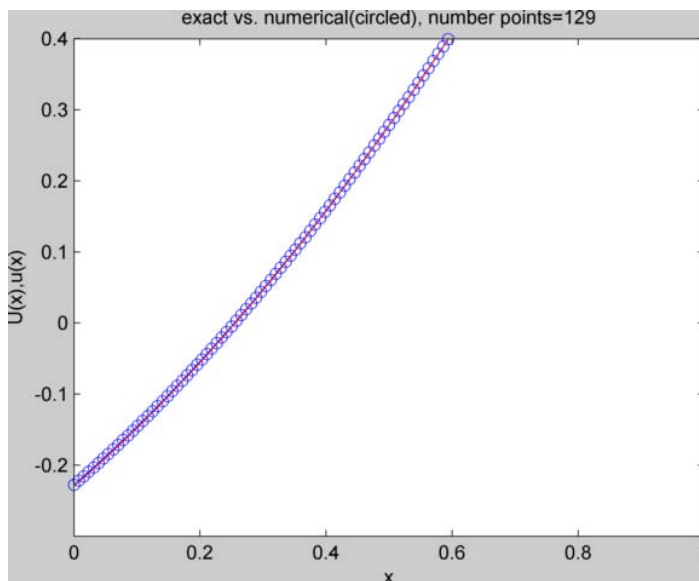
Figure 2.10: nma HW2 prob1 part b solution
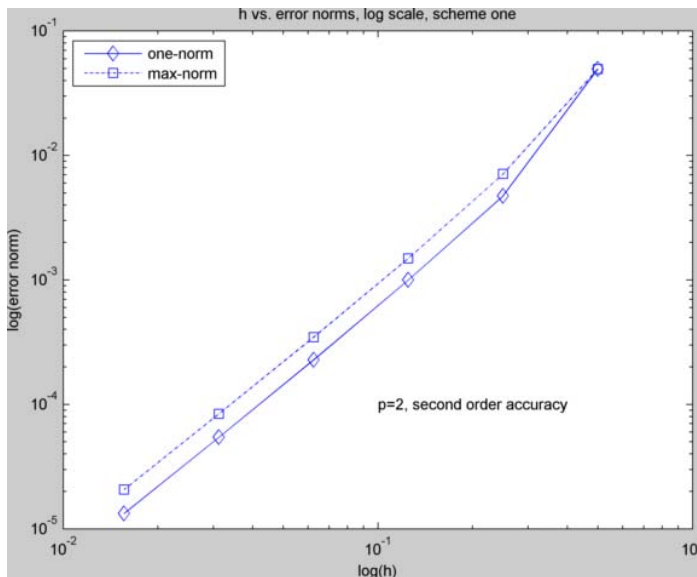
Figure 2.11: nma HW2 part b scheme one plot2

2.3.2.3.4   Scheme one   The error table is the following:

```
EDU>> nma_HW2_part_b_scheme_one
N          h      emax      ratio         e1      ratio
3    0.5000000 4.9560e-002  0.0000e+000 4.9560e-002 0.0000e+000
5    0.2500000 7.1036e-003  6.9766e+000 4.7358e-003 1.0465e+001
9    0.1250000 1.4925e-003  4.7596e+000 9.9728e-004 4.7487e+000
17   0.0625000 3.4734e-004  4.2969e+000 2.2786e-004 4.3768e+000
33   0.0312500 8.4008e-005  4.1346e+000 5.4367e-005 4.1912e+000
65   0.0156250 2.0668e-005  4.0646e+000 1.3271e-005 4.0967e+000
```

.
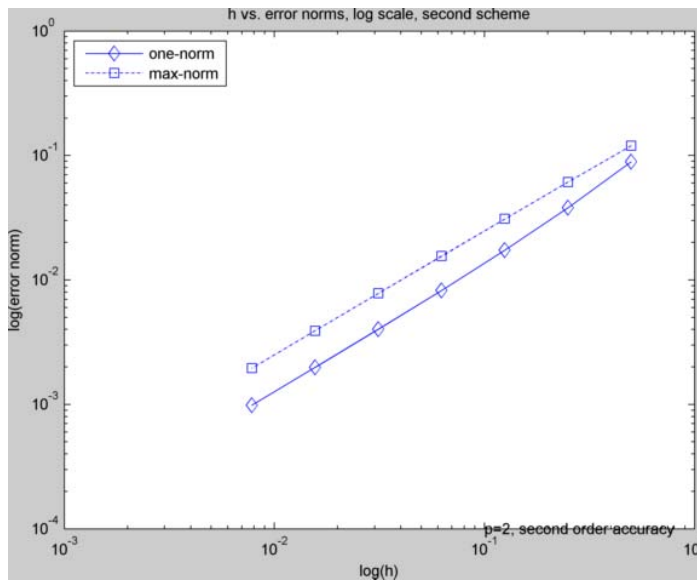
Figure 2.12: HW2 prob1 part b scheme 2

2.3.2.3.5 Scheme 2   And the error table is

```
EDU>> nma_HW2_prob1_part_b(2)
N            h         emax        ratio          e1        ratio
3      0.5000000 5.9960e-003 0.0000e+000 5.9960e-003 0.0000e+000
5      0.2500000 6.0976e-003 9.8333e-001 6.0976e-003 9.8333e-001
9      0.1250000 8.1787e-004 7.4554e+000 4.3938e-004 1.3878e+001
17     0.0625000 1.6346e-004 5.0034e+000 8.9562e-005 4.9058e+000
33     0.0312500 6.0030e-005 2.7230e+000 3.6203e-005 2.4739e+000
65     0.0156250 1.7602e-005 3.4103e+000 1.0956e-005 3.3043e+000
129    0.0078125 4.7387e-006 3.7146e+000 2.9860e-006 3.6693e+000
257    0.0039063 1.2278e-006 3.8594e+000 7.7785e-007 3.8388e+000
513    0.0019531 3.1240e-007 3.9302e+000 1.9841e-007 3.9204e+000
1025   0.0009766 7.8786e-008 3.9652e+000 5.0098e-008 3.9605e+000
```

.

Conclusion

Two different schemes were used for solving part(b). Both schemes had an LTE of $O\left(h^2\right)$, therefore they both were consistent numerical schemes. The result shows that $\|e\|$ had $O\left(h^2\right)$ implying $\left\|A^{-1}\right\|$ had order $O\left(1\right)$ and that both scheme were stable.

It is observed that the second scheme required more iterations to coverage to the exact solution compared to the first scheme. Attempt was made to find if any error was made in the derivation of the scheme or in the code, but none found. Therefore, this showed that when comparing 2 schemes with the same theoretical order of accuracy, this does not necessarily

imply they will have the same exact performance on the same numerical example. Therefore, before selecting a scheme, it would be useful to always run a number of numerical tests to compare schemes against each others on different numerical test problems.

2.3.3 Problem 2

Propose a discretization scheme for $u_{xx} = f$ with $u_x(0) - \alpha u(0) = g$ and $u(1) = b$

Answer:

This problem has Robin boundary conditions on the left side and Dirichlet boundary conditions on the right side. The condition for existence of a solution implies

$$u_x(1) - u_x(0) = \int_0^1 f(x)\,dx$$

Substituting the boundary conditions into the above results in

$$-\alpha u(0) - g = \int_0^1 f(x)\,dx$$

Hence any $f(x)$ which satisfies the above will make the problem a well posed one. In the following, 3 different schemes are presented using different methods of approximating the Robin boundary conditions. The last scheme resulted in an A matrix with the most desirable properties being tridiagonal and symmetric.

2.3.3.1 First scheme

$u_x(0)$ is approximated by 2 points centered difference by introducing an imaginary point U_{-1} resulting in

$$u_x(0) = \frac{U_{-1} - U_1}{2h}$$

Substituting the above in the Robin boundary condition gives

$$\frac{U_{-1} - U_1}{2h} - \alpha U_0 = g \tag{1}$$

$u_{xx}(0)$ is approximated using the standard 3 point formula

$$\frac{U_{-1} - 2U_0 + U_1}{h^2} = f_0 \tag{2}$$

U_{-1} is now eliminated From (1) and (2). From equation (1)

$$U_{-1} = 2h\left(g + \alpha U_0\right) + U_1$$

Substituting into (2)

$$\frac{\left[2h\left(g + \alpha U_0\right) + U_1\right] + 2U_0 + U_1}{h^2} = f_0$$

And simplifying

$$\frac{2U_0\left(1 + \alpha h\right) + 2U_1}{h^2} = f_0 - \frac{2g}{h}$$

This will be the equation to use at node $j = 0$ which is the first row in the A matrix.

For node $j = N$, $u_{xx}\left(N\right)$ is approximated using the standard 3 point formula

$$\frac{U_{N-1} - 2U_N + U_{N+1}}{h^2} = f_N$$

Since $U_{N+1} = b$, the above becomes

$$\frac{U_{N-1} + 2U_N}{h^2} = f_N - \frac{b}{h^2}$$

Which is the equation for node $j = N$. For all the remaining internal grid point, the standard 3 point formula is used to approximate u_{xx}. Therefore, the system which now contains $N + 1$ equations is completed

$$\frac{1}{h^2}\begin{pmatrix} 2\left(1 + \alpha h\right) & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ \cdot \\ \cdot \\ \cdot \\ U_{N-1} \\ U_N \end{pmatrix} = \begin{pmatrix} f_0 - \frac{2g}{h} \\ f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_{N-1} \\ f_N - \frac{b}{h^2} \end{pmatrix}$$

2.3.3.2   Second scheme

Approximating $u_x\left(0\right)$ by 3 points forward difference

$$u_x\left(0\right) = \frac{1}{h}\left(\frac{3}{2}U_0 - 2U_1 + \frac{1}{2}U_2\right)$$

Substituting the above in the Robin boundary condition gives

$$\frac{1}{h}\left(\frac{3}{2}U_0 - 2U_1 + \frac{1}{2}U_2\right) - \alpha U_0 = g$$

$$U_0\left(\frac{3}{2} - \alpha h\right) - 2U_1 + \frac{1}{2}U_2 = hg$$

Dividing both sides by h^2 to allow writing the matrix A with a common $\frac{1}{h^2}$ factored out

$$\frac{U_0\left(\frac{3}{2} - \alpha h\right) - 2U_1 + \frac{1}{2}U_2}{h^2} = \frac{g}{h}$$

For the right side, the same scheme is used as in the first scheme above

$$\frac{U_{N-1} + 2U_N}{h^2} = f_N - \frac{b}{h^2}$$

And the system becomes

$$
\frac{1}{h^2}
\begin{pmatrix}
\left(\frac{3}{2} - \alpha h\right) & -2 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2
\end{pmatrix}
\begin{pmatrix}
U_0 \\ U_1 \\ U_2 \\ \cdot \\ \cdot \\ \cdot \\ U_{N-1} \\ U_N
\end{pmatrix}
=
\begin{pmatrix}
\frac{g}{h} \\ f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_{N-1} \\ f_N - \frac{b}{h^2}
\end{pmatrix}
$$

2.3.3.3   Third scheme

Approximating $u_x(0)$ by 2 points forward difference which has an LTE of $O(h)$

$$
u_x(0) = \frac{1}{h}(U_1 - U_0)
$$

Substituting the above in the Robin boundary condition gives

$$
\frac{1}{h}(U_1 - U_0) - \alpha U_0 = g
$$
$$
U_1 - U_0 - \alpha h U_0 = hg
$$
$$
U_0(-1 - \alpha h) + U_1 = hg
$$

Dividing both sides by h^2 to allow writing the matrix A with a common $\frac{1}{h^2}$ factored out

$$
\frac{[U_0(-1 - \alpha h) + U_1]}{h^2} = \frac{g}{h}
$$

For the right side, the same scheme was used as in the first scheme above, resulting in

$$
\frac{1}{h^2}
\begin{pmatrix}
(-1 - \alpha h) & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2
\end{pmatrix}
\begin{pmatrix}
U_0 \\ U_1 \\ U_2 \\ \cdot \\ \cdot \\ \cdot \\ U_{N-1} \\ U_N
\end{pmatrix}
=
\begin{pmatrix}
\frac{g}{h} \\ f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_{N-1} \\ f_N - \frac{b}{h^2}
\end{pmatrix}
$$

This scheme has an LTE of $O(h)$ since an $O(h)$ approximation was used for $u_x(0)$

2.3.3.4   Order of accuracy

Each of the above 3 schemes is a consistent scheme because for the first 2 schemes the LTE is $O(h^2)$ and for the third scheme the LTE is $O(h)$. A consistent scheme is one in which $\|\tau\| \to 0$ as $h \to 0$. It expected that the final error norm $\|e\|$ will behave as $O(h^2)$ when using

the first 2 scheme and as $O(h)$ when using the third scheme. This is provided A is stable. Meaning that $\left\|A^{-1}\right\|$ is of order $O(1)$.

Considering the third scheme only since it is symmetric, and using the grid 2-norm, then given that

$$\|e\|_2 = \left\|A^{-1}\tau\right\|_2$$
$$\leq \left\|A^{-1}\right\|_2 \|\tau\|_2$$

And since for a symmetric matrix $\left\|A^{-1}\right\|_2 = \rho\left(A^{-1}\right) = \frac{1}{|\lambda_{\min}|}$ where $|\lambda_{\min}|$ is the smallest eigenvalue of A in absolute value, then showing that $|\lambda_{\min}| \geq 1$ would imply that $\|e\|_2 \leq \|\tau\|_2$.

There two analytical methods for finding the $\lambda_{\min}$ of A. By solving the eigenvalue problem $L\phi = \lambda\phi$ for the given boundary conditions or by using matrix algebra to solve for the eigenvalue of A directly. Attempt are made at both of these methods, but more time is needed to complete an analytical proof.

Therefore, to answer the question for this problem, a numerical method was used instead, where a refinement study was done using the third scheme found above in an attempt to show numerically that $\|e\|$ is of the same order as $\|\tau\|$ as expected. A well formed problem is constructed and used for the purpose of the numerical analysis part. The results and the conclusion follows.

2.3.3.5 Refinement study for the third scheme

The following numerical problem was select. $u_{xx} = f$ with $u_x(0) - \alpha u(0) = g$ and $u(1) = b$ with the following parameter values

$$\alpha = 1$$
$$g = 1$$
$$b = 1$$
$$f = \cos(x)$$

This problem is well posed, and has the following analytical solution

$$u(x) = \frac{1}{2}(1 + b - g - x + b\ x + g\ x + \cos(1) + x\ \cos(1) - 2\cos(x))$$

And it satisfies the necessary condition $-\alpha u(0) - g = \int\limits_0^1 f(x)\,dx$

The following are the results of the refinement study. The source code is in the appendix.
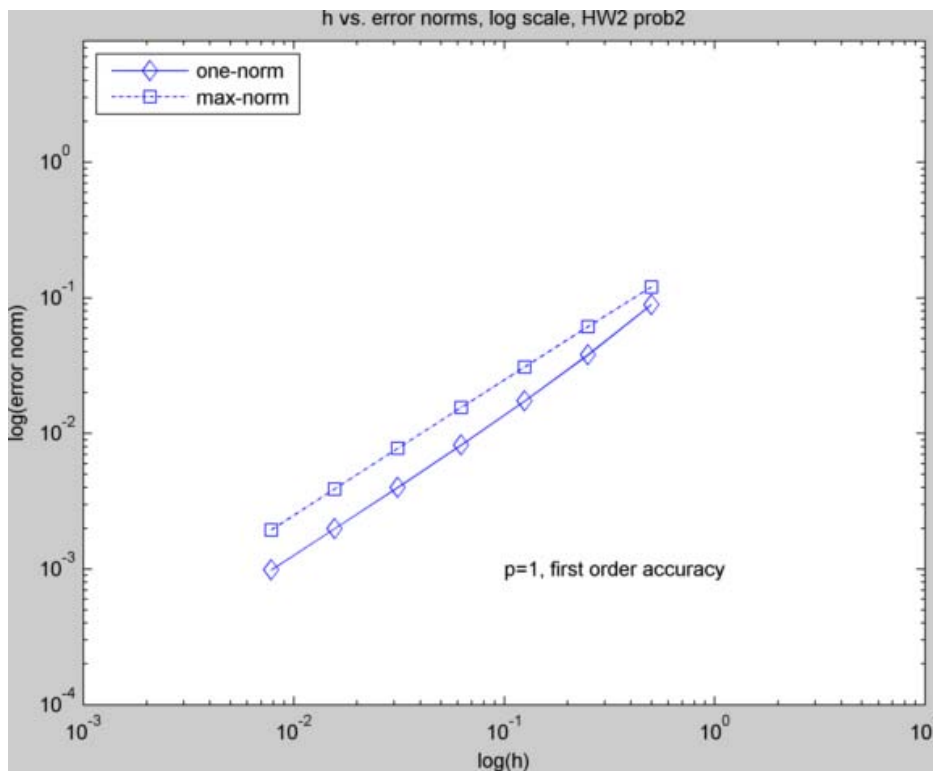
Figure 2.13: HW2 prob2

```
1  \begin{X301}
2  EDU>> close all; nma_HW2('hw2_prob2')
3  N            h         emax         ratio          e1          ratio
4  3       0.5000000  1.2015e-001  0.0000e+000  8.8980e-002  0.0000e+000
5  5       0.2500000  6.1299e-002  1.9601e+000  3.7962e-002  2.3439e+000
6  9       0.1250000  3.0950e-002  1.9806e+000  1.7318e-002  2.1921e+000
7  17      0.0625000  1.5550e-002  1.9904e+000  8.2379e-003  2.1023e+000
8  33      0.0312500  7.7938e-003  1.9952e+000  4.0129e-003  2.0529e+000
9  65      0.0156250  3.9016e-003  1.9976e+000  1.9798e-003  2.0269e+000
10 129     0.0078125  1.9520e-003  1.9988e+000  9.8324e-004  2.0136e+000
```

.

2.3.3.6 Conclusion

A scheme was implemented for Robin boundary conditions. LTE used was $O(h)$ since this specific scheme generated an A matrix with desirable form (symmetrical and tridiagonal) . The expectation of total error using this scheme was to be $O(h)$. This expectation was confirmed by doing a refinement study on the selected test problem. This result showed that the scheme is stable and therefore we conclude that $\|A\|$ is of order $O(1)$ based on this test case.

2.3.4   Problem 3

2.3.4.1   Part (a)

From definition

$$\|e\|_\infty = - \|B\tau\|_\infty \tag{1}$$

Where $B = A^{-1}$ was generated by the use of Green function as described on page 27 of the text book. Writing the product of the matrix B with the vector τ as

$$B\tau = b_1\tau_1 + b_1\tau_1 + \cdots + b_N\,\tau_N$$

where b_j is the jth column of matrix B, equation (1) becomes (all norms are to be taken as the max-norm, hence for clarify, the ∞ is dropped in what follows)

$$
\begin{aligned}
\|e\| &= -\|b_1\tau_1 + b_1\tau_1 + \cdots + b_N\tau_N\| \\
&\le -\left(\|b_1\tau_1\| + \|b_2\tau_2\| + \cdots + \|b_N\tau_N\|\right) \\
&\le -\left(\|b_1\|\,|\tau_1| + \|b_2\|\,|\tau_2| + \cdots + \|b_N\|\,|\tau_N|\right)
\end{aligned} \tag{3}
$$

$\|b_j\|$ is the maximum element in the jth column of the matrix B. These elements are located along the diagonal of B. Hence $\|b_j\| = |B_{jj}|$ and (3) becomes

$$\|e\| \le -\left(|B_{11}|\,|\tau_1| + |B_{22}|\,|\tau_2| + \cdots + |B_{NN}|\,|\tau_N|\right) \tag{4}$$

From equation 2.46 in the textbook,

$$
B_{ij} = \begin{cases} h(x_j - 1)x_i & i = 1, 2, \cdots j \\ h(x_i - 1)x_j & i = j, j+1, \cdots, N \end{cases} \tag{5}
$$

To answer the question, assume that $|\tau_1| = O(h^p)$, and assume for the moment, for generality, that the LTE at all the other points was different, hence for other points let $|\tau_j| = O(h^q)$. Equation (4) becomes

$$\|e\| \le -O(h^p)\,|B_{11}| \; - O(h^q)\sum_{k=2}^{N}|B_{kk}| \tag{6}$$

From (5), it is found that at point $j = 1$, $|B_{11}| = h(h-1)h = h^3 - h^2 = O(h^2)$, while at the internal points, that is, at points near the middle, B_{jj} is approximated by $O(h)$, hence (6) becomes

$$\|e\| \le -O(h^p)\,O(h^2) \; - O(h^q)\sum_{k=2}^{N}O(h)$$

But $\sum_{k=2}^{N}O(h) \sim (N-1)\times O(h) = O(1)$ since $N = \frac{1}{h}$, hence the above becomes

$$
\begin{aligned}
\|e\| &\le -O(h^p)\,O(h^2) \; - O(h^q) \\
&= -O(h^{p+2}) - O(h^q)
\end{aligned} \tag{7}
$$

Therefore, if $q = p$, i.e. if the LTE was the same at each grid point, including the first, the

above simplifies to

$$\|e\| \leq -O\left(h^{p+2}\right) - O\left(h^p\right)$$
$$\sim O\left(h^p\right)$$

Hence, having the LTE at the edge grid point be $O\left(h^p\right)$ resulted in $\|e\|_\infty$ having an order of accuracy $O\left(h^p\right)$ which is the expected. The smallest value of p that can be used for the edge point while still giving overall $\|e\| = O\left(h^2\right)$ can be found from (7)

$$\|e\| = O\left(h^2\right) = -O\left(h^{p+2}\right) - O\left(h^q\right)$$

Setting $p = 0$ and $q = 2$ results in

$$\|e\| = -O\left(h^2\right) - O\left(h^2\right)$$
$$\sim O\left(h^2\right)$$

Hence $p = 0$ or $O(1)$, is possible for the LTE at the edge while arriving at an overall $\|e\| \sim O\left(h^2\right)$.

2.3.4.2 Part (b)

From equation 2.46 in the textbook,

$$B_{ij} = \begin{cases} h(x_j - 1)x_i & i = 1, 2, \cdots j \\ h(x_i - 1)x_j & i = j, j+1, \cdots, N \end{cases} \tag{5}$$

To answer the question, assume $\left|\tau_{j=N/2}\right| = O\left(h^p\right)$, where j is the middle point, i.e. at $x_j = \frac{1}{2}$ and assume for other points $\left|\tau_j\right| = O\left(h^q\right)$, hence starting from

$$\|e\| \leq -\left(|B_{11}|\ |\tau_1| + |B_{22}|\ |\tau_2| + \cdots + |B_{NN}|\ |\tau_N|\right)$$

which was derived in part(a), then the above equation becomes

$$\|e\| \leq -O\left(h^q\right) \sum_{\substack{k=1 \\ k \neq N/2}}^{N} |B_{kk}| - O\left(h^p\right)\left|B_{N/2,N/2}\right| \tag{8}$$

At the middle point from (5), $\left|B_{N/2,N/2}\right|$ can be found to be $h\left(\frac{1}{2} - 1\right)\frac{1}{2} = \frac{-1}{4}h \sim O\left(h\right)$, while for the points away from the middle, $\left|B_{jj}\right|$ is approximated by $O\left(h^2\right)$, hence (8) becomes

$$\|e\| \leq -O\left(h^q\right)\left[(N-1) \times O\left(h^2\right)\right] - O\left(h^p\right)O\left(h\right)$$

But $(N-1) \times O\left(h^2\right)$ is $O(h)$ since $N = \frac{1}{h}$, hence the above becomes

$$\|e\| \leq -O\left(h^q\right)O\left(h\right) - O\left(h^{p+1}\right)$$
$$= -O\left(h^{q+1}\right) - O\left(h^{p+1}\right) \tag{9}$$

Therefore, if $q = p$, i.e. if the LTE was the same at each grid point, including the middle, then (9) becomes

$$\|e\| = -O\left(h^{p+1}\right) - O\left(h^{p+1}\right)$$
$$= O\left(h^{p+1}\right)$$

Hence, having the LTE at the middle grid point be $O(h^p)$ resulted in $\|e\|_\infty$ having an order of accuracy $O\left(h^{p+1}\right)$ which is higher than the expected $O(h^p)$. The smallest value of p that can be used while still giving $\|e\| = O\left(h^2\right)$ can be found from (9)

$$\|e\| = O\left(h^2\right) = -O\left(h^{q+1}\right) - O\left(h^{p+1}\right)$$

Setting $p = 1$ and $q = 1$ results in $\|e\| \sim O\left(h^2\right)$.

Hence $p = 1$ or $O(h)$ is possible for the LTE at the middle point while still resulting in an overall $\|e\| = O\left(h^2\right)$

2.3.4.3   Part (c)

To perform the numerical tests to verify the above conclusion, a numerical problem is used with a scheme which had an LTE of $O\left(h^2\right)$. The LTE at the edge point only was then modified to test part(a) and the LTE at the middle point only was modified to test part(b).

Refinement study was done to verify that the order of accuracy remained $O\left(h^2\right)$ after each modification.

For the numerical test, problem 1, part (a) was used for this test. That problem had an LTE of $O\left(h^2\right)$ at each grid point, and the standard 3-point central difference was used for approximating u_{xx}.

To modify the LTE at a specific grid point, given that the LTE is defined as

$$\tau_j = \frac{u(j-1) - 2u\left(j\right) + u\left(j+1\right)}{h^2} - f_j$$

Where u above is the exact solution sampled at the given grid points, then to force the LTE to be $O(h)$ in the middle, the above equation was modified for $x_j = \frac{1}{2}$(middle row of the system) to become

$$\tau_j = \frac{u(j-1) - 2u\left(j\right) + u\left(j+1\right)}{h^2} - f_j + h$$

And to force the LTE to be $O(1)$ at the first grid point, the LTE is modified for that point only (i.e. $x_j = h$, the first row of the system) to become

$$\tau_j = \frac{u(j-1) - 2u\left(j\right) + u\left(j+1\right)}{h^2} - f_j + 1$$

The refinement study which was done for problem 1, part a, is now repeated with the above modifications. The following are the results generated.

Part (A) refinement

Before making the LTE modification for the first point, the error table generated was

```
1  EDU>> close all; nma_HW2('hw2_prob1_parta')
2  N          h          emax          ratio          e1          ratio
3  3      0.5000000   4.3295e-003   0.0000e+000   2.1647e-003   0.0000e+000
4  5      0.2500000   1.0925e-003   3.9628e+000   6.8494e-004   3.1605e+000
5  9      0.1250000   2.7377e-004   3.9906e+000   1.8036e-004   3.7977e+000
6  17     0.0625000   6.8827e-005   3.9776e+000   4.5662e-005   3.9499e+000
7  33     0.0312500   1.7234e-005   3.9937e+000   1.1451e-005   3.9875e+000
8  65     0.0156250   4.3098e-006   3.9987e+000   2.8650e-006   3.9969e+000
9  129    0.0078125   1.0776e-006   3.9995e+000   7.1640e-007   3.9992e+000
```

.

One can see that $\|e\| = O\left(h^2\right)$ as the ratio is 4. Now the f vector was modified as described above, and here is the small code segment how this was done. Complete code listing is at the end.

```
1  f       = force(xcoordinates)';
2  f(1)    = f(1) - alpha/h^2 +1 ;
3  f(end) = f(end)-beta/h^2;
```

.

And now the program was run again

```
1  EDU>> close all; nma_HW2('hw2_prob3_1')
2  N          h          emax          ratio          e1          ratio
3  3      0.5000000   1.2067e-001   0.0000e+000   6.0335e-002   0.0000e+000
4  5      0.2500000   4.6119e-002   2.6165e+000   2.2753e-002   2.6518e+000
5  9      0.1250000   1.3566e-002   3.3997e+000   6.6556e-003   3.4186e+000
6  17     0.0625000   3.6481e-003   3.7185e+000   1.7854e-003   3.7278e+000
7  33     0.0312500   9.4426e-004   3.8635e+000   4.6157e-004   3.8681e+000
8  65     0.0156250   2.4010e-004   3.9328e+000   1.1730e-004   3.9350e+000
9  129    0.0078125   6.0530e-005   3.9666e+000   2.9563e-005   3.9678e+000
```

.

Examining the second table above shows that $\|e\|_\infty = O\left(h^2\right)$, since the ratio of the error norm still converged to 4, implying $p = 2$ for $\|e\|$. To compare the loglog plots, here are plots before and after the modification is made to the LTE.

LTE is O(h^2) on all points
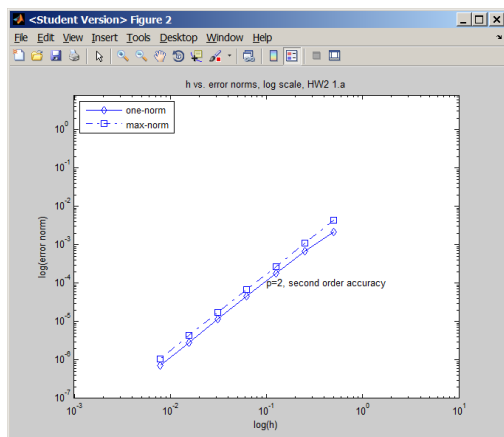
LTE is O(h^2) on all points, except at first point where it is O(1)
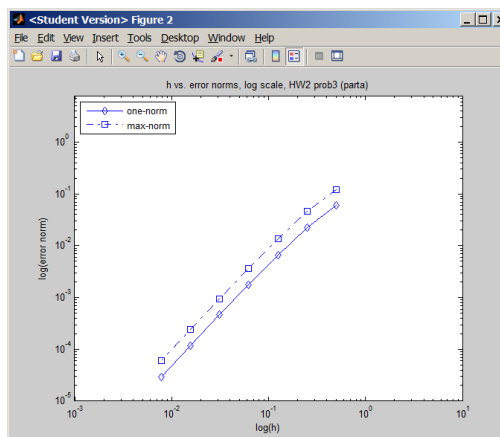
Figure 2.14: prob3 part a log

Part(b) refinement study

Before making the LTE modification for the middle point, the error table generated was

```
EDU>> close all; nma_HW2('hw2_prob1_parta')
N            h          emax        ratio           e1          ratio
3      0.5000000  4.3295e-003  0.0000e+000  2.1647e-003  0.0000e+000
5      0.2500000  1.0925e-003  3.9628e+000  6.8494e-004  3.1605e+000
9      0.1250000  2.7377e-004  3.9906e+000  1.8036e-004  3.7977e+000
17     0.0625000  6.8827e-005  3.9776e+000  4.5662e-005  3.9499e+000
33     0.0312500  1.7234e-005  3.9937e+000  1.1451e-005  3.9875e+000
65     0.0156250  4.3098e-006  3.9987e+000  2.8650e-006  3.9969e+000
129    0.0078125  1.0776e-006  3.9995e+000  7.1640e-007  3.9992e+000
```

.

One can see that $\|e\| = O\left(h^2\right)$ because the ratio is 4. Now the f vector was modified as described above, to force the middle point to have an LTE of $O(h)$, here is the small code segment showing the modification.

```
f                   = force(xcoordinates)';
f(1)                = f(1)-alpha/h^2;
middle_position     = round(length(f)/2);
f(middle_position)  = f(middle_position) + h;
f(end)              = f(end)-beta/h^2;
```

.

And now the program was run again

```
1  EDU>> close all; nma_HW2('hw2_prob3_2')
2  N            h         emax      ratio          e1       ratio
3  3      0.5000000 5.8171e-002 0.0000e+000 2.9085e-002 0.0000e+000
4  5      0.2500000 1.4532e-002 4.0028e+000 7.1276e-003 4.0807e+000
5  9      0.1250000 3.6325e-003 4.0007e+000 1.7728e-003 4.0206e+000
6  17     0.0625000 9.0808e-004 4.0002e+000 4.4262e-004 4.0052e+000
7  33     0.0312500 2.2702e-004 4.0000e+000 1.1062e-004 4.0013e+000
8  65     0.0156250 5.6754e-005 4.0000e+000 2.7653e-005 4.0003e+000
9  129    0.0078125 1.4189e-005 4.0000e+000 6.9130e-006 4.0001e+000
```

.

Examining the second table above, shows that $\|e\|_\infty = O\left(h^2\right)$, since the ratio of the error norm still converged to 4, implying $p = 2$ for $\|e\|$. To compare the loglog plots, here are plots before and after the modification is made to the LTE.

LTE is $O(h^2)$ on all points

LTE is $O(h^2)$ on all points, except at middle point where it is $O(h)$
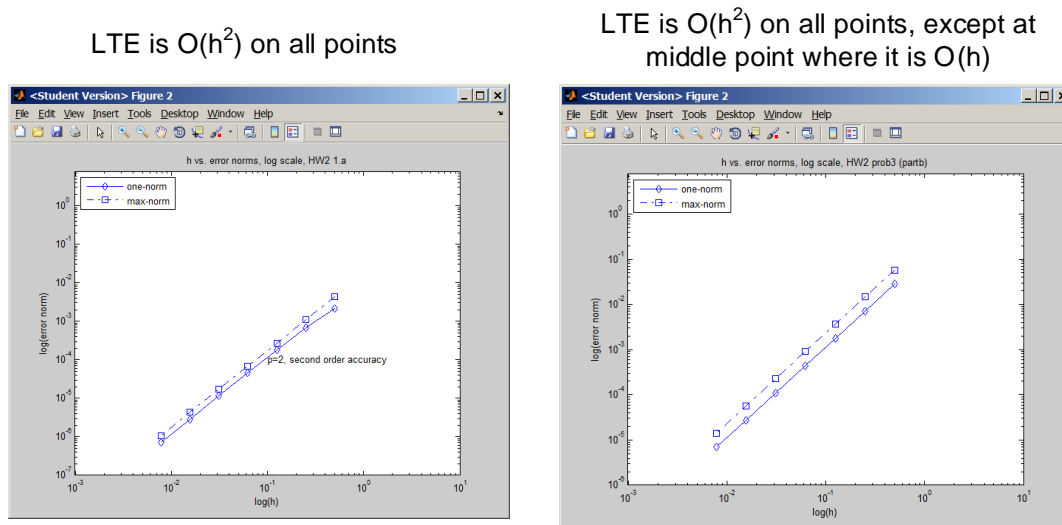
Figure 2.15: prob3 part blog

This concludes the refinement study verifying results from part(a) and part(c)

2.3.5   Source code written for this HW

```
1  %-------------------------------------------------------------
2  % HW 2  Math 228A, UC Davis, Fall 2010
3  % by Nasser M. Abbasi
4  %
5  % This .m files contains the main functions needed to solve HW2
6  %
7  % To Run:
8  %                From Matlab, type nma_HW2( problem_name )
```

```matlab
 9  %
10  % where problem_name is a string indicating which problem to run the
11  % refinment study for. The strings are
12  %
13  %           hw2_prob1_parta
14  %           hw2_prob1_partb_scheme_1
15  %           hw2_prob1_partb_scheme_2
16  %           hw2_prob2
17  %           hw2_prob3_1
18  %           hw2_prob3_2
19  %
20  % EXAMPLE
21  %
22  %            nma_HW2( 'hw2_prob1_parta' )
23  %
24  % make sure this m file is in your Matlab path.
25  %
26  function  nma_HW2()
27  close all;
28  problems={'hw2_prob1_parta';
29      'hw2_prob1_partb_scheme_1';
30      'hw2_prob1_partb_scheme_2';
31      'hw2_prob2';
32      'hw2_prob3_1';
33      'hw2_prob3_2'};
34  for i=1:2 %length(problems)
35      nma_HW2_sub( problems(i) )
36  end
37  fpritnf('completed...\n');
38
39  end
40
41  %--------------------------------------------------------
42  function  nma_HW2_sub( ID )
43
44  if nargin ~= 1
45      error('one argument is required');
46  end
47
48  N      = 7;                                 %number of iterations,
49  points = arrayfun( @(i) (2^i)+1 ,1:N );  % find number of grid points
50  h      = arrayfun( @(i) 1/(2^i) ,1:N );  % and corresponding spacings
51  data   = zeros(N,6);                        %allocate space for error table
52
53  %
54  % Main loop. For each number of points, find error norms and ratios
55  %
```

```matlab
56  for i = 1:N
57
58      if strcmp(ID,'hw2_prob1_partb_scheme_1')
59          [e,U,u,x] = error_vector_prob1_partb_scheme1(points(i),h(i));
60      elseif strcmp(ID,'hw2_prob1_partb_scheme_2')
61          [e,U,u,x] = error_vector_HW2_prob1_partb_scheme2(points(i),h(i));
62      elseif strcmp(ID,'hw2_prob1_parta')
63          [e,U,u,x] = error_vector_HW2_prob1_parta(points(i),h(i));
64      elseif strcmp(ID,'hw2_prob2')
65          [e,U,u,x] = error_vector_HW2_prob2(points(i),h(i));
66      elseif strcmp(ID,'hw2_prob3_1')
67          [e,U,u,x] = error_vector_HW2_prob3_1(points(i),h(i));
68      elseif strcmp(ID,'hw2_prob3_2')
69          [e,U,u,x] = error_vector_HW2_prob3_2(points(i),h(i));
70      else
71          error('invalid problem name');
72      end
73
74      % the columns of data are arranged in this order
75      %      npoints  h   e-max  ratio  e-1   ratio
76      %        (1)    (2)   (3)   (4)    (5)    (6)
77
78      data(i,1) = points(i);                      % number total points
79      data(i,2) = h(i);
80      data(i,3) = norm(e,inf);                    % e-max
81      data(i,5) = h(i)*norm(e,1);                 % e-1
82
83      if i>1
84          data(i,4) = data(i-1,3)/data(i,3);      %e-max ratio
85          data(i,6) = data(i-1,5)/data(i,5);      %e-1 ratio
86      end
87
88      %plot exact vs. approximate solution to verification
89      plot(x,U(:),'-o',x,u(:),'r');
90      title(sprintf('exact vs. numerical(circled), number points=%d', ...
91          points(i)));
92      axis([0 1 -.3 .4]);
93      xlabel('x'); ylabel('U(x),u(x)');
94      drawnow;
95      pause(1); % to enable animation to be seen
96  end
97
98  analysis_of_result(data,ID);          %generate error table/plots
99  end
100
101 %-----------------------------------------------------------------
102 % This function process the error table, formating it.
```

```matlab
103  % It accept the e-norm found at each spacing, the vector with the
104  % corresponding h spacing value, calculates the error ratio
105  % and print the result to the screen
106  %
107  function analysis_of_result(data,ID)
108  titles ={'N','h','emax','ratio','e1','ratio'};
109  fms    ={'d','.7f','.4e','.4e','.4e','.4e'};
110
111  wid       = 12;
112  fileID    = 1;
113  nma_format_matrix(titles,data,wid,fms,fileID,false);
114
115  figure;
116  set(0,'defaultaxesfontsize',8) ;
117  set(0,'defaulttextfontsize',8);
118
119  loglog(data(:,2),data(:,5),'-d'); hold on;
120  loglog(data(:,2),data(:,3),'-.s');
121
122  xlabel('log(h)'); ylabel('log(error norm)');
123  legend('one-norm','max-norm','Location','NorthWest');
124  axis equal;
125  grid off;
126
127  % plot the correct title based on which problem is being solved
128  if strcmp(ID,'hw2_prob1_partb_scheme_1')
129      text(10^-1,10^-4,'p=2, second order accuracy')
130      title('h vs. error norms, log scale, HW2 1.b scheme1');
131      export_fig nma_HW2_prob1_part_b_scheme_1.png
132  elseif strcmp(ID,'hw2_prob1_partb_scheme_2')
133      text(10^-1,10^-4,'p=2, second order accuracy')
134      title('h vs. error norms, log scale, HW2 1.b scheme2');
135      export_fig nma_HW2_prob1_part_b_scheme_2.png
136  elseif strcmp(ID,'hw2_prob1_parta')
137      text(10^-1,10^-4,'p=2, second order accuracy')
138      title('h vs. error norms, log scale, HW2 1.a');
139      export_fig nma_HW2_prob1_part_a.png
140  elseif strcmp(ID,'hw2_prob2')
141      text(10^-1,10^-3,'p=1, first order accuracy')
142      title('h vs. error norms, log scale, HW2 prob2');
143      export_fig nma_HW2_prob2.png
144  elseif strcmp(ID,'hw2_prob3_1')
145      title('h vs. error norms, log scale, HW2 prob3 (parta)');
146      export_fig nma_HW2_prob3_1.png
147  elseif strcmp(ID,'hw2_prob3_2')
148      title('h vs. error norms, log scale, HW2 prob3 (partb)');
149      export_fig nma_HW2_prob3_2.png
```

```matlab
150   end
151
152   end
153
154   %----------------------------------------------------------------
155   % Augment the A matrix
156   %
157   function [A,f] = augment_system(A,f)
158   f = f(:);
159   v = null(A');     % Null vector of adjoint of A
160   u = null(A);      % Null vector of A
161
162   %lambda = dot(v,f)/dot(v,v);  % used during testing to verify it is correct
163
164   % Build the A matrix the f matrix to force zero mean for the solution
165   A = [A     v;
166        u'    0];
167
168   f = [f ;
169        0];
170   end
171
172   %----------------------------------------------------------------
173   % this function builds A matrix, f vector, solves for U, the
174   % approximate solution, and determines the error vector e=U-u where
175   % u is the exact solution at the grid points
176   %
177   % For HW2, prob 1 part b, second scheme
178   %
179   function [e,U,u,xcoordinates] = ...
180       error_vector_HW2_prob1_partb_scheme2(total_number_of_points,h)
181
182       function f = force(x)
183           f = 2*(cos(pi*x)).^2;
184       end
185
186       function u = exactU(x)
187           % NOTE: This solution has a -1/6 as its additive constant.
188           u = (x.^2)/2 - cos(2*pi*x)/(4*pi^2) - 1/6;
189       end
190
191   % Boundary conditions, left and right
192   alpha = 0;
193   beta  = 1;
194   len   = 1;
195
196   xcoordinates = 0:h:len;                          %xcoordinates at points
```

```matlab
197
198  %   build the A matrix
199  A = nma_FDM_matrix_laplace_1D_Neumann_scheme_2(total_number_of_points);
200  A = A/h^2;
201
202  % build the f vector, make sure to adjust the first and last entries
203  f       = force(xcoordinates)';
204  f(1)   = alpha/h;
205  f(end) = beta/h;
206
207  [A,f] = augment_system(A,f);
208  U = A\f(:);                          % solve for U, the approximate solution
209  %lambda = U(end);                     % save lambda for reporting
210  U = U(1:end-1);                      % throw away the last entry, lambda
211
212  u = exactU( xcoordinates );      % find exact solution at the grid points
213
214  % U found above has a zero mean. This was by construction. Therefore, we
215  % have to shift it by the current mean of the sampled version of the exact
216  % solution which also has a zero mean, but only in the limit. Meaning as
217  % the number of grid points becomes very large.
218  % Therfore, at each iteration the mean of u will not yet be zero due to
219  % having small number of samples (grid points). The current mean of u must
220  % be used to adjust U to make the comparison between U and u applicable.
221  % When the number of grid points become very large, this adjustment become
222  % less important, since mean(u) will start to approach zero, and adding
223  % a zero to U will not affect the result.
224
225  U = U + mean(u);
226
227  % Now that we have found U, the numerical solution, we find the error
228  % vector.
229
230  u = u(:);   U=U(:);
231  e = U - u;
232  end  % end function(error_vector_HW2_prob1_partb_scheme2)
233
234  %----------------------------------------------------------------
235  % this function builds A matrix, f vector, solves for U, the approximate
236  % solution, and determines the error vector e=U-u where u is the exact
237  % solution at the grid points
238  %
239  % For HW2, prob 1 part b, first scheme
240  function [e,U,u,xcoordinates]=error_vector_prob1_partb_scheme1...
241      (total_number_of_points,h)
242
243      function f = force(x)
```

```
244            f = cos(pi*x).^2;
245            f = 2*f;
246        end
247
248        function u = exactU(x)
249            u = (x.^2)/2 - cos(2*pi*x)/(4*pi^2) - 1/6;
250        end
251
252  % Boundary conditions, left and right
253  alpha = 0;
254  beta  = 1;
255  len   = 1;
256
257  xcoordinates = 0:h:len;     %xcoordinates at points
258
259  A = nma_FDM_matrix_laplace_1D_Neumann_scheme_1(total_number_of_points);
260  A = A/h^2;
261
262  % build the f vector, make sure to adjust the first and last entries
263  f     = force(xcoordinates);
264  f(1)   = f(1)   + 2*alpha/h;
265  f(end) = f(end) - 2*beta/h;
266
267  USE_PINV = false;   % for testing to verify the augmented method against
268
269  if USE_PINV
270      U    = pinv(A)*f(:);
271      lambda = 0;
272  else
273      [A,f]  = augment_system(A,f);
274      U      = A\f(:);            % solve for U, the approximate solution
275      lambda = U(end);           % save lambda for reporting
276      U      = U(1:end-1);        % Not needed for finding error norm
277  end
278
279  u = exactU( xcoordinates );    % find exact solution at the grid points
280  U = U + mean(u);               % see note above
281
282  % Now that we have found U, the numerical solution, we find the error
283  % vector.
284
285  u = u(:);   U=U(:);
286  e = U - u;
287  end % end function(error_vector_prob1_partb_scheme1)
288
289  %-----------------------------------------------------------------------
290  % this function builds A matrix, f vector, solves for U, the approximate
```

```matlab
291  % solution, and determines the error vector e=U-u where u is the exact
292  % solution at the grid points
293  % HW2, prob 1, part a
294  %
295  function [e,U,u,xcoordinates]=error_vector_HW2_prob1_parta...
296      (total_number_of_points,h)
297
298      function f = force(x)
299          f = exp(x);
300      end
301
302      function u = exactU(x)
303          u = -1 + exp(x) + 2.*x - exp(1).*x;
304      end
305
306  % Boundary conditions, left and right
307  alpha = 0;
308  beta  = 1;
309  len   = 1;
310
311  number_internal_points = total_number_of_points-2;
312  xcoordinates           = h:h:len-h;      %xcoordinates at internal points
313
314  A = nma_FDM_matrix_laplace_1D_dirichlet(number_internal_points );
315  A = A/h^2;
316
317  % build the f vector, make sure to adjust the first and last entries
318  f      = force(xcoordinates)';
319  f(1)   = f(1)-alpha/h^2;
320  f(end) = f(end)-beta/h^2;
321
322  U = A\f;                          % solve for U, the approximate solution
323  u = exactU( xcoordinates )';      % find exact solution at the grid points
324  e = U - u;                        % find the total error vector e
325
326  end
327
328  %----------------------------------------------------------
329  % this function builds A matrix, f vector, solves for U, the approximate
330  % solution, and determines the error vector e=U-u where u is the exact
331  % solution at the grid points
332  % HW2 prob2
333  %
334  function [e,U,u,xcoordinates] = error_vector_HW2_prob2...
335      (total_number_of_points,h)
336
337      function f = force(x)
```

```matlab
338           f = cos(x);
339       end
340
341       function u = exactU(x,b,g)
342           u = (1/2)*(1+b-g-x+b*x+g*x+cos(1)+cos(1)*x-2*cos(x));
343       end
344
345 % Boundary conditions, left and right
346 alpha = 1;
347 b     = 1;
348 g     = 1;
349 len   = 1;
350 xcoordinates = 0:h:len-h;                          %xcoordinates at points
351
352 %  build the A matrix
353 A = nma_FDM_matrix_laplace_1D_robin(total_number_of_points-1,h,alpha);
354 A = A/h^2;
355
356 % build the f vector, make sure to adjust the first and last entries
357 f       = force(xcoordinates)';
358 f(1)    = g/h;
359 f(end) = f(end) - b/h^2;
360
361 U = A\f(:);                          % solve for U, the approximate solution
362 u = exactU( xcoordinates,b,g );   % find exact solution at the grid points
363 u = u(:);   U=U(:);
364 e = U - u;
365 end
366
367 %----------------------------------------------------------------------
368 % this function builds A matrix, f vector, solves for U, the approximate
369 % solution, and determines the error vector e=U-u where u is the exact
370 % solution at the grid points
371 % HW2, prob 1, part a
372 %
373 function [e,U,u,xcoordinates]=error_vector_HW2_prob3_1...
374     (total_number_of_points,h)
375
376       function f = force(x)
377           f = exp(x);
378       end
379
380       function u = exactU(x)
381           u = -1 + exp(x) + 2.*x - exp(1).*x;
382       end
383
384 % Boundary conditions, left and right
```

```matlab
385  alpha = 0;
386  beta  = 1;
387  len   = 1;
388
389  number_internal_points = total_number_of_points-2;
390  xcoordinates            = h:h:len-h;        %xcoordinates at internal points
391
392  A = nma_FDM_matrix_laplace_1D_dirichlet(number_internal_points );
393  A = A/h^2;
394
395  % build the f vector, make sure to adjust the first and last entries
396  f      = force(xcoordinates)';
397  f(1)   = f(1) - alpha/h^2 +1 ;       % make LTE O(1) instead of O(h^2)
398  f(end) = f(end)-beta/h^2;
399
400  U = A\f;                             % solve for U, the approximate solution
401  u = exactU( xcoordinates )';         % find exact solution at the grid points
402  e = U - u;                           % find the total error vector e
403
404  end
405
406  %-----------------------------------------------------------------------
407  % this function builds A matrix, f vector, solves for U, the approximate
408  % solution, and determines the error vector e=U-u where u is the exact
409  % solution at the grid points
410  % HW2, prob 1, part a
411  %
412  function [e,U,u,xcoordinates]=error_vector_HW2_prob3_2...
413      (total_number_of_points,h)
414
415      function f = force(x)
416          f = exp(x);
417      end
418
419      function u = exactU(x)
420          u = -1 + exp(x) + 2.*x - exp(1).*x;
421      end
422
423  % Boundary conditions, left and right
424  alpha = 0;
425  beta  = 1;
426  len   = 1;
427
428  number_internal_points = total_number_of_points-2;
429  xcoordinates            = h:h:len-h;        %xcoordinates at internal points
430
431  A = nma_FDM_matrix_laplace_1D_dirichlet(number_internal_points );
```

```
432   A = A/h^2;

433

434   % build the f vector, make sure to adjust the first and last entries
435   f                     = force(xcoordinates)';
436   f(1)                  = f(1)-alpha/h^2;
437   middle_position       = round(length(f)/2);
438   f(middle_position) = f(middle_position) + h; %Change LTE from O(h^2)to O(h)
439   f(end)                = f(end)-beta/h^2;

440

441   U = A\f;                              % solve for U, the approximate solution
442   u = exactU( xcoordinates )';          % find exact solution at the grid points
443   e = U - u;                            % find the total error vector e

444

445   end
```

```
1    function A = nma_FDM_matrix_laplace_1D_dirichlet(N )
2    %%
3    % nma_FDM_matrix_laplace_1D_dirichlet(N)
4    %
5    % returns the A matrix, which is the system finite difference matrix for
6    % numerical solution of 1-D laplace equation Uxx = f, with dirichlet
7    % boundary conditions on both sides of the element. Based on 3 point
8    % scheme    U'' =  U(j-1)-2*U(j)+U(j+1)
9    % notice that spacing h between the nodes is not used. The caller must
10   % divide by h^2 this A matrix abone return.
11   %
12   % INPUT:  N, the number of nodes
13   % OUTPUT: A, the matrix , see below
14   %
15   % EXAMPLE:
16   % A = nma_FDM_matrix_laplace_1D_dirichlet(6)
17   %
18   %      -2     1     0     0     0     0
19   %       1    -2     1     0     0     0
20   %       0     1    -2     1     0     0
21   %       0     0     1    -2     1     0
22   %       0     0     0     1    -2     1
23   %       0     0     0     0     1    -2
24   %
25   % A = A/h^2;  % h is space between points
26   % U = A\f;  % solve for U, where Uxx=f, need to set f as function before.
27   %
28
29   A = zeros(N);

30

31   for i=1:N
32       for j=1:N
33           if(i==j)
```

```
34              A(i,j) = -2;
35          else
36              if( i==j+1 || i==j-1 )
37                  A(i,j) = 1;
38              end
39          end
40      end
41  end
42
43  end
```

```
1  function  A = nma_FDM_matrix_laplace_1D_Neumann_scheme_1(N )
2  %----------------------------------------------------------------
3  % nma_FDM_matrix_laplace_1D_Neumann_scheme_1(N )
4  %
5  % returns the A matrix, which is the system finite difference matrix for
6  % numerical solution of 1-D laplace equation Uxx = f, with nuemman
7  % boundary conditions on both sides of the element. Based on 3 point
8  % scheme     U'' =  U(j-1)-2*U(j)+U(j+1)  for middle points and based on
9  % scheme  ((-2*U(0)+2*U(1))/(h^2))=f(0)+((2*a)/h) for the left most point,
10 % where a = U'(0), and h is the spacing. and for for the right most point
11 % ((2*U(N)-2*U(N+1))/(h^2)) = f(N+1)-((2*b)/h) where b=U'(1)
12 %
13 % notice that spacing h between the nodes is not used. The caller must
14 % divide by h^2 this A matrix abone return.
15 %
16 % INPUT:  N, the number of nodes
17 % OUTPUT: A, the matrix , see below
18 %
19 % EXAMPLE:
20 % A = nma_FDM_matrix_laplace_1D_Neumann_scheme_1(5)
21 %      -2     2     0     0     0
22 %       1    -2     1     0     0
23 %       0     1    -2     1     0
24 %       0     0     1    -2     1
25 %       0     0     0     2    -2
26 %
27 %
28 % A = A/h^2;  % h is space between points
29 % U = A\f;  % solve for U, where Uxx=f, need to set f as function before.
30 %
31 % copyright: Nasser M. Abbasi
32 % 10/18/2010
33
34 A = zeros(N);
35 for i=1:N
36     for j=1:N
37         if(i==j)
```

```
38              A(i,j) = -2;
39          else
40              if( i==j+1 || i==j-1 )
41                  A(i,j) = 1;
42              end
43          end
44      end
45  end
46
47  %fix A above for Von Neumann B.C. only 1st and last rows need fixed
48  A(1,2)      = 2;
49  A(end,end-1) = 2;
50  end
```

```
1  function A = nma_FDM_matrix_laplace_1D_Neumann_scheme_2(N)
2  %----------------------------------------------------------------
3  % A = nma_FDM_matrix_laplace_1D_Neumann_scheme_2(N)
4  %
5  % returns the A matrix, which is the system finite difference matrix for
6  % numerical solution of 1-D laplace equation Uxx = f, with nuemman
7  % boundary conditions on both sides of the element. Based on 3 point
8  % scheme    U'' =  U(j-1)-2*U(j)+U(j+1)  for middle points and the left
9  % and right most points, the following scheme is used
10 %    In this scheme, the Neumann boundary condition is approximated using
11 %
12 %        a =(1/h)((3/2)U(0)-2*U(1)+(1/2)U(2))
13 %        b =(1/h)((3/2)U(N+1)-2*U(N)+(1/2)U(N+1))
14 %
15 % where a=U'(0) and b=U'(1)
16 %
17 % notice that spacing h between the nodes is not used. The caller must
18 % divide by h^2 this A matrix abone return.
19 %
20 % INPUT:  N, the number of nodes
21 % OUTPUT: A, the matrix , see below
22 %
23 % EXAMPLE:
24 % A = nma_FDM_matrix_laplace_1D_Neumann_scheme_2(5)
25 %
26 %      1.5000   -2.0000    0.5000         0         0
27 %      1.0000   -2.0000    1.0000         0         0
28 %           0    1.0000   -2.0000    1.0000         0
29 %           0         0    1.0000   -2.0000    1.0000
30 %           0         0    0.5000   -2.0000    1.5000
31 %
32 %
33 % A = A/h^2;  % h is space between points
34 % U = A\f;  % solve for U, where Uxx=f, need to set f as function before.
```

```
35  %
36  % copyright: Nasser M. Abbasi
37  % 10/18/2010
38
39  A = zeros(N);
40  for i=1:N
41      for j=1:N
42          % as before, part(a), Dirichlet  B.C.
43          if(i==j)
44              A(i,j) = -2;
45          else
46              if( i==j+1 || i==j-1 )
47                  A(i,j) = 1;
48              end
49          end
50      end
51  end
52
53  %fix A above for Von Neumann B.C. only 1st and last rows need fixed
54  A(1,1) = 3/2;
55  A(1,2) = -2;
56  A(1,3) = 1/2;
57
58  A(end,end)   = (3/2);
59  A(end,end-1) =  -2;
60  A(end,end-2) = (1/2);
61
62  end
```

```
1   function A = nma_FDM_matrix_laplace_1D_robin(N,h,alpha)
2   %-------------------------------------------------------------
3   % A = nma_FDM_matrix_laplace_1D_robin(N,h)
4   %
5   % returns the A matrix, which is the system finite difference matrix for
6   % numerical solution of 1-D laplace equation Uxx = f, with nuemman
7   % boundary conditions on both sides of the element. Based on 3 point
8   %
9   % INPUT:  N, the number of nodes
10  %         h, spacing between nodes
11  %
12  % OUTPUT: A, the matrix , see below
13  %
14  %
15  % A = A/h^2;  % h is space between points
16  % U = A\f;  % solve for U, where Uxx=f, need to set f as function before.
17  %
18  % copyright: Nasser M. Abbasi
19  % 10/18/2010
```

```
20
21  A = zeros(N);
22  for i=1:N
23      for j=1:N
24          % as before, part(a), Dirichlet  B.C.
25          if(i==j)
26              A(i,j) = -2;
27          else
28              if( i==j+1 || i==j-1 )
29                  A(i,j) = 1;
30              end
31          end
32      end
33  end
34
35  %fix A above
36  A(1,1) = -1-alpha*h;
37
38  end
```

```
1   function [hd,bdy] =  nma_format_matrix(headings,data,wid,fms,fid,flag)
2   %   nma_format_matrix()
3   %
4   %   prints matrix of numerical data with headings in formatted way
5   %
6   %   INPUT:
7   %   headings: a cell array of strings for the headings of each column
8   %   data: a matrix, contains the numerical data
9   %   wid: how wide a field to use (see below for example)
10  %   fms: the formating cell string array to use for the numerical data
11  %        each string corresponds to formatting a field in the numberical
12  %        data
13  %   fid: the file id to print to. Use 1 for stdout
14  %   flag: if 0, then will not print anything, just format and return
15  %         result
16  %
17  %   Examples calling
18  %
19  %   titles={'number of projects','sales','profit'};
20  %   data=[2 rand(1) rand(1);
21  %         3 rand(1) rand(1)];
22  %
23  %   Suppose we want to format each numerical number above as %16.5E, then
24  %   in this case 10 will be the field width, and '.5E' is what to use for
25  %   the fms argument
26  %
27  %   wid    = 16;
28  %   fms    = {'d','.4E','.5E'};
```

```matlab
29  %   fileID = 1;
30  %   nma_format_matrix(titles,data,wid,fms,fileID,false);
31  %
32  %   version 1.0  10/15/2010
33  %   All blame to Nasser M. Abbasi
34  %   Thanks for help from matlab newsgroup: Ross W, Bruno Luong
35  %
36
37  %
38  % do some basic checking on input
39  %
40  if nargin ~= 6
41      error 'number of arguments must be 6'
42  end
43
44  if ~iscell(headings)
45      error 'headings must be cell array'
46  end
47
48  if ~iscell(fms)
49      error 'fms must be cell array'
50  end
51
52  if isempty(headings)
53      error 'headings can not be empty';
54  end
55
56  if isempty(data)
57      error 'data can not be empty';
58  end
59
60  if isempty(fms)
61      error 'fms can not be empty';
62  end
63
64  [nRowH,nColH] = size(headings);
65  if(nRowH>1)
66      error 'headings can not have more than one row';
67  end
68
69  [nRowFms,nColFms] = size(fms);
70  if(nRowFms>1)
71      error 'fms can not have more than one row';
72  end
73
74  if(nColH ~= nColFms)
75      error 'headings must have same number of columns as fms';
```

```matlab
76   end
77
78   [~,nCol] = size(data);
79   if(nColH ~= nCol)
80       error 'data must have same number of columns as headings';
81   end
82   %
83   % end basic checking on input, now do the formating and printing
84   %
85
86   fmt = arrayfun(@(x) ['%',num2str(wid),'s'],1:nCol,'UniformOutput',false);
87   if flag
88      fprintf(fid,[fmt{:} '\n'],headings{:});
89   end
90
91   hd=sprintf([fmt{:} '\n'],headings{:});
92
93   fmt = arrayfun(@(x) ['%',num2str(wid),fms{x}],1:nCol,'UniformOutput',false);
94   if flag
95      fprintf(fid,[fmt{:} '\n'],data');
96   end
97   bdy=sprintf([fmt{:} '\n'],data');
98
99   end
```

2.4 HW 3

2.4.1 residual error animation

Animation of the residual error R as it changes during iterative solution. Tolerance used for these is h^2, stopping criteria used is relative error < tolerance Only SOR was done.

2.4.1.1 SOR with $h = 2^{-7}$

These animations are large in size. (Click on any to see in actual size, will open in new window)

2.4.1.2 SOR with $h = 2^{-5}$

These animations are smaller in size. (Click on any to see in actual size, will open in new window)

2.4.2 Animation for density plot animations of solvers for problem 1

Animation plots below show solver as it updates each grid point by point in its main loop

The above shows each of the solvers (Jacobi, Gauss-Seidel, SOR) in the process of updating the grid during one iteration of the main loop.

Notice the how GS and SOR solvers update the solution immediately (left to right, down to top numbering is used), while Jacobi solver updates the solution only at the end each iterative step.

This one below was done for $h = 2^{-3}$. Clicking on it shows animation.

2.4.3 Animations of iterative solution

Animations of iterative solution (Click on image to see animation), Stopping criteria used is relative residual method, tolerance is h^2

2.4.4 Problem 1

Problem statement

1. Use Jacobi, Gauss-Seidel, and SOR (with optimal ω) to solve

$$\Delta u = -\exp\left(-(x - 0.25)^2 - (y - 0.6)^2\right)$$

on the unit square $(0,1) \times (0,1)$ with homogeneous Dirichlet boundary conditions. Find the solution for mesh spacings of $h = 2^{-5}$, 2^{-6}, and 2^{-7}. What tolerance did you use? What stopping criteria did you use? What value of ω did you use? Report the number of iterations it took to reach convergence for each method for each mesh.

Figure 2.16: Problem 1

Answer

Using the method of splitting, the iteration matrix T is found, for each method, for solving $Au = f$.

Let $A = M - N$, then $Au = f$ becomes

$$(M - N)u = f$$
$$Mu = Nu + f$$
$$u^{[k+1]} = M^{-1}Nu^{[k]} + M^{-1}f \tag{1}$$

2.4.4.1 Finding iterative matrix for the different solvers

The Jacobi method

For the Jacobi method $M = D$ and $N = L + U$, where D is the diagonal of A, L is the negative of the strictly lower triangle matrix of A and U is negative of the strictly upper triangle matrix of A. Hence(1) becomes

$$u^{[k+1]} = D^{-1}(L + U)u^{[k]} + D^{-1}f$$
$$u^{[k+1]} = Tu^{[k]} + C$$

Where T is called the Jacobi iteration matrix. Since $A = D - L - U$, hence $L + U = D - A$, therefore the iteration matrix T can be written as

$$T = D^{-1}(D - A)$$
$$= I - D^{-1}A$$

or

$$T = (I - D^{-1}A)$$
$$C = D^{-1}f$$

The Gauss-Seidel method

For Gauss-Seidel, $M = D - L$ and $N = U$, hence (1) becomes

$$u^{[k+1]} = (D - L)^{-1}U\,u^{[k]} + (D - L)^{-1}f$$

or

$$u^{[k+1]} = Tu^{[k]} + C$$

where

$$T = (D - L)^{-1} U$$
$$C = (D - L)^{-1} f$$

The SOR method

For SOR, $M = \frac{1}{\omega}(D - \omega L)$ and $N = \frac{1}{\omega}((1 - \omega)D + \omega U)$. Hence (1) becomes

$$
\begin{aligned}
u^{[k+1]} &= \left[\frac{1}{\omega}(D - \omega L)\right]^{-1}\left[\frac{1}{\omega}((1 - \omega)D + \omega U)\right] u^{[k]} + \left[\frac{1}{\omega}(D - \omega L)\right]^{-1} f \\
&= (D - \omega L)^{-1}((1 - \omega)D + \omega U) u^{[k]} + \omega (D - \omega L)^{-1} f \\
&= Tu^{[k]} + C
\end{aligned}
$$

where

$$T = (D - \omega L)^{-1}((1 - \omega)D + \omega U)$$
$$C = \omega (D - \omega L)^{-1}$$

Summary of iterative matrices used

This table summarizes the expression for the iterative matrix T and for the matrix C in the equation $u^{[k+1]} = Tu^{[k]} + C$ for the different methods used.

method	T	C
Jacobi	$(I - D^{-1}A)$	$D^{-1}f$
GS	$(D - L)^{-1} U$	$(D - L)^{-1} f$
SOR	$(D - \omega L)^{-1}((1 - \omega)D + \omega U)$	$\omega (D - \omega L)^{-1}$

The discretized algebraic equation resulting from approximating $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ with Dirichlet boundary conditions is based on the use of the standard 5 point Laplacian

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{h^2}\left(U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{i,j}\right)$$

which has local truncation error $O\left(h^2\right)$.

The notation used above is based on the following grid formating
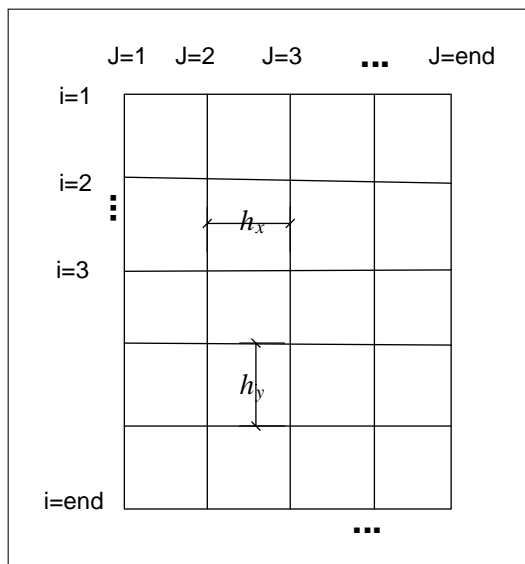
Figure 2.17: Grid notation

The derivation of the above formula is as follows: Consider a grid where the mesh spacing in the x-direction is h_x and in the y-direction is h_y. Then $\frac{\partial^2 u}{\partial x^2}$ is approximated, at position (i,j) by $\frac{U_{i-1,j}-2U_{i,j}+U_{i+1,j}}{h_x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ is approximated, at position (i,j), by $\frac{U_{i,j-1}-2U_{i,j}+U_{i,j+1}}{h_y^2}$ therefore $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} \approx \frac{U_{i-1,j}-2U_{i,j}+U_{i+1,j}}{h_x^2}+\frac{U_{i,j-1}-2U_{i,j}+U_{i,j+1}}{h_y^2}$, and since $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f_{i,j}$ at that position, this results in

$$\frac{U_{i-1,j}-2U_{i,j}+U_{i+1,j}}{h_x^2}+\frac{U_{i,j-1}-2U_{i,j}+U_{i,j+1}}{h_y^2}=f_{i,j}$$

Solving for $U_{i,j}$ from the above gives

$$U_{i,j}=\frac{1}{2\left(h_x^2+h_y^2\right)}\left(U_{i-1,j}h_y^2+U_{i+1,j}h_y^2+U_{i,j-1}h_x^2+U_{i,j+1}h_x^2\right)-\frac{h_x^2 h_y^2}{2\left(h_x^2+h_y^2\right)}f_{i,j}$$

$$=\frac{1}{2\left(h_x^2+h_y^2\right)}\left(\left(U_{i-1,j}+U_{i+1,j}\right)h_y^2+\left(U_{i,j-1}+U_{i,j+1}\right)h_x^2\right)-\frac{h_x^2 h_y^2}{2\left(h_x^2+h_y^2\right)}f_{i,j}$$

The following table shows the formula for updating $U_{i,j}$ using each of the 3 methods for the 2D case, under the assumption of uniform mesh spacing, i.e. when $h_x=h_y$ which simplifies the above formula as given below

method	Formula for updating $U_{i,j}$, uniform mesh
Jacobi	$U_{i,j}^{[k+1]}=\frac{1}{4}\left(U_{i-1,j}^{[k]}+U_{i+1,j}^{[k]}+U_{i,j-1}^{[k]}+U_{i,j+1}^{[k]}-h^2 f_{i,j}\right)$
GS	$U_{i,j}^{[k+1]}=\frac{1}{4}\left(U_{i-1,j}^{[k+1]}+U_{i+1,j}^{[k]}+U_{i,j-1}^{[k+1]}+U_{i,j+1}^{[k]}-h^2 f_{i,j}\right)$
SOR	$U_{i,j}^{[k+1]}=\frac{\omega}{4}\left(U_{i-1,j}^{[k+1]}+U_{i+1,j}^{[k]}+U_{i,j-1}^{[k+1]}+U_{i,j+1}^{[k]}-h^2 f_{i,j}\right)+(1-\omega)U_{i,j}^{[k]}$

note that for SOR, the general formula, using nonuniform mesh can be derived as follows

$$
U^{[k+1]}_{(gs)i,j} = \frac{1}{4}\left(U^{[k+1]}_{i-1,j} + U^{[k]}_{i+1,j} + U^{[k+1]}_{i,j-1} + U^{[k]}_{i,j+1} - h^2 f_{i,j}\right)
$$
$$
U^{[k+1]}_{(sor)i,j} = U^{[k]}_{i,j} + \omega\left(U^{[k+1]}_{(gs)i,j} - U^{[k]}_{i,j}\right)
$$

The second formula above can be simplified by substituing the first equation into it to yield

$$
U^{[k+1]}_{(sor)i,j} = U^{[k]}_{i,j} + \omega\left(\frac{1}{4}\left(U^{[k+1]}_{i-1,j} + U^{[k]}_{i+1,j} + U^{[k+1]}_{i,j-1} + U^{[k]}_{i,j+1} - h^2 f_{i,j}\right) - U^{[k]}_{i,j}\right)
$$
$$
= \frac{\omega}{4}\left(U^{[k+1]}_{i-1,j} + U^{[k]}_{i+1,j} + U^{[k+1]}_{i,j-1} + U^{[k]}_{i,j+1} - h^2 f_{i,j}\right) + (1-\omega)U^{[k]}_{i,j}
$$

Which is what shown in the table above. Using the same procedure, but for the general case, results in

$$
U^{[k+1]}_{(gs)i,j} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U^{[k+1]}_{i-1,j}h_y^2 + U^{[k]}_{i+1,j}h_y^2 + U^{[k+1]}_{i,j-1}h_x^2 + U^{[k]}_{i,j+1}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{i,j}
$$
$$
U^{[k+1]}_{(sor)i,j} = U^{[k]}_{i,j} + \omega\left(U^{[k+1]}_{(gs)i,j} - U^{[k]}_{i,j}\right)
$$

Hence, again, the second equation above becomes

$$
U^{[k+1]}_{(sor)i,j} = U^{[k]}_{i,j} + \omega\left(\frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U^{[k+1]}_{i-1,j}h_y^2 + U^{[k]}_{i+1,j}h_y^2 + U^{[k+1]}_{i,j-1}h_x^2 + U^{[k]}_{i,j+1}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{i,j} - U^{[k]}_{i,j}\right)
$$
$$
= \omega\left[\frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U^{[k+1]}_{i-1,j}h_y^2 + U^{[k]}_{i+1,j}h_y^2 + U^{[k+1]}_{i,j-1}h_x^2 + U^{[k]}_{i,j+1}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{i,j}\right] + (1-\omega)U^{[k]}_{i,j}
$$

Hence, for non-uniforma mesh the above update table becomes

method	Formula for updating $U_{i,j}$, non-uniform mesh
Jacobi	$U^{[k+1]}_{i,j} = \frac{1}{2\left(h_x^2+h_y^2\right)}\left(\left(U_{i-1,j} + U_{i+1,j}\right)h_y^2 + \left(U_{i,j-1} + U_{i,j+1}\right)h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2+h_y^2\right)}f_{i,j}$
GS	$U^{[k+1]}_{i,j} = \frac{1}{2\left(h_x^2+h_y^2\right)}\left(U^{[k+1]}_{i-1,j}h_y^2 + U^{[k]}_{i+1,j}h_y^2 + U^{[k+1]}_{i,j-1}h_x^2 + U^{[k]}_{i,j+1}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2+h_y^2\right)}f_{i,j}$
SOR	$U^{[k+1]}_{i,j} = \omega\left[\frac{1}{2\left(h_x^2+h_y^2\right)}\left(U^{[k+1]}_{i-1,j}h_y^2 + U^{[k]}_{i+1,j}h_y^2 + U^{[k+1]}_{i,j-1}h_x^2 + U^{[k]}_{i,j+1}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2+h_y^2\right)}f_{i,j}\right] + (1-\omega)U^{[k]}_{i,j}$

The residual formula is $R^{[k]} = f - Au^{[k]}$, which can be written using the above notations as

$$
R_{i,j} = f_{i,j} - \left(\frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h_x^2} + \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{h_y^2}\right)
$$

and for a uniform mesh the above becomes

$$
R_{i,j} = f_{i,j} - \frac{1}{h^2}\left(U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{i,j}\right)
$$

2.4.4.2 Tolerance used

The tolerance ϵ used is based on the global error in the numerical solution. For this problem $\|e\| = Ch^2$ where C is a constant C. Two different values for the constant were tried in the implementation: 0.1 and 1.

2.4.4.3 stopping criteria

The stopping criteria used was based on the use of the relative residual. Given

$$R^{[k]} = f - Au^{[k]}$$

The iterative process was stopped when the following condition was satisfied

$$\frac{\left\|R^{[k]}\right\|}{\|f\|} < \epsilon$$

Other possible stopping criterion are (but were not used) are

1. Absolute error: convergence achieved when $\left\|u^{[k+1]} - u^{[k]}\right\| < \epsilon$

2. Relative error: convergence achieved when $\frac{\left\|u^{[k+1]} - u^{[k]}\right\|}{\left\|u^{[k]}\right\|} < \epsilon$

The above two criterion are not used as they do not perform as well for Jacobi and Gauss-Seidel.

2.4.4.4 ω used for SOR

The ω_{opt} used is based on the relation to the spectral radius of the *Jacobi* iteration matrix. This relation was derived in class

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho_{Jacobian}^2}}$$

where for the 2D Poisson problem $\rho_{Jacobian}$ is given as $\cos(\pi h)$ where h is the grid spacing. Using this in the above and approximating for small h results in

$$\omega_{opt} \approx 2(1 - \pi h)$$

For the given h values in the problem, the following values were used for ω_{opt}

$h_x = h_y = h$	ω_{opt}
2^{-3}	1.2146
2^{-4}	1.6073
2^{-5}	1.8037
2^{-6}	1.9018
2^{-7}	1.9509

Notice that the above approximation formula is valid for small h and will not result in good convergence for SOR if used for large h.

Update 12/29/2010: After more analysis, the following formula is found to produce better results

$$t = \cos(\pi\ h_x) + \cos(\pi h_y)$$
$$poly(x) = x^2 t^2 - 16x + 16$$

Solve the above polynomial for x and then take the smaller root. This will be ω_{opt}, Using this results in

$h_x = h_y = h$	ω_{opt}
2^{-3}	1.4464
2^{-4}	1.6763
2^{-5}	1.821
2^{-6}	1.9064
2^{-7}	1.9509

2.4.4.5   Algorithm details

In this section, some of the details of the implementation are described for reference. The Matrix A is not used directly in the iterative method, but its structure is briefly described.

In the discussion below, updating the grid is described for the Jacobi method, showing how the new values of the dependent variable u are calculated.

Numbering system and grid updating

The numbering system used for the grid is the one described in class. The indices for the unknown $u_{i,j}$ are numbered row wise, left to right, bottom to top. This follows the standard Cartesian coordinates system. The reason for this, is to allow the use of the standard formula for the 5 point Laplacian. The following diagram illustrate the 5 point Laplacian borrowed from the text book, figure 5, page 61 "*Finite Difference Methods for Ordinary and Partial Differential Equations*" by Randall J. LeVeque
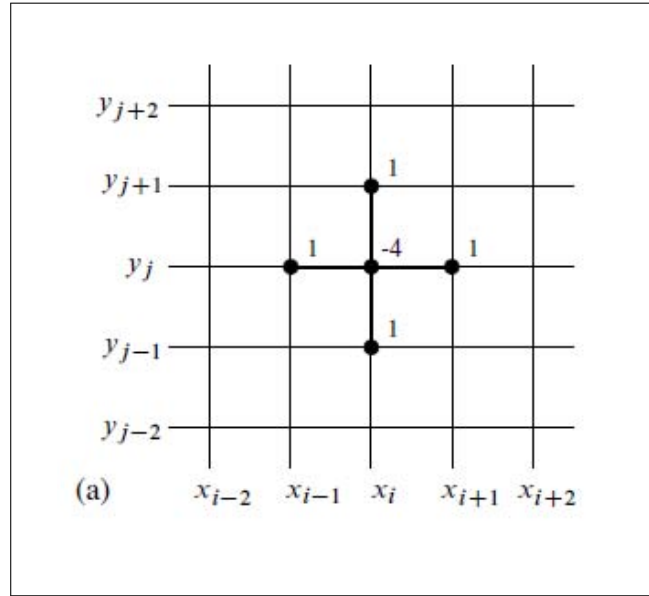
Figure 2.18: 5-point stencile for Laplacian at x(i,j)

Lower case n is used to indicate the number of unknowns along one dimension, and upper case N is used to indicate the total number of unknowns.
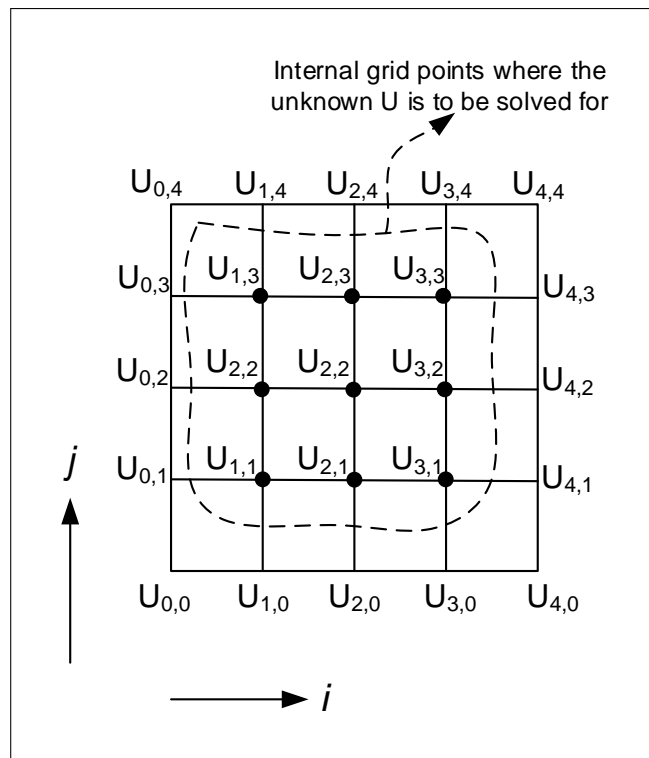
Figure 2.19: stencil move

In the diagram above, $n = 3$ is the number of unknowns on each one row or one column,

and since there are 3 internal rows, there will be 9 unknowns, all are located on internal grid points. There are a total of 25 grid points, 16 of which are on the boundaries and 9 internal.

To update the grid, the 5 point Laplacian is centered at each internal grid point and then moved left to right, bottom to top, each time an updated value of the unknown is generated and stored in an auxiliary grid.

In the Jacobian method, the new value at the location $x_{i,j}$ is not used in the calculation to update its neighbors when the stencil is moved, but

Only when the whole grid has been swept by the Laplacian will the grid be updated by copying the content of the auxiliary grid into it.

In the Gauss-Seidel and SOR, this not the case. Once a grid point have been updated, its new value is used immediately in the update of its neighbor as the Laplacian sweeps the grid. No auxiliary grid is required. In other words, the updates happens 'in place'.

Continuing this example, the following diagram shows how each grid point is updated (In a parallel computation, these operations can all be done at once for the Jacobi solver, but not for the Gauss-Seidel or the SOR solver, unless different numbering scheme is used such as black-red numbering).

$$U_{1,1} = \frac{1}{4}(U_{0,1} + U_{2,1} + U_{1,0} + U_{1,2} - h^2 f_{1,1})$$

$$U_{2,1} = \frac{1}{4}(U_{1,1} + U_{3,1} + U_{2,0} + U_{2,2} - h^2 f_{2,1})$$

$$U_{3,1} = \frac{1}{4}(U_{2,1} + U_{4,1} + U_{3,0} + U_{3,2} - h^2 f_{3,1})$$

$$U_{1,2} = \frac{1}{4}(U_{0,2} + U_{2,2} + U_{1,0} + U_{1,3} - h^2 f_{1,2})$$

$$U_{2,2} = \frac{1}{4}(U_{1,2} + U_{3,2} + U_{2,1} + U_{2,3} - h^2 f_{2,2})$$

$$U_{3,2} = \frac{1}{4}(U_{2,2} + U_{4,2} + U_{3,1} + U_{3,3} - h^2 f_{3,2})$$

$$U_{1,3} = \frac{1}{4}(U_{0,3} + U_{2,3} + U_{1,2} + U_{1,4} - h^2 f_{1,3})$$

$$U_{2,3} = \frac{1}{4}(U_{1,3} + U_{3,3} + U_{2,2} + U_{2,4} - h^2 f_{2,3})$$

And the last equation (no space to draw) results in:

$$U_{3,3} = \frac{1}{4}(U_{2,3} + U_{4,3} + U_{3,2} + U_{3,4} - h^2 f_{3,3})$$
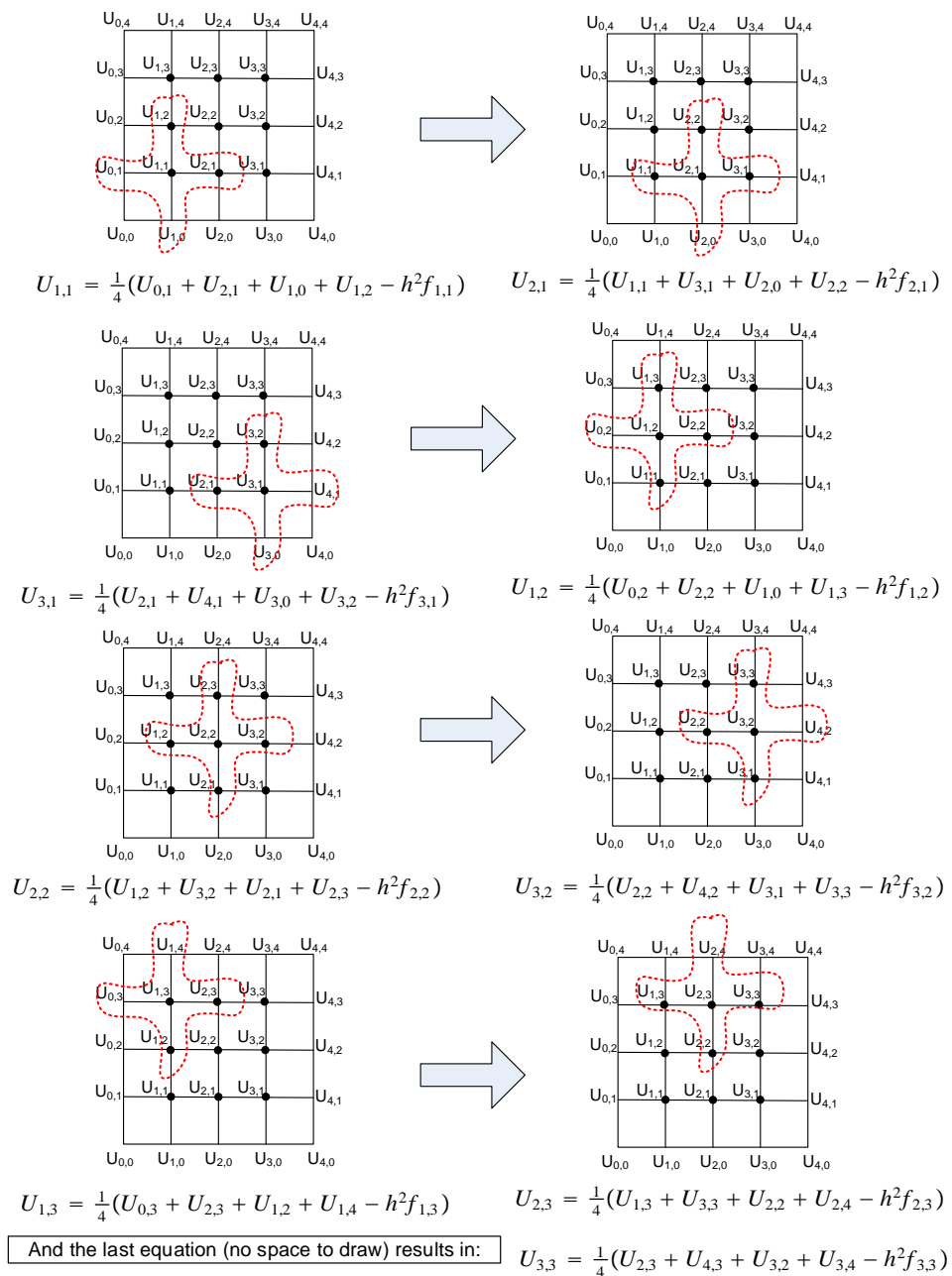
Figure 2.20: stencil move 2

Now that the grid has been updated once, the process is repeated again. This process is continued until convergence is achieved.

2.4.4.6 Structure of $Au = f$

The structure of $Au = f$ system matrix is now described. As an example, for number of unknowns = 9 the following characteristics of the matrix A can be seen
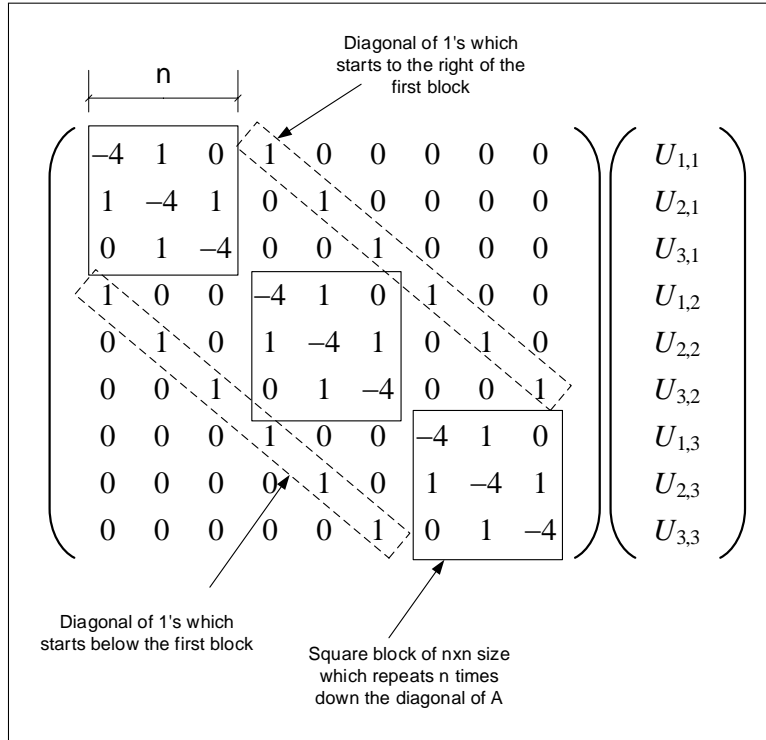
Figure 2.21: A structure

Structure of the A matrix for elliptic 2D PDE with Dirchillet boundary conditions for non-uniform mesh

This section was added at a later time here for completion, and is not required for the HW. Below the A matrix form is derived for the case for non-uniform mesh (this means h_x is not neccessarly the same as h_y) and also, the number of grid points in the x-direction is not the same as the number of grid points in the y-direction.

To ease the derivation, we will use a 5×5 grid as an example, hence this will generate 9 equations as there are 9 unknowns. From this derivation we will be able to see the form of the A matrix.

In addition, in this derivation, we will use i to represent row number, and j to represent column number, as this more closely matches the convention.

Figure 2.22: new grid

The unknowns are shown above in the circles. From earlier, we found that the discretization for the elliptic PDE at a node is

$$U_{i,j} = \frac{1}{2\left(h_x^2 + h_y^2\right)} \left(U_{i,j-1}h_y^2 + U_{i,j+1}h_y^2 + U_{i-1,j}h_x^2 + U_{i+1,j}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)} f_{i,j}$$

We will now write the 9 equations for each node, starting with $(2,2)$ to node $(4,4)$

$$U_{2,2} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{2,1}h_y^2 + U_{2,3}h_y^2 + U_{1,2}h_x^2 + U_{3,2}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{2,2}$$

$$U_{2,3} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{2,2}h_y^2 + U_{2,4}h_y^2 + U_{1,3}h_x^2 + U_{3,3}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{2,3}$$

$$U_{2,4} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{2,3}h_y^2 + U_{2,5}h_y^2 + U_{1,4}h_x^2 + U_{3,4}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{2,4}$$

$$U_{3,2} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{3,1}h_y^2 + U_{3,3}h_y^2 + U_{2,2}h_x^2 + U_{4,2}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{3,2}$$

$$U_{3,3} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{3,2}h_y^2 + U_{3,4}h_y^2 + U_{2,3}h_x^2 + U_{4,3}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{3,3}$$

$$U_{3,4} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{3,3}h_y^2 + U_{3,5}h_y^2 + U_{2,4}h_x^2 + U_{4,4}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{3,4}$$

$$U_{4,2} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{4,1}h_y^2 + U_{4,3}h_y^2 + U_{3,2}h_x^2 + U_{5,2}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{4,2}$$

$$U_{4,3} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{4,2}h_y^2 + U_{4,4}h_y^2 + U_{3,3}h_x^2 + U_{5,3}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{4,3}$$

$$U_{4,4} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{4,3}h_y^2 + U_{4,5}h_y^2 + U_{3,4}h_x^2 + U_{5,4}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{4,4}$$

Now, moving the knowns to the right, results in

$$2\left(h_x^2 + h_y^2\right)U_{2,2} - U_{2,3}h_y^2 - U_{3,2}h_x^2 = -h_x^2 h_y^2 f_{2,2} + U_{2,1}h_y^2 + U_{1,2}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{2,3} - U_{2,2}h_y^2 - U_{2,4}h_y^2 - U_{3,3}h_x^2 = -h_x^2 h_y^2 f_{2,3} + U_{1,3}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{2,4} - U_{2,3}h_y^2 - U_{3,4}h_x^2 = -h_x^2 h_y^2 f_{2,4} + U_{2,5}h_y^2 + U_{1,4}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{3,2} - U_{3,3}h_y^2 - U_{2,2}h_x^2 - U_{4,2}h_x^2 = -h_x^2 h_y^2 f_{3,2} + U_{3,1}h_y^2$$

$$2\left(h_x^2 + h_y^2\right)U_{3,3} - U_{3,2}h_y^2 - U_{3,4}h_y^2 - U_{2,3}h_x^2 - U_{4,3}h_x^2 = -h_x^2 h_y^2 f_{3,3}$$

$$2\left(h_x^2 + h_y^2\right)U_{3,4} - U_{3,3}h_y^2 - U_{2,4}h_x^2 - U_{4,4}h_x^2 = -h_x^2 h_y^2 f_{3,4} + U_{3,5}h_y^2$$

$$2\left(h_x^2 + h_y^2\right)U_{4,2} - U_{4,3}h_y^2 - U_{3,2}h_x^2 = -h_x^2 h_y^2 f_{4,2} + U_{4,1}h_y^2 + U_{5,2}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{4,3} - U_{4,2}h_y^2 - U_{4,4}h_y^2 - U_{3,3}h_x^2 = -h_x^2 h_y^2 f_{4,3} + U_{5,3}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{4,4} - U_{4,3}h_y^2 - U_{3,4}h_x^2 = -h_x^2 h_y^2 f_{4,4} + U_{4,5}h_y^2 + U_{5,4}h_x^2$$

Now, we can write $Ax = b$ as, letting $\beta = 2\left(h_x^2 + h_y^2\right)$

$$\begin{pmatrix} \beta & -h_y^2 & 0 & -h_x^2 & 0 & 0 & 0 & 0 & 0 \\ -h_y^2 & \beta & -h_y^2 & 0 & -h_x^2 & 0 & 0 & 0 & 0 \\ 0 & -h_y^2 & \beta & 0 & 0 & -h_x^2 & 0 & 0 & 0 \\ -h_x^2 & 0 & 0 & \beta & -h_y^2 & 0 & -h_x^2 & 0 & 0 \\ 0 & -h_x^2 & 0 & -h_y^2 & \beta & -h_y^2 & 0 & -h_x^2 & 0 \\ 0 & 0 & -h_x^2 & 0 & -h_y^2 & \beta & 0 & 0 & -h_x^2 \\ 0 & 0 & 0 & -h_x^2 & 0 & 0 & \beta & -h_y^2 & 0 \\ 0 & 0 & 0 & 0 & -h_y^2 & 0 & -h_y^2 & \beta & -h_y^2 \\ 0 & 0 & 0 & 0 & 0 & -h_x^2 & 0 & -h_y^2 & \beta \end{pmatrix} \begin{pmatrix} U_{22} \\ U_{23} \\ U_{24} \\ U_{32} \\ U_{33} \\ U_{34} \\ U_{42} \\ U_{43} \\ U_{44} \end{pmatrix} = \begin{pmatrix} -h_x^2 h_y^2 f_{2,2} + U_{2,1} h_y^2 + U_{1,2} h_x^2 \\ -h_x^2 h_y^2 f_{2,3} + U_{1,3} h_x^2 \\ -h_x^2 h_y^2 f_{2,4} + U_{2,5} h_y^2 + U_{1,4} h_x^2 \\ -h_x^2 h_y^2 f_{3,2} + U_{3,1} h_y^2 \\ -h_x^2 h_y^2 f_{3,3} \\ -h_x^2 h_y^2 f_{3,4} + U_{3,5} h_y^2 \\ -h_x^2 h_y^2 f_{4,2} + U_{4,1} h_y^2 + U_{5,2} h_x^2 \\ -h_x^2 h_y^2 f_{4,3} + U_{5,3} h_x^2 \\ -h_x^2 h_y^2 f_{4,4} + U_{4,5} h_y^2 + U_{5,4} h_x^2 \end{pmatrix}$$

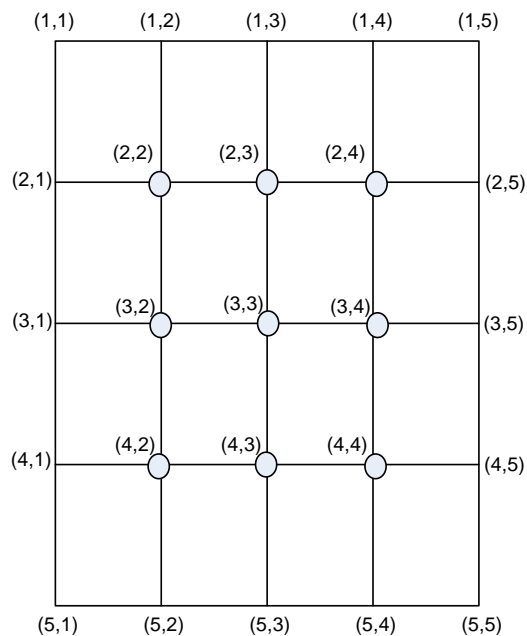Now we will do another case $n_x \neq n_y$
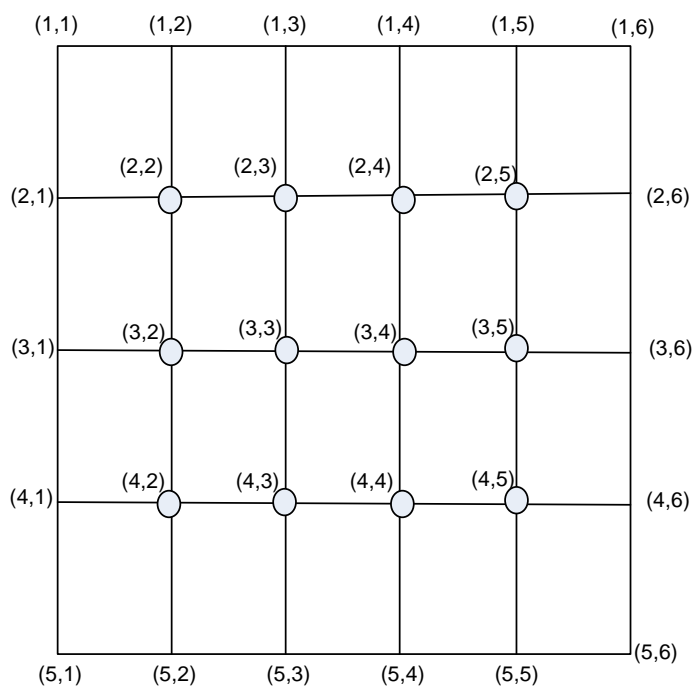
Figure 2.23: new grid 2

The unknowns are shown above in the circles. From earlier, we found that the discretization for the elliptic PDE at a node is

$$U_{i,j} = \frac{1}{2\left(h_x^2 + h_y^2\right)} \left(U_{i,j-1} h_y^2 + U_{i,j+1} h_y^2 + U_{i-1,j} h_x^2 + U_{i+1,j} h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)} f_{i,j}$$

We will now write the 12 equations for each node, starting with $(2, 2)$ to node $(4, 5)$

$$U_{2,2} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{2,1}h_y^2 + U_{2,3}h_y^2 + U_{1,2}h_x^2 + U_{3,2}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{2,2}$$

$$U_{2,3} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{2,2}h_y^2 + U_{2,4}h_y^2 + U_{1,3}h_x^2 + U_{3,3}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{2,3}$$

$$U_{2,4} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{2,3}h_y^2 + U_{2,5}h_y^2 + U_{1,4}h_x^2 + U_{3,4}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{2,4}$$

$$U_{2,5} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{2,4}h_y^2 + U_{2,6}h_y^2 + U_{1,5}h_x^2 + U_{3,5}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{2,5}$$

$$U_{3,2} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{3,1}h_y^2 + U_{3,3}h_y^2 + U_{2,2}h_x^2 + U_{4,2}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{3,2}$$

$$U_{3,3} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{3,2}h_y^2 + U_{3,4}h_y^2 + U_{2,3}h_x^2 + U_{4,3}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{3,3}$$

$$U_{3,4} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{3,3}h_y^2 + U_{3,5}h_y^2 + U_{2,4}h_x^2 + U_{4,4}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{3,4}$$

$$U_{3,5} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{3,4}h_y^2 + U_{3,6}h_y^2 + U_{2,5}h_x^2 + U_{4,5}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{3,4}$$

$$U_{4,2} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{4,1}h_y^2 + U_{4,3}h_y^2 + U_{3,2}h_x^2 + U_{5,2}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{4,2}$$

$$U_{4,3} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{4,2}h_y^2 + U_{4,4}h_y^2 + U_{3,3}h_x^2 + U_{5,3}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{4,3}$$

$$U_{4,4} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{4,3}h_y^2 + U_{4,5}h_y^2 + U_{3,4}h_x^2 + U_{5,4}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{4,4}$$

$$U_{4,5} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{4,4}h_y^2 + U_{4,6}h_y^2 + U_{3,5}h_x^2 + U_{5,5}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{4,5}$$

Now, moving the knowns to the right, results in

$$2\left(h_x^2 + h_y^2\right)U_{2,2} - U_{2,3}h_y^2 - U_{3,2}h_x^2 = -h_x^2 h_y^2 f_{2,2} + U_{2,1}h_y^2 + U_{1,2}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{2,3} - U_{2,2}h_y^2 - U_{2,4}h_y^2 - U_{3,3}h_x^2 = -h_x^2 h_y^2 f_{2,3} + U_{1,3}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{2,4} - U_{2,3}h_y^2 - U_{2,5}h_y^2 - U_{3,4}h_x^2 = -h_x^2 h_y^2 f_{2,4} + U_{1,4}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{2,5} - U_{2,4}h_y^2 + U_{3,5}h_x^2 = -h_x^2 h_y^2 f_{2,5} + U_{2,6}h_y^2 + U_{1,5}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{3,2} - U_{3,3}h_y^2 - U_{2,2}h_x^2 - U_{4,2}h_x^2 = -h_x^2 h_y^2 f_{3,2} + U_{3,1}h_y^2$$

$$2\left(h_x^2 + h_y^2\right)U_{3,3} - U_{3,2}h_y^2 - U_{3,4}h_y^2 - U_{2,3}h_x^2 - U_{4,3}h_x^2 = -h_x^2 h_y^2 f_{3,3}$$

$$2\left(h_x^2 + h_y^2\right)U_{3,4} - U_{3,3}h_y^2 - U_{3,5}h_y^2 - U_{2,4}h_x^2 - U_{4,4}h_x^2 = -h_x^2 h_y^2 f_{3,4}$$

$$2\left(h_x^2 + h_y^2\right)U_{3,5} - U_{3,4}h_y^2 + U_{2,5}h_x^2 + U_{4,5}h_x^2 = -h_x^2 h_y^2 f_{3,4} + U_{3,6}h_y^2$$

$$2\left(h_x^2 + h_y^2\right)U_{4,2} - U_{4,3}h_y^2 - U_{3,2}h_x^2 = -h_x^2 h_y^2 f_{4,2} + U_{4,1}h_y^2 + U_{5,2}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{4,3} - U_{4,2}h_y^2 - U_{4,4}h_y^2 - U_{3,3}h_x^2 = -h_x^2 h_y^2 f_{4,3} + U_{5,3}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{4,4} - U_{4,5}h_y^2 - U_{4,3}h_y^2 - U_{3,4}h_x^2 = -h_x^2 h_y^2 f_{4,4} + U_{5,4}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{4,5} - U_{4,4}h_y^2 - U_{3,5}h_x^2 = -h_x^2 h_y^2 f_{4,5} + U_{4,6}h_y^2 + U_{5,5}h_x^2$$

Now, we can write $Ax = b$ as, letting $\beta = 2\left(h_x^2 + h_y^2\right)$

$$
\begin{pmatrix}
\beta & -h_y^2 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-h_y^2 & \beta & -h_y^2 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -h_y^2 & \beta & -h_y^2 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -h_y^2 & \beta & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & 0 \\
-h_x^2 & 0 & 0 & 0 & \beta & -h_y^2 & 0 & 0 & -h_x^2 & 0 & 0 & 0 \\
0 & -h_x^2 & 0 & 0 & -h_y^2 & \beta & -h_y^2 & 0 & 0 & -h_x^2 & 0 & 0 \\
0 & 0 & -h_x^2 & 0 & 0 & -h_y^2 & \beta & -h_y^2 & 0 & 0 & -h_x^2 & 0 \\
0 & 0 & 0 & -h_x^2 & 0 & 0 & -h_y^2 & \beta & 0 & 0 & 0 & -h_x^2 \\
0 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & \beta & -h_y^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -h_y^2 & 0 & 0 & -h_y^2 & \beta & -h_y^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & -h_y^2 & \beta & -h_y^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & -h_y^2 & \beta
\end{pmatrix}
\begin{pmatrix}
U_{22} \\ U_{23} \\ U_{24} \\ U_{25} \\ U_{32} \\ U_{33} \\ U_{34} \\ U_{35} \\ U_{42} \\ U_{43} \\ U_{44} \\ U_{45}
\end{pmatrix}
=
\begin{pmatrix}
-h_x^2 h_y^2 f_{2,2} + U_{2,1}h_y^2 + U_{1,2}h_x^2 \\
-h_x^2 h_y^2 f_{2,3} + U_{1,3}h_x^2 \\
-h_x^2 h_y^2 f_{2,4} + U_{2,5}h_y^2 + U_{1,4}h_x^2 \\
-h_x^2 h_y^2 f_{2,5} + U_{2,6}h_y^2 + U_{1,5}h_x^2 \\
-h_x^2 h_y^2 f_{3,2} + U_{3,1}h_y^2 \\
-h_x^2 h_y^2 f_{3,3} \\
-h_x^2 h_y^2 f_{3,4} + U_{3,5}h_y^2 \\
-h_x^2 h_y^2 f_{3,4} + U_{3,6}h_y^2 \\
-h_x^2 h_y^2 f_{4,2} + U_{4,1}h_y^2 + U_{5,2}h_x^2 \\
-h_x^2 h_y^2 f_{4,3} + U_{5,3}h_x^2 \\
-h_x^2 h_y^2 f_{4,4} + U_{4,5}h_y^2 + U_{5,4}h_x^2 \\
-h_x^2 h_y^2 f_{4,5} + U_{4,6}h_y^2 + U_{5,5}h_x^2
\end{pmatrix}
$$

To create the above matrix, as sparse matrix, call the following Matlab code

```
1  A=lap2d(4,3,hx,hy);
2  %Where in the above, 4 is nx, which is the number of nodes in the x-
       direction.
3  %These are internal nodes. ny is number of nodes in the y-direction.
4  %hx is the spacing between nodes in the x-direction.
5  %hy is spacing between nodes in the y-direction
6
7  function L2 = lap2d(nx,ny,hx,hy)
8  Lx=lap1dy(nx,hx,hy);   %makes the MAIN block, only one block
9  Ly=lap1dx(ny,hx,hy);   %makes the off block, only one block
10
11  Ix=speye(nx);
```

```
12  Iy=speye(ny);
13
14  Lm=kron(Iy,Lx); %does the central diagonal
15  Lo=kron(Ly,Ix); %does the off diagonal
16  L2=Lm+Lo;
17
18  %-----------------------
19  function L=lap1dy(n,hx,hy)
20  e=ones(n,1);
21  T1=2*(hx^2+hy^2);
22  B=[-e*hy^2 (1/2)*T1*e -e*hy^2];
23  L=spdiags(B,[-1 0 1],n,n);
24
25  %-----------------------------------
26  %
27  %
28  function L=lap1dx(n,hx,hy)
29  e=ones(n,1);
30  T1=2*(hx^2+hy^2);
31  B=[-e*hx^2 (1/2)*T1*e -e*hx^2];
32  L=spdiags(B,[-1 0 1],n,n);
```

.

A matrix format for sparse storage for elliptic 2D with non-homegenous Nuemman boundary conditions

This section was added at later time, and not required for this HW.

To be able to forumlate the A matrix as sparse matrix, we need to find what the form will be in the case when one or more of the boundary conditions has Nuemman coditions on it. We will start as above, and start with assuming Nuemman conditions now exist on the left edge and find out what the form of the A matrix will turn out to be.
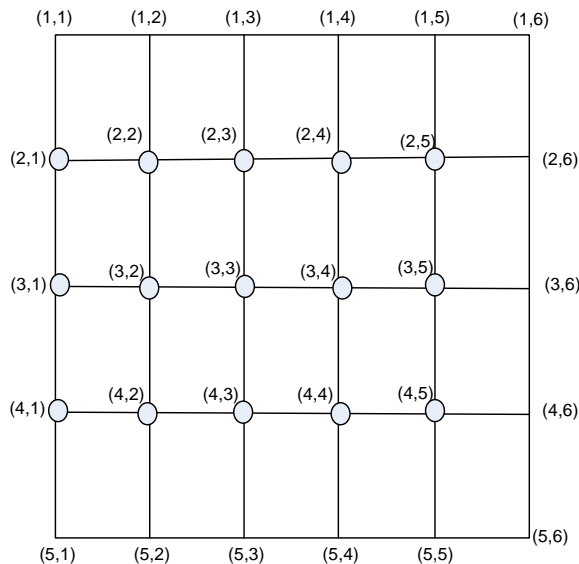
Figure 2.24: new grid 2 nuemman on left

The unknowns are shown above in the circles. From earlier, we found that the discretization for the elliptic PDE at an internal node is (note that in the above, i is the row number, and j is column number)

$$U_{i,j} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{i,j-1}h_y^2 + U_{i,j+1}h_y^2 + U_{i-1,j}h_x^2 + U_{i+1,j}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{i,j}$$

And for Nuemman, non-homogenouse boundary conditions, the left edge unknowns are given by

$$U_{i,1} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(2U_{i,2}\,h_y^2 + \left(U_{i-1,1} + U_{i+1,1}\right)h_x^2 - h_y^2 h_x g_{i,1}\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{i,1}$$

We will now write the 15 equations for each node, starting with $(2,1)$ to node $(4,5)$

$$U_{2,1} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(2U_{2,2}\,h_y^2 + \left(U_{1,1} + U_{3,1}\right)h_x^2 - h_y^2 h_x g_{2,1}\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{2,1}$$

$$U_{2,2} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{2,1}h_y^2 + U_{2,3}h_y^2 + U_{1,2}h_x^2 + U_{3,2}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{2,2}$$

$$U_{2,3} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{2,2}h_y^2 + U_{2,4}h_y^2 + U_{1,3}h_x^2 + U_{3,3}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{2,3}$$

$$U_{2,4} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{2,3}h_y^2 + U_{2,5}h_y^2 + U_{1,4}h_x^2 + U_{3,4}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{2,4}$$

$$U_{2,5} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{2,4}h_y^2 + U_{2,6}h_y^2 + U_{1,5}h_x^2 + U_{3,5}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{2,5}$$

$$U_{3,1} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(2U_{3,2}\,h_y^2 + \left(U_{2,1} + U_{4,1}\right)h_x^2 - h_y^2 h_x g_{3,1}\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{3,1}$$

$$U_{3,2} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{3,1}h_y^2 + U_{3,3}h_y^2 + U_{2,2}h_x^2 + U_{4,2}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{3,2}$$

$$U_{3,3} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{3,2}h_y^2 + U_{3,4}h_y^2 + U_{2,3}h_x^2 + U_{4,3}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{3,3}$$

$$U_{3,4} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{3,3}h_y^2 + U_{3,5}h_y^2 + U_{2,4}h_x^2 + U_{4,4}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{3,4}$$

$$U_{3,5} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{3,4}h_y^2 + U_{3,6}h_y^2 + U_{2,5}h_x^2 + U_{4,5}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{3,4}$$

$$U_{4,1} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(2U_{4,2}\,h_y^2 + \left(U_{3,1} + U_{5,1}\right)h_x^2 - h_y^2 h_x g_{4,1}\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{4,1}$$

$$U_{4,2} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{4,1}h_y^2 + U_{4,3}h_y^2 + U_{3,2}h_x^2 + U_{5,2}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{4,2}$$

$$U_{4,3} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{4,2}h_y^2 + U_{4,4}h_y^2 + U_{3,3}h_x^2 + U_{5,3}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{4,3}$$

$$U_{4,4} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{4,3}h_y^2 + U_{4,5}h_y^2 + U_{3,4}h_x^2 + U_{5,4}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{4,4}$$

$$U_{4,5} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{4,4}h_y^2 + U_{4,6}h_y^2 + U_{3,5}h_x^2 + U_{5,5}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{4,5}$$

Now, moving the knowns to the right, results in

$$2\left(h_x^2 + h_y^2\right)U_{2,1} - 2U_{2,2}\,h_y^2 - U_{3,1}h_x^2 = U_{1,1}h_x^2 - h_y^2 h_x g_{2,1} - h_x^2 h_y^2 f_{2,1}$$

$$2\left(h_x^2 + h_y^2\right)U_{2,2} - U_{2,3}h_y^2 - U_{3,2}h_x^2 = -h_x^2 h_y^2 f_{2,2} + U_{2,1}h_y^2 + U_{1,2}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{2,3} - U_{2,2}h_y^2 - U_{2,4}h_y^2 - U_{3,3}h_x^2 = -h_x^2 h_y^2 f_{2,3} + U_{1,3}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{2,4} - U_{2,3}h_y^2 - U_{2,5}h_y^2 - U_{3,4}h_x^2 = -h_x^2 h_y^2 f_{2,4} + U_{1,4}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{2,5} - U_{2,4}h_y^2 + U_{3,5}h_x^2 = -h_x^2 h_y^2 f_{2,5} + U_{2,6}h_y^2 + U_{1,5}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{3,1} - 2U_{3,2}\,h_y^2 - U_{4,1}h_x^2 - U_{2,1}h_x^2 = -h_y^2 h_x g_{3,1} - h_x^2 h_y^2 f_{3,1}$$

$$2\left(h_x^2 + h_y^2\right)U_{3,2} - U_{3,3}h_y^2 - U_{2,2}h_x^2 - U_{4,2}h_x^2 = -h_x^2 h_y^2 f_{3,2} + U_{3,1}h_y^2$$

$$2\left(h_x^2 + h_y^2\right)U_{3,3} - U_{3,2}h_y^2 - U_{3,4}h_y^2 - U_{2,3}h_x^2 - U_{4,3}h_x^2 = -h_x^2 h_y^2 f_{3,3}$$

$$2\left(h_x^2 + h_y^2\right)U_{3,4} - U_{3,3}h_y^2 - U_{3,5}h_y^2 - U_{2,4}h_x^2 - U_{4,4}h_x^2 = -h_x^2 h_y^2 f_{3,4}$$

$$2\left(h_x^2 + h_y^2\right)U_{3,5} - U_{3,4}h_y^2 + U_{2,5}h_x^2 + U_{4,5}h_x^2 = -h_x^2 h_y^2 f_{3,4} + U_{3,6}h_y^2$$

$$2\left(h_x^2 + h_y^2\right)U_{4,1} - 2U_{4,2}\,h_y^2 - U_{3,1}h_x^2 = U_{5,1}h_x^2 - h_y^2 h_x g_{4,1} - h_x^2 h_y^2 f_{4,1}$$

$$2\left(h_x^2 + h_y^2\right)U_{4,2} - U_{4,3}h_y^2 - U_{3,2}h_x^2 = -h_x^2 h_y^2 f_{4,2} + U_{4,1}h_y^2 + U_{5,2}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{4,3} - U_{4,2}h_y^2 - U_{4,4}h_y^2 - U_{3,3}h_x^2 = -h_x^2 h_y^2 f_{4,3} + U_{5,3}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{4,4} - U_{4,5}h_y^2 - U_{4,3}h_y^2 - U_{3,4}h_x^2 = -h_x^2 h_y^2 f_{4,4} + U_{5,4}h_x^2$$

$$2\left(h_x^2 + h_y^2\right)U_{4,5} - U_{4,4}h_y^2 - U_{3,5}h_x^2 = -h_x^2 h_y^2 f_{4,5} + U_{4,6}h_y^2 + U_{5,5}h_x^2$$

Now, we can write $Ax = b$ as, letting $\beta = 2\left(h_x^2 + h_y^2\right)$

$$\begin{pmatrix} \beta & -2h_y^2 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & -h_y^2 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -h_y^2 & \beta & -h_y^2 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -h_y^2 & \beta & -h_y^2 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -h_y^2 & \beta & 0 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & 0 & 0 \\ -h_x^2 & 0 & 0 & 0 & 0 & \beta & -2h_y^2 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & 0 \\ 0 & -h_x^2 & 0 & 0 & 0 & 0 & \beta & -h_y^2 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 \\ 0 & 0 & -h_x^2 & 0 & 0 & 0 & -h_y^2 & \beta & -h_y^2 & 0 & 0 & 0 & -h_x^2 & 0 & 0 \\ 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & -h_y^2 & \beta & -h_y^2 & 0 & 0 & 0 & -h_x^2 & 0 \\ 0 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & -h_y^2 & \beta & 0 & 0 & 0 & 0 & -h_x^2 \\ 0 & 0 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & 0 & \beta & -2h_y^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & \beta & -h_y^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_y^2 & 0 & 0 & 0 & -h_y^2 & \beta & -h_y^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & -h_y^2 & \beta & -h_y^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & -h_y^2 & \beta \end{pmatrix} \begin{pmatrix} U_{21} \\ U_{22} \\ U_{23} \\ U_{24} \\ U_{25} \\ U_{31} \\ U_{32} \\ U_{33} \\ U_{34} \\ U_{35} \\ U_{41} \\ U_{42} \\ U_{43} \\ U_{44} \\ U_{45} \end{pmatrix} = \begin{pmatrix} U_{1,1}h_x^2 - h_y^2 h_x g_{2,1} - h_x^2 h_y^2 f_{2,1} \\ -h_x^2 h_y^2 f_{2,2} + U_{2,1}h_y^2 + U_{1,2}h_x^2 \\ -h_x^2 h_y^2 f_{2,3} + U_{1,3}h_x^2 \\ -h_x^2 h_y^2 f_{2,4} + U_{2,5}h_y^2 + U_{1,4}h_x^2 \\ -h_x^2 h_y^2 f_{2,5} + U_{2,6}h_y^2 + U_{1,5}h_x^2 \\ -h_y^2 h_x g_{3,1} - h_x^2 h_y^2 f_{3,1} \\ -h_x^2 h_y^2 f_{3,2} + U_{3,1}h_y^2 \\ -h_x^2 h_y^2 f_{3,3} \\ -h_x^2 h_y^2 f_{3,4} + U_{3,5}h_y^2 \\ -h_x^2 h_y^2 f_{3,4} + U_{3,6}h_y^2 \\ U_{5,1}h_x^2 - h_y^2 h_x g_{4,1} - h_x^2 h_y^2 f_{4,1} \\ -h_x^2 h_y^2 f_{4,2} + U_{4,1}h_y^2 + U_{5,2}h_x^2 \\ -h_y^2 h_y^2 f_{4,3} + U_{5,3}h_x^2 \\ -h_x^2 h_y^2 f_{4,4} + U_{4,5}h_y^2 + U_{5,4}h_x^2 \\ -h_x^2 h_y^2 f_{4,5} + U_{4,6}h_y^2 + U_{5,5}h_x^2 \end{pmatrix}$$

The above is the matrix for the case of non-homogeneous Neumann boundary conditions on the left edge.

We will now do the same edge, but with homogeneous boundary conditions to see the difference. Recall, that when the edge is insulated, then

$$U_{i,1} = \frac{h_x h_y}{\left(h_x^2 + h_y^2\right)}\left(\frac{h_x}{2h_y}\left(U_{i-1,1} + U_{i+1,1}\right) - \frac{h_y}{h_x}U_{i,2}\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{i,1}$$

Using the above, we write the 15 equations starting with $(2,1)$ to node $(4,5)$

$$U_{2,1} = \frac{h_x h_y}{\left(h_x^2 + h_y^2\right)} \left(\frac{h_x}{2h_y}\left(U_{1,1} + U_{3,1}\right) - \frac{h_y}{h_x}U_{2,2}\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{2,1}$$

$$U_{2,2} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{2,1}h_y^2 + U_{2,3}h_y^2 + U_{1,2}h_x^2 + U_{3,2}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{2,2}$$

$$U_{2,3} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{2,2}h_y^2 + U_{2,4}h_y^2 + U_{1,3}h_x^2 + U_{3,3}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{2,3}$$

$$U_{2,4} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{2,3}h_y^2 + U_{2,5}h_y^2 + U_{1,4}h_x^2 + U_{3,4}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{2,4}$$

$$U_{2,5} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{2,4}h_y^2 + U_{2,6}h_y^2 + U_{1,5}h_x^2 + U_{3,5}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{2,5}$$

$$U_{3,1} = \frac{h_x h_y}{\left(h_x^2 + h_y^2\right)} \left(\frac{h_x}{2h_y}\left(U_{2,1} + U_{4,1}\right) - \frac{h_y}{h_x}U_{3,2}\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{3,1}$$

$$U_{3,2} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{3,1}h_y^2 + U_{3,3}h_y^2 + U_{2,2}h_x^2 + U_{4,2}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{3,2}$$

$$U_{3,3} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{3,2}h_y^2 + U_{3,4}h_y^2 + U_{2,3}h_x^2 + U_{4,3}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{3,3}$$

$$U_{3,4} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{3,3}h_y^2 + U_{3,5}h_y^2 + U_{2,4}h_x^2 + U_{4,4}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{3,4}$$

$$U_{3,5} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{3,4}h_y^2 + U_{3,6}h_y^2 + U_{2,5}h_x^2 + U_{4,5}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{3,4}$$

$$U_{4,1} = \frac{h_x h_y}{\left(h_x^2 + h_y^2\right)} \left(\frac{h_x}{2h_y}\left(U_{3,1} + U_{5,1}\right) - \frac{h_y}{h_x}U_{4,2}\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{4,1}$$

$$U_{4,2} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{4,1}h_y^2 + U_{4,3}h_y^2 + U_{3,2}h_x^2 + U_{5,2}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{4,2}$$

$$U_{4,3} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{4,2}h_y^2 + U_{4,4}h_y^2 + U_{3,3}h_x^2 + U_{5,3}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{4,3}$$

$$U_{4,4} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{4,3}h_y^2 + U_{4,5}h_y^2 + U_{3,4}h_x^2 + U_{5,4}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{4,4}$$

$$U_{4,5} = \frac{1}{2\left(h_x^2 + h_y^2\right)}\left(U_{4,4}h_y^2 + U_{4,6}h_y^2 + U_{3,5}h_x^2 + U_{5,5}h_x^2\right) - \frac{h_x^2 h_y^2}{2\left(h_x^2 + h_y^2\right)}f_{4,5}$$

Now, moving the knowns to the right, results in

$$\frac{\left(h_x^2 + h_y^2\right)}{h_x h_y} U_{2,1} - \frac{h_x}{2h_y} U_{3,1} + \frac{h_y}{h_x} U_{2,2} = \frac{h_x}{2h_y} U_{1,1} - h_x h_y f_{2,1}$$

$$2\left(h_x^2 + h_y^2\right) U_{2,2} - U_{2,3} h_y^2 - U_{3,2} h_x^2 = -h_x^2 h_y^2 f_{2,2} + U_{2,1} h_y^2 + U_{1,2} h_x^2$$

$$2\left(h_x^2 + h_y^2\right) U_{2,3} - U_{2,2} h_y^2 - U_{2,4} h_y^2 - U_{3,3} h_x^2 = -h_x^2 h_y^2 f_{2,3} + U_{1,3} h_x^2$$

$$2\left(h_x^2 + h_y^2\right) U_{2,4} - U_{2,3} h_y^2 - U_{2,5} h_y^2 - U_{3,4} h_x^2 = -h_x^2 h_y^2 f_{2,4} + U_{1,4} h_x^2$$

$$2\left(h_x^2 + h_y^2\right) U_{2,5} - U_{2,4} h_y^2 + U_{3,5} h_x^2 = -h_x^2 h_y^2 f_{2,5} + U_{2,6} h_y^2 + U_{1,5} h_x^2$$

$$\frac{\left(h_x^2 + h_y^2\right)}{h_x h_y} U_{3,1} - \frac{h_x}{2h_y} U_{2,1} - \frac{h_x}{2h_y} U_{4,1} + \frac{h_y}{h_x} U_{3,2} = -h_x h_y f_{3,1}$$

$$2\left(h_x^2 + h_y^2\right) U_{3,2} - U_{3,3} h_y^2 - U_{2,2} h_x^2 - U_{4,2} h_x^2 = -h_x^2 h_y^2 f_{3,2} + U_{3,1} h_y^2$$

$$2\left(h_x^2 + h_y^2\right) U_{3,3} - U_{3,2} h_y^2 - U_{3,4} h_y^2 - U_{2,3} h_x^2 - U_{4,3} h_x^2 = -h_x^2 h_y^2 f_{3,3}$$

$$2\left(h_x^2 + h_y^2\right) U_{3,4} - U_{3,3} h_y^2 - U_{3,5} h_y^2 - U_{2,4} h_x^2 - U_{4,4} h_x^2 = -h_x^2 h_y^2 f_{3,4}$$

$$2\left(h_x^2 + h_y^2\right) U_{3,5} - U_{3,4} h_y^2 + U_{2,5} h_x^2 + U_{4,5} h_x^2 = -h_x^2 h_y^2 f_{3,4} + U_{3,6} h_y^2$$

$$\frac{\left(h_x^2 + h_y^2\right)}{h_x h_y} U_{4,1} - \frac{h_x}{2h_y} U_{3,1} + \frac{h_y}{h_x} U_{4,2} = \frac{h_x}{2h_y} U_{5,1} - h_x h_y f_{4,1}$$

$$2\left(h_x^2 + h_y^2\right) U_{4,2} - U_{4,3} h_y^2 - U_{3,2} h_x^2 = -h_x^2 h_y^2 f_{4,2} + U_{4,1} h_y^2 + U_{5,2} h_x^2$$

$$2\left(h_x^2 + h_y^2\right) U_{4,3} - U_{4,2} h_y^2 - U_{4,4} h_y^2 - U_{3,3} h_x^2 = -h_x^2 h_y^2 f_{4,3} + U_{5,3} h_x^2$$

$$2\left(h_x^2 + h_y^2\right) U_{4,4} - U_{4,5} h_y^2 - U_{4,3} h_y^2 - U_{3,4} h_x^2 = -h_x^2 h_y^2 f_{4,4} + U_{5,4} h_x^2$$

$$2\left(h_x^2 + h_y^2\right) U_{4,5} - U_{4,4} h_y^2 - U_{3,5} h_x^2 = -h_x^2 h_y^2 f_{4,5} + U_{4,6} h_y^2 + U_{5,5} h_x^2$$

Now, we can write $Ax = b$ as, letting $\beta = 2\left(h_x^2 + h_y^2\right)$ and $\gamma = \frac{\left(h_x^2 + h_y^2\right)}{h_x h_y}$

$$\left(\begin{array}{cccccccccccccccc}
\gamma & \frac{h_y}{h_x} & 0 & 0 & 0 & -\frac{h_x}{2h_y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \beta & -h_y^2 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -h_y^2 & \beta & -h_y^2 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -h_y^2 & \beta & -h_y^2 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -h_y^2 & \beta & 0 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & 0 \\
-\frac{h_x}{2h_y} & 0 & 0 & 0 & 0 & \gamma & \frac{h_y}{h_x} & 0 & 0 & 0 & -\frac{h_x}{2h_y} & 0 & 0 & 0 \\
0 & -h_x^2 & 0 & 0 & 0 & 0 & \beta & -h_y^2 & 0 & 0 & 0 & -h_x^2 & 0 & 0 \\
0 & 0 & -h_x^2 & 0 & 0 & 0 & -h_y^2 & \beta & -h_y^2 & 0 & 0 & 0 & -h_x^2 & 0 \\
0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & -h_y^2 & \beta & -h_y^2 & 0 & 0 & 0 & -h_x^2 \\
0 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & -h_y^2 & \beta & 0 & 0 & 0 & -h_x^2 \\
0 & 0 & 0 & 0 & 0 & -\frac{h_x}{2h_y} & 0 & 0 & 0 & 0 & \gamma & \frac{h_y}{h_x} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & 0 & \beta & -h_y^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & -h_y^2 & \beta & -h_y^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & -h_y^2 & \beta & -h_y^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_x^2 & 0 & 0 & -h_y^2 & \beta
\end{array}\right)
\left(\begin{array}{c}
U_{21} \\ U_{22} \\ U_{23} \\ U_{24} \\ U_{25} \\ U_{31} \\ U_{32} \\ U_{33} \\ U_{34} \\ U_{35} \\ U_{41} \\ U_{42} \\ U_{43} \\ U_{44} \\ U_{45}
\end{array}\right)
=
\left(\begin{array}{c}
\frac{h_x}{2h_y} U_{1,1} - h_x h_y f_{2,1} \\
-h_x^2 h_y^2 f_{2,2} + U_{2,1} h_y^2 + U_{1,2} h_x^2 \\
-h_x^2 h_y^2 f_{2,3} + U_{1,3} h_x^2 \\
-h_x^2 h_y^2 f_{2,4} + U_{2,5} h_y^2 + U_{1,4} h_x^2 \\
-h_x^2 h_y^2 f_{2,5} + U_{2,6} h_y^2 + U_{1,5} h_x^2 \\
-h_x h_y f_{3,1} \\
-h_x^2 h_y^2 f_{3,2} + U_{3,1} h_y^2 \\
-h_x^2 h_y^2 f_{3,3} \\
-h_x^2 h_y^2 f_{3,4} + U_{3,5} h_y^2 \\
-h_x^2 h_y^2 f_{3,4} + U_{3,6} h_y^2 \\
\frac{h_x}{2h_y} U_{5,1} - h_x h_y f_{4,1} \\
-h_x^2 h_y^2 f_{4,2} + U_{4,1} h_y^2 + U_{5,2} h_x^2 \\
-h_x^2 h_y^2 f_{4,3} + U_{5,3} h_x^2 \\
-h_x^2 h_y^2 f_{4,4} + U_{4,5} h_y^2 + U_{5,4} h_x^2 \\
-h_x^2 h_y^2 f_{4,5} + U_{4,6} h_y^2 + U_{5,5} h_x^2
\end{array}\right)$$

<u>Storage requirements for the different solvers</u>

The total number of grid points along the x or the y direction is given by $\frac{length}{h} + 1$, where length is always 1, and hence n, the number of unknowns in one dimension is 2 less than the above number (since U is known at the boundaries).

It is important to note that the matrix A is not used explicitly to solve for the unknowns

in the iterative schemes. Storage is needed only for the internal grid points, which is the number of the unknowns n^2. An auxiliary grid is used to hold the updated values in the Jacobian method. In addition, an auxiliary grid is required for $f_{i,j}$, the force function. Hence, in total $3n^2$ storage is needed.

Comparing this to the storage needed in the case of direct solver, where the storage for A alone is n^4. This shows the main advantage of the iterative methods compared to the direct method when A is a dense matrix. (Use of sparse matrix become necessary if direct solver is to be used for large n problems).

The following table summarizes the above discussion for the h values given in this problem. In this calculations, double precision (8 bytes) per grid point is assumed. For single precision half of this storage would be needed.

h	n	number of unknowns n^2	storage	size of A n^4	storage for dense A
2^{-5}	30	900	30 (k Bytes)	$810,000$	6.4 MBytes
2^{-6}	62	3844	0.1 (MB)	$14,776,336$	118 MBytes
2^{-7}	126	15876	0.5 (MB)	$252,047,376$	2 GB

Loop algorithm for each method

This is a short description of the algorithm used by each method. In the following $u_{i,j}^{new}$ represents the new value (residing on the auxiliary grid) of the unknown, and $u_{i,j}^{current}$ is the current value. Initially $u^{current}$ is set to zero at each internal grid point. (any other initial value could also have been used).

Jacobi method algorithm

$k := 0$ (*counter*)

$\epsilon := Ch^2$ (*tolerance*)

$u^{current} := u^{new} := 0$ (*initialize storage*)

$f := f(x, y)$ (*initialize f on grid points*)

$CONVERGED := false$

$WHILE\ NOT\ CONVERGED$

    $LOOP\ i$

       $LOOP\ j$

$$u_{i,j}^{new} := \frac{1}{4}\left(u_{i-1,j}^{current} + u_{i+1,j}^{current} + u_{i,j-1}^{current} + u_{i,j+1}^{current} - h^2 f_{i,j}\right)$$

$$R_{i,j} = f_{i,j} - \frac{1}{h^2}\left(u_{i-1,j}^{current} + u_{i+1,j}^{current} + u_{i,j-1}^{current} + u_{i,j+1}^{current} - 4u_{i,j}^{current}\right) \quad (*residual*)$$

       $END\ LOOP\ j$

    $END\ LOOP\ i$

    $u^{current} := u^{new}$ (*update*)

    $k := k + 1$

    $IF \quad \left(\frac{\|R\|}{\|f\|} < \epsilon\right)\ THEN$ (*Norms are grid norms. see code*)

       $CONVERGED := true$

    $END\ IF$

$END\ WHILE$

Gauss-Seidel method

$k := 0$ (*counter*)

$\epsilon := Ch^2$ (*tolerance*)

$u^{current} := u^{new} := 0$ (*initialize storage*)

$f := f(x,y)$ (*initialize f on grid points*)

CONVERGED := false

WHILE NOT CONVERGED

    LOOP i

      LOOP j

$$u_{ij}^{new} := \frac{1}{4}\left(u_{i-1,j}^{new} + u_{i+1,j}^{current} + u_{i,j-1}^{new} + u_{i,j+1}^{current} - h^2 f_{i,j}\right)$$

$$R_{i,j} = f_{i,j} - \frac{1}{h^2}\left(u_{i-1,j}^{current} + u_{i+1,j}^{current} + u_{i,j-1}^{current} + u_{i,j+1}^{current} - 4u_{i,j}^{current}\right) \quad (*residual*)$$

      END LOOP j

    END LOOP i

    $u^{current} := u^{new}$ (*update*)

    $k := k + 1$

    IF $\quad \left(\frac{\|R\|}{\|f\|} < \epsilon\right)$ THEN $\quad$ (*Norms are grid norms. see code*)

      CONVERGED := true

    END IF

  END WHILE

SOR method

$k := 0$ (*counter*)

$\epsilon := Ch^2$ (*tolerance*)

$u^{current} := u^{new} := 0$ (*initialize storage*)

$f := f(x, y)$ (*initialize f on grid points*)

$CONVERGED := false$

WHILE NOT CONVERGED

    LOOP i

        LOOP j

$$u_{ij}^{new} := \frac{\omega}{4}\left(u_{i-1,j}^{new} + u_{i+1,j}^{current} + u_{i,j-1}^{new} + u_{i,j+1}^{current} - h^2 f_{i,j}\right) + (1 - \omega)\, u_{i,j}^{current}$$

$$R_{i,j} = f_{i,j} - \frac{1}{h^2}\left(u_{i-1,j}^{current} + u_{i+1,j}^{current} + u_{i,j-1}^{current} + u_{i,j+1}^{current} - 4u_{i,j}^{current}\right) \quad (*residual*)$$

        END LOOP j

    END LOOP i

$u^{current} := u^{new}$ (*update*)

$k := k + 1$

IF $\quad \left(\frac{\|R\|}{\|f\|} < \epsilon\right)$ *THEN*   (*Norms are grid norms. see code*)

        $CONVERGED := true$

    END IF

END WHILE

Notice that when $\omega = 1$, SOR becomes the same as *Gauss − Seidel*. When $\omega > 1$ the method is called overrelaxed and when $\omega < 1$ it is called underrelaxed.

2.4.4.7 Result of computation

The above 3 algorithms where implemented and ran for the 3 values of h given. The following tables summarizes the results obtained. The source code is in the appendix. This is a diagram illustrating the final converged solution from one of the runs.
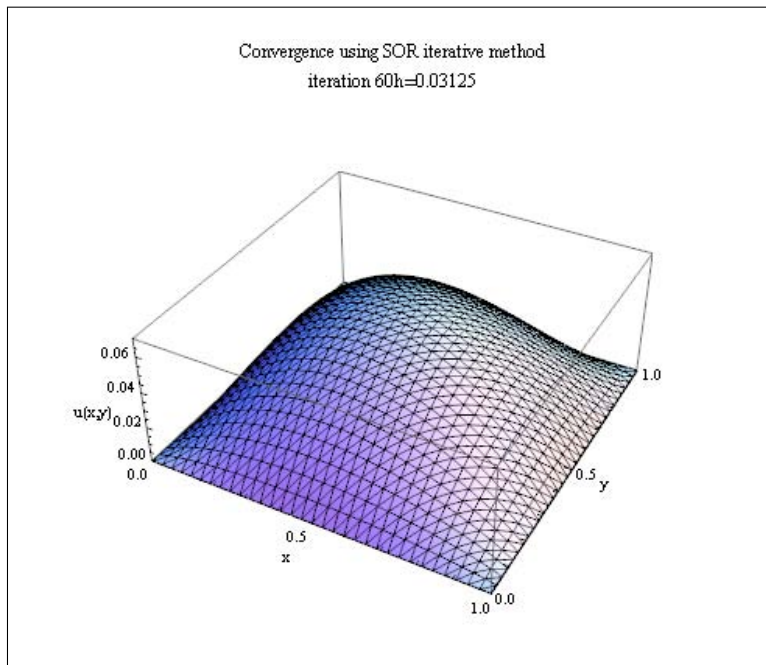
Figure 2.25: final convergedsolution

Number of steps to reach convergence

These table show the number of iterations to converge. The first was based on using $\varepsilon = 0.1h^2$ for tolerance and the second used $\varepsilon = h^2$

Number of iteration to reach convergence using tolerance $\varepsilon = 0.1h^2$

method	$h = 2^{-3}$ $n = 7$	$h = 2^{-4}$ $n = 15$	$h = 2^{-5}$ $n = 31$	$h = 2^{-6}$ $n = 63$	$h = 2^{-7}$ $n = 127$
Jacobi	82	400	1886	8689	note[4]
Gauss-Seidel	43	202	944	3446	19674
SOR	13	26	53	106	215

Number of iteration to reach convergence using tolerance $\varepsilon = h^2$

method	$h = 2^{-3}$ $n = 7$	$h = 2^{-4}$ $n = 15$	$h = 2^{-5}$ $n = 31$	$h = 2^{-6}$ $n = 63$	$h = 2^{-7}$ $n = 127$
Jacobi	52	281	1409	6779	31702
Gauss − Seidel	28	142	706	3391	15852
SOR	10	20	40	80	159

Convergence plots for each method

These error plots where generated only for tolerance $\varepsilon = 1 \times h^2$. They show how the log of the norm of the relative residual changes as the number of iterations changed until convergence

is achieved.

In these plots, the *yaxis* is $\log\left(\frac{\|R^{[k]}\|}{\|f\|}\right)$ or $\log\left(\frac{\|f-Au^{[k]}\|}{\|f\|}\right)$, and the *xaxis* is the k value (the iteration number).
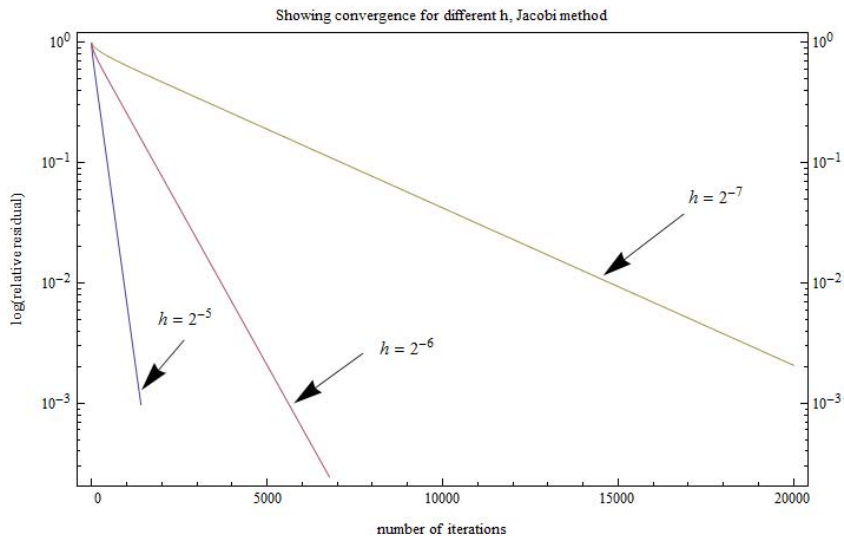
Figure 2.26: error plot Jacobi
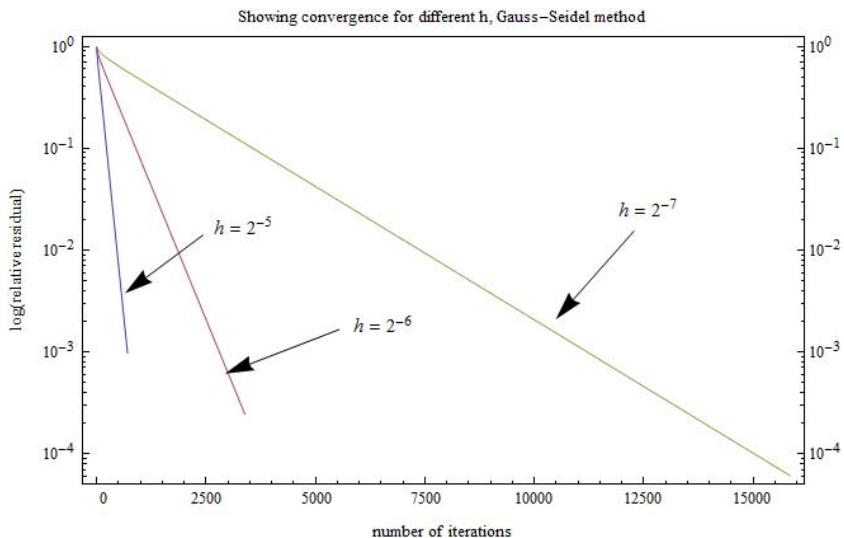
Figure 2.27: error plot GS
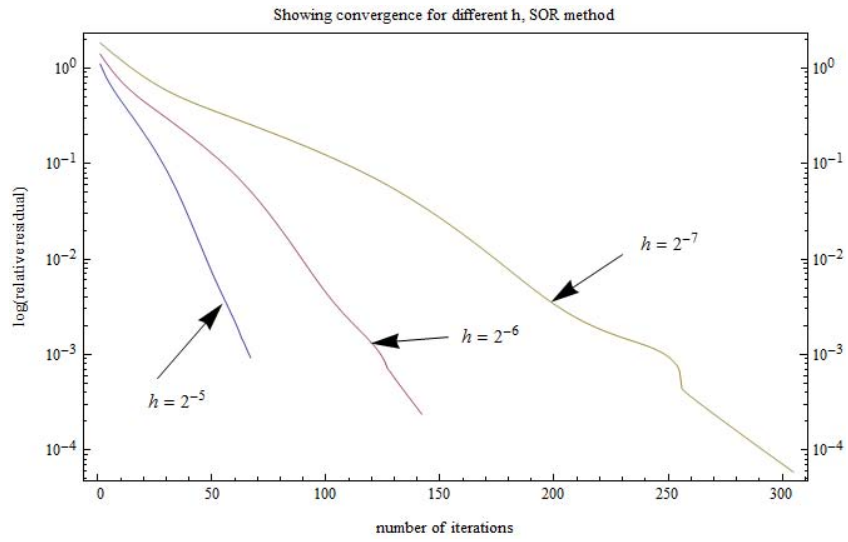
Figure 2.28: error plot SOR

Plots for comparing convergence of the 3 methods for $\epsilon = 1 \times h^2$ for different h values
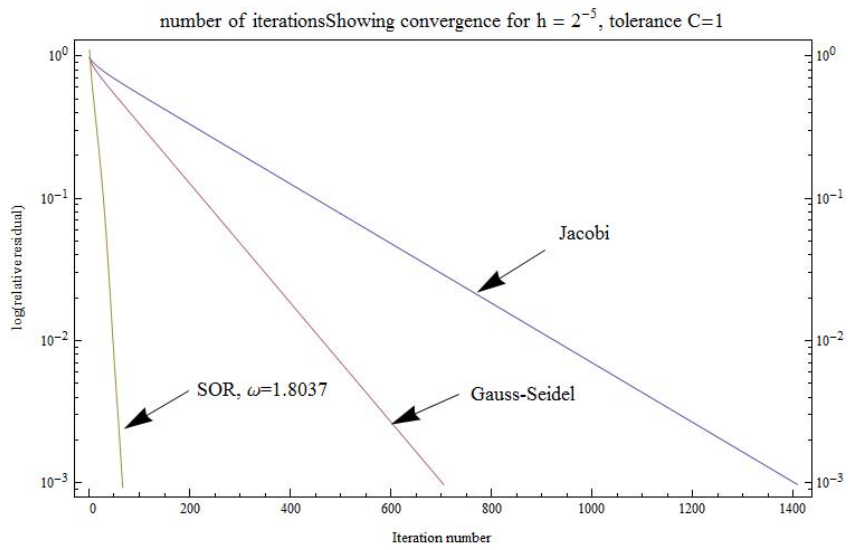
Figure 2.29: prob1 compare 3h 25
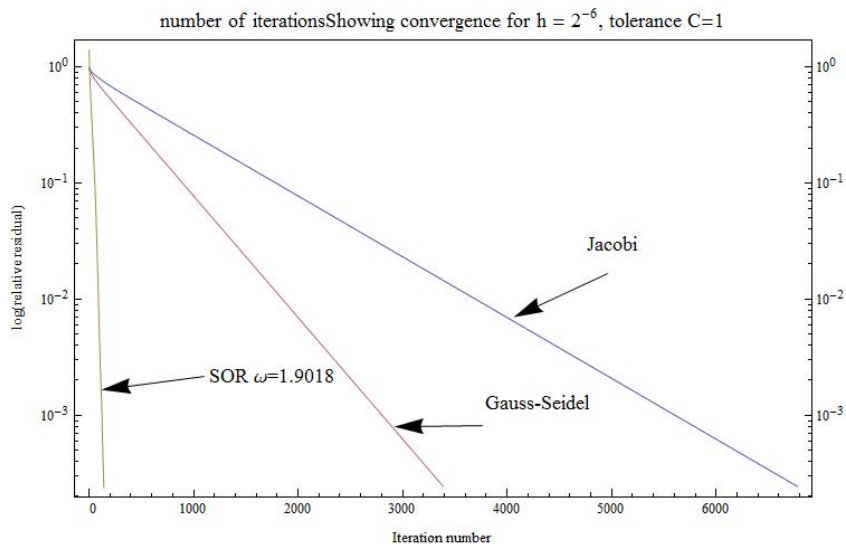
Figure 2.30: prob1 compare 3h 26
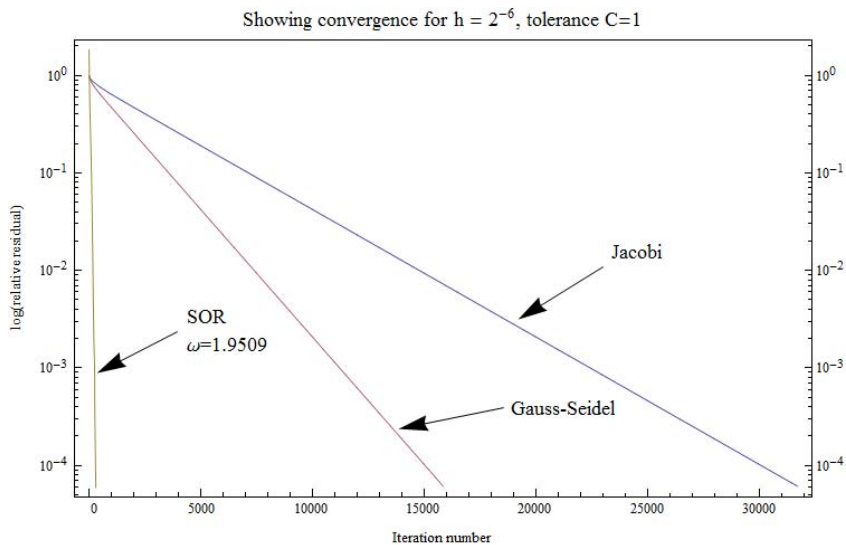
Figure 2.31: prob1 compare 3h 27

2.4.4.8 Conclusions and summary

1. SOR is the fastest iterative solver of the three solvers.

2. SOR method required calculation of an optimal ω to use. For this problem, this calculation was not difficult. In other problems it can be difficult to determine before hand the optimal ω. Some SOR methods use an adaptive ω where ω is readjusted as the solution progresses.

3. The use of relative residual to determine the condition of convergence required applying the matrix A without actually storing A.

4. Jacobi method required an additional auxiliary grid storage, hence its memory requirement was twice as much as Gauss-Seidel or SOR.

5. Jacobi method was the simplest to implement, but it was the slowest to converge.

6. Jacobi method is more suitable for use in parallel processing, where each grid point can be updated independent of the grid point next to it. This is not possible with Gauss-Seidel nor SOR due to the dependency of updates on its immediate grid points. However, if a red-black numbering is used, then it would be possible to implement these methods in parallel in 2 stages.

7. All methods are guaranteed to converge eventually, as the spectral radius of the iterative matrix for each method is less than one.

2.4.5 Problem 2

2. When solving parabolic equations numerically, one frequently needs to solve an equation of the form

$$u - \delta \Delta u = f,$$

where $\delta > 0$. The analysis and numerical methods we have discussed for the Poisson equation can be applied to the above equation. Suppose we are solving the above equation on the unit square with Dirichlet boundary conditions. Use the standard five point stencil for the discrete Laplacian.

(a) Analytically compute the eigenvalues of the Jacobi iteration matrix, and show that the Jacobi iteration converges.

(b) If $h = 10^{-2}$ and $\delta = 10^{-3}$, how many iterations of SOR would it take to reduce the error by a factor of 10^{-6}? How many iterations would it take for the Poisson equation? Use that the spectral radius of SOR is

$$\rho_{\text{sor}} = \omega_{\text{opt}} - 1,$$

where

$$\omega_{\text{opt}} = \frac{2}{1 + \sqrt{1 - \rho_J^2}},$$

and where ρ_J is the spectral radius of Jacobi.

Figure 2.32: Problem 2

2.4.5.1 Part(a)

Given

$$u - \delta \Delta u = f$$

And using standard 5 point Laplacian for the approximation of Δ, the above can be written as

$$u - \delta A u = f \tag{1}$$

Where A is the Jacobian matrix for 2D

$$\frac{1}{h^2} \begin{pmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \ddots & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{pmatrix}$$

Hence (1) becomes

$$(I - \delta A)\, u = f \tag{2}$$

Let

$$B = (I - \delta A) \tag{3}$$

Then (2) becomes

$$Bu = f \tag{3A}$$

To obtain the iterative matrix for the above system, the method of matrix splitting is used. Let $B = M - N$. Equation (3A) becomes

$$(M - N)\, u = f$$
$$Mu = Nu + f$$

M is selected so that it is invertible and such that M^{-1} is easy to compute, the iterative equation results

$$u^{[k+1]} = \left(M^{-1}N\right) u^{[k]} + M^{-1}f$$

Where iterative matrix T_j is

$$T_j = \left(M^{-1}N\right) \tag{3B}$$

For convergence it is required that the spectral radius of T_j be less than one. $\rho\left(T_j\right)$ is the largest eigenvalue of T_j in absolute terms. The largest eigenvalue of T_j is now found as follows.

For the Jacobi method let $M = D$, and $N = L + U$, where D is the diagonal of B, L is the negative of the strictly lower triangle matrix of B and U is negative of the strictly upper triangle matrix of B. (3B) becomes

$$T_j = D^{-1}\left(L + U\right)$$

But $B = D - L - U$, hence $L + U = D - B$ and the above becomes

$$T_j = D^{-1}(D - B)$$
$$= I - D^{-1}B \tag{4}$$

Now the spectral radius of T_j is determined. First D^{-1} is found. But first B needs to be determined. From (3)

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \frac{\delta}{h^2} \begin{pmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \ddots & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 1+\frac{4\delta}{h^2} & -\frac{\delta}{h^2} & 0 & -\frac{\delta}{h^2} & 0 & 0 & 0 \\ -\frac{\delta}{h^2} & 1+\frac{4\delta}{h^2} & -\frac{\delta}{h^2} & 0 & -\frac{\delta}{h^2} & 0 & 0 \\ 0 & -\frac{\delta}{h^2} & 1+\frac{4\delta}{h^2} & 0 & 0 & -\frac{\delta}{h^2} & 0 \\ -\frac{\delta}{h^2} & 0 & 0 & \ddots & 0 & 0 & -\frac{\delta}{h^2} \\ 0 & -\frac{\delta}{h^2} & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & -\frac{\delta}{h^2} & 0 & -\frac{\delta}{h^2} & 1+\frac{4\delta}{h^2} & -\frac{\delta}{h^2} \\ 0 & 0 & 0 & -\frac{\delta}{h^2} & 0 & -\frac{\delta}{h^2} & 1+\frac{4\delta}{h^2} \end{pmatrix}$$

Therefore, D the diagonal matrix of B is easily found

$$D = \begin{pmatrix} 1+\frac{4\delta}{h^2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1+\frac{4\delta}{h^2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1+\frac{4\delta}{h^2} & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\frac{4\delta}{h^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1+\frac{4\delta}{h^2} \end{pmatrix}$$

Now that D is known, the eigenvalue μ_{kl} of the iteration matrix T_j shown in (4) can be written down as

$$T_j = I - D^{-1}B$$
$$\Rightarrow$$
$$\mu_{kl} = 1 - \left(1 + \frac{4\delta}{h^2}\right)^{-1}\lambda_{kl} \tag{5}$$

where λ_{kl} is the eigenvalues of B. But from (3), $B = (I - \delta A)$, hence (5) becomes

$$\mu_{kl} = 1 - \left(1 + \frac{4\delta}{h^2}\right)^{-1}(1 - \delta v_{kl}) \tag{6}$$

Where v_{kl} is the eigenvalue of A, the standard Jacobi A matrix for 2D, with eigenvalues given in textbook (page 63, equation 3.15)

$$v_{kl} = \frac{2}{h^2}\left(\cos\left(k\pi h\right) + \cos\left(l\pi h\right) - 2\right)$$

Using this in (6) results in

$$\mu_{kl} = 1 - \left(1 + \frac{4\delta}{h^2}\right)^{-1}\left(1 - \frac{2\delta}{h^2}\cos\left(k\pi h\right) - \frac{2\delta}{h^2}\cos\left(l\pi h\right) + \frac{4\delta}{h^2}\right)$$

$$= 1 - \left(\frac{h^2}{h^2 + 4\delta}\right)\left(1 - \frac{2\delta}{h^2}\cos\left(k\pi h\right) - \frac{2\delta}{h^2}\cos\left(l\pi h\right) + \frac{4\delta}{h^2}\right)$$

$$= 1 - \left(\frac{h^2}{h^2 + 4\delta}\right) - \left(\frac{4\delta}{h^2 + 4\delta}\right) + \frac{2\delta}{h^2}\left(\frac{h^2}{h^2 + 4\delta}\right)\left\{\cos\left(k\pi h\right) + \cos\left(l\pi h\right)\right\}$$

$$= \left(\overbrace{\frac{h^2 + 4\delta - h^2 - 4\delta}{h^2 + 4\delta}}^{=0}\right) + \left(\frac{2\delta}{h^2 + 4\delta}\right)\left\{\cos\left(k\pi h\right) + \cos\left(l\pi h\right)\right\}$$

$$= \left(\frac{2\delta}{h^2 + 4\delta}\right)\left\{\cos\left(k\pi h\right) + \cos\left(l\pi h\right)\right\}$$

The largest value of the above occurs when $\cos\left(k\pi h\right) + \cos\left(l\pi h\right)$ is maximum, which is 2. Therefore

$$\rho\left(T_j\right) = \left(\frac{4\delta}{h^2 + 4\delta}\right)$$

Which is less than one for any $\delta > 0$.

Hence it is now shown that Jacobi iteration converges for this system.

2.4.5.2 Part (b)

Reducing the error by factor 10^{-6} implies

$$\left\|e^k\right\| < 10^{-6}\left\|e^0\right\| \tag{1}$$

but by definition

$$\left\|e^k\right\| = \rho_{sor}^k\left\|e^0\right\|$$

Hence (1) becomes

$$\rho_{sor}^k < 10^{-6}$$

Where the solution for k at equality is found (rounded to the largest integer if needed). Hence the above becomes, after taking logarithms of both sides

$$k \log \rho_{sor} = -6$$

$$k = \frac{-6}{\log \rho_{sor}} \tag{2}$$

But

$$\rho_{sor} = \omega_{opt} - 1$$

Where

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho_j^2}}$$

And $\rho_j = \left(\frac{4\delta}{h^2 + 4\delta}\right)$ from part(a), hence the above becomes

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \left(\frac{4\delta}{h^2 + 4\delta}\right)^2}}$$

Hence

$$\rho_{sor} = \frac{2}{1 + \sqrt{1 - \left(\frac{4\delta}{h^2 + 4\delta}\right)^2}} - 1$$

Substituting the numerical values $h = 10^{-2}, \delta = 10^{-3}$ in the above results in

$$\rho_{sor} = \frac{2}{1 + \sqrt{1 - \left(\frac{4\left(10^{-3}\right)}{\left(10^{-2}\right)^2 + 4\left(10^{-3}\right)}\right)^2}} - 1$$

$$= 0.64$$

Therefore, from (2)

$$k = \frac{-6}{\log(0.64)}$$

$$= 30.95$$

rounding up gives

$$k = 31$$

2.4.6　Problem 3

3. In this problem we compare the speed of SOR to a direct solve using Gaussian elimination. At the end of this assignment is MATLAB code to form the matrix for the 2D discrete Laplacian. The code for the 3D matrix is similar. Note that with 1 GB of memory, you can handle grids up to about 1000×1000 in 2D and $40 \times 40 \times 40$ in 3D with a direct solve. The range of grids you will explore depends on the amount of memory you have.

   (a) Solve the PDE from problem 1 using a direct solve. Put timing commands in your code and report the time to solve for a range of mesh spacings. Use SOR to solve on the same meshes and report the time and number of iterations. Comment on your results.

   (b) Repeat the previous part in three spatial dimensions for a range of mesh spacings. Change the right side of the equation to be a three dimensional Gaussian. Comment on your results.

Figure 2.33: Problem 3

2.4.6.1　Part(a)

To solve the problem using direct solver, the matrix A is constructed (sparse matrix), and the vector f is evaluated using the given function $f(x,y)$, this results in an $Au = f$ system, which is then solved using a direct solver.

Recall from problem (1) that the A matrix has the following form (as an example, for 3×3 internal grid, or for $h = 0.2$)

$$
\begin{pmatrix}
-4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4
\end{pmatrix}
\begin{pmatrix}
U_{1,1} \\ U_{2,1} \\ U_{3,1} \\ U_{1,2} \\ U_{2,2} \\ U_{3,2} \\ U_{1,3} \\ U_{2,3} \\ U_{3,3}
\end{pmatrix}
= h^2
\begin{pmatrix}
f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{1,2} \\ f_{2,2} \\ f_{3,2} \\ f_{1,3} \\ f_{2,3} \\ f_{3,3}
\end{pmatrix}
$$

The matrix A is set up, as sparse for the following set of values

h	internal grid size ($n = \frac{1}{h} - 1$)
2^{-5}	31×31
2^{-6}	63×63
2^{-7}	127×127
2^{-8}	255×255
2^{-9}	511×511
2^{-10}	1023×1023

A function is written to evaluate $f(x, y)$ at each of the internal grid points and reshaped to be column vector in the order shown above. Then the direct solver is called.

Next, the SOR solver is used for each of the above spacings. First ω was calculated for each h using $\omega_{opt} \approx 2(1 - \pi h)$ resulting in

h	ω_{opt}
2^{-5}	1.803 7
2^{-6}	1.901 8
2^{-7}	1.950 9
2^{-8}	1.975 5
2^{-9}	1.987 7
2^{-10}	1.993 9

Then the SOR solver which was written in problem (1) was called for each of the above cases. The next section shows the results obtained. The source code is in the appendix.

2.4.6.2 Result of computation

The following is an image of f(x,y) on the grid
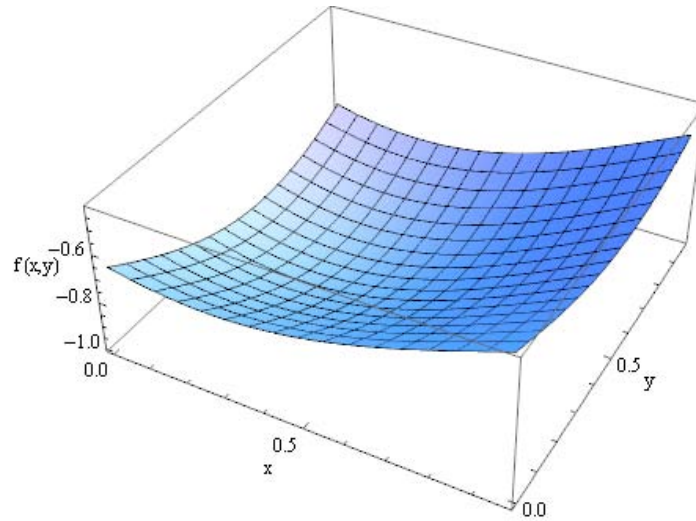
Figure 2.34: image of $f(x, y)$

And the solution obtained by direct solver on 2D is the following (implemented in Matlab and in Mathematica)
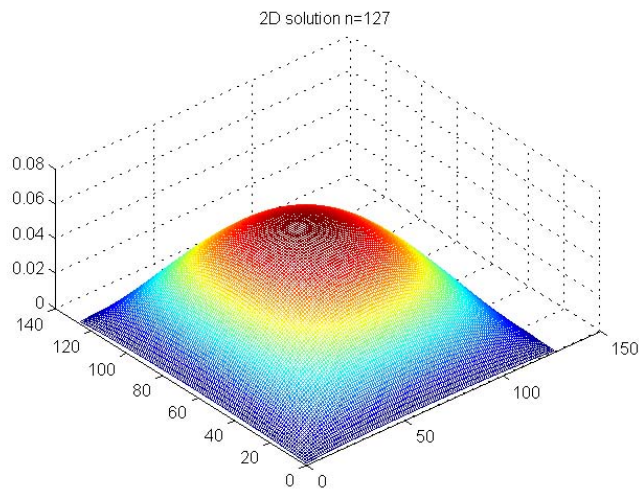
Figure 2.35: Solution by direct solver, Matlab

direct solver, n=63
cpu=0. sec
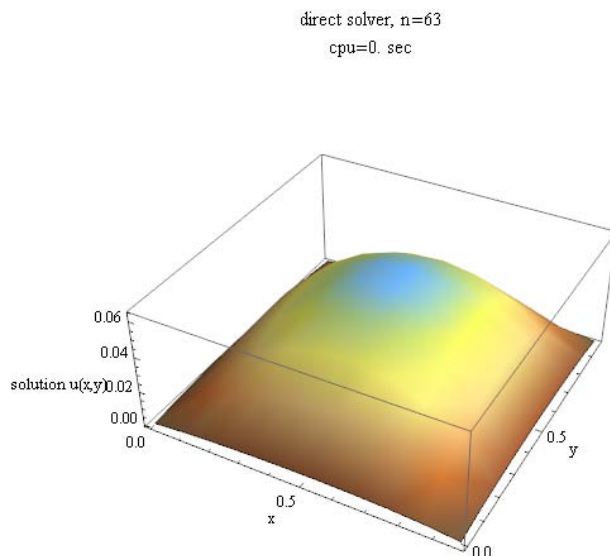
Figure 2.36: Solution by direct solver using Mathematica

The CPU results below are in seconds. The function cputime() was used to obtain cpu time used in Matlab. For SOR, the cpu time was that of the whole iterative loop until convergence was achieved. In Mathematica, the command Timing[] which measures the CPU time was used. These are the results obtained using Matlab 2010a and using Mathematica 7.0[5]

In this table, the grid size n represents the number of internal grid points in one dimension. For example, for $n = 31$, the grid size will be 31×31. The number of non zero elements shown in the table relates to storage used by the sparse matrix and was obtained in Matab by calling nnz(A).

h	n	N number of unknowns	number non zero elements	Direct Solver CPU MATLAB	Direct Solver CPU Mathematica	SOR Solver CPU	k SOR number of iterations
2^{-5}	31	961	4,681	0.015	0.016	0	68
2^{-6}	63	3,969	19,593	0.125	0.016	0.094	143
2^{-7}	127	16,129	80,137	0.250	0.063	0.6	306
2^{-8}	255	65,025	324,105	1.544	0.344	5.2	661
2^{-9}	511	261,121	1,303,561	5.538	1.90	48.9	1427
2^{-10}	1023	1,046,529	5,228,553	27.113	14.57	532	3088

These 2 plots illustrate the CPU difference, done on a normal scale and on log scale. (using

---

[5]Matlab 2010a, on windows 7, 64 bit OS, intel i7 930, with 8 GB installed physical RAM.

Matlab results only).
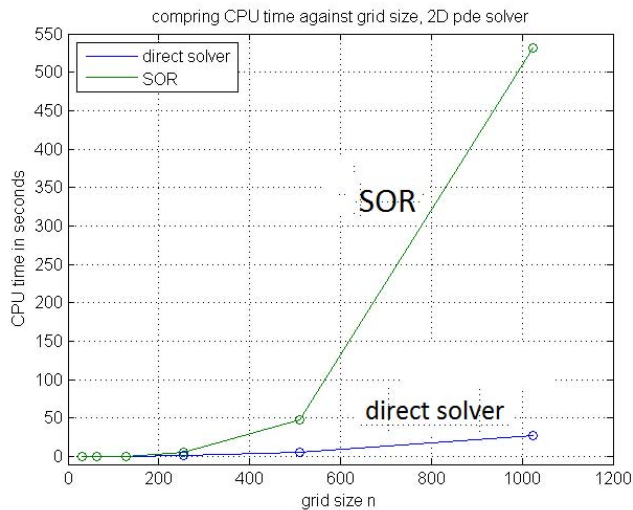
Figure 2.37: prob3 part a compare CPU normal scale
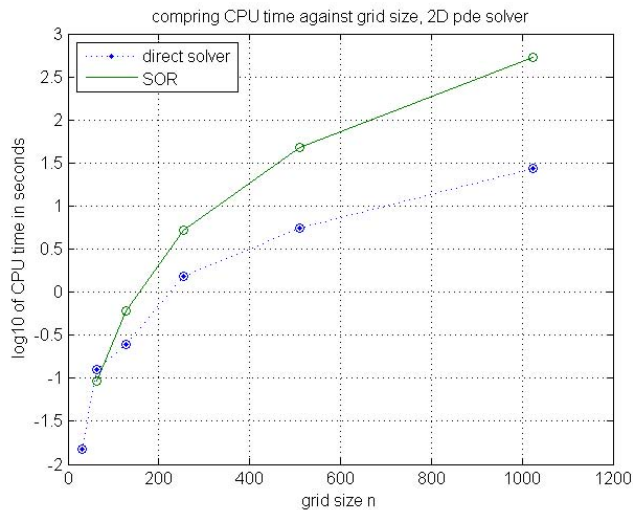
Figure 2.38: plot prob3 part a compare CPU

Comments on results obtained

CPU performance for SOR is given by

$$work = number\ iterations \times work\ per\ iteration$$

The number of iterations depends on the constant used for tolerance. Let k be the number

of iterations, and let the tolerance be Ch^2 where h is the spacings. Hence

$$k = \frac{\log\left(Ch^2\right)}{\log\rho_{sor}} = \frac{\log C + 2\log h}{\log\left(1 - 2\pi h\right)} \approx \frac{\log C + 2\log h}{-2\pi h} \approx O\left(h^{-1}\log h\right)$$

But $h = O\left(\frac{1}{n}\right)$ where n is the number of grid points in one dimension. Therefore

$$k = O\left(n\log n\right)$$

And since there are n^2 unknowns, the work per iteration is $O\left(n^2\right)$, hence for SOR performance, work becomes

$$CPU_{sor} = O\left(n^3\log n\right)$$

Expressing this in terms of the unknowns $N = n^2$ gives

$$CPU_{sor} = O\left(N^{\frac{3}{2}}\log N\right)$$

For direct solver, the work is proprprtional to (Nb) where b is the banwidth (when using nested dissection method)[6]. The bandwidth is n, hence for direct solver on 2D using sparse matrices, the performance is n^3

$$CPU_{direct} = O\left(N^{\frac{3}{2}}\right)$$

In summary

method	CPU (in terms of number of unknowns N)	CPU in terms of n
SOR 2D	$O\left(N^{\frac{3}{2}}\log N\right)$	$O\left(n^3\log n\right)$
direct solver 2D	$O\left(N^{\frac{3}{2}}\right)$	$O\left(n^3\right)$

For small number of unknowns, SOR was very competitive with direct solver but when the number of unknowns became larger than about $N \approx 100$, the direct solver is faster as the effect of the $\log n$ factor starts to take effect on the performance of *SOR*. The results shown in the plots above confirmed this performance analysis.

2.4.6.3 Part(b)

The goal is to solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\exp(-(x - 0.25)^2 - (y - 0.6)^2 - z^2)$$

On the unit cube. Referring to the following diagram made to help in setting up the 3D scheme to approximate the above PDE

---

[6]See textbook, page 68.
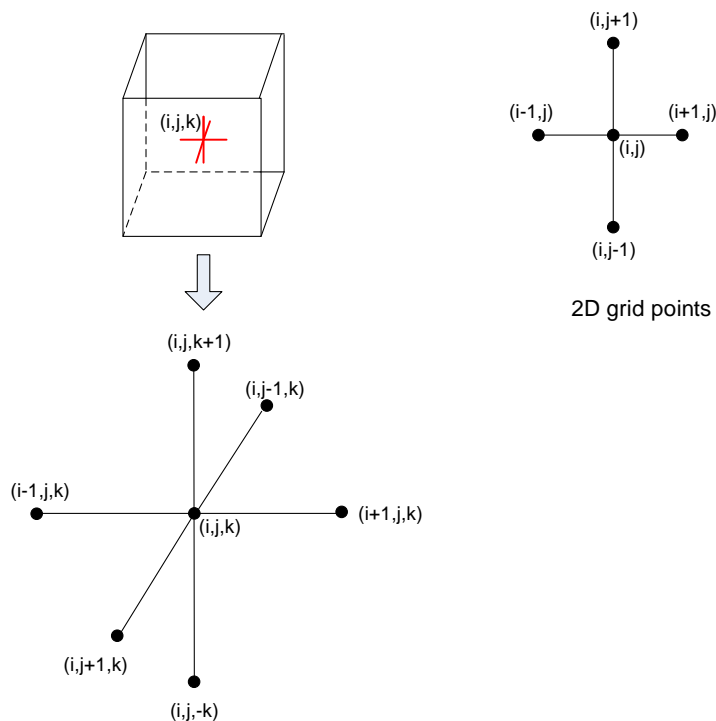
Figure 2.39: 3D axis

The discrete approximation to the PDE can be written as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{h^2}\left(U_{i-1,j,k} + U_{i+1,j,k} + U_{i,j-1},k + U_{i,j+1,k} + U_{i,k,k-1} + U_{i,j,k+1} - 6U_{i,j,k}\right)$$

Hence the SOR scheme becomes

$$U_{i,j,k}^{[k+1]} = \frac{\omega}{6}\left(U_{i-1,j,k}^{[k+1]} + U_{i+1,j,k}^{[k]} + U_{i,j-1,k}^{[k+1]} + U_{i,j+1,k}^{[k]} + U_{i,j,k-1}^{[k+1]} + U_{i,j,k+1}^{[k]} - h^2 f_{i,j}\right) + (1-\omega)\,U_{i,j,k}^{[k]}$$

For the direct solver, the A matrix needs to be formulated. From

$$\frac{1}{h^2}\left(U_{i-1,j,k} + U_{i+1,j,k} + U_{i,j-1},k + U_{i,j+1,k} + U_{i,k,k-1} + U_{i,j,k+1} - 6U_{i,j,k}\right) = f_{i,j,k}$$

And solving for $U_{i,j,k}$ results in

$$U_{i,j,k} = \frac{1}{6}\left(U_{i-1,j,k} + U_{i+1,j,k} + U_{i,j-1},k + U_{i,j+1,k} + U_{i,k,k-1} + U_{i,j,k+1} - h^2 f_{i,j,k}\right)$$

To help make the A matrix, using an example with $n = 2$, the following diagram is made with the standard numbering on each node
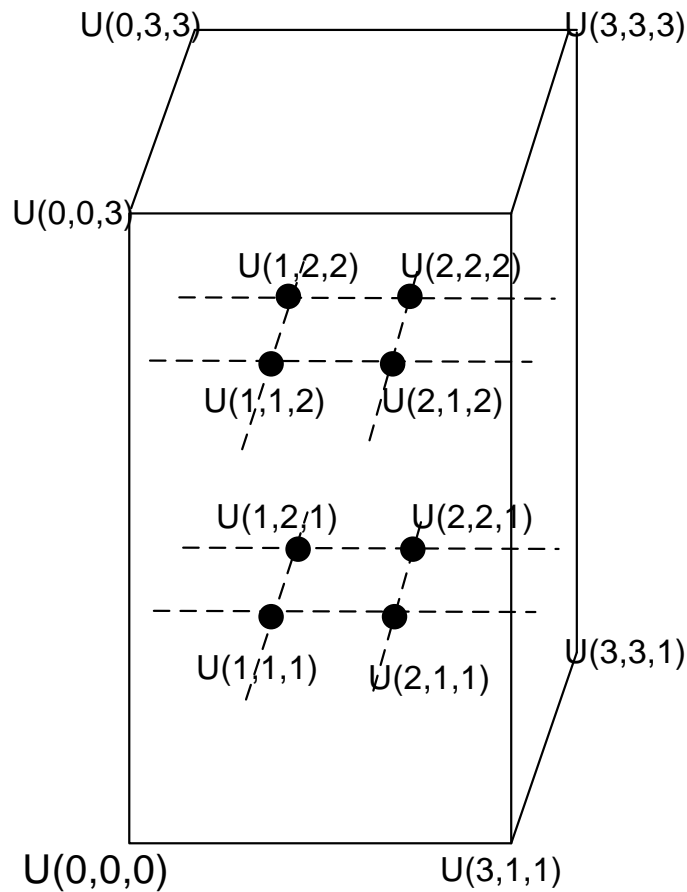
Figure 2.40: 3d grid example

By traversing the grid, left to right, then inwards into the paper, then upwards, the following A matrix results

Figure 2.41: repating A sructure n2

One can see the recursive pattern involved in these A matrices. Each A matrix contains

inside it a block on its diagonal which repeats n times. Each block in turn, contain inside it, on its diagonal, smaller block, which also repeats n times.

It is easier to see the pattern of building A by using numbers for the grid points, and label them in the same order as they would be visited, this allowed one to see the connection between each grid point to the other much easier. For example, for $n = 2$,
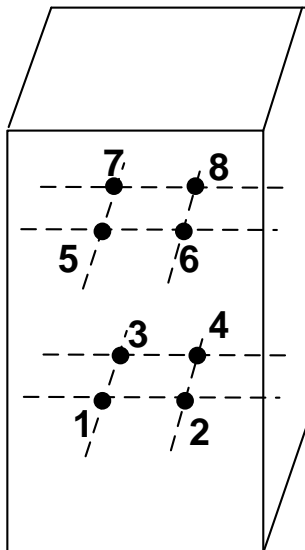
Figure 2.42: 3d grid n2 numbers

One can see now more easily the connections. grid point 1 has connection to only $2, 3, 5$ points. This means when looking at the A matrix, there will be a 1 in the first row, at columns $2, 3, 5$. Similarly, point 2 has connections only to $1, 4, 6$, which means in the second row, there will be a 1 at columns $1, 4, 6$. Extending the number of points to $n = 3$ to better see the pattern of A results in
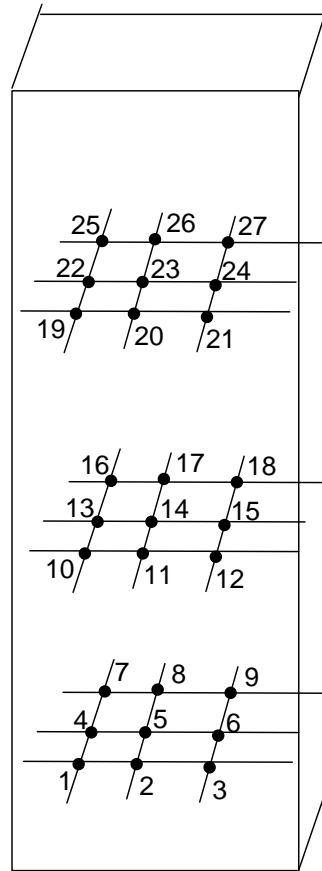
Figure 2.43: 3d grid n3 numbers

From the above, one can see clearly that, for example, point 1 is connected only to $2,4,10$ and 2 is connected to $1,3,5,11$ and so on. The above shows that each point will have a connection to a point which is numbered n^2 higher than the grid point itself. n^2 is the size of the grid in each surface. Hence, the general A matrix, for the above example, can now be written as
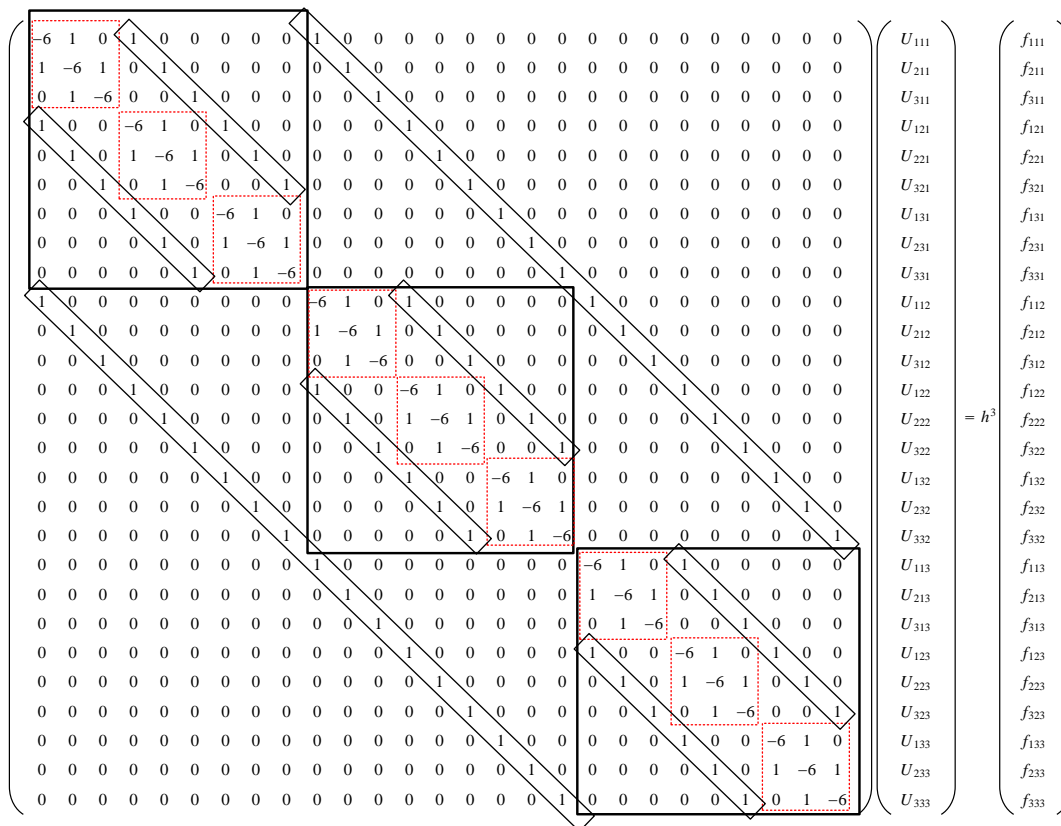
Figure 2.44: A sructure 3D

One can see the recursive structure again. There are $n = 3$ main repeating blocks on the diagonal, and each one of them in turn has $n = 3$ repeating blocks on its own diagonal. Here $n = 3$, the number of grid points along one dimension.

Now that the A structure is understood, the Matlab code for filling the sparse matrix is modified for the 3D case as follows

```matlab
function L3 = lap3d(n)

L2=lap2d(n,n);
e=ones(n^3,1);
L=spdiags([e e],[-n^2 n^2],n^3,n^3);
Iz=speye(n);

L3=kron(Iz,L2)+L;
end

%-------
function L2 = lap2d(nx,ny)

```

```
14 Lx=lap1d(nx);
15 Ly=lap1d(ny);
16
17 Ix=speye(nx);
18 Iy=speye(ny);
19
20 L2=kron(Iy,Lx)+kron(Ly,Ix);
21 end
22
23 function L=lap1d(n)
24 e=ones(n,1);
25 L=spdiags([e -3*e e],[-1 0 1],n,n);
26 end
```

.

To test, for example, for $n = 2$

```
1 EDU>> full(lap3d(2))
2 ans =
3      -6      1      1      0      1      0      0      0
4       1     -6      0      1      0      1      0      0
5       1      0     -6      1      0      0      1      0
6       0      1      1     -6      0      0      0      1
7       1      0      0      0     -6      1      1      0
8       0      1      0      0      1     -6      0      1
9       0      0      1      0      1      0     -6      1
10      0      0      0      1      0      1      1     -6
```

.

Using the above function, the solution was found using direct solver.

Result of computation

The results for the 3D solver are as follows. In this table, n represents the number of grid points in one dimension. Hence $n = 10$ represents a 3D space of $[10,10,10]$ points. The number of non zero elements in the table relates to the sparse matrix used for the direct solver and was obtained using Matab call nnz(A).

h	n	N total number unknowns (n^3)	make sparse CPU time	A\f CPU time	Total Direct Solver CPU	Total SOR Solver CPU	SOR number iterations
0.090909	10	1,000	0.047	0	0.047	0	23
0.047619	20	8,000	0.062	0.44	0.502	0.078	44
0.032258	30	27,000	0.016	3.90	3.90	0.405	65
0.027778	35	42,875	0.359	8.70	8.80	0.75	77
0.024390	40	64,000	0.328	21.20	21.50	1.29	88
0.021739	45	91,125	0.296	39.80	40.00	2.11	100
0.019608	50	125,000	0.624	84.20	84.80	3.24	112
0.017857	55	166,375	0.421	157.30	157.70	4.9	125
0.016393	60	216,000	0.889	244.10	244.20	7.17	138

For the direct solver, Matlab ran out of memory at $n = 65$ as shown below

```
EDU>> nma_HW3_problem_3_partB_direct_solver
.....

*****************************
grid is 3D [60,60,60]
h=0.016393
cpu time for making sparse matrix=1.060807 seconds
dimensions of A (sparse matrix) is [216000,216000]
nnz(A)= 1490400
cpu time for direct solver=240.693943 seconds
*****************************
grid is 3D [65,65,65]
h=0.015152
cpu time for making sparse matrix=0.826805 seconds
dimensions of A (sparse matrix) is [274625,274625]
nnz(A)= 1897025
??? Error using ==> mldivide
Out of memory. Type HELP MEMORY for your options.

Error in ==> nma_HW3_problem_3_partB_direct_solver at 54
    u = A\f;
```

.

This plot illustrates the CPU difference table on a log scale
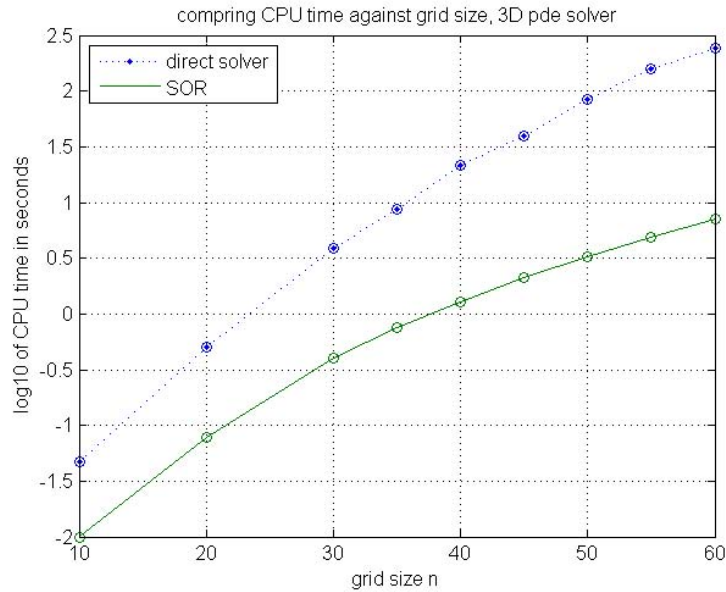
Figure 2.45: prob3 part B compare CPU log scale

Comments on results obtained

CPU performance for SOR is given by

$$\text{work} = \text{number iterations} \times \text{work per iteration}$$

The number of iterations depends on the constant used for tolerance. Let k be the number of iterations, and let the tolerance be Ch^2 where h is the spacings. Hence

$$k = \frac{\log\left(Ch^2\right)}{\log \rho_{sor}} = \frac{\log C + 2\log h}{\log\left(1 - 2\pi h\right)} \approx \frac{\log C + 2\log h}{-2\pi h} \approx O\left(h^{-1}\log h\right)$$

But $h = O\left(\frac{1}{n}\right)$ where n is the number of grid points in one dimension. Hence

$$k = O\left(n\log n\right)$$

And since there are n^3 unknowns (compared to n^2 in 2D), then work per iteration is $O\left(n^3\right)$, hence for SOR performance becomes

$$CPU_{sor} = O\left(n^4 \log n\right)$$

Expressing this in terms of $N = n^3$ as the number of unknowns, gives

$$CPU_{sor} = O\left(N^{\frac{4}{3}} \log N\right)$$

For direct solver, the work is proprrtional to (Nb) where b is the banwidth (when using nested dissection method)[7]. The bandwidth is n^2 in this case and not n as was with 2D.

---

[7]See textbook, page 68.

Hence the total cost is

$$CPU_{direct} = O(N \times n)$$
$$= O\left(n^5\right)$$
$$= O\left(N^{\frac{5}{3}}\right)$$

Hence

method	CPU (in terms of N)	CPU in terms of n
SOR 3D	$O\left(N^{\frac{4}{3}} \log N\right)$	$O\left(n^4 \log n\right)$
direct solver 3D	$O\left(N^{\frac{5}{3}}\right)$	$O\left(n^5\right)$

The above shows that SOR is faster than direct solver performance. The results shown in the plots above confirmed this analytical performance prediction showing SOR to be faster. To verify the above, a plot was made using the above complexity cost measure to determine if the resulting shape matches the one obtained from the actual runs above. The following plot shows the complexity cost made on sequence of values that represent n
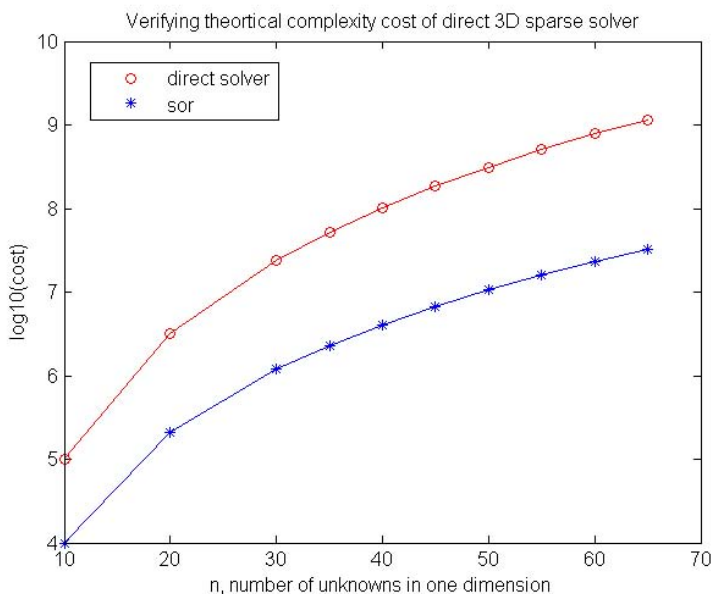
Figure 2.46: verify cost of 3D

The following matlab code was in part used to generate the above

```
n=[10 20 30 35 40 45 50 55 60 65];
plot(n,log10(n.^5),'ro',n,log10(n.^4.*log10(n)),'*');
```

It can be seen that the cost curves matches those produced with the actual runs, but for a scaling factor as can be expected.

Therefore one can conclude that in 3D SOR is faster than direct solver. This result was surprising as the expectation was that the direct solver will be faster than SOR in 3D as it was in 2D. Attempts were made to find any errors in the code that can explain this, and none were found.

2.4.7   Problem 4

3. In this problem we compare the speed of SOR to a direct solve using Gaussian elimination. At the end of this assignment is MATLAB code to form the matrix for the 2D discrete Laplacian. The code for the 3D matrix is similar. Note that with 1 GB of memory, you can handle grids up to about 1000×1000 in 2D and $40 \times 40 \times 40$ in 3D with a direct solve. The range of grids you will explore depends on the amount of memory you have.

   (a) Solve the PDE from problem 1 using a direct solve. Put timing commands in your code and report the time to solve for a range of mesh spacings. Use SOR to solve on the same meshes and report the time and number of iterations. Comment on your results.

   (b) Repeat the previous part in three spatial dimensions for a range of mesh spacings. Change the right side of the equation to be a three dimensional Gaussian. Comment on your results.

Figure 2.47: Problem 3

Periodic boundary conditions mean that the solution must be such that $u'(0) = u'(1)$ and $u(0) = u(1)$. As an example, the following is a solution to $u''(x) = f(x)$ with Periodic boundary conditions just for illustration
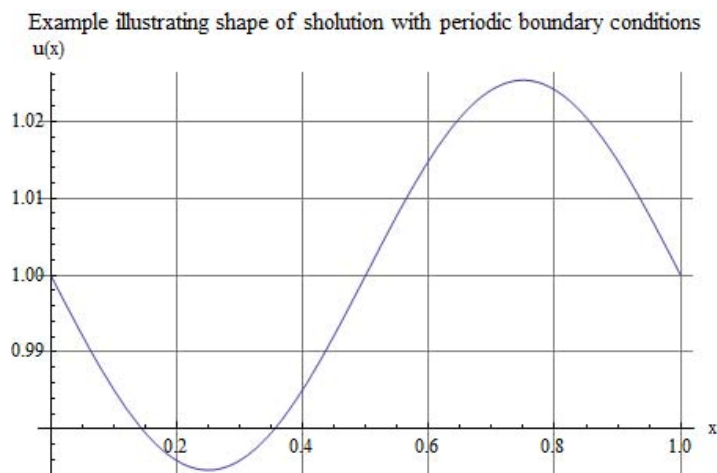
Figure 2.48: example periodic BC

2.4.7.1 part(a)

Using the standard numbering system
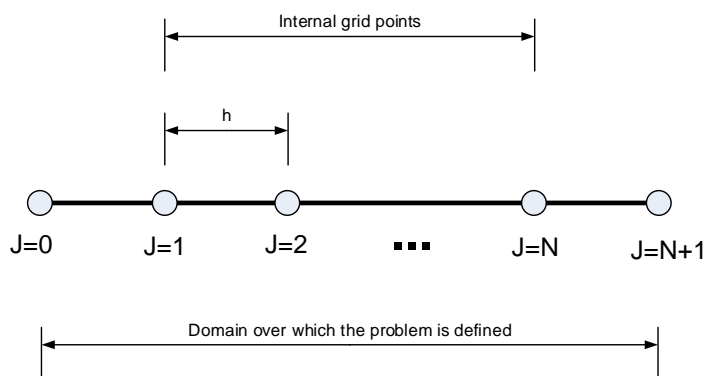
Figure 2.49: problem 4 part a scheme

In the above diagram, u_0 represents u at $x = 0$ and u_{N+1} represents u at $x = 1$. The 3 point discrete Laplacian for 1-D at x_0 is given by

$$u_0'' = \frac{u_{-1} - 2u_0 + u_1}{h^2} \tag{1}$$

where x_{-1} is an imaginary grid point to the left of x_0 in the diagram above.

Expanding u_{-1} about u_0 by Taylor results in $u_{-1} = u_0 - hu_0'$, hence

$$u_0' = \frac{u_0 - u_{-1}}{h} \tag{2}$$

Similarly, by Taylor expansion of u_N about u_{N+1} results in

$$u_N = u_{N+1} - hu_{N+1}'$$

Hence

$$u_{N+1}' = \frac{u_{N+1} - u_N}{h} \tag{3}$$

But $u_0' = u_{N+1}'$ from boundary conditions, hence (2)=(3) which results in

$$\frac{u_0 - u_{-1}}{h} = \frac{u_{N+1} - u_N}{h}$$

Solving now for u_{-1} from the above gives

$$u_{-1} = u_0 + u_N - u_{N+1}$$

But $u_{N+1} = u_0$, also from the boundary conditions, hence the above results in

$$u_{-1} = u_N$$

Use the above value of u_{-1} in (1) gives

$$u_0'' = \frac{u_N - 2u_0 + u_1}{h^2}$$

Similarly the derivation for u''_{N+1} results in

$$u''_{N+1} = \frac{u_N - 2u_{N+1} + u_1}{h^2}$$

For every other internal grid point $i = 1 \cdots N$ the standard 3 point central difference is used

$$u''_i = \frac{1}{h^2} (u_{i-1} - 2u_i + u_{i+1})$$

Therefore, the following set of equations are obtained

$$
\begin{aligned}
\tfrac{1}{h^2} (u_N - 2u_0 + u_1) &= f_0 & i &= 0 \\
\tfrac{1}{h^2} (u_{i-1} - 2u_i + u_{i+1}) &= f_i & i &= 1 \cdots N \\
\tfrac{1}{h^2} (u_N - 2u_{N+1} + u_1) &= f_{N+1} & i &= N+1
\end{aligned}
$$

And the system can now be put in the form $Au = f$ resulting in

$$
\frac{1}{h^2}
\begin{pmatrix}
-2 & 1 & 0 & 0 & \cdots & 1 & 0 \\
1 & -2 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & \vdots \\
0 & 0 & 0 & \cdots & \ddots & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 & -2 & 1 \\
0 & 1 & 0 & \cdots & 0 & 1 & -2
\end{pmatrix}
\begin{pmatrix}
u_0 \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N \\ u_{N+1}
\end{pmatrix}
=
\begin{pmatrix}
f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \\ f_{N+1}
\end{pmatrix}
\tag{4}
$$

The above A matrix is singular since $Ab = 0$ for b the vector 1^T. Hence the null space of A contains a vector other than the 0 vector meaning that A is singular.

To determine the dimension of the null space, the rank of A is determined. Removing the last column and the last row of A results in an $n-1$ by $n-1$ matrix

$$
A_{n-1} =
\begin{pmatrix}
-2 & 1 & 0 & 0 & \cdots & 1 \\
1 & -2 & 1 & 0 & \cdots & 0 \\
0 & 1 & \ddots & 1 & 0 & \cdots \\
0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & \cdots & 1 & -2
\end{pmatrix}
$$

The square matrix inside of A_{n-1} that extends from the first row to the one row before the last row is of size $n-2$

$$
A_{n-2} =
\begin{pmatrix}
-2 & 1 & 0 & 0 & \cdots \\
1 & -2 & 1 & 0 & \cdots \\
0 & 1 & \ddots & 1 & 0 \\
0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 1 & -2
\end{pmatrix}
$$

And this matrix is a full rank as it is the A matrix used for 1-D with Dirichlet boundary conditions and this matrix is known to be invertible (same one used in HW2).

Therefore, the rank of A can not be less than $n-2$ where n is the size of A.

In other words, the size of the null space of A can at most be 2. To determine if the size of the null space of A can be just one, the matrix A_{n-1} shown above has to be invertible as well.

One way to show that A_{n-1} is invertible, is to show that the last column of A_{n-1} is linearly independent to any of the remaining columns of A_{n-1}.

The last column of A_{n-1} is c_{n-1} $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ -2 \end{pmatrix}$ and this column is linearly independent with the first

column of A_{n-1} which is $c_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$ since $a \times c_{n-1} + b \times c_1 = 0$ only when a, b are both zero. The

same can be shown with all the other columns of A_{n-1}, they are all linearly independent to the last columns of c_{n-1}. Since all the other $n-2$ columns of A_{n-1} are linearly independent with each others (they make up the Dirichlet matrix known to be invertible) then A_{n-1} is invertible. This shows that the rank of A is $n-1$, hence the null space of A has dimension 1 only. In other words, only the and only the vector 1^T in the null of A. (Since A is symmetric, the null space of the adjoint of A is the same).

2.4.7.2   part (b)

In terms of looking at the conditions of solvability, the continuous case is considered and then the conditions are translated to the discrete case.

The pde $u''(x) = f(x)$ with periodic boundary conditions has an eigenvalue which is zero (the boundary conditions $u'(1) = u'(0)$ results in this), hence

$$0 = \int_0^1 f(x)\,dx$$

Is the solvability condition which results from the $u'(1) = u'(0)$ boundary conditions. (same argument that was carried out in part (d), problem 1, HW1 which had Neumann boundary conditions is used). Now, what solvability conditions does $u(0) = u(1)$ add to this if any?

Since

$$u''(x) = f(x)$$

Then integrating once gives

$$u'(1) - u'(0) = \int_0^1 f(x)\,dx + C_1$$

But since $u'(1) = u'(0)$, then the above implies that

$$0 = C_1$$

And integrating twice the PDE results in

$$u(1) - u(0) = C_2$$

But $u(1) - u(0) = 0$, hence $C_2 = 0$. So the only solvability condition is based on the fact that an eigenvalue is zero, which implies

$$\int_0^1 f(x)\,dx = 0$$

This is the same as was the case with Neumann boundary conditions. In the discrete case, this implies that solvability condition becomes the discrete integration (Riemman sum)

$$h \sum_{j=0}^{j=N+1} f\left(jh\right) = 0$$

For 2D, by extension of the above, there will be 2 eigenvalues of zero values, hence the discrete solvability condition becomes

$$h^2 * \sum_{i=0}^{i=N+1} \left(\sum_{j=0}^{j=N+1} f\left(ih, jh\right) \right) = 0$$

2.4.7.3 Part(c)

Since this is an iff problem, then the following needs to be shown

1. If v is in the null space of A then v is an eigenvector of T with eigenvalue 1

2. If v is an eigenvector of T with eigenvalue 1 then v is in the null space of A

Solving part(1)

Since v is in null space of A, then by definition

$$Av = 0$$

But $A = M - N$, hence the above becomes

$$(M - N)\,v = 0$$
$$Mv - Nv = 0$$

Since M is invertible by definition, then M^{-1} exists. Premultiply both sides by M^{-1}

$$M^{-1}Mv - M^{-1}Nv = M^{-1}0$$

But $M^{-1}0 = 0$ then the above becomes

$$Iv - M^{-1}Nv = 0$$
$$M^{-1}Nv = v$$
$$Tv = v$$

Therefore v is an eigenvector of T with an eigenvalue of 1.

Solving part (2)

Since v is an eigenvector of T with eigenvalue 1 then

$$Tv = \lambda v$$

With $\lambda = 1$, and since $T = M^{-1}N$, then the above becomes

$$M^{-1}Nv = v$$

Multiply both sides by M

$$Nv = Mv$$

Therefore

$$Mv - Nv = 0$$
$$(M - N)v = 0$$

Hence

$$Av = 0$$

Therefore v is in the null space of A.

2.4.8 References

1. Applied Mathematica by David Logan, chapter 8

2.4.9 Source code

```
1  function nma_build_HW3()
2
3  list = dir('*.m');
4
5  if isempty(list)
6      fprintf('no matlab files found\n');
7      return
8  end
9
10 for i=1:length(list)
11     name=list(i).name;
12     fprintf('processing %s\n',name)
13     p0 = fdep(list(i).name,'-q');
14     [pathstr, name_of_matlab_function, ext] = fileparts(name);
15
```

```matlab
16      %make a zip file of the m file and any of its dependency
17      p1=dir([name_of_matlab_function '.fig']);
18      if length(p1)==1
19          files_to_zip =[p1(1).name;p0.fun];
20      else
21          files_to_zip =p0.fun;
22      end
23
24      zip([name_of_matlab_function '.zip'],files_to_zip)
25
26  end
27
28  end
```

```matlab
1  function nma_cpu_plot_2D()
2  %
3  % to plot O(n^4) vs. O(n^3 log(n))
4  % used to verify the cost complexity for problem 3, part (b), HW3
5  % Math 228A
6  % Nasser M. Abbasi
7  %
8
9  close all;
10 n=10:1:40;
11 plot(n,n.^2,'r',n,n.^3.*log(n));
12 legend('direct solver','sor')
13
14 figure
15 close all;
16 n=10:1:4000;
17 plot(n,n.^(5/2),'r',n,n.^(3/2).*log(n));
18 legend('direct solver','sor')
19
20 figure
21 close all;
22 n=[10 20 30 35 40 45 50 55 60 65];
23 plot(n,log10(n.^5),'ro',n,log10(n.^4.*log10(n)),'*');
24 legend('direct solver','sor')
25 hold on
26 n=[10 20 30 35 40 45 50 55 60 65];
27 plot(n,log10(n.^5),'r-',n,log10(n.^4.*log10(n)),'-');
28 title('Verifying theortical complexity cost of direct 3D sparse solver');
29 xlabel('n, number of unknowns in one dimension');
30 ylabel('log10(cost)');
31
32 end
```

```matlab
1   function  nma_HW3_prob3_parta_SOR()
2   % file name: nma_HW3_prob3_parta_SOR.m
3   %
4   % This solves the SOR for 2D for HW3, problem 3, parta
5   % Math 228A, Fall 2010, UC Davis
6   % Nasser M. Abbasi
7   %
8
9   DOPLOTS = true; %set to false if do not want to see plots
10
11  % setup the spacings needed for the problem
12  %spacings = [2^-5 2^-6 2^-7 2^-8 2^-9 2^-10];
13  spacings = [2^-5 2^-6 2^-7 ];
14  omega    = arrayfun(@(i) 2*(1-pi*spacings(i)),1:length(spacings));
15
16  for m = 1:length(spacings)
17
18      h = spacings(m);
19
20      % evaluate f(x,y) on grid
21      [X,Y] = meshgrid(0:h:1, 0:h:1);
22      f       = -exp(-(X-0.25).^2-(Y-0.6).^2);
23      nPoints = size(f,1);
24      fv      = reshape(f,nPoints^2,1); % use grid vector norm
25      normf = sqrt(h) * norm(fv,2);
26
27      w = omega(m);
28
29      fprintf('******************\n');
30      fprintf('gridsize=[%d,%d]\n',nPoints-2,nPoints-2);
31
32      % initialize space (grid) for residual calculation and for solution
33      resid = zeros(nPoints,nPoints);
34      u       = zeros(nPoints,nPoints);
35      unew  = u;
36
37      done      = false; %flag set to true in loop below when it converges
38      tolerance = h^2;   % set tolerance
39      k         = 1;     % initialize iteration counter
40
41      t         = cputime;
42
43      while not(done)
44          for i = 2 : nPoints-1
45              for j = 2 : nPoints-1
46
47                  resid(i,j)= f(i,j) - 1/h^2 * ( u(i-1,j) + u(i+1,j) + ...
```

```matlab
48                      u(i,j-1) + u(i,j+1) - 4*u(i,j) );
49
50                  unew(i,j) = w/4* ( unew(i-1,j) + u(i+1,j) + unew(i,j-1)+...
51                      u(i,j+1) - h^2*f(i,j)) + (1-w)*u(i,j);
52              end
53          end
54
55          u       = unew;
56          residv = reshape(resid,nPoints^2,1); % use grid vector norm
57          normResidue = sqrt(h) * norm(residv,2);
58
59          if ( normResidue / normf) <tolerance
60              done = true;
61          else
62              k = k+1;
63          end
64
65          if DOPLOTS
66              subplot(1,2,1);
67              mesh(X ,Y ,resid );
68
69              subplot(1,2,2);
70              mesh(X ,Y ,u );
71
72              drawnow;
73          end
74
75      end
76
77
78      fprintf('cpu time for SOR=%f seconds\n',cputime-t);
79      fprintf('number of iterations = %d\n',k);
80
81  end
82
83  end
```

```matlab
1   function nma_HW3_prob3_partb_3d_SOR()
2
3   % file name: nma_HW3_prob3_partb_3d_SOR.m
4   %
5   % This does the SOR solver for 3D for HW3, problem 3, partB
6   % Math 228A, Fall 2010, UC Davis
7   % SOR 3D solver
8   %
9   % Nasser M. Abbasi
10  %
```

```matlab
11
12  DOPLOTS = true; %set to false if do not want to see plots
13
14  % setup the spacings needed for the problem
15  %gridsize = [10 20 30 35 40 45 50 55 60 65];
16  gridsize = [10 20 ];
17  spacings = arrayfun(@(i) 1/(gridsize(i)+1),1:length(gridsize));
18  omega    = arrayfun(@(i) 2*(1-pi*spacings(i)) ,1:length(spacings));
19
20  fprintf('----- Starting 3D SOR solver ---------\n');
21
22  for m = 1:length(spacings)
23
24      h = spacings(m);
25      nInternalGridPoints = gridsize(m);
26      nPoints = nInternalGridPoints+2;
27
28      fprintf('*****************************\n');
29      fprintf('grid is 3D [%d,%d,%d]\n',nInternalGridPoints, ...
30          nInternalGridPoints,nInternalGridPoints);
31
32      fprintf('h=%f\n',h);
33
34      [X,Y,Z] = meshgrid(0:h:1, 0:h:1,0:h:1);
35
36      % initialize space (grid) for residual calculation and for solution
37      u       = 0.*X+0.*Y+0.*Z;
38      unew    = u;
39      resid   = u;
40
41      % evaluate f(x,y,z) on grid
42      f       = -exp(-(X-0.25).^2-(Y-0.6).^2 - Z.^2);
43      normf   = sqrt(h)* norm( reshape(f(2:end-1,2:end-1,2:end-1),...
44          nInternalGridPoints^3,1),2);
45
46      w        = omega(m); %optimal w for SOR
47      done     = false; %flag set to true in loop below when it converges
48      tolerance = 0.1*h^2;   % set tolerance
49      k        = 1;     % initialize iteration counter
50
51      t        = cputime;
52
53      while not(done)
54          for i = 2 : nPoints-1
55              for j = 2 : nPoints-1
56                  for z = 2 : nPoints-1
57                      resid(i,j,z)= f(i,j,z) - 1/h^2 * ( u(i-1,j,z) + ...
```

```matlab
58                              u(i+1,j,z) + u(i,j-1,z) + u(i,j+1,z) - ...
59                              6*u(i,j,z) + u(i,j,z-1) + u(i,j,z+1));
60
61                      unew(i,j,z) = w/6* ( unew(i-1,j,z) + u(i+1,j,z) + ...
62                          unew(i,j-1,z) + u(i,j+1,z) + unew(i,j,z-1) + ...
63                          u(i,j,z+1)- h^2*f(i,j,z)) + (1-w)*u(i,j,z);
64                  end
65              end
66          end
67
68          u = unew;
69          if DOPLOTS
70              subplot(1,2,1);
71              mesh(X(:,:,nPoints-1),Y(:,:,nPoints-1),resid(:,:,nPoints-1));
72
73              subplot(1,2,2);
74              mesh(X(:,:,nPoints-1),Y(:,:,nPoints-1),u(:,:,nPoints-1));
75
76              drawnow;
77          end
78
79          % can't do norm on 3D, change to vector
80          residv = reshape(resid(2:end-1,2:end-1,2:end-1),...
81              nInternalGridPoints^3,1);
82          normResidue = sqrt(h) * norm(residv,2);
83
84          if (normResidue/normf) < tolerance
85              done = true;
86          else
87              k = k+1;
88          end
89
90      end
91
92      fprintf('cpu time for 3D SOR solver =%f seconds\n',cputime-t);
93      fprintf('number of iterations = %d\n',k);
94  end
95
96  end
```

```matlab
1  function nma_HW3_problem_3_part_A_graph_plot()
2  % This   used to generate plot to compare CPU time of SOR
3  % and direct solver for 2D problem
4  % file name nma_HW3_problem_3_part_A_graph_plot.m
5
6  close all;
7  x=[31 63 127 255 511 1023];
```

```matlab
 8  directCPU=[0.015 .125 .250 1.544 5.538 27.113];
 9  sorCPU=[0 0.094 0.6 5.2 48 532];
10
11  %plot(x,log10(directCPU),':.',x,log10(sorCPU));
12  plot(x,directCPU,'-',x,sorCPU);
13
14  title('compring CPU time against grid size, 2D pde solver');
15  xlabel('grid size n');
16  %ylabel('log10 of CPU time in seconds');
17  ylabel('CPU time in seconds');
18  hold on;
19  legend('direct solver','SOR','Location','NorthWest');
20
21  grid on
22  %plot(x,log10(directCPU),'o',x,log10(sorCPU),'o');
23  plot(x,directCPU,'o',x,sorCPU,'o');
24  ylim([-10 550]);
25
26  end
```

```matlab
 1  function nma_HW3_problem_3_part_B_graph_plot()
 2  % This  used to generate plot to compare CPU time of SOR
 3  % and direct solver for 3D problem
 4  % file name nma_HW3_problem_3_part_B_graph_plot.m
 5  % Nasser M. Abbasi
 6  % 11/8/2010
 7
 8  x=[10 20 30 35 40 45 50 55 60];
 9  directCPU=[0.047 0.5 3.9 8.8 21.50 40 84 157.7 244.4];
10  sorCPU=[0.01 0.078 0.405 0.75 1.29 2.11  3.24  4.9  7.17];
11
12  plot(x,log10(directCPU),':.',x,log10(sorCPU));
13  %plot(x,directCPU,'-',x,sorCPU);
14
15  title('compring CPU time against grid size, 3D pde solver');
16  xlabel('grid size n');
17  ylabel('log10 of CPU time in seconds');
18  %ylabel('CPU time in seconds');
19  hold on;
20  legend('direct solver','SOR','Location','NorthWest');
21
22  grid on
23  plot(x,log10(directCPU),'o',x,log10(sorCPU),'o');
24  %plot(x,directCPU,'o',x,sorCPU,'o');
25
26  end
```

```matlab
 1  function nma_HW3_problem_3_partA()
 2  %
 3  %   name: nma_HW3_problem_3_partA.m
 4  %   purpose: direct solver for HW3, problem 3, part A. 2D
 5  %   UC Davis math 228A
 6  %
 7  %   description of algorithm:
 8  %      This script when called, will find the solution to the problem
 9  %      Au=f as described in the HW, by using sparse matrix and direct
10  %      solver.  The script will solve the problem for the following h
11  %      spacings: 2^-5 2^-6 2^-7 2^-8 2^-9 2^-10
12  %
13  %      It will find the cpu time used and print to the screen the result
14  %      for each grid space.
15  %
16  % external functions called:
17  %      This scripts makes calls to nma_lap2d() to geberate
18  %      the sparse matrices.
19  %
20  % date written: 11/5/2010
21  % by: Nasser M. Abbasi
22  %
23
24  close all; clear all;
25
26  DOPLOTS  = true; %set to false if do not want to see plots
27  %spacings = [2^-5 2^-6 2^-7 2^-8 2^-9 2^-10];
28  spacings = [2^-5 2^-6 2^-7 ];
29
30  gridSize = arrayfun(@(i) 1/spacings(i)-1,1:length(spacings));
31
32  for i = 1:length(spacings)
33
34      h = spacings(i);
35      n = gridSize(i);
36
37      fprintf('n=%d\n',n);
38
39      % evaluate f(x,y) on grid and convert to vector
40      [X,Y] = meshgrid(h:h:1-h, h:h:1-h);
41      f     = -exp(-(X-0.25).^2-(Y-0.6).^2);
42      f     = reshape(f,n^2,1);
43
44      t = cputime;
45      A = nma_lap2d(n,n)./h^2;   %make the A matrix
46      fprintf('cpu time for making sparse matrix=%f seconds\n',cputime-t);
47      fprintf('nonzero elements=%d\n',nnz(A));
```

```
48
49
50      t = cputime;
51      u = A\f;
52      fprintf('cpu time for direct solver=%f seconds\n',cputime-t);
53
54      % plot solution if needed
55      if DOPLOTS
56          u=reshape(u,n,n);
57          mesh(u);
58          title(sprintf('2D solution n=%d',n));
59          drawnow;
60      end
61
62  end
63
64  end
```

```
1   function mma_HW3_problem_3_partB_direct_solver()
2   %
3   %   script file, name: mma_HW3_problem_3_partB_direct_solver.m
4   %
5   %   purpose: direct solver for HW3, problem 3, part B (3D).
6   %   UC Davis math 228A
7   %
8   %   description of algorithm:
9   %      This script when called, will find the solution to the problem
10  %      Au=f as described in the HW, by using sparse matrix and direct
11  %      solver.  The script will solve the problem for the following h
12  %      spacings: 2^-3, 2^-4, 2^-5 or grid size n=7,15,31
13  %
14  %      It will find the cpu time used and print to the screen the result
15  %      for each grid space.
16  %
17  % external functions called:
18  %      This makes calls to nma_lap3d() to generate
19  %      the sparse matrices.
20  %
21  % date written: 11/5/2010
22  % by: Nasser M. Abbasi
23  %
24  close all; clear all;
25  %gridsize = [10 20 30 35 40 45 50 55 60 65];
26  gridsize = [10 20 30];
27  spacings = arrayfun(@(i) 1/(gridsize(i)+1),1:length(gridsize));
28
29  fprintf('----- Starting 3D direct solver ---------\n');
```

```
30
31  DOPLOTS=true;   %set to false if do not see plot of solution
32
33  for i = 1:length(spacings)
34
35      h = spacings(i);
36      n = gridsize(i);
37
38      fprintf('******************************\n');
39      fprintf('grid is 3D [%d,%d,%d]\n',n,n,n);
40      fprintf('h=%f\n',h);
41
42      % evaluate f(x,y,z) on grid and convert to vector
43      [X,Y,Z] = meshgrid(h:h:1-h, h:h:1-h,h:h:1-h);
44      f       = -exp(-(X-0.25).^2-(Y-0.6).^2 - Z.^2);
45      f       = reshape(f,n^3,1);
46
47      t = cputime;
48      A = nma_lap3d(n)./h^2; % make the A matrix, sparse
49      fprintf('cpu time for making sparse matrix=%f seconds\n',cputime-t);
50      [nRow,nCol]=size(A);
51      fprintf('dimensions of A (sparse matrix) is [%d,%d]\n',nRow,nCol);
52      fprintf('nnz(A)= %d\n',nnz(A));
53
54      t = cputime;
55      u = A\f;
56      fprintf('cpu time for direct solver=%f seconds\n',cputime-t);
57
58      % plot solution if needed
59      if DOPLOTS
60          u=reshape(u,n,n,n);
61          mesh(u(:,:,n-1));
62          title(sprintf('3D solution, top surface only, n=%d',n));
63          hold on;
64      end
65
66  end
67
68  end
```

```
1  function nma_nnz_estimate()
2  % to estimate nnz() as function of n for 2D sparse matrix
3  % Nasser M. Abbasi
4  % HW3 math 228A
5
6  clear all;
7  close all;
```

```matlab
8  n=[3 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 200 300 400 500 600];
9  y=arrayfun(@(i) nnz(lap2d(n(i),n(i))),1:numel(n));
10 plot(n,y,'r');
11 hold on;
12 plot(n,n.^2.25);
13 title('estimating nnz order for 2D sparse matrix');
14 xlabel('n, number of grid points in one dimension');
15 ylabel('nnz, number of non-zero elements');
16 legend('Matlab nnz()','n^2.2')
17
18 end
```

```matlab
1  function nnz_estimate_3D()
2  % file name  nnz_estimate_3D.m
3  % to estimate nnz() as function of n for 3D sparse matrix
4  % Nasser M. Abbasi
5  % HW3, Math 228A
6
7  clear all;
8  close all;
9  n=[3 10 20 30 40 50 60 70 80 100 150 200];
10 y=arrayfun(@(i) nnz(lap3d(n(i))),1:numel(n));
11 plot(n,y,'r');
12 hold on;
13 plot(n,n.^3.35);
14 title('estimating nnz order for 3D sparse matrix');
15 xlabel('n, number of grid points in one dimension');
16 ylabel('nnz, number of non-zero elements');
17 legend('Matlab nnz()','n^3.35')
18
19 end
```

2.5 HW 4

2.5.1 Problem 1

1. Write a multigrid V-cycle code to solve the Poisson equation in two dimensions on the unit square with Dirichlet boundary conditions. Use full weighting for restriction, bilinear interpolation for prolongation, and red-black Gauss-Seidel for smoothing.

   Note: If you cannot get a V-cycle code working, write a simpler code such as a 2-grid code. You can also experiment in one dimension (do not use GSRB in 1D). You may turn in one of these simplified codes for reduced credit. You should state what your code does, and use your code for the second problem of this assignment.

Figure 2.50: Problem 1

The multigrid V cycle algorithm was implemented in Matlab 2010a. The documented source code is included in the appendix of this problem.

For relaxation, Gauss-Seidel with red-black (GSRB) ordering was used as the default. A relax() function was written to implement this method, in addition, this function can also implement relaxation using these solvers: Jacobi, Gauss-Seidel Lex, and SOR. Selecting the relaxation method is done via an argument option. These different methods are implemented in the relax() function for future numerical experimentation. GSRB is the one used for all the solutions below as required by the problem statement and is the default method. GSRB is known to have good smoothing rates and is suitable for parallelism as well.

For mapping from the fine mesh to coarse mesh, the full weighting method is used.

For mapping from coarse mesh to fine mesh, bilinear interpolation is used.

Additional auxiliary functions are written for performing the following: finding the norm (mesh norm), finding the residue and validating dimensions of the input.

The following diagram illustrates the call flow chart for a program making a call to the V cycle algorithm, such as the program written to solve problem 2 and 3. It shows the Matlab functions used, and the interface between them. This flow diagram also shows a full multigrid solver (FMG) function, which was implemented on top of the V cycle algorithm, but was not used to generate the results in problem 2 as the problem asked to use V cycle algorithm only. FMG cycle algorithm was implemented for future reference and to study its effect on reducing number of iterations. A note on this is can be found the end of this assignment.
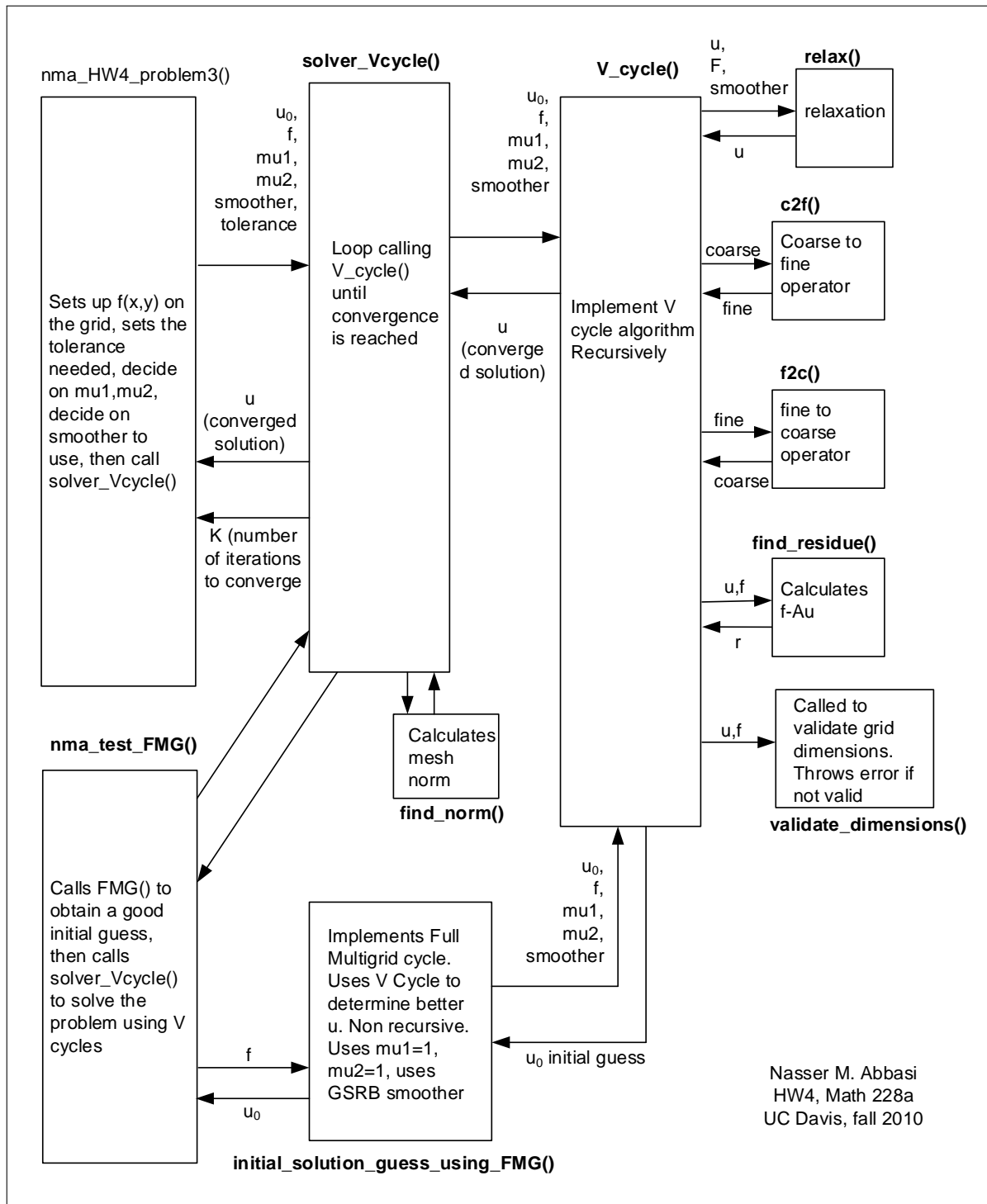
Figure 2.51: algorithm flow chart

2.5.1.1 Restriction and prolongation operators

The restriction operator I_h^{2h} (fine to coarse mesh) mapping uses full weighting, while the prolongation operator I_{2h}^h (coarse to fine mesh) uses bilinear interpolation.

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For illustration, the following diagram shows applying these operators for the 1D case for a mesh of 9 points. The edge points are boundary points and in this problem (Dirichlet homogeneous boundary conditions), these will always be zero.
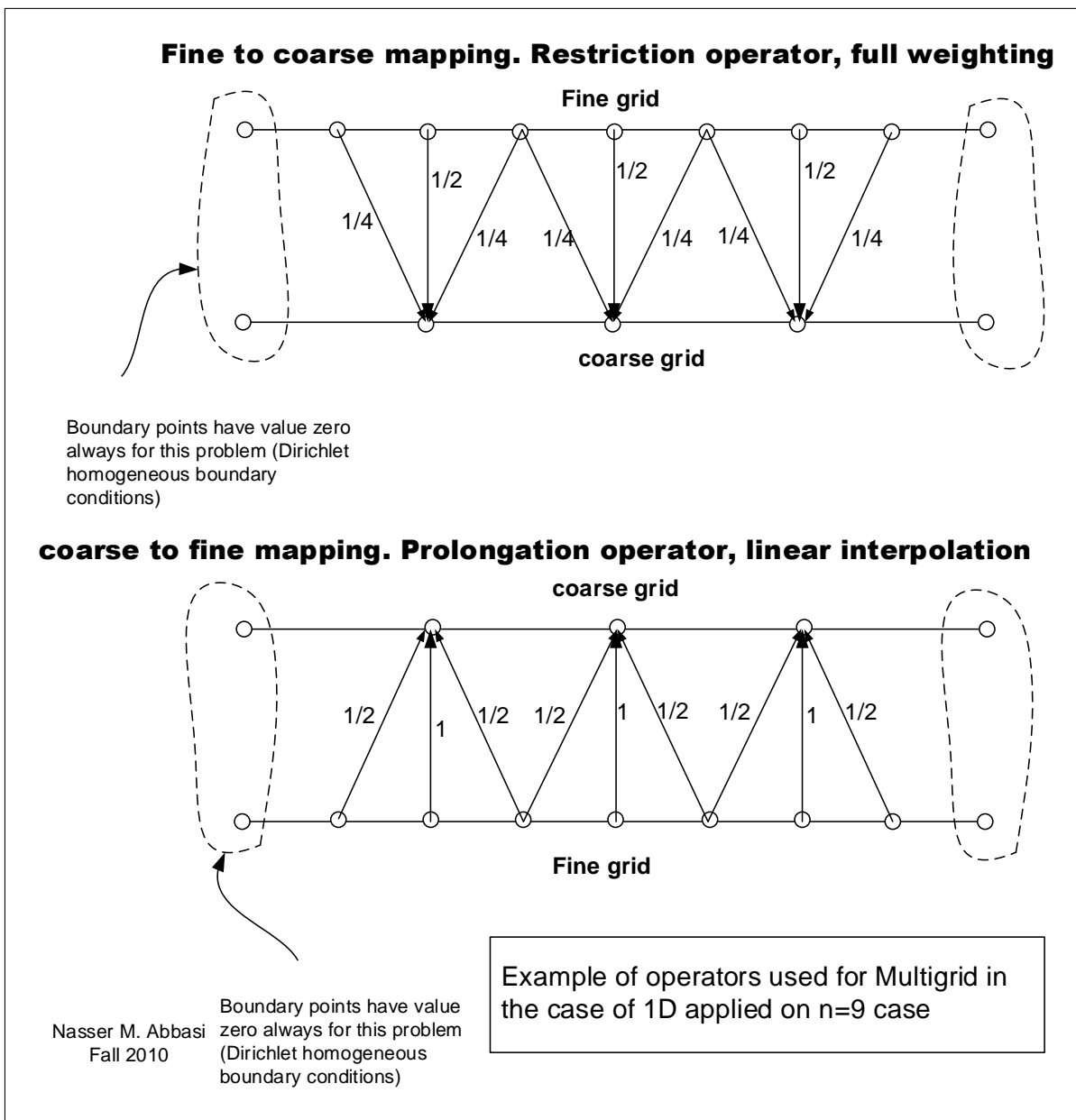
Figure 2.52: operator diagrams

2.5.1.2   V cycle algorithm

The multigrid V cycle algorithm is recursive in nature. The following is description of the algorithm

```
1   VCYCLE algorithm
2   ----------------
3   input:  u, f, mu1, mu2
4   output: u (more accurate u)
5
6   Let n be the number of grid points of u in one dimension
7
8   IF n = 3 THEN
9      find u by direct solution of 3x3 grid
10  ELSE
11     apply mu1 smoothing on u
12     residue = find residue (f-Au)
13     residue = apply fine-to-coarse mapping on residue
14     correction = CALL VCYCLE(ZERO,residue,mu1,mu2)
15     correction = apply coarse-to-fine mapping on correction
16     u = u + correction
17     apply mu2 smoothing on u
18  END IF
19
20  RETURN u
```

.

Problem 2 below also has a diagram which helps understand this algorithm more. The Matlab function shown below implements the above algorithm.

2.5.1.3 Multigrid V cycle function (V_cycle.m)

This function implements one multigrid V cycle. It is recursive function

2.5.1.4 Solver using V cycle (solver_Vcycle.m)

This function is an interface to V cycle algorithm to use for solving the 2D Poisson problem. It uses V_cycle.m in a loop until convergence is reached.

2.5.1.5 Relaxation or smoother function (relax.m)

This function implements Gauss-Seidel red-black solver. It is a little longer than needed as it also implements other solvers as was mentioned in the introduction. These are added for future numerical experimentation. The

algorithm is straight forward. If the sum of the row and column index adds to an even value, then the grid point is considered a red grid point, else it is black. The red grid points are smoothed first, then the black grid points are smoothed.

2.5.1.6   Find residual function (find_residue.m)

This function is called from a number of locations to obtain the residue mesh. The residue is defined as

$$r_{i,j} = f_{i,j} - \frac{1}{h^2}\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}\right)$$

2.5.1.7   Find norm function (find_norm.m)

This function is called from number of locations to obtain the norm of the mesh or any 2D matrix.

2.5.1.8   Validate boundary conditions (check_all_zero_boundaries.m)

An auxiliary function used by a number of functions to validate that input has consistent boundaries for this problem.

2.5.1.9   Coarse to fine prolongation operator 2D (c2f.m)

This function is the prolongation bilinear interpolation which implements coarse to fine grid mapping on 2D grid.

2.5.1.10   Fine to coarse full weight restriction operator 2D (f2c.m)

This function is the full weight restriction operator which implements the fine grid to coarse grid mapping on 2D grid.

2.5.1.11   Validate u and f have consistent dimensions(validate_dimensions.m)

An auxiliary function used by number of other function to validate that input dimensions are consistent.

2.5.1.12   Validate grid for consistent dimensions (validate_dimensions_1.m)

An auxiliary function used by number of other function to validate that a grid dimensions are consistent.

2.5.1.13   FMG solver (initial_solution_guess_using_FMG.m)

Implements a full multigrid cycle using V cycle algorithm as building block. Used to compare effect on solution only.

2.5.1.14   Restriction operator for 1D (f2c_1D.m)

This function is the full weight restriction operator which implements the fine grid to coarse grid mapping on 1D grid

2.5.1.15 Prolongation operator for 1D (c2f_1D.m)

This function is the prolongation bilinear interpolation which implements coarse to fine grid mapping on 1D grid

2.5.2 Problem 2

2. Numerically estimate the average convergence factor,

$$\left(\frac{\|e^{(k)}\|_\infty}{\|e^{(0)}\|_\infty}\right)^{1/k},$$

for different numbers of presmoothing steps, ν_1, and postsmoothing steps, ν_2, for $\nu = \nu_1 + \nu_2 \leq 4$. Be sure to use a small value of k because convergence may be reached very quickly. What test problem did you use? Report the results in a table, and discuss which choices of ν_1 and ν_2 give the most efficient solver.

Figure 2.53: Problem 2

The test problem used is

$$\nabla u = 0$$

with zero boundary conditions on the unit square. This has a known solution which is zero.

Initial guess is a random solution generated using matlab rand(). Hence $\|e^{(0)}\|$ has the same norm as the initial guess.

The V Cycle function written for problem 1 was used to generate the average convergence factor. For each combination of ν_1, ν_2, a table was generated which contained the following columns:

1. Cycle number.

2. The norm of the residue $\|r^{(k)}\| = \|f - Au^{(k)}\|$ after each cycle.

3. Ratio of the current residue norm to the previous residue norm $\frac{\|r^{(k+1)}\|}{\|r^{(k)}\|}$.

4. Error norm $\|e^{(k)}\| = \|u - u^{(k)}\| = \|u^{(k)}\|$ (since exact solution is zero).

5. The ratio of the current error norm to the previous error norm $\frac{\|e^{(k+1)}\|}{\|e^{(k)}\|}$.

6. The average convergence factor up to each cycle $\left(\frac{\|e^{(k)}\|_\infty}{\|e^{(0)}\|_\infty}\right)^{\left(\frac{1}{k}\right)}$.

The problem asked to generate result for $\nu \leq 4$. This solution extended this to $\nu \leq 8$ to use the results for future study if needed. The summary table below shows the final result

for $k = 15$. Each individual table generated for each combination of v_1, v_2 is listed in the appendix of this problem. The function HW4_problem2() was used to generate these tables and to calculate the work done for each solver combination of v_1, v_2.

2.5.2.1   Average convergence factor and work unit estimates

To determine the most efficient solver, the amount of work by each solver that achieves the same convergence is determined. The solver with the least amount of work is deemed the most efficient. The total amount of work for convergence is defined as

$$\text{WORK=number of Iterations for convergence } \times \text{work per iteration} \tag{1}$$

An iteration is one full V cycle. Each V cycle contains a number of levels. The same number of levels exist on the left side of the V cycle as on the right side of the V cycle. On the left side of the V cycle, work at each level consist of the following items

1. Work to perform v_1 number of pre-smooth operations.

2. Work needed to map to the next lower level of the grid.

3. Work needed to compute the residue.

The following diagram helps to illustrates this.

**The Multigrid V cycle stages applied to
initial grid of 33 X 33 size for illustration**

Most fine grid, total
number of grid points =
N = number of
unknowns

Fine grid, corrected
u solution after one
V cycle

pre-smooth
residue

33 X 33   post-smooth

I_h^{2h}

pre-smooth
residue

I_{2h}^h

17 X 17   post-smooth

I_h^{2h}

pre-smooth
residue

I_{2h}^h

9 X 9   post-smooth

Fine to
coarse
mapping

I_h^{2h}

pre-smooth
residue

5 X 5   post-smooth

I_{2h}^h

Coarse to
fine
mapping

I_h^{2h}

3 X 3

I_{2h}^h

I_{2h}^h coarse to fine operator
prolongation bilinear
interpolation

Most coarse grid
Direct solve

I_h^{2h} fine to coarse operator
Restriction full weight

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Math 228a, UC Davis

Figure 2.54: problem 2 v cycle shape

On the right side of the V cycle, work at each level consist of the following items

1. Work to perform v_2 number of post-smooth operations.

2. Work needed to map to the next lower level of the grid.

At each level, the work is proportional to the size of the grid at that level. For smoothing, it is estimated that 7 flops are needed to obtain an average of each grid point. (5 additions, one multiplication, one division) based on the use of the following formula

$$u_{i,j} = \frac{1}{4}\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - h^2 f\right)$$

Therefore work needed for smoothing is $7 \times (v_1 + v_2) \times N$ where N is the total number of grid points at that level. (Boundary grid points are not involved in this work, but for simplicity

of analysis, the total number of grid points N is used).

Work needed for finding the residual is also about $7N$. Work needed for mapping to the next grid level is about $6N$.

On the right branch of the cycle no residual calculation is required. To simplify the analysis, it is assumed that the same work is performed at each level on both sides of the V cycle.

Therefore, Letting N be the number of grid points at the most fine level (the number of unknowns), the total work per cycle is found to be

$$work\ per\ V\ cycle = (7(v_1 + v_2) + 13)N + (7(v_1 + v_2) + 13)\frac{N}{4} + (7(v_1 + v_2) + 13)\frac{N}{16} + \cdots$$

$$= (7(v_1 + v_2) + 13)N\left[1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^{L-1}}\right]$$

Where L is the number of levels. In the limit as L becomes very large, this becomes a geometric series whose sum is $\frac{(7(v_1+v_2)+13)N}{1-r}$ where $r = \frac{1}{4}$

Therefore, total amount of work from (1) becomes

$$W = M \times \frac{4}{3}(7(v_1 + v_2) + 13)N$$

Where M is number of iterations. Using the same tolerance ε for all solvers, $M = \frac{\log(\varepsilon)}{\log(\rho)}$. *Using the average convergence rate found as an estimate for the spectral radius ρ, W can now be found as a function of N*

$$W = \left(\frac{\log(\varepsilon)}{\log(\rho)}\right)\left(\frac{4}{3}(7(v_1 + v_2) + 13)N\right)$$

For the purpose of comparing the different solvers, the value of ε used is not important as long as it is the same value in all cases. Hence, for numerical computation, let $\varepsilon = 10^{-6}$ and the above becomes

$$W = \left(\frac{-6}{\log(\rho)}\right)\left(\frac{4}{3}(7(v_1 + v_2) + 13)N\right)$$

The program written for this problem uses the above equation to calculate the work done for each solver. The result is shown below. This table shows the work done by each solver for convergence based on the same tolerance. As mentioned above, changing the tolerance value will not change the result, as the effect will be to scale all result by the same amount.

2.5.2.2 Result

$v = (v_1, v_2)$	$v_1 + v_2$	$CF = \left(\frac{\|e^{(k)}\|_\infty}{\|e^{(0)}\|_\infty} \right)^{\left(\frac{1}{k}\right)}$	work $\frac{-6}{\log(\rho)} \times \frac{4}{3} \left(7 \left(v_1 + v_2 \right) + 13 \right) N$
$(0,1)$	1	0.374588	$213\, N$
$(0,2)$	2	0.202747	$177\, N$
$(0,3)$	3	0.132526	$174.9\, N$
$(0,4)$	4	0.098857	$182.7\, N$
$(1,0)$	1	0.323688	$182.4\, N$
$(1,1)$	2	0.116811	$129.4\, N$
$(1,2)$	3	0.079578	$138.4\, N$
$(1,3)$	4	0.060307	$150.5\, N$
$(1,4)$	5	0.049019	$164.1\, N$
$(2,0)$	2	0.171973	$158.5\, N$
$(2,1)$	3	0.079845	$138.6\, N$
$(2,2)$	4	0.060424	$150.6\, N$
$(2,3)$	5	0.049068	$164.2\, N$
$(2,4)$	6	0.041573	$178.4\, N$
$(3,0)$	3	0.117023	$163.6\, N$
$(3,1)$	4	0.060444	$150.6\, N$
$(3,2)$	5	0.049072	$164.2\, N$
$(3,3)$	6	0.041575	$178.4\, N$
$(3,4)$	7	0.036128	$192.6\, N$
$(4,0)$	4	0.088624	$174.6\, N$
$(4,1)$	5	0.049075	$164.2\, N$
$(4,2)$	6	0.041575	$178.4\, N$
$(4,3)$	7	0.036128	$192.6\, N$
$(4,4)$	8	0.031908	$206.7\, N$

2.5.2.3 Conclusion

From the above result, The least work was for the $(1,1)$ solver, followed by $(1,2)$ which had about the same as the $(2,1)$ solver. This result shows that using $v = 2$ or $v = 3$ *is the most efficient solver* in terms of least work required.

Notice that in full multigrid, the combination which makes up the value v is important (While for the case of the 2 level multigrid, this is not the case). For example, as shown in the above table, work for solver $v = (1,2)$ was $138\, N$ while work for solver $v = (3,0)$ was $163\, N$ even though they both add to same total number of smooth operations $v = 3$.

2.5.2.4   Appendix. Tables for each combination of $v = (v_1, v_2)$

The following tables are the result of running problem 2 program on the test problem. The fields for each table are described above. The last row in each table contain the result for $k = 15$. The value of the average convergence factor used is that for $k = 15$ under the column heading convergence factor. This below is link to the text file containing the tables as they are printed by the matlab function.

Result for $v_1 = 0, v_2 = 0$

```
 1  cycle  |residue|        ratio         |error|          ratio      convergence
              factor
 2    1    4.197646e+004   0.000000   3.791167e+000    0.000000     0.836431
 3    2    4.197672e+004   1.000006   3.618870e+000    0.954553     0.893542
 4    3    4.197682e+004   1.000002   3.571746e+000    0.986978     0.923661
 5    4    4.197685e+004   1.000001   3.557261e+000    0.995944     0.941224
 6    5    4.197686e+004   1.000000   3.552573e+000    0.998682     0.952445
 7    6    4.197687e+004   1.000000   3.551027e+000    0.999565     0.960141
 8    7    4.197687e+004   1.000000   3.550514e+000    0.999856     0.965717
 9    8    4.197687e+004   1.000000   3.550344e+000    0.999952     0.969931
10    9    4.197687e+004   1.000000   3.550287e+000    0.999984     0.973225
11   10    4.197687e+004   1.000000   3.550268e+000    0.999995     0.975870
12   11    4.197687e+004   1.000000   3.550262e+000    0.999998     0.978039
13   12    4.197687e+004   1.000000   3.550260e+000    0.999999     0.979850
14   13    4.197687e+004   1.000000   3.550259e+000    1.000000     0.981386
15   14    4.197687e+004   1.000000   3.550259e+000    1.000000     0.982704
16   15    4.197687e+004   1.000000   3.550259e+000    1.000000     0.983848
```

.

Result for $v_1 = 0, v_2 = 1$

```
 1  cycle  |residue|        ratio         |error|          ratio      convergence
              factor
 2    1    6.512884e+003   0.000000   1.841085e+000    0.000000     0.406191
 3    2    1.096932e+003   0.168425   7.221287e-001    0.392230     0.399150
 4    3    2.518487e+002   0.229594   2.950874e-001    0.408636     0.402287
 5    4    6.501181e+001   0.258138   1.172566e-001    0.397362     0.401050
 6    5    1.807445e+001   0.278018   4.592625e-002    0.391673     0.399157
 7    6    5.295406e+000   0.292977   1.774863e-002    0.386459     0.397012
 8    7    1.610302e+000   0.304094   6.756385e-003    0.380671     0.394635
 9    8    5.023475e-001   0.311959   2.532700e-003    0.374860     0.392107
10    9    1.594271e-001   0.317364   9.349623e-004    0.369156     0.389488
11   10    5.118833e-002   0.321077   3.401773e-004    0.363841     0.386844
12   11    1.656749e-002   0.323658   1.221213e-004    0.358993     0.384226
13   12    5.392540e-003   0.325489   4.331077e-005    0.354654     0.381670
14   13    1.762359e-003   0.326814   1.519371e-005    0.350807     0.379202
15   14    5.776815e-004   0.327789   5.278592e-006    0.347420     0.376839
16   15    1.897765e-004   0.328514   1.818208e-006    0.344449     0.374588
```

.

Result for $v_1 = 0, v_2 = 2$

	cycle	|residue|	ratio	|error|	ratio	convergence factor
1						
2	1	2.332824e+003	0.000000	1.130306e+000	0.000000	0.249375
3	2	1.658927e+002	0.071112	2.573654e-001	0.227695	0.238289
4	3	1.740638e+001	0.104926	5.746902e-002	0.223297	0.233183
5	4	2.213859e+000	0.127187	1.255196e-002	0.218413	0.229399
6	5	3.107344e-001	0.140359	2.675256e-003	0.213135	0.226050
7	6	4.627116e-002	0.148909	5.553596e-004	0.207591	0.222863
8	7	7.199976e-003	0.155604	1.123881e-004	0.202370	0.219813
9	8	1.162230e-003	0.161421	2.224051e-005	0.197890	0.216945
10	9	1.934709e-004	0.166465	4.319318e-006	0.194210	0.214293
11	10	3.300557e-005	0.170597	8.260186e-007	0.191238	0.211868
12	11	5.734479e-006	0.173743	1.559942e-007	0.188851	0.209664
13	12	1.009112e-006	0.175973	2.916007e-008	0.186930	0.207668
14	13	1.790692e-007	0.177452	5.405721e-009	0.185381	0.205863
15	14	3.194089e-008	0.178372	9.953326e-010	0.184126	0.204228
16	15	5.714226e-009	0.178900	1.822506e-010	0.183105	0.202747

.

Result for $v_1 = 0, v_2 = 3$

	cycle	|residue|	ratio	|error|	ratio	convergence factor
1						
2	1	1.279733e+003	0.000000	8.246208e-001	0.000000	0.181933
3	2	6.128118e+001	0.047886	1.239874e-001	0.150357	0.165393
4	3	4.197852e+000	0.068502	1.807404e-002	0.145773	0.158576
5	4	3.457379e-001	0.082361	2.513173e-003	0.139049	0.153451
6	5	3.161771e-002	0.091450	3.385354e-004	0.134704	0.149504
7	6	3.097433e-003	0.097965	4.446410e-005	0.131343	0.146311
8	7	3.196538e-004	0.103200	5.723743e-006	0.128727	0.143659
9	8	3.438009e-005	0.107554	7.254505e-007	0.126744	0.141427
10	9	3.818351e-006	0.111063	9.087146e-008	0.125262	0.139533
11	10	4.343261e-007	0.113747	1.128253e-008	0.124159	0.137913
12	11	5.025553e-008	0.115709	1.391556e-009	0.123337	0.136520
13	12	5.884932e-009	0.117100	1.707747e-010	0.122722	0.135313
14	13	6.948378e-010	0.118071	2.087896e-011	0.122260	0.134261
15	14	8.250871e-011	0.118745	2.545414e-012	0.121913	0.133339
16	15	9.836463e-012	0.119217	3.096536e-013	0.121652	0.132526

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Result for $v_1 = 0, v_2 = 4$

	cycle	|residue|	ratio	|error|	ratio	convergence
1						

	cycle	|residue|	ratio	|error|	ratio	convergence factor
2	1	8.351773e+002	0.000000	6.609090e-001	0.000000	0.145814
3	2	3.043485e+001	0.036441	7.256809e-002	0.109800	0.126532
4	3	1.541666e+000	0.050655	7.850138e-003	0.108176	0.120091
5	4	9.413409e-002	0.061060	8.032536e-004	0.102324	0.115379
6	5	6.459266e-003	0.068618	7.934903e-005	0.098785	0.111851
7	6	4.798607e-004	0.074290	7.650179e-006	0.096412	0.109116
8	7	3.773372e-005	0.078635	7.251982e-007	0.094795	0.106945
9	8	3.090474e-006	0.081902	6.795204e-008	0.093701	0.105192
10	9	2.605052e-007	0.084293	6.317243e-009	0.092966	0.103758
11	10	2.240698e-008	0.086014	5.841772e-010	0.092473	0.102570
12	11	1.954999e-009	0.087250	5.382824e-011	0.092144	0.101575
13	12	1.723247e-010	0.088146	4.948089e-012	0.091924	0.100734
14	13	1.530346e-011	0.088806	4.541247e-013	0.091778	0.100015
15	14	1.366630e-012	0.089302	4.163526e-014	0.091682	0.099395
16	15	1.225631e-013	0.089683	3.814695e-015	0.091622	0.098857

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Result for $v_1 = 1, v_2 = 0$

	cycle	|residue|	ratio	|error|	ratio	convergence factor
1						
2	1	6.179909e+003	0.000000	1.360828e+000	0.000000	0.300234
3	2	1.275576e+003	0.206407	4.199343e-001	0.308587	0.304382
4	3	3.536744e+002	0.277266	1.341956e-001	0.319563	0.309361
5	4	1.100024e+002	0.311027	4.345678e-002	0.323832	0.312917
6	5	3.621869e+001	0.329254	1.417374e-002	0.326157	0.315521
7	6	1.220133e+001	0.336879	4.640907e-003	0.327430	0.317475
8	7	4.148812e+000	0.340030	1.522676e-003	0.328099	0.318972
9	8	1.413642e+000	0.340734	5.000095e-004	0.328375	0.320132
10	9	4.811293e-001	0.340347	1.642075e-004	0.328409	0.321041
11	10	1.633133e-001	0.339437	5.390811e-005	0.328293	0.321759
12	11	5.524973e-002	0.338305	1.768674e-005	0.328091	0.322330
13	12	1.862458e-002	0.337098	5.798496e-006	0.327844	0.322786
14	13	6.255900e-003	0.335895	1.899475e-006	0.327581	0.323152
15	14	2.094091e-003	0.334739	6.217315e-007	0.327318	0.323448
16	15	6.986989e-004	0.333653	2.033475e-007	0.327067	0.323688

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Result for $v_1 = 1, v_2 = 1$

	cycle	|residue|	ratio	|error|	ratio	convergence factor
1						
2	1	1.030835e+003	0.000000	4.727264e-001	0.000000	0.103251
3	2	5.883607e+001	0.057076	5.305423e-002	0.112230	0.107647
4	3	3.801623e+000	0.064614	6.112766e-003	0.115217	0.110114
5	4	2.784357e-001	0.073241	7.123738e-004	0.116539	0.111686

6	5	2.222565e-002	0.079823	8.361417e-005	0.117374	0.112801
7	6	1.895919e-003	0.085303	9.862335e-006	0.117951	0.113643
8	7	1.697849e-004	0.089553	1.167283e-006	0.118358	0.114305
9	8	1.581453e-005	0.093144	1.384918e-007	0.118645	0.114839
10	9	1.525515e-006	0.096463	1.645918e-008	0.118846	0.115277
11	10	1.520761e-007	0.099688	1.958413e-009	0.118986	0.115643
12	11	1.563620e-008	0.102818	2.332146e-010	0.119083	0.115952
13	12	1.653635e-009	0.105757	2.778772e-011	0.119151	0.116215
14	13	1.792425e-010	0.108393	3.312230e-012	0.119198	0.116442
15	14	1.983378e-011	0.110653	3.949174e-013	0.119230	0.116639
16	15	2.231664e-012	0.112518	4.709496e-014	0.119253	0.116811

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Result for $v_1 = 1, v_2 = 2$

	cycle	|residue|	ratio	|error|	ratio	convergence factor
1						
2	1	3.491916e+002	0.000000	3.054693e-001	0.000000	0.066754
3	2	1.021899e+001	0.029265	2.334872e-002	0.076436	0.071431
4	3	4.299404e-001	0.042073	1.841565e-003	0.078872	0.073830
5	4	2.178631e-002	0.050673	1.473129e-004	0.079993	0.075325
6	5	1.245425e-003	0.057165	1.187207e-005	0.080591	0.076350
7	6	7.801009e-005	0.062637	9.606537e-007	0.080917	0.077093
8	7	5.235377e-006	0.067112	7.790676e-008	0.081098	0.077652
9	8	3.692823e-007	0.070536	6.325950e-009	0.081199	0.078087
10	9	2.697616e-008	0.073050	5.140286e-010	0.081257	0.078433
11	10	2.019954e-009	0.074879	4.178622e-011	0.081292	0.078714
12	11	1.539723e-010	0.076226	3.397763e-012	0.081313	0.078947
13	12	1.189244e-011	0.077238	2.763303e-013	0.081327	0.079143
14	13	9.277782e-013	0.078014	2.247589e-014	0.081337	0.079309
15	14	7.294259e-014	0.078621	1.828290e-015	0.081345	0.079453
16	15	5.769829e-015	0.079101	1.487324e-016	0.081351	0.079578

.

Result for $v_1 = 1, v_2 = 3$

	cycle	|residue|	ratio	|error|	ratio	convergence factor
1						
2	1	1.748348e+002	0.000000	2.273544e-001	0.000000	0.049683
3	2	4.020464e+000	0.022996	1.322844e-002	0.058184	0.053766
4	3	1.341199e-001	0.033359	7.939589e-004	0.060019	0.055774
5	4	5.362431e-003	0.039982	4.829726e-005	0.060831	0.056998
6	5	2.429919e-004	0.045314	2.956914e-006	0.061223	0.057819
7	6	1.205611e-005	0.049615	1.816104e-007	0.061419	0.058404
8	7	6.374272e-007	0.052872	1.117251e-008	0.061519	0.058839
9	8	3.520088e-008	0.055223	6.879119e-010	0.061572	0.059174
10	9	2.002565e-009	0.056890	4.237580e-011	0.061601	0.059439

11	10	1.162928e-010	0.058072	2.611069e-012	0.061617	0.059653
12	11	6.852108e-012	0.058921	1.609119e-013	0.061627	0.059830
13	12	4.079843e-013	0.059541	9.917484e-015	0.061633	0.059978
14	13	2.447990e-014	0.060002	6.112860e-016	0.061637	0.060104
15	14	1.477344e-015	0.060349	3.767977e-017	0.061640	0.060213
16	15	8.954816e-017	0.060614	2.322670e-018	0.061642	0.060307

.

Result for $v_1 = 1, v_2 = 4$

	cycle	|residue|	ratio	|error|	ratio	convergence factor
1						
2	1	1.042637e+002	0.000000	1.827858e-001	0.000000	0.039944
3	2	2.005234e+000	0.019232	8.672196e-003	0.047445	0.043533
4	3	5.546631e-002	0.027661	4.241753e-004	0.048912	0.045257
5	4	1.849251e-003	0.033340	2.101081e-005	0.049533	0.046290
6	5	6.998200e-005	0.037843	1.046739e-006	0.049819	0.046975
7	6	2.890955e-006	0.041310	5.229125e-008	0.049956	0.047459
8	7	1.267590e-007	0.043847	2.615867e-009	0.050025	0.047818
9	8	5.783896e-009	0.045629	1.309520e-010	0.050061	0.048092
10	9	2.709911e-010	0.046853	6.558096e-012	0.050080	0.048309
11	10	1.292323e-011	0.047689	3.285046e-013	0.050091	0.048485
12	11	6.237388e-013	0.048265	1.645757e-014	0.050098	0.048629
13	12	3.035654e-014	0.048669	8.245760e-016	0.050103	0.048750
14	13	1.486172e-015	0.048957	4.131659e-017	0.050106	0.048853
15	14	7.307172e-017	0.049168	2.070333e-018	0.050109	0.048942
16	15	3.604223e-018	0.049324	1.037465e-019	0.050111	0.049019

.

Result for $v_1 = 2, v_2 = 0$

	cycle	|residue|	ratio	|error|	ratio	convergence factor
1						
2	1	2.655117e+003	0.000000	7.196022e-001	0.000000	0.158763
3	2	2.807305e+002	0.105732	1.214403e-001	0.168760	0.163685
4	3	4.389822e+001	0.156371	2.105498e-002	0.173377	0.166854
5	4	7.595636e+000	0.173028	3.666160e-003	0.174123	0.168643
6	5	1.345020e+000	0.177078	6.386485e-004	0.174201	0.169740
7	6	2.391057e-001	0.177771	1.111764e-004	0.174081	0.170456
8	7	4.245722e-002	0.177567	1.933186e-005	0.173885	0.170941
9	8	7.518789e-003	0.177091	3.357127e-006	0.173658	0.171279
10	9	1.327337e-003	0.176536	5.821965e-007	0.173421	0.171515
11	10	2.335780e-004	0.175975	1.008285e-007	0.173186	0.171682
12	11	4.097873e-005	0.175439	1.743931e-008	0.172960	0.171798
13	12	7.168928e-006	0.174943	3.012563e-009	0.172746	0.171876
14	13	1.250905e-006	0.174490	5.198010e-010	0.172544	0.171928
15	14	2.177582e-007	0.174081	8.959148e-011	0.172357	0.171958

| 16 | 15 | 3.782731e-008 | 0.173712 | 1.542621e-011 | 0.172184 | 0.171973 |

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Result for $v_1 = 2, v_2 = 1$

	cycle	|residue|	ratio	|error|	ratio	convergence factor
1						
2	1	3.587387e+002	0.000000	3.008484e-001	0.000000	0.065744
3	2	1.067855e+001	0.029767	2.317076e-002	0.077018	0.071158
4	3	4.487287e-001	0.042021	1.842948e-003	0.079538	0.073848
5	4	2.256832e-002	0.050294	1.485356e-004	0.080597	0.075480
6	5	1.286179e-003	0.056990	1.204521e-005	0.081093	0.076571
7	6	8.087232e-005	0.062878	9.796869e-007	0.081334	0.077345
8	7	5.482293e-006	0.067789	7.979984e-008	0.081454	0.077919
9	8	3.922682e-007	0.071552	6.504963e-009	0.081516	0.078360
10	9	2.913427e-008	0.074271	5.304691e-010	0.081548	0.078708
11	10	2.219978e-009	0.076198	4.326818e-011	0.081566	0.078989
12	11	1.722107e-010	0.077573	3.529630e-012	0.081576	0.079221
13	12	1.353118e-011	0.078573	2.879518e-013	0.081581	0.079415
14	13	1.073247e-012	0.079317	2.349245e-014	0.081585	0.079580
15	14	8.572977e-014	0.079879	1.916672e-015	0.081587	0.079721
16	15	6.884988e-015	0.080310	1.563774e-016	0.081588	0.079845

.

Result for $v_1 = 2, v_2 = 2$

	cycle	|residue|	ratio	|error|	ratio	convergence factor
1						
2	1	1.448199e+002	0.000000	2.243955e-001	0.000000	0.049037
3	2	3.594191e+000	0.024818	1.315787e-002	0.058637	0.053622
4	3	1.246015e-001	0.034667	7.947099e-004	0.060398	0.055792
5	4	5.109282e-003	0.041005	4.857254e-005	0.061120	0.057079
6	5	2.356797e-004	0.046128	2.984395e-006	0.061442	0.057926
7	6	1.184050e-005	0.050240	1.838170e-007	0.061593	0.058521
8	7	6.316880e-007	0.053350	1.133526e-008	0.061666	0.058961
9	8	3.512166e-008	0.055600	6.994189e-010	0.061703	0.059297
10	9	2.008926e-009	0.057199	4.316969e-011	0.061722	0.059561
11	10	1.171965e-010	0.058338	2.664984e-012	0.061733	0.059775
12	11	6.933122e-012	0.059158	1.645328e-013	0.061739	0.059951
13	12	4.143073e-013	0.059758	1.015864e-014	0.061742	0.060098
14	13	2.494234e-014	0.060203	6.272422e-016	0.061745	0.060223
15	14	1.509931e-015	0.060537	3.872984e-017	0.061746	0.060331
16	15	9.179027e-017	0.060791	2.391465e-018	0.061747	0.060424

.

Result for $v_1 = 2, v_2 = 3$

```
1    cycle  |residue|           ratio       |error|            ratio   convergence
         factor
2     1   8.776592e+001     0.000000   1.811025e-001     0.000000      0.039576
3     2   1.834493e+000     0.020902   8.646274e-003     0.047742      0.043468
4     3   5.258689e-002     0.028666   4.246673e-004     0.049116      0.045274
5     4   1.790526e-003     0.034049   2.109371e-005     0.049671      0.046336
6     5   6.865074e-005     0.038341   1.052908e-006     0.049916      0.047030
7     6   2.859888e-006     0.041659   5.267602e-008     0.050029      0.047517
8     7   1.260987e-007     0.044092   2.638218e-009     0.050084      0.047876
9     8   5.776016e-009     0.045806   1.322046e-010     0.050111      0.048150
10    9   2.713848e-010     0.046985   6.626843e-012     0.050126      0.048365
11    10  1.297016e-011     0.047793   3.322268e-013     0.050133      0.048539
12    11  6.271114e-013     0.048350   1.665719e-014     0.050138      0.048683
13    12  3.056636e-014     0.048742   8.352047e-016     0.050141      0.048802
14    13  1.498397e-015     0.049021   4.187935e-017     0.050143      0.048904
15    14  7.375846e-017     0.049225   2.099994e-018     0.050144      0.048992
16    15  3.641929e-018     0.049376   1.053040e-019     0.050145      0.049068
```

.

Result for $v_1 = 2, v_2 = 4$

```
1    cycle  |residue|           ratio       |error|            ratio   convergence
         factor
2     1   5.968518e+001     0.000000   1.526966e-001     0.000000      0.033368
3     2   1.071291e+000     0.017949   6.187596e-003     0.040522      0.036772
4     3   2.628731e-002     0.024538   2.577593e-004     0.041657      0.038333
5     4   7.693324e-004     0.029266   1.085425e-005     0.042110      0.039244
6     5   2.541368e-005     0.033033   4.592064e-007     0.042307      0.039838
7     6   9.120864e-007     0.035890   1.946889e-008     0.042397      0.040254
8     7   3.457788e-008     0.037911   8.262608e-010     0.042440      0.040559
9     8   1.357937e-009     0.039272   3.508443e-011     0.042462      0.040792
10    9   5.454790e-011     0.040170   1.490141e-012     0.042473      0.040976
11    10  2.223580e-012     0.040764   6.330013e-014     0.042479      0.041124
12    11  9.153192e-014     0.041164   2.689178e-015     0.042483      0.041245
13    12  3.793199e-015     0.041441   1.142507e-016     0.042485      0.041347
14    13  1.579443e-016     0.041639   4.854164e-018     0.042487      0.041434
15    14  6.599536e-018     0.041784   2.062445e-019     0.042488      0.041508
16    15  2.764795e-019     0.041894   8.763145e-021     0.042489      0.041573
```

.

Result for $v_1 = 3, v_2 = 0$

```
1    cycle  |residue|           ratio       |error|            ratio   convergence
         factor
2     1   1.731895e+003     0.000000   4.772061e-001     0.000000      0.105284
3     2   1.332957e+002     0.076965   5.436461e-002     0.113923      0.109518
```

4	3	1.472030e+001	0.110433	6.363908e-003	0.117060	0.111977
5	4	1.734653e+000	0.117841	7.494297e-004	0.117762	0.113396
6	5	2.065908e-001	0.119096	8.847251e-005	0.118053	0.114312
7	6	2.463413e-002	0.119241	1.045852e-005	0.118212	0.114953
8	7	2.936045e-003	0.119186	1.237354e-006	0.118311	0.115427
9	8	3.497108e-004	0.119109	1.464715e-007	0.118375	0.115791
10	9	4.163098e-005	0.119044	1.734462e-008	0.118416	0.116080
11	10	4.953753e-006	0.118992	2.054331e-009	0.118442	0.116314
12	11	5.892482e-007	0.118950	2.433477e-010	0.118456	0.116507
13	12	7.006984e-008	0.118914	2.882724e-011	0.118461	0.116669
14	13	8.330007e-009	0.118881	3.414861e-012	0.118460	0.116806
15	14	9.900258e-010	0.118851	4.044995e-013	0.118453	0.116923
16	15	1.176348e-010	0.118820	4.790965e-014	0.118442	0.117023

.

Result for $v_1 = 3, v_2 = 1$

	cycle	|residue|	ratio	|error|	ratio	convergence
1						factor
2	1	1.806744e+002	0.000000	2.248328e-001	0.000000	0.049132
3	2	4.176399e+000	0.023116	1.317724e-002	0.058609	0.053662
4	3	1.375470e-001	0.032934	7.958156e-004	0.060393	0.055818
5	4	5.462156e-003	0.039711	4.864574e-005	0.061127	0.057100
6	5	2.477092e-004	0.045350	2.989565e-006	0.061456	0.057946
7	6	1.236926e-005	0.049935	1.841867e-007	0.061610	0.058541
8	7	6.603866e-007	0.053389	1.136149e-008	0.061685	0.058980
9	8	3.687804e-008	0.055843	7.012560e-010	0.061722	0.059316
10	9	2.122171e-009	0.057546	4.329681e-011	0.061742	0.059581
11	10	1.246363e-010	0.058731	2.673686e-012	0.061752	0.059795
12	11	7.424379e-012	0.059568	1.651230e-013	0.061759	0.059970
13	12	4.467406e-013	0.060172	1.019837e-014	0.061762	0.060118
14	13	2.707929e-014	0.060615	6.298990e-016	0.061765	0.060243
15	14	1.650363e-015	0.060946	3.890651e-017	0.061766	0.060350
16	15	1.009940e-016	0.061195	2.403156e-018	0.061767	0.060444

.

Result for $v_1 = 3, v_2 = 2$

	cycle	|residue|	ratio	|error|	ratio	convergence
1						factor
2	1	8.757448e+001	0.000000	1.810759e-001	0.000000	0.039570
3	2	1.833553e+000	0.020937	8.646696e-003	0.047752	0.043469
4	3	5.258255e-002	0.028678	4.247632e-004	0.049124	0.045278
5	4	1.791120e-003	0.034063	2.110165e-005	0.049679	0.046340
6	5	6.870264e-005	0.038357	1.053438e-006	0.049922	0.047035
7	6	2.863317e-006	0.041677	5.270844e-008	0.050035	0.047522
8	7	1.263069e-007	0.044112	2.640110e-009	0.050089	0.047881

9	8	5.788133e-009	0.045826	1.323120e-010	0.050116	0.048155
10	9	2.720719e-010	0.047005	6.632824e-012	0.050130	0.048370
11	10	1.300844e-011	0.047812	3.325556e-013	0.050138	0.048544
12	11	6.292148e-013	0.048370	1.667508e-014	0.050142	0.048687
13	12	3.068072e-014	0.048760	8.361702e-016	0.050145	0.048807
14	13	1.504560e-015	0.049039	4.193112e-017	0.050147	0.048909
15	14	7.408807e-017	0.049242	2.102755e-018	0.050148	0.048996
16	15	3.659443e-018	0.049393	1.054505e-019	0.050149	0.049072

.

Result for $v_1 = 3, v_2 = 3$

	cycle	|residue|	ratio	|error|	ratio	convergence factor
1						
2	1	5.945561e+001	0.000000	1.526825e-001	0.000000	0.033365
3	2	1.070297e+000	0.018002	6.187752e-003	0.040527	0.036772
4	3	2.627477e-002	0.024549	2.577899e-004	0.041661	0.038335
5	4	7.691734e-004	0.029274	1.085633e-005	0.042113	0.039246
6	5	2.541274e-005	0.033039	4.593210e-007	0.042309	0.039840
7	6	9.121526e-007	0.035894	1.947469e-008	0.042399	0.040256
8	7	3.458310e-008	0.037914	8.265419e-010	0.042442	0.040561
9	8	1.358230e-009	0.039274	3.509771e-011	0.042463	0.040794
10	9	5.456298e-011	0.040172	1.490758e-012	0.042475	0.040978
11	10	2.224324e-012	0.040766	6.332842e-014	0.042481	0.041125
12	11	9.156769e-014	0.041167	2.690464e-015	0.042484	0.041247
13	12	3.794885e-015	0.041443	1.143087e-016	0.042487	0.041349
14	13	1.580225e-016	0.041641	4.856765e-018	0.042488	0.041436
15	14	6.603117e-018	0.041786	2.063606e-019	0.042489	0.041510
16	15	2.766417e-019	0.041896	8.768302e-021	0.042490	0.041575

.

Result for $v_1 = 3, v_2 = 4$

	cycle	|residue|	ratio	|error|	ratio	convergence factor
1						
2	1	4.335915e+001	0.000000	1.321583e-001	0.000000	0.028880
3	2	6.849016e-001	0.015796	4.660361e-003	0.035263	0.031913
4	3	1.472864e-002	0.021505	1.688654e-004	0.036234	0.033293
5	4	3.797836e-004	0.025785	6.183105e-006	0.036616	0.034094
6	5	1.109514e-005	0.029214	2.274057e-007	0.036779	0.034615
7	6	3.520256e-007	0.031728	8.380416e-009	0.036852	0.034978
8	7	1.176545e-008	0.033422	3.091300e-010	0.036887	0.035245
9	8	4.060755e-010	0.034514	1.140828e-011	0.036904	0.035448
10	9	1.429858e-011	0.035212	4.211199e-013	0.036914	0.035608
11	10	5.099387e-013	0.035664	1.554713e-014	0.036919	0.035737
12	11	1.833989e-014	0.035965	5.740235e-016	0.036922	0.035843
13	12	6.634068e-016	0.036173	2.119492e-017	0.036923	0.035932

14	13	2.409612e-017	0.036322	7.826178e-019	0.036925	0.036007
15	14	8.778704e-019	0.036432	2.889877e-020	0.036926	0.036072
16	15	3.205646e-020	0.036516	1.067132e-021	0.036927	0.036128

.

Result for $v_1 = 4, v_2 = 0$

1	cycle	|residue| factor	ratio	|error|	ratio	convergence
2	1	1.305638e+003	0.000000	3.571047e-001	0.000000	0.078787
3	2	7.990465e+001	0.061200	3.078639e-002	0.086211	0.082415
4	3	6.837976e+000	0.085577	2.728251e-003	0.088619	0.084433
5	4	6.150176e-001	0.089941	2.433551e-004	0.089198	0.085600
6	5	5.555443e-002	0.090330	2.176670e-005	0.089444	0.086355
7	6	5.009766e-003	0.090178	1.949873e-006	0.089581	0.086885
8	7	4.509998e-004	0.090024	1.748412e-007	0.089668	0.087277
9	8	4.055972e-005	0.089933	1.568822e-008	0.089728	0.087580
10	9	3.645908e-006	0.089890	1.408358e-009	0.089772	0.087821
11	10	3.276772e-007	0.089875	1.264748e-010	0.089803	0.088017
12	11	2.945016e-008	0.089876	1.136069e-011	0.089826	0.088180
13	12	2.647044e-009	0.089882	1.020666e-012	0.089842	0.088317
14	13	2.379448e-010	0.089891	9.171024e-014	0.089853	0.088434
15	14	2.139097e-011	0.089899	8.241165e-015	0.089861	0.088536
16	15	1.923168e-012	0.089906	7.405975e-016	0.089866	0.088624

.

Result for $v_1 = 4, v_2 = 1$

1	cycle	|residue| factor	ratio	|error|	ratio	convergence
2	1	1.093773e+002	0.000000	1.812869e-001	0.000000	0.039616
3	2	2.098783e+000	0.019188	8.652527e-003	0.047728	0.043484
4	3	5.704834e-002	0.027182	4.249640e-004	0.049114	0.045285
5	4	1.882553e-003	0.032999	2.111018e-005	0.049675	0.046345
6	5	7.106054e-005	0.037747	1.053863e-006	0.049922	0.047039
7	6	2.943509e-006	0.041423	5.273130e-008	0.050036	0.047526
8	7	1.297051e-007	0.044065	2.641368e-009	0.050091	0.047884
9	8	5.950199e-009	0.045875	1.323812e-010	0.050118	0.048158
10	9	2.802074e-010	0.047092	6.636618e-012	0.050133	0.048373
11	10	1.342545e-011	0.047913	3.327617e-013	0.050140	0.048547
12	11	6.507751e-013	0.048473	1.668619e-014	0.050145	0.048690
13	12	3.179931e-014	0.048864	8.367653e-016	0.050147	0.048810
14	13	1.562661e-015	0.049141	4.196279e-017	0.050149	0.048912
15	14	7.710595e-017	0.049343	2.104432e-018	0.050150	0.048999
16	15	3.816096e-018	0.049492	1.055388e-019	0.050151	0.049075

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Result for $v_1 = 4, v_2 = 2$

cycle	|residue|	ratio	|error|	ratio	convergence factor
1	5.956504e+001	0.000000	1.526861e-001	0.000000	0.033366
2	1.070895e+000	0.017979	6.187900e-003	0.040527	0.036773
3	2.628769e-002	0.024547	2.577967e-004	0.041661	0.038335
4	7.695512e-004	0.029274	1.085667e-005	0.042113	0.039246
5	2.542668e-005	0.033041	4.593380e-007	0.042309	0.039841
6	9.127487e-007	0.035897	1.947552e-008	0.042399	0.040256
7	3.460995e-008	0.037918	8.265816e-010	0.042442	0.040561
8	1.359449e-009	0.039279	3.509957e-011	0.042464	0.040794
9	5.461792e-011	0.040177	1.490844e-012	0.042475	0.040978
10	2.226781e-012	0.040770	6.333235e-014	0.042481	0.041126
11	9.167659e-014	0.041170	2.690642e-015	0.042484	0.041247
12	3.799678e-015	0.041447	1.143166e-016	0.042487	0.041349
13	1.582320e-016	0.041644	4.857121e-018	0.042488	0.041436
14	6.612229e-018	0.041788	2.063764e-019	0.042489	0.041510
15	2.770358e-019	0.041897	8.769001e-021	0.042490	0.041575

.

Result for $v_1 = 4, v_2 = 3$

cycle	|residue|	ratio	|error|	ratio	convergence factor
1	4.335295e+001	0.000000	1.321581e-001	0.000000	0.028880
2	6.848903e-001	0.015798	4.660374e-003	0.035264	0.031913
3	1.472868e-002	0.021505	1.688664e-004	0.036235	0.033293
4	3.797890e-004	0.025786	6.183161e-006	0.036616	0.034094
5	1.109540e-005	0.029215	2.274083e-007	0.036779	0.034615
6	3.520365e-007	0.031728	8.380531e-009	0.036852	0.034978
7	1.176590e-008	0.033422	3.091348e-010	0.036887	0.035245
8	4.060938e-010	0.034514	1.140848e-011	0.036905	0.035448
9	1.429932e-011	0.035212	4.211278e-013	0.036914	0.035608
10	5.099674e-013	0.035664	1.554744e-014	0.036919	0.035737
11	1.834100e-014	0.035965	5.740358e-016	0.036922	0.035843
12	6.634495e-016	0.036173	2.119540e-017	0.036923	0.035932
13	2.409774e-017	0.036322	7.826365e-019	0.036925	0.036007
14	8.779315e-019	0.036432	2.889949e-020	0.036926	0.036072
15	3.205874e-020	0.036516	1.067159e-021	0.036927	0.036128

.

Result for $v_1 = 4, v_2 = 4$

cycle	|residue|	ratio	|error|	ratio	convergence

```
      factor
 2 | 1    3.318582e+001      0.000000    1.163430e-001      0.000000    0.025424
 3 | 2    4.677808e-001      0.014096    3.627130e-003      0.031176    0.028154
 4 | 3    8.965233e-003      0.019165    1.161573e-004      0.032025    0.029389
 5 | 4    2.073972e-004      0.023134    3.758056e-006      0.032353    0.030104
 6 | 5    5.448802e-006      0.026272    1.221050e-007      0.032492    0.030567
 7 | 6    1.551551e-007      0.028475    3.974903e-009      0.032553    0.030889
 8 | 7    4.638434e-009      0.029895    1.295102e-010      0.032582    0.031125
 9 | 8    1.427642e-010      0.030779    4.221525e-012      0.032596    0.031306
10 | 9    4.472442e-012      0.031327    1.376361e-013      0.032603    0.031447
11 |10    1.416727e-013      0.031677    4.487966e-015      0.032607    0.031561
12 |11    4.520389e-015      0.031907    1.463521e-016      0.032610    0.031655
13 |12    1.449490e-016      0.032066    4.772759e-018      0.032611    0.031734
14 |13    4.664276e-018      0.032179    1.556520e-019      0.032613    0.031801
15 |14    1.504811e-019      0.032262    5.076339e-021      0.032613    0.031858
16 |15    4.864498e-021      0.032326    1.655598e-022      0.032614    0.031908
```

.

2.5.3 Problem 3

3. Use your V-cycle code to solve

$$\Delta u = -\exp\left(-(x - 0.25)^2 - (y - 0.6)^2\right)$$

on the unit square $(0, 1) \times (0, 1)$ with homogeneous Dirichlet boundary conditions using a grid spacing of 2^{-7}. How many steps of pre and postsmoothing did you use? What tolerance did you use? How many cycles did it take to converge? Compare the amount of work needed to reach convergence with your solvers from Homework 3 taking into account how much work is involved in a V-cycle.

Figure 2.55: problem 3

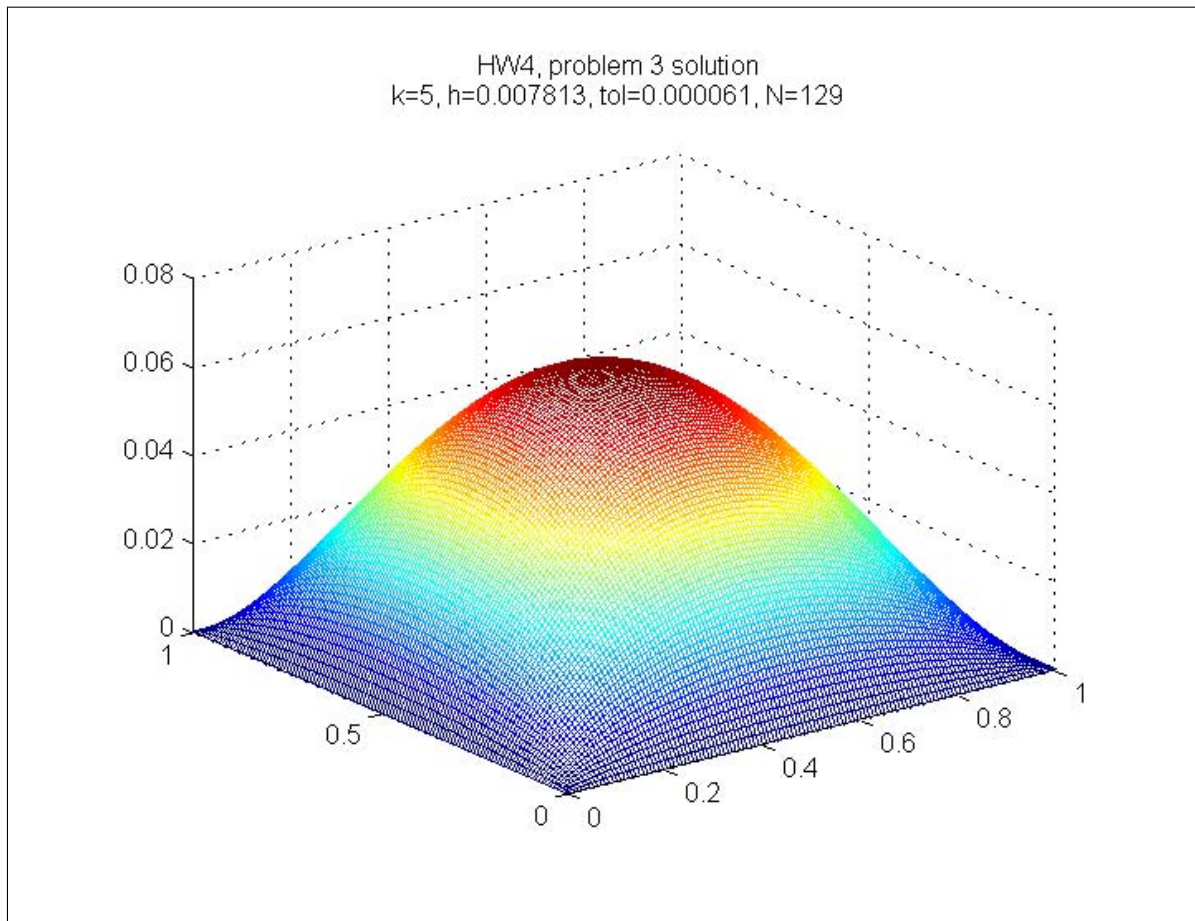This problem was solved using multigrid V cycle method. The following is the solution found

Figure 2.56: problem 3 solutions found earlier

Tolerance used is $h^2 = 0.000061$, the number of grid points along one dimension is $n = 129$, the spacing is 2^{-7}.

V cycle method converged in 5 iterations. The number of pre-smooth is 1 and the number of post smooth is 1. These are selected since in problem 2 it was found they lead to the most efficient solver.

Hence, the amount of work done

$$W = \left(\frac{\log(\varepsilon)}{\log(\rho)} \right) \left(\frac{3}{4} \left(7 \left(v_1 + v_2 \right) + 13 \right) N \right)$$

From the table in problem 2, $\rho = 0.116811$ for the solver $(1,1)$, and given that $N = (n-2)^2 = 127^2 = 16\,129$, hence the above becomes

$$W = \left(\frac{\log_{10}(0.000061)}{\log_{10}(0.116811)} \right) \left(\frac{3}{4} \left(7(1+1) + 13 \right) 16\,129 \right)$$

$$= 1.476\,2 \times 10^6 \text{ operations}$$

The above is compared with the solvers used in HW3, for same tolerance as above and

155

same h, from HW3, the results were the following

method	Number of iterations
Jacobi	31702
Gauss-Seidel	15852
SOR	306

To compare work between all methods, it is required to find the work per iteration for the Jacobi, GS and SOR.

Work per iteration in these methods required only one smooth operation, and one calculation for the residue. No mapping between different grid sizes was needed. Hence, assuming about 13 flops to calculate the averaging and residue per one grid point, work per iteration for the above solver becomes

$$Work\ Per\ iteration = (6N + 7N) = 13N$$

where N is the total number of grid points which is 16129 in this example.

Therefore, total work can be found for all the methods, including the multigrid solver. The following table summarizes the result

method	Number of iterations	W (flops)
Jacobi	31702	$31702 \times 13 \times 16129 = 6.6472 \times 10^9$
Gauss-Seidel	15852	$15852 \times 13 \times 16129 = 3.3238 \times 10^9$
SOR	306	$306 \times 13 \times 16129 = 6.4161 \times 10^7$
Multigrid V Cycle	5	1.4762×10^6

Using the multigrid as a base measure, and normalizing other solvers relative to it, the above becomes

method	work
Jacobi	$\frac{6.6472 \times 10^9}{1.4762 \times 10^6} = 4502.9$
Gauss-Seidel	$\frac{3.3238 \times 10^9}{1.4762 \times 10^6} = 2251.6$
SOR	$\frac{6.4161 \times 10^7}{1.4762 \times 10^6} = 43.464$
Multigrid V Cycle	1

The above shows clearly that Multigrid is the most efficient solver. SOR required about 44 times as much work, GS over 2251 more work, and Jacobi about 4500 more work.

2.5.4 Note on using Full Multigrid cycle (FMG) to improve convergence

It was found that using FMG to determine a better initial guess solution before initiating the V cycle algorithm resulted in about 40% reduction in the number of iterations needed to convergence by the V cycle algorithm.

The following is a plot of one of the tests performed showing the difference in number of iterations needed to converge. All other parameters are kept the same. This shows that with FMG cycle, convergence reached in 4 iterations, while without FMG, it was reached in 7 cycles. The cost of the FMG cycle itself was not taken into account. It is estimated that the FMG correction cycle adds about $\frac{1}{2}$ the cost of one V cycle to the total cost.
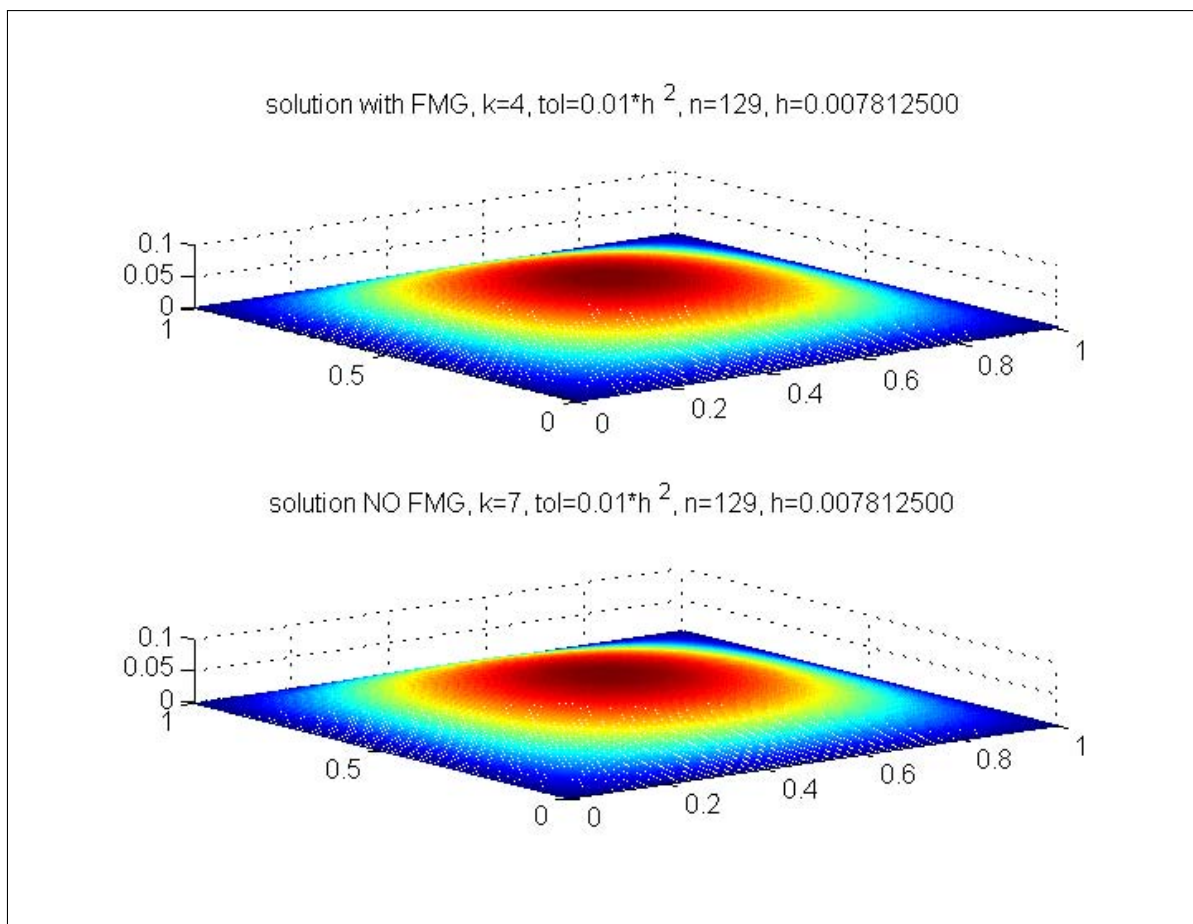

Figure 2.57: compare with FMG

The following is a small function written to compare convergence when using FMG and without using FMG which generated the above result. (see code on web page)

2.5.5 References

[1] Thor Gjesdal, Analysis of a new Red-Black ordering for Gauss-Seidel smoothing in cell-centered multigrid. ref no. CMR-93-A200007, November 1993.

[2] William L. Briggs, A multigrid tutorial, W-7405-Eng-48.

[3] Robert Guy, Lecture notes, Math 228a, Fall 2010. Mathematics dept. UC Davis.

[4] Jim Demmel, Lecture notes, CS267, 1996. UC Berkeley.

[5] P. Wesseling, A survey of Fourier smoothing analysis results. ISNM 98, Multigrid methods II page 105-106.

2.5.6   Source code listing

```matlab
function nma_build_HW4()

list = dir('*.m');

if isempty(list)
    fprintf('no matlab files found\n');
    return
end

for i=1:length(list)
    name=list(i).name;
    fprintf('processing %s\n',name)
    p0 = fdep(list(i).name,'-q');
    [pathstr, name_of_matlab_function, ext] = fileparts(name);

    %make a zip file of the m file and any of its dependency
    p1=dir([name_of_matlab_function '.fig']);
    if length(p1)==1
        files_to_zip =[p1(1).name;p0.fun];
    else
        files_to_zip =p0.fun;
    end

    zip([name_of_matlab_function '.zip'],files_to_zip)

end

end
```

```matlab
%---------------------------------------------------------
% This function is the prologation operator for 1D
% it takes 1D coarse grid of spacing 2h and generate 1D fine
% grid of spacing h by linear interpolation
%
% INPUT:
%   c the coarse 1D grid, spacing 2h
% OUTPUT:
%   f the fine 1D grid, spacing h
%
% EXAMPLE:
% nma_c2f_1D( [0 1 0] )
%         0    0.5000    1.0000    0.5000         0
```

```matlab
14  %
15  % Nasser M. Abbasi
16  % Math 228a, UC Davis fall 2010
17  function  f = nma_c2f_1D( c )
18  if ndims(c) > 2
19      error('nma_c2f_1D:: input number of dimensions too large');
20  end
21
22  [n,nCol]  = size(c(:));
23  if nCol>1
24      error('nma_c2f_1D::input must be a vector');
25  end
26
27  valid_grid_points = log2(n-1);
28  if  round(valid_grid_points) ~= valid_grid_points
29      error('nma_c2f_1D:: invalid number of grid points value');
30  end
31
32  if n < 3
33      error('length of coarse grid must be at least 3');
34  end
35
36  fine_n     = 2*(n-2) + 3;
37  f          = zeros(fine_n ,1);
38  f(1:2:end) = c;
39  indx       = 2:2:fine_n;
40  f(indx)    = ( f(indx-1) + f(indx+1) )/2;
41  f          = f(:)';
42  end
```

```matlab
1   %-------------------------------------------
2   % function for debugging
3   %
4   function nma_DEBUG(msg)
5   debug = false;
6   if debug
7
8       fprintf(msg);
9       fprintf('\n');
10  end
11  end
```

```matlab
1   %---------------------------------------------------------
2   % This function is the restriction full weight operator
3   % for mapping 1D fine grid to 1D coarse grid
4   %
```

```matlab
 5  % INPUT:
 6  %    f the fine 1D grid, spacing h
 7  % OUTPUT:
 8  %    c the corase 1D grid, spacing 2h
 9  %
10  % EXAMPLE
11  %nma_c2f_1D( [0 1 0] )
12  %          0     0.5000     1.0000     0.5000          0
13  %nma_f2c_1D(ans)
14  %          0     0.7500          0
15  %
16  % Nasser M. Abbasi
17  % Math 228a, UC Davis fall 2010

18
19  function c = nma_f2c_1D( f )
20  if ndims(f) > 2
21      error('nma_f2c_1D:: input number of dimensions too large');
22  end
23
24  [n,nCol]  = size(f(:));
25  if nCol>1
26      error('nma_f2c_1D::input must be a vector');
27  end
28
29  valid_grid_points = log2(n-1);
30  if  round(valid_grid_points) ~= valid_grid_points
31      error('nma_f2c_1D:: invalid number of grid points value');
32  end
33
34  if n < 5
35      error('length of fine grid cant be smaller than 5');
36  end
37
38  if 2*floor(n/2) == n
39      error('length of fine grid must be odd number');
40  end
41
42  c          = zeros((n+1)/2,1);
43  indx       = 3:2:n-2;
44  c(2:end-1) = (1/4)*f(indx-1) + (1/2)*f(indx) + (1/4)*f(indx+1);
45  c          = c(:)';
46  end
```

```matlab
1  function  nma_HW4_residue_animation
2
3  myforce = @(X,Y) -exp( -(X-0.25).^2 - (Y-0.6).^2 ); % RHS of PDE
4  N = 6;          % number of levels
```

```
5   n = 2^N+1;      % total number of grid points, always odd number
6   h = 1/(n-1);    % mesh spacing
7   [X,Y] = meshgrid(0:h:1,0:h:1);  % for plotting solution
8
9   % use a problem whose solution is known Lu=0, since B.C. are zero
10  f      = myforce(X,Y);    %  RHS of the problem
11  u      = 0.*X;  % initial guess of solution, use random
12
13  %-- select number of cycle to run. choose number not too large
14  number_of_cycles =10;
15
16  METHOD  = 1;   % GSRB
17  mu1=2; mu2=1;
18  k=0;
19
20
21  for i = 1:number_of_cycles
22
23      k = k+1;
24
25      u = nma_V_cycle(u,f,mu1,mu2,METHOD);  %--- CALL V CYCLE
26      the_residue  = nma_find_residue(u,f);
27
28      mesh(X,Y, u);  drawnow; title(sprintf('k=%d\n',k));
29      pause(.01);
30  end
31  end
```

```
1   %----------------------------------------
2   %This function perform full mutligrid cycle
3   %to determine a good initial guess to use
4   %for initial solution to the V cycle interative
5   %solver
6   %
7   % INPUT
8   %   f: 2D grid, the force on the orginal fine grid
9   % OUTPUT:
10  %   u: estimated solution on the fine grid obtained
11  %       by running FMG cycle once
12  %
13  % Example use: see nma_test_FMG.m
14  %
15  % by Nasser M. Abbasi
16  % Math 228a, UC Davis, Fall 2010
17
18  function u = nma_initial_solution_guess_using_FMG( f )
19
```

```matlab
20  nma_validate_dimensions_1(f); %asserts dimensions make sense
21      [n,~] = size(f);
22      if n<3
23          error('initial_solution_guess_using_FMG:: size of f too small');
24      end
25
26      mu1 = 1;
27      mu2 = 1;
28      smoother = 1; %GSRB
29
30      k = 3;
31      while k <= n
32          fk = f;
33          [current_size,~] = size(fk);
34          while current_size ~= k
35                  fk = nma_f2c(fk);
36                  [current_size,~] = size(fk);
37          end
38
39          if k == 3
40              u       = zeros(3,3);
41              u(2,2) = -0.25*(0.5)^2*fk(2,2);
42          else
43            u = nma_c2f(u);
44            u = nma_V_cycle(u,fk,mu1,mu2,smoother);
45          end
46          k = 2*k - 1;
47      end
48
49  end
```

```matlab
1  %--------------------------------------------------------
2  % This function solves problem 2 in HW4
3  %
4  %LOGIC
5  %
6  %  LOOP over all combinations of mu1 and mu2
7  %       run V cycle 15 times on the problem
8  %       record the result in table, which contains
9  %       average convergence factor
10 %  END LOOP
11 %
12 % By Nasser M. Abbasi
13 % Math 228A, UC Davis
14 function nma_math_228_fall_2010_HW4_problem2
15
16 N = 6;           % number of levels
```

```
17  n = 2^N+1;      % total number of grid points, always odd number
18  h = 1/(n-1);   % mesh spacing
19
20  [X,Y] = meshgrid(0:h:1,0:h:1);
21  % use a problem whose solution is known Lu=0, since B.C. are zero
22
23  %force = @(X,Y) -2* ( (1-6*X.^2).*Y.^2.*(1-Y.^2)+(1-6*Y.^2).*X.^2.*(1-X.^2)); % RHS of PDE
24  %force = @(X,Y) -exp( -(X-0.25).^2 - (Y-0.6).^2 );  % RHS of PDE
25  %exact = @(X,Y) (X.^4-X.^2).*(Y.^4-Y.^2);
26  %f = force(X,Y);
27
28  table = zeros(20,6);
29  f      = zeros(n,n);    %  RHS of the problem
30  exact = zeros(n,n);    %  exact solution
31
32  initial_guess = rand(n,n);  % initial guess of solution, use random
33  initial_guess(:,1)=0; initial_guess(:,end)=0; %initialize B.C. to zero
34  initial_guess(1,:)=0; initial_guess(end,:)=0;
35
36  %-- select mu1 and mu2, these are the number of pre-smooth and
37  %-- number of post smooth to be used inside the V cycle function
38  mu1_values = 0:2;
39  mu2_values = 0:2;
40
41  %-- select number of cycle to run. choose number not too large
42  number_cycles = 10;
43
44  %-- select where to send the result, either to stdout or to a file
45  %fileID = fopen('table_result_work','w');
46  fileID  = 1;
47  METHOD  = 1;   % GSRB
48
49  for i = 1:length(mu1_values)
50
51      mu1 = mu1_values(i);
52      for j = 1:length(mu2_values)
53          mu2   = mu2_values(j);
54
55          table = run_V_cycle(initial_guess, ...
56              exact,...
57              f,...
58              mu1,...
59              mu2,...
60              number_cycles,...
61              h,...
62              METHOD);
63
```

```matlab
64            print_table(table,mu1,mu2,fileID);
65        end
66  end
67  %fclose(fileID);
68  end
69
70  %-------------
71  % This function runs the V cycle for k times, recording
72  % the average convergence rate at each step k. In the end
73  % it returns table containing the results found
74  %
75  %
76  function table = run_V_cycle(u,exact,f,mu1,mu2,number_of_cycles,h,METHOD)
77
78  %-- initialization of data and storage
79  table = zeros(number_of_cycles,7);  %initialize table for result collection
80
81  last_error_norm   = 0;
82  last_residue_norm = 0;
83  initial_error_norm = nma_find_norm(u-exact);
84  k=0;
85  [X,Y] = meshgrid(0:h:1,0:h:1);  % for plotting solution
86
87  for i = 1:number_of_cycles
88
89      k = k+1;
90
91      u = nma_V_cycle(u,f,mu1,mu2,METHOD);  %--- CALL V CYCLE
92
93      e = exact-u ;
94      norm_error   = nma_find_norm(e);
95      the_residue  = nma_find_residue(u,f);
96      norm_residue = nma_find_norm(the_residue);
97
98      % fill in table for analysis
99      table(k,1) = k;
100     table(k,2) = norm_residue;
101     table(k,4) = norm_error;
102     table(k,6) = (norm_error/initial_error_norm)^(1/k);
103
104     if k > 1
105         table(k,3) = norm_residue/last_residue_norm;
106         table(k,5) = norm_error/last_error_norm;
107         table(k,7) = -6/log10(table(k,6))*(3/4)*(7*(mu1+mu2)+13);
108     end
109
110     last_error_norm   = norm_error;
```

```matlab
111        last_residue_norm = norm_residue;
112
113        mesh(X,Y, u);  drawnow; title(sprintf('k=%d\n',k));
114    end
115    end
116    %---------------------------------------------
117    % This function is called by HW4 problem 2 solver
118    % to format the results and print it. THe results
119    % are included in the final report.
120    %
121    % this also makes plots of the data
122    %
123    function print_table(table,mu1,mu2,fileID)
124
125    fprintf(fileID,'mu1=%d, mu2=%d\n',mu1,mu2);
126
127    % figure;
128    % subplot(5,1,1);
129    % plot(table(:,3),'-o');
130    % title('residual ratio');
131    %
132    % subplot(5,1,2);
133    % plot(table(:,5),'-o');
134    % title('error ratio');
135    %
136    % subplot(5,1,3);
137    % plot(table(:,2),'-o');
138    % title('residual norm');
139    %
140    % subplot(5,1,4);
141    % plot(table(:,4),'-o');
142    % title('error norm');
143    %
144    % subplot(5,1,5);
145    % plot(table(:,6),'-o');
146    % title('RAO(M)');
147
148    titles={'V-cycle','|residue|','ratio','|error|','ratio','C.F','work'};
149    wid    = 16;
150    fms    = {'d','.6e','.6f','.6e','.6f','.6f','.1f'};
151
152    nma_format_matrix(titles,table,wid,fms,fileID,true);
153
154    end
```

```matlab
1    %----------------------------------------------------------
2    % This function solves problem 3 in HW4
```

```matlab
3   % It uses the code generated in problem 1, which implements
4   % V cycle.
5   % By Nasser M. Abbasi
6   % Math 228A, UC Davis
7
8   function nma_math_228_fall_2010_HW4_problem3
9
10  %----------- INITIALIZATION SECTION -------------
11  myforce = @(X,Y) -exp( -(X-0.25).^2 - (Y-0.6).^2 ); % RHS of PDE
12  h       = 2^-7;          % mesh spacing
13  n       = 1/h+1;         % total number of grid points, always odd number
14  tol     = 1*h^2;         % tolerance used
15  [X,Y]   = meshgrid(0:h:1,0:h:1); % coordinates
16  f       = myforce(X,Y); % evaluate the force on the grid
17  u       = zeros(n,n);    % initial guess of solution
18  mu1     = 1;             % mu1 number of pre-smooth
19  mu2     = 1;             % mu2 number of post-smooth
20  METHOD  = 1;             % relaxation method for multigrid: Gause-Seidel B/R
21
22  [k, u ] = nma_solver_Vcycle(u,f,mu1,mu2,METHOD,tol,false);
23
24  [X,Y] = meshgrid(0:h:1,0:h:1); % coordinates
25  mesh(X,Y, u);  drawnow; title(sprintf( ...
26     ('HW4, problem 3 solution\nk=%d, h=%f, tol=%f, N=%d\n',k,h,tol,n));
27
28  end
```

```matlab
1   %---------------------------------------
2   %This function compares the performance of multigrid
3   % with and without a FMG initial cycle by solving HW3 poisson
4   % problem and comparing the number of iterations needed
5   % to converge when using V cycle.
6   %
7   % by Nasser M. Abbasi
8   % Math 228a, UC Davis, Fall 2010
9   function nma_math_228_fall_2010_HW4_test_FMG()
10  %----------- INITIALIZATION SECTION -------------
11  myforce = @(X,Y) -exp( -(X-0.25).^2 - (Y-0.6).^2 ); % RHS of PDE
12  h       = 2^-7;          % mesh spacing
13  n       = 1/h+1;         % total number of grid points, always odd number
14  tol     = 0.01*h^2;      % tolerance used
15  [X,Y]   = meshgrid(0:h:1,0:h:1); % coordinates
16  f       = myforce(X,Y); % evaluate the force on the grid
17  mu1     = 1;             % mu1 number of pre-smooth
18  mu2     = 1;             % mu2 number of post-smooth
19  METHOD  = 1;             % relaxation method for multigrid: Gause-Seidel B/R
20
```

```matlab
21
22 %----------- LOGIC SECTION -------------------
23
24 % Solve by doing initial FMG correction
25 u     = nma_initial_solution_guess_using_FMG(f);
26 [k,u] = nma_solver_Vcycle(u,f,mu1,mu2,METHOD,tol,false);
27 subplot(2,1,1);
28 mesh(X,Y, u);
29 title(sprintf('solution with FMG, k=%d, tol=%s, n=%d, h=%0.9f\n',k, ...
30             '0.01*h^2',n,h));
31
32 % Solve without doing an initial FMG correction
33 u     = zeros(n);
34 [k,u] = nma_solver_Vcycle(u,f,mu1,mu2,METHOD,tol,false);
35 subplot(2,1,2);
36 mesh(X,Y, u);
37 title(sprintf('solution NO FMG, k=%d, tol=%s, n=%d, h=%0.9f\n',...
38             k,'0.01*h^2',n,h));
39
40 end
```

2.6 HW 5

2.6.1 Introduction

Math 228A
Homework 5
Due Friday, 12/10/10, 4:00 P.M.

Homework must be turned in to Arcade before the deadline. You may email him a pdf file or put a hard copy in his mailbox.

Exam week office hours:
Bob, Monday 12-1
Arcade, Tuesday & Wednesday 1:30-2:20

1. Write a program to solve the discrete Poisson equation on the unit square using preconditioned conjugate gradient. Set up a test problem and compare the number of iterations and efficiency of using (i) no preconditioning and (ii) SSOR preconditioning. Run your tests for different grid sizes. How does the number of iterations scale with the number of unknowns as the grid is refined?

   Note that there are two typos in the PCG algorithm in our textbook. See your class notes, another textbook, or the author's webpage for the correct algorithm.

   SSOR preconditioning Symmetric SOR (SSOR) consists of one forward sweep of SOR followed by one backward sweep of SOR. For the discrete Poisson equation, one step of SSOR is

   $$u_{i,j}^{k+1/2} = \frac{\omega}{4}(u_{i-1,j}^{k+1/2} + u_{i,j-1}^{k+1/2} + u_{i+1,j}^{k} + u_{i,j+1}^{k} - h^2 f_{i,j}) + (1 - \omega)u_{i,j}^{k}$$
   $$u_{i,j}^{k+1} = \frac{\omega}{4}(u_{i-1,j}^{k+1/2} + u_{i,j-1}^{k+1/2} + u_{i+1,j}^{k+1} + u_{i,j+1}^{k+1} - h^2 f_{i,j}) + (1 - \omega)u_{i,j}^{k+1/2}.$$

   It can be shown that one step of SSOR in matrix form is equivalent to

   $$\frac{1}{\omega(2 - \omega)}(D - \omega L)D^{-1}(D - \omega U)(\boldsymbol{u}^{k+1} - \boldsymbol{u}^{k}) = \boldsymbol{f},$$

   where $A = D - L - U$.

   For the constant coefficient problem, this suggests the preconditioner.

   $$M = (D - \omega L)(D - \omega U).$$

   Note: If you are interested, experiment with incomplete Cholesky factorization preconditioning and multigrid preconditioning. Incomplete Cholesky preconditioning requires that you form the matrix. Vary the amount of fill (in MATLAB use `cholinc` and vary the drop tolerance). Obviously, a factorization with more elements results in fewer iterations of CG, but it is more expensive to compute and to apply the preconditioner. To use MG as a preconditioner, the product $M^{-1}r$ is computed by applying one V-cycle with zero initial guess with right hand side r. If the smoother is symmetric and the number of pre and post smoothing steps are the same, this preconditioner is symmetric positive definite and may be used with CG.

Figure 2.58: problem description

The test problem used is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

on the unit square $[(0,1),(0,1)]$ with zero boundary conditions.

The above problem is solved using the numerical method of conjugate gradient iterative solver. The mesh spacings used is

$$h = \left\{ \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128} \right\}$$

and the tolerance used to check for convergence is

$$\varepsilon = 10^{-6}$$

The solver terminates when the mesh norm of the residual becomes smaller than the above quantity using the following check

$$\sqrt{h} \left\| r^{(k)} \right\|_2 < \varepsilon$$

Where in the above, $r^{(k)}$ is the residual at the kth iteration and h is the current value of the mesh spacing.

The reason for using *zero* as the driving force on the RHS of the pde, is to allow the calculation and tracking of the error at each iteration as the exact solution u_{exact} for this problem is now known, which is zero. Now the error at each iteration k to be found using

$$e^{(k)} = \left\| u_{exact} - u^{(k)} \right\|$$
$$= \left\| u^{(k)} \right\|$$

Where in the above $u^{(k)}$ represents the approximate solution at the kth iteration.

A Matlab function *CG.m* was written to implement conjugate gradient solver. One of the parameters this function accepts is the name of the preconditioner to use. The following preconditioners are supported

NONE, SSOR, MultiGrid, IncompleteCholesky

When NONE is specified, then no preconditioning is done.

For each preconditioner, the solver was run to find the solution to the above test problem. The initial guess for the solution $u^{(0)}$ used was generated using Matlab rand() function.

For each preconditioner the following plots were generated

1. Plot of error $\left\| e^{(k)} \right\|$ per iteration k which showed how the rate of error reduction per iteration. The plot was generated in log and linear scale.

2. Plot of the residual $\left\| r^{(k)} \right\|$ per iteration which showed the rate of residual norm reduction per iteration, and also plotted in log and linear scale. The initial residual is defined as $r^{(0)} = f - Au^{(0)}$ and each subsequent iteration, the residual is defined as $r^{(k)} = r^{(k-1)} - \alpha A p^{(k-1)}$. The algorithm below illustrates this in more details.

3. The spectrum of the eigenvalues of A and the spectrum of the eigenvalues of $M^{-1}A$ are plotted using matlab's scatter() command to better see the effect on the condition number value when multiplying A by M^{-1}, where M is the preconditioner matrix.

4. Plot of the final solution found on a 3D mesh plot. The final solution was verified to be close to zero, which is the same as the exact solution.

In addition to the above plots, for each mesh spacing h, the actual result table is printed which tabulates the above values at each iteration. This table was used to generate the above plots. The printed tables also show the ratio of the value of norm of the residual at the current iteration to its value at the previous iteration, similarly for the error norm.

Due to the large size of these tables, the tables for all the spacings and for each solver are available in the appendix.

Conjugate gradient algorithm description

The idea of conjugate gradient is to use preconditioning matrix to speed up the convergence of the conjugate gradient method. The original problem

$$Ax = f$$

is transformed to a new problem

$$M^{-1}Ax = M^{-1}f$$

such that $M^{-1}A$ has a smaller condition number than A. For most iterative solvers, the rate of convergence increases as the condition number of the system matrix A decreases.

The conjugate gradient method works only on symmetric positive definite A matrix, and its speed of convergence is affected by the distribution of the eigenvalues of the A matrix. The estimate of convergence is more accurate if the distribution of eigenvalues is uniform. For the discrete 2D Poisson problem, this is the case, as verified by plots of the spectrum generated below for each case.

The preconditioning is used to modify the spectrum of A so that the eigenvalues of the new system matrix $M^{-1}A$ become more clustered together causing the condition number to become smaller and thus increasing the convergence rate.

The following table was generated to show the effect of preconditioning on lowering the condition number. It shows the condition number for CG (in other words, for the A matrix only), and then the condition number for $M^{-1}A$ for different solvers as mesh spacing is changed. It also shows below the condition number value, in a box, the maximum eigenvalue and the minimum eigenvalue. Notice that in the following table, if one tries to apply the $\frac{|\max \lambda|}{|\min \lambda|}$ to determine the condition number, then the result will not match the condition number as shown. The above formula do not apply in this case, as these are sparse matrices and the condition number was found by estimating its value using the Matlab function condest() and not by applying the above formula.

solver	$h = \frac{1}{16}, N = 225$	$h = \frac{1}{32}, N = 961$	$h = \frac{1}{64}, N = 3969$	$h = \frac{1}{128}, N = 16129$
NONE	$150, (7.923, 0.0768)$	$603, (7.9807, 0.01926)$	$2413, (7.9995, 0.0048)$	$9655, (7.9987, 0.001205)$
SSOR	$33, (0.25, 0.018203)$	$128, (0.25, 0.004748)$	$511, (0.25, 0.0012)$	
IncCholesky $\varepsilon = 10^{-2}$	$32, (2.365, 0.2279)$	$108, (2.44, 0.0585)$	$422, (2.537, 0.0146)$	
IncCholesky $\varepsilon = 10^{-3}$	$48, (2.337, 0.508)$	$153, (2.526, 0.1603)$	$373, (2.592, 0.041756)$	

This diagram below reflects the above table result to clearly show the reduction of the condition number as a result of preconditioning.
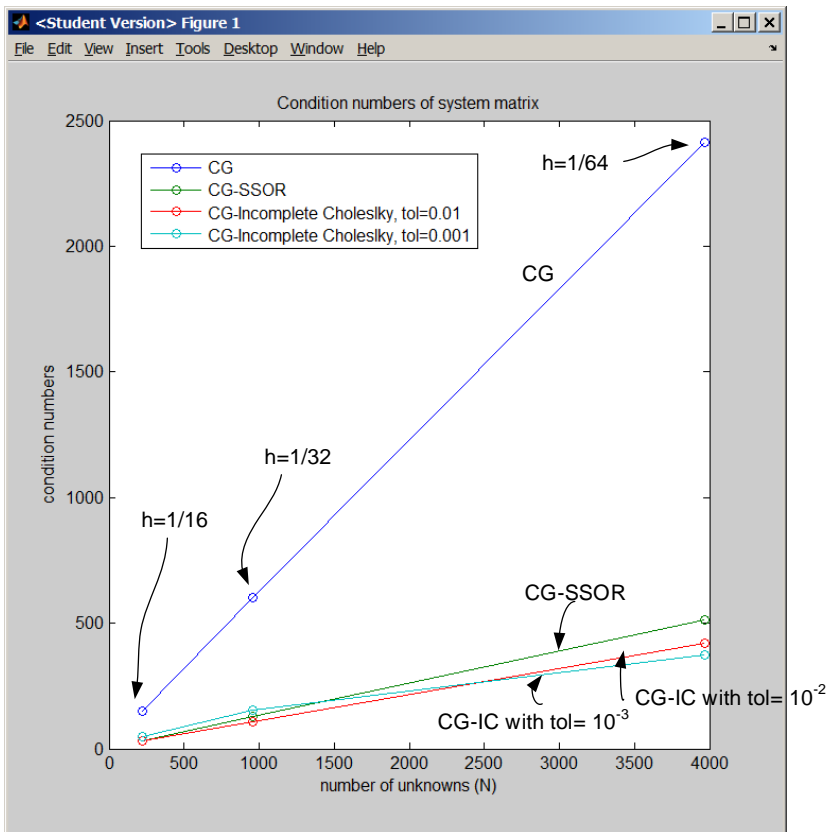

Figure 2.59: compare condition numbers

CG Algorithm pseudocode

The following is the algorithm used for the implementation of conjugate gradient with precondition-

ing.

> Input : $A, f, tol, preconditionSolverName, dropTol$
>
> Output : $\tilde{u}$ approximate solution to $Ax = 0$
>
> $u_0 = \text{rand}()$ (*initial guess of solution* *)
>
> $r_0 = f - Au_0$ (*initial residual* *)
>
> $z_0 \leftarrow CALL \ \ preconditionSolver(r_0, A, preconditionSolverName, dropTol)$
>
> $p_0 = z_0$
>
> FOR $k = 1, 2, 3, \cdots$
>
> $\qquad e_{k-1} = \sqrt{h}\, \|u_{k-1}\|_2$ (* the error since u_{exact} is known to be zero* *)
>
> $\qquad \omega_{k-1} = Ap_{k-1}$
>
> $\qquad \alpha_{k-1} = \dfrac{z_{k-1}^T r_{k-1}}{p_{k-1}^T \omega_{k-1}}$
>
> $\qquad u_k = u_{k-1} + \alpha_{k-1}\, p_{k-1}$
>
> $\qquad r_k = r_{k-1} - \alpha_{k-1}\omega_{k-1}$
>
> $\qquad IF \ \left(\sqrt{h}\,\|r_k\|_2\right) < tol \ THEN$
>
> $\qquad\qquad RETURN \ u_k$
>
> $\qquad END \ IF$
>
> $\qquad z_k \leftarrow CALL \ \ preconditionSolver(r_k, A, preconditionSolverName, dropTol)$
>
> $\qquad \beta_{k-1} = \dfrac{z_k^T r_k}{z_{k-1}^T r_{k-1}}$
>
> $\qquad p_k = z_k + \beta_{k-1} p_{k-1}$
>
> END FOR

The algorithm for the function *preconditionSolver*() is as follows

> Input : $r, A, preconditionSolverName, dropTol$
>
> Output : z approximate solution to $Mz = r$
>
> CASE preconditionSolverName IS
>
> $\qquad WHEN \ NONE \ z \leftarrow r$ $\qquad$ // no preconditioning
>
> $\qquad WHEN \ MultiGrid$
>
> $\qquad\qquad \mu_1 = \mu_2 = 1$ (* presmoother and postsmoother* *)
>
> $\qquad\qquad z \leftarrow CALL \ VCYCLE(zeroInitialGuess, r)$
>
> $\qquad\qquad\quad$ //VCYCLE is one implemented in HW4 but changed to do
>
> $\qquad\qquad\quad$ //one forward Gauss − Seidel/red − black followed by
>
> $\qquad\qquad\quad$ //one reverse Gauss − Seidel/red − black
>
> $\qquad WHEN \ \ SSOR$
>
> $\qquad\qquad z \leftarrow CALL \ SOR \ forward \ followed \ by \ SOR \ in \ reverse$
>
> $\qquad WHEN \ IncompleteCholesky$
>
> $\qquad\quad R = cholinc(A, dropTol)$
>
> $\qquad\quad z \leftarrow R\backslash(R^T\backslash r)$
>
> END CASE
>
> RETURN z

2.6.2 Solvers efficiency and iterations count

In addition to finding the number of iterations needed for convergence by each solver, the problem also asked to compare the efficiency of each solver. This is done by finding the work needed by each solver to converge.

Work needed is defined as

$$Work = NumberOfIterations \times WorkPerIteration$$

Before determining the work for each solver, the following table lists the *cputime* used by each solver for the different spacings. The cpu time is measured using Matlab cputime function, and measures only the call to CG() and does not include any other calls such as plotting.

preconditioning	$h = \frac{1}{16}$ $N = 225$	$h = \frac{1}{32}$ $N = 961$	$h = \frac{1}{64}$ $N = 3969$	$h = \frac{1}{128}$ $N = 16129$
NONE	0.19	0.37	13.48	556.6
Multigrid	0.34	0.6	13.48	566
SSOR	0.23	0.5	13.3	564.6
Incomplete Cholesky $\varepsilon = 10^{-2}$	0.16	0.34	13.8	559
Incomplete Cholesky $\varepsilon = 10^{-3}$	0.19	0.5	13.3	559

Surprisingly, no appreciable difference can be seen between the different solvers in terms of cpu time. It was expected that NONE would have the largest CPU time as it has the lowest efficiency. This result can be attributed to using small number of N values, which was not large enough in the limit to reflect the difference. One needs to use much larger values of N to see the effect of preconditioning on CPU time difference. Due to memory limitation, this was not possible to implement at this time. Now the work per iteration is analyzed.

2.6.2.1 Work per iteration

All solvers perform similar work per iterations except for the step needed to apply the preconditioning to determine z_k. The only difference between not using preconditioning and using one, is in the step to solve for z in $Mz = r$. Using work per iteration as $O(N)$ for the base CG with no preconditioning, then the following can be defined for work per iteration for each solver:

1. No preconditioner is applied: no extra work is needed, as z is the same as r hence $O(N)$

2. *multigrid* : work needed to determine z adds an extra cost of one V cycle. Work for one V cycle was found from HW4 to be $\frac{4}{3}C \times N$ where N is number of unknowns and C is a constant estimated to be $(7(v_1 + v_2) + 13)$ where v_1, v_2 are the numbers of pre smooth and post smooth operations. These are both *one* in this case. The smoothing is done twice (forward and reverse), hence the above becomes $(2 \times 7(v_1 + v_2) + 13)$, resulting in work per iteration of $\frac{4}{3}(2 \times 7(v_1 + v_2) + 13)N$ or about $55N$. Adding the $O(N)$ from the above, this is still results in $O(N)$.

3. *SSOR* : The cost is twice one SOR step. One step of SOR work is $7N$, where N is number of unknowns, since it takes about 7 flop operations to smooth one grid point, and there are N of these. Hence for SSOR work is twice that or $14N$. As above, this is still an order of N.

2.6.2.2 Number of iterations

From lectures notes, it was found that the error rate in conjugate gradient (with no preconditioning) behaves as

$$\|e_k\|_A \le 2 \frac{\sqrt{\Bbbk(A)} - 1}{\sqrt{\Bbbk(A)} + 1} \|e_0\|_A$$

Where $\Bbbk(A)$ is the condition number of A, it was shown that $\Bbbk(A) = O\left(h^{-1}\right)$ where h is the mesh spacing. Hence, for fixed tolerance, which is the case here, the number of iterations is $O\left(h^{-1}\right) = O\left(N^{1/2}\right)$ where N is number of unknowns.

The results of the numerical experiment done agrees with the above, as shown below, for the case of NONE (which is the case of conjugate gradient with no preconditioning).

The following table was generated from result of running the program. In this table, N is the number of unknowns, $\Bbbk\left(M^{-1}A\right)$ is the condition number of $M^{-1}A$, and $\Bbbk(A)$ is the condition number of A. Since sparse matrices are used, the Matlab function condest() was used to find the condition numbers. In this table I.C. means Incomplete Cholesky

preconditioner	$h = \frac{1}{16}$ $N = 225$	$\Bbbk\left(M^{-1}A\right)$	$\Bbbk(A)$	$h = \frac{1}{32}$ $N = 961$	$\Bbbk\left(M^{-1}A\right)$	$\Bbbk(A)$
NONE	42	N/A	150	82	N/A	603
Multigrid	4		150	4		603
SSOR	18	33	150	30	128	603
IC $\varepsilon = 10^{-2}$	7	32	150	13	108	603
IC $\varepsilon = 10^{-3}$	4	48	150	6	153	603

preconditioner	$h = \frac{1}{64}$ $N = 3969$	$\Bbbk\left(M^{-1}A\right)$	$\Bbbk(A)$	$h = \frac{1}{128}$ $N = 16129$	$\Bbbk\left(M^{-1}A\right)$	$\Bbbk(A)$
NONE	157	N/A	2413	291	N/A	9655
Multigrid	4		2413	4		9655
SSOR	56	511	2413	103	Memory problem	9655
IC $\varepsilon = 10^{-2}$	22	422	2413	39	Memory problem	9655
IC $\varepsilon = 10^{-3}$	8	373	2413	14	Memory problem	9655

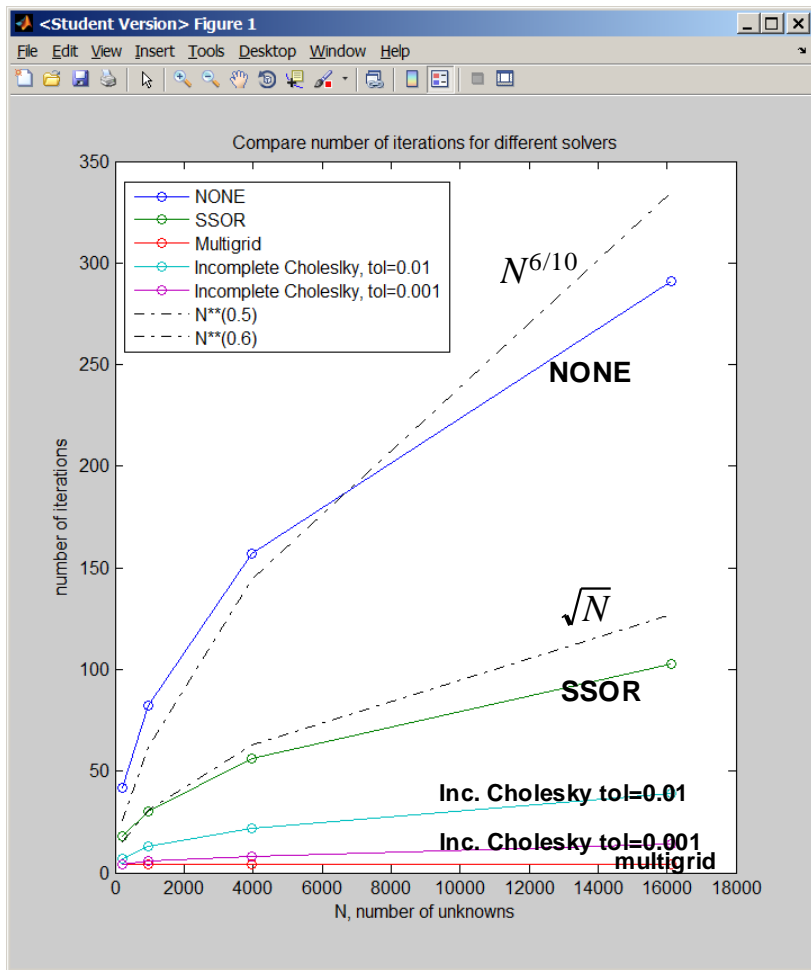The following is a plot that represents the above results.

Figure 2.60: iterations plot

From the above one can see that multigrid has $O(1)$ for the number of iterations. The number of iterations was 4 for all the cases from $h = \frac{1}{16}$ to $h = \frac{1}{128}$. For SSOR, the number of iterations grew sublinear in terms of N, from the above one can estimate this to be $O\left(N^{1/4}\right)$, while for no preconditioning, the number of iterations grew as approximately as $O\left(N^{\frac{1}{2}}\right)$ as predicted by earlier analytical result.

2.6.3 Discussion of results and conclusions

The use of preconditioning on A caused a reduction of the number of iterations to convergence to the same fixed tolerance when compared to convergence with no preconditioning. This was due to reduction of the condition number of the system matrix as can be seen in the above table. By reducing the largest eigenvalue, the rate of convergence increased. However, preconditioning also adds an extra cost per iteration. The extra work however, was also of order N and hence the final efficiency was governed by the number of iterations for large N.

Therefore, this is the result of work efficiency for the main solvers, using the formula of

$$\text{Work} = \text{NumberOfIterations} \times \text{WorkPerIteration}$$

1. NONE: $O\left(N^{1/2}\right) \times O\left(N\right) = O\left(N^{3/2}\right)$

2. SSOR: $O\left(N^{1/4}\right) \times O\left(N\right) = O\left(N^{5/4}\right)$

3. Multigrid: $O\left(1\right) \times O\left(N\right) = O\left(N\right)$

Incomplete Cholesky was not added as it was hard to estimate from the curve above the number of iterations and since depend on the drop tolerance values.

The above analysis showed that **Multigrid is most efficient**, followed by Incomplete Cholesky (but these depend on the tolerance drop term), followed by SSOR, and finally by CG with no preconditioning

In conclusion, using Multigrid for preconditioner for conjugate gradient seems to be the most effective solver.

2.6.4 Appendix
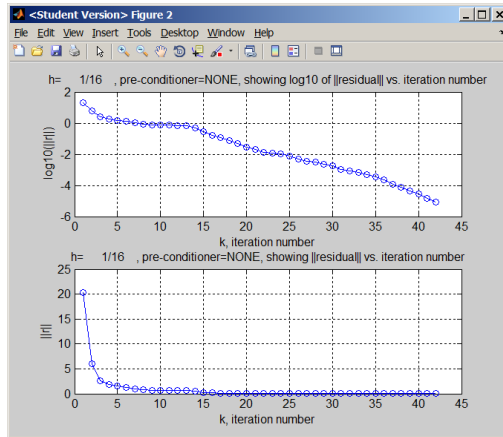
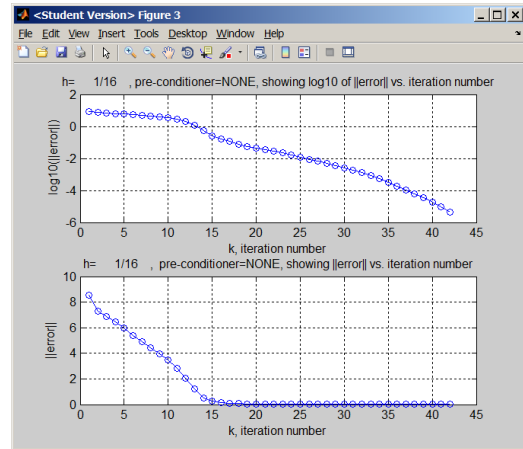#### 2.6.4.1 Result for CG with no preconditioner

Plots h=1/16
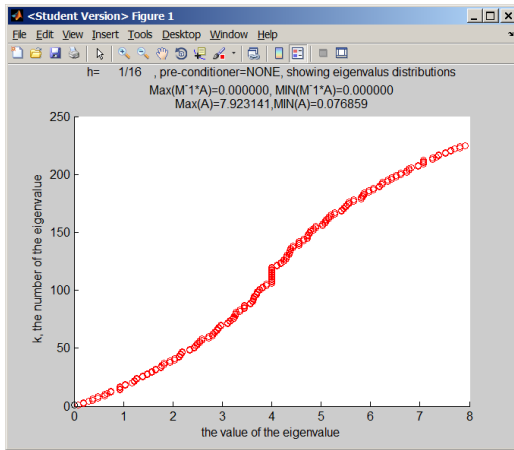
Eigenvalues of A
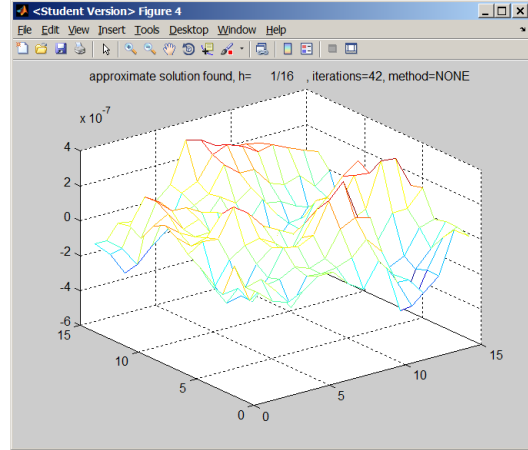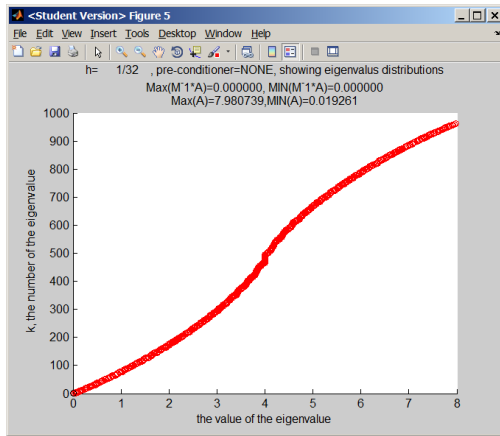
Final solution

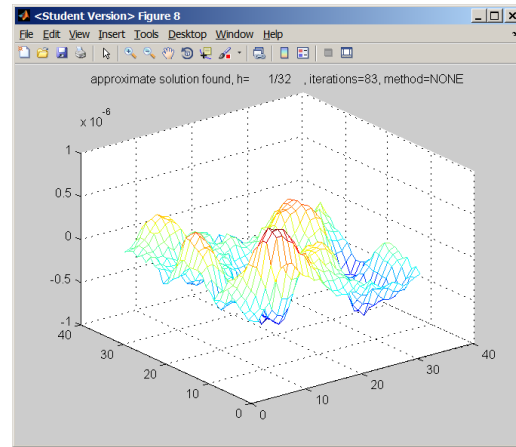**Residual per iteration**

ERROR per iteration
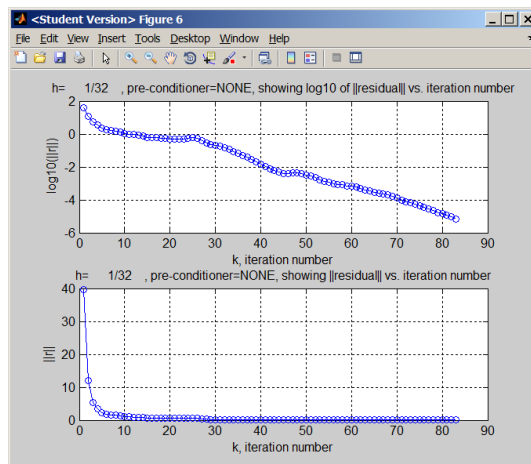
Figure 2.61: solver none plots 16
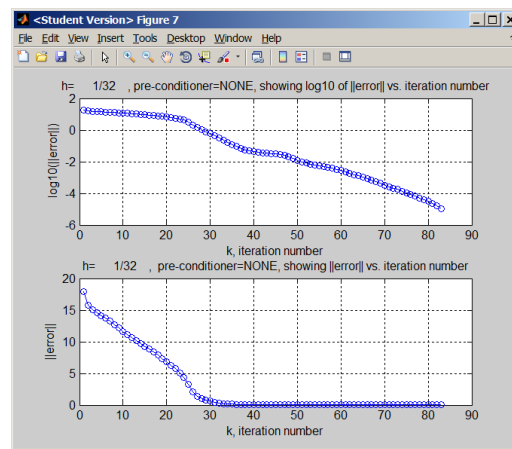
Plots for h=1/32

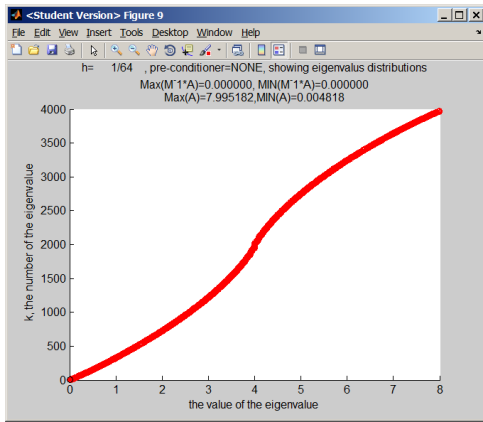**Eigenvalues of A**

Final solution

Residual per iteration

ERROR per iteration

Figure 2.62: solver none plots 32

Plot for h=1/64

Eigenvalues of A

Final solution

Residual per iteration

ERROR per iteration

Figure 2.63: solver none plots 64

Plots for h=1/128

Eigenvalues of A

Final solution

Residual per iteration
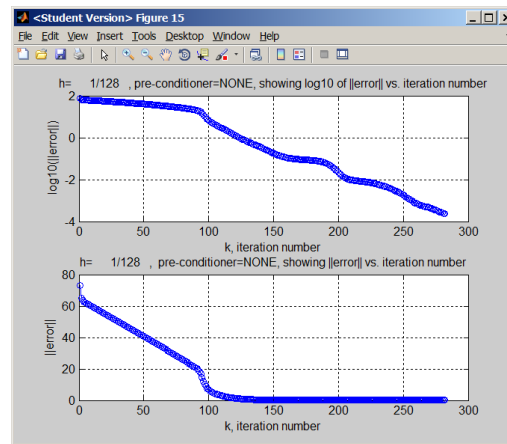
ERROR per iteration

Figure 2.64: solver none plots 128

2.6.4.2   Result for CG with Multigrid preconditioner

Plots for h=1/16
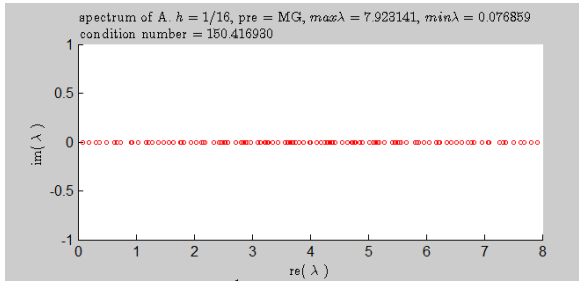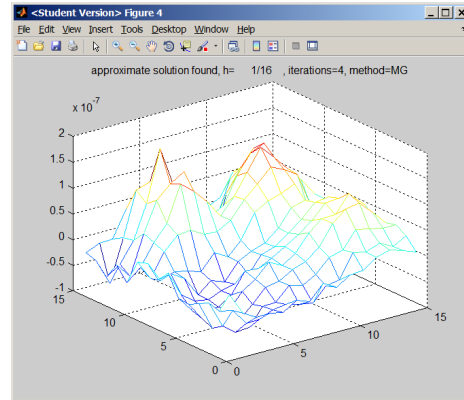
Eigenvalues of A
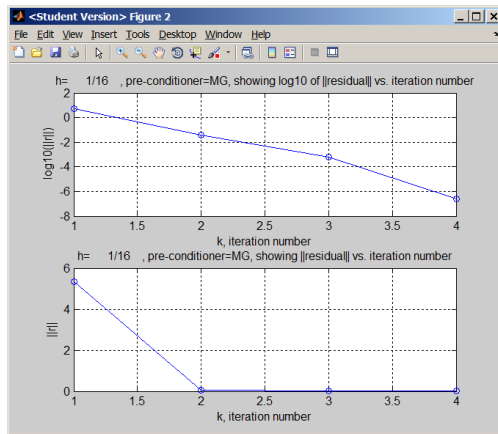
Final solution

Residual per iteration

ERROR per iteration

Figure 2.65: solver MG plots 16

Plots for h1/32
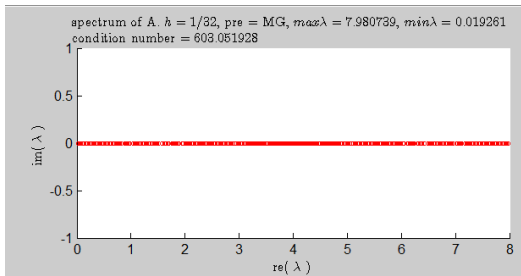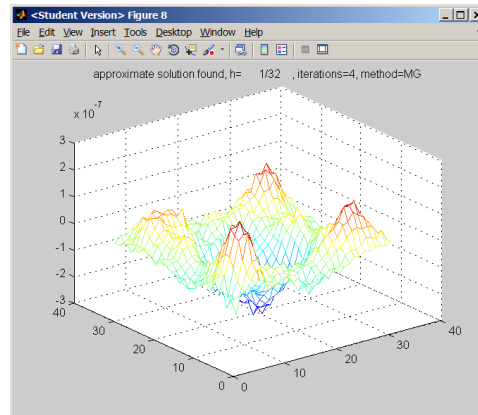
Eigenvalues of A
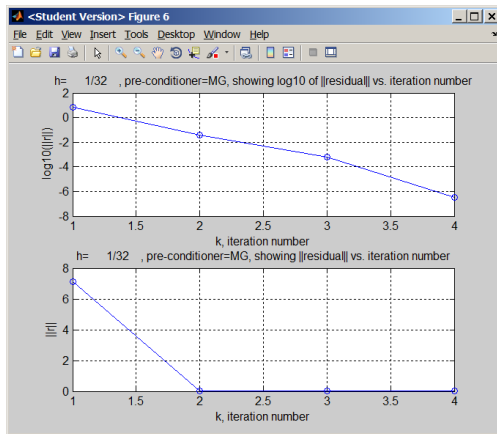
Final solution

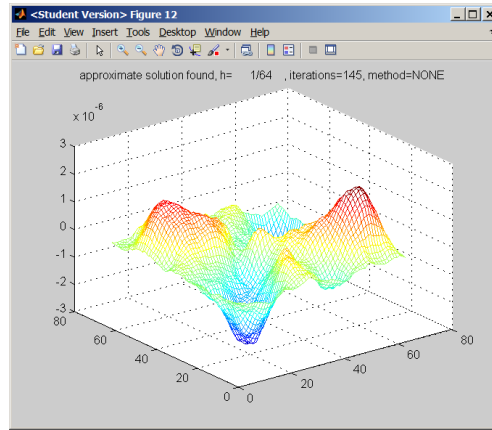**Residual per iteration**
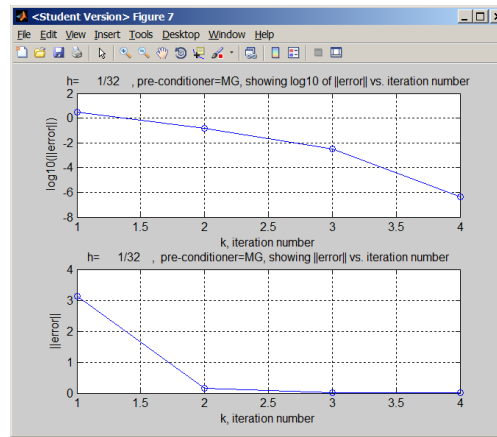
ERROR per iteration
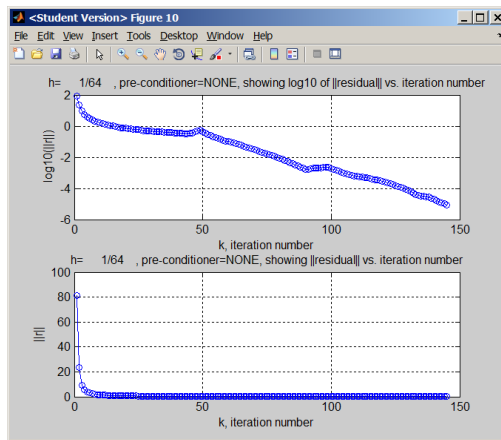
Figure 2.66: solver MG plots 32
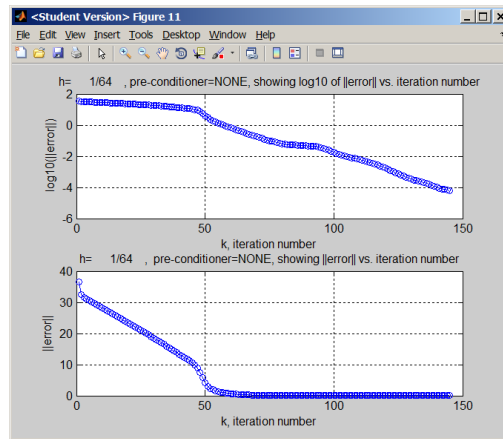
Plots for h=164

Eigenvalues of A

Final solution

Residual per iteration

ERROR per iteration
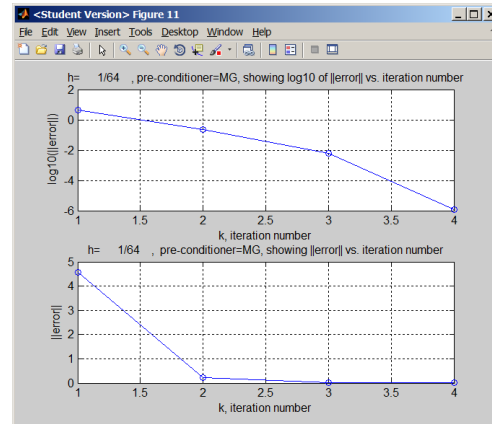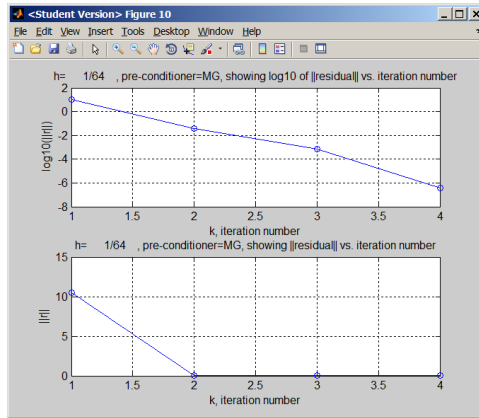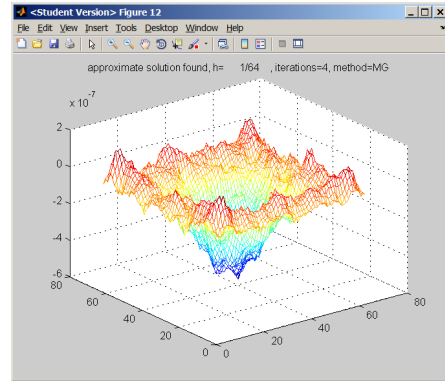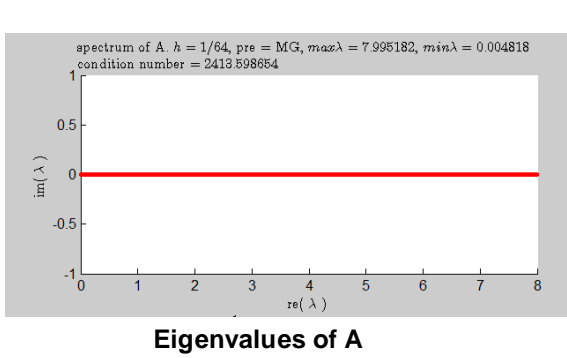
Figure 2.67: solver MG plots 64

Plots for h=1/128
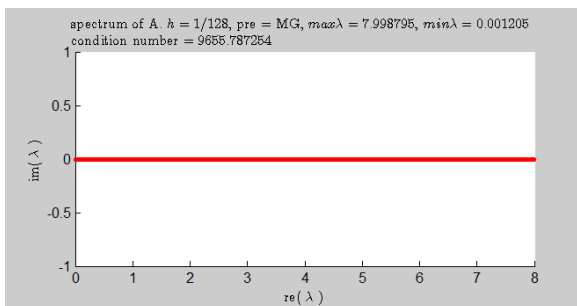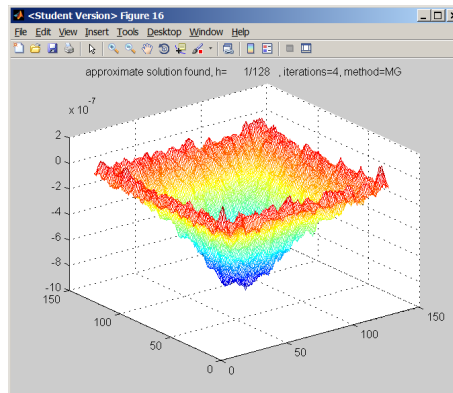
Eigenvalues of A
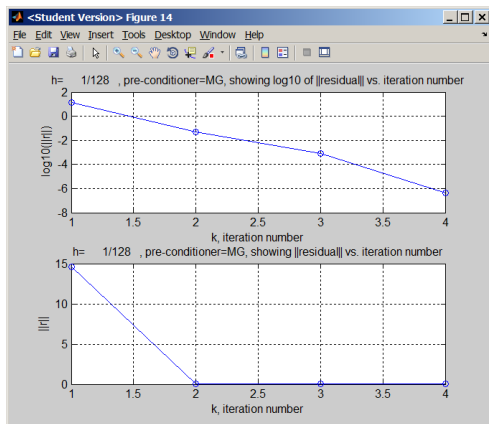
Final solution

Residual per iteration
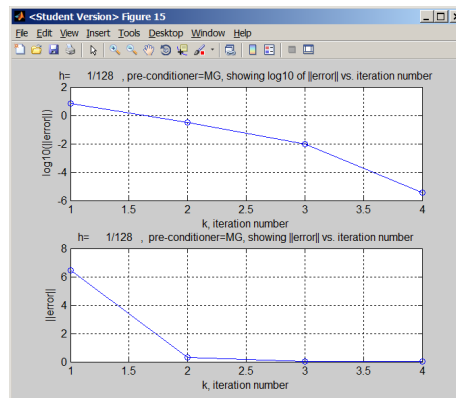
ERROR per iteration

Figure 2.68: solver MG plots 128

2.6.4.3 Result for CG with SSOR preconditioner

Plots for h=1/16
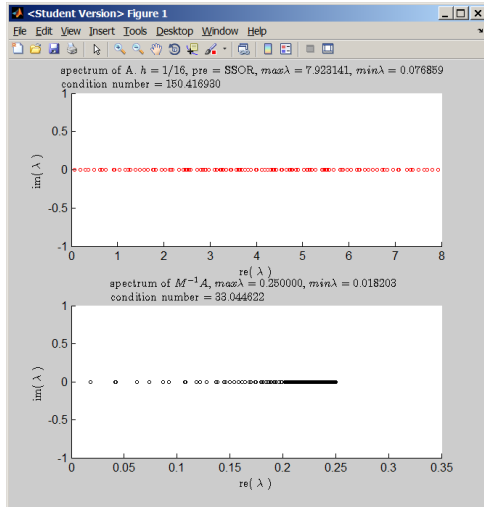
Eigenvalues of A and M⁻¹*A
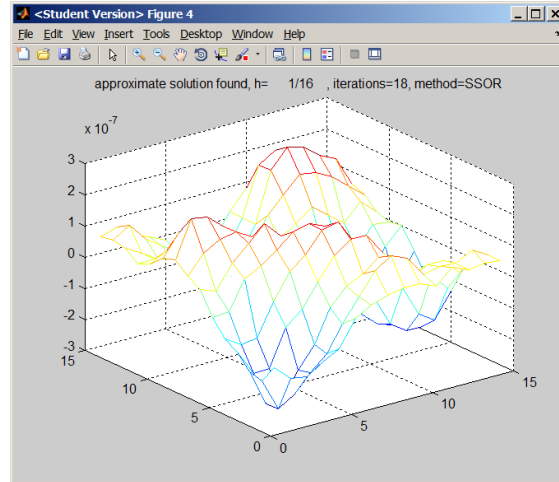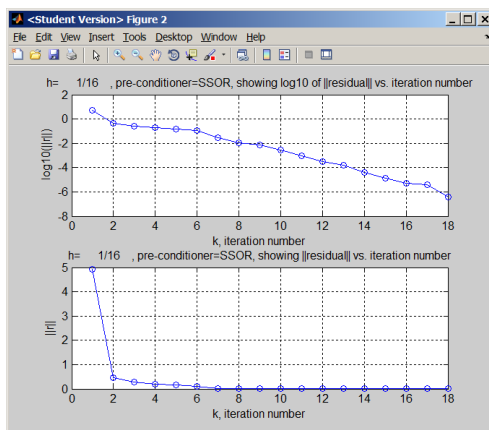
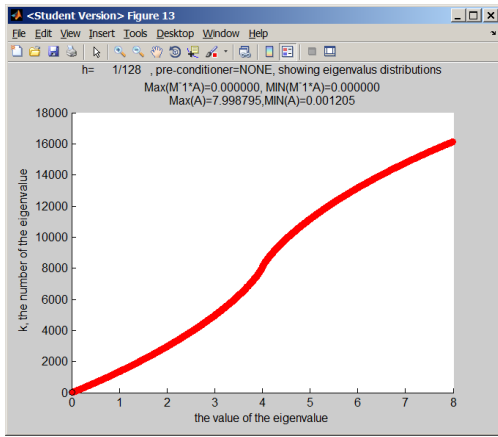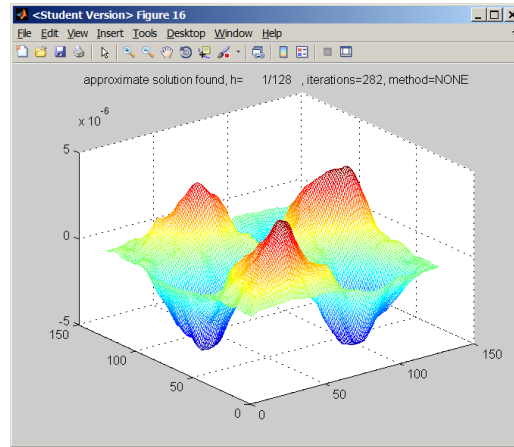
Final solution

Residual per iteration
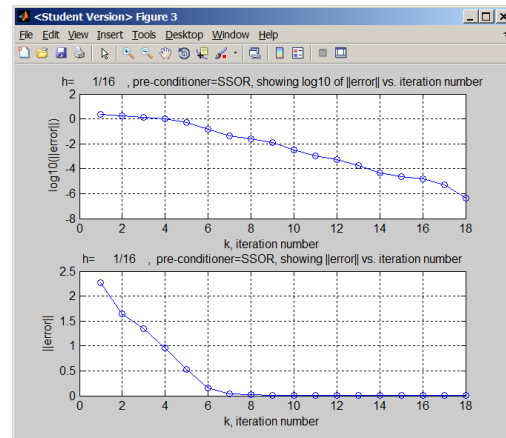
ERROR per iteration

Figure 2.69: solver SSOR plots 16

Plots for h=1/32
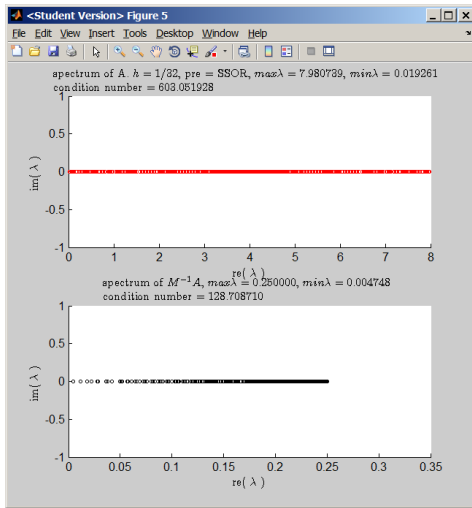
Eigenvalues of A and M^{-1}*A
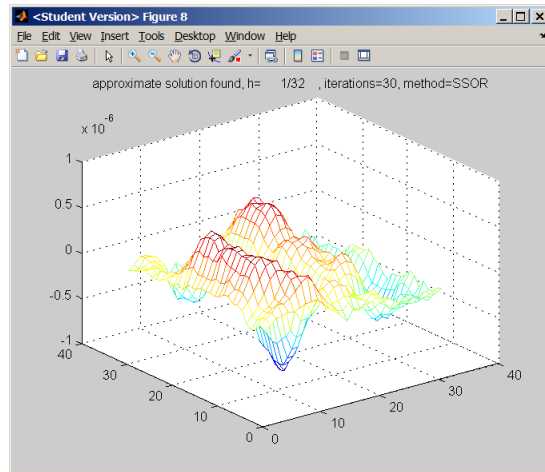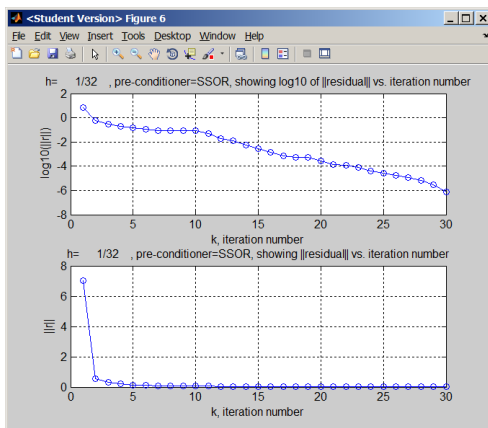
Final solution

Residual per iteration
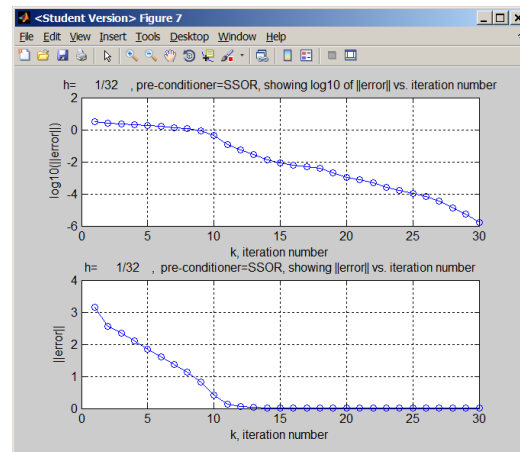
ERROR per iteration

Figure 2.70: solver SSOR plots 32

Plots for h=1/64
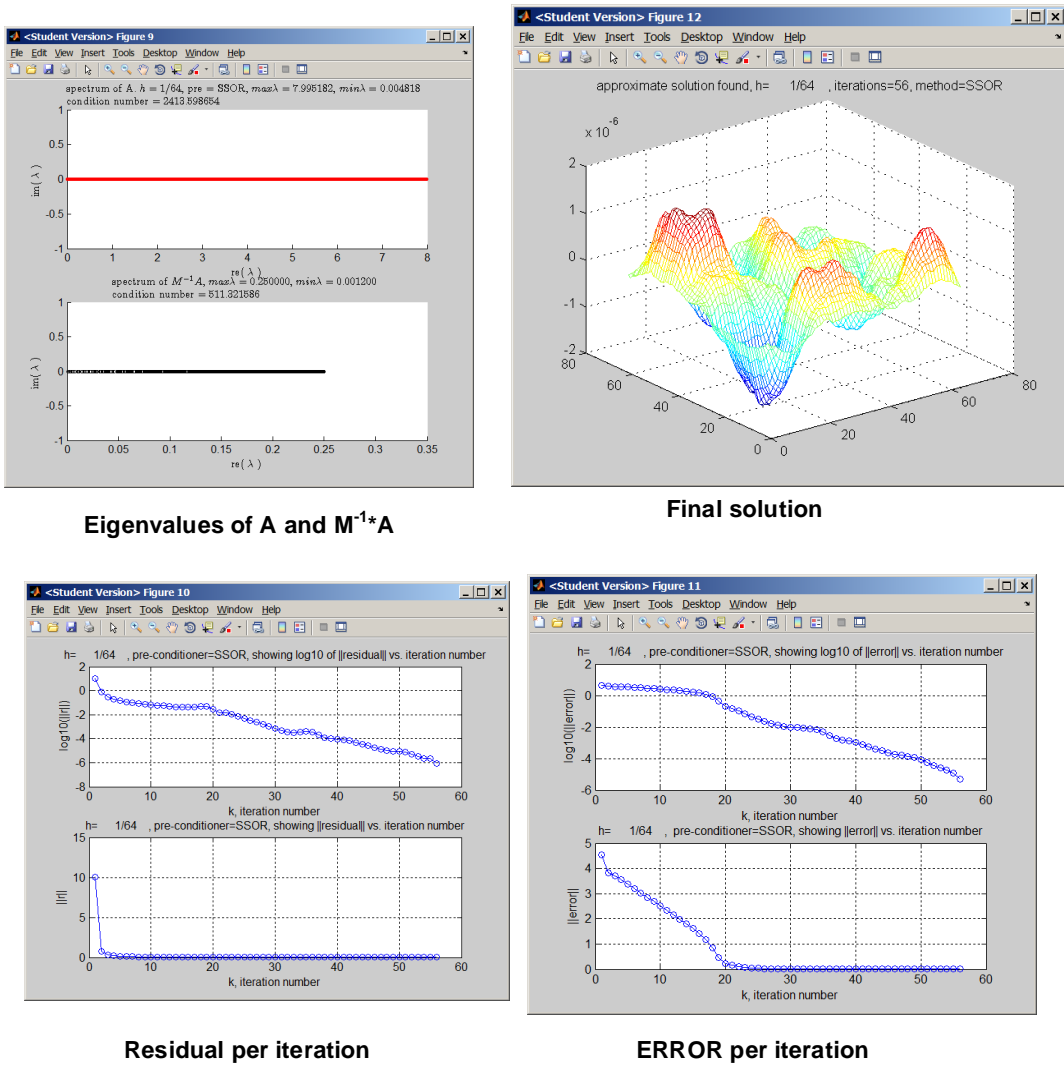
Eigenvalues of A and M^{-1}*A

Final solution

Residual per iteration

ERROR per iteration

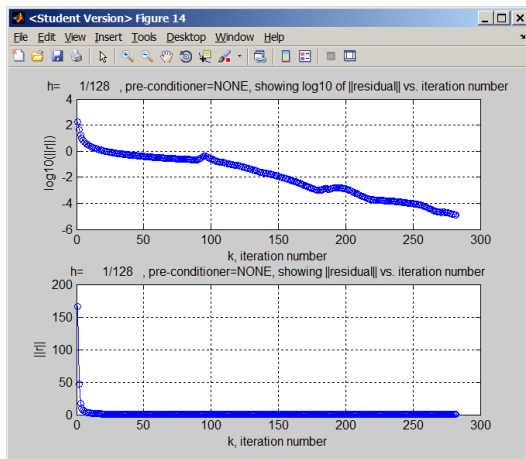Figure 2.71: solver SSOR plots 64

2.6.4.4 Plots for h=1/128
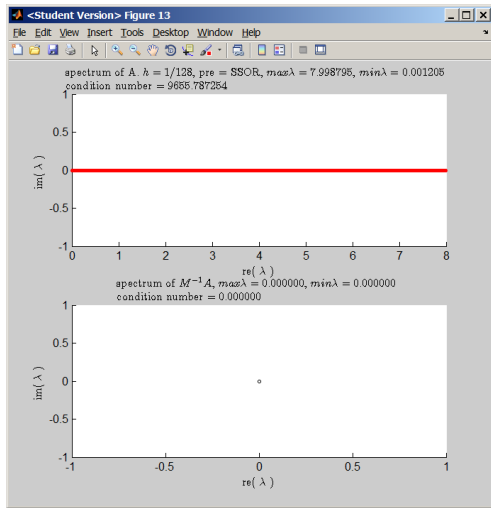
Eigenvalues of A
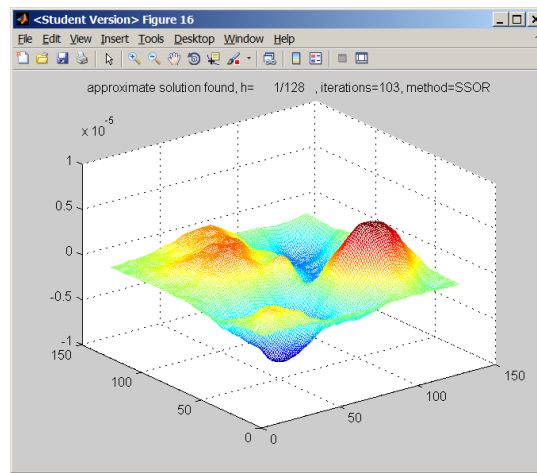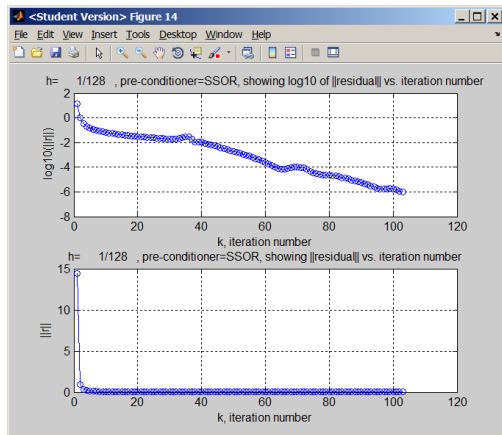
Final solution

Residual per iteration
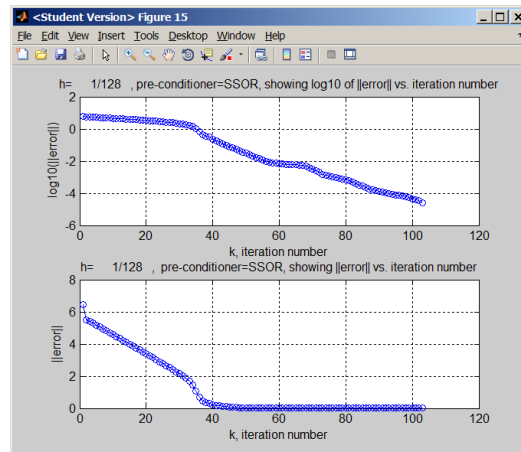
ERROR per iteration

Figure 2.72: solver SSOR plots 128

2.6.4.5 Result for CG with incomplete cholesky preconditioner $\varepsilon = 10^{-2}$

Plots for h=1/16
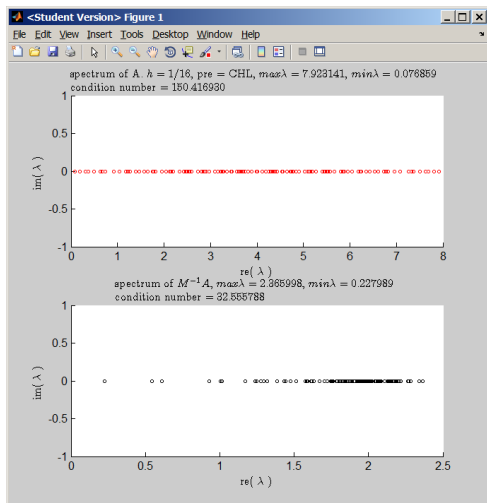
Eigenvalues of A and M^{-1}*A
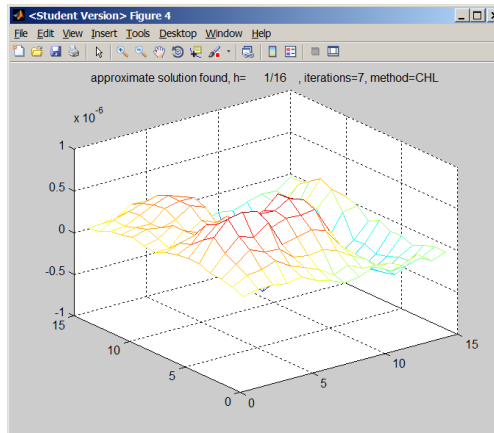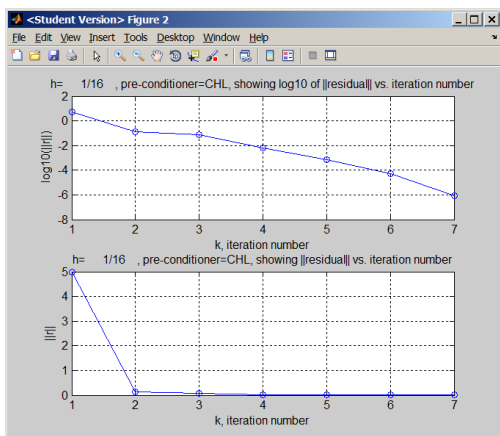
Final solution

Residual per iteration
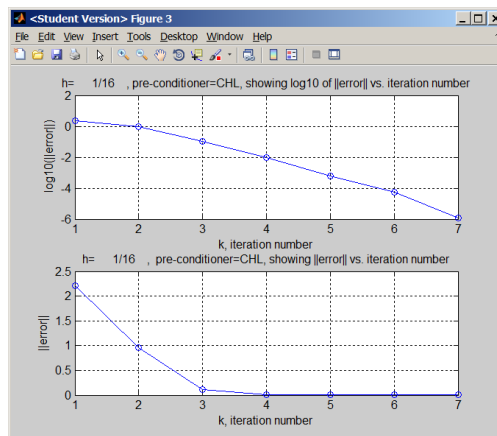
ERROR per iteration

Figure 2.73: solver incomplete cholesky plots 16

<u>Plots for h=1/32</u>
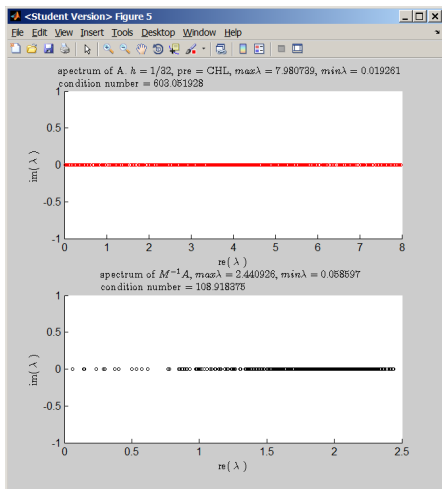
Eigenvalues of A and M⁻¹*A
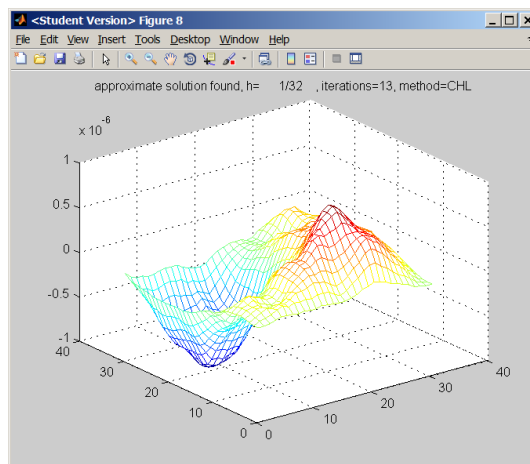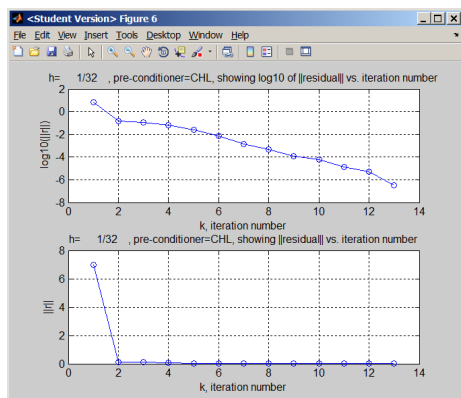
Final solution

Residual per iteration
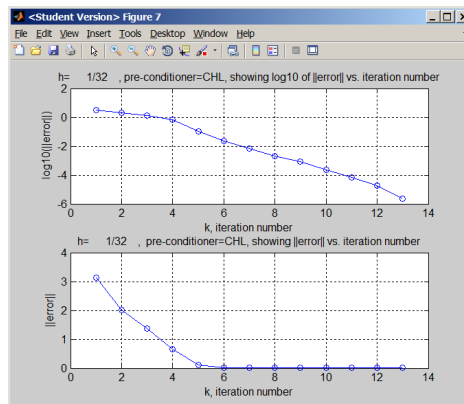
ERROR per iteration

Figure 2.74: solver incomplete cholesky plots 32

Plots for h=1/64
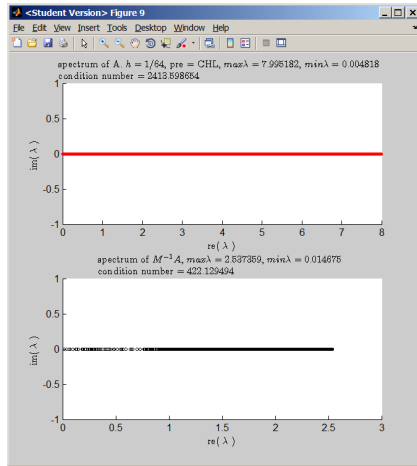
Eigenvalues of A and M⁻¹*A
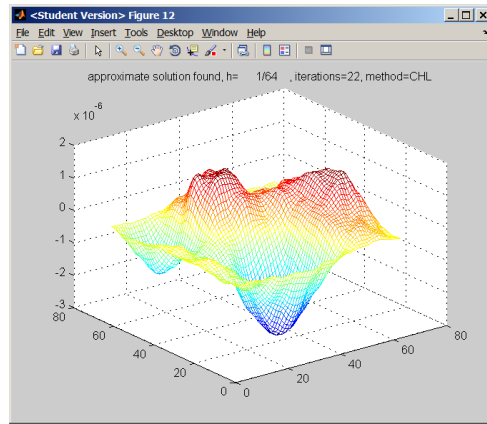
Final solution
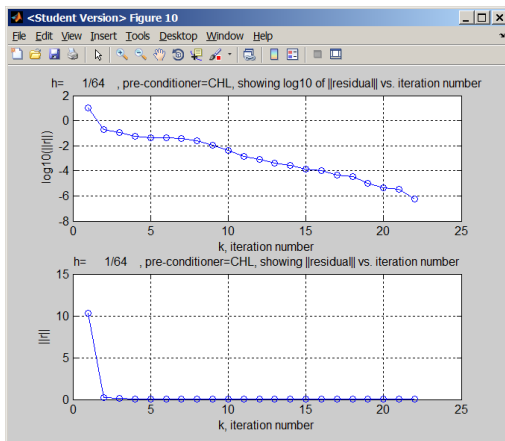
Residual per iteration
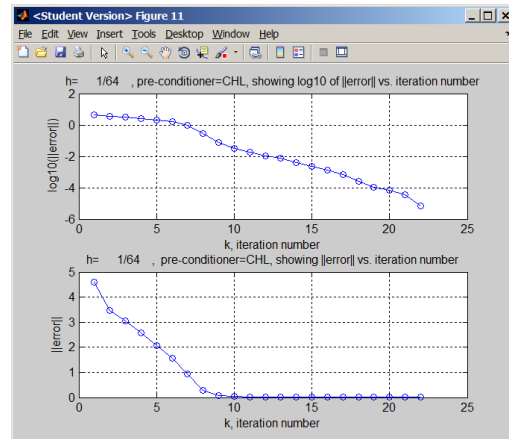
ERROR per iteration

Figure 2.75: solver incomplete cholesky plots 64

Plots for h=1/128

Eigenvalues of A

Final solution
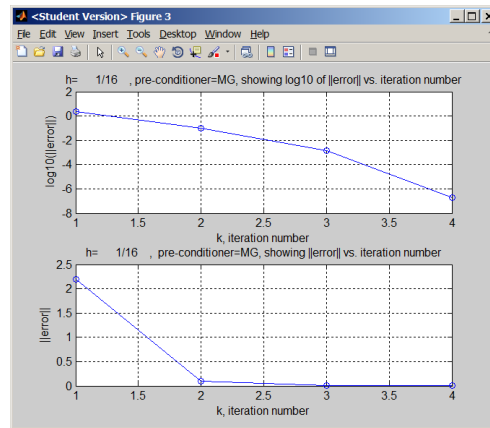

Residual per iteration

ERROR per iteration
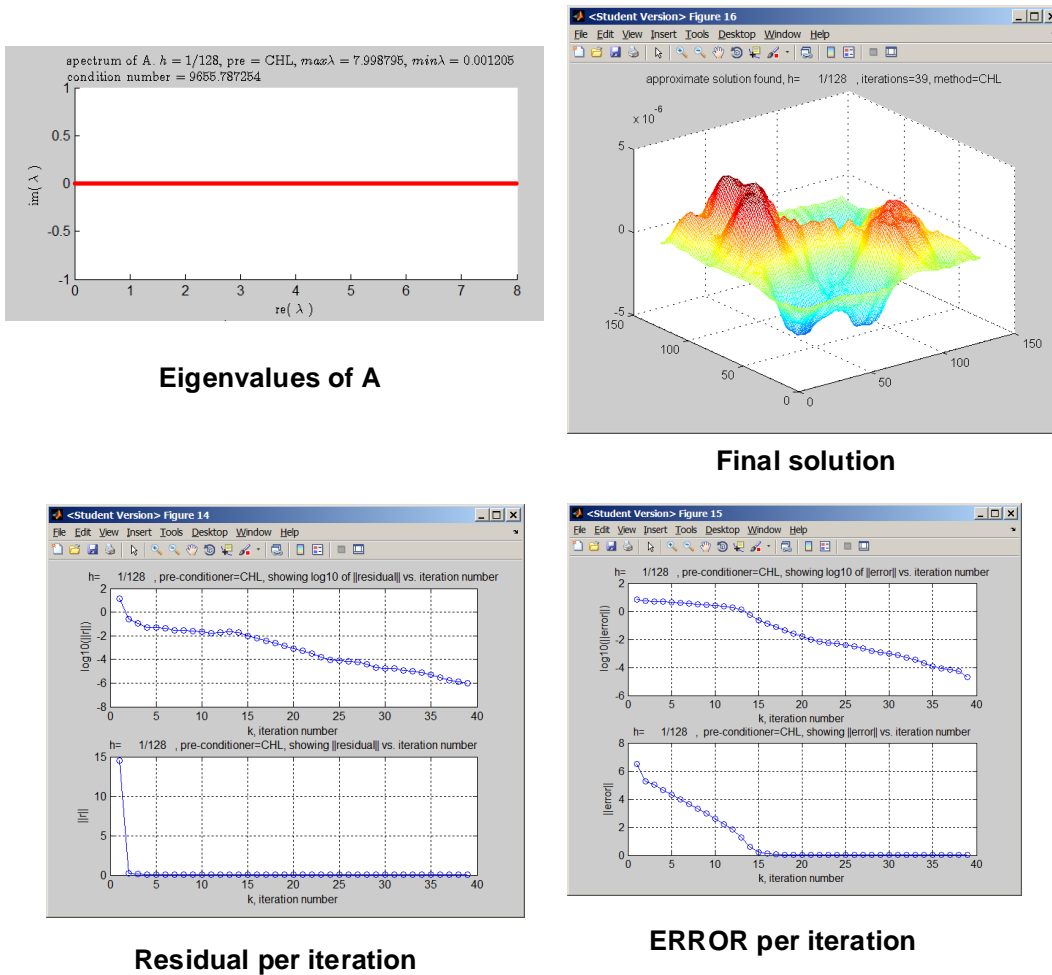
Figure 2.76: solver incomplete cholesky plots 128

2.6.5 Result for CG with incomplete cholesky preconditioner
$\varepsilon = 10^{-3}$

Plots for h=1/16
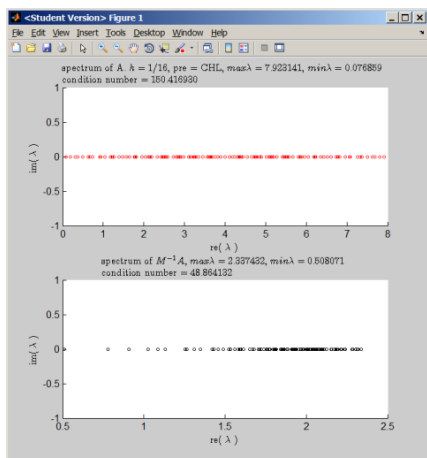
Eigenvalues of A and M^{-1}*A
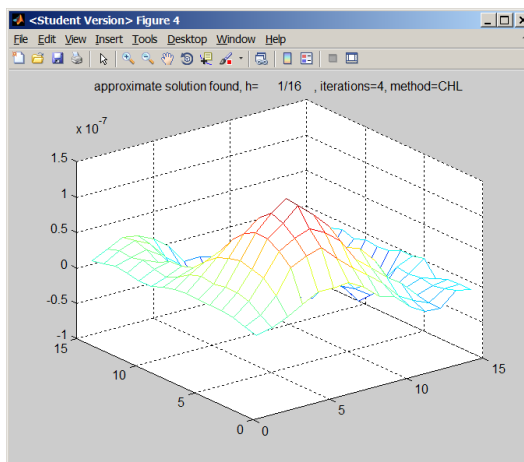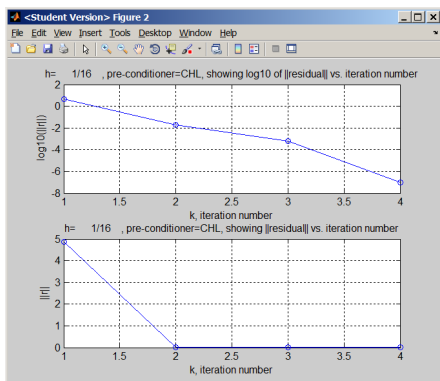
Final solution

Residual per iteration
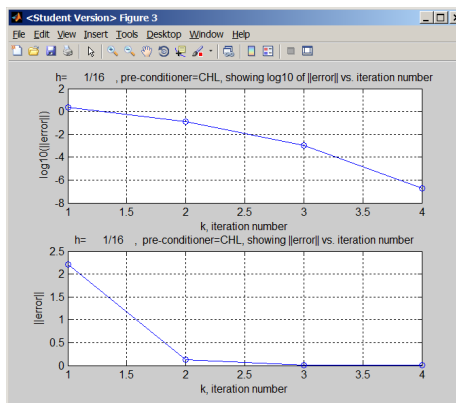
ERROR per iteration

Figure 2.77: solver CG with incomplete cholesky preconditioner 16

underlinePlots for h=1/32
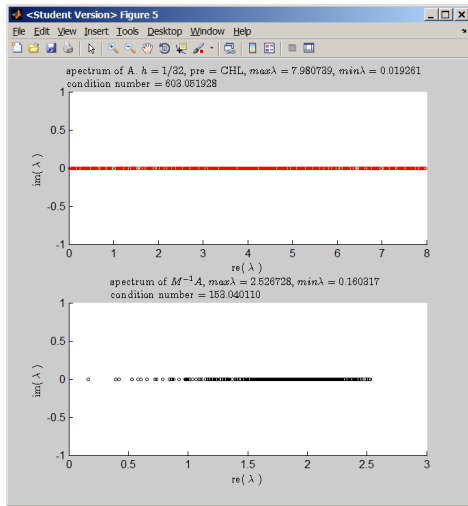
Eigenvalues of A and M^{-1}*A
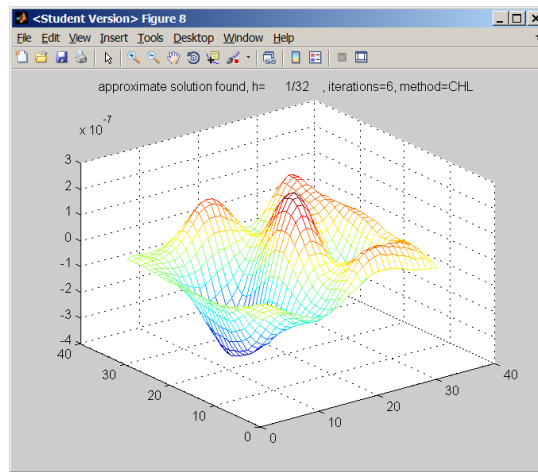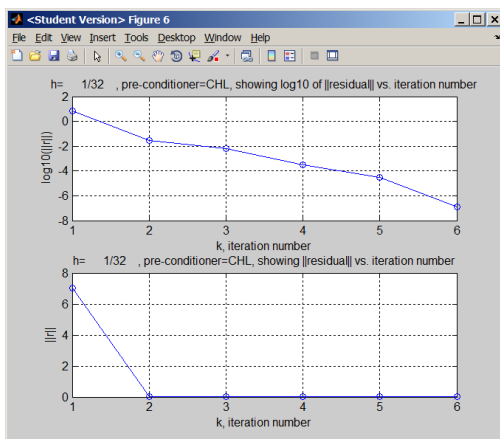
Final solution

Residual per iteration
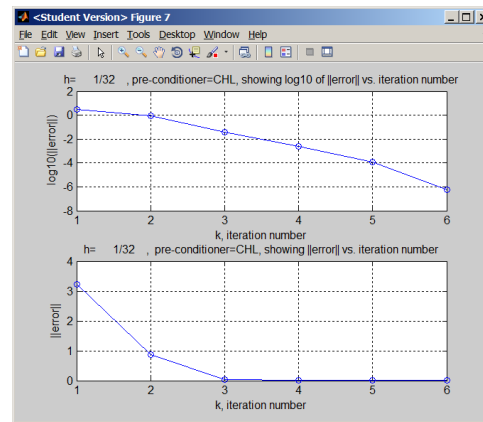
ERROR per iteration

Figure 2.78: solver CG with incomplete cholesky preconditioner 32

Plots for $h = \frac{1}{64}$
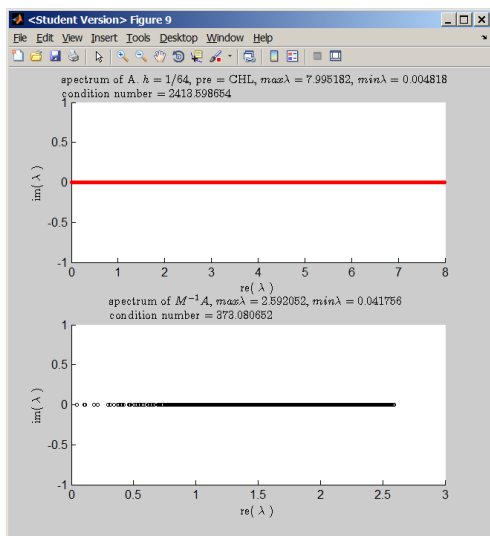
Eigenvalues of A and M⁻¹*A
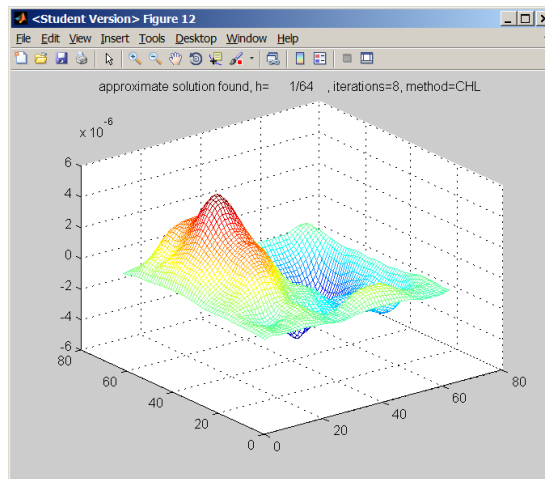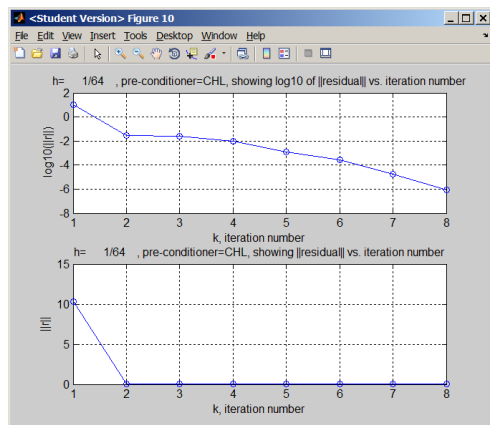
Final solution

Residual per iteration
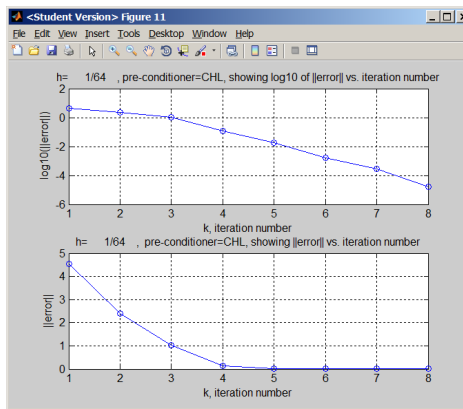
ERROR per iteration

Figure 2.79: solver CG with incomplete cholesky preconditioner 64

Plots for h=1/128
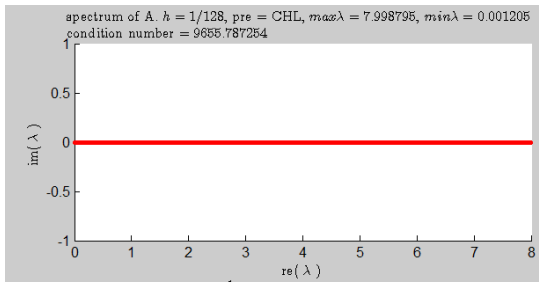
Eigenvalues of A
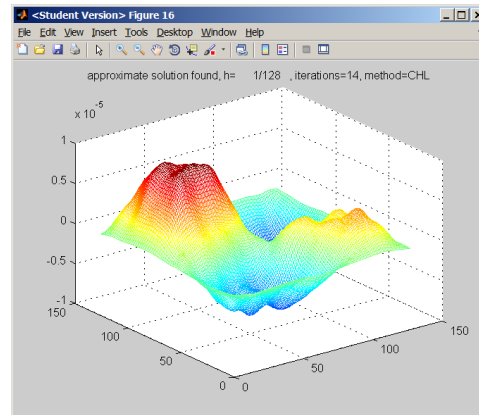
Final solution
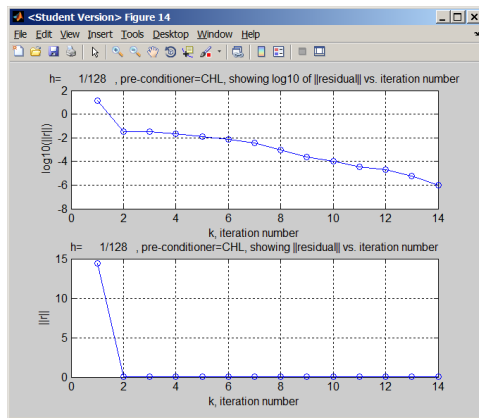
Residual per iteration
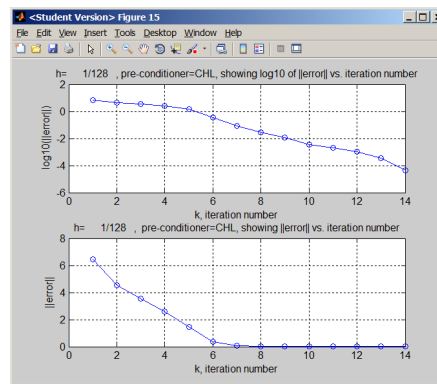
ERROR per iteration

Figure 2.80: solver CG with incomplete cholesky preconditioner 128

2.6.5.1　References

1. R. J. LeVeque. Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems. SIAM, 2007.

2.6.5.2　Computation Tables

table for CG with no preconditioner

table for CG with no preconditioner $h = \frac{1}{16}$

Tolerance=10^{-6} method=NONE, Iterations = 42, condition number A=150.4167

	k	|e|	ratio	|r|	ratio
1					
2	1	2.1830924	0.0000000	4.7210582	0.0000000
3	2	1.8864972	0.8641399	1.4184825	0.3004586
4	3	1.7843339	0.9458449	0.6424897	0.4529416
5	4	1.6861080	0.9449509	0.4396146	0.6842360
6	5	1.5849468	0.9400032	0.3336348	0.7589256
7	6	1.4553992	0.9182637	0.3163522	0.9481991
8	7	1.2957415	0.8902998	0.2736152	0.8649068
9	8	1.1409611	0.8805468	0.2420287	0.8845587
10	9	0.9977301	0.8744646	0.2038050	0.8420696
11	10	0.8499980	0.8519318	0.2051285	1.0064937
12	11	0.6651595	0.7825425	0.1974297	0.9624684
13	12	0.4864479	0.7313251	0.1747116	0.8849309
14	13	0.2960128	0.6085191	0.1779989	1.0188155
15	14	0.1451740	0.4904313	0.1199788	0.6740421
16	15	0.0935858	0.6446460	0.0678391	0.5654261
17	16	0.0756963	0.8088434	0.0408147	0.6016397
18	17	0.0635298	0.8392727	0.0298095	0.7303614
19	18	0.0529339	0.8332136	0.0218161	0.7318495
20	19	0.0429735	0.8118324	0.0195722	0.8971481
21	20	0.0322133	0.7496102	0.0163238	0.8340275
22	21	0.0237202	0.7363457	0.0127609	0.7817369
23	22	0.0157525	0.6640959	0.0107769	0.8445245
24	23	0.0090001	0.5713465	0.0081136	0.7528675
25	24	0.0049041	0.5448943	0.0056646	0.6981578
26	25	0.0029273	0.5968981	0.0032193	0.5683292
27	26	0.0021383	0.7304863	0.0017564	0.5455764
28	27	0.0016007	0.7485962	0.0012297	0.7001280
29	28	0.0011635	0.7268656	0.0008660	0.7042553
30	29	0.0008013	0.6886615	0.0006405	0.7396406
31	30	0.0005365	0.6695259	0.0005187	0.8097440
32	31	0.0003581	0.6674555	0.0002751	0.5304362
33	32	0.0002692	0.7517878	0.0001920	0.6979471
34	33	0.0001902	0.7066626	0.0001491	0.7762729
35	34	0.0001158	0.6085937	0.0001143	0.7665146
36	35	0.0000648	0.5601207	0.0000746	0.6531095
37	36	0.0000377	0.5807215	0.0000446	0.5983047
38	37	0.0000238	0.6316540	0.0000268	0.5993857
39	38	0.0000137	0.5773472	0.0000177	0.6619224
40	39	0.0000077	0.5599609	0.0000104	0.5863875
41	40	0.0000043	0.5545265	0.0000060	0.5767504

| 42 | 41 | 0.0000023 | 0.5432272 | 0.0000034 | 0.5745691 |
| 43 | 42 | 0.0000005 | 0.2307583 | 0.0000009 | 0.2689011 |

Table CG with no preconditioner for $h = \frac{1}{32}$

Tolerance=10^{-6} method=NONE, Iterations =82, condition number A=603

	k	|e|	ratio	|r|	ratio
1					
2	1	3.1348156	0.0000000	7.2393385	0.0000000
3	2	2.7330660	0.8718427	2.0520496	0.2834582
4	3	2.6175107	0.9577196	0.9150415	0.4459159
5	4	2.5284570	0.9659777	0.6101126	0.6667595
6	5	2.4421554	0.9658679	0.4386752	0.7190070
7	6	2.3586462	0.9658051	0.3380251	0.7705588
8	7	2.2766645	0.9652421	0.2853482	0.8441627
9	8	2.1887867	0.9614006	0.2476061	0.8677331
10	9	2.1039473	0.9612391	0.2079568	0.8398696
11	10	2.0254995	0.9627140	0.1877177	0.9026765
12	11	1.9387591	0.9571758	0.1699200	0.9051889
13	12	1.8539303	0.9562459	0.1545855	0.9097544
14	13	1.7682926	0.9538075	0.1400435	0.9059291
15	14	1.6834581	0.9520246	0.1278646	0.9130349
16	15	1.6022568	0.9517652	0.1160407	0.9075285
17	16	1.5217100	0.9497291	0.1094328	0.9430547
18	17	1.4395506	0.9460085	0.1027356	0.9388016
19	18	1.3562155	0.9421104	0.0942611	0.9175109
20	19	1.2759187	0.9407934	0.0898504	0.9532076
21	20	1.1866336	0.9300229	0.0872327	0.9708661
22	21	1.0934573	0.9214784	0.0826619	0.9476025
23	22	1.0024063	0.9167311	0.0788741	0.9541777
24	23	0.8992085	0.8970499	0.0830311	1.0527040
25	24	0.7631293	0.8486678	0.0970541	1.1688879
26	25	0.5530580	0.7247239	0.1109983	1.1436745
27	26	0.3359324	0.6074089	0.0961082	0.8658534
28	27	0.2191717	0.6524281	0.0639449	0.6653430
29	28	0.1626895	0.7422922	0.0484804	0.7581592
30	29	0.1228014	0.7548210	0.0407252	0.8400342
31	30	0.0917790	0.7473771	0.0335094	0.8228158
32	31	0.0700724	0.7634914	0.0273614	0.8165289
33	32	0.0540805	0.7717795	0.0214580	0.7842445
34	33	0.0432646	0.8000041	0.0163945	0.7640291
35	34	0.0364310	0.8420503	0.0126036	0.7687676
36	35	0.0309924	0.8507149	0.0103112	0.8181153
37	36	0.0268480	0.8662776	0.0080396	0.7796959
38	37	0.0236146	0.8795683	0.0066579	0.8281358
39	38	0.0209965	0.8891317	0.0052897	0.7945024
40	39	0.0191928	0.9140930	0.0040810	0.7714996
41	40	0.0177298	0.9237770	0.0033815	0.8285992
42	41	0.0164277	0.9265582	0.0027478	0.8126066
43	42	0.0154290	0.9392033	0.0020770	0.7558581
44	43	0.0144978	0.9396484	0.0018150	0.8738906
45	44	0.0134250	0.9260010	0.0016909	0.9316005
46	45	0.0121487	0.9049280	0.0017116	1.0122194
47	46	0.0103132	0.8489185	0.0019431	1.1352723

47	0.0076740	0.7440911	0.0020798	1.0703591
48	0.0052118	0.6791481	0.0017795	0.8556085
49	0.0036674	0.7036863	0.0012802	0.7194204
50	0.0028794	0.7851283	0.0009362	0.7313235
51	0.0023421	0.8133830	0.0007764	0.8292528
52	0.0019119	0.8163410	0.0006295	0.8108465
53	0.0016151	0.8447753	0.0004538	0.7207790
54	0.0014025	0.8683421	0.0003797	0.8368191
55	0.0011871	0.8464182	0.0003408	0.8975492
56	0.0009760	0.8221407	0.0002934	0.8609202
57	0.0007699	0.7888788	0.0002736	0.9323579
58	0.0005719	0.7428557	0.0002359	0.8624696
59	0.0004284	0.7489784	0.0001812	0.7678420
60	0.0003281	0.7659409	0.0001493	0.8242995
61	0.0002531	0.7714986	0.0001175	0.7865263
62	0.0002060	0.8136716	0.0000861	0.7334368
63	0.0001797	0.8726498	0.0000546	0.6339584
64	0.0001644	0.9147896	0.0000414	0.7578420
65	0.0001500	0.9121801	0.0000324	0.7822544
66	0.0001359	0.9062892	0.0000282	0.8704977
67	0.0001196	0.8799064	0.0000263	0.9342924
68	0.0001024	0.8564000	0.0000246	0.9353246
69	0.0000828	0.8079831	0.0000233	0.9453090
70	0.0000620	0.7488319	0.0000226	0.9703960
71	0.0000452	0.7287425	0.0000167	0.7383479
72	0.0000366	0.8101289	0.0000122	0.7285159
73	0.0000299	0.8162470	0.0000101	0.8335502
74	0.0000239	0.8015867	0.0000085	0.8430243
75	0.0000186	0.7787371	0.0000074	0.8676639
76	0.0000140	0.7525703	0.0000060	0.8134688
77	0.0000104	0.7435247	0.0000050	0.8225143
78	0.0000076	0.7246112	0.0000039	0.7889123
79	0.0000055	0.7232358	0.0000030	0.7583253
80	0.0000040	0.7287703	0.0000023	0.7618901
81	0.0000029	0.7237151	0.0000017	0.7706604
82	0.0000015	0.5224390	0.0000010	0.5519726

Table for CG with no preconditioner $h = \frac{1}{64}$

Tolerance=10^{-6} method=NONE, Iterations = 157, condition number A=2413

k	|e|	ratio	|r|	ratio
1	4.5618175	0.0000000	10.2041977	0.0000000
2	4.0484423	0.8874626	2.9223637	0.2863884
3	3.9296490	0.9706570	1.1769802	0.4027494
4	3.8598809	0.9822457	0.7112536	0.6043038
5	3.7995456	0.9843686	0.4805222	0.6755989
6	3.7431529	0.9851580	0.3718902	0.7739293
7	3.6863983	0.9848378	0.2978303	0.8008555
8	3.6292174	0.9844887	0.2541673	0.8533964
9	3.5685722	0.9832897	0.2226181	0.8758725
10	3.5069581	0.9827342	0.2001179	0.8989292
11	3.4429589	0.9817508	0.1826321	0.9126225
12	3.3782370	0.9812017	0.1713830	0.9384053

14	13	3.3146909	0.9811896	0.1522840	0.8885599
15	14	3.2564054	0.9824160	0.1328174	0.8721691
16	15	3.2008849	0.9829504	0.1224334	0.9218176
17	16	3.1455152	0.9827018	0.1137015	0.9286800
18	17	3.0882964	0.9818094	0.1060581	0.9327769
19	18	3.0308365	0.9813943	0.0996429	0.9395123
20	19	2.9722551	0.9806716	0.0956821	0.9602499
21	20	2.9114870	0.9795549	0.0903023	0.9437739
22	21	2.8528462	0.9798588	0.0851183	0.9425936
23	22	2.7941605	0.9794291	0.0812703	0.9547925
24	23	2.7330511	0.9781296	0.0777886	0.9571584
25	24	2.6717953	0.9775870	0.0762112	0.9797225
26	25	2.6080786	0.9761521	0.0724646	0.9508390
27	26	2.5466310	0.9764395	0.0682339	0.9416164
28	27	2.4869367	0.9765595	0.0647979	0.9496439
29	28	2.4272004	0.9759800	0.0620220	0.9571603
30	29	2.3688169	0.9759461	0.0598978	0.9657516
31	30	2.3065758	0.9737248	0.0596884	0.9965033
32	31	2.2416075	0.9718334	0.0576136	0.9652406
33	32	2.1770400	0.9711959	0.0558695	0.9697274
34	33	2.1148235	0.9714215	0.0530152	0.9489108
35	34	2.0513467	0.9699848	0.0526473	0.9930604
36	35	1.9830861	0.9667240	0.0520339	0.9883494
37	36	1.9164540	0.9663998	0.0501573	0.9639341
38	37	1.8482870	0.9644306	0.0495531	0.9879541
39	38	1.7802122	0.9631687	0.0470089	0.9486572
40	39	1.7180330	0.9650720	0.0434971	0.9252956
41	40	1.6579511	0.9650286	0.0428035	0.9840528
42	41	1.5950758	0.9620765	0.0419171	0.9792931
43	42	1.5314235	0.9600945	0.0407596	0.9723845
44	43	1.4665547	0.9576415	0.0403056	0.9888615
45	44	1.3990502	0.9539707	0.0399968	0.9923390
46	45	1.3251017	0.9471438	0.0414126	1.0353981
47	46	1.2399275	0.9357225	0.0446116	1.0772483
48	47	1.1274063	0.9092518	0.0518707	1.1627170
49	48	0.9608210	0.8522402	0.0624979	1.2048781
50	49	0.7395289	0.7696844	0.0669797	1.0717122
51	50	0.5283032	0.7143780	0.0585964	0.8748381
52	51	0.3950155	0.7477062	0.0442497	0.7551608
53	52	0.3199062	0.8098571	0.0353741	0.7994194
54	53	0.2652143	0.8290380	0.0311022	0.8792360
55	54	0.2243041	0.8457467	0.0261245	0.8399568
56	55	0.1942000	0.8657888	0.0227046	0.8690920
57	56	0.1684643	0.8674782	0.0210588	0.9275151
58	57	0.1453235	0.8626370	0.0187644	0.8910446
59	58	0.1265923	0.8711071	0.0167577	0.8930590
60	59	0.1097734	0.8671408	0.0154345	0.9210396
61	60	0.0947833	0.8634453	0.0138393	0.8966502
62	61	0.0823391	0.8687086	0.0123950	0.8956317
63	62	0.0713607	0.8666691	0.0111566	0.9000953
64	63	0.0615239	0.8621533	0.0099908	0.8954998
65	64	0.0531284	0.8635401	0.0089677	0.8976025
66	65	0.0460486	0.8667428	0.0078037	0.8701995
67	66	0.0401432	0.8717562	0.0068652	0.8797327

68	67	0.0349347	0.8702530	0.0061224	0.8918082
69	68	0.0305072	0.8732635	0.0053390	0.8720462
70	69	0.0267317	0.8762420	0.0047623	0.8919769
71	70	0.0234572	0.8775033	0.0041094	0.8628988
72	71	0.0207112	0.8829387	0.0036173	0.8802590
73	72	0.0181979	0.8786512	0.0032920	0.9100613
74	73	0.0159488	0.8764068	0.0028970	0.8800222
75	74	0.0140658	0.8819366	0.0025342	0.8747771
76	75	0.0125049	0.8890299	0.0021879	0.8633171
77	76	0.0111985	0.8955273	0.0019189	0.8770658
78	77	0.0101148	0.9032251	0.0016804	0.8757329
79	78	0.0091865	0.9082280	0.0015038	0.8948967
80	79	0.0084158	0.9160976	0.0013190	0.8770684
81	80	0.0078196	0.9291617	0.0011071	0.8393543
82	81	0.0073758	0.9432389	0.0009256	0.8360613
83	82	0.0070385	0.9542812	0.0007869	0.8502076
84	83	0.0067629	0.9608446	0.0006653	0.8453718
85	84	0.0065430	0.9674723	0.0005542	0.8330332
86	85	0.0063606	0.9721362	0.0004631	0.8356598
87	86	0.0062041	0.9753943	0.0003904	0.8429988
88	87	0.0060628	0.9772213	0.0003301	0.8455336
89	88	0.0059316	0.9783588	0.0002793	0.8462168
90	89	0.0058038	0.9784528	0.0002434	0.8714097
91	90	0.0056707	0.9770673	0.0002200	0.9036392
92	91	0.0055215	0.9736957	0.0002182	0.9922456
93	92	0.0053270	0.9647725	0.0002329	1.0671837
94	93	0.0050613	0.9501202	0.0002563	1.1003357
95	94	0.0047075	0.9300953	0.0002754	1.0744267
96	95	0.0043063	0.9147708	0.0002692	0.9778062
97	96	0.0039172	0.9096464	0.0002592	0.9626214
98	97	0.0035245	0.8997507	0.0002627	1.0135967
99	98	0.0030928	0.8774993	0.0002717	1.0343359
100	99	0.0026357	0.8522177	0.0002652	0.9760185
101	100	0.0022317	0.8467214	0.0002384	0.8988006
102	101	0.0019165	0.8587692	0.0002124	0.8912480
103	102	0.0016658	0.8691736	0.0001896	0.8925872
104	103	0.0014748	0.8853198	0.0001562	0.8237657
105	104	0.0013425	0.9102889	0.0001341	0.8584851
106	105	0.0012338	0.9190797	0.0001229	0.9163269
107	106	0.0011364	0.9210448	0.0001110	0.9029318
108	107	0.0010529	0.9264833	0.0000981	0.8837443
109	108	0.0009822	0.9329029	0.0000884	0.9015361
110	109	0.0009173	0.9338643	0.0000805	0.9108633
111	110	0.0008605	0.9381542	0.0000706	0.8763963
112	111	0.0008106	0.9420035	0.0000648	0.9185117
113	112	0.0007612	0.9390822	0.0000609	0.9395127
114	113	0.0007143	0.9383693	0.0000557	0.9139756
115	114	0.0006696	0.9374446	0.0000519	0.9329107
116	115	0.0006260	0.9348144	0.0000481	0.9264502
117	116	0.0005854	0.9351858	0.0000443	0.9199678
118	117	0.0005460	0.9326180	0.0000424	0.9576760
119	118	0.0005044	0.9239461	0.0000413	0.9738142
120	119	0.0004627	0.9172921	0.0000391	0.9473491
121	120	0.0004208	0.9094667	0.0000381	0.9752708

122	121	0.0003784	0.8992550	0.0000372	0.9763676
123	122	0.0003372	0.8911407	0.0000350	0.9403920
124	123	0.0002996	0.8885168	0.0000322	0.9205549
125	124	0.0002665	0.8895264	0.0000296	0.9186267
126	125	0.0002382	0.8936247	0.0000267	0.9014817
127	126	0.0002145	0.9005476	0.0000235	0.8810575
128	127	0.0001949	0.9087432	0.0000205	0.8702270
129	128	0.0001793	0.9195943	0.0000191	0.9333716
130	129	0.0001645	0.9175205	0.0000165	0.8627026
131	130	0.0001529	0.9299627	0.0000145	0.8810031
132	131	0.0001428	0.9334836	0.0000126	0.8674787
133	132	0.0001338	0.9370194	0.0000113	0.8986161
134	133	0.0001252	0.9354832	0.0000105	0.9248925
135	134	0.0001161	0.9280017	0.0000101	0.9641272
136	135	0.0001061	0.9132398	0.0000104	1.0276371
137	136	0.0000941	0.8872630	0.0000108	1.0403204
138	137	0.0000809	0.8591660	0.0000106	0.9825534
139	138	0.0000692	0.8553820	0.0000095	0.9003622
140	139	0.0000600	0.8674816	0.0000080	0.8360480
141	140	0.0000535	0.8925543	0.0000066	0.8225625
142	141	0.0000486	0.9079807	0.0000057	0.8611308
143	142	0.0000443	0.9113730	0.0000052	0.9135594
144	143	0.0000402	0.9066825	0.0000048	0.9354110
145	144	0.0000361	0.8975802	0.0000045	0.9304066
146	145	0.0000323	0.8956690	0.0000041	0.9034467
147	146	0.0000288	0.8917449	0.0000038	0.9321324
148	147	0.0000255	0.8865527	0.0000034	0.9105714
149	148	0.0000226	0.8841495	0.0000031	0.9002286
150	149	0.0000200	0.8862625	0.0000028	0.8885652
151	150	0.0000177	0.8849574	0.0000025	0.8993699
152	151	0.0000155	0.8766179	0.0000024	0.9491487
153	152	0.0000133	0.8572703	0.0000022	0.9393268
154	153	0.0000113	0.8482148	0.0000020	0.9135275
155	154	0.0000095	0.8451107	0.0000018	0.8915801
156	155	0.0000081	0.8501849	0.0000015	0.8470707
157	156	0.0000070	0.8664622	0.0000013	0.8602920
158	157	0.0000054	0.7724505	0.0000010	0.7331119

Table for CG with no preconditioner $h = \frac{1}{128}$

Tolerance=10^{-6} method=NONE, Iterations = 291, condition number A=9655

	k	|e|	ratio	|r|	ratio
1					
2	1	6.5216900	0.0000000	14.4577169	0.0000000
3	2	5.8211800	0.8925877	4.0764931	0.2819597
4	3	5.6842162	0.9764715	1.5799015	0.3875639
5	4	5.6198883	0.9886831	0.8968116	0.5676377
6	5	5.5711714	0.9913313	0.5778805	0.6443723
7	6	5.5285647	0.9923523	0.4303215	0.7446548
8	7	5.4869080	0.9924652	0.3341897	0.7766047
9	8	5.4468461	0.9926986	0.2785535	0.8335192
10	9	5.4044559	0.9922175	0.2424388	0.8703491
11	10	5.3616029	0.9920708	0.2146822	0.8855108
12	11	5.3191442	0.9920810	0.1862262	0.8674507

13	12	5.2782868	0.9923188	0.1677396	0.9007302
14	13	5.2363954	0.9920634	0.1524839	0.9090513
15	14	5.1941353	0.9919295	0.1420818	0.9317825
16	15	5.1513426	0.9917614	0.1293917	0.9106846
17	16	5.1099665	0.9919679	0.1195224	0.9237248
18	17	5.0683715	0.9918600	0.1119117	0.9363248
19	18	5.0255870	0.9915585	0.1066056	0.9525867
20	19	4.9819254	0.9913121	0.0994276	0.9326674
21	20	4.9398031	0.9915450	0.0934835	0.9402165
22	21	4.8972751	0.9913908	0.0895284	0.9576918
23	22	4.8541439	0.9911928	0.0849488	0.9488477
24	23	4.8107070	0.9910516	0.0811183	0.9549077
25	24	4.7673483	0.9909871	0.0782140	0.9641979
26	25	4.7234798	0.9907981	0.0742769	0.9496615
27	26	4.6807711	0.9909582	0.0714271	0.9616324
28	27	4.6372353	0.9906990	0.0684405	0.9581873
29	28	4.5948580	0.9908615	0.0651097	0.9513333
30	29	4.5530996	0.9909119	0.0621499	0.9545408
31	30	4.5112658	0.9908120	0.0601683	0.9681159
32	31	4.4687574	0.9905773	0.0584916	0.9721339
33	32	4.4262843	0.9904955	0.0566503	0.9685198
34	33	4.3834625	0.9903256	0.0547127	0.9657966
35	34	4.3405454	0.9902093	0.0529867	0.9684541
36	35	4.2983731	0.9902841	0.0507715	0.9581929
37	36	4.2562271	0.9901949	0.0496163	0.9772479
38	37	4.2138748	0.9900493	0.0481310	0.9700646
39	38	4.1708118	0.9897807	0.0472816	0.9823505
40	39	4.1266023	0.9894003	0.0466718	0.9871047
41	40	4.0823351	0.9892727	0.0454224	0.9732297
42	41	4.0380267	0.9891463	0.0441588	0.9721808
43	42	3.9946964	0.9892694	0.0426145	0.9650276
44	43	3.9522103	0.9893644	0.0409349	0.9605877
45	44	3.9106277	0.9894786	0.0398300	0.9730069
46	45	3.8684976	0.9892268	0.0392197	0.9846780
47	46	3.8257317	0.9889451	0.0385564	0.9830880
48	47	3.7830507	0.9888437	0.0372870	0.9670762
49	48	3.7403946	0.9887244	0.0366904	0.9839996
50	49	3.6974958	0.9885310	0.0359622	0.9801539
51	50	3.6549289	0.9884876	0.0352283	0.9795926
52	51	3.6113640	0.9880805	0.0345719	0.9813650
53	52	3.5675354	0.9878637	0.0340535	0.9850064
54	53	3.5231205	0.9875503	0.0336276	0.9874943
55	54	3.4787130	0.9873954	0.0329058	0.9785343
56	55	3.4345867	0.9873153	0.0323069	0.9817996
57	56	3.3901581	0.9870643	0.0316810	0.9806276
58	57	3.3461400	0.9870159	0.0306893	0.9686978
59	58	3.3036285	0.9872954	0.0298158	0.9715352
60	59	3.2612089	0.9871597	0.0294984	0.9893541
61	60	3.2175113	0.9866008	0.0293399	0.9946276
62	61	3.1732557	0.9862454	0.0288105	0.9819557
63	62	3.1292108	0.9861200	0.0282986	0.9822332
64	63	3.0851351	0.9859147	0.0277461	0.9804746
65	64	3.0412129	0.9857633	0.0274147	0.9880581
66	65	2.9964828	0.9852920	0.0267788	0.9768022

67	66	2.9522621	0.9852425	0.0265317	0.9907736
68	67	2.9083386	0.9851221	0.0258464	0.9741706
69	68	2.8648102	0.9850332	0.0254640	0.9852068
70	69	2.8199937	0.9843562	0.0255037	1.0015562
71	70	2.7742200	0.9837682	0.0252014	0.9881474
72	71	2.7287776	0.9836197	0.0245978	0.9760481
73	72	2.6844169	0.9837434	0.0237590	0.9659003
74	73	2.6412743	0.9839285	0.0235033	0.9892381
75	74	2.5968246	0.9831711	0.0234332	0.9970183
76	75	2.5517313	0.9826352	0.0230106	0.9819663
77	76	2.5070630	0.9824949	0.0225744	0.9810401
78	77	2.4633246	0.9825539	0.0220316	0.9759583
79	78	2.4188791	0.9819571	0.0219368	0.9956962
80	79	2.3736882	0.9813174	0.0216703	0.9878504
81	80	2.3279918	0.9807488	0.0217378	1.0031152
82	81	2.2808231	0.9797385	0.0215495	0.9913383
83	82	2.2336755	0.9793287	0.0212204	0.9847292
84	83	2.1862533	0.9787694	0.0208521	0.9826415
85	84	2.1387752	0.9782833	0.0206835	0.9919146
86	85	2.0909984	0.9776616	0.0203045	0.9816770
87	86	2.0438264	0.9774405	0.0199742	0.9837320
88	87	1.9962370	0.9767155	0.0198473	0.9936479
89	88	1.9483597	0.9760162	0.0195917	0.9871222
90	89	1.8995855	0.9749666	0.0195412	0.9974214
91	90	1.8476324	0.9726503	0.0202739	1.0374967
92	91	1.7887809	0.9681476	0.0215423	1.0625600
93	92	1.7179957	0.9604282	0.0239573	1.1121081
94	93	1.6232429	0.9448469	0.0281514	1.1750654
95	94	1.4854374	0.9151048	0.0338549	1.2025994
96	95	1.2901182	0.8685107	0.0389859	1.1515605
97	96	1.0644065	0.8250457	0.0390577	1.0018411
98	97	0.8691133	0.8165238	0.0342085	0.8758440
99	98	0.7330221	0.8434137	0.0280872	0.8210594
100	99	0.6386873	0.8713069	0.0245142	0.8727891
101	100	0.5637403	0.8826547	0.0221816	0.9048483
102	101	0.5029967	0.8922489	0.0199448	0.8991569
103	102	0.4540556	0.9027011	0.0177812	0.8915224
104	103	0.4150739	0.9141476	0.0160900	0.9048868
105	104	0.3813251	0.9186921	0.0152002	0.9446998
106	105	0.3512848	0.9212212	0.0140495	0.9243007
107	106	0.3250302	0.9252614	0.0132617	0.9439243
108	107	0.3006703	0.9250533	0.0127023	0.9578196
109	108	0.2782780	0.9255254	0.0118652	0.9340934
110	109	0.2583980	0.9285605	0.0111782	0.9421066
111	110	0.2399280	0.9285211	0.0107487	0.9615730
112	111	0.2224847	0.9272978	0.0101863	0.9476738
113	112	0.2067140	0.9291155	0.0095446	0.9370026
114	113	0.1922750	0.9301500	0.0090068	0.9436612
115	114	0.1789377	0.9306343	0.0085970	0.9544985
116	115	0.1659538	0.9274389	0.0084277	0.9803081
117	116	0.1532174	0.9232532	0.0081486	0.9668770
118	117	0.1413960	0.9228457	0.0076599	0.9400255
119	118	0.1307287	0.9245573	0.0071955	0.9393847
120	119	0.1209090	0.9248852	0.0067734	0.9413369

121	120	0.1120288	0.9265547	0.0063259	0.9339284
122	121	0.1039599	0.9279741	0.0059769	0.9448358
123	122	0.0964983	0.9282266	0.0056345	0.9427016
124	123	0.0896672	0.9292102	0.0053026	0.9411039
125	124	0.0833125	0.9291304	0.0049828	0.9396944
126	125	0.0776324	0.9318209	0.0046406	0.9313226
127	126	0.0724280	0.9329618	0.0043473	0.9367799
128	127	0.0676853	0.9345182	0.0040083	0.9220193
129	128	0.0635380	0.9387263	0.0036864	0.9197089
130	129	0.0597311	0.9400845	0.0034773	0.9432675
131	130	0.0560418	0.9382361	0.0033230	0.9556383
132	131	0.0525073	0.9369301	0.0031496	0.9478249
133	132	0.0491855	0.9367361	0.0029710	0.9432938
134	133	0.0460666	0.9365890	0.0027967	0.9413153
135	134	0.0431344	0.9363485	0.0026568	0.9499861
136	135	0.0403771	0.9360763	0.0024800	0.9334519
137	136	0.0378184	0.9366311	0.0023383	0.9428438
138	137	0.0354510	0.9374012	0.0021922	0.9375558
139	138	0.0332424	0.9376994	0.0020377	0.9294867
140	139	0.0312191	0.9391363	0.0019028	0.9338130
141	140	0.0293648	0.9406007	0.0017822	0.9366054
142	141	0.0275654	0.9387233	0.0017140	0.9617729
143	142	0.0258209	0.9367154	0.0016189	0.9444659
144	143	0.0242216	0.9380629	0.0015084	0.9317937
145	144	0.0227180	0.9379219	0.0014412	0.9554332
146	145	0.0212899	0.9371386	0.0013528	0.9386613
147	146	0.0199738	0.9381832	0.0012813	0.9471259
148	147	0.0187271	0.9375794	0.0012066	0.9417288
149	148	0.0175963	0.9396166	0.0011173	0.9259961
150	149	0.0165930	0.9429837	0.0010360	0.9271850
151	150	0.0156685	0.9442818	0.0009697	0.9359917
152	151	0.0148093	0.9451670	0.0009143	0.9429451
153	152	0.0140387	0.9479640	0.0008446	0.9236841
154	153	0.0133481	0.9508081	0.0007828	0.9269097
155	154	0.0127077	0.9520237	0.0007435	0.9498209
156	155	0.0121136	0.9532515	0.0007005	0.9420813
157	156	0.0115697	0.9551005	0.0006583	0.9397314
158	157	0.0110828	0.9579138	0.0006133	0.9317308
159	158	0.0106451	0.9605075	0.0005721	0.9327397
160	159	0.0102507	0.9629485	0.0005351	0.9353541
161	160	0.0099019	0.9659712	0.0005003	0.9348993
162	161	0.0095977	0.9692748	0.0004553	0.9101756
163	162	0.0093373	0.9728771	0.0004165	0.9146442
164	163	0.0091078	0.9754214	0.0003875	0.9305895
165	164	0.0089051	0.9777442	0.0003573	0.9218909
166	165	0.0087271	0.9800029	0.0003282	0.9185575
167	166	0.0085684	0.9818138	0.0003015	0.9188234
168	167	0.0084309	0.9839583	0.0002719	0.9016052
169	168	0.0083109	0.9857609	0.0002449	0.9008619
170	169	0.0082058	0.9873564	0.0002206	0.9008608
171	170	0.0081135	0.9887523	0.0001955	0.8858579
172	171	0.0080298	0.9896858	0.0001764	0.9025415
173	172	0.0079505	0.9901290	0.0001610	0.9127841
174	173	0.0078748	0.9904714	0.0001459	0.9060863

175	174	0.0078020	0.9907636	0.0001330	0.9115998
176	175	0.0077322	0.9910514	0.0001201	0.9027629
177	176	0.0076646	0.9912555	0.0001102	0.9181004
178	177	0.0075980	0.9913082	0.0000994	0.9021361
179	178	0.0075343	0.9916206	0.0000903	0.9080043
180	179	0.0074697	0.9914240	0.0000843	0.9338806
181	180	0.0073991	0.9905492	0.0000836	0.9916163
182	181	0.0073144	0.9885474	0.0000873	1.0443408
183	182	0.0072065	0.9852455	0.0000944	1.0807980
184	183	0.0070661	0.9805186	0.0001041	1.1030488
185	184	0.0068861	0.9745304	0.0001130	1.0857231
186	185	0.0066684	0.9683929	0.0001181	1.0445829
187	186	0.0064350	0.9649952	0.0001170	0.9906712
188	187	0.0062072	0.9645942	0.0001121	0.9587357
189	188	0.0059900	0.9650072	0.0001088	0.9702163
190	189	0.0057630	0.9621089	0.0001137	1.0446340
191	190	0.0054891	0.9524693	0.0001251	1.1007868
192	191	0.0051459	0.9374773	0.0001347	1.0766640
193	192	0.0047715	0.9272480	0.0001344	0.9978854
194	193	0.0044069	0.9235887	0.0001313	0.9767634
195	194	0.0040493	0.9188441	0.0001316	1.0022126
196	195	0.0036749	0.9075451	0.0001353	1.0284495
197	196	0.0032688	0.8895021	0.0001380	1.0193791
198	197	0.0028661	0.8768005	0.0001344	0.9740698
199	198	0.0024955	0.8707062	0.0001284	0.9558213
200	199	0.0021550	0.8635418	0.0001229	0.9570292
201	200	0.0018567	0.8615758	0.0001137	0.9250923
202	201	0.0016166	0.8706679	0.0001009	0.8877237
203	202	0.0014368	0.8887817	0.0000897	0.8888228
204	203	0.0012963	0.9022463	0.0000810	0.9033006
205	204	0.0011838	0.9131675	0.0000720	0.8884030
206	205	0.0010994	0.9286959	0.0000635	0.8813002
207	206	0.0010326	0.9392711	0.0000578	0.9112359
208	207	0.0009772	0.9463790	0.0000524	0.9068811
209	208	0.0009315	0.9532468	0.0000471	0.8981476
210	209	0.0008948	0.9606123	0.0000418	0.8876419
211	210	0.0008645	0.9660678	0.0000379	0.9073230
212	211	0.0008375	0.9687479	0.0000348	0.9164967
213	212	0.0008140	0.9720156	0.0000308	0.8855352
214	213	0.0007936	0.9749646	0.0000281	0.9143270
215	214	0.0007742	0.9755578	0.0000264	0.9377285
216	215	0.0007555	0.9758417	0.0000243	0.9211895
217	216	0.0007382	0.9770402	0.0000225	0.9238694
218	217	0.0007210	0.9766651	0.0000216	0.9599043
219	218	0.0007031	0.9752798	0.0000205	0.9529276
220	219	0.0006856	0.9749951	0.0000197	0.9592020
221	220	0.0006669	0.9727617	0.0000196	0.9934542
222	221	0.0006474	0.9706996	0.0000188	0.9599738
223	222	0.0006279	0.9699750	0.0000182	0.9670444
224	223	0.0006080	0.9682281	0.0000178	0.9778561
225	224	0.0005876	0.9665155	0.0000170	0.9586794
226	225	0.0005677	0.9661052	0.0000167	0.9784560
227	226	0.0005461	0.9619817	0.0000169	1.0115654
228	227	0.0005230	0.9576032	0.0000169	1.0017682

229	228	0.0004988	0.9538113	0.0000169	0.9996731
230	229	0.0004738	0.9498157	0.0000167	0.9877224
231	230	0.0004487	0.9470574	0.0000163	0.9743509
232	231	0.0004240	0.9450280	0.0000160	0.9872214
233	232	0.0003992	0.9414172	0.0000157	0.9805466
234	233	0.0003747	0.9386238	0.0000154	0.9774359
235	234	0.0003508	0.9362417	0.0000150	0.9738462
236	235	0.0003281	0.9354138	0.0000143	0.9557997
237	236	0.0003072	0.9361931	0.0000135	0.9455793
238	237	0.0002887	0.9398553	0.0000125	0.9270502
239	238	0.0002722	0.9427474	0.0000119	0.9470597
240	239	0.0002566	0.9425761	0.0000115	0.9644387
241	240	0.0002423	0.9445275	0.0000106	0.9283254
242	241	0.0002296	0.9476198	0.0000099	0.9320668
243	242	0.0002183	0.9506384	0.0000093	0.9384921
244	243	0.0002078	0.9516694	0.0000088	0.9457131
245	244	0.0001979	0.9526251	0.0000083	0.9406033
246	245	0.0001890	0.9550989	0.0000078	0.9394368
247	246	0.0001807	0.9558598	0.0000074	0.9484184
248	247	0.0001727	0.9559939	0.0000072	0.9728821
249	248	0.0001647	0.9537430	0.0000070	0.9768819
250	249	0.0001571	0.9536549	0.0000067	0.9498763
251	250	0.0001499	0.9544458	0.0000064	0.9553444
252	251	0.0001433	0.9558753	0.0000060	0.9370424
253	252	0.0001374	0.9585791	0.0000056	0.9316401
254	253	0.0001320	0.9606996	0.0000052	0.9306494
255	254	0.0001272	0.9637858	0.0000048	0.9190719
256	255	0.0001228	0.9655622	0.0000044	0.9347832
257	256	0.0001188	0.9668738	0.0000044	0.9940088
258	257	0.0001143	0.9625490	0.0000043	0.9723134
259	258	0.0001102	0.9637233	0.0000041	0.9537761
260	259	0.0001063	0.9649779	0.0000038	0.9226164
261	260	0.0001030	0.9687202	0.0000035	0.9165359
262	261	0.0001000	0.9705944	0.0000032	0.9210341
263	262	0.0000972	0.9725780	0.0000030	0.9262509
264	263	0.0000947	0.9740029	0.0000027	0.9186089
265	264	0.0000923	0.9752465	0.0000025	0.9217604
266	265	0.0000902	0.9762555	0.0000023	0.9262124
267	266	0.0000880	0.9765195	0.0000022	0.9434152
268	267	0.0000859	0.9755254	0.0000021	0.9638640
269	268	0.0000836	0.9736855	0.0000021	0.9837715
270	269	0.0000812	0.9707770	0.0000021	0.9930902
271	270	0.0000785	0.9675646	0.0000021	1.0173215
272	271	0.0000755	0.9611090	0.0000022	1.0450837
273	272	0.0000719	0.9527471	0.0000023	1.0645178
274	273	0.0000678	0.9424429	0.0000024	1.0398748
275	274	0.0000632	0.9323311	0.0000025	1.0201026
276	275	0.0000585	0.9260217	0.0000024	0.9839147
277	276	0.0000541	0.9241385	0.0000023	0.9540136
278	277	0.0000500	0.9240218	0.0000022	0.9530567
279	278	0.0000462	0.9245567	0.0000021	0.9527424
280	279	0.0000426	0.9230466	0.0000020	0.9612610
281	280	0.0000394	0.9231135	0.0000019	0.9543941
282	281	0.0000364	0.9245357	0.0000018	0.9222267

283	282	0.0000339	0.9301992	0.0000016	0.9061832
284	283	0.0000317	0.9376877	0.0000015	0.9005013
285	284	0.0000299	0.9433252	0.0000013	0.9244644
286	285	0.0000283	0.9433824	0.0000013	0.9573062
287	286	0.0000266	0.9404446	0.0000013	0.9814851
288	287	0.0000249	0.9364105	0.0000012	0.9666827
289	288	0.0000232	0.9340297	0.0000012	0.9701371
290	289	0.0000216	0.9308285	0.0000012	0.9738573
291	290	0.0000200	0.9257747	0.0000011	0.9852032
292	291	0.0000171	0.8523176	0.0000010	0.8703063

tables for CG with Multigrid preconditioner

```
h =      1/16    eps=0.000001    method=MG
k=4, cond A=150.416930
                k           |e|          ratio              |r|          ratio
                1      2.1998630     0.0000000       5.3253083     0.0000000
                2      0.0956633     0.0434860       0.0363016     0.0068168
                3      0.0014271     0.0149181       0.0006556     0.0180607
                4      0.0000002     0.0001388       0.0000002     0.0003796


h =      1/32           eps=0.000001    method=MG
k=4,   cond A=603.051928
                k           |e|          ratio              |r|          ratio
                1      3.1204692     0.0000000       7.1238671     0.0000000
                2      0.1461191     0.0468260       0.0360066     0.0050544
                3      0.0031325     0.0214382       0.0006083     0.0168931
                4      0.0000004     0.0001403       0.0000003     0.0005447


h =      1/64     ,n=65   eps=0.000001    method=MG
k=4, cond A=2413.598654
                k           |e|          ratio              |r|          ratio
                1      4.5764854     0.0000000      10.4620417     0.0000000
                2      0.2180254     0.0476404       0.0388724     0.0037156
                3      0.0058026     0.0266142       0.0006767     0.0174076
                4      0.0000012     0.0002040       0.0000004     0.0005660


h =      1/128    ,n=129  eps=0.000001    method=MG
k=4, cond A=9655.787254
                k           |e|          ratio              |r|          ratio
                1      6.4563811     0.0000000      14.6329466     0.0000000
                2      0.3174234     0.0491643       0.0482766     0.0032992
                3      0.0094526     0.0297790       0.0008327     0.0172495
                4      0.0000034     0.0003597       0.0000004     0.0005216
```

Tables for CG with SSOR preconditioner

Tables for CG with SSOR preconditioner h=1/16

	k	|e|	ratio	|r|	ratio
1	k	|e|	ratio	|r|	ratio
2	1	2.2693639	0.0000000	4.9322226	0.0000000

	k	|e|	ratio	|r|	ratio
3	2	1.6468398	0.7256835	0.4605221	0.0933701
4	3	1.3513665	0.8205816	0.2730178	0.5928442
5	4	0.9606275	0.7108564	0.1964313	0.7194818
6	5	0.5252730	0.5468019	0.1561994	0.7951855
7	6	0.1534885	0.2922070	0.1084332	0.6941974
8	7	0.0416077	0.2710804	0.0275143	0.2537445
9	8	0.0262239	0.6302655	0.0103129	0.3748198
10	9	0.0123709	0.4717420	0.0068234	0.6616409
11	10	0.0033437	0.2702897	0.0029164	0.4274017
12	11	0.0011039	0.3301359	0.0008844	0.3032476
13	12	0.0005418	0.4908182	0.0003122	0.3530697
14	13	0.0001777	0.3279708	0.0001683	0.5389642
15	14	0.0000471	0.2650202	0.0000421	0.2501429
16	15	0.0000242	0.5133066	0.0000142	0.3362731
17	16	0.0000147	0.6099509	0.0000054	0.3833065
18	17	0.0000053	0.3600554	0.0000040	0.7457782
19	18	0.0000004	0.0818492	0.0000004	0.0934337

Tables for CG with SSOR preconditioner h=1/32

	k	|e|	ratio	|r|	ratio
1	k	|e|	ratio	|r|	ratio
2	1	3.1617307	0.0000000	7.0150453	0.0000000
3	2	2.5617890	0.8102490	0.5539254	0.0789625
4	3	2.3542794	0.9189982	0.2962589	0.5348354
5	4	2.1065890	0.8947914	0.1986559	0.6705483
6	5	1.8518713	0.8790853	0.1422168	0.7158951
7	6	1.6156673	0.8724512	0.1133796	0.7972310
8	7	1.3789468	0.8534844	0.0910121	0.8027200
9	8	1.1345409	0.8227590	0.0843711	0.9270315
10	9	0.8224095	0.7248831	0.0847173	1.0041038
11	10	0.4180835	0.5083641	0.0825150	0.9740034
12	11	0.1235549	0.2955268	0.0482795	0.5850996
13	12	0.0551450	0.4463198	0.0203644	0.4218024
14	13	0.0278858	0.5056810	0.0122853	0.6032745
15	14	0.0128799	0.4618815	0.0057640	0.4691758
16	15	0.0081751	0.6347162	0.0026993	0.4683099
17	16	0.0058303	0.7131826	0.0013998	0.5185673
18	17	0.0047541	0.8154064	0.0006696	0.4783424
19	18	0.0038683	0.8136734	0.0005167	0.7716602
20	19	0.0021487	0.5554644	0.0005517	1.0677534
21	20	0.0011032	0.5134135	0.0002821	0.5114078
22	21	0.0007772	0.7045585	0.0001444	0.5119507
23	22	0.0005067	0.6519440	0.0001267	0.8768386
24	23	0.0002475	0.4884356	0.0000763	0.6020652
25	24	0.0001620	0.6545659	0.0000398	0.5221399
26	25	0.0001077	0.6648438	0.0000249	0.6246745
27	26	0.0000674	0.6260178	0.0000172	0.6916799
28	27	0.0000335	0.4974052	0.0000121	0.7029690
29	28	0.0000133	0.3976940	0.0000067	0.5550799
30	29	0.0000054	0.4034871	0.0000029	0.4265622
31	30	0.0000016	0.2990445	0.0000007	0.2473623

Tables for CG with SSOR preconditioner h=1/64

	k	|e|	ratio	|r|	ratio
1	k	|e|	ratio	|r|	ratio

209

1	4.5490436	0.0000000	10.0541553	0.0000000
2	3.8362026	0.8432987	0.7280693	0.0724148
3	3.7003001	0.9645737	0.3053137	0.4193471
4	3.5423025	0.9573014	0.1926717	0.6310615
5	3.3783165	0.9537064	0.1437188	0.7459256
6	3.1995620	0.9470877	0.1180847	0.8216375
7	3.0218250	0.9444496	0.1001557	0.8481680
8	2.8564678	0.9452790	0.0828611	0.8273226
9	2.6921619	0.9424794	0.0706543	0.8526838
10	2.5198220	0.9359846	0.0632266	0.8948726
11	2.3462801	0.9311293	0.0583966	0.9236077
12	2.1638926	0.9222652	0.0533587	0.9137299
13	1.9814273	0.9156773	0.0489931	0.9181837
14	1.8029129	0.9099062	0.0433065	0.8839308
15	1.6149689	0.8957553	0.0422280	0.9750970
16	1.4112614	0.8738629	0.0412639	0.9771686
17	1.1749024	0.8325193	0.0429641	1.0412016
18	0.8528075	0.7258539	0.0493421	1.1484509
19	0.4518308	0.5298157	0.0484776	0.9824796
20	0.2132808	0.4720368	0.0278557	0.5746100
21	0.1501244	0.7038816	0.0146585	0.5262282
22	0.1111903	0.7406542	0.0140617	0.9592857
23	0.0662040	0.5954116	0.0111642	0.7939485
24	0.0437040	0.6601420	0.0069109	0.6190224
25	0.0303526	0.6945044	0.0050944	0.7371500
26	0.0221190	0.7287351	0.0033023	0.6482254
27	0.0165263	0.7471507	0.0024898	0.7539705
28	0.0129979	0.7865014	0.0016588	0.6662198
29	0.0110448	0.8497327	0.0011217	0.6762264
30	0.0098942	0.8958298	0.0007244	0.6457952
31	0.0091291	0.9226731	0.0004742	0.6545639
32	0.0084946	0.9304946	0.0003511	0.7405437
33	0.0076879	0.9050339	0.0003237	0.9219407
34	0.0065529	0.8523626	0.0003485	1.0763733
35	0.0047519	0.7251674	0.0003915	1.1234523
36	0.0028376	0.5971538	0.0003422	0.8742031
37	0.0018887	0.6656051	0.0002004	0.5855785
38	0.0015408	0.8157954	0.0001243	0.6204067
39	0.0012973	0.8419573	0.0001008	0.8108109
40	0.0010583	0.8157939	0.0000853	0.8463996
41	0.0007990	0.7549702	0.0000787	0.9219365
42	0.0005529	0.6919610	0.0000666	0.8468646
43	0.0003932	0.7111390	0.0000455	0.6826150
44	0.0003029	0.7704465	0.0000339	0.7445688
45	0.0002333	0.7702420	0.0000257	0.7580561
46	0.0001916	0.8211070	0.0000175	0.6821221
47	0.0001600	0.8349544	0.0000136	0.7744449
48	0.0001358	0.8490333	0.0000095	0.7032869
49	0.0001119	0.8239944	0.0000090	0.9472819
50	0.0000825	0.7372094	0.0000083	0.9171382
51	0.0000541	0.6554525	0.0000073	0.8784722
52	0.0000349	0.6460945	0.0000048	0.6616077
53	0.0000248	0.7089250	0.0000034	0.7101718
54	0.0000182	0.7361531	0.0000023	0.6772298

| 56 | 55 | 0.0000124 | 0.6811877 | 0.0000021 | 0.9093606 |
| 57 | 56 | 0.0000048 | 0.3868446 | 0.0000009 | 0.4194420 |

Tables for CG with SSOR preconditioner h=1/128

	k	|e|	ratio	|r|	ratio
1					
2	1	6.4457756	0.0000000	14.4540779	0.0000000
3	2	5.5139993	0.8554439	0.9798537	0.0677908
4	3	5.4088913	0.9809380	0.3403982	0.3473970
5	4	5.3026393	0.9803560	0.2034492	0.5976800
6	5	5.1853961	0.9778897	0.1483529	0.7291887
7	6	5.0654954	0.9768772	0.1202717	0.8107138
8	7	4.9451401	0.9762402	0.1012563	0.8418966
9	8	4.8245020	0.9756047	0.0858063	0.8474169
10	9	4.7066959	0.9755817	0.0722637	0.8421718
11	10	4.5874237	0.9746590	0.0649999	0.8994827
12	11	4.4629279	0.9728615	0.0605125	0.9309626
13	12	4.3391818	0.9722724	0.0545367	0.9012460
14	13	4.2186913	0.9722320	0.0488740	0.8961687
15	14	4.1021192	0.9723677	0.0440151	0.9005828
16	15	3.9854067	0.9715483	0.0412976	0.9382589
17	16	3.8680259	0.9705473	0.0382945	0.9272820
18	17	3.7513422	0.9698338	0.0359069	0.9376523
19	18	3.6295795	0.9675415	0.0344123	0.9583746
20	19	3.5086283	0.9666763	0.0316764	0.9204971
21	20	3.3903746	0.9662963	0.0299359	0.9450520
22	21	3.2691604	0.9642475	0.0290988	0.9720394
23	22	3.1436705	0.9616140	0.0277443	0.9534507
24	23	3.0186217	0.9602220	0.0262770	0.9471139
25	24	2.8957417	0.9592927	0.0248096	0.9441578
26	25	2.7732455	0.9576978	0.0238982	0.9632612
27	26	2.6506297	0.9557862	0.0223533	0.9353568
28	27	2.5323184	0.9553648	0.0216356	0.9678923
29	28	2.4045866	0.9495594	0.0212703	0.9831139
30	29	2.2772037	0.9470250	0.0200174	0.9410971
31	30	2.1483420	0.9434123	0.0196727	0.9827790
32	31	2.0149086	0.9378901	0.0192478	0.9784036
33	32	1.8686594	0.9274165	0.0199859	1.0383474
34	33	1.6895903	0.9041724	0.0217554	1.0885392
35	34	1.4442876	0.8548153	0.0258210	1.1868754
36	35	1.0931431	0.7568736	0.0302817	1.1727536
37	36	0.6837108	0.6254540	0.0283683	0.9368158
38	37	0.4418863	0.6463058	0.0184521	0.6504463
39	38	0.3518847	0.7963243	0.0114809	0.6221984
40	39	0.3006824	0.8544912	0.0105466	0.9186222
41	40	0.2372194	0.7889368	0.0114586	1.0864744
42	41	0.1840178	0.7757283	0.0085694	0.7478555
43	42	0.1528291	0.8305127	0.0068315	0.7971989
44	43	0.1237482	0.8097158	0.0067367	0.9861302
45	44	0.0985025	0.7959918	0.0053549	0.7948776
46	45	0.0801615	0.8138015	0.0046556	0.8694177
47	46	0.0645647	0.8054324	0.0038580	0.8286741
48	47	0.0538188	0.8335634	0.0030348	0.7866214
49	48	0.0443159	0.8234283	0.0027050	0.8913447

50	49	0.0369617	0.8340510	0.0021314	0.7879483
51	50	0.0306704	0.8297872	0.0018818	0.8828969
52	51	0.0251646	0.8204853	0.0015780	0.8385403
53	52	0.0206586	0.8209407	0.0013350	0.8459954
54	53	0.0170433	0.8249991	0.0010816	0.8101736
55	54	0.0140243	0.8228592	0.0009494	0.8778358
56	55	0.0116435	0.8302376	0.0007679	0.8087564
57	56	0.0099038	0.8505901	0.0006226	0.8107796
58	57	0.0087211	0.8805767	0.0004813	0.7731395
59	58	0.0078802	0.9035759	0.0003957	0.8221169
60	59	0.0072607	0.9213899	0.0003211	0.8113631
61	60	0.0068350	0.9413714	0.0002460	0.7660613
62	61	0.0065434	0.9573309	0.0001854	0.7536923
63	62	0.0063343	0.9680457	0.0001386	0.7475188
64	63	0.0061677	0.9736986	0.0001019	0.7352154
65	64	0.0060189	0.9758793	0.0000781	0.7662600
66	65	0.0058564	0.9730031	0.0000697	0.8929152
67	66	0.0056418	0.9633473	0.0000711	1.0197906
68	67	0.0053083	0.9408880	0.0000812	1.1420576
69	68	0.0047907	0.9024899	0.0000954	1.1756096
70	69	0.0040810	0.8518580	0.0001007	1.0553830
71	70	0.0033234	0.8143729	0.0000974	0.9667177
72	71	0.0026225	0.7891023	0.0000930	0.9553683
73	72	0.0019626	0.7483464	0.0000878	0.9441616
74	73	0.0014856	0.7569965	0.0000692	0.7874535
75	74	0.0012467	0.8391488	0.0000479	0.6925092
76	75	0.0011161	0.8952418	0.0000366	0.7631700
77	76	0.0010177	0.9118797	0.0000315	0.8615686
78	77	0.0009285	0.9123426	0.0000269	0.8534135
79	78	0.0008523	0.9179524	0.0000225	0.8376764
80	79	0.0007739	0.9080042	0.0000224	0.9961889
81	80	0.0006799	0.8785639	0.0000216	0.9629637
82	81	0.0005881	0.8648625	0.0000201	0.9290428
83	82	0.0004882	0.8301801	0.0000208	1.0351272
84	83	0.0003913	0.8015060	0.0000176	0.8481698
85	84	0.0003242	0.8285988	0.0000142	0.8060784
86	85	0.0002702	0.8334427	0.0000129	0.9107954
87	86	0.0002246	0.8311510	0.0000109	0.8432222
88	87	0.0001916	0.8532194	0.0000089	0.8125177
89	88	0.0001647	0.8592511	0.0000077	0.8643326
90	89	0.0001439	0.8739769	0.0000063	0.8266711
91	90	0.0001263	0.8773506	0.0000057	0.8934908
92	91	0.0001118	0.8853776	0.0000047	0.8270530
93	92	0.0001006	0.8995960	0.0000040	0.8506773
94	93	0.0000920	0.9145468	0.0000031	0.7799253
95	94	0.0000856	0.9305279	0.0000025	0.8191927
96	95	0.0000800	0.9350646	0.0000021	0.8149292
97	96	0.0000752	0.9394217	0.0000017	0.8363557
98	97	0.0000698	0.9282142	0.0000017	0.9814936
99	98	0.0000632	0.9062999	0.0000017	1.0138043
100	99	0.0000547	0.8654208	0.0000018	1.0641346
101	100	0.0000461	0.8418280	0.0000016	0.8914509
102	101	0.0000391	0.8481096	0.0000014	0.8598060
103	102	0.0000338	0.8637455	0.0000011	0.8164981

104	103	0.0000254	0.7524275	0.0000009	0.8216432

2.6.5.3 Table CG with incomplete cholesky

Table CG with incomplete cholesky preconditioner $\epsilon = 10^{-2}$, $h = \frac{1}{16}$

	k	|e|	ratio	|r|	ratio
1					
2	1	2.2117568	0.0000000	4.9847338	0.0000000
3	2	0.9605500	0.4342928	0.1236319	0.0248021
4	3	0.1101739	0.1146987	0.0740761	0.5991666
5	4	0.0098178	0.0891114	0.0064913	0.0876300
6	5	0.0006397	0.0651529	0.0007141	0.1100128
7	6	0.0000566	0.0885013	0.0000495	0.0693702
8	7	0.0000012	0.0210676	0.0000009	0.0178494

Table CG with incomplete cholesky preconditioner $\epsilon = 10^{-2}$, $h = \frac{1}{32}$

	k	|e|	ratio	|r|	ratio
1					
2	1	3.1411335	0.0000000	6.9640117	0.0000000
3	2	2.0214097	0.6435287	0.1420103	0.0203920
4	3	1.3623817	0.6739760	0.1149900	0.8097304
5	4	0.6575397	0.4826399	0.0682371	0.5934171
6	5	0.1011955	0.1539003	0.0247907	0.3633033
7	6	0.0225210	0.2225494	0.0069472	0.2802356
8	7	0.0067767	0.3009055	0.0015084	0.2171254
9	8	0.0019548	0.2884659	0.0004569	0.3028904
10	9	0.0008448	0.4321526	0.0001217	0.2664539
11	10	0.0002326	0.2753055	0.0000595	0.4889375
12	11	0.0000681	0.2927750	0.0000127	0.2125976
13	12	0.0000188	0.2754409	0.0000052	0.4094558
14	13	0.0000022	0.1195568	0.0000003	0.0640489

Table CG with incomplete cholesky preconditioner $\epsilon = 10^{-2}$, $h = \frac{1}{64}$

	k	|e|	ratio	|r|	ratio
1					
2	1	4.6047158	0.0000000	10.3310633	0.0000000
3	2	3.4648962	0.7524669	0.1937203	0.0187512
4	3	3.0608082	0.8833766	0.1211410	0.6253400
5	4	2.5757427	0.8415237	0.0563599	0.4652418
6	5	2.0650650	0.8017358	0.0413553	0.7337726
7	6	1.5529332	0.7520021	0.0446463	1.0795781
8	7	0.9346100	0.6018353	0.0369208	0.8269618
9	8	0.2831630	0.3029745	0.0265590	0.7193503
10	9	0.0732691	0.2587522	0.0112939	0.4252372
11	10	0.0321756	0.4391431	0.0039872	0.3530414
12	11	0.0178107	0.5535469	0.0013842	0.3471708
13	12	0.0110120	0.6182812	0.0008543	0.6171309
14	13	0.0073396	0.6665038	0.0004321	0.5057645
15	14	0.0039218	0.5343410	0.0002636	0.6101306
16	15	0.0023472	0.5984915	0.0001345	0.5102407
17	16	0.0013514	0.5757587	0.0001004	0.7467383
18	17	0.0007088	0.5244614	0.0000461	0.4586731
19	18	0.0002407	0.3396233	0.0000342	0.7426365
20	19	0.0001089	0.4524637	0.0000103	0.3000293

21	20	0.0000698	0.6407427	0.0000045	0.4406339
22	21	0.0000343	0.4918772	0.0000034	0.7510210
23	22	0.0000068	0.1977073	0.0000005	0.1584427

Table CG with incomplete cholesky preconditioner $\epsilon = 10^{-2}$, $h = \frac{1}{400}$

	k	|e|	ratio	|r|	ratio
1					
2	1	6.4885732	0.0000000	14.5067213	0.0000000
3	2	5.2612798	0.8108531	0.2641937	0.0182118
4	3	5.0111119	0.9524511	0.1164715	0.4408564
5	4	4.6620201	0.9303364	0.0500341	0.4295820
6	5	4.3222195	0.9271130	0.0496910	0.9931436
7	6	3.9989939	0.9252177	0.0423171	0.8516054
8	7	3.6528553	0.9134436	0.0289715	0.6846290
9	8	3.2996211	0.9032991	0.0270517	0.9337335
10	9	2.9620134	0.8976829	0.0260842	0.9642352
11	10	2.6075430	0.8803279	0.0207816	0.7967142
12	11	2.2325951	0.8562064	0.0176972	0.8515802
13	12	1.8177887	0.8142044	0.0202955	1.1468181
14	13	1.2782021	0.7031632	0.0227175	1.1193341
15	14	0.5725682	0.4479481	0.0191123	0.8413068
16	15	0.2181351	0.3809767	0.0101596	0.5315716
17	16	0.1256071	0.5758222	0.0059499	0.5856484
18	17	0.0759987	0.6050512	0.0036252	0.6092867
19	18	0.0443639	0.5837452	0.0024628	0.6793574
20	19	0.0240991	0.5432141	0.0014339	0.5822123
21	20	0.0156123	0.6478383	0.0008604	0.6000823
22	21	0.0094219	0.6034940	0.0005069	0.5891044
23	22	0.0066705	0.7079759	0.0002955	0.5830443
24	23	0.0054586	0.8183167	0.0001677	0.5673221
25	24	0.0047031	0.8615927	0.0000873	0.5205250
26	25	0.0040218	0.8551456	0.0000745	0.8537794
27	26	0.0030834	0.7666791	0.0000688	0.9227595
28	27	0.0021837	0.7082150	0.0000606	0.8814875
29	28	0.0014407	0.6597450	0.0000421	0.6948456
30	29	0.0011682	0.8108344	0.0000212	0.5036729
31	30	0.0009944	0.8512118	0.0000176	0.8296109
32	31	0.0007347	0.7388340	0.0000182	1.0336879
33	32	0.0005166	0.7031852	0.0000114	0.6264216
34	33	0.0003559	0.6888947	0.0000107	0.9380414
35	34	0.0002062	0.5794708	0.0000081	0.7584751
36	35	0.0001239	0.6006305	0.0000052	0.6377109
37	36	0.0000837	0.6759086	0.0000032	0.6095393
38	37	0.0000659	0.7871051	0.0000017	0.5390215
39	38	0.0000539	0.8184570	0.0000013	0.7405635
40	39	0.0000202	0.3752176	0.0000009	0.7459009

Tables for CG with incomplete cholesky preconditioner $\epsilon = 10^{-3}$

1	h =	1/16	,n=17	eps=0.000001	method=CHL
2		k	|e|	ratio	|r|	ratio
3		1	2.2113015	0.0000000	4.8698707	0.0000000
4		2	0.1268927	0.0573837	0.0196604	0.0040372
5		3	0.0011025	0.0086886	0.0005919	0.0301065

```
6             4        0.0000002      0.0001684      0.0000001      0.0001565
7
8   h =      1/32     ,n=33    eps=0.000001   method=CHL
9            k              |e|        ratio           |r|          ratio
10           1        3.2189095      0.0000000      7.0357482      0.0000000
11           2        0.8589679      0.2668506      0.0306470      0.0043559
12           3        0.0363666      0.0423375      0.0064895      0.2117505
13           4        0.0025347      0.0696990      0.0003268      0.0503545
14           5        0.0001276      0.0503368      0.0000319      0.0976751
15           6        0.0000006      0.0047450      0.0000001      0.0037884
16  h =      1/64     ,n=65    eps=0.000001   method=CHL
17           k              |e|        ratio           |r|          ratio
18           1        4.5281055      0.0000000     10.2891303      0.0000000
19           2        2.3962171      0.5291876      0.0305161      0.0029659
20           3        1.0190972      0.4252942      0.0239474      0.7847488
21           4        0.1236226      0.1213060      0.0102118      0.4264245
22           5        0.0186393      0.1507755      0.0012096      0.1184530
23           6        0.0016236      0.0871065      0.0002558      0.2114490
24           7        0.0002756      0.1697321      0.0000182      0.0713307
25           8        0.0000163      0.0592016      0.0000009      0.0502623
26  h =      1/128    ,n=129   eps=0.000001   method=CHL
27           k              |e|        ratio           |r|          ratio
28           1        6.4658650      0.0000000     14.3821767      0.0000000
29           2        4.5500543      0.7037039      0.0349350      0.0024290
30           3        3.5640524      0.7832989      0.0316566      0.9061590
31           4        2.5832848      0.7248167      0.0229909      0.7262586
32           5        1.4701895      0.5691163      0.0119903      0.5215250
33           6        0.3578766      0.2434221      0.0077001      0.6421915
34           7        0.0850789      0.2377325      0.0036799      0.4779069
35           8        0.0274688      0.3228626      0.0009809      0.2665421
36           9        0.0121215      0.4412808      0.0002445      0.2492235
37          10        0.0034751      0.2866930      0.0001041      0.4258668
38          11        0.0019516      0.5615987      0.0000341      0.3280329
39          12        0.0010720      0.5492590      0.0000211      0.6186532
40          13        0.0003595      0.3353743      0.0000058      0.2766025
41          14        0.0000439      0.1220343      0.0000010      0.1636738
```

Chapter 3

Study notes

3.1 note on finding expressions for centered difference of higher order

3.1.1 Introduction

To find centered difference approximation for $f'(x)$, we can processed as follows.
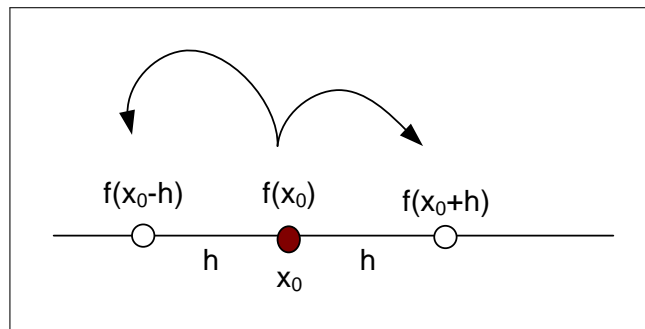

Figure 3.1: centered difference

3.1.2 approximation for $f'(x_o)$

Since

$$f(x_o + h) = f(x_o) + hf'(x_o) + O(h)$$

Then

$$f'(x_o) \approx \frac{1}{h}\left[f(x_o + h) - f(x_o)\right] \tag{1}$$

But we also know that

$$f(x_o - h) = f(x_o) - hf'(x_o) + O(h)$$

The trick is to find $f(x_o)$ from the above and plug it in (1). From the above we find

$$f(x_o) \approx f(x_o - h) + hf'(x_o)$$

substituting the above in (1) gives

$$f'(x_o) \approx \frac{1}{h}\left[f(x_o + h) - \left(f(x_o - h) + hf'(x_o)\right)\right]$$

$$\approx \frac{f(x_o + h) - f(x_o - h)}{h} - f'(x_o)$$

Hence

$$f'(x_o) \approx \frac{f(x_o + h) - f(x_o - h)}{2h}$$

3.1.3   approximation for $f''(x_o)$

We can do the same trick to find centered difference approximation for $f''(x_o)$. Since

$$f(x_o + h) = f(x_o) + hf'(x_o) + \frac{h^2}{2}f''(x_o) + O\left(h^2\right)$$

Then

$$f''(x_o) \approx \frac{2}{h^2}\left(f(x_o + h) - f(x_o) - hf'(x_o)\right) \tag{2}$$

But we also know that

$$f(x_o - h) = f(x_o) - hf'(x_o) + \frac{h^2}{2}f''(x_o) + O\left(h^2\right)$$

$$hf'(x_o) \approx -f(x_o - h) + f(x_o) + \frac{h^2}{2}f''(x_o)$$

$$f'(x_o) \approx \frac{f(x_o) - f(x_o - h)}{h} + \frac{h}{2}f''(x_o)$$

Sustituting the above into (2), we find

$$f''(x_o) \approx \frac{2}{h^2}\left[f(x_o + h) - f(x_o) - h\left(\frac{f(x_o) - f(x_o - h)}{h} + \frac{h}{2}f''(x_o)\right)\right]$$

$$\approx \frac{2}{h^2}\left[f(x_o + h) - f(x_o) - \left(f(x_o) - f(x_o - h) + \frac{h^2}{2}f''(x_o)\right)\right]$$

$$\approx \frac{2}{h^2}\left[f(x_o + h) - 2f(x_o) + f(x_o - h) - \frac{h^2}{2}f''(x_o)\right]$$

$$\approx \frac{2}{h^2}\left(f(x_o + h) - 2f(x_o) + f(x_o - h)\right) - f''(x_o)$$

$$2f''(x_o) \approx \frac{2}{h^2}\left(f(x_o + h) - 2f(x_o) + f(x_o - h)\right)$$

Solving for $f''(x_0)$ from the above gives

$$f''(x_o) \approx \frac{f(x_o + h) - 2f(x_o) + f(x_o - h)}{h^2}$$

This method can be used to find approximations for higher derivatives.

3.2　Generating Error Table, Handout of oct 8,2010 for the $u_{xx} = -\sin(3\pi x)$

Generating finite difference error table, Handout of oct 8,2010

Math 228A, UC Davis

Nasser M. Abbasi, 10/11/2010

Problem: Generate error table for finite difference approximation to solution of
$u_{xx} = \sin(3\,\pi\,x)$, u(0)=1,u(1)=0

```
In[6]:= len = 1; (*length of element *)
      h = len / 16; (*Initial spacing *)
      α = 1; (* left Boundary conditions *)
      β = 0; (* right Boundary conditions *)

      (*generate spacings by halving it each time *)
      nIter = 7;
      hValues = Table[h / 2^i, {i, 0, nIter}];

      (*---{nPoints,h,errorNorm2,errorNorm1,errorNormInfinity,errorNormFrobenius}--*)
      tbl = Table[ process[N[hValues[[i]]], len, x], {i, nIter + 1}];

      (*we are done, format the table *)
      tbl = Table[{
          tbl[[i, 1]],
          se[tbl[[i, 2]]],
          se[tbl[[i, 3]]], If[i == 1, 0, se[ tbl[[i - 1, 3]] / tbl[[i, 3]] ]],
          se[tbl[[i, 4]]], If[i == 1, 0, se[ tbl[[i - 1, 4]] / tbl[[i, 4]] ]],
          se[tbl[[i, 5]]], If[i == 1, 0, se[ tbl[[i - 1, 5]] / tbl[[i, 5]] ]],
          se[tbl[[i, 6]]], If[i == 1, 0, se[ tbl[[i - 1, 6]] / tbl[[i, 6]] ]]
          }, {i, 1, Length[tbl]}];
      heading =
        {"pts", "h", "‖e‖₂", "ratio", "‖e‖₁", "ratio", "‖e‖∞", "ratio", "‖e‖frobenius", "ratio"};

      Framed[TableForm[tbl, TableHeadings → {None, heading},
        TableSpacing → {1, 3}], ImageSize → 870]
```
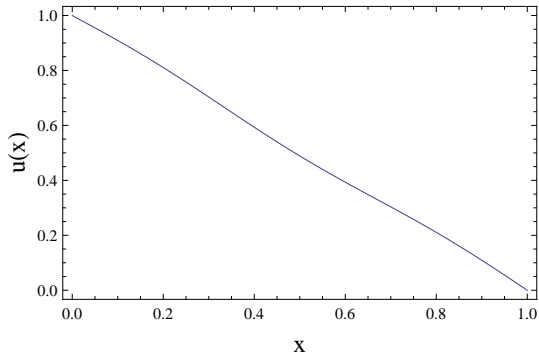
219

2 │ handout1_table.nb

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

solution to $u_{xx}=-\sin(3\pi x)$, $u(0)=1$, $u(1)=0$

$$u(x) = -x + \frac{\sin(3\pi x)}{9\pi^2} + 1$$

Out[15]=

pts	h	$\|e\|_2$	ratio	$\|e\|_1$	ratio	$\|e\|_\infty$
15.	6.2500e−2	2.3423e−4	0	2.1020e−4	0	3.3125e−4
31.	3.1250e−2	5.7795e−5	4.0527e	5.1992e−5	4.0429e	8.1734e−5
63.	1.5625e−2	1.4402e−5	4.0131e	1.2964e−5	4.0106e	2.0367e−5
127.	7.8125e−3	3.5975e−6	4.0033e	3.2387e−6	4.0027e	5.0876e−6
255.	3.9063e−3	8.9919e−7	4.0008e	8.0955e−7	4.0007e	1.2717e−6
511.	1.9531e−3	2.2479e−7	4.0002e	2.0238e−7	4.0002e	3.1790e−7
1023.	9.7656e−4	5.6196e−8	4.0000e	5.0594e−8	4.0000e	7.9473e−8
2047.	4.8828e−4	1.4049e−8	4.0000e	1.2648e−8	4.0000e	1.9868e−8

■ **Function process**

```
process[h_, len_, x_] :=
  Module[{de, sol, solExact, pExact, u, f, A, approxSolution, data, pApprox, i,
    nPoints, error, errorNorm2, errorNorm1, errorNormInfinity, errorNormFrobenius},
          len
    nPoints = ─── - 1;
           h
    de = u''[x] == -Sin[3 Pi x];
    sol = First@DSolve[{de, u[0] == 1, u[1] == 0}, u[x], x];
    solExact = Simplify[(u[x] /. sol)];
    pExact = Plot[solExact, {x, 0, 1},
      Frame → True,
      PlotRange → All,
      FrameLabel → {{Style["u(x)", 14], None},
        {Style["x", 14], Column[{Style["solution to u_xx=-sin(3πx), u(0)=1,u(1)=0", 14],
            Style[Row[{"u(x) = ", solExact}], 14]}, Alignment → Center]}}
     ];

    A = makeA[nPoints];


    f = makeF[nPoints, α, β, rhs, len];
                            1
    approxSolution = LinearSolve[── A, f];
                            h²
    data = Table[{(i - 1) * h,
        If[i == 1, α, If[i == nPoints + 2, β, approxSolution[[i - 1]]]]}, {i, 1, nPoints + 2}];
    pApprox = ListPlot[data, Joined → True, AxesOrigin → {0, 0},
      Frame → True, PlotStyle → Red, PlotRange → All];

    If[h == 0.0625,
     {Print[MatrixForm[A]];
      Print[pExact];
     }];

    error = Table[approxSolution[[i]] - solExact /. x → (i * h), {i, 1, nPoints}];
    errorNorm2 = Norm[error, 2] * √h ;
    errorNorm1 = Norm[error, 1] * h;
    errorNormInfinity = Norm[error, Infinity];
    errorNormFrobenius = Norm[error, "Frobenius"] * √h ;

    {nPoints, h, errorNorm2, errorNorm1, errorNormInfinity, errorNormFrobenius}

   ];
```

■ **Function to format decimal numbers like the handout**

```
In[2]:= se[n_] := Module[{},
          Style[ScientificForm[N[n, $MachinePrecision], {6, 4},
            NumberFormat → (Row[{#1, "e", #3}] &), NumberPadding → {"", "0"}], 14]];
```

■ **Function to construct A matrix**

```
In[3]:=
      makeA[nPoints_] := Module[{A},
        fill[i_, j_] := If[i == j, -2, If[i == j + 1 || i == j - 1, 1, 0]];
        A = Table[fill[i, j], {i, nPoints}, {j, nPoints}]
        ];
```

■ **Function to construct the f matrix**

```
In[4]:= makeF[nPoints_, α_, β_, f_, len_] := Module[{h, rhs, i},
```

$$h = \frac{len}{nPoints + 1};$$

$$rhs = Table\left[If\left[i == 1, f[i*h] - \frac{\alpha}{h^2}, If\left[i == nPoints, f[i*h] - \frac{\beta}{h^2}, f[i*h]\right]\right], \{i, nPoints\}\right]$$

```
        ];
```

■ **Function to evaluate the rhs at a grid point**

```
In[5]:= rhs[x_] := Module[{}, -Sin[3 π x]];
```

3.3 generate table 1 in textbook on approximation of derivatives

HTML

3.4 looking at eigenvalues of weighted jacobian iteration matrix

HTML