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HW # 7

Math 121B

NASSER ABBASI

UCB extension

ch 12

$\boxed{22.1}$ verify equations (22.2) (22.3) (22.4) and (22.8)

22.2 $(D-x)(D+x)y = \left(\frac{d}{dx}-x\right)(y'+xy) = y''-x^2y+y$ —— ⓐ

and $(D+x)(D-x)y = y''-x^2y-y.$ —— ⓑ

note $D=\frac{d}{dx}$

eq a is $\left(\frac{d}{dx}-x\right)\left(\frac{d}{dx}+x\right)y$

$$= \left(\frac{d}{dx}-x\right)\left(\frac{dy}{dx}+xy\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{d}{dx}(xy) - x\frac{dy}{dx} - x^2y$$

$$= \frac{d^2y}{dx^2} + \left(x\frac{dy}{dx}+y\frac{dx}{dx}\right) - x\frac{dy}{dx} - x^2y$$

$$= y'' + \underline{xy'} + y - \underline{xy'} - x^2y$$

$$= y''+y-x^2y$$ verified ok.

for b

$$\left(\frac{d}{dx}+x\right)\left(\frac{d}{dx}-x\right)y =$$

$$= \left(\frac{d}{dx}+x\right)\left(\frac{d}{dx}y - xy\right)$$ =1

$$= \frac{d}{dy}\left(\frac{dy}{dx}\right) - \frac{d}{dx}(xy) + x\frac{dy}{dx} - x^2y$$

$$= \frac{d^2y}{dx^2} - \left(x\frac{d}{dx}y + y\frac{d}{dx}x\right) + x\frac{dy}{dx} - x^2y$$

$$= y'' - xy' - y + xy' - x^2y$$

$$= y'' - y - x^2y$$ verifed ok.

for 22.3 ⟶

(22.3) need to verify $(D-x)(D+x)y_n = -2ny_n$.

since $(D-x)(D+x)y = y'' - x^2y + y$ from 22.2

then $(D-x)(D+x)y_n = y_n'' - x^2y_n + y_n$

but $y_n'' - x^2y_n = -(2n+1)y_n$ from 22.1

so $(D-x)(D+x)y_n = -(2n+1)y_n + y_n$

so $(D-x)(D+x)y_n = -2ny_n - y_n + y_n = \boxed{-2ny_n}$

to verify (22.4)

$$(D+x)(D-x)y_n = -2(n+1)y_n.$$

since $(D+x)(D-x)y = y'' - x^2y - y$ from 22.2

Then $(D+x)(D-x)y_n = y_n'' - x^2y_n - y_n$

so $(D+x)(D-x)y_n = -(2n+1)y_n - y_n$

$= -2ny_n - y_n - y_n$

$= -2ny_n - 2y_n = \boxed{-2(n+1)y_n}$

to verify 22.8

$$y_{m-1} = (D+x)y_m$$

from (22.4) $(D+x)(D-x)y_n = -2(n+1)y_n$

from (22.5) $(D+x)(D-x)[(D+x)y_m] = -2m[(D+x)y_m]$

if $y_n = (D+x)y_m$ and $(n+1) = m$, then equations are identical. then $n = m-1$ from .

so $\boxed{y_{m-1} = (D+x)y_m}$ by replacing n by $m-1$ in this.

Ch 12

$\boxed{22.4}$ using 22.12 find Hermite poly given in 22.13, then use 22.17b to find $H_3(x)$ and $H_4(x)$.

22.12 is $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$

22.17b is $H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)$.

22.13 is $H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$.

from 22.12, let $n=0$

$$H_0(x) = (-1)^0 e^{x^2} \frac{d^0}{dx^0} e^{-x^2}$$

$$= 1 \, e^{x^2} \left(e^{-x^2}\right) = \boxed{1}$$

let $n=1$

$$H_1(x) = (-1)^1 e^{x^2} \frac{d^1}{dx^1} e^{-x^2}$$

$$= (-1) e^{x^2} \left(-2x e^{-x^2}\right)$$

$$= -e^{x^2}\left(-2x e^{-x^2}\right)$$

$$= \boxed{2x}$$

let $n=2$

$$H_2(x) = (-1)^2 e^{x^2} \frac{d^2}{dx^2} \left(e^{-x^2}\right)$$

$$= 1 \, e^{x^2} \frac{d}{dx}\left(-2x e^{-x^2}\right)$$

$$= e^{x^2}\left(-2x\left(-2x e^{-x^2}\right) + e^{-x^2}(-2)\right)$$

$$= e^{x^2}\left(4x^2 e^{-x^2} - 2e^{-x^2}\right)$$

$$= \boxed{4x^2 - 2}.$$

now use 22.17b to find H_3 and H_4 $\rightarrow$

from 22.17b $H_{n+1}(x) = 2x H_n - 2n H_{n-1}$

let $\underline{n=2}$

so $H_3 = 2x H_2 - 2(2) H_1$

$= 2x(4x^2-2) - 4(2x)$

$= 8x^3 - 4x - 8x$

$= \boxed{8x^3 - 12x}$

let $n=3$

$H_4 = 2x H_3 - 2(3) H_2$

$= 2x(8x^3 - 12x) - 6(4x^2-2)$

$= 16x^4 - 24x^2 - 24x^2 + 12$

$H_4 = \boxed{16x^4 - 48x^2 + 12}$

Ch 12

$\boxed{22.5}$ Solve Hermit DE

$$y'' - 2xy' + 2py = 0$$

by power series.

let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$

$\rightarrow -2xy' = -2x a_1 - (2\cdot 2)a_2 x^2 - (2\cdot 3)a_3 x^3 - (2\cdot 4)a_4 x^4 - \cdots$

$\rightarrow y'' = 2a_2 + (2\cdot 3)a_3 x + (3\cdot 4)a_4 x^2 + \cdots$

$\rightarrow 2py = 2pa_0 + 2pa_1 x + 2pa_2 x^2 + \cdots$

set up Table

	x^0	x^1	x^2	x^n
y''	$2a_2$	$6a_3$	$12a_4$	$(n+1)(n+2)a_{n+2}$
$-2xy'$	—	$-2a_1$	$-4a_2$	$-2(n)a_n$
$2py$	$2pa_0$	$2pa_1$	$2pa_2$	$2pa_n$

from first Column, $2pa_0 + 2a_2 = 0$

so $a_2 = -\frac{2pa_0}{2} = \boxed{-pa_0}$

recursive equation

$$a_{n+2}(n+1)(n+2) - a_n 2n + a_n 2p = 0$$

$$a_{n+2} = \frac{a_n(2n-2p)}{(n+1)(n+2)}$$

$\rightarrow$

$n=1$ $\quad \underbrace{a_{1+2}}_{3} = -a_1 \dfrac{(2\times 1 - 2P)}{(1+1)(1+2)}$

$n=2$ $\quad \underbrace{a_{2+2}}_{4} = -a_2 \dfrac{(2\times 2 - 2P)}{(2+1)(2+2)} = +Pa_0 \dfrac{(2\times 2 - 2P)}{(2+1)(2+2)}$

$n=3$ $\quad \underbrace{a_{3+2}}_{5} = -a_3 \dfrac{(2\times 3 - 2P)}{(3+1)(3+2)} = -\left(-a_1 \dfrac{(2\times 1 - 2P)}{(1+1)(1+2)} \dfrac{2\cdot 3 - 2P}{(3+1)(3+2)}\right)$

$$= a_1 \frac{(2\cdot 1 - 2P)(2\cdot 3 - 2P)}{(1+1)(1+2)(1+3)(3+2)}$$

$n=4$ $\quad \underbrace{a_{4+2}}_{6} = -a_4 \dfrac{(2\cdot 4 - 2P)}{(4+1)(4+2)} = -\left[Pa_0 \dfrac{(2\cdot 2 - 2P)}{(2+1)(2+2)}\right] \dfrac{(2\cdot 4 - 2P)}{(4+1)(4+2)}$

$$= -Pa_0 \frac{(2\cdot 2 - 2P)(2\cdot 4 - 2P)}{(2+1)(2+2)(4+1)(4+2)}$$

So $y = a_0 + a_1 x + (-Pa_0)x^2 + \left(-a_1 \dfrac{(2\times 1 - 2P)}{(1+1)(1+2)}\right)x^3 + \cdots$

$$y = a_0\left(1 - Px^2 + P\frac{(2\times 2 - 2P)}{(2+1)(2+2)}x^4 - P\frac{(2\cdot 2 - 2P)(2\cdot 4 - 2P)}{(2+1)(2+2)(4+1)(4+2)}x^6 + \cdots\right)$$
$$+ a_1\left(x - \frac{2\times 1 - 2P}{2\cdot 3}x^3 + \frac{(2\cdot 1 - 2P)(2\cdot 3 - 2P)}{2\cdot 3\cdot 4\cdot 5}x^5 - \cdots\right)$$

$$y = a_0\left(1 - Px^2 + P\frac{(2\cdot 2 - 2P)}{3\cdot 4}x^4 - P\frac{(2\cdot 2 - 2P)(2\cdot 4 - 2P)}{3\cdot 4\cdot 5\cdot 6}x^6 + \cdots\right)$$
$$+ a_1\left(x - \frac{(2\cdot 1 - 2P)}{2\cdot 3}x^3 + \frac{(2\cdot 1 - 2P)(2\cdot 3 - 2P)}{2\cdot 3\cdot 4\cdot 5}x^5 - \cdots\right)$$

To make denomator a factorial, multiply by $\frac{2}{2}$ the a_0 series:

$$y = a_0\left(\frac{2}{2}\times 1 - \frac{2Px^2}{2} + \frac{2(\overbrace{2\cdot 2-2P}^{2(2-P)})P}{4!}x^4 - \frac{2P(\overbrace{2\cdot 2-2P}^{2(2-P)})(\overbrace{2\cdot 4-2P}^{2(4-P)})}{6!}x^6 + \cdots\right)$$

$$+ a_1\left(x - \frac{(\overbrace{2\cdot 1-2P}^{2(1-P)})}{3!}x^3 - \frac{(\overbrace{2\cdot 1-2P}^{2(1-P)})(\overbrace{2\cdot 3-2P}^{2(3-P)})}{5!}x^5 - \cdots\right)$$

So The general Term for $\underline{a_0}$ series is:

$$(a_0)\ \frac{2^{2n}\ P\left(\overbrace{(2-P)(4-P)(6-P)\cdots((2n-2)-P)}^{\text{one of these is zero for even } P}\right)x^{2n}}{2n!}$$

where I have extracted '2' from each product to get the 2^{2n} part. for example, the above for x^6 is $x^{2\times 3}$, ie $n=3$ is

$$\frac{2^6 P\left((2-P)(4-P)\right)x^6}{6!} \quad \text{etc...}$$

so we see clearly now that if P is positive even, then the series terminates. For example, if $P=N$, Then the $(N-P)$ term will cause The term to be zero, and each term after that to be zero as well..

for The $\underline{a_1}$ series, general term is:

$$(a_1)\quad 2^{2n}\ \frac{\overbrace{(1-P)(3-P)(5-P)(7-P)\cdots(2n+1)-P)}^{\text{one of these will be zero for odd } P}}{(2n+1)!}\ x^{2n+1}$$

here we see if P is odd, Then one of the terms in the product $(1-P)(3-P)\cdots((2n-1)-P)$ will be zero. and so the whole term is zero. and series terminates.

so, since P can be either odd or even, then solution will only contain the a_0 or a_1 series.

to find H_0, H_1 and H_2.

the polynomials are

$$y = a_0\left(1 - Px^2 + \frac{2(4-2P)Px^4}{24} - \cdots\right) + a_1\left(x - \frac{(2-2P)}{6}x^3 + \cdots\right)$$

$$y = a_0\left(1 - Px^2 + \tfrac{1}{12}(4P-2P^2)x^4 - \cdots\right) + a_1\left(x - \frac{(1-P)}{3}x^3 + \cdots\right)$$

$$= a_0\left(1 - Px^2 + \tfrac{1}{6}(2P-P^2)x^4 - \cdots\right) + a_1\left(x - \left(\frac{1-P}{3}\right)x^3 + \cdots\right)$$

so now look for polynomials with highest order term as

$$(2x)^0, \quad (2x)^1, \quad (2x)^2$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$a_0 \times 1 \qquad a_1 x \qquad a_0(-Px^2) + a_0 \times 1$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$a_0 \qquad a_1 x \qquad -Pa_0x^2 + a_0 \times 1$$

$$(H_0) \qquad (H_1) \qquad (H_2)$$

let $a_0 = 1$
we get
$\boxed{H_0 = 1}$

let $a_1 = 2$
we get
$\boxed{H_1 = 2x}$

let $a_0 = -2$, $P = 2$
we get
$\boxed{4x^2 - 2}$

Ch 12

$\boxed{22.7}$ Prove that the functions $H_n(x)$ are orthogonal on $(-\infty, \infty)$ w.r.t. weight function e^{-x^2}.

Hermit DE is $y'' - 2xy' + 2ny = 0$ —— ①

to show they are orthogonal, need to show that

$$\int_{-\infty}^{\infty} e^{-x^2} H_n H_m \, dx = 0 \qquad n \neq m.$$

where H_n is solution to $y'' - 2xy' + 2ny = 0$

and H_m is solution to $y'' - 2xy' + 2my = 0$

using hint, I write ① as

$$e^{x^2} \frac{d}{dx}\left(e^{-x^2} y'\right) + 2ny = 0$$

since H_n is solution to ↗, then can write

$$\boxed{\begin{aligned} &e^{x^2} \frac{d}{dx}\left(e^{-x^2} H_n'\right) + 2n H_n = 0 \\ \text{and} \quad &e^{x^2} \frac{d}{dx}\left(e^{-x^2} H_m'\right) + 2m H_m = 0 \end{aligned}}$$

multiply first equation by H_m, second by H_n and subtract from each others $\Longrightarrow$

$$H_m e^{x^2} \frac{d}{dx}\left(e^{-x^2} H_n'\right) + 2n H_n H_m - H_n e^{x^2} \frac{d}{dx}\left(e^{-x^2} H_m'\right) - 2m H_m H_n = 0$$

divide by e^{x^2} $\Longrightarrow$

$$H_m \frac{d}{dx}\left(e^{-x^2} H_n'\right) + (2n-2m) H_n H_m e^{-x^2} - H_n \frac{d}{dx}\left(e^{-x^2} H_m'\right) = 0$$

or $$H_m \frac{d}{dx}\left(e^{-x^2} H_n'\right) - H_n \frac{d}{dx}\left(e^{-x^2} H_m'\right) + 2(n-m) H_n H_m e^{-x^2} = 0$$

integrate $\int_{-\infty}^{\infty} \Rightarrow$

equation (A)

$$\overset{①}{\int_{-\infty}^{\infty} H_m \frac{d}{dx}\left(e^{-x^2} H_n'\right) dx} - \overset{②}{\int_{-\infty}^{\infty} H_n \frac{d}{dx}\left(e^{-x^2} H_m'\right) dx} + \overset{③}{2(n-m)\int_{-\infty}^{\infty} H_n H_m e^{-x^2} dx} = 0$$

now need to show that and integrals are zero to be able to proof the orthogonality. look at <u>first integral</u>: ①

$$\int_{-\infty}^{\infty} H_m \frac{d}{dx}\left(e^{-x^2} H_n'\right) dx = \int_{-\infty}^{\infty} H_m \left(e^{-x^2} H_n'' - H_n' 2x e^{-x^2}\right) dx$$

$$= \int_{-\infty}^{\infty} H_m e^{-x^2} H_n'' - 2x H_m H_n' e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-x^2}\left(H_m H_n'' - 2x H_m H_n'\right) dx$$

look at <u>second integral</u>: ②

$$\int_{-\infty}^{\infty} H_n \frac{d}{dx}\left(e^{-x^2} H_m'\right) dx = \int_{-\infty}^{\infty} H_n \left(e^{-x^2} H_m'' - 2x e^{-x^2} H_m'\right) dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2}\left(H_n H_m'' - 2x H_n H_m'\right) dx$$

so ① − ② since

$$\int_{-\infty}^{\infty} e^{-x^2}\left(H_m H_n'' - 2x H_m H_n' - H_n H_m'' + 2x H_n H_m'\right) dx$$

$$\int_{-\infty}^{\infty} e^{-x^2}\left(H_m H_n'' - H_n H_m'' + 2x\left(H_n H_m' - H_m H_n'\right)\right) dx \qquad (*)$$

$\longrightarrow$

$$\frac{d}{dx} H_m(e^{-x^2}H_n') = H_m\left(-2xe^{-x^2}H_n' + e^{-x^2}H_n''\right) + H_m'\left(e^{-x^2}H_n'\right)$$

$$= e^{-x^2}\left(-2x H_m H_n' + \underline{H_m H_n''} + H_m' H_n'\right) \quad — ①$$

$$\frac{d}{dx} H_n(e^{-x^2}H_m') = H_n\left(-2xe^{-x^2}H_m' + e^{-x^2}H_m''\right) + \left(e^{-x^2}H_m'\right)H_n'$$

$$= e^{-x^2}\left(-2x H_n H_m' + \underline{H_n H_m''} + H_m' H_n'\right) \quad — ②$$

① − ② since

$$e^{-x^2}\left(-2x H_m H_n' + \underline{H_m H_n''} + \cancel{H_m' H_n'} + 2x H_n H_m' - \underline{H_n H_m''} - \cancel{H_m' H_n'}\right)$$

$$e^{-x^2}\left(H_m H_n'' - H_n H_m'' + 2x\left(H_n H_m' - H_m H_n'\right)\right) \quad —— ③$$

looking at ③ and at integral $(*)$ in last page

This shows I can write integral $(*)$ as

$$\int_{-\infty}^{\infty}\left[\frac{d}{dx} H_m(e^{-x^2}H_n') - \frac{d}{dx} H_n(e^{-x^2}H_m')\right] dx$$

$$\int_{-\infty}^{\infty} \frac{d}{dx}\left(H_m(e^{-x^2}H_n') - H_n(e^{-x^2}H_m')\right) dx$$

using Fundamental theory of Calculus

$$\int \frac{d}{dx} g(x)\, dx = g(x)$$, Then value of

above integral is

$$\left[H_m(e^{-x^2}H_n') - H_n(e^{-x^2}H_m')\right]_{-\infty}^{\infty} \rightarrow$$

$$= \left[e^{-x^2}\left(H_m H_n' - H_n H_m' \right) \right]_{-\infty}^{\infty}$$

since x^2 is always positive, the e^{-x^2} at ∞ is zero and e^{-x^2} at $-\infty$ is zero also.

So for the integral we set $[0-0] = 0$.

hence, looking 2 pages ago at equation labeled (A)

it shows that

$$\boxed{2(n-m)\int_{-\infty}^{\infty} H_n H_m e^{-x^2}\, dx = 0}$$ is what left,

since the other 2 integrals are zero as shown above.

So this means that if $n \neq m$, then

$$\int_{-\infty}^{\infty} H_n H_m e^{-x^2}\, dx = 0$$

but this is the definition of orthogonal functions w.r.t. weight e^{-x^2}. so H_n, H_m are orthogonal w.r.t. e^{-x^2}.

QED

ch 12

$\boxed{22.8}$ in the generating function 22.16, expand the exponential in power series and collect powers of h to obtain the first few Hermite polynomials. verify

$$\frac{\partial^2 \phi}{\partial x^2} - 2x\frac{\partial \phi}{\partial x} + 2h\frac{\partial \phi}{\partial h} = 0 .$$

substitute the series in 22.16 into this identity to prove that $H_n(x)$ in 22.16 satisfy 22.14.

Solution

eq 22.16 (generating function for Hermite poly) is

$$\Phi(x,h) = e^{2xh-h^2} = \sum_{n=0}^{\infty} H_n(x)\frac{h^n}{n!} .$$

expand exp function in power series around 0. we get

$$e^{\xi} = 1 + \xi + \frac{\xi^2}{2!} + \frac{\xi^3}{3!} + \cdots$$

so
$$e^{2xh-h^2} = 1 + (2xh-h^2) + \frac{(2xh-h^2)^2}{2} + \frac{(2xh-h^2)^3}{3!} + \cdots$$

$$= 1 + (2xh-h^2) + \frac{(4x^2h^2+h^4-4xh^3)}{2} + \frac{(2xh-h^2)(4x^2h^2+h^4-4xh^3)}{6} \cdots$$

$$= 1 + (2xh-h^2) + \frac{4x^2h^2}{2} + \frac{h^4}{2} - \frac{4xh^3}{2} + \frac{8x^3h^3}{3!} + \frac{2xh^5}{3!} - \frac{8x^2h^4}{3!} - \frac{4x^2h^4}{3!}$$
$$- \frac{h^6}{3!} + \frac{4xh^5}{3!} + \cdots$$

$$= h^0(1) + h^1(2x) + h^2(-1+2x^2) + h^3\left(-2x + \frac{4}{3}x^3\right) + \cdots$$

$\longrightarrow$

so from 22.16 we get

$$h^0(1) + h'(2x) + h^2(-1+2x^2) + h^3\left(-2x + \tfrac{4}{3}x^3\right) + \cdots$$

$$= H_0(x)\, h^0 + H_1(x)\, h' + H_2(x)\, \frac{h^2}{2} + H_3(x)\, \frac{h^3}{3!} + \cdots$$

equating coefficients of h, we get

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$\frac{H_2(x)}{2} = (-1+2x^2)$$

$$\frac{H_3(x)}{3!} = -2x + \frac{4}{3}x^3$$

$$\vdots$$

so
$$\boxed{\begin{aligned} H_0 &= 1 \\ H_1 &= 2x \\ H_2 &= -2+4x^2 \\ H_3 &= -12x+8x^3 \end{aligned}}$$

now Verify identity $\dfrac{\partial^2\phi}{\partial x^2} - 2x\dfrac{\partial \phi}{\partial x} + 2h\dfrac{\partial \phi}{\partial h} = 0.$

using expansion we got (using first 3 terms of expansion only)

$\phi(x,h) = 1 + 2xh + (-1+2x^2)h^2$

find $\dfrac{\partial \phi}{\partial x}$, $\dfrac{\partial^2 \phi}{\partial x^2}$, $\dfrac{\partial \phi}{\partial h}$, and substitute to see if identity valid

$\longrightarrow$

$$\frac{\partial\phi}{\partial x} = 2h + 4xh^2$$

$$\frac{\partial^2\phi}{\partial x^2} = 4h^2$$

$$\frac{\partial\phi}{\partial h} = 2x - 2h + 4x^2 h$$

so $\frac{\partial^2\phi}{\partial x^2} - 2x\frac{\partial\phi}{\partial x} + 2h\frac{\partial\phi}{\partial h} =$

$$= (4h^2) - 2x(2h + 4xh^2) + 2h(2x - 2h + 4x^2 h)$$

$$= \cancel{4h^2} \cancel{-4xh} \cancel{-8x^2h^2} + \cancel{+4xh} \cancel{-4h^2} \cancel{+8x^2h^2}$$

$$= \boxed{0}$$

hence Hermite polynomials generated by generating function satisfy Hermite DE 22.14 $y'' - 2xy' + 2ny = 0$

now to verify that highest term in $H_n(x)$ is $(2x)^n$.

looking at

$H_0 = 1 \longrightarrow n=0$, highest term $(2x)^0 = 1$ ok.

$H_1 = 2x \longrightarrow n=1,\ (2x)^1 = 2x$ ok

$H_2 = -2 + 4x^2 \longrightarrow n=2,\ (2x)^2 = 4x^2$ ok

$H_3 = -12x + 8x^3 \longrightarrow n=3,\ (2x)^3 = 8x^3$ ok.

all verified.

Ch 12 4/5

$\boxed{22.12}$ using Leibniz rule (section 3) carry out the differentiation in 22.18 to obtain 22.19

22.18 : $L_n(x) = \frac{1}{n!} e^x \frac{d^n}{dx^n}(x^n e^{-x})$

22.19 : $L_n(x) = 1 - nx + \frac{n(n-1)}{2!}\frac{x^2}{2!} - \frac{n(n-1)(n-2)}{3!}\frac{x^3}{3!} + \cdots + \frac{(-1)^n x^n}{n!}$

$= \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{x^m}{m!}$ Laguerre polynomials.

Leibniz rule is used to differentiate a product

$$\frac{d^n}{dx^n}(uv) = u v^{(n)} + n u^{(1)} v^{(n-1)} + \frac{n(n-1)}{2!} u^{(2)} v^{(n-2)} + \cdots$$

where $v^{(n)}$ means $\frac{d^n}{dx^n} v$

so using this rule, looking at $\frac{1}{n!} e^x \frac{d^n}{dx^n}(x^n e^{-x})$,

let $u = x^n$

let $v = e^{-x}$

so $\frac{d^n}{dx^n}(uv) = x^n \frac{d^n}{dx^n} e^{-x} + n \frac{d}{dx} x^n \frac{d^{n-1}}{dx^{n-1}} e^{-x} + \frac{n(n-1)}{2!} \frac{d^2}{dx^2} x^n \frac{d^{n-2}}{dx^{n-2}} e^{-x} + \cdots$

$+ \frac{d^n}{dx^n} x^n \frac{d^0}{dx^0} e^{-x}$

now $\frac{d^m}{dx^m} e^{-x} = -e^{-x}$ or $+e^{-x}$. if m is even, then we get $+e^{-x}$. if m is odd, we get $-e^{-x}$. so $\boxed{\frac{d^m}{dx^m}(e^{-x}) = (-1)^m e^{-x}}$

so $\frac{d^n}{dx^n}(x^n e^{-x}) = x^n e^{-x} - n\left(\frac{d}{dx} x^n\right) e^{-x} + \frac{n(n-1)}{2!}\left(\frac{d^2}{dx^2} x^n\right) e^{-x} - \cdots$

$+ \frac{d^n}{dx^n} x^n e^{-x}$

now $\frac{d^m}{dx^m} x^n = n(n-1)(n-2)\cdots(n-m+1)\, x^{n-m}$.

for example $\frac{d^{3 \to m}}{dx^3} x^{5 \to n} = \frac{d^2}{dx^2}(5x^4) = \frac{d}{dx}(5 \cdot 4 \cdot x^3) = \underset{n}{5} \cdot 4 \cdot \underset{m}{3}\, x^2$

so $\frac{d^n}{dx^n}(x^n e^{-x}) = x^n e^{-x} - n\left[n x^{n-1}\right] e^{-x} + \frac{n(n-1)}{2!}\left[n(n-1) x^{n-2}\right] e^{-x} - \cdots$

$+ \left[n(n-1)(\cdots)(1)\, x^0\right] e^{-x}$

hence $\frac{1}{n!} e^{x}\left[\frac{d^n}{dx^n}(x^n e^{-x})\right]$

here you've taken n derivatives of e^{-x} $\Rightarrow$ so $(-1)^n$

here you haven't taken any derivatives of e^{-x} $\Rightarrow$ so +.

$= \frac{1}{n!} e^{x}\left[x^n e^{-x} - n\left[n x^{n-1}\right] e^{-x} + \cdots + \frac{\cancel{n!}}{\cancel{n!}}\left[n!\, x^0\right] e^{-x}\right]$

$= \boxed{\frac{x^n}{n!} - \frac{n}{(n-1)!} x^{n-1} + \frac{n(n-1)}{(n-2)!} x^{n-2} + \cdots \pm 1}$

Notice depending on if n is even or odd

I can write above as

$- \frac{x^n}{n!} + \frac{n}{(n-1)!} x^{n-1} - \frac{n(n-1)}{(n-2)!} + \cdots + (-1)^n$

or

$+ \frac{x^n}{n!} - \frac{n}{(n-1)!} x^{n-1} + \cdots + (-1)^n$

the book puts the sign in 22.19 on the last term only.

Ch 12

$\boxed{22.13}$ using 22.19, verify 22.20 and also find L_3 and L_4.

$$22.19: \quad L_n(x) = 1 - nx + \frac{n(n-1)}{2!}\frac{x^2}{2!} - \frac{n(n-1)(n-2)}{3!}\frac{x^3}{3!} + \cdots + \frac{(-1)^n x^n}{n!}$$

$$= \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{x^m}{m!}$$

Laguerre polynomials.

22.20

$$L_0(x) = 1$$
$$L_1(x) = 1 - x$$
$$L_2(x) = 1 - 2x + \frac{x^2}{2}$$

need to use 22.19 to find 22.20.

for L_0, put $n=0$ in the sum operation. this results in

$$\boxed{L_0(x) = 1}$$

for L_1, put $n=1$, we get

$$\sum_{m=0}^{1} (-1)^m \binom{n}{m} \frac{x^m}{m!} = (-1)^0 \binom{1}{0} \frac{x^0}{0!} + (-1)^1 \binom{1}{1} \frac{x^1}{1!}$$

$$= \boxed{1 - x} \qquad \text{since } \binom{1}{0} = 1.$$

note that $\binom{n}{m} = \frac{n!}{(n-m)! - m!}$, and $0! = 1$, $1! = 1$.

For L_2, put $n=2$. so

$$\sum_{m=0}^{2} (-1)^m \binom{n}{m} \frac{x^m}{m!} = (-1)^0 \binom{2}{0} \frac{x^0}{0!} + (-1)^1 \binom{2}{1} \frac{x^1}{1!} + (-1)^2 \binom{2}{2} \frac{x^2}{2!}$$

$$= 1 - \frac{2!}{(2-1)!\,1!} x^1 + \frac{2!}{(2-2)!\,2!} \frac{x^2}{2!}$$

$$= \boxed{1 - 2x + \frac{x^2}{2}}$$

so all verified

→

now use 22.19 to find L_3 and L_4

For L_3, put $n=3$.

so $\displaystyle\sum_{m=0}^{3} (-1)^m \binom{3}{m} \frac{x^m}{m!} = (-1)^0 \binom{3}{0} \frac{x^0}{0!} + (-1)^1 \binom{3}{1} \frac{x^1}{1!} + (-1)^2 \binom{3}{2} \frac{x^2}{2!}$

$+ (-1)^3 \binom{3}{3} \frac{x^3}{3!}$

$$= 1 - \frac{3!}{(3-1)!\,1!} x^1 + \frac{3!}{(3-2)!\,2!} \frac{x^2}{2!} - \frac{3!}{(3-3)!\,3!} \frac{x^3}{3!}$$

$$= 1 - \frac{3!}{2!} x + \frac{3!}{2!} \frac{x^2}{2!} - \frac{3!}{3!} \frac{x^3}{3!}$$

$$L_3 = \boxed{1 - 3x + \frac{3}{2} x^2 - \frac{x^3}{6}}$$

For L_4, put $n=4$. we get

$L_4 = \displaystyle\sum_{m=0}^{4} (-1)^m \binom{4}{m} \frac{x^m}{m!} = (-1)^0 \binom{4}{0} \frac{x^0}{0!} + (-1)^1 \binom{4}{1} \frac{x^1}{1!} + (-1)^2 \binom{4}{2} \frac{x^2}{2!}$

$+ (-1)^3 \binom{4}{3} \frac{x^3}{3!} + (-1)^4 \binom{4}{4} \frac{x^4}{4!}$

$$= 1 - \frac{4!}{(4-1)!\,1!} x + \frac{4!}{(4-2)!\,2!} \frac{x^2}{2!} - \frac{4!}{(4-3)!\,3!} \frac{x^3}{3!} + \frac{4!}{(4-4)!\,4!} \frac{x^4}{4!}$$

$$= 1 - \frac{4!}{3!} x + \frac{4!}{2!\,2!} \frac{x^2}{2!} - \frac{4!}{3!} \frac{x^3}{3!} + \frac{4!}{4!} \frac{x^4}{4!}$$

$$= 1 - 4x + \frac{4 \times 3 \times \cancel{2}}{\cancel{2} \times 2} \frac{x^2}{\cancel{2}} - 4 \frac{x^3}{6} + \frac{x^4}{24}$$

$$L_4 = \boxed{1 - 4x + 3x^2 - \frac{2}{3} x^3 + \frac{x^4}{24}}$$

Ch 12

$\boxed{22.15}$ solve the Laguerre DE $xy'' + (1-x)y' + Py = 0$

by power series.

let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

$\longrightarrow Py = Pa_0 + Pa_1 x + Pa_2 x^2 + Pa_3 x^3 + \cdots$

$\longrightarrow y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$

$\longrightarrow -xy' = -a_1 x - 2a_2 x^2 - 3a_3 x^3 - 4a_4 x^4 - \cdots$

$y'' = 2a_2 + 2\cdot 3\, a_3 x + 3\cdot 4\, a_4 x^2 + \cdots$

$\rightarrow xy'' = 2a_2 x + 2\cdot 3\, a_3 x^2 + 3\cdot 4\, a_4 x^3 + \cdots$

So table is

	x^0	x^1	x^2	x^3	x^n
Py	Pa_0	Pa_1	Pa_2	Pa_3	Pa_n
y'	a_1	$2a_2$	$3a_3$	$4a_4$	$(n+1)a_{n+1}$
$-xy'$	—	$-a_1$	$-2a_2$	$-3a_3$	$-na_n$
xy''	—	$2a_2$	$2\cdot 3\, a_3$	$3\cdot 4\, a_4$	$n(n+1)a_{n+1}$

from first column, we get

$\boxed{Pa_0 = -a_1}$ or $\boxed{a_1 = -Pa_0}$

from general recursive formula

$(n+1)a_{n+1} + n(n+1)a_{n+1} = na_n - Pa_n$

$(n+1 + n(n+1))\, a_{n+1} = a_n(n-P)$

$a_{n+1} = -a_n \dfrac{(P-n)}{(n+1) + n(n+1)} = \boxed{-\dfrac{a_n(P-n)}{(n+1)^2}} \rightarrow$

let me look at few terms

$\underline{n=1}$

$$a_2 = -\frac{a_1(P-1)}{4} = -\frac{(P-1)}{4}(-Pa_0) = \frac{P(P-1)}{4}a_0$$

$\underline{n=2}$

$$a_3 = -\frac{a_2(P-2)}{9} = \frac{-(P-2)}{9}\left(\frac{P(P-1)}{4}\right)a_0$$

$$= \frac{-P(P-1)(P-2)}{4\cdot 9}a_0$$

$n=3$

$$a_4 = \frac{-a_3(P-3)}{16} = \frac{-(P-3)}{16}\left(\frac{-P(P-1)(P-2)}{4.9}\right)a_0$$

$$= \frac{P(P-1)(P-2)(P-3)}{4.9.16}a_0$$

so $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

$$= a_0 - Pa_0 x + \frac{P(P-1)}{4}a_0 x^2 - \frac{P(P-1)(P-2)}{4.9}a_0 x^3 + \frac{P(P-1)(P-2)(P-3)}{4.9.16}a_0 x^4$$

$+ \cdots$

$$= a_0\left(1 - Px + \frac{P(P-1)}{4}x^2 - \frac{P(P-1)(P-2)}{4.9}x^3 + \cdots\right)$$

so, if P is an integer, when it is equal to n, the factor $(P-n)$ will become zero, and hence that term, and each term after it (since those will also include the $(P-n)$ factor) will all be zero. hence the a_0 series terminates.

$\longrightarrow$

to find L_0, L_1, L_2 and L_3

set $p = 0, 1, 2$, or 3 in the a_0 series and stop when we reach the term with $(p-n)$ when $n = p$.

So from $y = a_0\left(1 - px + \frac{p(p-1)}{4}x^2 - \frac{p(p-1)(p-2)}{4 \cdot 9}x^3 + \cdots\right)$

for $p=0$ we get $y = a_0 = 1$.

so $L_0(x) = \boxed{1}$

for $p=1$ we get

$$y = a_0(1-x) = \boxed{1-x} \text{ for } a_0 = 1$$

for $p=2$, we set

$$y = a_0\left(1 - 2x + \frac{2(2-1)}{4}x^2\right)$$

$$= a_0\left(1 - 2x + \frac{x^2}{2}\right)$$

$$L_2 = \boxed{1 - 2x + \frac{x^2}{2}} \text{ for } a_0 = 1$$

For $p = 3$

$$L_3 = y = a_0\left(1 - 3x + \frac{3(3-1)}{4}x^2 - \frac{3(3-1)(3-2)}{4 \cdot 9}x^3\right)$$

$$= 1 - 3x + \frac{3}{2}x^2 - \frac{\cancel{3} \times \cancel{2}}{{}_2\cancel{4} \times \cancel{9}_3}x^3$$

$$= \boxed{1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3}$$

this is an eigenvalue problem, since for different parameter of the DE, we find the corresponding solution (eigenfunction)

QED

ch 12

$\boxed{22.18}$ verify the recursion relation 22.24

22.24: $L'_{n+1} - L'_n + L_n = 0$

$(n+1)L_{n+1} - (2n+1-x)L_n + nL_{n-1} = 0$

$xL'_n - nL_n + nL_{n-1} = 0$

from 22.23: $\Phi(x,h) = \dfrac{e^{-xh/(1-h)}}{1-h} = \sum_{n=0}^{\infty} L_n(x)h^n$

differentiate w.r.t. x

so $\dfrac{\partial\phi}{\partial x} = \dfrac{1}{1-h} e^{-xh/(1-h)}\left(\dfrac{(-h)}{(1-h)}\right) = \sum_{n=0}^{\infty} L'_n(x)h^n$

$\dfrac{\partial\phi}{\partial x} = -\dfrac{h}{(1-h)^2} e^{-\frac{xh}{1-h}} = \sum_n^{\infty} L'_n h^n$

$\dfrac{\partial\phi}{\partial x} = -\dfrac{h}{1-h} \underbrace{\dfrac{e^{-\frac{xh}{1-h}}}{1-h}}_{\Phi} = \sum_n^{\infty} L'_n h^n$

so $\boxed{\dfrac{\partial\phi}{\partial x} = -\dfrac{h}{1-h}\phi = \sum_n^{\infty} L'_n h^n}$

i.e $\boxed{h\phi = (h-1)\dfrac{\partial\phi}{\partial x}}$

so from above I write

$h \underbrace{\sum L_n h^n}_{\phi} = (h-1) \underbrace{\sum L'_n h^n}_{\frac{\partial\phi}{\partial x}}$ $\rightarrow$

$$h\left[L_0 h^0 + L_1 h^1 + L_2 h^2 + \cdots\right] = (h-1)\left[L_0' h^0 + L_1' h^1 + L_2' h^2 + \cdots\right]$$

$$L_0 h^1 + L_1 h^2 + L_2 h^3 + \cdots + L_n h^{n+1} + \cdots = L_0' h + L_1' h^2 + \cdots + L_n' h^{n+1}$$
$$- \left(L_0' h^0 + L_1' h^1 + \cdots + L_n' h^n + L_{n+1}' h^{n+1} + \cdots\right)$$

so by equating coefficients of h^{n+1} we have

$$L_n = L_n' - L_{n+1}'$$

i.e. $\boxed{L_{n+1}' - L_n' + L_n = 0}$ which is part (a).

now to find part (b):

differentiate 22.23 w.r.t. h

$$\frac{\partial \phi}{\partial h} = \frac{1}{1-h} e^{\frac{-xh}{1-h}} \left(\frac{d}{dh}\left(\frac{-xh}{1-h}\right)\right) + e^{-\frac{xh}{1-h}} \frac{d}{dh}\left(\frac{1}{1-h}\right) = \sum_{n=0}^{\infty} n h^{n-1} L_n$$

$$\frac{\partial \phi}{\partial h} = \frac{1}{1-h} e^{\frac{-xh}{1-h}} \left(-\left(\frac{x}{1-h}(1) + h\left(-\frac{1}{(1-h)^2}(-1)\right)\right)\right) + e^{\frac{-xh}{1-h}}\left(-\frac{1}{(1-h)^2}(-1)\right) = \downarrow$$

$$\frac{\partial \phi}{\partial h} = \frac{1}{1-h} e^{\frac{-xh}{1-h}} \left(-\left(\frac{x}{1-h} + \frac{h}{(1-h)^2}\right)\right) + e^{\frac{-xh}{1-h}}\left(\frac{1}{(1-h)^2}\right) = \downarrow$$

$$\frac{\partial \phi}{\partial h} = \frac{-1}{1-h} e^{\frac{-xh}{1-h}} \left(\frac{x(1-h)+h}{(1-h)^2}\right) + e^{\frac{-xh}{1-h}} \frac{1}{(1-h)^2} = \downarrow$$

$$= \frac{-e^{\frac{-xh}{1-h}}\left(x(1-h)+h\right)}{(1-h)^3} + \frac{e^{\frac{-xh}{1-h}}(1-h)}{(1-h)^3} = \downarrow$$

$\longrightarrow$

$$\frac{\partial \phi}{\partial h} = \frac{e^{-\frac{xh}{1-h}}\left[1-h - x(1-h) - h\right]}{(1-h)^3} = \sum_{n=0}^{\infty} n h^{n-1} L_n$$

$$= \frac{e^{-\frac{xh}{1-h}}\left[1-h-x+h-h\right]}{(1-h)^3} = \downarrow$$

$$\boxed{\frac{\partial \phi}{\partial h} = \frac{e^{-\frac{xh}{1-h}}\left[1-x-h\right]}{(1-h)^3} = \sum_{n=0}^{\infty} n h^{n-1} L_n}$$

but $\dfrac{e^{-\frac{xh}{1-h}}}{1-h} = \phi$ so above can be rewritten as

$$\frac{\partial \phi}{\partial h} = \phi \, \frac{(1-x-h)}{(1-h)^2} = \sum n h^{n-1} L_n$$

or $\boxed{(1-h)^2 \dfrac{\partial \phi}{\partial h} = \phi(1-x-h)}$

so $(1-h)^2 \left(\sum n h^{n-1} L_n\right) = \left(\sum L_n h^n\right) \left[1-x-h\right]$

or $(1-h)^2 \left[0 + h^0 L_1 + 2h^1 L_2 + 3h^2 L_3 + \cdots + (n+1) h^n L_{n+1} + \cdots\right]$

$$= \left[L_0 h^0 + L_1 h^1 + \ldots L_n h^n + \cdots\right]\left[1-x-h\right]$$

expand and equate coeff of h^n:

$$(1-2h+h^2)\left[h^0L_1 + 2h^1L_2 + \cdots + (n+1)h^nL_{n+1} + \cdots\right]$$

$$= \left[L_0h^0 + L_1h^1 + \cdots + L_nh^n + \cdots\right]\left[1-x-h\right]$$

$$\boxed{\begin{aligned}&(h^0L_1 + \cdots + (n+1)L_{n+1}h^n + \cdots) - 2(h^1L_1 + \cdots + (n)h^nL_n + \cdots) \\ &+ (h^2L_1 + \cdots + (n-1)h^nL_{n-1} + \cdots) \\ &= (L_0h^0 + \cdots + L_nh^n + \cdots) - (xL_0h^0 + \cdots + xL_nh^n + \cdots) - (L_0h^1 + \cdots L_{n-1}h^n + \cdots)\end{aligned}}$$

so looking at h^n only terms

$$(n+1)L_{n+1} - 2nL_n + (n-1)L_{n-1} = L_n - xL_n - L_{n-1}$$

$$(n+1)L_{n+1} + L_n(-2n-1+x) + L_{n+1}((n-1)+1) = 0$$

$$\boxed{(n+1)L_{n+1} - L_n(2n+1-x) + L_{n+1}(n) = 0}$$

which is part (b).

now to show part (c):

(a) $h\phi = (h-1)\dfrac{\partial\phi}{\partial x}$

(b) $(1-h-x)\phi = (1-h)^2\dfrac{\partial\phi}{\partial h}$ $\longrightarrow$

$$x\left(\frac{\partial\phi}{\partial x}\right) + h\phi - h(1-h)\frac{\partial\phi}{\partial h} =$$

From (a) From (b)

$$x\left[\frac{h\phi}{h-1}\right] + h\phi - h\cancel{(1-h)}\left[\frac{(1-h-x)\phi}{(1-h)^{\cancel{2}}}\right]$$

$$= \frac{-xh\phi}{1-h} + h\phi - \frac{h(1-h-x)\phi}{(1-h)}$$

$$= \frac{-xh\phi + h\phi(1-h)}{1-h} - \frac{h(1-h-x)\phi}{1-h}$$

$$= \frac{-xh\phi + h\phi - h^2\phi - h(1-h-x)\phi}{1-h}$$

$$= \frac{\cancel{-xh\phi} + \cancel{h\phi} - \cancel{h^2\phi} - \cancel{h\phi} + \cancel{h^2\phi} + \cancel{x\phi h}}{1-h} = 0$$

so $$\boxed{x\frac{\partial\phi}{\partial x} + h\phi - h(1-h)\frac{\partial\phi}{\partial h} = 0}$$

sub for $\frac{\partial\phi}{\partial x} = \sum L'_n h^n$

for $\frac{\partial\phi}{\partial h} = \sum L_n n h^{n-1}$ into above and equate

Coeff of h^n $\longrightarrow$

$$x\left[\sum L_n' h^n\right] + h\left[\sum L_n h^n\right] - h(1-h)\left[\sum L_n n h^{n-1}\right] = 0$$

pick terms only wit $h^n \Rightarrow$

$$x\sum L_n' h^n + \left(\sum L_n h^{n+1}\right) - h + h^2\left(\sum L_n n h^{n-1}\right) = 0$$

$$x\sum L_n' h^n + \sum L_n h^{n+1} - \sum L_n n h^n + \sum L_n n h^{n+1} = 0$$

so

$$x L_n' + L_{n-1} - nL_n + (n-1)L_{n-1} = 0$$

$$x L_n' + L_{n-1}(1-1+n) - nL_n = 0$$

$$\boxed{x L_n' - nL_n + nL_{n-1} = 0}$$

which is part (c)

QED

ch 12

$\boxed{22.27}$ given $f_n(x) = x^{l+1} e^{-\frac{x}{2n}} L^{2l+1}_{n-l-1}\left(\frac{x}{n}\right)$

for $l=1$, show that $f_2(x) = x^2 e^{-x/4}$

$f_3(x) = x^2 e^{-\frac{x}{6}}\left(4 - \frac{x}{3}\right)$

$f_4(x) = x^2 e^{-\frac{x}{8}}\left(10 - \frac{5x}{4} + \frac{x^2}{32}\right)$

First Find L^3_0, L^3_1, L^3_2.

using 22.25: $L^k_n(x) = (-1)^k \frac{d^k}{dx^k} L_{n+k}(x)$.

so $L^3_0 = (-1)^3 \frac{d^3}{dx^3} L_{0+3}(x) = -\frac{d^3}{dx^3} L_3$

but $L_3 = 1 - 3x + \frac{3}{2}x^2 - \frac{x^3}{6}$ (from problem 22.13 we just did in this HW set)

So $L^3_0 = -\frac{d^3}{dx^3}\left(1 - 3x + \frac{3}{2}x^2 - \frac{x^3}{6}\right)$

$= -\frac{d^2}{dx^2}\left(-3 + 3x - \frac{3x^2}{6}\right)$

$= -\frac{d}{dx}\left(3 - \frac{2}{2}x\right)$

$= -(-1) = \underline{1}$

now, $K = 3$, so since $K = 2l+1$, then $2l = 2$, i.e $l = 1$

So for $\underline{\underline{n=2}}$ $f_n(x) = x^{l+1} e^{-\frac{x}{2n}} L^{2l+1}_{n-l-1}\left(\frac{x}{n}\right)$

i.e $\boxed{f_2(x)} = x^2 e^{-\frac{x}{4}} L^{2+1}_{2-2}\left(\frac{x}{2}\right)$

$= x^2 e^{-\frac{x}{4}} L^3_0\left(\frac{x}{2}\right) \Longrightarrow \boxed{x^2 e^{-\frac{x}{4}}}$ since $L^3_0 = 1$ only.

To find $f_3(x)$. here $n=3$.

so $f_3(x) = x^2 e^{-\frac{x}{6}} L^3_{3-1-1}\left(\frac{x}{3}\right)$

$$= x^2 e^{-\frac{x}{6}} L^3_1\left(\frac{x}{3}\right). \quad \text{———} \quad (1)$$

now find $L^3_1(x)$, then replace x by $\frac{x}{3}$ in result.

$L^3_1(x)$ is found from 22.25

$$L^k_n(x) = (-1)^k \frac{d^k}{dx^k} L_{n+k}(x)$$

so $L^3_1(x) = (-1)^3 \frac{d^3}{dx^3} L_{3+1}(x) = -\frac{d^3}{dx^3} L_4(x)$.

But $L_4(x) = 1 - 4x + 3x^2 - \frac{2}{3}x^3 + \frac{x^4}{24}$ (from 22.13).

so $L^3_1(x) = -\frac{d^2}{dx^2}\left(-4 + 6x - 2x^2 + \frac{4x^3}{24}\right)$

$$= -\frac{d}{dx}\left(6 - 4x + \frac{3x^2}{6}\right)$$

$$= -\left(-4 + \frac{2x}{2}\right) = -(-4+x) = \boxed{4-x}$$

so $\boxed{L^3_1\left(\frac{x}{3}\right) = 4 - \left(\frac{x}{3}\right)}$

So from (1) above

$$\boxed{f_3(x) = x^2 e^{-\frac{x}{6}}\left(4 - \frac{x}{3}\right)}$$

$\longrightarrow$

now to find $f_4(x)$.

here $n=4$, so

$$f_4(x) = x^2 e^{-\frac{x}{8}} L^{3}_{4-1-1}\left(\frac{x}{4}\right)$$

$$= x^2 e^{-\frac{x}{8}} L^{3}_{2}\left(\frac{x}{4}\right).$$

find $L^3_2(x)$ from 22.25, and replace result x by $\frac{x}{4}$:

from 22.25

$$L^K_n(x) = (-1)^K \frac{d^K}{dx^K} L_{n+K}(x)$$

so $L^3_2(x) = (-1)^3 \dfrac{d^3}{dx^3} L_5(x)$.

need to find $L_5(x)$ first. (using 22.19)

$$L_5(x) = \sum_{m=0}^{5} (-1)^m \binom{5}{m} \frac{x^m}{m!}$$

$$L_5(x) = (-1)^0\binom{5}{0}\frac{x^0}{1} + (-1)^1\binom{5}{1}\frac{x}{1!} + (-1)^2\binom{5}{2}\frac{x^2}{2!} + (-1)^3\binom{5}{3}\frac{x^3}{3!} + (-1)^4\binom{5}{4}\frac{x^4}{4!} + (-1)^5\binom{5}{5}\frac{x^5}{5!}$$

$$= 1 - \frac{5!}{(5-1)!\,1!}x + \frac{5!}{(5-2)!\,2!}\frac{x^2}{2!} - \frac{5!}{(5-3)!\,3!}\frac{x^3}{3!} + \frac{5!}{(5-4)!\,4!}\frac{x^4}{4!} - \frac{5!}{5!}\frac{x^5}{5!}$$

$$= 1 - \frac{5!}{4!}x + \frac{5!}{3!\,2!}\frac{x^2}{2!} - \frac{5!}{2!\,3!}\frac{x^3}{3!} + \frac{5!}{4!}\frac{x^4}{4!} - \frac{x^5}{5!}$$

$$= 1 - \frac{120}{24}x + \frac{120}{6\times 2}\frac{x^2}{2} - \frac{120}{2\times 6}\frac{x^3}{6} + \frac{120}{24}\frac{x^4}{24} - \frac{x^5}{120}$$

$\rightarrow$

$$L_5(x) = 1 - 5x + 5x^2 - \frac{\cancel{120}^{\cancel{60}\,10}}{2\times 36_{6}} x^3 + \frac{\cancel{120}^{\cancel{60}\,\cancel{30}\,15}}{576_{\cancel{288}\,\cancel{144}\,72}} x^4 - \frac{x^5}{120}$$

$$L_5(x) = 1 - 5x + 5x^2 - \frac{10}{6}x^3 + \frac{15}{72}x^4 - \frac{x^5}{120}$$

So now can find $L_2^3(x)$.

$$L_2^3 = -\frac{d^3}{dx^3} L_5(x)$$

$$= -\frac{d^2}{dx^2}\left(-5 + 10x - \frac{10}{2}x^2 + \frac{4\times 15}{72}x^3 - \frac{5x^4}{120}\right)$$

$$= -\frac{d}{dx}\left(10 - 10x + \frac{3\times 4\times 15}{72}x^2 - \frac{5\times 4 x^3}{120}\right)$$

$$= -\left(-10 + \frac{\cancel{2\cdot 3\cdot 4}\cdot 15}{\cancel{72}_{36\,9}}x - \frac{\cancel{3\cdot 5\cdot 4}\,x^2}{\cancel{120}_{30\,10\,2}}\right)$$

$$\boxed{L_2^3 = 10 - \frac{3\cdot 15}{9}x + \frac{x^2}{2}}$$

now replace x by $\left(\frac{x}{4}\right)$. $\Longrightarrow$

$$L_2^3 = 10 - 5\left(\frac{x}{4}\right) + \frac{1}{2}\left(\frac{x}{4}\right)^2$$

$$\boxed{L_2^3 = 10 - \frac{5}{4}x + \frac{x^2}{32}}$$

So $$\boxed{f_4(x) = x^2 e^{-\frac{x}{8}}\left(10 - \frac{5}{4}x + \frac{x^2}{32}\right)}$$

now to proof the orthogonality $\longrightarrow$

to show f_2, f_3, f_4 are orthogonal over $(0, \infty)$,

need to show that $\int_0^\infty f_a f_b \, dx = 0$

for each combination. i.e. (f_2, f_3), (f_2, f_4), (f_3, f_4).

for f_2, f_3

$$\int_0^\infty f_2 f_3 \, dx = \int_0^\infty x^2 e^{-\frac{x}{4}} \; x^2 e^{-\frac{x}{6}} \left(4 - \frac{x}{3}\right) dx$$

$$= \int_0^\infty x^4 e^{-\frac{x}{4} - \frac{x}{6}} \left(4 - \frac{x}{3}\right) dx$$

$$= \int_0^\infty x^4 e^{\frac{-6x-4x}{24}} \left(4 - \frac{x}{3}\right) dx = \int_0^\infty x^4 e^{\frac{-10x}{24}} \left(4 - \frac{x}{3}\right) dx$$

$$= 4 \int_0^\infty x^4 e^{-\frac{5}{12}x} \, dx - \frac{1}{3} \int_0^\infty x^5 e^{-\frac{5}{12}x} \, dx$$

need to use Γ integral, which is

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x} \, dx$$

so let $\boxed{\frac{5}{12} x = u}$, so $\frac{5}{12} dx = du$ or $\boxed{dx = \frac{12}{5} du}$

and $x = \frac{12}{5} u$. so $x^4 = \left(\frac{12}{5} u\right)^4 = \left(\frac{12}{5}\right)^4 u^4$

when $x = 0$, $u = 0$

when $x = \infty$, $u = \infty$. hence integrals are

$$= 4 \int_0^\infty \left(\frac{12}{5}\right)^4 u^4 e^{-u} \left(\frac{12}{5}\right) du - \frac{1}{3} \int_0^\infty \left(\frac{12}{5}\right)^5 u^5 e^{-u} \left(\frac{12}{5}\right) du$$

→

$$= 4\left(\frac{12}{5}\right)^4\left(\frac{12}{5}\right)\int_0^\infty u^4 e^{-u}\,du - \frac{1}{3}\left(\frac{12}{5}\right)^5\left(\frac{12}{5}\right)\int_0^\infty u^5 e^{-u}\,du$$

$$= 4\left(\frac{12}{5}\right)^5\left[\Gamma(4+1)\right] - \frac{1}{3}\left(\frac{12}{5}\right)^6\left[\Gamma(5+1)\right]$$

$$= 4\left(\frac{12}{5}\right)^5 4! - \frac{1}{3}\left(\frac{12}{5}\right)^6 5!$$

$$= \left(\frac{12}{5}\right)^5\left[4\times 4! - \frac{1}{\cancel{3}}\left(\frac{\cancel{12}}{5}\right)5!\right]$$

$$= \left(\frac{12}{5}\right)^5\left[4(4\cdot3\cdot2) - \frac{4}{\cancel{5}}(\cancel{5}\cdot4\cdot3\cdot2)\right]$$

$$= \left(\frac{12}{5}\right)^5\left[4(4\cdot3\cdot2) - 4(4\cdot3\cdot2)\right] = 0$$

hence this show f_2, f_3 are orthognal over $(0,\infty)$.

using similar steps f_2, f_4 and f_3, f_4 can be shown to be orthogonal. I do not think these need to be done as nothing new needs to be shown. it will be just same steps as above.

QED

Ch 13

$\boxed{1.1}$ Assume from electricity the equations $\bar{\nabla}\cdot\bar{D}=\rho$ and $\bar{D}=-\epsilon\bar{\nabla}\phi$, show that electrostatic potential satisfies Laplace equation in charge free region and satisfies poisson's equation in region with charge density ρ.

Laplace equation $\nabla^2 u=0$

poisson's equation $\nabla^2 u = f(x,y,z)$.

in a charge free region, $\rho=0$. i.e $\bar{\nabla}\cdot D=0$

so $\bar{\nabla}\cdot(-\epsilon\bar{\nabla}\phi)=0$

$$\left(\frac{\partial}{\partial x}\bar{i}+\frac{\partial}{\partial y}\bar{j}+\frac{\partial}{\partial z}\bar{k}\right)\cdot\left(-\epsilon\left(\bar{i}\frac{\partial}{\partial x}\phi+\bar{j}\frac{\partial}{\partial y}\phi+\bar{k}\frac{\partial}{\partial z}\phi\right)\right)=0$$

so $$-\epsilon\left(\frac{\partial^2}{\partial x^2}\phi+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\right)=0$$

i.e $$\boxed{\frac{\partial^2}{\partial x^2}\phi+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0}$$ since $\epsilon\neq 0$.

this is Laplace equation.

in region with charge density ρ. we have

$\bar{\nabla}\cdot\bar{D}=\rho(x,y,z) \longrightarrow$ density is assumed a function of position. i.e it can change depending on part of region we are in.

so $\bar{\nabla}\cdot\left(-\epsilon\bar{\nabla}\phi(x,y,z)\right)=\rho(x,y,z)$

$$\Rightarrow -\epsilon\left(\frac{\partial^2}{\partial x^2}\phi+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2}{\partial z^2}\phi\right)=\rho(x,y,z)$$

i.e $$\frac{\partial^2}{\partial x^2}\phi+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=-\frac{\rho(x,y,z)}{\epsilon}=f(x,y,z).$$

this is poisson's equation. QED.

Ch 13

$\boxed{1.2}$ Show that the expression $u = \sin(x - vt)$ describing a sinusoidal wave satisfies the wave equation. Show that in general $u = f(x - vt)$ and $u = f(x + vt)$ satisfy the wave equation 1.4 where f is any function with a second derivative.

wave equation $\nabla^2 u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}$

to show that $u = \sin(x - vt)$ satisfy wave equation, sub into the wave equation:

$$\frac{\partial u}{\partial x} = \cos(x - vt) \qquad \frac{\partial u}{\partial t} = \cos(x - vt)(-v)$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin(x - vt) \qquad \frac{\partial^2 u}{\partial t^2} = -\sin(x - vt)(v^2)$$

so $\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}}$ which is the wave equation

if $u = f(x + vt)$, then

$$\frac{\partial u}{\partial x} = f' \qquad \frac{\partial u}{\partial t} = f' v$$

$$\frac{\partial^2 u}{\partial x^2} = f'' \qquad \frac{\partial^2 u}{\partial t^2} = f'' v^2$$

so $\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}}$

for $u = f(x - vt)$, we get

$$\frac{\partial u}{\partial x} = f' \qquad \frac{\partial u}{\partial t} = -v f'$$

$$\frac{\partial^2 u}{\partial x^2} = f'' \qquad \frac{\partial^2 u}{\partial t^2} = v^2 f''$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}}$$

QED.

ch 13

$\boxed{1.3}$ Assume from electricity the following equations which are valid in free space.

$$\bar{\nabla}\cdot\bar{E}=0 \qquad \bar{\nabla}\cdot\bar{H}=0$$

$$\bar{\nabla}\times\bar{E} = -\mu\frac{\partial\bar{H}}{\partial t}, \quad \bar{\nabla}\times\bar{H} = \epsilon\frac{\partial\bar{E}}{\partial t}$$

from them show that any component of $\bar{E}$ or $\bar{H}$ satisfies the wave equation (1.4) with $v=(\epsilon\mu)^{-1/2}$.

wave equation is $\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$

identity (c) is $\bar{\nabla}\times(\bar{\nabla}\times\bar{\nabla}) = \bar{\nabla}(\bar{\nabla}\cdot\bar{\nabla}) - \nabla^2\bar{\nabla}$

in $\bar{\nabla}\times\bar{H} = \epsilon\frac{\partial\bar{E}}{\partial t}$, let $v=(\epsilon\mu)^{-1/2}$. so $v^2 = \frac{1}{\epsilon\mu}$

i.e $\epsilon = \frac{1}{\mu v^2}$. so $\boxed{\bar{\nabla}\times\bar{H} = \frac{1}{\mu v^2}\frac{\partial\bar{E}}{\partial t}}$

so
$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \frac{1}{\mu v^2}\left(\bar{i}\frac{\partial E_x}{\partial t} + \bar{j}\frac{\partial E_y}{\partial t} + \bar{k}\frac{\partial E_z}{\partial t}\right)$$

$$i\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) - \bar{j}\left(\frac{\partial}{\partial x}H_z - \frac{\partial}{\partial z}H_x\right) + \bar{k}\left(\frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x\right) = \frac{1}{\mu v^2}(\qquad)$$

so
$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \frac{1}{\mu v^2}\frac{\partial E_x}{\partial t}$$

$$\frac{\partial}{\partial x}H_z - \frac{\partial}{\partial z}H_x = \frac{1}{\mu v^2}\frac{\partial E_y}{\partial t}$$

$$\frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x = \frac{1}{\mu v^2}\frac{\partial E_z}{\partial t}$$

Sorry, do not see how to continue with this problem.