University Course

MATH 121 A
Mathematical Tools for the Physical Sciences

UC BERKELEY
Spring 2004

My Class Notes

Nasser M. Abbasi

Spring 2004

Contents

Chapter 1

Introduction

This is my web page for course MATH 121 A, Mathematical methods in physical sciences, that I took in spring 2004 at UC Berkeley. Hard course. Lots of HWs.

This is the syllabus

Math 121A, Spring 2004
Fraydoun Rezakhanlou

Homework from *Mathematical Methods in Physical Sciences*, by Boas, second edition

Homework set 1. **Due Monday Feb. 2**
Chapter 1: 1.5, 2.(6-7), 4.(2,4 8), 5.(1,8)
6.(1-3, 5,7,8,11,12,15-18,21,33-35), 7.(2,3,5,6)
9.(1,3,7,12,21), 10.(1,3,15,19,20)
13.(2,6,8,13,14,16,17,33,37,40)

Homework set 2. **Due Monday, Feb. 9**
Chapter 1: 14(5,6,9), 15.(5,11,16,18), 16.(3,18,22)
Chapter 2: 4.(7,14), 5.(6, 13, 20,32,39,45,51,53,55,56,60,62)
6.(4,10,12), 7.(3,12), 8.(1,2)

Homework set 3. **Due Wednesday, Feb. 18**
Chapter 2: 9.(2,12,19,24,27,28), 10.(18,22,28), 11.(5,6,11,18) 14.(6.9,19,24(b)
15.(3, 17), 16.11, 17.(7,14,17,23,24,30,32)

Homework set 4. **Due Monday, Feb. 23**
Chapter 4: 1.(6,7,23,24), 2.(3,6,8) 4.(2,8-10,13)

The **first midterm** is Wednesday, Feb. 25 , in class and will cover Chapter 1, Chapter 2 and Sections 1-4 of Chapter 4.

Homework set 5. **Due Monday, March 1**
Chapter 4: 5.(3,4), 6.(3,5,6), 7.(3,5,8,15,19), 8.(1-3)

Homework set 6. **Due Monday, March 8**
Chapter 4: 9.(2-8), 10.(2,7,10), 11.(1,3,6-9), 12.(1-4,7,8)
Chapter 14: 1.(6,12)

Homework set 7. **Due Monday, March 15**
Chapter 14: 2.(22,23,34,37,40, 55,56,60), 3.(3b,5,17-20,23), 4.(6,7,9,10)
5.1

Homework set 8. **Due Monday, March 29**
Chapter 14: 6.(5-9,15,19,23,31) , 7.(4,5,12,18,20,29,30a,34,45), 8.(4,5,15),

The **second midterm** is on Friday, April 2, in class and will cover Sections 5-12 of Chapter 4 and Sections 1-8 of Chapter 14.

Homework set 9. **Due Monday, April 5**
Chapter 14: 9.(2,3,4,7), 10.(5,6,11,13)
Chapter 7: 3.(4,6)

Homework set 10. **Due Monday, April 12**
Chapter 7: 4.(5, 12,13,15), 5.(2,5,8,10), 6.5, 7.(4,12,13) 8.12

Homework set 11. **Due Monday, April 19**
Chapter 7: 9.(2,6,10), 11.(7,10), 12.(8.13)
Chapter 15: 2.(2-5,9,11,17,18,21-23), 3.(4,6,8,11,24,25,29,30)

Homework set 12. **Due Monday, April 26**
Chapter 15: 4.(3,5,7,12,18,20,21,23,25) , 5.(1,2,4,10,22) , 6.(2,4,5,9), 7.(7,9,11)

Homework set 13. **Due Monday, May 3**
Chapter 7: 8.(2,3,12,15,17)
Chapter 9: 2.(1,3,6), 3.(2,4,6,9), 5.(2,6) 6.(1,2,5).

Final Exam On Friday 5-8 pm, May 21

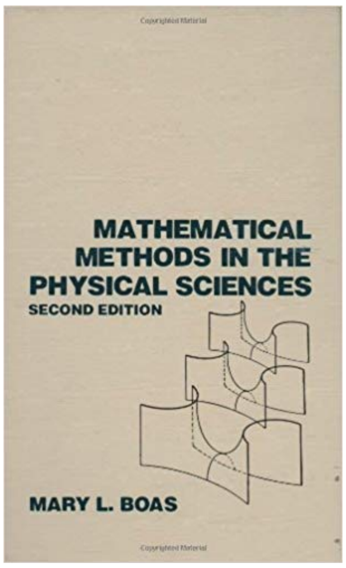
No make-up exams, make sure you can make all the exams.
Grading Policy: HWs (25 points), Midterms (each 20 points), Final (35 points)

Professor Fraydoun Rezakhanlou, UC Berkeley Math department, gave the course. A very good teacher. Teaches without using notes, all from memory, which I thought was really amazing.

This course was lots of work. About 30-40 problems for HW each week. But only 2 random problems are graded. So it is possible to solve 39 of out 40 problems correctly and get only 50%, if the problem missed happened to be selected to grade.

Exams as closed notes and closed book. No cheat sheet either. No calculators.

Textbook was Mary Boas, second edition. This seems to be the standard book for this type of course are most universities, at least the ones I know about. It is a a very good book, but more detailed worked examples would have been nice. So another book such as the problem solvers type book might be useful to have.



For some reason this course (and Math121B) did not have discussion period, this was unfortunate since discussion periods can be really useful. I was told this is because of budget cuts in the California university system.

This below is a picture of Evans hall. It is a big tall building full of very smart people. The math department is on the 9th floor. The course was in room 75, which is on the ground floor on Evans hall.

1.0.1 Course description

These are the Course outline handouts. Page 1, and page 2. it contains the problems to solve for each HW.

Spring 2004 (January-May 2004)

Course description: Functions of a complex variable, Fourier series, finite-dimensional linear systems. Infinite-dimensional linear systems, orthogonal expansions, special functions, partial differential equations arising in mathematical physics. Intended for students in the physical sciences who are not planning to take more advanced mathematics courses.

Units: 4

Book: MATHEMATICAL METHODS IN PHYSICAL SCI, BOAS. 2nd edition chapters 1,2,4,7,9,14,15

Instructor: Professor Fraydoun Rezakhanlou, Math department, UC Berkeley

Position: Associate Professor

E-mail: `rezakhan@math.berkeley.edu` To reduce spam, this address is javascript encoded.

Phone: +1 (510) 642-2838

Office: 815 Evans Hall

Research: Probability theory, Partial differential equations

Chapter 2

My notes, study notes

Near the end of the course I started to write down some study notes to have ready for the final exam, Here you find my own study notes for this course and Math 121B I took at the same time. I should have started this earlier in the course.

2.1 Misc. notes

1. it is OVER !! finals finished.
2. math 121A. review HW 11 done 11 PM wed. review of HW 12...finished 3:20 AM. (4 hrs per HW to go over!). getting ready to start on HW 13...6 AM, almost finished HW 13 most of the rest I know, about calculus of variations, studied that before, but need to finish HW13 (may be one more hr). Then start on the HW 10 (Fourier series). will do after wake up. now going to sleep. 1 pm Thursday.. finishing HW 13 now.... 3:00 PM finished HW 13. This contained important stuff. 8 hrs study only today
3. now midnight Thursday. 1 hr study. went over some problem from midterm2, and Lagrange equations physics problem derivations. need to finish review of midterm 2, then back to HW review. it is 4 AM now, read notes and finished midterm, starting on HW 9, getting tired, will not be able to review everything before finals, need to try to concentrate on last stuff only... 4:40 AM finished HW 10 (Fourier series, easy stuff), now starting on HW 9...5:20 AM Friday, finished HW 9. HW 8 is on Laurent series, so important... 6:40 AM, ok finished. going to eat something and sleep and wake up for the exam.
4. finals for math 121B over. I made 3 very stupid mistakes., can't believe I did those. blow away 3 fairly easy questions I could have full credit on. but I think I can pull a B in the course. keep fingers crossed.
5. Practice more chapter 7 Fourier series tricks (odd/even) stuff
6. Make sure I remember ds^2 in all coordinates
7. learn better how to evaluate this:$\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right)$
8. HW's for 121B went over since midterm exam: HW5 chapter 12, HW 6 done, HW 7 stop here. Saturday night.., HW 11 done, HW 12 working on..finished. Now studying probability distribution, last HW
9. write down the same space for the 2 die, with the sum, some problems use it.

questions:

1. Why did we use series method to find solution to Legendre ODE, but used generalized series method to find solution to Bessel ODE? how to know when to use which? Answer: if ODE has something like $(1-x)y''$, then at $x = 0$ we'll have problem, then use the generalized power series)

2. The Legendre ODE is solved using series method, assuming l is an integer. We get one solution which is Legendre function of first kind $p_l(x)$. What if l is not an integer? A: Legendre P_l is only defined for integer l? YES? No, there are tables for non-integer, but these cases are not important.

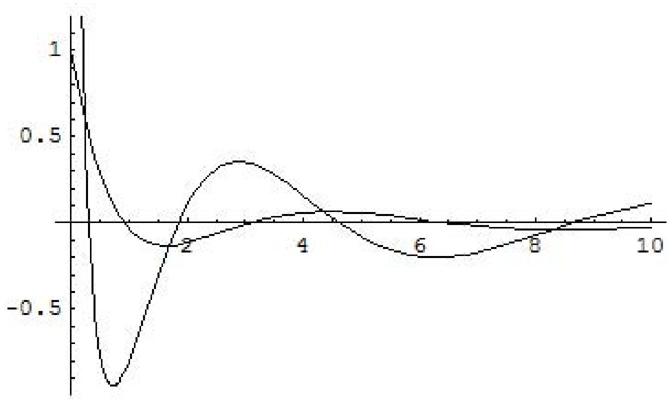
3. What if we get a legendre ODE and we want to find solution for $x > 1$? Since legendre functions are only defined for x less than one (to have convergence). Physics example? usually x is the cosine of an angle so it is ≤ 1.

4. What if l is not an integer in the legendre ODE? how to get a solution? this is special cases, not important, look up handbooks.

5. problem I solved in HW#6, chapter 12, 16.3. check my solution. I claimed that the second solution is N_p but since I found P NOT to be an integer, hence the second solution is one containing log and not a combination of J_{-p}. When I solved it in mathematica, I get this solution (notice complex number?), could this second solution be converted to log function? answer: OK, the solution I did will turn out to have log in it if I put p=integer and use L'hospital's rule to evaluate.

```
eq = x y''[x] + 2 y'[x] + 4 y[x] == 0
sol = DSolve[eq, y[x], x]
allY = y[x] /. sol;
allY = allY /. {C[1] → 1, C[2] → 2}
Plot[{Evaluate[Re[allY]], Evaluate[Im[allY]]}, {x, 0, 10}]
```

$$\left\{\left\{y[x] \to \frac{\text{BesselJ}[1, 4\sqrt{x}]\, C[1]}{2\sqrt{x}} - \frac{i\, \text{BesselY}[1, 4\sqrt{x}]\, C[2]}{\sqrt{x}}\right\}\right\}$$



6. When solving for equation 16.1 on page 516, we seem to only take the positive root for the variables, why? see for example page 516. $b = 2$ but it is really $b = \pm 2$ answer: OK, any of these will give a good solution, just pick one.

7. on page 528, can I just set $n = 0$ always to solve for the indical equation as shown in the example? is it better to solve this using the $\sum$ directly as shown in the example instead of setting up a table? table seems more clear, but the example method seems shorter.

8. How to solve chapter 16, 4.1 part (c) using Bayes rule? I write: Let A=event first chair is empty, let B=event second chair is empty. We need to find $P(AB) = P(A)P_A(B) = (\frac{1}{10})(\frac{1}{9}) = \frac{1}{90}$, but the answer should be $\frac{1}{45}$, what is it I am doing wrong? wrong. $P(A) = 1/5$ *not* 1/10.

9. for problem HW 12, chapter 16, 4.8, part b. It says given 2 cards drawn from deck, if you know one is an ace, what it the chance the BOTH are an ace? I know how to solve by the book. but why can I not say the following: since we KNOW that one card is an ace, then the chance that both cards are an ace is just the chance the second card being an ace (since we know the first is an ace). So this should give $\frac{3}{51}$

10. random variable is defined as a function on the sample space. however, it is multi-valued. for example, if x= sum of 2 die throw, then more than one event can give the same random variable. is this OK? I thought a function must be single valued? answer: I am wrong. it is NOT multivalued.
11. check that my solution for chapter 16, 5.1 MATH 121B is correct, I have solution on paper. this is the last HW

2.2 Table summary of topics to study

2.2.1 Math 121A

ch.	title	topics	Exam
1	series	infinite series, power series,def. of convergence, tests for convergence, test for alternating series, power series, binomial series	1
2	complex numbers	finding circle of convergence (limit test), Euler formula power and roots of complex numbers, log, inverse log	1
4	partial differentiation	total diffenertials, chain rule, implicit differentiation partial diff for max and minumum, Lagrange muktipliers, change of variables Leibniz rule for differnetiation of integrals	1
14	complex functions	Def. of analytic fn, Cauchy-Riemann conditions, laplace equation, contour integrals, Laurent series, Residue theorm, methods of finding residues, pole type, evaluating integrals by residue, Mapping, conformal	1
7	Fourier series	expansion of function in sin and cosin, complex form, how to find coeff, Dirichlet conditions, different intervals, even/odd, Parseval's	2

15	Laplace/Fourier transforms	Laplce transform, table, how to use Laplace to solve ODE, Methods of finding inverse laplace, partial fraction, convolution, sum of residues, Fourier transform, sin/-consine transforms, Direc Delta Green method to solve ODE using impluse	F
9	Calculus of variations	Euler equation solving, Setting up Lagrange equations, KE, PE Solving Euler with constrainsts	F

2.2.2 Math 121B

ch.	title	topics	Exam
11	Special functions	Gamma, Debta, Error function	1
12	Series solution to ODE	Legendre, Bessel, orthogonality	1
13	PDE	separation of variables, Laplace (steady state), Heat (diffusion), Wave equation. Laplce in different coordinates, Laplacian, Wave in different coord.Poission equation	2
16	Probability	Baye's formula, how to find probability, methods of counting, Random variable concept, mean, Var, SD, distributions (Binomial, Gauss, Poisson)	final

2.3 Math 121 A notes

2.3.1 Chapter 1. Series

$a + ar + ar^2 + \cdots + ar^n + \cdots = \frac{a(1-r^n)}{1-r}$, Now, if $|r| < 1$, then the above is convergent, hence we get $S_n = \frac{a}{1-r}$. Always start by looking for a constant term a here, and then a term that is multiplied each time, r here.

2.3.2 Chapter 14. Complex functions

2.3.2.1 How to find the residue?

seek book page 598

2.3.3 Chapter 7. Fourier Series

Expand a periodic function (must be periodic) in sin and cos functions.

Let the function angular velocity be ω, which is defined as angles (radian) per second, i.e. $\omega = \frac{2\pi}{T}$ where T is the period in time, which is the time to make 2π angle.

$$\begin{aligned} f(x) = \frac{1}{2}a_0 + a_1 \cos \omega x + a_2 \cos 2\omega x + \\ \cdots + b_1 \sin \omega x + b_2 \sin 2\omega x + \cdots \end{aligned}$$

So, for a function whose period is 2π, i.e. $\omega = 1$, the above can be written as

$$\begin{aligned} f(x) = \frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \\ \cdots + b_1 \sin x + b_2 \sin 2x + \cdots \end{aligned}$$

Now, to find a_n and b_n

$$\begin{aligned} a_n &= \frac{2}{T}\int_T f(x) \cos \omega n x \, dx \\ b_n &= \frac{2}{T}\int_T f(x) \sin \omega n x \, dx \end{aligned}$$

So, I only need to remember ONE formula

note: Remember, when finding a_n, for a_0, do it separately, set $n = 0$ in the integral first and integrate that, do not set $n = 0$ in the result, leave that for $n \neq 0$. For b_n we do not need to worry about this, since for sin series it starts at $n = 1$

note: *When will this expansion converge to* $f(x)$? when the function meet the Dirichlet conditions. Basically it needs to be periodic of period 2π, single valued, has finite number of jumps. At jumps, the series converges to average of the function there.

In these kind of problems, we are given a function $f(x)$ and asked to find its F. series. So need to apply the above formulas to find the coefficients. Need to know some tricks for quickly evaluating the integrals.

Now there is a complex form of all the above equations.

$$f(x) = c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + c_2 e^{2ix} + c_{-2} e^{-2ix} + \cdots = \sum_{-\infty}^{\infty}$$

$$c_n = \frac{1}{T}\int_T f(x)\, e^{-inx\omega}\, dx$$

Now, ω, is the angular velocity. i.e. $\theta = \omega t$, so for ONE period T, $\theta = 2\pi$, hence $\omega = \frac{2\pi}{T}$, so c_n can be written as

$$c_n = \frac{1}{T}\int_T f(x)\, e^{-inx\frac{2\pi}{T}}\, dx$$

Notice that in this chapters we use distance for period (i.e. wave length λ) instead of time as period T. it does not matter, they are the same, choose one. i.e. we can say that the function repeats every λ, or the function repeats every one period T.

When using λ for period, say $-l, l$ or $-\pi, \pi$ the above equation becomes

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-inx\frac{2\pi}{2l}}\, dx = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-inx\frac{\pi}{l}}\, dx$$

note: Above integral for c_n is for negative n as well as positive n. In non-complex exponential expansion, there is no negative n, only positive.

note: $c_{-n} = \bar{c}_n$

note: there is a relation between the a_n, b_n, and the c_n which is

$a_n = c_{-n} + c_n$ and $b_n = i(-c_{-n} + c_n)$

IF given $f(x)$, defined over $(0, L)$, The algorithm to find Fourier series is this:

```
IF asked to find a(n) i.e. the COSIN series, THEN
extend f(x) so that it is EVEN (this makes b(n)=0)
and period now is 2L
ELSE
IF asked to find b(n), i.e. the SIN series, THEN
extend f(x) to be ODD (this makes a(n)=0)
and period now is 2L
ELSE we want the standard SIN/COSIN
period remains L, and use the c(n) formula
(and remember to do the c(0) separatly for the DC term)
END IF
END IF
```

2.3.3.1 Parseval's theorem for fourier series

This theory gives a relation between the average of the square of $f(x)$ over a period and the fourier coefficients. Physically, it says that this:

the total energy of a wave is the sum of the energies of the individual harmonics it carries

Average of $\left[f(x)\right]^2 = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_1^\infty a_n^2 + \frac{1}{2}\sum_1^\infty b_n^2$ over ONE period.

In complex form, Average of $\left|f(x)^2\right| = \sum_{-\infty}^{\infty} |c_n|^2$. Think of this like pythagoras theorem.

For example, given $f(x) = x$, then $\left[f(x)\right]^2 = \frac{1}{2}\int_{-1}^{1} x^2 dx = \frac{1}{3}$, then $\frac{1}{3} = \sum_{-\infty}^{\infty} |c_n|^2$

In the above we used the standard formula for average of a function, which is

average of $f(x) = \frac{1}{T}\int_T f(x)\, dx$, here we should need to square $f(x)$

2.3.4 Chapter 15. Integral transforms (Laplace and Fourier transforms)

2.3.4.1 Laplace and Fourier transforms definitions

$$F\,f(x) = F(p) = \int_0^{\infty} f(x)\,e^{-px}\,dx \qquad p>0$$
$$F\,g(x) = g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,e^{-i\alpha x}\,dx$$

Associate Fourier with $\frac{1}{2\pi}$. (mind pic: Fourier=Fraction i.e.→ $\frac{1}{2\pi}$) and Fourier goes from $-\infty$ to $+\infty$ (mind pic: Fourier=whole Floor), Fourier imaginary exponent, Laplace real exponent.

Note: Laplace transform is linear operator, hence $L\left[f(t)+g(t)\right] = Lf(t) + Lg(t)$ and $L\left[c\,f(t)\right] = c\,Lf(t)$

2.3.4.2 Inverse Fourier and Laplace transform formulas

(We do not really use the inverse Laplace formula directly (called Bromwich integral), we find inverse Laplace using other methods, see below)

$$f(x) = \frac{1}{2\pi i}\int_{c-i\,\infty}^{c+i\,\infty} F(z)\,e^{zt}\,dz \qquad t>0 \qquad \text{Inverse laplace}$$
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\alpha)\,e^{i\alpha x}\,d\alpha \qquad \text{Inverse fourier}$$

The Fourier transform has 2 other siblings to it (which Laplace does not), these are the sin and cos transform and inverse transform. I'll add these later but I do not think we will get these in the exam.

Note: To get the inverse Laplace transform the main methods are

1. using partial fractions to break the expression to smaller ones we can lookup in tables
2. Use Convolution. i.e. given $Y = L\left(f_1\right)L\left(f_2\right) \to y = \int_0^t g(t-\tau)\,f(\tau)\;d\tau = g\otimes f$ use this as an alternative to partial fraction decomposition if easier. mind pic: t one time, τ 2 times.
3. Use the above integral (Bromwich) directly (hardly done)
4. To find $f(t)$ from the Laplace transform, instead of using the above formula, we can write

   $f(t)$ = sum of residues of $F(z)e^{zt}$ at all poles. For example given $F(z)$, we multiply it by e^{zt}, and then find all the poles of the resulting function (i.e. the zeros of the denominator), then add these.

Note: To find Fourier transform , $g(\alpha)$, must carry the integration (i.e. apply the integral directly, no tables like with Laplace).

Note: we use Laplace transform as a technique to solve ODE. Why do we need Fourier transform? To represent an arbitrary function (must be periodic or extend to be period if not) as a sequence of sin/cosine functions. And why do we do this? To make it easier to analyze it and find what frequency components it has. For continuous function, use fourier transform (integral).

note: Function must satisfy Dirichlet conditions to use in fourier transform or Fourier series.

note: Fourier series expansion of a function will accurately fit the function as more terms are added. But in places where there is a jump, it will go to the average value of the function at the jump.

question: When do we use fourier series, and when to use fourier transform? Why do we need F. transform if we can use F. Series? We use F. transform for continuous frequencies. What does this really mean?

2.3.4.3 Using Laplace transform to solve ODE

Remember

$$\begin{aligned} L(y) &= Y \\ L(y') &= pY - y_0 \\ L(y'') &= p^2Y - py_0 - y_0' \end{aligned}$$

note: p has same power as order of derivative. do not mix up where the p goes in the y'' equation. remember the y_0' has no p with it. mind pic: think of the y_0 as the senior guy since coming from before so it is the one who gets the p.

note: if $y_0 = y_0' = 0$ (which most HW problem was of this sort), then the above simplifies to

$$\begin{aligned} L(y') &= pY \\ L(y'') &= p^2Y \end{aligned}$$

So given an ODE such as $y'' + 4y' + y = f(t) \rightarrow (p^2 + 4p + 4)Y = L(f(t))$

i.e. just replace y'' by p^2, etc... This saves lots of time in exams. Now we get an equation with Y in terms of p, now solve to find $y(t)$ from Y using tables. Notice that solution of ODE this way gives a particular solution, since we used the boundary conditions already.

For an ODE such as

$$Ay'' + By' + Cy = h(t)$$

its Laplace transform can be written immediately as

$$\begin{aligned} Ap^2Y + BpY + CY &= L\,h(t) \\ Y &= \frac{L\,h(t)}{Ap^2 + Bp + C} \end{aligned}$$

whenever the B.C. are $y_0' = 0$ and $y_0 = 0$

2.3.4.4 Partial fraction decomposition

When denominator is linear time quadratic or quadratic time quadratic PFD is probably needed.

This is how to do PFD for common cases

$$\frac{1}{(x+c)\left(x^2+x+6\right)}=\frac{A}{(x+c)}+\frac{Bx+C}{\left(x^2+x+6\right)} \quad \text{(quadratic in denominator case)}$$

$$\frac{1}{\left(x^2+3x+4\right)\left(x^2+x+6\right)}=\frac{Ax+B}{\left(x^2+3x+4\right)}+\frac{Cx+D}{\left(x^2+x+6\right)} \quad \text{(quadratic in denominator case)}$$

$$\frac{1}{(x+c)(x+d)}=\frac{A}{(x+1)}+\frac{B}{(x+d)}$$

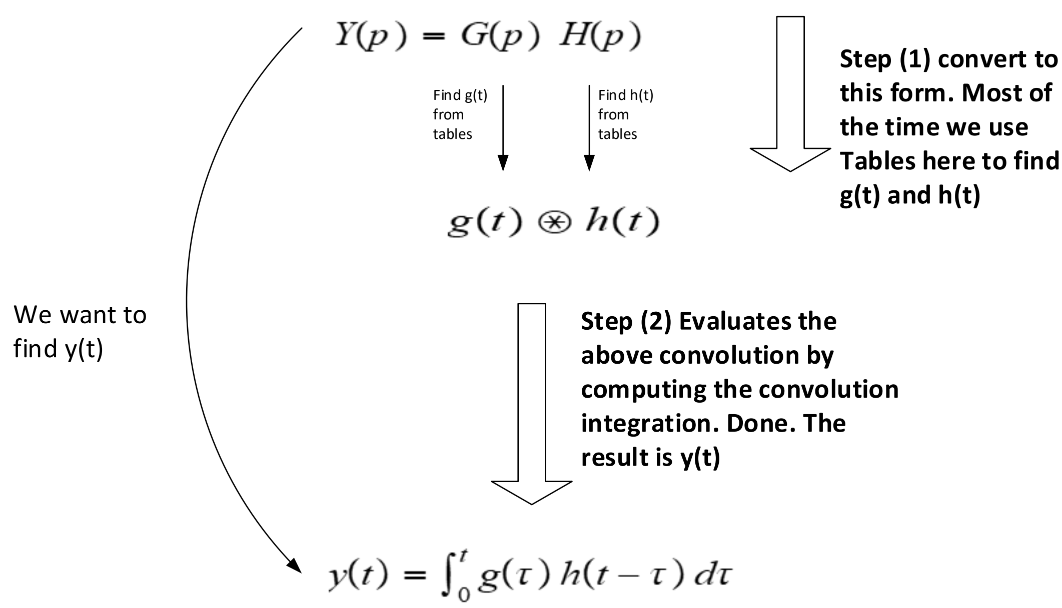
$$\frac{x^2+\ x+b}{(x+c)(x-d)^2}=\frac{A}{(x+1)}+\frac{B}{(x-d)}+\frac{B}{(x-d)^2} \quad \text{(repeated roots case)}$$

we get some equations which we solve for A, B, etc... This part can be time consuming in exam.

2.3.4.5 convolution

Main use of convolution in this class is to find the inverse laplace transform.

If we are given the transform itself (i.e. frequency domain) function, and asked to find the inverse, i.e. the time domain function. Then look at the function given, if it made of 2 functions multiplied by each others, then good chance we use convolution.



Finding the inverse Laplace transform using convolution

Example:

Given this equation

$$Y(p)=G(p)\ \ H(p)$$

We first find the inverse of $G(p)$ and $H(p)$ separately. i.e. we find $g(t)$ and $h(t)$. we usually do this by looking up tables. Once we do this step, the next step is to take the convolution of these 2 time domain functions.

The result, will be $y(t)$, i.e. the inverse of $Y(z)$.

Notice that you **can NOT** just say $y(t)=g(t)\ h(t)$, DO NOT DO THIS. But we must use convolution to find $y(t)$:

$$
\begin{aligned}
y(t) &= g(t) \circledast h(t) \\
y(t) &= \int_0^t g(\tau)\, h(t-\tau)\, d\tau \\
&= \int_0^t g(t-\tau)\, h(\tau)\, d\tau
\end{aligned}
$$

Notice, choose the simpler function to put the $(t-\tau)$ in. It does not matter if it is the f or the h. remember, the τ occur 2 times in the integral, the t one time.

The above means

$$
\begin{aligned}
\mathscr{L} y &= \mathscr{L} g(t)\ \mathscr{L} h(t) = \mathscr{L}\left[\, g(t)\ \circledast h(t\,)\right] \\
y &= g(t)\ \circledast h(t\,)
\end{aligned}
$$

The above comes when we want to solve an ODE. Usually we know $g(t)$ which is the transfer function, and $h(t)$ is given (the forcing function of the ODE).

For Fourier transform, convolution can be used as well. it is very similar equation:

$$
F\left(g(t)\right)\ \ F\left(h(t)\right) = \frac{1}{2\pi}\, \mathscr{F}\left[\, g(t)\ \circledast h(t\,)\right]
$$

So difference is the $\frac{1}{2\pi}$

2.3.4.6 Parseval's theorem

(total energy in a signal equal the sum of the energies in the harmonics that make up the signal).

$$
\int_{-\infty}^{\infty} \left|g(\alpha)\right|^2 d\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|f(x)\right|^2 dx
$$

2.3.4.7 Dirac delta and Green function for solving ODE

Dirac delta function is a function defined for t, who has an area of 1 and zero width and ∞ value at t. (not a real function). Used to represent an impulse force being applied at t.

When multiplied with any other function inside an integral will given that other function at the time the impulse was applied. i.e. $\int f(t)\ \delta(t-t_0)\ dt = f(t_0)$, here t_0 is the time the impulse is applied.

note: Fourier transform of delta function: $g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(x-x_0) e^{-i\alpha x}\, dx = e^{-i\alpha x_0}$

note: Green function $G(t,t')$ is the response of a system (solution of an ODE) when the force (input) is an impulse at time $t=t'$

How to use Green function to solve an ODE? Given $G(t,t')$, $y(t) = \int_0^{\infty} G(t,t')\ \ f(t')\ dt'$ Where $f(t)$ is the force on the system (the RHS to the ODE). Usually we are given the Green function and asked to solve the ODE. So just need to apply the above integral.

Question: ask about if the above is correct for the finals or is it possible we need to find G as well?

2.3.4.7.1 Solving an ODE using green method Here we are given an ODE, with a forcing function (i.e. nonhomogeneous ODE). And given 2 solutions to it, and asked to find the particular solution.

Example, $y'' - y = f(t)$ and solutions are y_1, y_2 then the particular solution is $y_p = y_2 \int \frac{y_1 f(t)}{W} dt - y_1 \int \frac{y_2 f(t)}{W} dt$ where $W = \begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix}$

2.3.5 Chapter 2. Complex Numbers

note: When given a problem such as evaluate $(-2-2i)^{\frac{1}{5}}$, always start by finding the length of the complex number, then extract it out before converting to the $re^{in\theta}$ form. For example, $(-2-2i)^{\frac{1}{5}} = 2\sqrt{2}\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$, the reason is that now the stuff inside the brackets has length ONE. So we now get $2\sqrt{2}\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = 2\sqrt{2}\,e^{-\frac{3}{4}\pi i}$ and only now apply the last raising of power to get $\left(2\sqrt{2}\,e^{-\frac{3}{4}\pi i}\right)^{\frac{1}{5}} = 2^{\frac{3}{10}} e^{\frac{\frac{3}{4}\pi i + 2n\pi}{5}}$ for $n = 0, 1, 2, 3, \cdots$ make sure not to forget the $2n\pi$, I seem to forget that.

2.3.6 Chapter 9. Calculus of variations

2.3.6.1 Euler equation

How to construct Euler equation $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$. If integrand does not depend on x then change to y. Example $\int_{x_2}^{x_1} y'^2\, y\, dx \to \int_{y_2}^{y_1} \frac{1}{x'^2}\, y \left(x'\, dy\right) \to \int_{y_2}^{y_1} \frac{1}{x'}\, y \;\; dy$ this is done by making the substitution $y' = \frac{1}{x'}$ and $dx = x'\, dy$. Now Euler equation changes from $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ to $\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0$.

Normally, $\frac{\partial F}{\partial y}$ will be zero. Hence we end up with $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ and this means $\frac{\partial F}{\partial y'} = c$, and so we only need to do ONE integral (i.e. solve a first order ODE). If I find myself with a 2 order ODE (for this course!) , I have done something wrong since all problems we had are of this sort.

2.3.6.2 Lagrange equations

are just Euler equations, but one for each dimension.

F is now called L. where $L = T - V$ where $T = K.E.$ and $V = P.E.$, $T = \frac{1}{2} m v^2$, $V = mgh$

So given a problem, need to construct L ourselves. Then solve the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$$

The tricky part is finding v^2 for different coordinates. This is easy if you know ds^2, so just remember those

$$
\begin{aligned}
ds^2 &= dr^2 + r^2 d\theta^2 && \text{(polar)}\\
ds^2 &= dr^2 + r^2 d\theta^2 + dz^2 && \text{(cylindrical)}\\
ds^2 &= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 && \text{(spherical)}
\end{aligned}
$$

So to find v^2 just divide by dt^2 and it follows right away the following

$$
\begin{aligned}
v^2 &= \dot{r}^2 + r^2\dot{\theta}^2 && \text{(polar)}\\
v^2 &= \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 && \text{(cylindrical)}\\
v^2 &= \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2 && \text{(spherical)}
\end{aligned}
$$

To help remember these: Note ds^2 all start with $dr^2 + r^2 d\theta^2$ for each coordinates system. So just need to remember the third terms. (think of polar as subset to the other two). Also see that each variable is squared. So the only hard think is to remember the last term for the spherical.

Remember that in a system with particles, need to find the KE and PE for each particle, and then sum these to find the whole system KE and PE, and this will give one L for the whole system before we start using the Lagrange equations.

2.3.6.3 Solving Euler-Lagrange with constraints

The last thing to know is this chapter is how to solve constraint problems. This is just like solving for Euler, expect now we have an additional integral to deal with.

So in these problems we are given 2 integrals instead of one. One of these will be equal to some number say l.

So we need to minimize $I = \int_{x_2}^{x_1} F(x, y', y)\,dx$ subject to constraint that $g = \int_{x_2}^{x_1} G(x, y', y)\,dx = l$

Follow the same method as Euler, but now we write

$$
\frac{d}{dx}\left(\frac{\partial}{\partial y'}(F + \lambda G)\right) - \frac{\partial}{\partial y}(F + \lambda G) = 0
$$

So replace F by $F + \lambda G$

This will give as equation with 3 unknowns, 2 for integration constants, and one with λ, we solve for these given the Boundary conditions, and l but we do not have to do this, just need to derive the equations themselves.

Some integrals useful to know in solving the final integrals for the Euler problems are these

$$
\int \frac{c}{\sqrt{y^2 - c^2}}\,dy = c\cosh^{-1}\left(\frac{y}{c}\right) + k
$$

$$
\int \frac{c}{\sqrt{1 - c^2\,y^2}}\,dy = c\sin^{-1}\left(c\,y\right) + k
$$

$$
\int \frac{c}{y\sqrt{y^2 - c^2}}\,dy = \frac{1}{c}\cos^{-1}\left(\frac{c}{y}\right) + k
$$

2.4 MATH 121B Notes

2.4.1 Chapter 12. Series solution of ODE and special functions

Bessel ODE	$x^2 y'' + x y' + \left(x^2 - p^2\right)y = 0$ defined for INTEGER and NON integer P
first solution	$y_1 = J_p(x) = \sum \frac{-1^n}{\Gamma(n+1)\Gamma\left(n+p+1\right)}\left(\frac{x}{2}\right)^{2n+p}$ (for p an integer or not)
second solution	$y_2 = N_p(x) = Y_p(x) = \frac{\cos\left(\pi p\right) J_p(x) - J_{-p}}{\sin \pi p}$ (note: p here is NOT an integer)
second solution	y_x contains a log function. note: p here IS an integer.
Orthogonality	$\int_0^1 x\, J_p(ax)\, J_p(bx)\, dx = \begin{cases} 0 & if\ a \neq b \\ \frac{1}{2}J_{p+1}^2(a) = \frac{1}{2}J_{p-1}^2(a) = \frac{1}{2}J_p'^2(a) & if\ a = b \end{cases}$ a, b are zeros of J_p
recursive formula	$\frac{d}{dx}\left[x^p J_p\right] = x^p J_{p-1}$, $\frac{d}{dx}\left[\frac{1}{x^p}J_p\right] = -\frac{1}{x^p}J_{p+1}$, $J_{p-1} + J_{p+1} = \frac{2p}{x}J_p$, $J_{p-1} - J_{p+1} = 2J_p'$
	$J_p' = -\frac{p}{x}J_p + J_{p-1} = \frac{p}{x}J_p - J_{p+1}$ NOTICE: No Rodrigues formula for Bessel func, since not polyn.
notes:	We used a *generalized power series* method to find the solutions. IF p is NOT an integer, then J_p and J_{-p} (or N_p) are two independent solutions IF p is an integer, then J_p and J_{-p} are NOT two independent solutions, use log for y_2 J_p is called Bessel function of first kind, and Y_p is called second kind. p is called the ORDER. IF $p = n + \frac{1}{2}$, a special case, we get spherical bessel functions $j_n(x)$ and $y_n(x)$ $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x) = x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right)$

Legendre ODE	$\left(1-x^2\right) y'' - 2x\, y' + l(l+1)y = 0$ defined for INTEGER l only
first solution	$y_1 = P_l(x)$ examples: $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}\left(3x^2-1\right), P_3(x) = \frac{1}{2}\left(5x^3-3x\right)$ $P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right),\ P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$
second sol.	We do not use this. Called Legendre polynomials of second kind $Q_l(x)$
Orthogonality	$\int_{-1}^{1} P_l(x)\ P_m(x)\, dx = 0$ if $m \neq n$, also $\int_{-1}^{1} P_l(x)\ \times \left(\text{any poly degree} < l\right) dx = 0$
Normalization	$\int_{-1}^{1} [P_l(x)]^2 dx = \frac{2}{2l+1}$
Generating function	$\Phi(x,h) = \frac{1}{\sqrt{(1-2xh+h^2)}}, \lvert h\rvert < 1,\ \Phi(x,h) = P_0(x) + hP_1(x) + h^2P_2(x) + \cdots = \sum_{l=0}^{\infty} h^l P_l(x)$
recursive formula	later, see book page 491
Rodrigues	$P_l = \frac{1}{2^l l!} \frac{d^l}{dx^l}\left(x^2-1\right)^l$
notes:	we used series method to find the solution (not generalized series method). x must be less than 1, this is needed to have convergence. Hence Legendre solution only defined over $-1,1$. Also, l is assumed to be a non-negative integer.l is called the ORDER of legendre poly.

Associated Legendre	$\left(1-x^2\right) y'' - 2x\, y' + \left[l(l+1) - \frac{m^2}{1-x^2}\right] y = 0$
first solution	$y_1 = P_l^m(x) = \left(1-x^2\right)^{\frac{m}{2}} \frac{d^m}{dx^m}\left(p_l(x)\right)$
second solution	do not use
Orthogonality	did not cover, but should be the same as Legendre polynomials P_l
Normalization	$\int_{-1}^{1} \left[P_l^m(x)\right]^2 dx = \frac{2}{2l+1}\left(\frac{l+m!}{l-m!}\right)$

example using recursive formula for Legendre: $\Phi(x,h) = \frac{1}{\sqrt{(1-2xh+h^2)}}$, let $y = 2xh - h^2$ then $\Phi(x,h) = \frac{1}{\sqrt{(1-y)}} = 1 + \frac{1}{2}y + \frac{\frac{1}{2}\frac{3}{2}}{2!}y^2 + \cdots$, then sub back for y, and simplify we get $\Phi = 1 + xh + h^2\left(\frac{3}{2}x^2 - \frac{1}{2}\right) + \cdots = P_0 + hP_1 + hP_2 + \cdots$, hence $P_0 = 1$, $P_1 = x, P_2 = \left(\frac{3}{2}x^2 - \frac{1}{2}\right)$,etc..

Series solution: $y = a_0 + a_1 x + a_2 x^2 + \cdots$

Generalized series solution: $y = a_0 x^s + a_1 x^{s+1} + a_2 x^{s+2} + \cdots$ solve for s, we get indicial eq. for each s we solve again to find the a_0 and the a_1 solutions. Final solution is the sum of the solutions for both s values. Will only get 2 solutions in total (for second order ODE).

2.4.1.1 Leibniz Rule for differentiation of product

$$\begin{aligned}\frac{d^n}{dx^n}(fg) &= \begin{pmatrix} n \\ 0 \end{pmatrix}\frac{d^0}{dx^0}f\frac{d^n}{dx^n}g + \begin{pmatrix} n \\ 1 \end{pmatrix}\frac{d}{dx}f\,\frac{d^{n-1}}{dx^{n-1}}g + \begin{pmatrix} n \\ 2 \end{pmatrix}\frac{d^2}{dx^2}f\,\frac{d^{n-2}}{dx^{n-2}}g + \cdots + \begin{pmatrix} n \\ n \end{pmatrix}\frac{d^n}{dx^n}f\frac{d^0}{dx^0}g \\ &= \frac{d^0}{dx^0}f\frac{d^n}{dx^n}g + n\,\frac{d}{dx}f\,\frac{d^{n-1}}{dx^{n-1}}g + \frac{n\times n-1}{2!}\frac{d^2}{dx^2}f\,\frac{d^{n-2}}{dx^{n-2}}g + \cdots + \frac{d^n}{dx^n}f\frac{d^0}{dx^0}g\end{aligned}$$

For example $\frac{d^9}{dx^9}(x\ \sin x) = x\ \frac{d^9}{dx^9}\sin x + 9\times\frac{d}{dx}x\ \frac{d^3}{dx^3}\sin x$ + rest is ZERO terms, so we get $\frac{d^9}{dx^9}(x\ \sin x) = x\ \cos x + 9\sin x$ this is much faster than actually differentiating 9 times !

This can be remembered since it is the same form as the binomial expansion

$$\begin{aligned}(f+g)^n &= \begin{pmatrix} n \\ 0 \end{pmatrix}f^0\ g^n + \begin{pmatrix} n \\ 1 \end{pmatrix}f^1\ g^{n-1} + \begin{pmatrix} n \\ 2 \end{pmatrix}f^2\ g^{n-1} + \cdots + \begin{pmatrix} n \\ n \end{pmatrix}f^n\ g^0 \\ (f+g)^9 &= g^9 + 9\,f\,g^8 + \frac{9\times 8}{2!}f^2\ g^7 + \cdots + f^9\end{aligned}$$

2.4.1.2 Finding second solution for ODE

when the indicial equation gives only one value for s (from the generalized power series method), we can find the second solution by assuming

$$y_2 = y_1\ln(x) + \sum_{n=0}^{\infty} b_n\ x^{n+s}$$

Then find y', y'', from these, and sub back into ODE and set $n = 0$ to solve for the new indicial equation, find s from it (should get one solution), then most likely you'll find $b_n = 0$ for all $n > 0$ (for the HW's we did), and so just need to use $b_0 x^{n+s}$ and this gives the complete solution. $y = A\,y_1 + B\,y_2 = A\,y_1 + \left(y_1\ln(x) + \sum_{n=0}^{\infty} b_n\ x^{n+s}\right)$

note: IF when solving the indicial equation, 2 values for s that differ by an integer from each others (say 4, 6), then must use the value 6, also when we solve for the second solution, s there must come out to be the first s which we did not use for the first solution, i.e. 4 in this example (so I really do not need to solve for s again!, expect I need to find the recursive formula).

2.4.2 Chapter 16. Probability

Let A, B are 2 successive events.

$P_A(B)$ is the probability that B will happen KNOWING that A has already happened.

$P(AB)$ is the prob. that A and B will both happen

$$\begin{aligned}P(AB) &= P(A)P_A(B) \\ &= P(B)P_B(A)\end{aligned}$$

Or

$$P_A(B) = \frac{P(AB)}{P(A)}$$

If A and B are independent, then $P_A(B) = P(B)$

Then it follows that

$$P(AB) = P(A)\,(B) \qquad \text{If A,B independent}$$

Probability that A OR B will happen is:

$$\begin{aligned} P(A+B) &= P(A) + \ (B) - P(AB) \\ P(A+B) &= P(A) + \ (B) \qquad \text{IF A,B are mutually exclusive} \end{aligned}$$

This means that $P(AB)\ = 0$ if they are mutually exclusive (obvious)

$$P(A+B+C) = P(A) + P(B) + P(C) - \{P(AB) + P(AC) + P(BC)\} + P(ABC)$$

note: P_r^n = number of permutations (arrangements) or n things taken r at a time. $P_r^n = \frac{n!}{(n-r)!}$ Here order is important. i.e. ABC is DIFFERENT from CAB, hence this number will be larger than the one below.

$$\begin{pmatrix} n \\ r \end{pmatrix} = C_r^n = \frac{n!}{(n-1)!\ r!}$$

Number of combinations OR selections of n things r at a time. here order is NOT important. so ABC is counted the same as CAB, hence this number will be smaller.

note: In how many ways can 10 people be seated on a bench with 4 seats?

A)$\begin{pmatrix} 10 \\ 4 \end{pmatrix} 4! = \frac{10!}{6!4!} 4! = \frac{10!}{6!} = 10 \times 9 \times 8 \times 7$

To understand this: $\begin{pmatrix} 10 \\ 4 \end{pmatrix}$ is the number of ways 4 people can be selected out of 10. ONCE those 4 people have been selected, then there are 4! different ways they can be arranged on the bench. Hence the answer is we multiply these together.

note: Find number of ways of putting r particles in n boxes according to the 3 kinds of statistics.

Answer

1. For Maxwell-Blotzman (MB) it is n^r
2. For Fermi-Dirac (FM), it is ${}_nC_r$
3. For Bose-Einstein (BE) it is ${}_{n+1}C_r$

note: If asked this: there is box A which has 5 red balls and 6 black balls, and box B which has 5 red balls and 8 white balls, what is *the prob. of picking a red ball*? Answer:

P(pick box A) P(pick red ball from it) + P(pick box B) P(pick red ball from it)

note: If we get a problem such as 2 boxes A,B, and more than more try picking balls, it is easier to draw a tree diagram and pull the chances out the tree than having to calculate them directly in the exam. Tree can be drawn in 2 minutes and will have all the info I need.

note: write down the cancer chance problem.

note: random variable x is a function defined on the sample space (for the example, the sum of 2 die throw). The probability density is the probability of each random variable.

average or mean of a random variable $\mu = \sum x_i\ P_i$ where P_i is the probability of the random variable.

The Variance Var measures the spread of the random variable around the average, also called dispersion defined as

$$Var(x) = \sum \left(x_i - \mu\right)^2 P_i$$

Standard deviation is another measure of the dispersion, defined as $\sigma(x) = \sqrt{Var(x)}$

Distribution function is just a histogram of the probability density. it tells one what the probability of a random variable being less than a certain x value. see page 711.

2.4.3 Chapter 13. PDE

PDE	*solution*	*equation*	*notes*
Laplace	$u(x,y,z)$	$\nabla^2 u = 0$	describes steady state (no time) of region with no source for example, gravitional potential with no matter, electrostatic potential with no charge, or *steady state* Temp. distribution
Poisson	$u(x,y,z)$	$\nabla^2 u = f(x,y,z)$	Same as Laplace, i.e. sescribes steady state, howevere here the source of the field is present. $f(x,y,z)$ is called the source density. i.e. it is a function that describes the density distribution of the source of the potential.
Diffusion or heat equation	$u(t,x,y,z)$	$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$	Here u is usually the temperature T function. Now time is involved. So this equation is alive.
Wave equation	$u(t,x,y,z)$	$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$	Here u is the position of a point on the wave at time t. Notice the wave equation has second derivative w.r.t. time while the diffusion is first derivative w.r.t. time
Helmholtz equation	$F(x,y,z)$	$\nabla^2 F + k^2 F = 0$	The diffusion and wave equation generate this. This is the SPACE only solution of the wave and heat equations. i.e. $u = F(x,y,z)T(t)$ is the solution for both heat and wave eq.

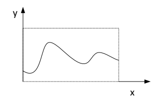
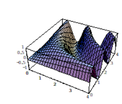
Each of these equations has a set of candidate solutions, which we start with and try to fit the boundary and initial condition into to eliminate some solution of this set that do not fit until we are left with the one candidate solution. We then use this candidate solution to find the general solution, which is a linear combination of it. We use fourier series expansion in this part of the solution.

In table below I show for each equation what the set of candidate solutions are. Use these to start the solution with unless the question asks to start at an earlier stage, which is the separation of variables.

So the algorithm for solving these PDE is

```
Select THE PDE to use ---->
Obtain set of candidate solution ---->
Eliminate those that do not fit ----->
obtain the general solution by linear combination
(use orthogonality principle here)
```

PDE	candidate solutions	notes
$\nabla^2 u = 0$	$u(x,y) = \begin{cases} e^{ky} \cos kx \\ e^{ky} \sin kx \\ e^{-ky} \cos kx \\ e^{-ky} \sin kx \end{cases}$	for 2 dimensions
$\nabla^2 u = f(x,y,z)$	$u(x,y,z) = -\frac{1}{4\pi} \int\int\int \frac{f(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2+}} dx'\, dy'\, dz'$	$f(x',y',z')$ is a function that describes mass density distribution evaluated at point x',y',z'. The point x,y,z is where we are calculating the
		potential u itself
$\nabla^2 u = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}$	$u(t,x) = \begin{cases} e^{-k^2\alpha^2 t} \cos kx \\ e^{-k^2\alpha^2 t} \sin kx \end{cases}$	for one space dimension

$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$	$Y = XT$, where $X(x) = \begin{cases} \cos kx \\ \sin kx \end{cases}$ $T(t) = \begin{cases} \cos \omega t \\ \sin \omega t \end{cases}$ $Y(x,t) = \begin{cases} \cos kx\ \cos\omega t \\ \cos kx\ \sin\omega t \\ \sin kx\ \sin\omega t \\ \sin kx\ \cos\omega t \end{cases}$	v is the wave velosity, 1D 
	$Z = XYT$, where $X(x) = \begin{cases} \cos k_x x \\ \sin k_x x \end{cases}$ $Y(x) = \begin{cases} \cos k_y x \\ \sin k_y x \end{cases}$ $T(t) = \begin{cases} \cos \omega t \\ \sin \omega t \end{cases}$	2D case in rectangular co-ord 
$\nabla^2 F + k^2 F = 0$		

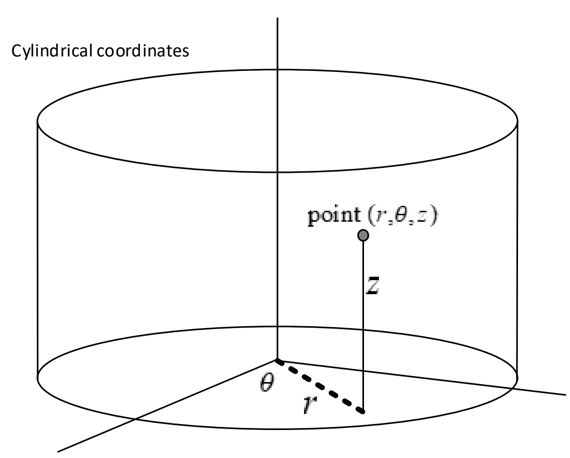
Now the solutions in different coordinates systems

$$X(x) = \begin{cases} \cos kx \\ \sin kx \end{cases}$$

$$T(t) = \begin{cases} \cos \omega t \\ \sin \omega t \end{cases}$$

2.4.3.1 Laplace equation in cylindrical coordinates

The Laplacian in cylindrical is $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$, the solution can be written as $u = R(r)\Theta(\theta)Z(z)$



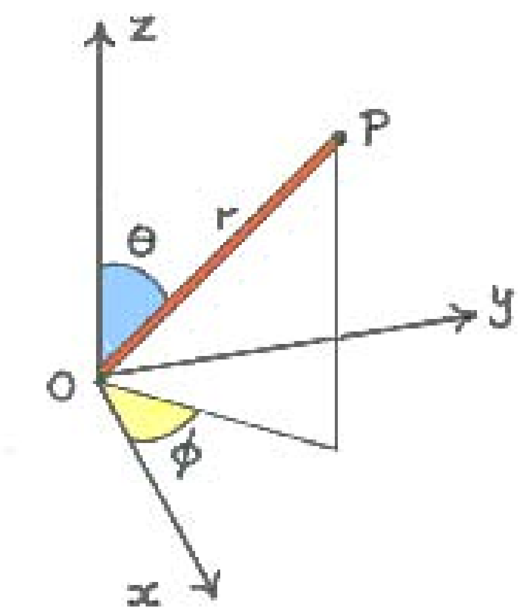
$Z(z) = \begin{cases} e^{kz} \\ e^{-kz} \end{cases}$, we quickly eliminate the e^{kz} since we do not want the potential to blow up as z becomes larger. $\Theta(\theta) = \begin{cases} \sin n\theta \\ \cos n\theta \end{cases}$, $R(r) = J_n(kr)$ where $J_n(kr)$ is Bessel function of order n, we do not use $N_n(kr)$ solutions since we origin is on base of cylinder. see book for more details, all problems we will get will be like this. We find k from boundary conditions, it will turn out to be the zeros of J_n. From above, the set of candidate solutions for Laplace on cylindrical is

$$u(r,\theta,z)=\begin{cases} J_n(kr)\ \sin n\theta\, e^{-kz} \\ \\ J_n(kr)\ \cos n\theta\, e^{-kz} \end{cases}$$

Now usually we eliminate the θ dependency if boundary condition is such that it is not dependent of angle. So we get $u(r,\theta,z)=J_0(k_m r)\ \ e^{-k_m z}$ and from this we need to solve $u=\sum_{m=1}^{\infty} c_m\, J_0(k_m r)\ \ e^{-k_m z}$, now we use boundary condition to find c_m, for example if given that base $(z=0)$ was at temp (or potential) $=100$, then we need to solve $100=\sum_{m=1}^{\infty} c_m\, J_0(k_m r)$ and here to use orthogonality of bessel functions to expand RHS.

2.4.3.2 Laplace equation in spherical coordinates

The Laplacian in spherical is $\nabla^2 u=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\frac{\partial^2 u}{\partial\phi^2}=0$. Separate using $u=R(r)\Theta(\theta)\Phi\big(\phi\big)$



The solutions are $\Phi=\begin{cases}\sin m\phi\\ \cos m\phi\end{cases}$ and $\Theta=P_l^m(\cos\theta)$ and $R(r)=\begin{cases} r^l \\ r^{-l-1}\end{cases}$ here l is an integer (came from separation of constants by setting $k=l(l+1)$), Here P_l^m is the associated Legendre function.

Now we quickly discard solution r^{-l-1} because we want solution inside the sphere, so our set of candidate solutions are $u=R(r)\Theta(\theta)\Phi\big(\phi\big)=r^l\quad P_l^m(\cos\theta)\begin{cases}\sin m\phi\\ \cos m\phi\end{cases}$. For symmetry w.r.t. ϕ we set $m=0$ and solution reduces to $r^l\quad P_l(\cos\theta)$ and then the general solution is $u=\sum c_l\, r^l\quad P_l(\cos\theta)$

2.4.3.3 Wave equation in polar coordinates

2.5 General equations

$$
\begin{aligned}
\sin nx &= \frac{e^{inx} - e^{-inx}}{2i} \\
\cos nx &= \frac{e^{inx} + e^{-inx}}{2}
\end{aligned}
$$

$$
\begin{aligned}
\int \frac{\sin x}{\cos x} dx &= \ln(\sin x) \\
\csc x &= \frac{1}{\sin x} \\
\text{average value of } f(x) \text{ over } [b,a] &= \frac{\int_a^b f(x)dx}{b-a}
\end{aligned}
$$

$$
\begin{aligned}
\cos^2 kx &= \frac{1 + \cos(2k)}{2} \\
\sin^2 kx &= \frac{1 - \cos(2k)}{2}
\end{aligned}
$$

$$
\begin{aligned}
\sin A \ \sin B &= \frac{1}{2}[\ \cos(A-B) - \cos(A+B)] \\
\cos A \ \cos B &= \frac{1}{2}[\ \cos(A-B) + \cos(A+B)] \\
\sin A \ \cos B &= \frac{1}{2}[\ \sin(A-B) + \sin(A+B)]
\end{aligned}
$$

I need a geometric way to visualize these equations, but for now for the exam remember them as follows: they all start with $A - B$, and when the functions being multiplied are different on the LHS, we get sin on the RHS, else we get cos (think of cos as nicer, since even function :).

$$
\begin{aligned}
\int \tanh(x) \ &= \ln(\cosh \ x) \\
\int \tan x &= -\ln(\cos \ x)
\end{aligned}
$$

$\int_a^b \cos^2 kx \ dx \ = \ \frac{b-a}{2}$. if $k(b-a)$ is an integer multiple of π. (the same for $\sin^2 kx$), for example $\int_{-1}^1 \cos^2 \pi x \ dx \ = 1$, $\int_{-1}^1 \cos^2 5\pi x \ dx \ = 1$, $\int_{-5}^1 \cos^2 7\pi x \ dx \ = 3$, $\int_{-1}^1 \sin^2 \pi x \ dx \ = 1$, etc... this can be very useful so remember it!

$\int_a^b \cos kx \ dx = 0$ if over a complete period. same for $\sin x$, for example $\int_{-\pi}^{\pi} \cos kx \ dx = 0$

$$
\begin{aligned}
\sinh x &= -i \ \sin(ix) \\
\cosh x &= \cos(ix) \\
\tanh x &= -i \ \tan(ix)
\end{aligned}
$$

$$
\begin{aligned}
e^{\ln z} &= z \\
z^b &= e^{b \ln z}
\end{aligned}
$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

$$\begin{aligned}
\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad -1 < x \leq 1 \\
(1+x)^p &= 1 + px + \frac{p\left(p-1\right)}{2!}x^2 + \cdots + \qquad |x| < 1
\end{aligned}$$

Leibinz rule for differentiation of integrals

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\,dt = f(x,v(x))\,\frac{d}{dx}v(x) - f(x,u(x))\,\frac{d}{dx}u(x) + \int_u^v \frac{\partial}{\partial x} f(x,t)\,dt$$

example:

$$\begin{aligned}
\frac{d}{dx}\int_x^{2x} \frac{e^{xt}}{t}\,dt &= \frac{e^{x(2x)}}{2x}\,\frac{d}{dx}(2x) - \frac{e^{x(x)}}{x}\,\frac{d}{dx}(x) + \int_x^{2x} \frac{\partial}{\partial x}\left(\frac{e^{xt}}{t}\right)\,dt \\
&= \frac{e^{2x^2}}{x} - \frac{e^{x^2}}{x} + \int_x^{2x} \frac{te^{xt}}{t}\,dt \\
&= \frac{e^{2x^2}}{x} - \frac{e^{x^2}}{x} + \left[\frac{e^{xt}}{x}\right]_x^{2x}
\end{aligned}$$

To help remember the above 2 formulas, notice that when $+x$ we get a $-$ shown (i.e. terms flip flop), but when we have $-x$ the series is all positive terms. These are very important to remember for problems when finding Laurent expansion of a function.

Expansion of cos **and** sin **around a point different than** 0

expand $\cos(z)$ around a, we get

$$\left(\cos(a) - \frac{\cos(a)(z-a)^2}{2!} + \frac{\cos(a)(z-a)^4}{4!} - \cdots\right) + \left(-\sin(a)(z-a) + \frac{\sin(a)(z-a)^3}{3!}\ \cdots\right)$$

For example to expand $\cos(x)$ about π we get

$$\left(\cos(\pi) - \frac{\cos(\pi)(z-\pi)^2}{2!} + \frac{\cos(\pi)(z-\pi)^4}{4!} - \cdots\right) + \overbrace{\left(-\sin(\pi)(z-\pi) + \frac{\sin(a)(z-\pi)^3}{3!} \quad \cdots\right)}^{=0}$$
$$= -1 + \frac{1}{2}(\pi - z)^2 - \frac{1}{24}(\pi - z)^4 + \cdots$$

so above is easy to remember. The $\cos(z)$ part is the same as around zero, but it has $\cos(a)$ multiplied to it, and the sin part is the same as the $\sin(z)$ about zero but has $\sin(a)$ multiplied to it, and the signs are reversed.

For expansion of $\sin(z)$ use

$$\left(\sin(a) - \frac{\sin(a)(z-a)^2}{2!} + \frac{\sin(a)(z-a)^4}{4!} - \cdots\right) + \left(\cos(a)(z-a) - \frac{\cos(a)(z-a)^3}{3!} \quad \cdots\right)$$

This is the same as the expansion of $\cos(z)$ but the roles are reversed and notice the cos part start now with positive not negative term. SO all what I need to remember is that expansion of $\cos(z)$ starts with $\cos(a)$ terms while expansion of $\sin(z)$ start with the $\sin(a)$ term. This is faster than having to do Taylor series expansion to find these series in the exam.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$
$$\Gamma(P+1) = P\Gamma(P)$$

Chapter 3

HWs

3.1 HW table

This is list of all the HWs. All problems are from Mary Boas, second edition.

HW	notes
HW 1 chapter one 1.5, 2.6, 2.7, 4.2, 4.4, 5.8, 6.1, 6.2, 6.3, 6.5, 6.7, 6.8, 6.11, 6.12, 6.15, 6.16, 6.17, 6.18, 6.21, 6.27, 6.33, 6.34, 6.35, 7.2, 7.3, 7.5, 7.6, 9.1, 9.3, 9.7, 9.12, 9.21, 10.1, 10.3, 10.15, 10.19, 10.20, 13.2, 13.6, 13.8, 13.13, 13.14, 13.16, 13.17, 13.33, 13.37, 13.40	Infinite series, power series, convergence, interval of, etc.
HW 2 chapter one 14.9, 15.5, 15.11, 15.16, 15.18, 16.3, 16.22 chapter 2 4.7, 4.14, 5.6, 5.13, 5.20, 5.32, 5.39, 5.45, 5.51, 5.53, 5.55, 5.56, 5.60, 5.62, 6.4, 6.10, 6.12, 7.3, 7.12, 8.1, 8.2	Rest of chapter one, on infinite series and power series. And chapter 2, complex numbers, complex series, circle of convergence.
HW 3 chapter 2 9.2, 9.12, 9.19, 9.24, 9.27, 9.28, 10.18, 10.22, 10.28, 11.5, 11.6, 11.11, 11.18, 14.6, 14.9, 14.14, 14.22, 14.24, 15.13, 16.7, 16.11, 17.14, 17.17, 17.23, 17.23, 17.24, 17.30, 17.32	Rest of chapter 2, complex numbers, Euler formulas, log and powers of complex numbers, complex roots, inverse trig and some applications.
HW 4 chapter 4 1.6, 1.7, 1.23, 1.24, 2.3, 2.6, 2.8, 4.2, 4.8, 4.9, 4.10, 4.13	Partial differentiation, power series in 2 variables.
HW 5 chapter 4 5.3, 5.4, 6.3, 6.5, 6.6, 7.3, 7.5, 7.8, 7.15, 7.19, 8.1, 8.2, 8.3	More Partial differentiation, Chain rule, min and max problems.

HW 6 chapter 4 9.2, 9.3, 9.4, 9.5, 9.6, 9.7, 9.8, 10.2, 10.7, 10.10, 11.1, 11.3, 11.6, 11.7, 11.8, 11.9, 12.1, 12.2, 12.4, 12.7, 12.8	More Partial differentiation, min and max problems.
HW 7 chapter 14 1.6, 1.12, 2.22, 2.23, 2.34, 2.37, 2.40, 2.55, 2.56, 2.60, 3.2, 3.5, 3.17, 3.18, 3.19, 3.20, 3.23, 4.6, 4.7, 4.9, 4.10, 5.1	Function of complex variables. Analytic functions, cotour integration, Laurent series, Residue
HW 8 chapter 14 6.5, 6.6, 6.7, 6.8, 6.9, 6.15, 6.19, 6.23, 6.31, 7.5, 7.12, 7.18, 7.20, 7.29, 7.30, 7.34, 7.45, 8.4, 8.5, 8.15	Function of complex variables. Methods of finding residues, integration using residues, residues at infinity.
HW 9 chapter 14 9.2, 9.3, 9.4, 9.7, 10.5, 10.6, 10.11, 10.13 chapter 7 3.4, 3.6	Function of complex variables. Mapping, applications of conformal mapping. Start of chapter 3. Linear equations, vectors and matrices.
HW 10 chapter 7 4.2, 4.5, 4.8, 4.10, 4.12, 4.13, 4.15, 5.2, 5.4, 5.5, 5.8, 5.10, 6.5, 7.4, 7.7, 7.12, 8.12	Fourier series.
HW 11 chapter 15 2.2, 2.3, 2.4, 2.5, 2.9, 2.11, 2.17, 2.18, 2.21, 2.22, 2.23, 3.4, 3.6, 3.8, 3.11, 3.24, 3.25, 3.29, 3.30	Integral transforms. Laplace transform, Solution of ODE using Laplace.
HW 12 chapter 15 4.3, 4.5, 4.7, 4.12, 4.18, 4.20, 4.21, 4.23, 4.25, 5.1, 5.2, 5.10, 5.22, 6.2, 6.4, 6.5, 6.9, 7.7, 7.9, 7.11	Integral transforms. Fourier transform, Convolution, Inverse Laplace, Dirac delta.
HW 13 chapter 15 (Integral transforms) 8.2, 8.3, 8.12, 8.15, 8.17 chapter 9 (Calculus of Variations) 2.1, 2.3, 2.6, 3.1, 3.2, 3.4, 3.6, 3.9, 5.2, 5.6, 6.1, 6.2, 6.5	Integral transforms: Green functions. Calculus of Variations: Euler equation, Using Euler equation, Isoperimetric problems

3.2 HW 1

Math 121 A

HW # 1

3/3 Very Good Work!

Student : NASSER ABBASI

Spring 2004

UC Berkeley.

Feb 2nd.

(1)

Chapter 1

use $S=\frac{a}{1-r}$ to find fraction equivalent to

1.5 $s=0.58333\cdots$

the above can be written as

$s=0.58+\underbrace{\frac{3}{1000}+\frac{3}{10000}+\cdots}$

$\downarrow$

$a+ar+\cdots$ for $a=\frac{3}{1000}$

$r=\frac{1}{10}$

Since $r<1$ then convergent and the sum is

$\frac{a}{1-r}=\frac{\frac{3}{1000}}{1-\frac{1}{10}}=\frac{\frac{3}{1000}}{\frac{9}{10}}=\frac{3}{900}$ ✓

hence $0.58333\cdots=0.58+\frac{3}{900}$

$=\frac{58}{100}+\frac{3}{900}=\frac{522+3}{900}=\boxed{\frac{525}{900}}=\frac{7}{12}$ ✓

2.6 write series in form $a_1+a_2+\cdots$

$\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$

$=\frac{(1!)^2}{(2)!}+\frac{(2!)^2}{4!}+\frac{(3!)^2}{6!}+\frac{(4!)^2}{8!}+\cdots$

$=\frac{2}{2}+\frac{2^2}{4\times3\times2}+\frac{(3\times2)^2}{(6\times5\times4\times3\times2)}+\frac{(4\times3\times2)^2}{8\times7\times6\times5\times4\times3\times2}+\cdots$

$=\boxed{1+\frac{1}{6}+\frac{1}{20}+\frac{1}{70}+\cdots}$

(2)

chap 1

2.7 write in abberviated $\sum$ Form

$$\frac{1}{3}+\frac{2}{5}+\frac{4}{7}+\frac{8}{9}+\frac{16}{11}+\cdots$$

denominator behaves as $2n+3$

numerator as 2^n

hence series $= \boxed{\sum_{n=0}^{\infty}\frac{2^n}{2n+3}}$

for example, $n=0 \Rightarrow \frac{2^0}{2(0)+3}=\frac{1}{3}$ ok

$n=1 \Rightarrow \frac{2^1}{2(1)+3}=\frac{2}{5}$ ok

$n=2 \Rightarrow \frac{2^2}{2(2)+3}=\frac{4}{7}$ ok

$n=3 \Rightarrow \frac{2^3}{2(3)+3}=\frac{8}{9}$ ok

etc...

③

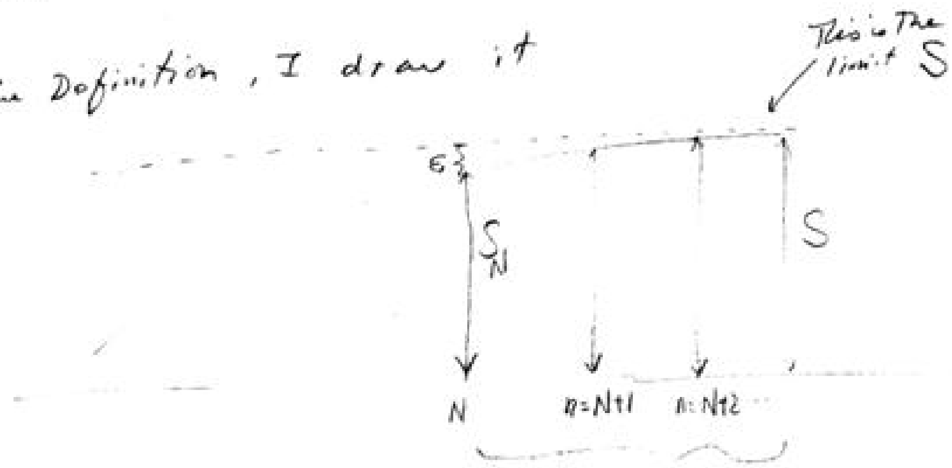
Chapter 1

4.2) A careful math. definition of a Convergent Series with Sum S is: Given any small positive number called ε, it is possible to find an integer N so that $|S-S_n|<\epsilon$ for every $n\geq N$.

Select some ε and corresponding N for

$$\sum_{n=1}^{\infty}\frac{1}{5^n}$$

To visulize the Definition, I draw it



For all $n\geq N$, we have $|S-S_n|<\varepsilon$

series is Convergent by ratio test

$$\frac{|a_{n+1}|}{|a_n|}=\frac{\frac{1}{5^{n+1}}}{\frac{1}{5^n}}=\frac{5^n}{5^{n+1}}=5^{n-(n+1)}=5^{-1}=\frac{1}{5}<1$$

let $\varepsilon=10^{-7}$

need to find the limiting sum S

$$\sum_{n=1}^{\infty}\frac{1}{5^n}=\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\cdots$$

, this is a geometric series with $a=\frac{1}{5}$ and $r=\frac{1}{5}$

hence. $S=\frac{a}{1-r}=\frac{\frac{1}{5}}{1-\frac{1}{5}}=\frac{\frac{1}{5}}{\frac{4}{5}}=\frac{1}{4}$ $\Longrightarrow$

(4)

and $S_n = \frac{a(1-r^n)}{1-r} = \frac{\frac{1}{5}\left(1-\frac{1}{5^n}\right)}{1-\frac{1}{5}} = \frac{\frac{1}{5}\left(1-\frac{1}{5^n}\right)}{\frac{4}{5}}$

$= \frac{1}{4}\left(1-\frac{1}{5^n}\right)$

Now I can find N.

$|S-S_n| = \frac{1}{4} - \frac{1}{4}\left(1-\frac{1}{5^n}\right) = \frac{1}{4}\left(1-\left(1-\frac{1}{5^n}\right)\right)$

$= \frac{1}{4}\left(\frac{1}{5^n}\right)$

so need $\frac{1}{4}\frac{1}{5^n} < 10^{-7}$, solve for n.

$\frac{1}{4}\frac{1}{5^n} < \frac{1}{10^7}$

i.e. $(4)\,5^n > 10^7$

i.e. $\log(4\,5^n) > \log 10^7$

i.e. $\log 4 + \log 5^n > 7$

i.e. $0.6 + n(\log 5) > 7$

i.e. $0.6 + n(0.698) > 7$

i.e. $0.698\,n > 6.4$

i.e. $n > 9.16$ i.e. $n = 10$

so $\boxed{N = 10}$ To test this $\Rightarrow$

(5)

for $n=10$, $S_n = \frac{a(1-r^n)}{1-r} = \frac{1}{4}\left(1-\frac{1}{5^n}\right)$

$= \frac{1}{4}\left(1-\frac{1}{5^{10}}\right) = 0.2499999744$

so $|S - S_n| = 2.56 \times 10^{-8}$

which is smaller than $\varepsilon = 10^{-7}$

ok

(6)

Chapter 1

4.2

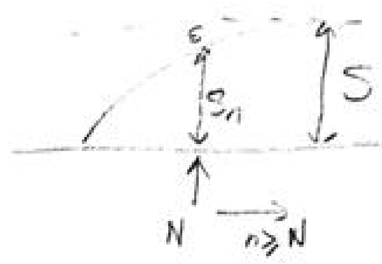
$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

since by comparison test, this series is convergent ($\frac{1}{n!} < \frac{1}{2^n}$ for $n>3$).

so $S-S_n$ for this series is smaller than $S-S_n$ for the geometric series which is

$$\frac{a}{1-r} - \frac{a(1-r^n)}{1-r} = \frac{a-a(1-r^n)}{1-r} = \frac{a-a+ar^n}{1-r}$$

$$= \frac{ar^n}{1-r}$$



hence need $\left|\frac{ar^n}{1-r}\right| \leq \varepsilon$

let $\varepsilon = 10^{-7}$. for geometric series, $a=\frac{1}{2}$, $r=\frac{1}{2}$.

so need $\frac{\frac{1}{2}\frac{1}{2}^n}{\frac{1}{2}} \leq \frac{1}{10^7}$

i.e. $\left(\frac{1}{2}\right)^n \leq 10^{-7}$ or $\frac{1}{2^n} \leq \frac{1}{10^7}$

or $2^n > 10^7$ or $n \log 2 > 7$

or $n > \frac{7}{0.3}$ or $n > 23.2$

i.e. $\boxed{N = 24}$

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chapter 11 use preliminary test to decide if divergent or more testing required.

5.1

$$\frac{1}{2} - \frac{4}{5} + \frac{9}{10} - \frac{16}{17} + \frac{25}{26} - \frac{36}{37} + \cdots$$

in preliminary test, find a_n. if $a_n \neq 0$ for $n \to \infty$ then divergent, else more testing needed to see if convergent.

The above series is $\displaystyle \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^2+1}$

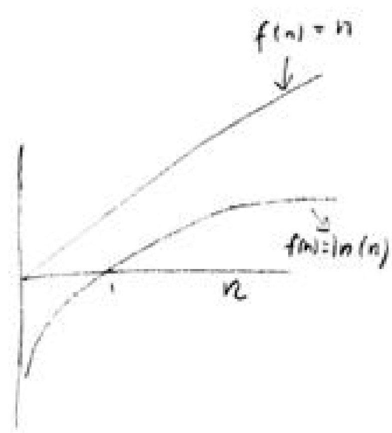
so $a_n = \dfrac{n^2}{n^2+1}$

$\lim_{n \to \infty} a_n = 1$ or $a_n = -1$ depending on sign

i.e. $a_n \neq 0$ as $n \to \infty$.

hence **divergent**

5.8 $\displaystyle \sum_{n=1}^{\infty} \frac{\ln n}{n}$



for large n
ln(n) grows much more slowly than n. see graph.

hence $\lim_{n \to \infty} a_n = \dfrac{\ln(n)}{n} \longrightarrow 0$

hence **more testing is needed** to see if convergent

(8)

chapter 1

6.1 Show that $n! > 2^n$

$S_1 = n! = 1 + 2 + 2\cdot 3 + 2\cdot 3\cdot 4 + 2\cdot 3\cdot 4\cdot 5 + \cdots$
$S_2 = 2^n = 1 + 2 + 2\cdot 2 + 2\cdot 2\cdot 2 + 2\cdot 2\cdot 2\cdot 2 + \cdots$

looking at above 2 sequences, each term in S_1 is being multiplied by a factor larger than the factor that the corresponding term in S_2 is being multiplied with. (This is starting at $n=3$)

This implies the sum of terms of S_1 is larger than sum of terms of S_2. QED

6.2 Prove that harmonic series $\sum_1^\infty \frac{1}{n}$ is divergent by comparing it to series $1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + \cdots$

write out $\sum \frac{1}{n}$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20} + \cdots$

4 terms, 8 terms

$= 1 + \frac{1}{2} + \frac{7}{12} + \frac{533}{840} + \frac{95549}{144144} + \cdots$

$= 1 + \frac{1}{2} + 0.58 + 0.634 + 0.662 + \cdots$

so we see that $\sum \frac{1}{n}$ can be rewritten as a series whose each term (after $n=2$) is larger than $\frac{1}{2}$ by collecting 4 terms, then 8 terms, then 16 terms, etc. and since there is ∞ number of terms, then we can keep doing this as we please so by comparison test to series $1 + \frac{1}{2} + \frac{1}{2} + \cdots$ which is divergent, we conclude that $\sum \frac{1}{n}$ is divergent. QED.

(1)

chapter 1

6.3 prove convergence of $\sum \frac{1}{n^2}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \cdots$$

$$= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256}$$

combine, starting from $n=2$, in sequence $\sum \frac{1}{n^2}$, 2 terms, then 4 terms, then 8 terms, etc... to get

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) - 1 = \frac{13}{36} + \frac{26581}{176400} + \cdots + \cdots$$

less than $\frac{1}{2}$ + less than $\frac{1}{4}$ + less than $\frac{1}{8}$ + ...

hence, this is a series whose each term a_n, $n = 1 \ldots \infty$, is smaller than corresponding term in geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$, which we know is convergent (since $r = \frac{1}{2} < 1$)

hence $\boxed{\sum \frac{1}{n^2} \text{ is convergent by comparison test}}$

note the term $a_1 = 1$ in $\frac{1}{n^2}$ was ignored. this of course does not affect the convergence test.

(10)

chapter 1

6.5 (a) Test for convergence using comparison test

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{1} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} + \cdots$$

but $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$

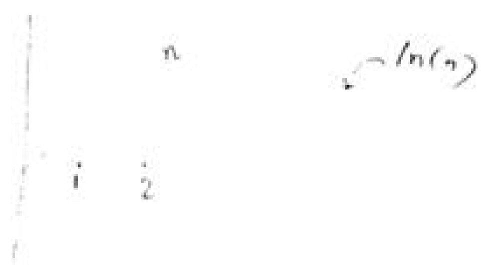
since $\sqrt{n} < n$, then $\frac{1}{\sqrt{n}} > \frac{1}{n}$

and since $\sum \frac{1}{n}$ diverges (see problem solution 6.2), then this implies that $\boxed{\sum^{\infty} \frac{1}{\sqrt{n}} \text{ diverges}}$, since each term in $\sum \frac{1}{\sqrt{n}}$ is larger than each corresponding term in $\sum \frac{1}{n}$.

6.5 (b) $\sum_{n=2}^{\infty} \frac{1}{\ln n} = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \frac{1}{\ln 5} + \frac{1}{\ln 6} + \cdots$

again, compare to $\sum_{n=2}^{\infty} \frac{1}{n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$

each term in $\sum \frac{1}{\ln n}$ is larger than each corresponding term in $\sum \frac{1}{n}$ which is divergent. hence $\sum_{2}^{\infty} \frac{1}{\ln n}$ **diverges**. This is because $\ln(n)$ is smaller than n for all positive n's.



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chapter 1

6.7 use integral test to find if series diverge or converges

$$S = \sum_{n=2}^{\infty} \frac{1}{n \ln(n)} = \frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \frac{1}{4\ln 4} + \cdots$$

since all terms are positive and $a_{n+1} < a_n$, then can use integral test

$$I = \int^{\infty} \frac{1}{n\ln(n)}\,dn = \ln(\ln(n))\Big|^{\infty} = \ln(\ln(\infty)) = \ln(\infty) = \infty$$

since Integral diverges, then $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges

6.8 $S = \sum_{n=1}^{\infty} \frac{n}{n^2+4}$

$$I = \int^{\infty} \frac{x}{x^2+4}\,dx = \frac{1}{2}\ln(x^2+4)\Big|^{\infty} = \frac{1}{2}\ln(\infty) = \infty$$

hence series diverges

6.11 $\sum_{1}^{\infty} \frac{1}{n(1+\ln n)^{3/2}}$

$$I = \int^{\infty} \frac{1}{x(1+\ln(x))^{3/2}}\,dx = \frac{-2}{\sqrt{1+\ln(x)}}\Bigg|^{\infty}$$

as $x \to \infty$, $\frac{1}{\sqrt{1+\ln(x)}} \to 0$ hence converges

hence series converges

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chapter 1

6.12 $S = \sum_{1}^{\infty} \frac{n}{(n^2+1)^2}$

$$I = \int^{\infty} \frac{x}{(x^2+1)^2}\,dx = -\frac{1}{2(1+x^2)}\Big|^{\infty}$$

when $x=\infty$, $I \to 0$. hence **Series Converges**

6.15 use integral test to prove the following so-called p-series test.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is } \begin{cases} \text{convergent if } p>1 \\ \text{divergent if } p\le 1 \end{cases}$$

$$I = \int^{\infty} \frac{1}{x^p}\,dx = \int^{\infty} x^{-p}\,dx = \frac{x^{-p+1}}{-p+1}\Big|^{\infty}$$

$$= \left(\frac{1}{1-p}\right)\left(\frac{1}{x^{1+p}}\right)\Big|^{\infty}$$

when $p>1$, then $\frac{1}{x^{1+p}} \to 0$ as $x \to \infty$ since x^{1+p} grows larger and larger. hence $I \to 0$. hence **Converges**

when $p<1$, then $1+p$ is negative, and so $\frac{1}{x^{1+p}} \to \infty$ as x increases since denomenator $\to 0$ now.

hence **divergent series**

when $p=1$ $\frac{1}{1-p} = \frac{1}{0} = \infty$. so I diverges.

hence **sequence diverges** also

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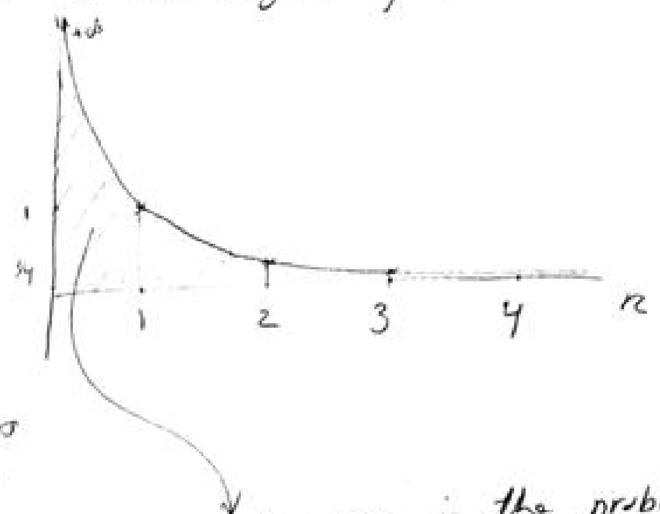
chapter 1

6.16

$\sum \frac{1}{n^2}$ using integral Test on $\int_0^\infty \frac{1}{n^2} dn$ results in ∞. however this is wrong. the reason is that there is a pole at $n=0$, and so lower limit must start at a point to the right of 0.



$n=0 \quad f(n) = \frac{1}{n^2} = \infty$

$n=1 \quad f(n) = \frac{1}{n^2} = 1$

$n=2 \quad f(n) = \frac{1}{4}$

$n=3 \quad f(n) = \frac{1}{9}$

this area is the problem. needs not be considered in the integral. should start for $n=1$

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Chapter 1

6.17

use integral test for divergence or convergence;

$$\sum_{n=0}^{\infty} e^{-n^2}$$

$$I = \int^{\infty} e^{-n^2}\, dn$$

can't evaluate integral. but area under $f(x) = e^{-x^2}$ is smaller than area under $f(x) = e^{-x}$, since e^{-x^2} approaches zero faster. So if I can show that $\int^{\infty} e^{-x}\, dx$ is finite, then this means $\int^{\infty} e^{-x^2}\, dx$ is finite as well.

$$I = \int^{\infty} e^{-n}\, dn = -e^{-n} \Big|^{\infty}$$

as $n \to \infty$ $e^{-n} \to 0$. hence integral converges.

hence $\int^{\infty} e^{-x^2}\, dx$ converges as well. hence

$$\sum_{n=0}^{\infty} e^{-n^2}$$ **converges**

chapter 1 (15)

6.18 use ratio test to find if series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}.$$

$$a_n = \frac{2^n}{n^2}, \qquad a_{n+1} = \frac{2^{n+1}}{(n+1)^2}$$

$$\text{so } \rho_n = \frac{|a_{n+1}|}{|a_n|} = \frac{\frac{2^{n+1}}{(n+1)^2}}{\frac{2^n}{n^2}} = \frac{2^{n+1}\, n^2}{2^n\,(n+1)^2} = 2\,\frac{n^2}{(n+1)^2} \to 2$$

since denominator has an extra 'n' factor, this converges ×

$$\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2/n^2}{(n+1)^2/n^2} = 1$$

$$\lim_{n\to\infty} \rho_n = \frac{2\,n^2}{n^2+2n+1} \Rightarrow 0 \quad \times$$

larger than n^2

1/2

hence sequence **converges** ×

6.21 $\sum_{n=0}^{\infty} \frac{5^n (n!)^2}{(2n)!}$ $\qquad a_n = \frac{5^n (n!)^2}{(2n)!} \qquad a_{n+1} = \frac{5^{n+1}((n+1)!)^2}{(2(n+1))!}$

$$\text{so } \rho_n = \left|\frac{a_{n+1}}{a_n}\right| = \frac{5^{n+1}((n+1)!)^2\,(2n)!}{5^n (n!)^2\,(2(n+1))!} = \frac{5\,((n+1)!)^2\,(2n)!}{(n!)^2\,(2(n+1))!}$$

$$\text{now } (n+1)! = 1\times 2\times \cdots \times n \times (n+1) = (n+1)\, n!$$

$$\text{so } \rho_n = \frac{5\,((n+1)\,n!)^2\,(2n)!}{(n!)^2\,(2(n+1))!} = \frac{5\,(n+1)^2\,(2n)!}{(2(n+1))!}$$

$$\text{now } (2(n+1))! = (2n+2)! = 1\times 2\times \cdots \times n \times (n+1) \times \cdots \times 2n \times (2n+1)\times(2n+2)$$
$$= (2n!)\,(2n+1)\,(2n+2)$$

$$\text{so } \rho_n = \frac{5(n+1)^2\,(2n)!}{(2n)!\,(2n+1)(2n+2)} = \frac{5\,(n^2+2n+1)}{4n^2+6n+2} \Longrightarrow$$

(16)

divid by n^2 :

$$\rho_n = \frac{5\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}{4+\frac{6}{n}+\frac{2}{n^2}}$$

$$\lim_{n\to\infty} \rho_n = \boxed{\frac{5}{4}}$$

>1 hence $\boxed{\text{diverges}}$

$\boxed{6.27}$ use ratio test

$$\sum_{n=0}^{\infty} \frac{100^n}{n^{200}}$$

$$a_n = \frac{100^n}{n^{200}} \quad ; \quad a_{n+1} = \frac{100^{n+1}}{(n+1)^{200}}$$

$$\rho_n = \left|\frac{a_{n+1}}{a_n}\right| = \frac{100^{n+1}\, n^{200}}{100^n\,(n+1)^{200}} = 100\,\frac{n^{200}}{(n+1)^{200}}$$

now $\lim_{n\to\infty} \frac{n^{200}}{(n+1)^{200}} = 1$

so $\lim_{n\to\infty} \rho_n = (100)(1) = 100 > 1$

hence $\boxed{\text{diverges}}$

chapter 1 (17)

6.33 use special comparison test for convergence or divergence

$$\sum_{n=5}^{\infty} \frac{1}{2^n - n^2}$$

First need to find the comparison series. looking at

$2^n - n^2$ as $n \to \infty$ and looking at the log, we have

$\log 2^n = n \log 2$

$\log n^2 = 2 \log n$

since $\log n$ grows more slowly than n, then

2^n is the dominant term in denominator. So

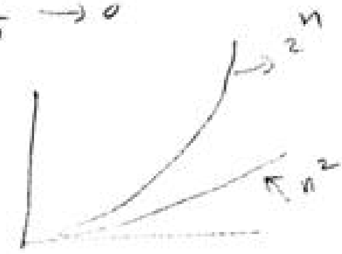
compare with $\sum_{n=5}^{\infty} \frac{1}{2^n}$.

this is a convergent sequence, since geometric with $r = \frac{1}{2} < 1$

hence use test (a)

$$\frac{\frac{1}{2^n - n^2}}{\frac{1}{2^n}} = \frac{2^n}{2^n - n^2} = \frac{1}{1 - \frac{n^2}{2^n}}$$

$\lim_{n \to \infty} \frac{1}{1 - \frac{n^2}{2^n}} = 1$ since $\lim_{n \to \infty} \frac{n^2}{2^n} \to 0$



since n^2 grow slower than 2^n

since this is a finite limit, then

$$\sum_{n=5}^{\infty} \frac{1}{2^n - n^2}$$ **Converges**

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chapter 1

6.34 use special comparison test to find if convergent or divergent.

$$s = \sum_{n=1}^{\infty} \frac{n^2+3n+4}{n^4+7n^3+6n-3}$$

need to find a comparison sequence. looking at s. as $n \to \infty$, numerator $\to n^2$. for denominator, as $n \to \infty$ it goes as n^4, so use $\frac{n^2}{n^4}$ or $\frac{1}{n^2}$ as comparison sequence. to find if $\sum \frac{1}{n^2}$ converges, use integral test

$$\int^{\infty} \frac{1}{n^2}\, dn = -\frac{1}{2n}\Big|^{\infty} \longrightarrow 0 \quad \text{as } n \to \infty.$$

so $\sum \frac{1}{n^2}$ converges. so use test (a).

$$\frac{\frac{n^2+3n+4}{n^4+7n^3+6n-3}}{\frac{1}{n^2}} = \frac{(n^2)(n^2+3n+4)}{n^4+7n^3+6n-3}$$

$$= \frac{n^4+3n^3+4n^2}{n^4+7n^3+6n-3} \text{. divide by } n^4 \longrightarrow \frac{1+\frac{3}{n}+\frac{4}{n^2}}{1+\frac{7}{n}+\frac{6}{n^3}-\frac{3}{n^4}}$$

as $n \to \infty$ above goes to 1

this is a finite limit. hence **convergent**

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chapter 1

6.35 use special comparison test on

$$\sum_{n=3}^{\infty} \frac{(n-\ln n)^2}{5n^4-3n^2+1}$$

need to first find the comparison series.
in denominator, it goes as $5n^4$ for large n.
in numerator, for large n, $n > \ln n$. hence it goes as n^2.

so need to use $\frac{n^2}{5n^4}$ or $\frac{1}{n^2}$.

$\sum \frac{1}{n^2}$ is convergent by integral test. (see previous problem 6.34)

so use test (a)

$$\frac{\frac{(n-\ln n)^2}{5n^4-3n^2+1}}{\frac{1}{n^2}} = \frac{n^2(n-\ln n)^2}{5n^4-3n^2+1}$$

$$= \frac{n^2(n^2-2n\ln n - \ln^2 n)}{5n^4-3n^2+1} = \frac{n^4-2n^3\ln n - n^2\ln^2 n}{5n^4-3n^2+1}$$

$$= \frac{1-\frac{2\ln n}{n}-\frac{\ln^2 n}{n^2}}{5-\frac{3}{n}+\frac{1}{n^4}} = \frac{1-\frac{2\ln n}{n}-\left(\frac{\ln n}{n}\right)^2}{5-\frac{3}{n}+\frac{1}{n^4}}$$

as $n \to \infty$ $\frac{\ln n}{n} \to 0$ since $n > \ln n$. so limit $\to$ **$\frac{1}{5}$**

This is finite, hence **converges**

chapter 1 (20)

7.2 test the following alternate series for convergence

- $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$. First see if abs. convergent since if so, no need to do more testing since an alternate series that is abs. convergent is convergent

so look at $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ (comparison test)

$\log 2^n = n \log 2$
$\log n^2 = 2 \log n$ } since n grows faster than $\log n$, then $\frac{\log 2^n}{\log n^2} \longrightarrow \infty$ as $n \to \infty$

so $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ diverges. so series is not abs. convergent.

- it can still be convergent however.

An alternate series converges if $\frac{|a_{n+1}|}{|a_n|} \leq 1$ and $\lim_{n\to\infty} a_n = 0$.

$|a_{n+1}| = \left| \frac{2^{n+1}}{(n+1)^2} \right|$; $|a_n| = \left| \frac{2^n}{n^2} \right|$

$\frac{|a_{n+1}|}{|a_n|} = \left| \frac{(2^{n+1})(n^2)}{(2^n)(n+1)^2} \right| = \left| \frac{(2)(n^2)}{n^2+2n+1} \right| = \left| \frac{\frac{2}{n^2} 2}{1+\frac{2}{n}+\frac{1}{n^2}} \right|$

as $n \to \infty$ above $\to$ 2. hence $\frac{|a_{n+1}|}{|a_n|} > 1$. hence

- **diverges**

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chapter 1

7.3 use alternate series test

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

First do the absolute convergance test.

look at $\sum_{n=1}^{\infty} \frac{1}{n^2}$ this is a convergant series by integral test (see 6.34).

hence since abs. convergent, $\sum_{1}^{\infty} \frac{(-1)^n}{n^2}$ is **convergent**

7.5 $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

first look at $\sum_{n=2}^{\infty} \frac{1}{\ln n}$. use comparison test with $\sum \frac{1}{n}$. since $\sum \frac{1}{n}$ diverges, $\frac{\frac{1}{\ln n}}{\frac{1}{n}} = \frac{n}{\ln n} \to \infty$ for Large n, hence this is not absolutely convergent. Then need to do more testing.

look at $\frac{|a_{n+1}|}{|a_n|} = \frac{\left|\frac{1}{\ln(n+1)}\right|}{\left|\frac{1}{\ln(n)}\right|} = \left|\frac{\ln(n)}{\ln(n+1)}\right|$ hence

$|a_{n+1}| \leq |a_n|$. so now look at $\lim_{n\to\infty} a_n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln(n)} = 0$. hence **Converges Conditionally**

Chapter 1 (22)

7.6 test alternate series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+5}$$

this is not absolutely convergent, since $\lim_{n\to\infty} \frac{n}{n+5} \to 1$

so use $\frac{|a_{n+1}|}{|a_n|}$ test.

$$\frac{\left|\frac{n+1}{(n+1)+5}\right|}{\left|\frac{n}{(n+5)}\right|} = \frac{(n+1)(n+5)}{(n+6)(n)} = \frac{n^2+6n+5}{n^2+6n}$$

so $\frac{|a_{n+1}|}{|a_n|} \geq 1$ i.e. $|a_{n+1}| \geq |a_n|$

hence **diverges**

9.1 test for convergence or divergence

$$\sum_{n=1}^{\infty} \frac{n-1}{(n+2)(n+3)}$$

preliminary test: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n-1}{n^2+5n+6} = \lim_{n\to\infty} \frac{\frac{1}{n^2}-\frac{1}{n}}{1+\frac{5}{n}+\frac{6}{n^2}} = \frac{0}{1} = 0$.

so must test more.

try ratio test

$$\rho_n = \frac{|a_{n+1}|}{|a_n|} = \frac{\frac{n}{(n+3)(n+4)}}{\frac{n-1}{(n+2)(n+3)}} = \frac{n(n+2)(n+3)}{(n-1)(n+3)(n+4)} = \frac{n^3+6n^2+8n}{n^3+6n^2+5n-12} = \frac{1+\frac{6}{n}+\frac{8}{n^2}}{1+\frac{6}{n}+\frac{5}{n^2}-\frac{12}{n^3}}$$

$\lim_{n\to\infty} \rho_n = \frac{1}{1} = 1$. hence use different test $\longrightarrow$

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try comparison test

compare with $\sum \frac{1}{n}$ which diverges.

$$\frac{n-1}{(n+2)(n+3)} \overset{?}{\geq} \frac{1}{n}$$

ie $$\frac{\frac{n-1}{(n+2)(n+3)}}{\frac{1}{n}} \overset{?}{\geq} 1 \quad ; \quad \frac{(n-1)(n)}{n^2+5n+6} \overset{?}{\geq} 1$$

$$\frac{n^2-n}{n^2+5n+6} \overset{?}{\geq} 1$$ NO by looking at numerator and denominator.

so need to try against $\sum \frac{1}{n^2}$ for convergence (since $\sum \frac{1}{n^2}$ converges by integral test)

$$\frac{n-1}{(n+2)(n+3)} \overset{?}{\leq} \frac{1}{n^2}$$

$$\frac{\frac{n-1}{(n+2)(n+3)}}{\frac{1}{n^2}} \overset{?}{\leq} 1$$

$$\frac{(n-1)(n^2)}{n^2+5n+6} \overset{?}{\leq} 1 \qquad \frac{n^3-n^2}{n^2+5n+6} \overset{?}{\leq} 1 \qquad \frac{n-\frac{1}{n}}{\frac{1}{n}+\frac{5}{n^2}+\frac{6}{n^3}} \overset{?}{\leq} 1$$

as $n \to \infty$ this ratio is n. hence test is not usefull. need to try other test

try integral test ⟶

(27)

$$\int^{\infty} \frac{n-1}{(n+2)(n+3)}\,dn = -3\ln(2+n) + 4\ln(3+n)\Big|^{\infty}$$

$$= 4\ln(\infty) - 3\ln(\infty) = \infty - \infty = 0$$

hence finite limit. hence **converges**

9.3 test for convergence or divergence

$$\sum_{1}^{\infty} \frac{1}{n^{\ln 3}}$$

preme test $\lim_{n\to\infty} \frac{1}{n^{\ln 3}} = 0$ hence must try other tests.

$\log 3 = 1.098 = C$ some constant.

try integral test

$$\int^{\infty} \frac{1}{n^{C}}\,dn = \frac{n^{1-C}}{1-C}\Big|^{\infty}$$

since $C = \ln 3 > 1$, then n^{1-C} goes to zero as $n\to\infty$

since $1-C$ is negative.

hence $I \longrightarrow 0$ as $n\to\infty$.

hence finite limit. hence **converges**

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chapter 1

9.7 test for convergence or divergence

$$\sum_{n=0}^{\infty} \frac{(2n)!}{3^n (n!)^2}$$

prem. test. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(2n)!}{3^n (n!)^2}$

$$= \lim_{n\to\infty} \frac{1\times 2\times \cdots \times n \times (n+1)\times (n+2)\times \cdots \times 2n}{3^n (1\times 2\times \cdots \times n)^2} = \frac{(n+1)(n+2)\cdots (2n)}{3^n\; n!}$$

$$= \lim_{n\to\infty} \frac{n\,(1\times 2\times 3\times \cdots \times n)}{3^n\; n!} = \lim_{n\to\infty} \frac{n}{3^n} = 0$$ since 3^n grows faster than n.

hence need more testing.

try ratio testing

$$a_{n+1} = \frac{(2(n+1))!}{3^{n+1} ((n+1)!)^2}, \quad a_n = \frac{(2n)!}{3^n (n!)^2}$$

$$\rho_n = \frac{a_{n+1}}{a_n} = \frac{(2(n+1))!\; 3^n (n!)^2}{3^{n+1} ((n+1)!)^2\; (2n)!} = \frac{(2n+2)!\; (n!)^2}{3\,(1\times 2\times \cdots \times n\times (n+1))^2\; (1\times 2\times \cdots \times n\times (n+1)\cdots 2n)}$$

$$\rho_n = \frac{1\times 2\times \cdots \times n\times (n+1)\times \cdots \times 2n\times (2n+1)\times (2n+2)\; (1\times 2\times \cdots \times n)^2}{3\,(1\times 2\times \cdots \times n\times (n+1))^2\; (1\times 2\times \cdots \times n\times (n+1)\times \cdots \times 2n)}$$

$$= \frac{(n+2)\times (n+3)\times \cdots \times 2n\times (2n+1)(2n+2)}{3\,(n+1)^2\; ((n+2)(n+3)\cdots (2n))} = \frac{(2n+1)(2n+2)}{3\,(n+1)^2}$$

$$\rho_n = \frac{4n^2+6n+2}{3n^2+6n+3} \qquad \lim_{n\to\infty} \rho_n = \frac{4+\frac{6}{n}+\frac{2}{n^2}}{3+\frac{6}{n}+\frac{3}{n^2}} = \frac{4}{3} > 1$$ **diverges**

Chapter 1 test for convergence or divergence (26)

9.12

$$\sum_{n=2}^{\infty} \frac{1}{n^2-n}$$

prem. test

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2-n} = 0$ since $n^2 > n$ for $n \geq 2$.

so more testing needed.

try comparison test with $\sum_{n=2}^{\infty} \frac{1}{n^2}$ which is convergent by integral test.

is $\frac{1}{n^2-n} \overset{?}{\leq} \frac{1}{n^2}$ No so try another test.

try integral test

$$\int^{\infty} \frac{1}{n^2-n}\, dn = \ln(n-1) - \ln(n) \Big|^{\infty}$$

$$= \ln(\infty) - \ln(\infty) = \infty - \infty = 0$$

so finite limit. hence **Convergent**

9.21 $\sum_{n=1}^{\infty} a_n$ if $a_{n+1} = \frac{n}{2n+3} a_n$

$a_1 = \frac{1}{2+3} a_0$

$a_2 = \frac{2}{4+3} a_1 = \frac{2}{4+3} \frac{1}{2+3} a_0$

$a_3 = \frac{3}{6+3} a_2 = \frac{3}{6+3} \frac{2}{4+3} \frac{1}{2+3} a_0$

$a_4 = \frac{4}{8+3} a_3 = \frac{4}{8+3} \frac{3}{6+3} \frac{2}{4+3} \frac{1}{2+3} a_0$

so $a_n = \frac{n!}{\prod_{n=1}^{\infty} (2n+3)}$ $\Longrightarrow$

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prem. test for $a_n = \frac{n!}{\prod_{n=1}^{\infty}(2n+3)}$

this goes to zero as $n \to \infty$. (because denominator is larger than numerator.

so need more testing.

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chapter 1

10.1 find interval of convergence. be sure to investigate end points.

$$\sum_{n=0}^{\infty} (-1)^n x^n$$ use ratio test.

$$\rho = \frac{|a_{n+1}|}{|a_n|} = \left|\frac{x^{n+1}}{x^n}\right| = |x|$$

so **converges for $|x| < 1$**

-1 0 +1

at $x = -1$, series is $\sum (-1)^n (-1)^n = \sum_{n=0}^{\infty} +1^{2n} = 1 + (\cdot)^2 + (1)^4 + (1)^6 + \cdots$

$= 1 + 1 + 1 + 1 \cdots$

This diverges at $x = -1$

at $x = +1$

series is $\sum_{n=0}^{\infty} (-1)^n (+1)^n = \sum_{n=0}^{\infty} (-1)^{2n}$.

$= -1^0 + (-1)^2 - (-1)^4 + (-1)^6 + \cdots$

$= 1 + 1 + 1 + 1 \cdots$ as above. so

diverges at $x = +1$

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chapter 1

10.3 find interval of convergence.

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n(n+1)}$$

$$\rho_n = \frac{|a_{n+1}|}{|a_n|} = \left| \frac{\frac{x^{n+1}}{(n+1)(n+2)}}{\frac{x^n}{n(n+1)}} \right| = \left| \frac{x^{n+1}}{x^n} \frac{n(n+1)}{(n+1)(n+2)} \right|$$

$$\rho_n = \left| \frac{x\, n}{n+2} \right| = \left| \frac{x}{1+\frac{2}{n}} \right|$$

$\lim_{n\to\infty} \rho_n = |x|$ so **converges for $|x| < 1$**

at $x=+1$, series is $\sum_{n=1}^{\infty} \frac{(-1)^n (+1)^n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n(n+1)}$ ← this removes -ve sign from all terms.

$$= \frac{-1^2}{2} + \frac{(-1)^4}{2\times 3} + \frac{(-1)^6}{3\times 4} + \cdots = \frac{1}{2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

now series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. to find if convergent, use integral test:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1) + Bn}{(n)(n+1)} \Rightarrow A(n+1) + Bn = 1 \Rightarrow A=1,\ B=-1$$

$$\text{so } \int^{\infty} \frac{1}{n} - \frac{1}{n+1}\, dn = \ln(n) - \ln(1+n) \Big|^{\infty} = \infty - \infty = 0$$

hence **converges at +1**

at $x=-1$, series is $\sum \frac{(-1)^n (-1)^n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{(+1)^{2n}}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

same as above, so **converges at $x = -1$**

← conj... [illegible]

(30)

chapter 1 find radius of convergence.

10.15

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n}$$

$$\rho_n = \frac{|a_{n+1}|}{|a_n|} = \left| \frac{\frac{(x-2)^{n+1}}{3^{n+1}}}{\frac{(x-2)^n}{3^n}} \right| = \left| \frac{(x-2)^{n+1}\, 3^n}{3^{n+1}\,(x-2)^n} \right|$$

$$\rho_n = \left| \frac{(x-2)}{3} \right| \quad \text{so } \lim_{n\to\infty} \rho_n = \left| \frac{x-2}{3} \right|$$

so converges for $\left| \frac{x-2}{3} \right| < 1$ i.e. $\left| \frac{1}{3}(x-2) \right| < 1$

i.e. $|x-2| < 3$ i.e. $-3 < x-2 < 3$ or $\boxed{-1 < x < 5}$

at $\boxed{x = +5}$ $\sum_{n=1}^{\infty} \frac{-3^n}{3^n} = \sum_{n=1}^{\infty} -1 = -1-1-1-1\cdots$

so $\boxed{\text{divergent at } x = +5}$

at $x = -1$ $\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=1}^{\infty} -1 = -1-1-1\cdots$

so $\boxed{\text{divergent at } x = -1}$

?? ask about this. this should be $-1+1-1+1-1+1$ which is convergent to [illegible] chapter 1 find where series converge (31)

10.19

$$\sum_0^\infty 8^{-n}(x^2-1)^n$$

let $y=x^2-1$ then $\sum_0^\infty 8^{-n} y^n$

$$\rho_n = \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{8^{-(n+1)} y^{n+1}}{8^{-n} y^n}\right| = \left|\frac{y}{8}\right|$$

so $\lim_{n\to\infty} \rho_n = \left|\frac{y}{8}\right|$ so converges for $\left|\frac{y}{8}\right|<1$

or $|y|<8$

i.e. $|x^2-1|<8$?

or $|x^2|<9$

or $|x|<3$

test at -3

$$\sum_0^\infty 8^{-n}(9-1)^n = \sum_0^\infty 8^{-n}8^n = \sum_0^\infty 8^0 = \sum_{n=0}^\infty 1$$

$= 1+1+1+1+\cdots$ diverges at -3

test at $+3$

$$\sum_0^\infty 8^{-n}(9-1)^n$$ this is the same as above.

so diverges at $+3$

(32)

Chapter 1

10.20 test for convergence or divergence:

$$\sum_0^\infty (-1)^n \frac{2^n}{n!}(x^2+1)^{2n}$$

let $y = x^2+1$ so series is $\sum_0^\infty (-1)^n \frac{2^n}{n!} y^{2n}$

$$\rho_n = \left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{2^{n+1} y^{2(n+1)}}{(n+1)!}}{\frac{2^n y^{2n}}{n!}} = \left|\frac{2^{n+1} y^{2(n+1)}}{(n+1)!\ 2^n y^{2n}} n!\right|$$

$$= \left|\frac{2 y^2}{\underbrace{1\times2\times3\times4\times\cdots\times n}_{n!}\times n+1} n!\right| = \left|\frac{2y^2}{n+1}\right|$$

so $\lim_{n\to\infty} \rho_n = 0$

so converges for all values of y.

but $y = x^2+1$ i.e. $x^2 = y-1$

or $x = \pm\sqrt{y-1}$

since converges for any value of y, x can attain any values as well.

hence converges for any value of X

(33)

chapt. 1

13.2 expand in power series

$e^x \sin(x)$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$(e^x \sin x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!}$$

$$x^2 + 0 - \frac{x^4}{3!} + 0 + \frac{x^6}{5!} + 0 - \frac{x^8}{7!} + \cdots$$

$$\frac{x^3}{2!} + 0 - \frac{x^5}{2!\,3!} + 0 + \frac{x^7}{2!\,5!} + 0$$

$$+ \frac{x^4}{3!} + 0 - \frac{x^6}{3!\,3!} + 0 + 0$$

$$+ \frac{x^5}{4!} + 0 - \frac{x^7}{4!\,3!} + 0$$

$$= x + x^2 - \frac{x^3}{6} + \frac{x^3}{2!} + \frac{x^5}{5!} - \frac{x^5}{2!\,3!} + \frac{x^5}{4!} + \frac{x^6}{5!} - \frac{x^6}{3!\,3!}$$

$$= x + x^2 + \frac{-x^3 + 3x^3}{6} + \frac{x^5}{30} + \frac{x^6}{90} + \cdots$$

$$= \boxed{x + x^2 + \frac{x^3}{3} + \frac{x^5}{30} + \frac{x^6}{90} + \cdots}$$

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Chapter 1

13.6 Find power series for $\frac{e^x}{1-x}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \quad \text{for all } x$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \quad -1 < x < 1$$

$$\frac{e^x}{1-x} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\right)\left(1 + x + x^2 + x^3 + x^4 + x^5 + \cdots\right)$$

$$\begin{aligned}
= 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \\
x + x^2 + x^3 + x^4 + x^5 + \cdots \\
\frac{x^2}{2!} + \frac{x^3}{2!} + \frac{x^4}{2!} + \frac{x^5}{2!} + \cdots \\
\frac{x^3}{3!} + \frac{x^4}{3!} + \frac{x^5}{3!} + \cdots \\
\frac{x^4}{4!} + \frac{x^5}{4!} + \cdots \\
\frac{x^5}{5!} + \cdots
\end{aligned}$$

$$1 + 2x + x^2 + \frac{x^2}{2} + 2x^3 + \frac{x^3}{2!} + \frac{x^3}{3!} + 2x^4 + \frac{x^4}{2!} + \frac{x^4}{3!} + \frac{x^4}{4!} + 2x^5 + \frac{x^5}{2!} + \frac{x^5}{3!} + \frac{x^5}{4!} + \frac{x^5}{5!} + \cdots$$

$$\boxed{= 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \frac{65}{24}x^4 + \frac{163}{60}x^5 + \cdots}$$

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Chapter 1

13.8 Find power series for $\sec x = \frac{1}{\cos x}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$

hence $(\sec x)(\cos x) = 1$

so assume $\sec x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

then $(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) = 1$

$$\Rightarrow \quad a_0 + 0 - \frac{a_0}{2}x^2 + 0 + \frac{a_0}{4!}x^4 + 0 + \cdots$$

$$a_1 x + 0 - \frac{a_1}{2}x^3 + 0 + \frac{a_1}{4!}x^5 + \cdots$$

$$a_2 x^2 + 0 - \frac{a_2}{2}x^4 + 0 + \cdots$$

$$a_3 x^3 + 0 - a_3\frac{x^5}{2} + \cdots$$

$$a_4 x^4 + 0 - \frac{a_4}{2}x^6 + \cdots$$

$$a_0 + a_1 x + x^2\left(a_2 - \frac{a_0}{2}\right) + x^3\left(a_3 - \frac{a_1}{2}\right) + x^4\left(\frac{a_0}{4!} - \frac{a_2}{2} + a_4\right) + \cdots = 1$$

by comparing terms:

hence $a_0 = 1$

$a_1 = 0$

$a_2 - \frac{a_0}{2} = 0 \implies a_2 = \frac{1}{2}$

$a_3 - \frac{a_1}{2} = 0 \implies a_3 = 0$

$\frac{a_0}{4!} - \frac{a_2}{2} + a_4 = 0 \implies a_4 = -\frac{a_0}{4!} + \frac{a_2}{2} = -\frac{1}{4!} + \frac{1}{4} = \frac{5}{24}$

hence $\boxed{\sec(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots}$

chapter 1 (36)

13.13 Find power series for

$\sin x^2$.

let $y = x^2$

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots$$

so $\sin x^2 = (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots$

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

13.14

$\frac{\sin\sqrt{x}}{\sqrt{x}}$ $x>0$

let $y=\sqrt{x}$, so $\frac{\sin\sqrt{x}}{\sqrt{x}} = \frac{\sin y}{y} = \frac{y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots}{y}$

$$= 1 - \frac{y^2}{3!} + \frac{y^4}{5!} - \frac{y^6}{7!} + \cdots$$

but $y = x^{1/2}$. so

$$\frac{\sin\sqrt{x}}{\sqrt{x}} = 1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \cdots$$

chapter 1 (37)

13.16 Find power series for

$\sin[\ln(1+x)]$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad -1 < x \leq 1$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad$ for all x.

let $\ln(1+x) = y$.

$\sin(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!} + \cdots$

$\sin(\ln(\cdots)) = \underbrace{\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right]}_{\ln(1+x)} - \frac{1}{3!}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right]^3 + \frac{1}{5!}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right]^5 + \cdots$

then write it as

$= \ln(1+x) - \frac{1}{3!}\ln^3(1+x) + \frac{1}{5!}\left(\ln^5(1+x)\right) + \cdots$

$= \ln(1+x)\left[1 - \frac{1}{3!}\ln^2(1+x) + \frac{1}{5!}\ln^4(1+x) + \cdots\right]$

looking at only 3 terms in $\ln(1+x)$.

$= \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)\left[1 - \frac{1}{3!}\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)^2 + \frac{1}{5!}\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)^5 - \cdots\right]$

$\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)^2 = \left(\left(x - \frac{x^2}{2}\right) + \frac{x^3}{3}\right)^2 = \left(x - \frac{x^2}{2}\right)^2 + \frac{x^6}{3} + 2\,\frac{x^3}{3}\left(x - \frac{x^2}{2}\right)$

$= x^2 + \frac{x^4}{2} - 2\,\frac{x^3}{2} + \frac{x^6}{3} + \frac{2}{3}\left(x^4 - \frac{x^5}{2}\right) = x^2 + \frac{1}{2}x^4 - x^3 + \frac{1}{3}x^6 + \frac{2}{3}x^4 - \frac{1}{3}x^5$

$= x^2 - x^3 + \frac{7}{6}x^4 - \frac{1}{3}x^5 - \frac{1}{3}x^6$

so $\sin(\ln(1+x)) = \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)\left(1 - \frac{1}{3!}\left(x^2 - x^3 + \frac{7}{6}x^4 - \frac{1}{3}x^5 + \frac{1}{3}x^6\right)\right)$

$= \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)\left(1 - \frac{x^2}{6} + \frac{x^3}{6} - \frac{7}{36}x^4 + \frac{x^5}{18} + \frac{x^6}{18}\right) \Longrightarrow$

(38)

$$\sin(\ln(1+x)) = x + 0 - \frac{x^3}{6} + \frac{x^4}{6} - \frac{7}{36}x^5 + \frac{x^6}{18} + \frac{x^7}{18}$$

$$-\frac{x^2}{2} + 0 - \frac{x^4}{12} - \frac{x^5}{12} + \frac{7}{72}x^6 - \frac{x^7}{36} + \cdots$$

$$\frac{x^3}{3} + 0 - \frac{x^5}{18} + \frac{x^6}{18} - \frac{7x^7}{3\times 36} + \cdots$$

$$\boxed{x - \frac{x^2}{2} + \frac{1}{6}x^3 + \cdots}$$

these are the terms I can be sure about since I omitted $\frac{1}{5!}\ln^5(1+x)$ earlier as multiplication was becoming messy.

is there a more direct method to do this?

(39)

Chapter 1

13.17 expand in power series $\int_0^x \cos t^2\, dt$.

$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!}$

so $\cos t^2 = 1 - \frac{t^4}{2!} + \frac{t^8}{4!} - \frac{t^{12}}{6!} + \frac{t^{16}}{8!} - \cdots$

hence $\int_0^x \cos t^2\, dt = \int_0^x 1 - \frac{t^4}{2} + \frac{t^8}{4!} - \frac{t^{12}}{6!} + \frac{t^{16}}{8!} \cdots \, dt$

$$= t\Big|_0^x - \frac{1}{2}\left[\frac{t^5}{5}\right]_0^x + \frac{1}{4!}\left[\frac{t^9}{9}\right]_0^x - \frac{1}{6!}\left[\frac{t^{13}}{13}\right]_0^x + \cdots$$

$$= (x-0) - \frac{1}{2}\left(\frac{x^5}{5}\right) + \frac{1}{4!}\left(\frac{x^9}{9}\right) - \frac{1}{6!}\left(\frac{x^{13}}{13}\right) + \cdots$$

$$= x - \frac{x^5}{10} + \frac{x^9}{4!\,9} - \frac{x^{13}}{6!\,13} + \cdots$$

$$\boxed{= x - \frac{x^5}{10} + \frac{x^9}{216} - \frac{x^{13}}{9360} + \cdots}$$

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chapter 1

13.33 Find power series for $\ln(\cos x)$

$$\ln \cos x = -\int_0^x \tan u \, du.$$

but $\tan u = u + \frac{u^3}{3} + \frac{2u^5}{15} + \frac{17u^7}{315} + \cdots$

$$\text{so } -\int_0^x \tan u \, du = -\int_0^x u + \frac{u^3}{3} + \frac{2}{15}u^5 + \frac{17}{315}u^7 + \cdots \, du$$

$$= -\left(\left[\frac{u^2}{2}\right]_0^x + \frac{1}{3}\left[\frac{u^4}{4}\right]_0^x + \frac{2}{15}\left[\frac{u^6}{6}\right]_0^x + \frac{17}{315}\left[\frac{u^8}{8}\right]_0^x + \cdots \right)$$

$$= -\left(\frac{x^2}{2} + \frac{1}{3}\frac{x^4}{4} + \frac{2}{15}\frac{x^6}{6} + \frac{17}{315}\frac{x^8}{8} + \cdots \right)$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \cdots$$

13.37 expand e^x about $a=3$

i.e. expand $e^{(x-3+3)} = e^{(x-3)}e^3$

$$= e^3\left(1 + (x-3) + \frac{(x-3)^2}{2!} + \frac{(x-3)^3}{3!} + \frac{(x-4)^4}{4!} + \cdots\right)$$

$$= e^3 + e^3(x-3) + \frac{e^3}{2}(x-3)^2 + \frac{e^3}{3!}(x-3)^3 + \frac{e^3}{4!}(x-4)^4 + \cdots$$

✓

ch 1

1. 13.40

$f(x) = \sqrt{x}$ about point $a = 25$

$$\sqrt{x} = \sqrt{x - 25 + 25} = \sqrt{25\left(1 + \frac{x-25}{25}\right)} = 5\sqrt{1 + \frac{x-25}{25}}$$

expand. $\left(1 + \frac{x-25}{25}\right)^{1/2}$. let $z = \frac{x-25}{25}$

$$\text{so } (1+z)^{1/2} \equiv (1+z)^{p} = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

$$\text{so } (1+z)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{\frac{1}{4}}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{1}{(4)2!}x^2 + \frac{3}{2^3\,3!}x^3 + \cdots$$

← These should be 'z' not 'x'

So answer is

$$\sqrt{x} = 5\left(1 + \frac{\left(\frac{x-25}{25}\right)}{2} - \frac{\left(\frac{x-25}{25}\right)^2}{2^2\,2!} + \frac{3}{2^3\,3!}\left(\frac{x-25}{25}\right)^3 - \cdots\right)$$

$$\sqrt{x} = 5\left(1 + \frac{1}{2}\left(\frac{x-25}{25}\right) + \cdots + \frac{n}{2^n\,n!}\left(\frac{x-25}{25}\right)^n + \cdots\right)$$

→ a different solution →

Chapter 1 this is another way to solve 13.40 (41)

13.40 expand $f(x)=\sqrt{x}$ about point $a=25$

First find power series for $x^{1/2}$

$f(x) = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f''(x) = -\frac{1}{4}x^{-3/2}$

$f'''(x) = \frac{3}{8}x^{-5/2}$

$f''''(x) = -\frac{15}{16}x^{-7/2}$

So power series for $x^{1/2}$ at $x=25$ is

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \frac{1}{3!}(x-a)^3 f'''(a) + \cdots$$

$$= \sqrt{25} + (x-25)\frac{1}{2}\frac{1}{\sqrt{25}} + \frac{1}{2!}(x-25)^2\left(-\frac{1}{4}\frac{1}{(\sqrt{25})^3}\right) + \frac{1}{3!}(x-25)^3\left(\frac{3}{8}\frac{1}{(\sqrt{25})^5}\right)$$

$$= 5 + \frac{(x-25)}{10} - \frac{(x-25)^2}{2\times 4\times 5^3} + \frac{(x-25)^3}{2\times 8\times 5^5} - \cdots$$

$$\boxed{= 5 + \frac{1}{10}(x-25) - \frac{1}{1000}(x-25)^2 + \frac{(x-25)^3}{50000} - \cdots}$$

3.3 HW 2

2/2

HW #2

Math 121A

NASSER ABBASI

(UCB extension)

①

ch 1

14.5 show that $1-\cos x = \frac{x^2}{2}$ with an error less than 0.003 for $|x| < \frac{1}{2}$

Power series expansion for $\cos x$

$$1-\cos x = 1-\left[1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right]$$

$$= \frac{x^2}{2}-\frac{x^4}{4!}+\frac{x^6}{6!}-\cdots$$

alternating series with $|a_{n+1}| < |a_n|$ since $|x| < 1$

apply 14.3 rule

$|S-(a_1+a_2+\cdots)| \leq |a_{n+1}|$ i.e. convergent

here $n=1$ since we want to keep first term $\frac{x^2}{2}$.

So $|S-a_1| \leq |a_2|$

this is $(1-\cos x)$; this is $\frac{x^2}{2}$; this is $\frac{x^4}{4!}$

i.e. this says that error is less than the first neglected term. see theory (14.3).

So $\frac{x^4}{4!}$ is max error. which is largest at $x=\frac{1}{2}$.

hence error max value is $\frac{\left(\frac{1}{2}\right)^4}{1\times2\times3\times4} = \frac{\frac{1}{16}}{2\cdot3\cdot4} = \frac{1}{16\cdot2\cdot3\cdot4} = \boxed{0.0026}$

which is less than 0.003.

(2)

Ch 1

14.6 show that $\ln(1-x) = -x$ with an error less than 0.0056 for $|x| < 0.1$

expand $\ln(1-x)$ in power series

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

this is not an alternating series, so use theorem 14.4, page 31

$$\left| \ln(1-x) + x \right| < \left| \frac{\frac{x^2}{2}}{1-|x|} \right|$$

(x: First term 'N'; $\frac{x^2}{2}$: second term 'N+1')

so max $|error| \leq \left| \frac{\left(\frac{0.1}{2}\right)^2}{1-0.1} \right| \leq 0.00277$

so this is < 0.0056. QED.

Ch 1 ③

14.9 Find sum of $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots$

Series $=\sum_{1}^{\infty}\frac{1}{n(n+1)}$

$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{(n+1)}$, hence Series $=\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots\right)-\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)$

take one term from each series, we get

$=\frac{1}{1}+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\frac{1}{4}-\frac{1}{4}+\frac{1}{5}-\frac{1}{5}\cdots$

$=1$

hence $\boxed{S \text{ for } \sum_{1}^{\infty}\frac{1}{n(n+1)}=1}$

Now to find the Reminder, rewrite as

$1+\frac{1}{2}\quad\underbrace{-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\frac{1}{4}-\frac{1}{4}+\frac{1}{5}-\frac{1}{5}\cdots}_{\text{alternating Series}}$

alternating Series $\Rightarrow \sum_{n=2}^{\infty}(-1)^n\frac{1}{(n)}$

So apply theorem 14.3, which say that for alternating series, the reminder after n terms is $\leq |(n+1)^{th}$ term.$|$

hence, the $|a_{n+1}|$ term is $\frac{1}{(n+1)}$ from

~~for $n=200$, $|a_n|=\frac{1}{200}=0.005$~~

~~Reminder $=|a_{n+1}|=\frac{1}{201}=0.00497512$~~

For $n=200$, $a_n=\frac{1}{n(n+1)}=\frac{1}{200(201)}=0.0000248$ ← This a_n from original Series.

while Reminder $=\frac{1}{201}=0.004975$ which $>a_n$ hence not reliable estimate as much smaller.

(4)

Ch 1

15.5 use power series to evaluate the function at the given point.

$$\frac{d^4}{dx^4} \ln(1+x^3) \quad \text{at } x=0.2$$

find power series for $\ln(1+x^3)$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

replace x by x^3

$$\ln(1+x^3) = x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots$$

so $f' = 3x^2 - \frac{6x^5}{2} + \frac{9x^8}{3} - \frac{12x^{11}}{4} + \cdots$

$f^2 = 6x - \frac{30x^4}{2} + \frac{72x^7}{3} - \frac{132x^{10}}{4} + \cdots$

$f^3 = 6 - \frac{120x^3}{2} + \frac{504x^6}{3} - \frac{1320x^9}{4} + \cdots$

$f^4 = 0 - \frac{360x^2}{2} + \frac{3024x^5}{3} - \frac{11880x^8}{4} + \cdots$

$f^4(x=0.2) = -(0.2)^2 \frac{360}{2} + (0.2)^5 \frac{3024}{3} - (0.2)^8 \frac{11880}{4} + \cdots$

$= -6.8850$

easier using series method, since eliminates need to do complicated differentiation many times.

5

Ch 1

15.11 use power Series to evaluate $\int_0^1 \cos x^2\,dx$.

expand $\cos x^2$ in power Series

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

replace x by x^2

$$\cos x^2 = 1 - \frac{x^4}{2} + \frac{x^8}{4!} - \frac{x^{12}}{6!}$$

integrate term by term

$$\int_0^1 \cos x^2\,dx = \int_0^1 1\,dx - \int_0^1 \frac{x^4}{2}\,dx + \int_0^1 \frac{x^8}{4!} - \int_0^1 \frac{x^{12}}{6!} \cdots$$

$$= [x]_0^1 - \frac{1}{2}\left[\frac{x^5}{5}\right]_0^1 + \frac{1}{4!}\left[\frac{x^9}{9}\right]_0^1 - \frac{1}{6!}\left[\frac{x^{13}}{13}\right]_0^1 \cdots$$

$$= 1 - \frac{1}{2}\frac{1}{5} + \frac{1}{4!}\frac{1}{9} - \frac{1}{6!}\frac{1}{13} \cdots$$

$$= 1 - \frac{1}{10} + \frac{1}{4!\,9} - \frac{1}{6!\,13} = 0.9045$$

this is easier than integrating $\cos x^2$ directly which does not have simple integral.

(6)

Ch 1

15.16 use power series to evaluate $\lim_{x\to 0} \frac{\tan x - x}{x^3}$

expand $\tan x$ in power series

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots}{1 - \frac{x^2}{2!} - \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$$

$$\text{So } \lim_{x\to 0} \frac{\left(\frac{x - \frac{x^3}{3!} + \cdots}{1 - \frac{x^2}{2!} - \cdots}\right) - x}{x^3} = \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \cdots\right) - x\left(1 - \frac{x^2}{2!} + \cdots\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{\overbrace{\left(x - \frac{x^3}{3!} + \cdots\right)}^{A} - \overbrace{\left(x - \frac{x^3}{2!} + \cdots\right)}^{B}}{x^3}$$

first term x in B cancel first term x in B, we get

$$\lim_{x\to 0} \frac{\left(-\frac{x^3}{3!} + \cdots\right) - \left(-\frac{x^3}{2!} + \cdots\right)}{x^3}$$

divid numerator and denominator by x^3

~~[illegible]~~

$$\lim_{x\to 0} \frac{\left(-\frac{1}{3!} + \overbrace{\cdots\cdots}^{\text{terms with } x \text{ in numerator}}\right) - \left(-\frac{1}{2!} + \cdots\right)}{1}$$

hence when $x=0$, we are left with $-\frac{1}{3!} + \frac{1}{2!} = \frac{1}{2} - \frac{1}{6}$

$$= \boxed{\frac{1}{3}}$$

ch 1 ⑦

15.18 evaluate using power series

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$$

$$\lim_{x\to 0}\left(\frac{(e^x-1)-x}{xe^x-x}\right)$$

but $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

so $$\lim_{x\to 0}\left(\frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)-1-x}{\left(x+x^2+\frac{x^3}{2!}+\frac{x^4}{3!}+\cdots\right)-x}\right)=\frac{\left(\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)}{\left(x^2+\frac{x^3}{2!}+\frac{x^4}{3!}+\cdots\right)}$$

divide by x^2

$$\lim_{x\to 0}\left(\frac{\frac{1}{2}+\frac{x}{3!}+\cdots}{1+\frac{x}{2!}+\cdots}\right)=\boxed{\frac{1}{2}}$$

(8)

ch 1

16.3 Show that $\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ is convergent.

prem. Test: $\lim_{n\to\infty} a_n = 0$ hence can be convergent.

try integral test: $\int^{\infty} \frac{1}{n^{3/2}}\, dn = -\frac{2}{\sqrt{n}}\Big|^{\infty} = 0$

hence convergent

what is wrong with the following proof that it is divergent?

$$\frac{1}{\sqrt{8}} + \frac{1}{\sqrt{27}} + \cdots > \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{36}} + \cdots$$

$$> \frac{1}{3} + \frac{1}{6} + \cdots$$

$$> \frac{1}{3}\left(1 + \frac{1}{2} + \cdots\right)$$

→ harmonic. diverges.

answer the problem with this proof is that not enough terms were considered.

The question is

$$\frac{1}{n^{3/2}} \overset{?}{>} \frac{1}{3n} \quad \text{for all terms.}$$

it looks to be true for first few, but try for $n=10$ we get

$$\frac{1}{\sqrt{10^3}} \overset{?}{>} \frac{1}{30} \quad \text{or} \quad \frac{1}{1000} \overset{?}{>} \frac{1}{900}$$

here we see that it is not true. $\frac{1}{1000} < \frac{1}{900}$

hence the "proof" for divergence is faulty as it did not consider enough terms.

(9)

ch 1

16.18 Find Maclaurian Series for $\arctan x = \int_0^x \frac{du}{1+u^2}$

MacLaurin series is taylor series expanded about the origin.

expand $\frac{1}{1+u^2}$ in ~~power series~~ Taylor series

$f(u) = \frac{1}{1+u^2} \Rightarrow f(0) = 1$

$f'(u) = \frac{-2u}{(1+u^2)^2} \Rightarrow f'(0) = 0$

$f''(u) = \frac{8u^2}{(1+u^2)^3} - \frac{2}{(1+u^2)^2} \Rightarrow f''(0) = -2$

$f'''(u) = \frac{-48u^3}{(1+u^2)^4} + \frac{24u}{(1+u^2)^3} \Rightarrow f'''(0) = 0$

$f''''(u) = \frac{384u^4}{(1+u^2)^5} - \frac{288u^2}{(1+u^2)^4} + \frac{24}{(1+u^2)^3} \Rightarrow f''''(0) = 24$

could have used formula ? $(1+x)$ $|x|<1$

$$\text{so } f(u) = f(\bar{u}) + f'(\bar{u})(u-\bar{u}) + \frac{f''(\bar{u})(u-\bar{u})^2}{2!} + \frac{f'''(\bar{u})(u-\bar{u})^3}{3!} + \frac{f''''(\bar{u})(u-\bar{u})^4}{4!} + \cdots$$

$$= 1 + 0 - \frac{2u^2}{2!} + 0 + \frac{24u^4}{24} - \cdots$$

$$\boxed{f(u) = 1 - u^2 + u^4 - u^6 + u^8 - \cdots}$$

$$\text{so } \arctan x = \int_0^x f(u)\,du = \int_0^x 1 - \int_0^x u^2 + \int_0^x u^4 - \int_0^x u^6 \cdots$$

$$\boxed{= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots}$$

(19)

ch 1

16.22 use series you know to show that

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$$

from problem 18, we found that $\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots$

So for $x=1$, we get the same series.

hence $\boxed{\arctan 1 = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots}$

So, the angle whose $\tan = 1$ is 45°

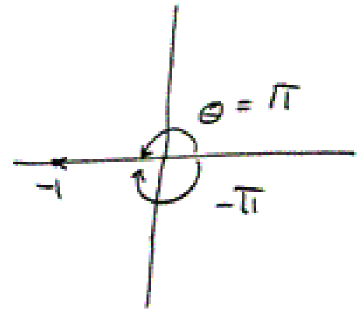
but $45^\circ = \frac{\pi}{4}$ as required to show.

(11)

Ch 2

4.7 Plot the following numbers in complex plane. for each, give the numerical value of Re(z) and Im(z), its mod, and one value of the angle ϕ. Lable each plotted point in five ways as in figure 3.3. find and plot the complex conjugate.

$z = -1$ or $z = -1 + 0i$

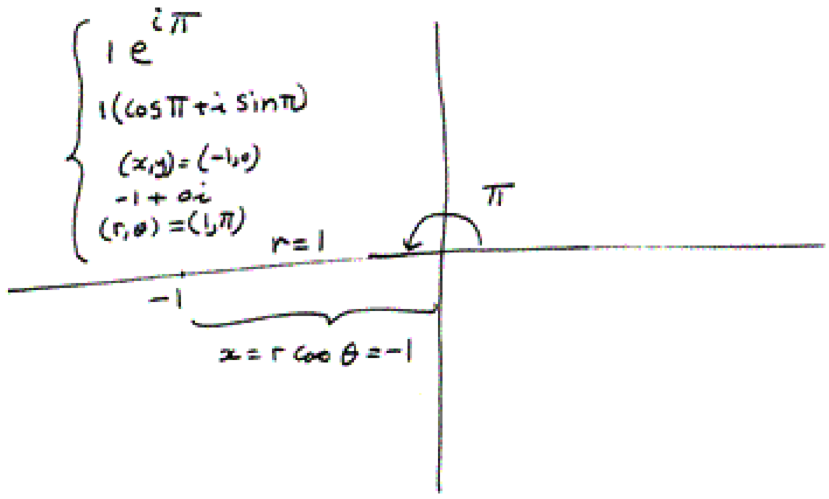


$Re(z) = -1$

$Im(z) = 0$

$|z| = 1$

$\theta = 180^\circ$



$\bar{z} = -1$ because $\bar{z} = r(\cos(-\theta) + i\sin(-\theta))$

$\bar{z} = 1(\cos(-\pi) + i\sin(-\pi))$

$\bar{z} = 1(\cos(-\pi))$

but $\cos(-\pi) = \cos(\pi)$

so $\bar{z} = -1$

(12)

Ch 2

4.14 Plot $z = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$Re(z) = 2\cos\frac{\pi}{4} = 2\left(\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}$

$Im(z) = 2\sin\frac{\pi}{4} = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$

$|z| = \sqrt{Re^2(z) + Im^2(z)} = \sqrt{\frac{4}{2}+\frac{4}{2}} = \sqrt{4} = 2$

$\theta = \frac{\pi}{4}$

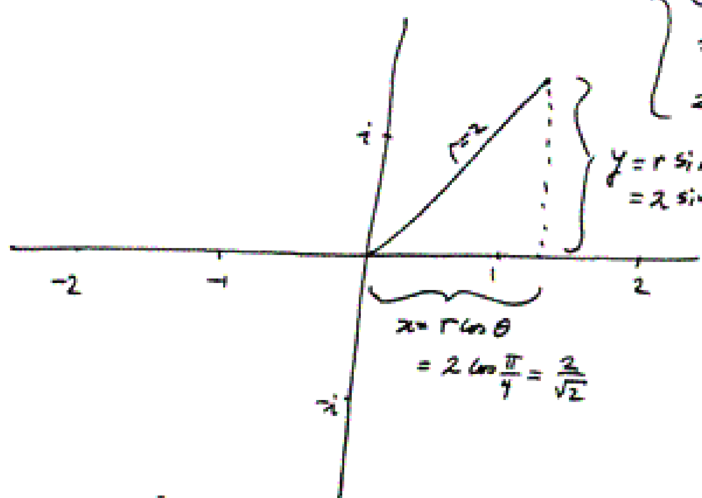
$(x,y) = \frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}$

$\frac{2}{\sqrt{2}} + i\frac{2}{\sqrt{2}}$

$(r,\theta) = (2, \frac{\pi}{4})$

$2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

$2e^{i\frac{\pi}{4}}$



$y = r\sin\theta = 2\sin\frac{\pi}{4} = \frac{2}{\sqrt{2}}$

$x = r\cos\theta = 2\cos\frac{\pi}{4} = \frac{2}{\sqrt{2}}$

5.6 Plot $\left(\frac{1+i}{1-i}\right)^2$

$= \left(\frac{1+i}{1-i}\frac{1-i}{1-i}\right)^2 = \left(\frac{2i}{2}\right)^2 = i^2 = -1$

this is the same as problem **4.7** which I solved already. nothing more to do.

Ch 2 (13)

5.13 Plot $5\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$

$r = 5$

$\theta = \frac{2\pi}{5} = 72^\circ$

$\operatorname{Re}(z) = 5\cos\frac{2\pi}{5} = 1.545$

$\operatorname{Im}(z) = 5\sin\frac{2\pi}{5} = 4.755$

$|z| = \sqrt{\operatorname{Re}^2(z) + \operatorname{Im}^2(z)} = 5$

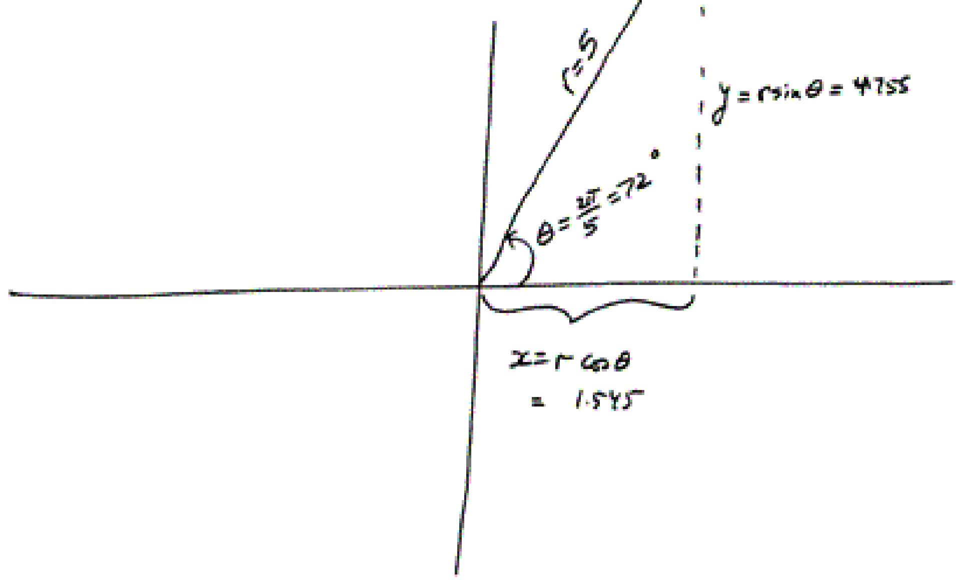
$(x,y) = (1.545, 4.755)$

$1.545 + i\,4.755$

$(r,\theta) = \left(5, \frac{2\pi}{5}\right)$

$5\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$

$5e^{i\frac{2\pi}{5}}$



5.20 Find in rectangular form $(a+ib)$. $1/z^2$

when $z = 2-3i$

$$\frac{1}{(2-3i)^2} = \frac{1}{4-12i-9} = \frac{1}{-5-12i} \quad \text{multiply by complex conjugate}$$

$$= \frac{1}{(-5-12i)}\frac{(-5+12i)}{(-5+12i)} = \frac{-5+12i}{25-12^2 i^2} = \frac{-5+12i}{25+144} = \frac{-5+12i}{169} = \boxed{\frac{-5}{169} + \frac{12}{169}i}$$

when $z = x+iy$

$$\frac{1}{(x+iy)^2} = \frac{1}{x^2+2ixy-y^2} = \frac{1}{(x^2-y^2+i(2xy))} \frac{(x^2-y^2-i(2xy))}{(x^2-y^2-i\,2xy)}$$

$$= \frac{x^2-y^2-i\,2xy}{(x^2-y^2)^2+(2xy)^2} = \frac{x^2-y^2}{(x^2-y^2)^2+(2xy)^2} - i\frac{2xy}{(x^2-y^2)^2+(2xy)^2}$$

(14)

ch 2

5.32 find mod of $(2-3i)^4$

$|z| = r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$

$z = 2-3i$

$|z| = \sqrt{z\bar{z}} = \sqrt{(2-3i)(2+3i)} = \sqrt{4-9i^2} = \sqrt{13}$

but $|z^4| = |z|^4 \Rightarrow |z|^4 = (\sqrt{13})^4 = (13)(13) = \boxed{169}$

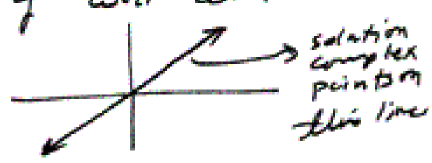
this means $|(x+iy)^n| = |(x+iy)|^n$

This can be better seen from $(re^{i\theta})^4 = r^4 e^{i4\theta}$ (r^4 → length.)

5.39 Solve for all possible values of the real numbers x, y.

$x+iy = y+ix$

$\left.\begin{array}{l} x = y \\ y = x \end{array}\right\}$ any real value of x, y will work. as long as $x=y$



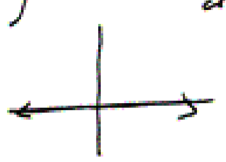
5.45

$(x+iy)^2 = (x-iy)^2$

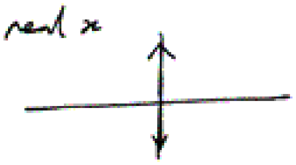
$x^2 + i^2y^2 + 2xyi = x^2 + i^2y^2 - 2xyi$

$x^2 - y^2 + i\,2xy = x^2 - y^2 - i\,2xy.$

so $\left.\begin{array}{l} x^2-y^2 = x^2-y^2 \\ -xy = xy \end{array}\right\} \Rightarrow$ $x=0$, any real y or $y=0$, any real x



or

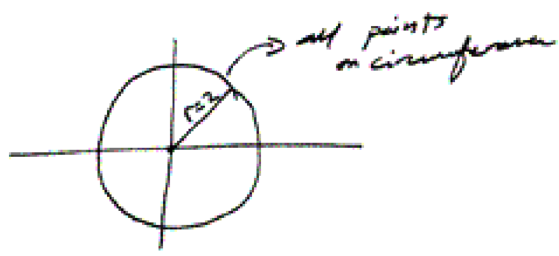


ch 2

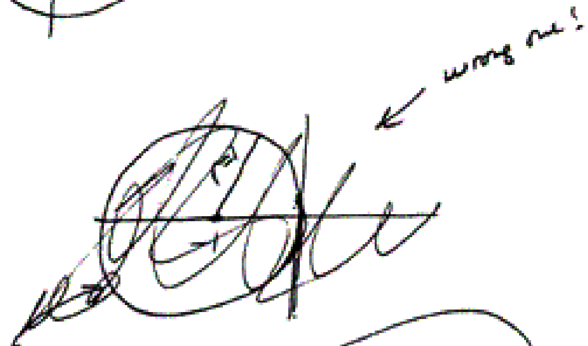
5.51 describe geometrically the set of complex points satisfying

$|z| = 2.$

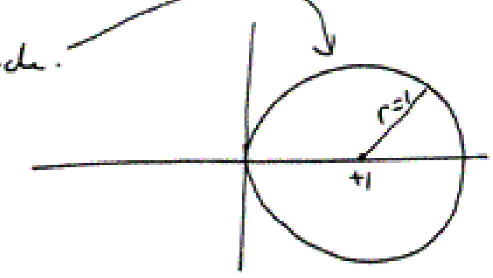
this is a complex number whose length is 2.
then this describes all the points on the circumference of a circle whose radius = 2 centered at origin.



5.53 $|z-1| = 1$



this is a circle centered at
$z = 1 + 0i$
and radius = 1
so all points on circumference of this circle.



Can also solve this as follows
let $z = x + iy$. Then ✓

$|x+iy-1| = 1 \Rightarrow |(x-1)+iy| = 1$

$\Rightarrow (x-1)^2 + y^2 = 1^2$

$\Rightarrow x^2 - 2x + 1 + y^2 = 1 \Rightarrow x^2 + y^2 - 2x = 0$

$\Rightarrow \underbrace{(x-1)^2 + (y-0)^2 = 1^2}$

general form of a circle equation, where

$(x-x_0)^2 + (y-y_0)^2 = r^2$ ✓, centre at (x_0, y_0) and radius $= r$

Ch 2 (16)

5.55 describe geometrically

$z - \bar{z} = 5i$

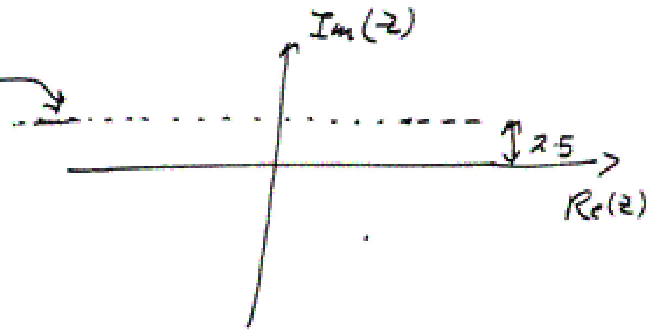
let $z = x + iy$.

then $(x+iy) - (x - iy) = 5i$

i.e. $2iy = 5i \Rightarrow \boxed{y = 2.5}$

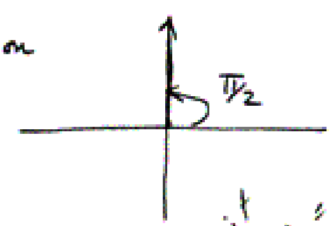
and any x real value will work.

so any complex number on this line



5.56 angle of $z = \frac{\pi}{2}$

so any complex number on the the positive Im(z) axis will satisfy this equation.



5.60 $|z - 1 + i| = 2$

$|z - (1 - i)| = 2$

This is a circle, radius = 2 Centered at $(1, -1)$ (my drawing is not perfect !)

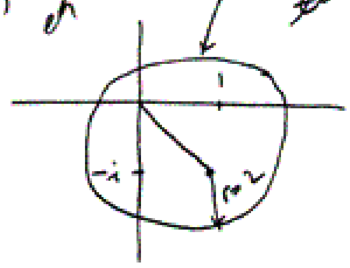
can also do it

$|(x+iy) - 1 + i| = 2$

$(x-1)^2 + (y+1)^2 = 4$

equation of circle $(x - x_0)^2 + (y - y_0)^2 = r^2$

any point on this circle.



(16.1)

ch 2

5.62 Describe geometrically

$$|z+1| + |z-1| = 8$$

$z = x + iy$. then $z+1 = (x+1) + iy$ and $z-1 = (x-1) + iy$.

Then $\sqrt{(x+1)^2 + y^2} + \sqrt{(x-1)^2 + y^2} = 8$ (1)

need to solve the above to find an equation in x, y.

$$\sqrt{(x+1)^2 + y^2} = 8 - \sqrt{(x-1)^2 + y^2}$$

square both sides to remove the $\sqrt{\ }$

$$(x+1)^2 + y^2 = \left(8 - \sqrt{(x-1)^2 + y^2}\right)^2$$

$$(x+1)^2 + y^2 = 64 - 16\sqrt{(x-1)^2 + y^2} + (x-1)^2 + y^2)$$

$$x^2 + 2x + 1 + y^2 = 64 - 16\sqrt{(x-1)^2 + y^2} + x^2 - 2x + 1 + y^2$$

$$4x = 64 - 16\sqrt{(x-1)^2 + y^2}$$

or $16\sqrt{(x-1)^2 + y^2} = 64 - 4x \Rightarrow 4\sqrt{(x-1)^2 + y^2} = 16 - x$

square again

$$16\left((x-1)^2 + y^2\right) = (16 - x)^2$$

$$16\left(x^2 - 2x + 1 + y^2\right) = 256 - 32x + x^2$$

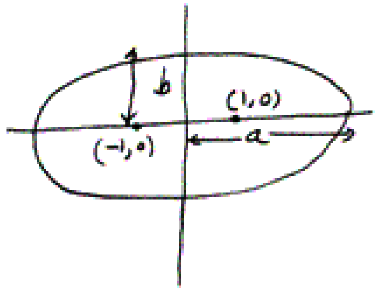
$$16 + 16x^2 - 32x + 16y^2 = 256 - 32x + x^2$$

$15x^2 + 16y^2 = 240 \Rightarrow$ but $240 = 15 \times 16$

so $\boxed{\frac{x^2}{16} + \frac{y^2}{15} = 1}$ this is an equation of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Longrightarrow$

16-2



so $a = \sqrt{16} = 4$

$b = \sqrt{15}$

The foci of an ellipse are at $x = \pm\sqrt{a^2 - b^2}$

$= \pm\sqrt{16-15} = \pm 1$

So foci is $(1,0)$ and $(-1,0)$.

ch 2

6.4 test for convergence $\sum \left(\frac{1-i}{1+i}\right)^n$

to test for convergence of complex series, use the ratio test.

$$\rho_n = \left| \frac{\left(\frac{1-i}{1+i}\right)^{n+1}}{\left(\frac{1-i}{1+i}\right)^n} \right| = \left| \frac{(1-i)^{n+1}}{(1+i)^{n+1}} \frac{(1+i)^n}{(1-i)^n} \right|$$

$$= \left| \frac{(1-i)}{1+i} \right| = \left| \frac{(1-i)(1-i)}{(1+i)(1-i)} \right| = \left| \frac{1-2i+1}{1+1} \right|$$

-1

$$= \left| \frac{2-2i}{2} \right| = |1-i| \qquad \frac{-2i}{2} = |-i|$$

so $\rho = 1$ not convergent since must be <1

$$\rho = \lim_{n\to\infty} \rho_n = |1-i| = \sqrt{1^2+1^2} = \sqrt{2} > 1$$

since $\rho > 1$, hence series **not convergent**

6.10 $\sum \left(\frac{1+i}{1-i\sqrt{3}}\right)^n$

$$\rho_n = \left| \frac{\left(\frac{1+i}{1-i\sqrt{3}}\right)^{n+1}}{\left(\frac{1+i}{1-i\sqrt{3}}\right)^n} \right| = \left| \frac{(1+i)^{n+1}}{(1-i\sqrt{3})^{n+1}} \frac{(1-i\sqrt{3})^n}{(1+i)^n} \right| = \left| \frac{(1+i)}{1-i\sqrt{3}} \right|$$

$$= \left| \frac{(1+i)(1+i\sqrt{3})}{(1-i\sqrt{3})(1+i\sqrt{3})} \right| = \left| \frac{1+i\sqrt{3}+i+i^2\sqrt{3}}{1-i^2\sqrt{3}^2} \right| = \left| \frac{1-\sqrt{3}+i(1+\sqrt{3})}{1+3} \right|$$

$$\rho = \lim_{n\to\infty}\rho_n = \left| \frac{1-\sqrt{3}+i(1+\sqrt{3})}{4} \right| = \sqrt{\left(\frac{1-\sqrt{3}}{4}\right)^2 + \left(\frac{1+\sqrt{3}}{4}\right)^2} = \frac{1}{4}\sqrt{(1-\sqrt{3})^2+(1+\sqrt{3})^2}$$

$$= \frac{1}{4}\sqrt{(1+3-2\sqrt{3})+(1+3+2\sqrt{3})} = \frac{1}{4}\sqrt{8} = \frac{1}{4}\sqrt{2^2\cdot 2} = \frac{2}{4}\sqrt{2}$$

$$= \frac{1}{2}\sqrt{2} < 1$$ hence series is **convergent**

ch 2

6.12 test for convergence

$$\sum \frac{(3+2i)^n}{n!}$$

use ratio test.

$$\rho_n = \left| \frac{\frac{(3+2i)^{n+1}}{(n+1)!}}{\frac{(3+2i)^n}{n!}} \right| = \left| \frac{(3+2i)^{n+1} \; n!}{(n+1)! \; (3+2i)^n} \right|$$

$$= \left| \frac{(3+2i) \; n!}{n! \; (n+1)} \right| = \left| \frac{3+2i}{n+1} \right|$$

$$\rho = \lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \left| \frac{3+2i}{n+1} \right| = 0$$

hence series is **convergent**

ch 2 (16.3)

7.3 Find circle of convergence for

$$1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots$$

method of solution: find a_n, a_{n+1}. then find $\rho_n=\left|\frac{a_{n+1}}{a_n}\right|$, then find $\lim_{n\to\infty}\rho_n$.

$$\text{Series}=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n+1)!}$$

so

$$\rho_n=\left|\frac{\frac{z^{2n+1}}{(2n+2)!}}{\frac{z^{2n}}{(2n+1)!}}\right|=\left|\frac{z\,(2n+1)!}{(2n+2)!}\right|=\left|\frac{z}{2n+2}\right|$$

$$\rho=\lim_{n\to\infty}\left|\frac{z}{2n+2}\right|=0$$

so this series converges for all values of z. so $R=\frac{1}{\rho}=\infty$

i.e. circle is the whole z plane.

7.12 Find circle of convergence $\sum_{n=0}^{\infty}\frac{(n!)^2 z^n}{(2n)!}$

$$\rho_n=\left|\frac{\frac{(n+1!)^2 z^{n+1}}{(2(n+1))!}}{\frac{(n!)^2 z^n}{(2n)!}}\right|=\left|\frac{(n+1!)^2 z\,(2n)!}{(2(n+1))!\,(n!)^2}\right|=\left|\frac{(n+1!)^2 z}{(2n+1)(n!)^2 *(2n+2)}\right|$$

$$=\left|\frac{(n!\,(n+1))^2 z}{(2n+1)(n!)^2 *(2n+2)}\right|=\left|\frac{(n+1)^2 z}{(2n+2)(2n+1)}\right|=\left|\frac{(n^2+2n+1)z}{(2n+2)(2n+1)}\right|$$

$$=\left|\frac{(n^2+2n+1)z}{4n^2+6n+2}\right|=\left|\frac{1+\frac{2}{n}+\frac{1}{n^2}\ z}{4+\frac{6}{n}+\frac{2}{n}}\right|$$

$\rho=\lim_{n\to\infty}\rho_n=\left|\frac{z}{4}\right|$ so $\left|\frac{z}{4}\right|<1$ or $|z|<4$. So circle with radius $R=\frac{1}{\rho}=4$.

i.e. all z points (inside) circle with radius $=4$, centered at $(0,0)$.

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ch 2 [8.1] show from using power series for e^z that

$$e^{z_1} e^{z_2} = e^{z_1+z_2}$$

method of solution: express e^{z_1}, e^{z_2} in power series, Then do long multiplication, then collect all terms.

$$e^{z_1}e^{z_2} = \left(1+\frac{z_1}{1!}+\frac{z_1^2}{2!}+\frac{z_1^3}{3!}+\cdots\right)\left(1+\frac{z_2}{1!}+\frac{z_2^2}{2!}+\frac{z_2^3}{3!}+\cdots\right)$$

$$= 1 + z_2 + \frac{z_2^2}{2!} + \frac{z_2^3}{3!} + \cdots$$

$$z_1 + z_1 z_2 + \frac{z_1 z_2^2}{2!} + \frac{z_1 z_3^3}{3!} + \cdots$$

$$\frac{z_1^2}{2!} + \frac{z_1^2 z_2}{2!} + \frac{z_1^2 z_2^2}{2!\,2!} + \frac{z_1^2 z_2^3}{2!\,3!} + \cdots$$

$$\frac{z_1^3}{3!} + \frac{z_1^3 z_2}{3!} + \frac{z_1^3 z_2^2}{3!\,2!} + \frac{z_1^3 z_2^3}{3!\,3!} + \cdots$$

etc.

Now add each column, we get

$$1 + (z_2+z_1) + \left(\frac{z_2^2}{2} + z_1 z_2 + \frac{z_1^2}{2}\right) + \left(\frac{z_2^3}{3!} + \frac{z_1 z_2^2}{2!} + \frac{z_1^2 z_2}{2!} + \frac{z_1^3}{3!}\right)$$

First col, Second col, Third column, Fourth column.

etc.. but above can be reduced to:

$$= 1 + (z_2+z_1) + \frac{(z_1+z_2)^2}{2} + \frac{(z_1+z_2)^3}{3!} + \cdots$$

$\Rightarrow$

(18)

in column 2 gives $(z_2 + z_1)$

column 3 gives $\frac{(z_1+z_2)^2}{2}$

column 4 gives $\frac{(z_1+z_2)^3}{3!}$

column n gives $\frac{(z_1+z_2)^{n-1}}{(n-1)!}$

but this is the same as $e^{(z_1+z_2)}$

hence $\boxed{e^{z_1} e^{z_2} = e^{z_1+z_2}}$

8.2 Show from power series that $\frac{d}{dz} e^z = e^z$

$$\frac{d}{dz}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right)$$

$$= \frac{d}{dz}1 + \frac{d}{dz}z + \frac{d}{dz}\frac{z^2}{2!} + \cdots$$

$$= 0 + 1 + \frac{2z}{2!} + \frac{3z^2}{3\times 2} + \frac{4z^3}{4\times 3\times 2} + \cdots$$

$$= 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots = e^z.$$

3.4 HW 3

HW # 3

Math 121 A

NASSER ABBASI

UCB extension.

2/2

①

ch 2

9.2 express in form $x + iy$.

$e^{i\frac{\pi}{2}}$

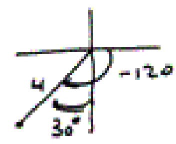
hence $x = 0$

$y = 1$

so $e^{i\frac{\pi}{2}} = \boxed{0 + i}$

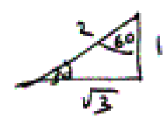
9.12 $4e^{-\frac{8}{3}i\pi}$

length $= 4$ angle $-\frac{8}{3}\pi = -120^\circ$



so $x = -4\cos 60^\circ = -4\left(\frac{1}{2}\right) = -2$

$y = -4\cos 30^\circ = -4\frac{\sqrt{3}}{2} = -2\sqrt{3}$

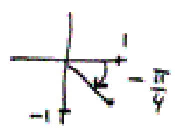


so $z = \boxed{-2 - 2\sqrt{3}\,i}$

9.19 $z_1 = (1 - i)^8$

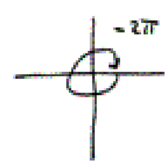
$z = 1 - i \Rightarrow r = \sqrt{2}$

$\theta = -\frac{\pi}{4}$



so $z = \sqrt{2}\,e^{-\frac{\pi}{4}i}$

so $z^8 = \left(\sqrt{2}\,e^{-\frac{\pi}{4}i}\right)^8 = 2^4 e^{-2\pi i} = 16\,e^{-2\pi i}$



so $x = 16$

$y = 0$

i.e. $\boxed{16 + 0i}$

(2)

Ch 2

9.24 express in $x+iy$ form

$$\frac{(1-i\sqrt{3})^{21}}{(i-1)^{38}}$$

$z_1 = 1-i\sqrt{3}$

so $r = \sqrt{1+3} = 2$

$\theta = -60^\circ = -\frac{\pi}{3}$

$z_1 = 2e^{-\frac{\pi}{3}i}$

$z_2 = i-1$

so $r_2 = \sqrt{2}$

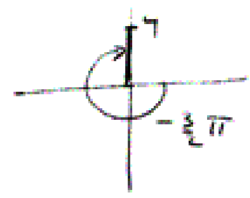
$\theta_2 = 90^\circ + 45^\circ = \frac{3}{4}\pi$

so $z_2 = \sqrt{2}\, e^{\frac{3}{4}i\pi}$

so $z = \frac{z_1^{21}}{z_2^{38}} = \frac{(2e^{-\frac{\pi}{3}i})^{21}}{(\sqrt{2}\,e^{\frac{3}{4}i\pi})^{38}} = \frac{2^{21}\, e^{-\frac{21}{3}i\pi}}{2^{19}\, e^{\frac{3}{2}\pi i(19)}}$

$= 2^2\, e^{-7i\pi - \frac{57}{2}\pi i} = 4\, e^{\left(\frac{-14-57}{2}\right)i\pi} = 4\, e^{-\frac{71}{2}i\pi}$

$= 4\, e^{-35\frac{1}{2}i\pi} = 4\, e^{(-34\pi - \frac{3}{2}\pi)i} = 4\, e^{-\frac{3}{2}\pi i}$



so $x = 0$

$y = 4$

i.e. number is $\boxed{0+4i}$

③

ch 2

9.27 show that for any real y, $|e^{iy}|=1$. hence show that $|e^{z}|=e^{x}$ for every complex z.

$$|e^{iy}| = |\cos y + i\sin y| = \sqrt{\cos^2 y + \sin^2 y} = \sqrt{1} = 1$$

$e^{z} = e^{x+iy}$ where $z=x+iy$.

hence $|e^{z}| = |e^{x+iy}| = |e^{x} e^{iy}| = |e^{x}|\,|e^{iy}|$

but $|e^{x}| = e^{x}$ since real x.

and $|e^{iy}| = 1$ from above.

hence $|e^{z}| = e^{x}$

9.28 show that absolute value of a product of two complex numbers is equal to the product of the abs values.

let the two complex numbers be z_1, z_2

we need to show that $|z_1 z_2| = |z_1|\,|z_2|$.

write z as $re^{i\theta}$.

so $|z_1 z_2| = |r_1 e^{i\theta_1}\, r_2 e^{i\theta_2}| = |r_1 r_2\, e^{i(\theta_1+\theta_2)}|$

$= r_1 r_2$ since this is the length of $z_1 z_2$.

$|z_1|\,|z_2| = |r_1 e^{i\theta_1}|\,|r_2 e^{i\theta_2}| = r_1 r_2$

(with $|r_1 e^{i\theta_1}| \to r_1$ and $|r_2 e^{i\theta_2}| \to r_2$)

hence $|z_1 z_2| = |z_1|\,|z_2|$.

Now, show that abs value of quotient of two complex numbers is the quotient of the abs. values →

4

we need to show that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

$$\left|\frac{z_1}{z_2}\right| = \left|\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right| = \left|\frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}\right| = \frac{r_1}{r_2}$$

$$\frac{|z_1|}{|z_2|} = \frac{|r_1 e^{i\theta_1}|}{|r_2 e^{i\theta_2}|} = \frac{r_1}{r_2} \qquad \text{QED.}$$

10.18 Find all values of roots and plot them

$\sqrt{i}$

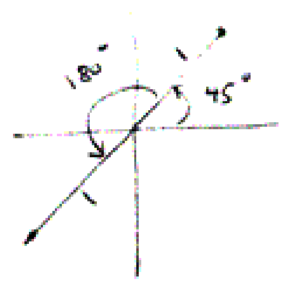
$\theta = \frac{\pi}{2}$, $r = 1$

so $z = e^{i\frac{\pi}{2}}$

so $z^{1/2} = 1^{1/2} \left(e^{i\frac{\pi}{2}}\right)^{1/2} = e^{i\left(\frac{\pi}{2} + 2\pi k\right)\frac{1}{2}}$ $\qquad k = 0, 1$

so roots $= e^{i\left(\frac{\pi}{4}\right)}, \; e^{i\left(\frac{\pi}{4} + \frac{2\pi}{2}\right)}$

$= e^{i\frac{\pi}{4}}, \; e^{i\left(\frac{5}{4}\pi\right)}$



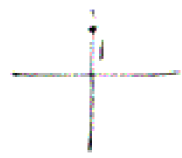
we need to show that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ (4)

$$\left|\frac{z_1}{z_2}\right| = \left|\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right| = \left|\frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}\right| = \frac{r_1}{r_2}$$

$$\frac{|z_1|}{|z_2|} = \frac{|r_1 e^{i\theta_1}|}{|r_2 e^{i\theta_2}|} = \frac{r_1}{r_2} \qquad \text{QED.}$$

10.18 Find all values of roots and plot them

$\sqrt{i}$



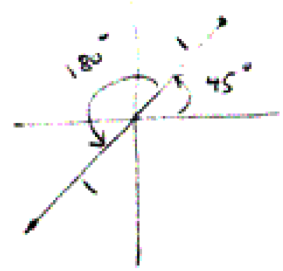
$\theta = \frac{\pi}{2}$, $r = 1$

so $z = e^{i\frac{\pi}{2}}$

so $z^{1/2} = i^{1/2} (e^{i\frac{\pi}{2}})^{1/2} = e^{i\left(\frac{\pi}{2} + 2\pi k\right)\frac{1}{2}}$ $k = 0, 1$

so roots $= e^{i\left(\frac{\pi}{4}\right)}$, $e^{i\left(\frac{\pi}{4} + \frac{2\pi}{2}\right)}$

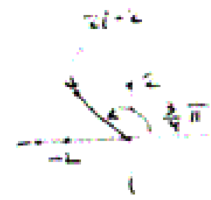
$= e^{i\frac{\pi}{4}}$, $e^{i\left(\frac{5}{4}\pi\right)}$



(5)

Ch 2
[10.22] Find roots of

$\sqrt[3]{2i-2}$



$r=\sqrt{2^2+2^2}=2\sqrt{2}$

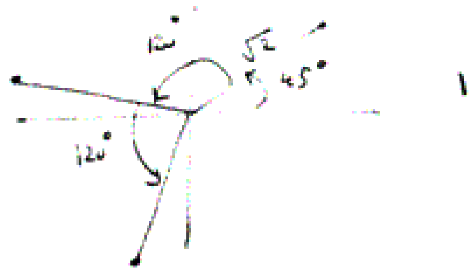
$\theta=\frac{3}{4}\pi$

so $\left(2\sqrt{2}\, e^{i\left(\frac{3}{4}\pi\right)}\right)^{1/3} = \sqrt{2}\, e^{i\left(\frac{3}{4}\pi+2\pi k\right)^{1/3}}$ $k=0,1,2$

So roots

$\sqrt{2}\, e^{i\left(\frac{1}{4}\pi\right)}$, $\sqrt{2}\, e^{i\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)}$, $\sqrt{2}\, e^{i\left(\frac{\pi}{4}+\frac{4\pi}{3}\right)}$

each root is $\frac{2\pi}{3}=120^\circ$ away from previous root.



Ch 2

10.28 Find formula for $\sin 3\theta$ and $\cos 3\theta$.

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta.$$

so put $n=3$ here.

$$\begin{aligned}
\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\
&= (\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta) \\
&= (\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta)(\cos\theta + i\sin\theta) \\
&= \cos^3\theta + i\cos^2\theta\sin\theta - \sin^2\theta\cos\theta - i\sin^3\theta \\
&\quad + 2i\cos^2\theta\sin\theta - 2\cos\theta\sin^2\theta \\
&= \cos^3\theta - 3\sin^2\theta\cos\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)
\end{aligned}$$

so by equating real parts to real parts and imaginary parts to imaginary parts we get

$$\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$$

$$\sin 3\theta = -\sin^3\theta + 3\cos^2\theta\sin\theta$$

(7)

ch 2

11.5 Find in $x+iy$ form

$$z = e^{i\frac{\pi}{4} + \frac{\ln 2}{2}} = \left(e^{\ln 2}\right)^{1/2} e^{i\frac{\pi}{4}}$$

but $e^{\ln x} = x$

so $z = 2^{1/2} e^{i\frac{\pi}{4}} = \sqrt{2}\, e^{i\frac{\pi}{4}}$

so $x = \sqrt{2}\cos\frac{\pi}{4} = \sqrt{2}\,\frac{1}{\sqrt{2}} = 1$

$y = \sqrt{2}\sin\frac{\pi}{4} = \sqrt{2}\,\frac{1}{\sqrt{2}} = 1$

so $z = \boxed{1+i}$

Ch 2
11.6 Find in $x+iy$ form

$\sin i$

since $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

Then $\sin i = \dfrac{e^{i^2} - e^{-i^2}}{2i} = \dfrac{e^{-1} - e^{1}}{2i}$

$= \dfrac{\frac{1}{e} - e}{2i} \dfrac{(-2i)}{(-2i)} = \dfrac{-2i\left(\frac{1}{e} - e\right)}{4}$

$= -\frac{i}{2}\left(\frac{1}{e} - e\right)$

so $x = 0$
$y = -\frac{1}{2}\left(\frac{1}{e} - e\right) \approx 1.1752$

so $\sin i = \boxed{1.1752\, i}$

ch 2 4.4 (9)

$$\int_{-\pi}^{\pi} \cos 2x \cos 3x \, dx$$

$$= \int_{-\pi}^{\pi} \left(\frac{e^{2xi}+e^{-2xi}}{2}\right)\left(\frac{e^{3xi}+e^{-3xi}}{2}\right) dx$$

$$= \frac{1}{4}\int_{-\pi}^{\pi} e^{5xi} + e^{-xi} + e^{xi} + e^{-5xi} \, dx$$

$$= \frac{1}{4}\left(\int_{-\pi}^{\pi} e^{5xi}\,dx + \int_{-\pi}^{\pi} e^{-xi}\,dx + \int_{-\pi}^{\pi} e^{xi}\,dx + \int_{-\pi}^{\pi} e^{-5xi}\,dx\right)$$

but $\int_{-\pi}^{\pi} e^{nxi}\,dx = \frac{1}{ni}\left[e^{nxi}\right]_{-\pi}^{\pi} = \frac{1}{ni}\left[e^{n\pi i} - e^{-n\pi i}\right]$

and $e^{n\pi i} = \cos n\pi + i\sin n\pi$ ($\sin n\pi = 0$)

and $e^{-n\pi i} = \cos -n\pi + i\sin -n\pi = \cos n\pi - i\sin n\pi$ ($\sin n\pi = 0$)

So $e^{n\pi i} = e^{-n\pi i}$

So $\int_{-\pi}^{\pi} e^{nxi}\,dx = 0$ for any integer $n \neq 0$.

hence $\frac{1}{4}\left(\int_{-\pi}^{\pi} e^{5xi}\,dx + \cdots\cdots\right) = \frac{1}{4}(0+0+0+0) = 0$

(10)

Ch 2

11.18 evaluate $\int e^{(a+ib)x}\,dx$ to show that

$$\int e^{ax}\sin bx\,dx = e^{ax}\frac{(a\sin bx - b\cos bx)}{a^2+b^2}$$

$$\int e^{ax}\sin bx\,dx = \int e^{ax}\left(\frac{e^{bix}-e^{-bix}}{2i}\right)dx$$

$$= \frac{1}{2i}\int e^{ax}e^{bix} - e^{ax}e^{-bix}\,dx$$

$$= \frac{1}{2i}\int e^{x(a+bi)} - e^{x(a-bi)}\,dx = \frac{1}{2i}\left[\frac{1}{a+bi}e^{x(a+bi)} - \frac{1}{a-bi}e^{x(a-bi)}\right]$$

$$= \frac{1}{2i}\left[\frac{(a-bi)e^{x(a+bi)} - (a+bi)e^{x(a-bi)}}{(a+bi)(a-bi)}\right]$$

$$= \frac{1}{2i}\left[\frac{(a-bi)e^{ax}e^{xib} - (a+bi)e^{ax}e^{-xbi}}{a^2+b^2}\right]$$

$$= \frac{e^{ax}}{a^2+b^2}\left[\frac{(a-bi)e^{xib} - (a+bi)e^{-xbi}}{2i}\right]$$

$$= \frac{e^{ax}}{a^2+b^2}\left[\frac{ae^{xib} - bie^{xib} - ae^{-xbi} - bie^{-xbi}}{2i}\right]$$

$$= \frac{e^{ax}}{a^2+b^2}\left[\frac{a(e^{ixb}-e^{-xbi})}{2i} - \frac{bi(e^{xib}+e^{-xbi})}{2i}\right]$$

$$= \frac{e^{ax}}{a^2+b^2}\left[a\sin xb - b\cos x\right]$$

ch 2

14.6 Find one value of following in $x+iy$ form

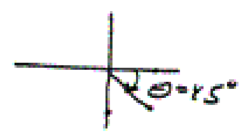
$$\ln\left(\frac{1-i}{\sqrt{2}}\right)$$

$\ln w$ is a multivalued function. we are asked to find one value.

first express $\frac{1-i}{\sqrt{2}}$ in polar.

$z = \frac{1-i}{\sqrt{2}}$, $r = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2}+\frac{1}{2}} = \sqrt{\frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$

$\theta = -\frac{\pi}{4}$.



so $\ln\left(\frac{1}{2}\sqrt{2}\, e^{-i\frac{\pi}{4}}\right)$

$= \operatorname{Ln}\frac{\sqrt{2}}{2} + \ln e^{-i\left(\frac{\pi}{4}\right)}$

$= \operatorname{Ln}\frac{\sqrt{2}}{2} - i\left(\frac{\pi}{4} \pm 2n\pi\right)$

So $\ln w = \operatorname{Ln}\frac{\sqrt{2}}{2} - i\frac{\pi}{4}$, $\operatorname{Ln}\frac{\sqrt{2}}{2} - i\frac{9}{4}\pi$, $\operatorname{Ln}\frac{\sqrt{2}}{2} - i\frac{17}{4}\pi$, etc...

pick the first one

$$\ln\left(\frac{1-i}{\sqrt{2}}\right) = \boxed{\operatorname{Ln}\frac{\sqrt{2}}{2} - i\frac{\pi}{4}}$$

ch 2 [14.9] Find the value of $(-1)^i$ in the form $x+iy$.

12

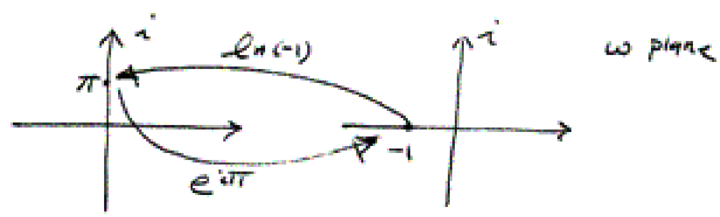
$(-1)^i = e^{i\ln(-1)}$

since $e^{i\ln(-1)} = (e^{\ln(-1)})^i = (-1)^i$

so $(e^{\ln(-1)})^i = (e^{\pi i})^i = e^{-\pi} = -1$??

notice that $\ln(-1)$ is not defined in the real line.

but in complex plane, $\ln(-1) = i\pi$ (or $i(\pi \pm 2n\pi)$)



This is because $e^{i\pi} = e^{z} = e^{0+i\pi} = e^{0} e^{i\pi}$

$= 1 (\cos\pi + i\sin\pi) = -1$

So since $e^{i\pi} = -1$ Then by definition, $\ln(-1) = i\pi$

so $(-1)^i = e^{i\ln(-1)} = e^{i(i(\pi \pm 2n\pi))} = e^{-(\pi \mp 2n\pi)}$

$= e^{-\pi}$ or $e^{-3\pi}$ or $e^{-4\pi}, \ldots$ or $e^{-\pi}, e^{\pi}, e^{3\pi}, \ldots$

↓

pick this $= \boxed{-1}$??

see next page

let $\omega = (-1)^i$

so $\ln(\omega) = \ln(-1)^i$

$= i \ln(-1)$

$= i \ln\left(e^{i(\pi \pm 2\pi k)}\right)$

~~$= i \ln(e^{i})$~~

$\ln(\omega) = i\left(i(\pi \pm 2\pi k)\right)$

$\ln(\omega) = -(\pi \pm 2\pi k)$

so $\omega = e^{-(\pi \pm 2\pi k)}$

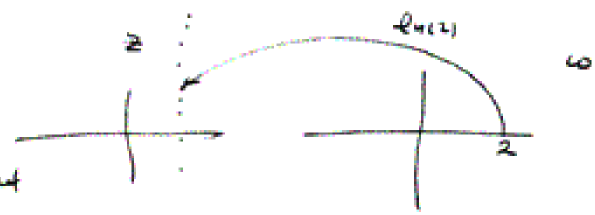
so $\omega = e^{-\pi}$ or $e^{-\pi+2\pi}$ or $e^{-\pi-2\pi}$ or ...

$\boxed{\omega = e^{-\pi} \text{ or } e^{\pi} \text{ or } e^{-3\pi} \text{ or } e \cdots}$

(13)

ch 2 express in $x+iy$:

p 14.14 $\cos(\pi + i\ln 2)$



first find $\ln(2)$, a multivalued function in complex plane.

let $z = \ln(2)$.

$$e^z = e^{\ln(2)} = e^{\ln 2 e^{i(0\pm 2n\pi)}} = e^{\text{Ln}(2) + \ln e^{i(0\pm 2n\pi)}}$$

so $z = \text{Ln}(2) + i(0 \pm 2n\pi)$

$$\text{so } \cos(\pi + i\ln 2) = \cos(\pi + i(\text{Ln}(2) + i(0\pm 2n\pi)))$$

$$= \cos(\pi + i\,\text{Ln}(2) - (0 \pm 2n\pi)) \qquad n = 0,1,2,\dots$$

$$= \cos(\pi \mp 2n\pi + i\,\text{Ln}(2))$$

$$= \cos(\pi(1\mp 2n) + i\,\text{Ln}(2)) \qquad n = 0,1,2,\dots$$

let $z = \pi(1\mp 2n) + i\,\text{Ln}(2)$.

$$\text{so } \cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{i(\pi(1\mp 2n) + i\,\text{Ln}(2))} + e^{-i(i\,\text{Ln}(2) + \pi(1\mp 2n))}}{2}$$

$$= \frac{e^{i\pi(1\mp 2n)}e^{-\text{Ln}(2)} + e^{-i(\pi(1\mp 2n))}e^{\text{Ln}(2)}}{2}$$

now $e^{i\pi(1\mp 2n)} = \cos\pi(1\mp 2n) + i\sin(\pi(1\mp 2n))$ $\qquad n = 0,1,2,\dots$

$= -1$ for any n

also $e^{-i(\pi(1\mp 2n))} = \cos -\pi(1\mp 2n) - i\sin(\pi(1\mp 2n))$

$= \cos\pi(1\mp 2n) - i\sin(\pi(1\mp 2n)) = -1$ for any n

$$\text{so } \cos z = \frac{-e^{-\text{Ln}(2)} - e^{\text{Ln}(2)}}{2} = -\frac{1}{2}\left(e^{\text{Ln}(2)} + e^{-\text{Ln}(2)}\right)$$

$$= \boxed{-1.25} \text{ or } \boxed{-\frac{5}{4}} \Rightarrow [illegible]$$

14

ch 2

14.22 find in $x+iy$ form

$$\sin\left(i \ln\left(\frac{\sqrt{3}+i}{2}\right)\right)$$

let $z = \frac{\sqrt{3}+i}{2}$. $|z| = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1$

so $z = 1\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$ i.e. $\cos\theta = \frac{\sqrt{3}}{2}$, $\sin\theta = \frac{1}{2}$

so $\theta = 30^\circ = \frac{\pi}{6}$

so $z = 1 e^{i\frac{\pi}{6}}$

so $e^{\ln(1\cdot e^{i\frac{\pi}{6}})} = e^{\text{Ln}(1) + \ln(e^{i\pi/6})} = e^{\text{Ln}(1) + i(\pi/6 \pm 2n\pi)}$

so $\sin\left(i \ln\left(\frac{\sqrt{3}+i}{2}\right)\right) = \sin\left(i\left(\text{Ln}(1) + i(\pi/6 \pm 2n\pi)\right)\right)$

$\text{Ln}(1) = 0$

so $= \sin(-(\pi/6 \pm 2n\pi))$ $n = 0, 1, 2, \ldots$ $-1, -2, \ldots$

for $n=0$

$= \sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = \boxed{-\frac{1}{2}}$

or $\boxed{-\frac{1}{2} + 0\,i}$

ch 2 15

14.24 (b) Show that $(a^b)^c$ can have more values than a^{bc}.

$(i^i)^i$ and i^{-1}

so need to show that $(i^i)^i$ can have more values than i^{-1}.

$$(i^i)^i = \left(\left(e^{\ln i}\right)^i\right)^i = \left(\left(e^{\ln 1 e^{i(\frac{\pi}{2} \pm 2n\pi)}}\right)^i\right)^i$$

$$= \left(\left(e^{\ln(1) + \ln e^{i(\frac{\pi}{2} \pm 2n\pi)}}\right)^i\right)^i \qquad n = 0, 1, 2, \ldots$$

$$= \left(\left(e^{\ln(1)}\, e^{\ln e^{i(\pi/2 \pm 2n\pi)}}\right)^i\right)^i$$

$$= \left(\left(e^{i(\frac{\pi}{2} \pm 2n\pi)}\right)^i\right)^i = \left(e^{-(\frac{\pi}{2} \pm 2n\pi)}\right)^i$$

$$= e^{-i(\frac{\pi}{2} \pm 2n\pi)}$$

$$= \cos\left(\frac{\pi}{2} \pm 2n\pi\right) - i \sin\left(\frac{\pi}{2} \pm 2n\pi\right)$$

but $\cos\left(\frac{\pi}{2} \pm 2n\pi\right) = 0$ for any n.

so $(i^i)^i = -i \sin\left(\frac{\pi}{2} \pm 2n\pi\right)$

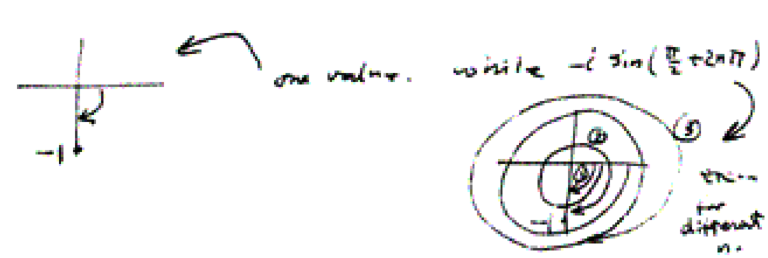
which is $-i\sin\left(\frac{\pi}{2}\right)$, $-i\sin\left(\frac{\pi}{2} + 2\pi\right)$, $-i\sin\left(\frac{\pi}{2} + 4\pi\right), \ldots$

$-i\sin\left(\frac{\pi}{2} - 2\pi\right)$, $-i\sin\left(\frac{\pi}{2} - 4\pi\right), \ldots$

while $i^{-1} = -i\sin\left(\frac{\pi}{2}\right)$

This can be seen better if plotted.

$i^{-1} = -i$



(16

ch 2
15.3 Find in $x+iy$ form.

$\cosh^{-1}(1/2)$.

$\cosh z = \frac{1}{2}$

$\frac{e^z+e^{-z}}{2} = \frac{1}{2} \Rightarrow e^z + e^{-z} = 1$

let $u = e^z \Rightarrow u + u^{-1} = 1$ multiply by u

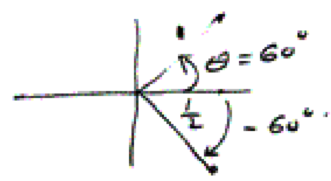
$u^2+1-u=0$

$u^2-u+1=0 \Rightarrow u = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ where $a=1$, $b=-1$, $c=1$

so $u = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

$|u| = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$

$u = 1\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\right)$ $\theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$



so $z = \ln(1) + i\left(\pm\frac{\pi}{3} \pm 2n\pi\right)$

$= \ln(1) \cdot i\left(\pm\frac{\pi}{3} \pm 2n\pi\right) = i\left(\pm\frac{\pi}{3} + 2n\pi\right)$ $n=0,1,2,\ldots$

so $\cosh^{-1}(z)$ is multivalued function. like $\ln(z)$ is.

(17

Ch. 2 show that $\tan z$ can never take value of $\pm i$

assume $\tan z = i$

i.e. $\frac{\sin z}{\cos z} = i$

i.e. $$\frac{\frac{e^{iz}-e^{-iz}}{2i}}{\frac{e^{iz}+e^{-iz}}{2}} = i$$

i.e. $$\frac{-i\left(e^{iz}-e^{-iz}\right)}{e^{iz}+e^{-iz}} = i$$

i.e. $$\frac{-e^{iz}+e^{-iz}}{e^{iz}+e^{-iz}} = 1$$

i.e. $-e^{iz}+e^{-iz}-e^{iz}-e^{-iz}=0$

i.e. $-2e^{iz}=0$

$e^{iz}=0$

$\Rightarrow$ iz has no value which can make $e^{iz}=0$.

$e^{x}=0$ has no solution.

Hence $\tan z$ can not be i.

if I let $\tan z = -i$, then my difference is that I get

$e^{iz}-e^{-iz}-e^{iz}-e^{-iz}=0$

or $-2e^{-iz}=0$

or $e^{-iz}=0$ $\Longrightarrow$ again, not possible

QED

ch 2
16.11

prove that $\cos\theta + \cos 3\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}$

and $\sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$.

write $e^{i\theta} + e^{i3\theta} + \cdots + e^{i(2n-1)\theta} = \frac{a(1-r^n)}{1-r}$

where $a = e^{i\theta}$, $r = e^{i2\theta}$, and use Euler relationship.

$$(\cos\theta + i\sin\theta) + (\cos 3\theta + i\sin 3\theta) + \cdots (\cos(2n-1)\theta + i\sin(2n-1)\theta) = \frac{e^{i\theta}(1-e^{i2n\theta})}{1-e^{i2\theta}}$$

so $(\cos\theta + \cos 3\theta + \cdots + \cos(2n-1)\theta) + i(\sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta) =$

looking at RHS: denominator is:

$$1-e^{i2\theta} = \underbrace{e^{i\theta}e^{-i\theta}}_{1} - \underbrace{e^{i\theta}e^{i\theta}}_{e^{i2\theta}} = e^{i\theta}(e^{-i\theta} - e^{i\theta})$$

$$= \frac{(2i)}{(2i)} e^{i\theta}(e^{-i\theta} - e^{i\theta})$$

$$= (2i)e^{i\theta}\frac{(e^{-i\theta} - e^{i\theta})}{2i}$$

$$= (-2i)e^{i\theta}\frac{(e^{i\theta} - e^{-i\theta})}{2i} = (-2i)e^{i\theta}\sin\theta$$

numerator is: $e^{i\theta}(1-e^{i2n\theta}) = e^{i\theta}\left[\underbrace{e^{in\theta}e^{-in\theta}}_{1} - \underbrace{(e^{in\theta}e^{in\theta})}_{e^{i2n\theta}}\right]$

$$= e^{i\theta}\left[e^{in\theta}(e^{-in\theta} - e^{in\theta})\right]$$

$$= e^{i\theta}\left[\frac{(-2i)}{(2i)}e^{in\theta}(e^{-in\theta} - e^{in\theta})\right] = e^{i\theta}\left[(-2i)e^{in\theta}\frac{(e^{in\theta} - e^{-in\theta})}{2i}\right]$$

$$= e^{i\theta}\left[-2i\,e^{in\theta}\sin n\theta\right]$$

$$\text{Hence RHS} = \frac{e^{i\theta}\left[-2i\,e^{in\theta}\sin n\theta\right]}{-2i\,e^{i\theta}\sin\theta} = e^{in\theta}\frac{\sin n\theta}{\sin\theta}$$

$$= (\cos n\theta + i\sin n\theta)\frac{\sin n\theta}{\sin\theta} = \frac{\cos n\theta\sin n\theta}{\sin\theta} + i\frac{\sin n\theta\sin n\theta}{\sin\theta} \Rightarrow$$

now equate real parts and imaginary parts ⇒

$$\cos\theta + \cos 3\theta + \cdots + \cos(2n-1)\theta = \frac{\cos n\theta \sin n\theta}{\sin\theta} \quad (1)$$

and $$\sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta = \frac{\sin n\theta \sin n\theta}{\sin\theta} \quad (2)$$

$$\sin n\theta \cos n\theta = \frac{1}{2}\sin 2n\theta$$

so (1) becomes $$\boxed{\cos\theta + \cos 3\theta + \cdots = \frac{\sin 2n\theta}{2\sin\theta}}$$

and (2) is $$\sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$$

Ch 2

17.14 Find the circle of convergence of series

$$\sum \frac{(z-2i)^n}{n}$$

$$\rho_n = \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(z-2i)^{n+1}}{n+1}}{\frac{(z-2i)^n}{n}}\right| = \left|\frac{(z-2i)\,n}{n+1}\right|$$

$$\rho = \lim_{n\to\infty} \rho_n = |z-2i|$$

so converges for $|z-2i| < 1$

let $z = x+iy$.

then we want $|x+iy-2i| < 1$

$$|x+i(y-2)| < 1$$

$$\sqrt{x^2+(y-2)^2} < 1$$

$$x^2+(y-2)^2 < 1$$

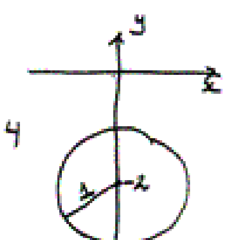
$$x^2+y^2+4-4y < 1$$

$$x^2+y^2-4y < -3 \quad (1)$$

equation of circle can be written as

$$(x-x_0)^2+(y-y_0)^2 = r^2$$

So x^2+y^2-4y can be written as $(x-0)^2+(y-2)^2-4$



so (1) becomes

$$(x-0)^2+(y-2)^2-4 < -3$$

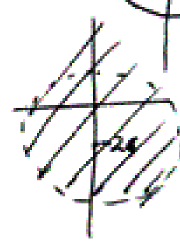
$$(x-0)^2+(y-2)^2 < +1$$

$(0,+2)$

So center is $(0,-2)$

and radius $r^2 < +1$

$\Rightarrow r < 1$



ch 2

17.17 verify $\arcsin z = -i \ln\left(iz \pm \sqrt{1-z^2}\right)$

let $\arcsin z = w$

so $\sin w = z$

$$z = \frac{e^{iw} - e^{-iw}}{2i}$$ ✓

now need to find w in terms of to get answer required.

e^{iw} is a complex number. let $e^{iw} = \alpha$

so $z = \dfrac{\alpha - \alpha^{-1}}{2i} = \dfrac{\alpha^2 - 1}{2i\alpha}$ ✓

so $z(2i\alpha) = \alpha^2 - 1$

$\alpha^2 - 1 - \alpha(2iz) = 0$ ✓ $\Rightarrow \alpha = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{2iz \pm \sqrt{-4z^2 - 4(-1)}}{2}$$

$$= \frac{2iz \pm 2\sqrt{1-z^2}}{2}$$

so $\alpha = iz \pm \sqrt{1-z^2}$

but $\alpha = e^{iw}$

so $e^{iw} = iz \pm \sqrt{1-z^2}$ ✓

so $\ln e^{iw} = \ln\left(iz \pm \sqrt{1-z^2}\right)$

$iw = \ln\left(iz \pm \sqrt{1-z^2}\right)$

$w = \frac{1}{i} \ln\left(iz \pm \sqrt{1-z^2}\right)$ ✓

$$\boxed{w = -i \ln\left(iz \pm \sqrt{1-z^2}\right)}$$ ✓

ch 2
17.23 Verify $\cos iz = \cosh z$.

$$\cos iz = \frac{e^{i(iz)} + e^{-i(iz)}}{2} = \frac{e^{-z} + e^{+z}}{2} = \cosh(z).$$

17.24 verify $\cosh iz = \cos z$.

$$\cosh(iz) = \frac{e^{-(iz)} + e^{(iz)}}{2} = \frac{e^{iz} + e^{-iz}}{2} = \cos(z)$$

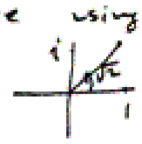
17.30 write series for $e^{x(1+i)}$. Write $1+i$ in the $re^{i\theta}$ form and obtain the powers of $(1+i)$. Then show for example that $e^x \cos x$ series has no x^2 term, no x^6 term, etc. and a similar result for $e^x \sin x$. Find a formula for the general term for each series.

power series for $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

so $e^{x(1+i)} = 1 + (x+ix) + \frac{(x+ix)^2}{2!} + \frac{(x+ix)^3}{3!} + \cdots$

but I need to rewrite using $e^{i\theta}$. so above is not useful.

$(1+i) = \sqrt{2}\, e^{i\frac{\pi}{4}}$



so $e^{x(1+i)} = e^{x\sqrt{2}e^{i\frac{\pi}{4}}} = 1 + x\sqrt{2}e^{i\frac{\pi}{4}} + \frac{x^2(\sqrt{2}e^{i\frac{\pi}{4}})^2}{2!} + \cdots$

$$= 1 + x\sqrt{2}e^{i\frac{\pi}{4}} + \frac{x^2}{2!}\,2e^{i\frac{\pi}{2}} + \frac{x^3}{3!}\,2\sqrt{2}\,e^{i\frac{3}{4}\pi} + \cdots \frac{(x\sqrt{2}e^{i\frac{\pi}{4}})^n}{n!}$$

now $e^{x(1+i)} = e^{x}e^{xi} = e^{x}(\cos x + i\sin x)$

So $e^{x}(\cos x + i \sin x) = \sum_{n=0}^{\infty} \frac{(x\sqrt{2}e^{i\frac{\pi}{4}})^n}{n!}$ $\Longrightarrow$

ch 2

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$$\cos iz = \frac{e^{i(iz)} + e^{-i(iz)}}{2} = \frac{e^{-z} + e^{+z}}{2} = \cosh(z).$$

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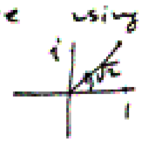


so $e^{x(1+i)} = e^{x\sqrt{2}e^{i\frac{\pi}{4}}} = 1 + x\sqrt{2}e^{i\frac{\pi}{4}} + \frac{x^2(\sqrt{2}e^{i\frac{\pi}{4}})^2}{2!} + \cdots$

$$= 1 + x\sqrt{2}e^{i\frac{\pi}{4}} + \frac{x^2}{2!}\, 2e^{i\frac{\pi}{2}} + \frac{x^3}{3!}\, 2\sqrt{2}\, e^{i\frac{3}{4}\pi} + \cdots \frac{(x\sqrt{2}e^{i\frac{\pi}{4}})^n}{n!}$$

now $e^{x(1+i)} = e^x e^{xi} = e^x(\cos x + i\sin x)$

So $e^x(\cos x + i\sin x) = \sum_{n=0}^{\infty} \frac{(x\sqrt{2}e^{i\frac{\pi}{4}})^n}{n!}$ $\Longrightarrow$

$$e^x \cos x + i\, e^x \sin x = \sum_0^\infty \frac{x^n\, 2^{\frac{n}{2}} \left(e^{i\frac{n\pi}{4}}\right)}{n!}$$

$$e^x \cos x + i\, e^x \sin x = \sum_0^\infty \frac{x^n\, 2^{\frac{n}{2}}}{n!} \left(\cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4}\right)$$

$$= \sum_0^\infty \left(\frac{(x\sqrt{2})^n}{n!} \cos \frac{n\pi}{4}\right) + i \left(\frac{(x\sqrt{2})^n}{n!} \sin \frac{n\pi}{4}\right)$$

Compare real parts and imaginary parts.

$$\boxed{\begin{aligned} e^x \cos x &= \sum_0^\infty \frac{(x\sqrt{2})^n}{n!} \cos \frac{n\pi}{4} \\ e^x \sin x &= \sum_0^\infty \frac{(x\sqrt{2})^n}{n!} \sin \frac{n\pi}{4} \end{aligned}}$$

$n=2,\ \theta = 90^\circ$
$n=6,\ \theta = 270$
$n=10,\ \theta = 90^\circ + 360$
etc...

now when $n = 2, 6, 10, \ldots$ ~~etc~~, then $\cos \frac{n\pi}{4} = 0$

hence $e^x \cos x$ is represented by series with no $n=2, 6, 10, \ldots$ i.e with no $x^2, x^6, x^{10} \ldots$

Similarly, looking at the $e^x \sin x$ series and asking when will $\sin \frac{n\pi}{4}$ be zero. this happen at $\theta = 0, 180, 360$, etc... i.e at $n = 0, 4, 8, 12 \ldots$

so the $e^x \sin x$ series has no $x^4, x^8, x^{12}, \ldots$ term

QED

Ch 2
17.32 Use series you know to show that

$$\sum_{n=0}^{\infty} \frac{(1+i\pi)^n}{n!} = -e$$

$$
\begin{aligned}
-e &= e(-1)\\
&= e(\cos\pi + i\sin\pi)\\
&= e\, e^{i\pi} \quad \text{, using Euler formula}\\
&= e^{1+i\pi}\\
&= 1 + (1+i\pi) + \frac{(1+i\pi)^2}{2!} + \frac{(1+i\pi)^3}{3!} + \cdots\\
&= \sum_{0}^{\infty} \frac{(1+i\pi)^n}{n!}
\end{aligned}
$$

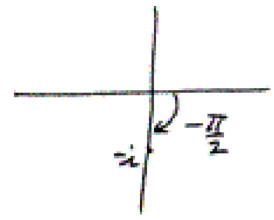
10

Ch 2

17-2 Find one or more values of each of the following complex number

$(-i)^i$

ω plane



$$(-i)^i = \left(e^{\ln(-i)}\right)^i$$

$$= \left(e^{\ln\left(1\cdot e^{i\left(-\frac{\pi}{2}\pm 2n\pi\right)}\right)}\right)^i$$

$$= \left(e^{\ln(1) + \ln\left(e^{i\left(-\frac{\pi}{2}\pm 2n\pi\right)}\right)}\right)^i$$

$$= \left(e^{\ln(1)}\, e^{\ln\left(e^{i\left(-\frac{\pi}{2}\pm 2n\pi\right)}\right)}\right)^i$$

$$= \left((e^0)\left(e^{i\left(-\frac{\pi}{2}\pm 2n\pi\right)}\right)\right)^i$$

$$= e^{-\left(-\frac{\pi}{2}\pm 2n\pi\right)} = e^{\frac{\pi}{2}\mp 2n\pi} \qquad n=0,1,2,\ldots$$

So $(-i)^i = e^{\pi/2}$ or $e^{\pi/2-2\pi}$ or $e^{\pi/2+2\pi}$ or $e^{\pi/2-4\pi}$ or ....

$= e^{\pi/2}$ or $e^{-\frac{3}{2}\pi}$ or $e^{\frac{5}{2}\pi}$ or ...

3.5 HW 4

1/1

HW # 4

Math 121A

Nasser Abbasi

UCB extension

4

ch 4

1.6 for $u = e^x \cos y$

(a) verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

(b) verify that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(a) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\sin y\, e^x\right) = -\sin y\, e^x$

$\frac{\partial^2}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(e^x \cos y\right) = -\sin y\, e^x$ ✓

hence the same.

(b) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(e^x \cos y\right) = e^x \cos y$

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-e^x \sin y\right) = -e^x \cos y$ ✓

so add to give $e^x \cos y - e^x \cos y = 0$

1.7 if $z = x^2 + 2y^2$, $x = r\cos\theta$, $y = r\sin\theta$, find $\left(\frac{\partial z}{\partial x}\right)_y$

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + 2y^2\right) = 2x$

1.23 find $\frac{\partial^2 z}{\partial r \partial \theta} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial \theta}\right) = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial \theta}\left(x^2 + 2y^2\right)\right)$

$= \frac{\partial}{\partial r}\left(\frac{\partial}{\partial \theta}\left(r^2\cos^2\theta + 2r^2\sin^2\theta\right)\right)$

$= \frac{\partial}{\partial r}\left(r^2(-2\cos\theta\sin\theta) + 2r^2(2\sin\theta\cos\theta)\right)$

$= \frac{\partial}{\partial r}\left(-2r^2\cos\theta\sin\theta + 4r^2\sin\theta\cos\theta\right)$

$= -4r\cos\theta\sin\theta + 8r\sin\theta\cos\theta$

$= \boxed{4r\cos\theta\sin\theta}$

(13)

Ch 4

[2.6] Find maclaurin series for e^{x+y}.

$e^{x+y} = e^x e^y$

so can find expansion for exp function for each independent variable x, y and multiply both series.

or I can let $x+y = z$, expand in z, then subtitute back for $z = x+y$.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$= 1 + (x+y) + \frac{(x+y)^2}{2!} + \frac{(x+y)^3}{3!} + \frac{(x+y)^4}{4!} \cdots$$

expand $(x+y)^a$ using binomial:

$$(x+y)^a = x^a + x^{a-1}\binom{a}{1} y + x^{a-2}\binom{a}{2} y^2 + \cdots + y^a$$

$$\text{so } (x+y)^3 = x^3 + \binom{3}{1} x^2 y + \binom{3}{2} x y^2 + \binom{3}{3} x^0 y^3$$

$$= x^3 + \frac{3!}{2!} x^2 y + \frac{3!}{2!} x y^2 + y^3 = x^3 + 3x^2 y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$

$$= x^4 + \frac{4!}{3!} x^3 y + \frac{4!}{2!\,2!} x^2 y^2 + \frac{4!}{1!\,3!} x y^3 + y^4$$

$$= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4$$

$$\text{So } e^z = 1 + (x+y) + \frac{1}{2}\left(x^2 + 2xy + y^2\right) + \frac{1}{6}\left(x^3 + 3x^2 y + 3xy^2 + y^3\right)$$

$$+ \frac{1}{12}\left(x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4\right) + \cdots$$

$$= 1 + (x+y) + \frac{x^2}{2} + xy + \frac{y^2}{2} + \frac{x^3}{6} + \frac{1}{2}x^2 y + \frac{1}{2}xy^2 + \frac{1}{6}y^3$$

$$+ \frac{1}{12}x^4 + \frac{1}{3}x^3 y + \frac{1}{2}x^2 y^2 + \frac{1}{3}xy^3 + \frac{1}{12}y^4 + \cdots \longrightarrow$$

$$= 1 + (x+y) + \left(xy + \frac{x^2}{2} + \frac{y^2}{2}\right) + \left(\frac{x^3}{6} + \frac{1}{2}x^2y + \frac{1}{2}xy^2 + \frac{1}{6}y^3\right)$$
$$+ \left(\frac{1}{12}x^4 + \frac{1}{3}x^3y + \frac{1}{2}x^2y^2 + \frac{1}{3}xy^3 + \frac{1}{12}y^4\right) + \cdots$$

ch 4

1.24 Find $\frac{\partial^2 z}{\partial x \partial y}$ when $z = x^2 + 2y^2$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(4y) = 0$$

2.3 Find a two-variable maclaurin series for

$$\frac{\ln(1+x)}{1+y}$$

this is a function in 2 variables x, y.

expanding in maclaurin series of $f(x,y)$ is (about $z=0$).

expand $\ln(1+x)$: $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad -1 < x \le 1$

expand $(1+y)^{-1}$: $1 + (-1)y + \frac{(-1)(-1-1)}{2!}y^2 + \frac{(-1)(-2)(-3)}{3!}y^3 \qquad |y| < 1$

so we have

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)\left(1 - y + y^2 - y^3 + y^4 - \cdots\right)$$

multiply each term in the x series by the y series:

$$= \left(x - xy + xy^2 - xy^3 + \cdots\right) + \left(-\frac{x^2}{2} + \frac{x^2 y}{2} - \frac{x^2 y^2}{2} + \frac{x^2 y^3}{2} - \cdots\right) + \left(\frac{x^3}{3} - \frac{x^3 y}{3} + \frac{x^3 y^2}{3} \cdots\right)$$

This can be rearranged as an increasing powers of x or y or by increasing powers of xy together. book did not say. I assume the third option:

$$x - \left(xy + \frac{x^2}{2}\right) + \left(xy^2 + \frac{x^3}{3}\right) + \left(-xy^3 - \frac{x^2y^2}{2} - \frac{x^3 y}{3}\right) + \cdots$$

(exponents add to 2) (exponents add to 3) (exponents add to 4)

15

Ch 4

2.8 Find a 2 variable maclaurin series for $e^x \cos y$ and $e^x \sin y$ by finding series for $e^z = e^{x+iy}$ and taking real and imaginary parts.

need to expand $e^z = e^{x+iy}$.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$ From Table page 24

$$= 1 + (x+iy) + \frac{(x+iy)^2}{2} + \frac{(x+iy)^3}{3!} + \cdots$$

$$= 1 + (x+iy) + \frac{1}{2}(x^2 + 2ixy - y^2) + \frac{1}{2\cdot 3}(x^3 + 3ix^2y - 3xy^2 - iy^3) + \cdots$$

Collect real part and imaginary part:

$$= \left(1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{1}{2\cdot 3}x^3 - \frac{3}{2\cdot 3}xy^2 + \cdots\right) + i\left(y + xy + \frac{3}{2\cdot 3}x^2y - \frac{1}{2\cdot 3}y^3 - \cdots\right)$$

$$= \left(1 + x + \frac{1}{2}(x^2 - y^2) + \frac{1}{2\cdot 3}(x^3 - 3xy^2) + \cdots\right) + i\left(y + xy + \frac{1}{2\cdot 3}(3x^2y - y^3) + \cdots\right)$$

Since $e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$
$= e^x \cos y + i\,(e^x \sin y)$

Then $e^x \cos y = \left(1 + x + \frac{1}{2}(x^2 - y^2) + \frac{1}{6}(x^3 - 3xy^2) + \cdots\right)$

and $e^x \sin y = \left(y + xy + \frac{1}{6}(3x^2y - y^3) + \cdots\right)$

6

ch 4

4-2 use differential to show that for large n and small a

$$\sqrt{n+a} - \sqrt{n} \cong \frac{a}{2\sqrt{n}}$$

Consider $n = x$, so we have $\sqrt{x+dx} - \sqrt{x}$

This is a differential of $\sqrt{x}$ ✓

i.e $d(\sqrt{x}) = d(x^{1/2}) = \frac{1}{2} x^{-1/2} dx$

replace back x by 'n', and dx by 'a' we get

$$= \boxed{\frac{1}{2}\frac{a}{\sqrt{n}}}$$ ✓

Find approx value of $\sqrt{10^{12}+15} - \sqrt{10^{12}}$

here $a = 15$, $n = 10^{12}$

So result $= \frac{1}{2}\frac{a}{\sqrt{n}} = \frac{1}{2}\frac{15}{(10^{12})^{1/2}} = \frac{1}{2}\frac{15}{10^{6}}$

$$= \boxed{7.5 \times 10^{-6}}$$

✓

ch 4 (7

4.8 About how much (in percent) does an error of 1% in a and b affect a^2b^3?

let $R = a^2b^3$

so we want to find $\frac{dR}{R}$

but $dR = \frac{\partial R}{\partial a} da + \frac{\partial R}{\partial b} db$ since R is function of (a,b).

so $dR = 2ab^3\, da + 3b^2a^2\, db$

$$\frac{dR}{R} = \frac{2ab^3}{R} da + \frac{3b^2a^2}{R} db = \frac{2ab^3}{a^2b^3} da + \frac{3b^2a^2}{a^2b^3} db$$

$$= \frac{2}{a} da + \frac{3}{b} db = 2\frac{da}{a} + 3\frac{db}{b}$$

(1%) (1%)

$$= 2(0.01) + 3(0.01) = 0.05$$

so R is affected **5%**

ch 4 (8

4.9 Show that the approximate error $\frac{df}{f}$ of a product $f=gh$ is the sum of the approximate relative errors of the factors.

$$df = \frac{\partial f}{\partial g}dg + \frac{\partial f}{\partial h}dh$$

$$\frac{df}{f} = \frac{\frac{\partial f}{\partial g}dg}{f} + \frac{\frac{\partial f}{\partial h}dh}{f} = \frac{\frac{\partial f}{\partial g}dg}{gh} + \frac{\frac{\partial f}{\partial h}dh}{gh}$$

$$\frac{df}{f} = \frac{\partial f}{\partial g}\frac{dg}{g}\frac{1}{h} + \frac{\partial f}{\partial h}\frac{dh}{h}\frac{1}{g} \quad (1)$$

but $\frac{\partial f}{\partial g} = h$

and $\frac{\partial f}{\partial h} = g$

So (1) becomes

$$\frac{df}{f} = h\frac{dg}{g}\frac{1}{h} + g\frac{dh}{h}\frac{1}{g}$$

$$\boxed{\frac{df}{f} = \frac{dg}{g} + \frac{dh}{h}}$$

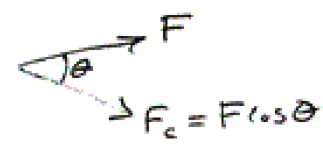
which means the result required.

(9

ch 4

4.10 a force of 500 nt in measured with possible error of 1 nt. Its component in direction 60° away from its Line of action is required, where the angle is subject to an error of 0.5°. what is approx the Largest possible error in the component?

F_c = Component = $F\cos\theta$



so $dF_c = \dfrac{\partial F_c}{\partial F} dF + \dfrac{\partial F_c}{\partial \theta} d\theta$

$dF_c = \cos\theta \, dF - F\sin\theta \, d\theta$

For $\theta = 60$, $d\theta = 0.5°$
and for $F = 500$, error = 1 i.e $dF = 1$

so $dF_c = (\cos 60°)(\pm 1) - (500)\sin(60°)\,(\pm 0.5°)$

$dF_c = \underbrace{\frac{1}{2}(\pm 1)}_{A} - \underbrace{500\frac{\sqrt{3}}{2}(\pm 0.5°)}_{B}$

convert to radian
$\dfrac{180°}{0.5°} = \dfrac{\pi}{r}$
$\Rightarrow \boxed{r = \frac{0.5}{180}\pi} = 8.726 \times 10^{-3}$

There are 4 possible values (all +ve, all −ve, A +ve B −ve, A −ve B +ve)

(1) $dF_c = \frac{1}{2} - 500\frac{\sqrt{3}}{2}(+r) = -3.2787$ nt

(2) $dF_c = -\frac{1}{2} + 500\frac{\sqrt{3}}{2}(-r) = +3.2787$ nt

(3) $dF_c = \frac{1}{2} + 500\frac{\sqrt{3}}{2}(-r) = +4.2784$ nt

(4) $dF_c = -\frac{1}{2} - 500\frac{\sqrt{3}}{2}(+r) = -4.2784$ nt

So largest +ve error is $\boxed{4.278}$ nt, and largest −ve error is $\boxed{-4.278}$ nt, so error range is 2×4.278 or 8.556 nt

(10

Ch 4

4.13 without using calculator, estimate the change in length of space diagonal of a box whose dimensions are changed from 200 x 200 x 100 to 201 x 202 x 99

$L =$ length of space diagonal $= \sqrt{x^2+y^2+z^2}$

so $dL = \frac{\partial L}{\partial x} dx + \frac{\partial L}{\partial y} dy + \frac{\partial L}{\partial z} dz$

$\frac{\partial L}{\partial x} = \frac{1}{2} \frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}$; same for y, z.

so $\boxed{dL = \frac{x}{\sqrt{x^2+y^2+z^2}} dx + \frac{y}{\sqrt{x^2+y^2+z^2}} dy + \frac{z}{\sqrt{x^2+y^2+z^2}} dz}$

now $dx = 201-200 = 1$
$dy = 202-200 = 2$
$dz = 99-100 = -1$

but $\sqrt{x^2+y^2+z^2} = \sqrt{200^2+200^2+100^2} = \sqrt{4\times10^4+4\times10^4+10^4} = 100\sqrt{9} = 300$

so $dL = \frac{200}{300}(1) + \frac{200}{300}(2) + \frac{100}{300}(-1)$ (with dx, dy, dz marked above the factors)

$dL = \frac{2}{3} + \frac{4}{3} - \frac{1}{3} = \boxed{\frac{5}{3}}$ so space diagonal has increased by $\frac{5}{3}$ or $\boxed{1.666}$

So new length $= 300 + \frac{5}{3} = 301.666$

3.6 HW 5

HW # 5
Math 121 A

Nasser Abbasi

UCB extension

2/2

ch 4

5.3 given $r = e^{-p^2-q^2}$, $p = e^s$, $q = e^{-s}$

find $\frac{dr}{ds}$

$$dr = \frac{\partial r}{\partial p} dp + \frac{\partial r}{\partial q} dq \Rightarrow \frac{dr}{ds} = \frac{\partial r}{\partial p}\frac{dp}{ds} + \frac{\partial r}{\partial q}\frac{dq}{ds}$$

$$\frac{\partial r}{\partial p} = \left(e^{-p^2-q^2}\right)(-2p)$$

$$\frac{\partial r}{\partial q} = e^{(-p^2-q^2)}(-2q)$$

$$\frac{dp}{ds} = e^s$$

$$\frac{dq}{ds} = -e^{-s}$$

so
$$\frac{dr}{ds} = \left(-2pe^{-p^2-q^2}\right)\left(e^s\right) + \left(-2qe^{-p^2-q^2}\right)\left(-e^{-s}\right)$$

$$= -2e^s p e^{-p^2-q^2} + 2e^{-s} q e^{-p^2-q^2}$$

$$= 2r\left[-e^s p + e^{-s} q\right]$$

$$\frac{dr}{ds} = 2r\left[-p^2 + q^2\right]$$

Any one of these 2 is a solution

(2

Ch4

5.4 given $x = \ln(u^2 - v^2)$, $u = t^2$, $v = \cos t$

find $\frac{dx}{dt}$.

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

$$\frac{dx}{dt} = \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}$$

note rule for $\frac{d}{dx} \ln(f(x)\, g(x)) = \frac{g(x) f'(x) + f(x) g'(x)}{f(x)\, g(x)}$

So $\frac{\partial x}{\partial u} = \frac{2u}{u^2 - v^2}$

$\frac{\partial x}{\partial v} = \frac{-2v}{u^2 - v^2}$

so $\frac{dx}{dt} = \frac{2u}{u^2 - v^2}(2t) + \left(\frac{-2v}{u^2 - v^2}\right)(-\sin t)$

$$= \frac{4t\, u}{u^2 - v^2} + \frac{2v \sin t}{u^2 - v^2} = \boxed{\left(\frac{2}{u^2 - v^2}\right)\left(2tu + v \sin t\right)}$$

I probably need to express everything in terms of t:

$$\frac{dx}{dt} = \left(\frac{2}{t^4 - \cos^2 t}\right)\left(2t(t^2) + \cos t \sin t\right)$$

$$\boxed{\frac{dx}{dt} = \frac{4t^3 + 2\cos(t)\sin(t)}{t^4 - \cos^2(t)}}$$

→ This looks better :) wasn't it

Ch 4

6.3 if $x^y = y^x$ find $\frac{dy}{dx}$ at $(2,4)$.

$$y \ln(x) = x \ln(y)$$

$$\frac{\ln(x)}{x} = \frac{\ln(y)}{y}$$

take differentials:

$$\left[\frac{1}{x}\frac{1}{x} + \left(-1\frac{1}{x^2}\ln(x)\right)\right] dx = \left[\frac{1}{y}\frac{1}{y} + \left(-1\frac{1}{y^2}\ln(y)\right)\right] dy$$

$$\left[\frac{1}{x^2} - \frac{1}{x^2}\ln(x)\right] dx = \left(\frac{1}{y^2} - \frac{1}{y^2}\ln(y)\right] dy$$

$$\frac{dy}{dx} = \frac{\left(\frac{1-\ln(x)}{x^2}\right)}{\left(\frac{1-\ln(y)}{y^2}\right)} = \frac{y^2(1-\ln(x))}{x^2(1-\ln(y))}$$

at $x=2$, $y=4$ $\Rightarrow$

$$\frac{dy}{dx} = \frac{16(1-\ln(2))}{4(1-\ln(4))} = 4\,\frac{(1-\ln(2))}{1-\ln(4)} = -3.1774$$

ch 4 **6.5**

if $xy^3 - yx^3 = 6$, find slope at $(1,2)$ and equation of tangent line at this point.

differentiate equation implicitly w.r.t x

$$\left(x \frac{d}{dx}(y^3) + \frac{d}{dx}(x)\, y^3\right) - \left(y \frac{d}{dx}(x^3) + \frac{dy}{dx} x^3\right) = 0$$

$$x\left(3y^2 \frac{dy}{dx}\right) + y^3 - \left(y(3x^2) + x^3 \frac{dy}{dx}\right) = 0$$

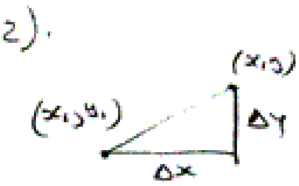
$$3xy^2 \frac{dy}{dx} + y^3 - 3yx^2 - x^3 \frac{dy}{dx} = 0$$

let $x=1$, $y=2$ $\Rightarrow$

$$3(1)(4)\frac{dy}{dx} + 8 - 3(2)(1) - (1)\frac{dy}{dx} = 0$$

$$12\frac{dy}{dx} + 8 - 6 - \frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-2}{11}}$$ the slope at point $(1,2)$.



equation of line is $\frac{y - y_1}{x - x_1} = \text{slope}$

so $y - y_1 = \frac{-2}{11}(x - x_1) \Rightarrow y - 2 = -\frac{2}{11}(x - 1)$

$y - 2 = -\frac{2}{11}x + \frac{2}{11} \Rightarrow y + \frac{2}{11}x = \frac{24}{11}$

$$\Rightarrow \boxed{2x + 11y = 24}$$

(5

ch 4

6.6

Find $\frac{d^2y}{dx^2}$ at (1,2). where $xy^3 - yx^3 = 6$

implicit diff of $xy^3 - yx^3 - 6 = 0$ w.r.t. x $\Rightarrow$

$$3xy^2 \frac{dy}{dx} + y^3 - 3yx^2 - x^3 \frac{dy}{dx} = 0$$

implicit diff again $\Rightarrow$

$$\left(3x \frac{d}{dx}\left(y^2 \frac{dy}{dx}\right) + 3y^2 \frac{dy}{dx}\right) + \frac{d}{dx}\left(y^3\right) - \left(3y \frac{d}{dx}(x^2) + 3x^2 \frac{d}{dx} y\right)$$

$$- \left(x^3 \frac{d}{dx}\left(\frac{dy}{dx}\right) + 3x^2 \frac{dy}{dx}\right) = 0$$

$$\left(3x\left(y^2 \frac{d^2y}{dx^2} + 2y\left[\frac{dy}{dx}\right]^2 + y^2 \frac{dy}{dx}\right) + 3y^2 \frac{dy}{dx} - \left(3y(2x) + 3x^2 \frac{dy}{dx}\right)\right.$$

$$- \left(x^3 \frac{d^2y}{dx^2} + 3x^2 \frac{dy}{dx}\right) = 0$$

$$3xy^2 y'' + 6xy[y']^2 + 3y^2 y' + 3y^2 y' - 6yx - 3x^2 y' - x^3 y'' - 3x^2 y' = 0$$

$$y''(3xy^2 - x^3) = 3x^2 y' + 3x^2 y' + 6yx - 3y^2 y' - 3y^2 y' - 6xy[y']^2$$

$$y''(3xy^2 - x^3) = y'(3x^2 + 3x^2 - 3y^2 - 3y^2) - 6xy[y']^2 + 6yx$$

at (1,2) $y' = \frac{-2}{11}$ from problem 6.5.

$\Rightarrow$

(6

at $x=1$, $y=2$ $\Rightarrow$

$$y''(3(4)-1) = \frac{-2}{11}\left(6-6(4)\right) - 6(2)\left(-\frac{2}{11}\right)^2 + 6(2)$$

$$y''(11) = \frac{-2}{11}(6-24) - 12\left(\frac{4}{121}\right) + 12$$

$$y''(11) = -\frac{2}{11}(-18) - \frac{48}{121} + 12$$

$$y''(11) = \frac{36}{11} - \frac{48}{121} + 12$$

$$y'' = \frac{3.272 - 0.3967 + 12}{11} = \boxed{1.352}$$

(7

ch 4

7.3 if $z = x e^{-y}$ and $x = \cosh(t)$, $y = \cos(s)$, find $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial y}\frac{dy}{ds} = (-x e^{-y})(-\sin(s)) = \cosh(t) e^{-\cos(s)} \sin(s)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} = (e^{-y})(\sinh(t)) = e^{-\cos(s)} \sinh t$$

7.5 if $u = x^2 y^3 z$ and $x = \sin(s+t)$, $y = \cos(s+t)$, $z = e^{st}$, find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$.

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} + \frac{\partial u}{\partial z}\frac{dz}{ds}$$

$$= (2xy^3z)\cos(s+t) + (3x^2y^2z)(-\sin(s+t)) + x^2y^3 t e^{st}$$

$$= 2xy^3z\cos(s+t) - 3x^2y^2z\sin(s+t) + x^2y^3 t e^{st}$$

or $= 2xy^4z - 3x^3y^2z + x^2y^3tz$ See next page

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= (2xy^3z)\cos(s+t) + (3x^2y^2z)(-\sin(s+t)) + (x^2y^3)\, s e^{st}$$

$$= 2xy^3z\cos(s+t) - 3x^2y^2z\sin(s+t) + x^2y^3 s e^{st}$$

or $\Rightarrow 2xy^4z - 3x^3y^2z + x^2y^3z\, s$ → see next page

Note I think I should re write these answers more in terms of 's' and 't' as much as possible instead as I did. So please see next page →

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$$\frac{\partial u}{\partial s} = 2\sin(s+t)\cos^4(s+t)\,e^{st} - 3\sin^3(s+t)\cos^2(s+t)\,e^{st} + \sin^2(s+t)\cos^3(s+t)\,t\,e^{st}$$

$$\frac{\partial u}{\partial t} = 2\sin(s+t)\cos^4(s+t)\,e^{st} - 3\sin^3(s+t)\cos^2(s+t)\,e^{st} + \sin^2(s+t)\cos^3(s+t)\,s\,e^{st}$$

ch 4

7.8 if $xs^2+yt^2=1$ and $x^2s+y^2t=xy-4$

find $\frac{\partial x}{\partial s}$, $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial s}$, $\frac{\partial y}{\partial t}$ at $(x,y,s,t)=(1,-3,2,-1)$

from first equation, take derivative w.r.t. 's' and then w.r.t. 't' ⇒

$$\left(x\,2s+\frac{\partial x}{\partial s}s^2\right)+\frac{\partial y}{\partial s}t^2=0 \qquad (1)$$

$$\frac{\partial x}{\partial t}s^2+\left(2ty+t^2\frac{\partial y}{\partial t}\right)=0 \qquad (2)$$

From second equation, do the same

$$\left(x^2+s\,2x\frac{\partial x}{\partial s}\right)+2y\frac{\partial y}{\partial s}t=\left(x\frac{\partial y}{\partial s}+y\frac{\partial x}{\partial s}\right) \quad \text{—} \ (3)$$

$$2x\frac{\partial x}{\partial t}s+2y\frac{\partial y}{\partial t}t+y^2=\left(x\frac{\partial y}{\partial t}+y\frac{\partial x}{\partial t}\right) \quad \text{—} \ (4)$$

4 equations, 4 unknowns.

rewrite (3) and (4)

$$\frac{\partial x}{\partial s}(2xs-y)+\frac{\partial y}{\partial s}(2yt-x)=-x^2 \quad \text{—} \ (5)$$

$$\frac{\partial x}{\partial t}(2xs-y)+\frac{\partial y}{\partial t}(2yt-x)=-y^2 \quad \text{—} \ (6)$$

rewrite (1) and (2)

$$\frac{\partial x}{\partial s}(s^2)+\frac{\partial y}{\partial s}(t^2)=-2xs \quad \text{—} \ (7)$$

$$\frac{\partial x}{\partial t}(s^2)+\frac{\partial y}{\partial t}(t^2)=-2ty \quad \text{—} \ (8)$$

now solve (5) and (7) together for $\frac{\partial x}{\partial s}$, $\frac{\partial y}{\partial s}$

and solve (6) and (8) together for $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial t}$

→

(10

solving (5) and (7) $\Rightarrow$

$$\underbrace{\begin{pmatrix} 2xs-\frac{y}{c} & 2yt-x \\ s^2 & t^2 \end{pmatrix}}_{2\times 2} \underbrace{\begin{pmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \end{pmatrix}}_{2\times 1} = \underbrace{\begin{pmatrix} -x^2 \\ -2xs \end{pmatrix}}_{2\times 1} \qquad \text{—— (9)}$$

solving (6) and (8) $\Rightarrow$

$$\begin{pmatrix} 2xs-\frac{y}{c} & 2yt-x \\ s^2 & t^2 \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} -y^2 \\ -2\frac{y}{c}t \end{pmatrix} \qquad \text{—— (10)}$$

Solve 9 by Cramer Rule

- says that given $\overbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \overbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}^{C}$ then

$x_i = \dfrac{|A_i|}{|A|}$ where A_i is A matrix with the i^{th} column replaced by C vector.

so
$$\frac{\partial x}{\partial s} = \frac{\begin{vmatrix} -x^2 & 2yt-x \\ -2xs & t^2 \end{vmatrix}}{\begin{vmatrix} 2xs-y & 2yt-x \\ s^2 & t^2 \end{vmatrix}} = \frac{(-x^2t^2)-((2yt-x)(-2xs))}{t^2(2xs-y)-(s^2)(2yt-x)} = \boxed{\frac{x^2(-t^2-2s)+4yxts}{x(2st^2+s^2)+y(-t^2-2ts^2)}}$$

$$\frac{\partial y}{\partial s} = \frac{\begin{vmatrix} 2xs-y & -x^2 \\ s^2 & -2xs \end{vmatrix}}{\begin{vmatrix} 2xs-y & 2yt-x \\ s^2 & t^2 \end{vmatrix}} = \frac{(2xs-y)(-2xs)-(-x^2)(s^2)}{t^2(2xs-y)-(s^2)(2yt-x)} = \boxed{\frac{-3x^2s^2+2xsy}{x(2st^2+s^2)+y(-t^2-2ts^2)}}$$

Similarly, solve (10) by Cramer rule to get $\frac{\partial x}{\partial t}$ and $\frac{\partial y}{\partial t}$ $\Rightarrow$

$$\frac{\partial x}{\partial t} = \frac{\begin{vmatrix} -y^2 & 2yt-x \\ -2yt & t^2 \end{vmatrix}}{\begin{vmatrix} 2xs-y & 2yt-x \\ s^2 & t^2 \end{vmatrix}} = \frac{(-y^2t^2) - (2yt-x)(-2yt)}{x(2st^2+s^2)+y(-t^2-2ts^2)} \quad -|A|$$

$$= \frac{-y^2t^2 - (-4y^2t^2+2ytx)}{|A|} = \boxed{\frac{3y^2t^2 - 2ytx}{x(2st^2+s^2)+y(-t^2-2ts^2)}}$$

$$\frac{\partial y}{\partial t} = \frac{\begin{vmatrix} 2xs-y & -y^2 \\ s^2 & -2yt \end{vmatrix}}{\begin{vmatrix} 2xs-y & 2yt-x \\ s^2 & t^2 \end{vmatrix}} = \frac{(2xs-y)(-2yt) - (-y^2)(s^2)}{|A|}$$

$$= \boxed{\frac{-4xsyt + 2y^2t + y^2s^2}{x(2st^2+s^2)+y(-t^2-2ts^2)}}$$

now, at $(x,y,s,t) = (1,-3,2,-1)$ we get

$$\frac{\partial x}{\partial s} = \frac{(1)^2(-(-1)^2 - 2(2)) + 4(-3)(1)(-1)(2)}{(1)(2(2)(-1)^2)+(2)^2) + (-3)(-(-1)^2 - 2(-1)(2)^2)} = \frac{(-1-4)+24}{8-3(-1+8)} = \boxed{\frac{19}{-13}}$$

$$\frac{\partial y}{\partial s} = \frac{-3(1)^2(2)^2 + 2(1)(2)(-3)}{-13} = \frac{-12-12}{-13} = \frac{-24}{-13} = \boxed{\frac{24}{13}}$$

$$\frac{\partial x}{\partial t} = \frac{3(-3)^2(-1)^2 - 2(-3)(-1)(1)}{-13} = \frac{27-6}{-13} = \boxed{\frac{21}{-13}}$$

$$\frac{\partial y}{\partial t} = \frac{-4(1)(2)(-3)(-1) + 2(-3)^2(-1) + (-3)^2(2)^2}{-13} = \frac{-24-18+36}{-13} = \boxed{\frac{6}{13}}$$

(12

ch 4

7.15 given $x^2u - y^2v = 1$ and $x + y = uv$

find $\left(\frac{\partial x}{\partial u}\right)_v$, $\left(\frac{\partial x}{\partial u}\right)_y$

$\left(\frac{\partial x}{\partial u}\right)_v$ means that $x(u,v)$. i.e x is function of u,v only.

so take derivative w.r.t. u, we get from first equation:

$$\frac{\partial}{\partial u}(x^2 u) - \frac{\partial}{\partial u}(y^2 v) = 0$$

$$\left(x^2 + 2x\frac{\partial x}{\partial u}u\right) - \left(2y\frac{\partial y}{\partial u}v\right) = 0$$

$$\frac{\partial x}{\partial u}(2xu) + \frac{\partial y}{\partial u}(-2yv) = -x^2 \quad \text{——} \ (1)$$

From second equation:

$$\frac{\partial}{\partial u}(x+y) = \frac{\partial}{\partial u}(uv)$$

$$\frac{\partial x}{\partial u} + \frac{\partial y}{\partial u} = v \quad \text{——} \ (2)$$

Solve (1), (2) for $\frac{\partial x}{\partial u}$.

from (2) $\frac{\partial y}{\partial u} = v - \frac{\partial x}{\partial u}$

plug in (1) $\Rightarrow$ $2xu\frac{\partial x}{\partial u} + \left[v - \frac{\partial x}{\partial u}\right](-2yv) = -x^2$

$$2xu\frac{\partial x}{\partial u} - 2v^2 y + \frac{\partial x}{\partial u}2yv = -x^2$$

$$\frac{\partial x}{\partial u}\left[2xu + 2yv\right] = -x^2 + 2v^2 y$$

$$\boxed{\left(\frac{\partial x}{\partial u}\right)_v = \frac{-x^2 + 2v^2 y}{2xu + 2yv}} \longrightarrow$$

(13

note at This point I have the feeling we should be solving this using 'differentials', not taking partial derivatives as I did. even though answer should be the same. (I am a little confused as which method to apply and when).

for example, I redo $\left(\frac{\partial x}{\partial u}\right)_v$ using differentials:

given $\begin{cases} x^2u - y^2v = 1 \\ x + y = uv \end{cases}$

$$\Rightarrow \begin{cases} 2x\,dx\,u + x^2\,du - (2y\,dy\,v + y^2\,dv) = 0 \\ dx + dy = du\,v + u\,dv \end{cases}$$

since $\left(\frac{\partial x}{\partial u}\right)_v$ → v means constant. so $dv = 0$

So above equations become

$$\Rightarrow \begin{cases} 2x\,dx\,u + x^2\,du - 2y\,dy\,v = 0 & \text{— (1)} \\ dx + dy = du\,v & \text{— (2)} \end{cases}$$

from (1) $dy = v\,du - dx$

Plug in (2) ⇒ $2xu\,dx + u^2\,du - 2y(v\,du - dx)v = 0$

$$dx\,(2xu + 2yv) = du\,(-x^2 + 2yv^2)$$

so $\left(\frac{dx}{du}\right)_v = \boxed{\dfrac{-x^2 + 2v^2y}{2xu + 2yv}}$

This is $\left(\frac{\partial x}{\partial u}\right)$ really since partial derivative.

which is the same I got. Now I do $\left(\frac{\partial x}{\partial u}\right)_y$ →

(14

take differentials:

$$\begin{cases} 2x\,dx\,u + x^2\,du - 2y\,dy\,v - y^2\,dv = 0 \\ dx + dy = du\,v + u\,dv. \end{cases}$$

since we want $\left(\frac{\partial x}{\partial u}\right)_y \to$ mean $dy=0$.

so

$$\Rightarrow \begin{cases} 2x\,dx\,u + x^2\,du - y^2\,dv = 0 & -① \\ dx = du\,v + u\,dv & -② \end{cases}$$

from ②, $dv = \dfrac{dx - du\,v}{u}$

plug in ①

$$\Rightarrow 2x\,dx\,u + x^2\,du - y^2\left(\frac{dx - du\,v}{u}\right) = 0$$

$$dx\left(2xu - \frac{y^2}{u}\right) = du\left(-x^2 - y^2\frac{v}{u}\right)$$

so $$\left(\frac{dx}{du}\right)_y = \frac{-x^2 - y^2\frac{v}{u}}{2xu - \frac{y^2}{u}} = \boxed{\frac{-ux^2 - y^2 v}{2xu^2 - y^2}}$$

$\left(\frac{\partial x}{\partial u}\right)_y$ since partial.

15

ch 4

7.19

if $z = r + s^2$

$x + y = s^3 + r^3 - 3$

$xy = s^2 - r^2$

find $\left(\frac{\partial x}{\partial z}\right)_s, \left(\frac{\partial x}{\partial z}\right)_r, \left(\frac{\partial x}{\partial z}\right)_y$ at $(r,s,x,y,z) = (-1,2,3,13)$

$\left(\frac{\partial x}{\partial z}\right)_s$ means $x(z,s)$ and we take $ds = 0$ (constant):

$$\Rightarrow \left\{\begin{array}{l} dz = dr + 2s\,ds \\ dx + dy = 3s^2\,ds + 3r^2\,dr \\ x\,dy + y\,dx = 2s\,ds - 2r\,dr \end{array}\right\} A$$

Since $ds = 0$, Then we get

$$\left\{\begin{array}{ll} dz = dr & \text{—— (1)} \\ dx + dy = 3r^2\,dr & \text{—— (2)} \\ x\,dy + y\,dx = -2r\,dr & \text{—— (3)} \end{array}\right.$$

From (1) sub dz into (3) $\Rightarrow$ $x\,dy + y\,dx = -2r(dz)$. —— (4)

From (4) $\rightarrow$ $dy = \frac{-2r\,dz - y\,dx}{x}$ —— (5)

From (5) sub dy into (2) $\Rightarrow$ $dx + \left(\frac{-2r\,dz}{x} - \frac{y\,dx}{x}\right) = 3r^2\,dr$ —— (6)

From (1) sub dz into (6) to eliminate dr

$$x\,dx - 2r\,dz - y\,dx = 3r^2 x\,dz$$

$$dx(x - y) = dz(3r^2 x + 2r)$$

$$\boxed{\left(\frac{\partial x}{\partial z}\right)_s = \frac{3r^2 x + 2r}{x - y}}$$

(16

now do $\left(\frac{\partial x}{\partial z}\right)_r$

set $dr=0$ in set of equations marked 'A' in previous page.

$$\Rightarrow \begin{cases} dz = 2s\, ds & \text{--- (1)} \\ dx+dy = 3s^2 ds & \text{--- (2)} \\ x\,dy + y\,dx = 2s\,ds & \text{--- (3)} \end{cases}$$

from (1) $ds = \frac{dz}{2s}$

replace ds in (2) and (3) by above value

$$\Rightarrow \begin{cases} dx+dy = 3s^2\left(\frac{dz}{2s}\right) = dx+dy = \frac{3}{2}s\,dz & \text{--- (4)} \\ x\,dy + y\,dx = 2s\left(\frac{dz}{2s}\right) = x\,dy + y\,dx = dz & \text{--- (5)} \end{cases}$$

from (5), $dy = \frac{dz - y\,dx}{x}$

replace into (4)

$$dx + \left(\frac{dz - y\,dx}{x}\right) = \frac{3s\,dz}{2}$$

$$x\,dx + dz - y\,dx = \frac{3s\,dz\,x}{2}$$

$$dx(x-y) = dz\left(\frac{3sx}{2} - 1\right)$$

$$\boxed{\left(\frac{\partial x}{\partial z}\right)_r = \frac{3sx-2}{2(x-y)}}$$

(17

Now do $\left(\frac{\partial x}{\partial z}\right)_y$.

replace $dy=0$ into 'A' equations shown before.

$$\Rightarrow \begin{cases} dz = dr + 2s\,ds & \text{--- (1)} \\ dx = 3s^2 ds + 3r^2 dr & \text{--- (2)} \\ y\,dx = 2s\,ds - 2r\,dr & \text{--- (3)} \end{cases}$$

From (1) $dr = dz - 2s\,ds$

replace dr from (2) and (3) $\Rightarrow$

$$\Rightarrow \begin{cases} dx = 3s^2 ds + 3r^2 (dz - 2s\,ds) \\ y\,dx = 2s\,ds - 2r\,(dz - 2s\,ds) \end{cases}$$

$$\Rightarrow \begin{cases} dx = 3s^2 ds + 3r^2 dz - 6r^2 s\,ds & \text{--- (4)} \\ y\,dx = 2s\,ds - 2r\,dz + 4rs\,ds & \text{--- (5)} \end{cases}$$

From (5) Find ds and replace into (4)

$$ds\,(2s + 4rs) = y\,dx + 2r\,dz$$

$$ds = \left(\frac{y\,dx + 2r\,dz}{2s + 4rs}\right)$$

plug into (4) $\Rightarrow$ $dx = 3s^2\left(\frac{y\,dx + 2r\,dz}{2s+4rs}\right) + 3r^2 dz - 6r^2 s\left(\frac{y\,dx + 2r\,dz}{2s+4rs}\right)$

$$dx = \frac{3s^2 y\,dx + 6s^2 r\,dz}{2s+4rs} + 3r^2 dz - \frac{6r^2 s y\,dx + 12r^3 s\,dz}{2s+4rs}$$

$$dx\,(2s+4rs) = 3s^2 y\,dx + 6s^2 r\,dz + 6sr^2 dz + 12r^3 s\,dz - 6r^2 s y\,dx - 12r^3 s\,dz$$

$$dx\,(2s + 4rs - 3s^2 y + 6r^2 s y) = dz\,(6s^2 r + 6sr^2)$$

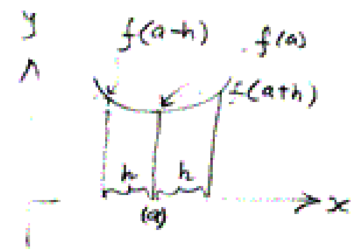
$$\text{so } \left(\frac{\partial x}{\partial z}\right)_y = \frac{6s^2 r + 6sr^2}{2s + 4rs - 3s^2 y + 6r^2 s y} = \boxed{\frac{6r(s+r)}{2 + 4r - 3sy + 6r^2 y}}$$

(18

Ch4

8.1 Use the Taylor series about $x=a$ to verify the familiar "second derivative test" for a maximum or minimum point. That is, show that if $f'(a)=0$ and then $f''(a)>0$ implies a min point at $x=a$ and $f''<0$ implies a maximum point.

let me first look at the min case. Consider diagram: here $f(a)$ is a min and need to show that this implies $f''(a)>0$.



since it is a min. Then $\frac{f(a+h)+f(a-h)}{2} > f(a)$ —— (1)

where h is some small distance away from 'a' on either side.

but $f(a+h) = f(a) + h f'(a) + \frac{h^2}{2} f''(a) + \cdots$ ← ignore rest.

and $f(a-h) = f(a) + h f'(a) + \frac{h^2}{2} f''(a)$.

so from (1) we get

$$\frac{\left[f(a)+h f'(a)+\frac{h^2}{2}f''(a)\right] + \left[f(a)+h f'(a)+\frac{h^2}{2}f''(a)\right]}{2} > f(a)$$

$$\frac{2f(a)+2h f'(a)+h^2 f''(a)}{2} > f(a)$$

$$f(a)+h f'(a)+\frac{h^2}{2}f''(a) > f(a).$$

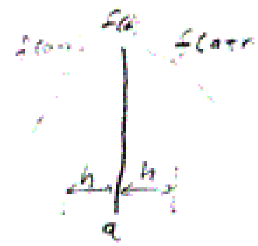
$$h f'(a)+\frac{h^2}{2}f''(a) > 0$$

but $f'(a)=0 \Rightarrow \frac{h^2}{2}f''(a)>0$ and since $h\neq 0$, $\Rightarrow$ $\boxed{f''(a)>0}$

(19)

now similarly I show that for a max, it implies

$f''(a) < 0$.



here we have $\frac{f(a-h)+f(a+h)}{2} < f(a)$

hence $\frac{\left(f(a)+hf'(a)+h^2\frac{f''(a)}{2}\right)+\left(f(a)+hf'(a)+h^2\frac{f''(a)}{2}\right)}{2} < f(a)$

i.e. as before we get

$$hf'(a) + \frac{h^2}{2}f''(a) < 0$$

but $f'(a) \Rightarrow \frac{h^2}{2}f''(a) < 0$

as $h \neq 0$, $\Rightarrow$ $f''(a) < 0$ QED

(20

ch 4

8.2 use a 2 variable Taylor series prove the following "second derivative tests" for max or min points of functions of 2 variables. If $f_x = f_y = 0$ at (a,b) then

(a,b) is min point if at (a,b) $f_{xx} > 0$, $f_{yy} > 0$ and $f_{xx} f_{yy} > f_{xy}^2$

(a,b) is max point if at (a,b) $f_{xx} < 0$, $f_{yy} < 0$ and $f_{xx} f_{yy} > f_{xy}^2$

(a,b) is neither min nor max if $f_{xx} f_{yy} < f_{xy}^2$

given $f(x,y)$ at a point, (a,b), expand $f(x,y)$ near (a,b) using Taylor. The first 3 terms are

$$f(x,y) = f(a,b) + \left[f_x(a,b)\,h + f_y(a,b)\,k\right] + \frac{1}{2!}\left[f_{xx}(a,b)\,h^2 + 2f_{xy}(a,b)\,hk + f_{yy}(a,b)\,k^2\right]$$

if (a,b) is min, Then $f(x,y)$ at all points close to it are greater than $f(a,b)$.

i.e. $$\frac{f(x,y)_+ + f(x,y)_-}{2} > f(a,b)$$

where $f(x,y)_+$ is one point and $f(x,y)_-$ is another point around (a,b).

Then we have $$\frac{2f(a,b) + 2\left[f_x h + f_y k\right] + \left[f_{xx} h^2 + 2f_{xy} hk + f_{yy} k^2\right]}{2} > f(a,b)$$

now $f_x = f_y = 0$ at (a,b), so above reduces to

$$f(a,b) + \left[f_{xx} h^2 + 2f_{xy}\,hk + f_{yy}\,k^2\right] > f(a,b)$$

or $$f_{xx} h^2 + 2f_{xy}\,hk + f_{yy}\,k^2 > 0$$

write $f_{xx} = A$, $f_{xy} = B$, $f_{yy} = C$

$$\Rightarrow \quad Ah^2 + 2Bhk + Ck^2 > 0 \longrightarrow$$

(2)

rewrite as $A\left(h+\frac{Bk}{A}\right)^2+\left(C-\frac{B^2}{A}\right)k^2>0$

since $\left(h+\frac{Bk}{A}\right)^2$ is always positive, and k^2 is always positive,

So we have $\left[A\times\text{something positive}+\left(C-\frac{B^2}{A}\right)\times\text{something positive}>0\right]$

or $\left[A\times\text{something positive}+\left(\frac{CA-B^2}{A}\right)\times\text{something positive}>0\right]$

This means from looking at (the first term) A is positive.

and this means $\frac{CA-B^2}{A}$ is positive. i.e. $CA-B^2$ is positive

i.e. $CA>B^2$ and since B^2 is always positive (since squared), Then CA is positive, and since A positive, then C is positive

Then
$$\left.\begin{array}{l}C>0\\A>0\\CA>B^2\end{array}\right\}\implies\begin{array}{l}f_{yy}>0\\f_{xx}>0\\f_{yy}f_{xx}>f_{xy}^2\end{array}$$

$\longrightarrow$

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now to show that (a,b) is a max point if at (a,b), $f_{xx}<0$, $f_{yy}<0$ and $f_{xx}f_{yy}>f_{xy}^2$, same argument is used.

This leads to

$$A\left(h+\frac{BK}{A}\right)^2+\left(C-\frac{B^2}{A}\right)k^2<0$$

again, since $\left(h+\frac{BK}{A}\right)^2$ is always positive, and k^2 is always positive, then

A is negative and $\left(C-\frac{B^2}{A}\right)$ is negative.

i.e. $\frac{CA-B^2}{A}<0$

since $A<0$, then $CA-B^2>0$ to keep expression <0.

so $CA-B^2>0 \Rightarrow CA>B^2$. but B^2 is positive always, so CA must be positive. but $A<0 \Rightarrow C<0$.

i.e.

$$\left.\begin{array}{l} A<0 \\ C<0 \\ CA>B^2 \end{array}\right\} \Longrightarrow \begin{array}{l} f_{xx}<0 \\ f_{yy}<0 \\ f_{yy}f_{xx}>f_{xy}^2 \end{array}$$

⟶

now need to show that (a,b) is neither a max nor (23
a min if $f_{xx} f_{yy} < f_{xy}^2$.

looking at $A\left(h+\frac{Bk}{A}\right)^2 + \left(\frac{CA-B^2}{A}\right)k^2 = 0$

2 cases

assume $\boxed{\begin{matrix}A<0\\C>0\end{matrix}}$, Then since $\left(h+\frac{Bk}{A}\right)^2$ is always positive, Then

$\left(\frac{CA-B^2}{A}\right)k^2$ must be positive (to have 0 as total result).

but $k^2>0$, $\Rightarrow \frac{CA-B^2}{A}>0$.

But $A<0$, Then $CA-B^2<0 \Rightarrow \boxed{CA<B^2}$

assume $\boxed{\begin{matrix}A>0\\C<0\end{matrix}}$ Then since $\left(h+\frac{Bk}{A}\right)^2>0$, Then $\left(\frac{CA-B^2}{A}\right)k^2<0$.

since $A>0 \Rightarrow (CA-B^2)<0$ (since k^2 always positive)

so $\boxed{CA<B^2}$ again.

hence in both cases

$$f_{xx} f_{yy} < f_{xy}^2$$

QED

(24

ch 4

8.3 find min and max

$f(x,y) = x^2 + y^2 + 2x - 4y + 10$

$f_x = 2x + 2$
$f_y = 2y - 4$
$f_{xx} = 2$
$f_{yy} = 2$
$f_{xy} = 0$

at min or max $\left.\begin{array}{l} f_x = 0 = 2x+2 \\ f_y = 0 = 2y-4 \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} x = -1 \\ y = 2 \end{array}}$

now need to find if $(1,2)$ is max or min or saddle.

$f_{xx} > 0$
$f_{yy} > 0$

but $f_{xx} f_{yy} = 4 > \underbrace{f_{xy}^2}_{0}$

so $f_{xx} > 0$, $f_{yy} > 0$, $f_{xx} f_{yy} > f_{xy}^2 \Rightarrow$ min point.

3.7 HW 6

HW # 6

Math 121A

Nasser Abbasi

UCB extension

2/2 → score changed.

Please Review the grade on this HW.

I have correctly solved 22 problems and just forgot to solve 2 simple problem. for this I lost 50% of the grade?

This is NOT Fair.

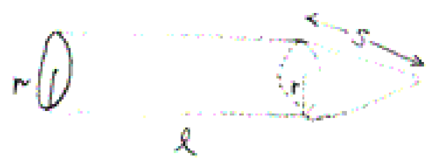
Please check the grading policy. I get time/money to only grade 2–3 problems a week carefully. If a student misses 1 of those problems, then it's bad luck maybe. On the other hand, if I grade based on number of problems attempted, then everyone will get full marks, which defeats the purpose of grading homework. But: 1) 2 points per homework is very little, so I agree that even 1/2 a point is a great %.

2) I have discussed with the instructor, and we shall amend things if necessary.

Thanks for your feedback!

ch 4

9.2 what proportions will maximize the volume of a projectile in the form of a circular cylinder with one conical end and one flat end if the surface area is given?



$s^2 = r^2 + h^2$

so $h = \sqrt{s^2 - r^2}$

Volume of cone $= \frac{1}{3}$ base $\times h = \frac{1}{3}(\pi r^2)\sqrt{s^2-r^2}$

so Total V = Cylinder Volume + Cone Volume

$$\boxed{V = \pi r^2 l + \frac{1}{3}(\pi r^2)\sqrt{s^2-r^2}}$$

now need to find total surface Area. this is the constraint.

Surface area of cylinder $= 2\pi r l + \pi r^2$

Surface area of cone $= \frac{\text{base circumference}}{2} \times s = \frac{1}{2} 2\pi r s = \pi r s$

So $\boxed{A = (2\pi r l + \pi r^2) + \pi r s}$ → lagrange multiplier

So our maximizer function $\boxed{F = V + \lambda A}$ → ϕ

$$\boxed{F = \pi r^2 l + \frac{1}{3}\pi r^2 \sqrt{s^2 - r^2} + \lambda \pi (2rl + r^2 + rs)}$$

So F is function of r, l, s.

→

$$\frac{\partial F}{\partial r} = 0 = 2r\pi l + \frac{1}{3}\pi r^2\left(\frac{1}{2}\frac{1}{(s^2-r^2)^{1/2}}\cdot(-2r)\right) + \frac{1}{3}2\pi r(s^2-r^2)^{1/2} + 2\lambda\pi l + 2r\lambda\pi + s\lambda\pi$$

$$\frac{\partial F}{\partial r}: 0 = 2r\pi l - \frac{\pi r^2}{3}\frac{r}{\sqrt{s^2-r^2}} + \frac{2\pi r}{3}\sqrt{s^2-r^2} + 2\lambda\pi l + 2r\lambda\pi + s\lambda\pi$$

$$\frac{\partial F}{\partial r}: 0 = \pi\left(2rl - \frac{r^2}{3}\frac{r}{\sqrt{s^2-r^2}} + \frac{2r}{3}\sqrt{s^2-r^2} + 2\lambda l + 2r\lambda + s\lambda\right) \text{ --- (1)}$$

$$\frac{\partial F}{\partial s} = 0 = \frac{1}{3}\pi r^2\frac{1}{2}\frac{1}{\sqrt{s^2-r^2}}2s + r\lambda\pi$$

$$= \pi\left(\frac{1}{3}r^2\frac{s}{\sqrt{s^2-r^2}} + r\lambda\right) \text{ --- (2)}$$

$$\frac{\partial F}{\partial l} = 0 = \pi r^2 + 2r\pi\lambda = \pi(r^2 + 2r\lambda) \text{ --- (3)}$$

start by eliminating λ.

From (3) $0 = \pi r^2 + 2\pi r\lambda \Rightarrow \lambda = \frac{-\pi r^2}{2\pi r} = -\frac{r^2}{2r} = \boxed{-\frac{r}{2}}$

Plug this value for λ into (2)

$$\Rightarrow 0 = \pi\left(\frac{1}{3}r^2\frac{s}{\sqrt{s^2-r^2}} - \frac{r}{2}r\right) = \pi\left(\frac{r^2 s}{3\sqrt{s^2-r^2}} - \frac{r^2}{2}\right)$$

so $0 = \pi\left(\frac{2r^2 s - 3\sqrt{s^2-r^2}\,r^2}{6\sqrt{s^2-r^2}}\right) \Rightarrow 2r^2 s - 3\sqrt{s^2-r^2}\,r^2 = 0$

$\Rightarrow 2s - 3\sqrt{s^2-r^2} = 0 \Rightarrow 2s = 3\sqrt{s^2-r^2} \Rightarrow 4s^2 = 9(s^2-r^2)$

$\Rightarrow 4s^2 = 9s^2 - 9r^2 \Rightarrow 9r^2 = 5s^2 \Rightarrow \boxed{\frac{r}{s} = \frac{\sqrt{5}}{3}}$ or $\boxed{r = \frac{\sqrt{5}}{3}s}$

Plug the value for r into (1). also plug value for λ into (1)

This leaves (1) in terms of l, s only:

$$\lambda = -\frac{r}{2} = \boxed{-\frac{1}{2}\frac{\sqrt{5}}{3}s}$$

so (1) becomes $\longrightarrow$

$$0=\pi\left(2\left(\frac{\sqrt{5}}{3}s\right)l-\frac{\left(\frac{\sqrt{5}}{3}s\right)^2}{3}\frac{\left(\frac{\sqrt{5}}{3}s\right)}{\sqrt{s^2-\left(\frac{\sqrt{5}}{3}s\right)^2}}+\frac{2}{3}\left(\frac{\sqrt{5}}{3}s\right)\sqrt{s^2-\left(\frac{\sqrt{5}}{3}s\right)^2}+2\left(\frac{-\sqrt{5}}{6}s\right)l\right.$$

$$\left.+2\left(\frac{\sqrt{5}}{3}s\right)\left(\frac{-\sqrt{5}}{6}s\right)+s\left(\frac{\sqrt{5}}{6}s\right)\right)$$

divide by $\left(\frac{\sqrt{5}}{3}s\right)\Rightarrow$

$$0=\pi\left(2l-\frac{\left(\frac{\sqrt{5}}{3}s\right)^2}{3}\frac{1}{\sqrt{s^2-\frac{5}{9}s^2}}+\frac{2}{3}\sqrt{s^2-\frac{5}{9}s^2}-2l\frac{1}{2}-\frac{\sqrt{5}}{3}s-\frac{1}{2}s\right)$$

$$0=2l-\frac{\frac{5}{9}s^2}{3}\frac{1}{\sqrt{s^2-\frac{5}{9}s^2}}+\frac{2}{3}\sqrt{s^2-\frac{5}{9}s^2}-l-\frac{\sqrt{5}}{3}s-\frac{1}{2}s$$

$$0=2l-\frac{5s^2}{9\cdot 3}\frac{1}{\sqrt{\frac{4}{9}s^2}}+\frac{2}{3}\sqrt{\frac{4}{9}s^2}-l-\frac{\sqrt{5}}{3}s-\frac{1}{2}s$$

$$=2l-\frac{5s^2}{9\cdot 3}\frac{1}{\frac{2}{3}s}+\frac{2}{3}\cdot\frac{2}{3}s-l-\frac{\sqrt{5}}{3}s-\frac{1}{2}s$$

$$=2l-\frac{5s}{9\cdot 2}+\frac{4}{9}s-l-\frac{\sqrt{5}}{3}s-\frac{1}{2}s=l+s\left(\frac{4}{9}-\frac{\sqrt{5}}{3}-\frac{1}{2}-\frac{5}{18}\right)$$

$$0=l+s\left(\frac{8-6\sqrt{5}-9-5}{18}\right)\Rightarrow 0=l+s\left(\frac{-6-6\sqrt{5}}{18}\right)\Rightarrow 0=l-s\left(\frac{1+\sqrt{5}}{3}\right)$$

$$l=s\left(\frac{1+\sqrt{5}}{3}\right)\Longrightarrow\boxed{\frac{l}{s}=\frac{1+\sqrt{5}}{3}}$$

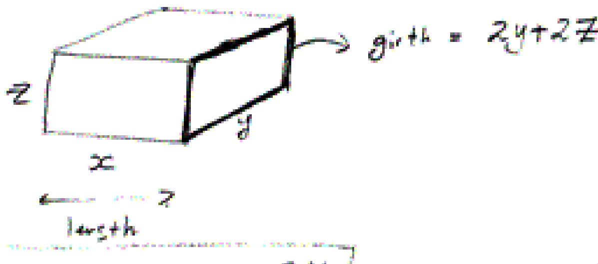
So proportions to maximize volume are

$$\boxed{\frac{r}{s}=\frac{\sqrt{5}}{3}}\quad\text{and}\quad\boxed{\frac{l}{s}=\frac{1+\sqrt{5}}{3}}$$

$$\text{or}\ \boxed{l:r:s\equiv 1+\sqrt{5}:\sqrt{5}:3}$$

Ch 4

9.3 Find largest box that can be shipped by parcel post (length plus girth = 84 in)



So $\boxed{\phi^{(x,y,z)} = 2y+2Z + x = 84}$ ——— (1)

(girth: $2y+2Z$; length: x)

$$V = xyz$$

we want to maximize V subject to $\phi = 84$.

$$\boxed{F = V + \lambda \phi}$$

$$F = xyz + \lambda(2y+2z+x)$$

$\frac{\partial F}{\partial x} = 0 = yz + \lambda$ ——— (2)

$\frac{\partial F}{\partial y} = 0 = xz + 2\lambda$ ——— (3)

$\frac{\partial F}{\partial z} = 0 = xy + 2\lambda$ ——— (4)

equations (1)–(4) are now solved for λ, x, y, z.

eliminate λ from equations (2)–(3). this will give 2 new equations in x, y, z. use these 2 new equations with eq (1) to solve for x, y, z.

from (4) $\lambda = -\frac{xy}{2}$ plug in (3)

$\Rightarrow 0 = xz + 2\left(-\frac{xy}{2}\right) = xz - xy = 0$ ——— (5)

plug λ in (2) $\Rightarrow 0 = yz - \frac{xy}{2}$ ——— (6)

(5), (6) } solve (5), (6) and (1) $\Longrightarrow$

$$xz - xy = 0 \quad \text{--- (5)}$$

$$yz - \frac{xy}{2} = 0 \quad \text{--- (6)}$$

$$2y + 2z + x = 84 \quad \text{--- (1)}$$

From (5) $x(z-y) = 0$ so $z = y$

From (6) $z = \frac{x}{2}$

$\Rightarrow y = \frac{x}{2}$

so from (1) $2\left(\frac{x}{2}\right) + 2\left(\frac{x}{2}\right) + x = 84$

$$x + x + x = 84$$

$$3x = 84$$

$$\boxed{x = 28}$$

so $\boxed{z = \frac{28}{2} = 14}$

and $\boxed{y = \frac{28}{2} = 14}$

$\therefore$ max. Volume $= xyz = 28 \times 14 \times 14$

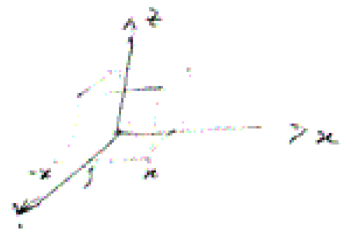
$$= \boxed{5488 \text{ in}^2}$$

16

Ch 4

9.4 Find largest box (with faces parallel to coordinate axes) that can be inscribed in $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{25}=1$

$$\phi(x,y,z) = \frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{25}=1 \quad \text{——} (1)$$



$$\text{Volume} = (2x)(2y)(2z) = 6xyz$$

so $\boxed{F = V + \lambda \phi}$

$$F = xyz + \lambda\left(\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{25}\right)$$

$$\frac{\partial F}{\partial x} = 0 = 6yz + \frac{\lambda x}{2} \quad \text{——} (2)$$

$$\frac{\partial F}{\partial y} = 0 = 6xz + \frac{2y}{9}\lambda \quad \text{——} (3)$$

$$\frac{\partial F}{\partial z} = 0 = 6xy + \frac{2z}{25}\lambda \quad \text{——} (4)$$

solve (1),(2),(3),(4) for λ, x, y, z.

First eliminate λ

from (4) $\lambda = -6xy\left(\frac{25}{2z}\right)$

Plug in (3) $\Rightarrow$ $0 = 6xz + \frac{2y}{9}\left(-6xy\frac{25}{2z}\right)$

$$0 = xz - \frac{25xy^2}{9z}$$

$$0 = z - \frac{25y^2}{9z} \quad \text{——} (5)$$

Plug λ into (2)

$$0 = 6yz + \frac{x}{2}\left(-6xy\frac{25}{2z}\right)$$

$$0 = yz - \frac{x^2 y\, 25}{4z}$$

$$0 = z - \frac{x^2}{z}\frac{25}{4} \quad \text{——} (6)$$

now use (5),(6) and (1) to find x, y, z $\longrightarrow$

(7

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1 \quad \text{———} \quad (1)$$

$$0 = z - \frac{25y^2}{9z} \quad \text{———} \quad (5)$$

$$0 = z - \frac{x^2}{z}\frac{25}{4} \quad \text{———} \quad (6)$$

from (5) $9z^2 - 25y^2 = 0 \Rightarrow z^2 = \frac{25}{9}y^2$

from (6) $4z^2 - 25x^2 = 0 \Rightarrow z^2 = \frac{25}{4}x^2$

$\frac{25}{9}y^2 = \frac{25}{4}x^2$

or $4y^2 = 9x^2$

or $y^2 = \frac{9}{4}x^2$

so (1) becomes $\frac{x^2}{4} + \frac{1}{9}\left(\frac{9}{4}x^2\right) + \frac{1}{25}\left(\frac{25}{4}x^2\right) = 1$

$$\frac{x^2}{4} + \frac{1}{4}x^2 + \frac{1}{4}x^2 = 1$$

$$\frac{3}{4}x^2 = 1 \Rightarrow x^2 = \frac{4}{3} \Rightarrow \boxed{x = \frac{2}{\sqrt{3}}}$$

so $y^2 = \frac{9}{4} \times \frac{4}{3} = 3 \Rightarrow \boxed{y = \sqrt{3}}$ i.e. $\boxed{2y = \frac{6}{\sqrt{3}}}$ i.e. $\boxed{2x = \frac{4}{\sqrt{3}}}$

so $z^2 = \frac{25}{4} \times \frac{4}{3} = \frac{25}{3} \Rightarrow \boxed{z = \frac{5}{\sqrt{3}}}$ i.e. $\boxed{2z = \frac{10}{\sqrt{3}}}$

So largest Box $= 6xyz$

$$= 6\left(\frac{2}{\sqrt{3}} \times \sqrt{3} \times \frac{5}{\sqrt{3}}\right) = \boxed{\frac{60}{\sqrt{3}}} \cong 34.64 \quad m^3$$

(8

ch 4

9.5 Find the point on $2x+\frac{3}{2}y+z-11=0$ for which $4x^2+y^2+z^2$ is a min.

$\phi(x,y,z) = 2x+\frac{3}{2}y+z=11$ —— ①

$f(x,y,z) = 4x^2+y^2+z^2$

so $\boxed{F=f+\lambda\phi}$ $F = 4x^2+y^2+z^2+\lambda(2x+\frac{3}{2}y+z)$

$\frac{\partial F}{\partial x}=0=8x+2\lambda$ —— ②

$\frac{\partial F}{\partial y}=0=2y+\frac{3}{2}\lambda$ —— ③

$\frac{\partial F}{\partial z}=0=2z+\lambda$ —— ④

From ④ $\lambda=-2z$. plug in ③ ⇒ $2y=\frac{6z}{2}$ or $y=3z$ —— ⑤

plug in ② ⇒ $8x=4z$ or $x=\frac{1}{2}z$ —— ⑥

solve ①, ⑤, ⑥ for x, y, z.

From ⑤, ⑥ ⇒ $x=\frac{1}{4}y$. so ① becomes $2x+3(4x)+4(x)=11$

$2x+12x+4x=11$ → $18x=11$ ⇒ $\boxed{x=\frac{11}{18}}$

so $z=4\times\frac{11}{18}=\frac{44}{18}=\boxed{\frac{22}{9}}$

$\boxed{y=\frac{22}{9}}$

so point is $(x,y,z)=\left(\frac{11}{18},\frac{22}{9},\frac{22}{9}\right)$

Not correct arithmetic error !

Ch 4

19.6 A box has 3 of its faces in the coordinate planes and one vertex on the plane $2x+3y+4z=6$. Find max Volume.

$\Phi(x,y,z) = 2x+3y+4z=6$ —①

$V(x,y,z) = xyz$

$F = V + \lambda \Phi$

$F = xyz + \lambda(2x+3y+4z)$

$\frac{\partial F}{\partial x}=0 = yz + 2\lambda$ — ②

$\frac{\partial F}{\partial y}=0 = xz + 3\lambda$ — ③

$\frac{\partial F}{\partial z}=0 = xy + 4\lambda$ — ④

solve for λ from (1),(2),(3)

From ④ $\lambda = -\frac{xy}{4}$ Plug in ③ $\Rightarrow xz + 3\left(-\frac{xy}{4}\right)=0$

$\Rightarrow xz - \frac{3}{4}xy = 0$ or $z - \frac{3}{4}y = 0$ — ⑤

Plug λ in ② $\Rightarrow yz + 2\left(-\frac{xy}{4}\right) = 0 \Rightarrow z - \frac{x}{2} = 0$ — ⑥

use ①,⑤,⑥ to solve for x,y,z

$2x+3y+4z=6$ ①

$z - \frac{3}{4}y = 0$ ⑤

$z - \frac{x}{2} = 0$ ⑥

From ⑤,⑥ $\Rightarrow -\frac{3}{4}y = -\frac{x}{2} \Rightarrow \frac{3}{2}y = x \Rightarrow y = \frac{2}{3}x$

From ⑥ $z = \frac{x}{2}$. So ① becomes $2x + 3\left(\frac{2}{3}x\right) + 4\left(\frac{x}{2}\right) = 6$

$\Rightarrow 2x + 2x + 2x = 6 \Rightarrow \boxed{x=1}$. so $\boxed{y=\frac{2}{3}}$ and $\boxed{z=\frac{1}{2}}$

So Max $V = xyz = 1 \times \frac{2}{3} \times \frac{1}{2} = \boxed{\frac{1}{3}}$ m^3

(10

Ch 4

9.7 repeat problem 6 if plane is $ax+by+cz=d$.

$\phi(x,y,z) = ax+by+cz=d$ ——— (1)

$f(x,y,z) = xyz$

$F = f(x,y,z)+\lambda\phi(x,y,z)$

$F = xyz+\lambda(ax+by+cz)$

$\frac{\partial F}{\partial x} = yz + a\lambda = 0$ ——— (2)

$\frac{\partial F}{\partial y} = xz + b\lambda = 0$ ——— (3)

$\frac{\partial F}{\partial z} = xy + c\lambda = 0$ ——— (4)

From (4) $\lambda = -\frac{xy}{c}$. Plug in (3) $\Rightarrow xz + b\left(-\frac{xy}{c}\right) = 0 \Rightarrow z - \frac{by}{c} = 0 \Rightarrow cz - by = 0$ ——— (5)

Plug λ into (2) $\Rightarrow yz + a\left(-\frac{xy}{c}\right) = 0 \Rightarrow z - \frac{ax}{c} = 0 \Rightarrow cz - ax = 0$ ——— (6)

Solve (5), (6) and (1) for $x\,y\,z$.

$$ax+by+cz = d \quad \text{——— (1)}$$
$$cz - by = 0 \quad \text{——— (5)}$$
$$cz - ax = 0 \quad \text{——— (6)}$$

from (5),(6) $\Rightarrow -by = -ax \Rightarrow y = \frac{a}{b}x$

from (6) $z = \frac{ax}{c}$.

so (1) becomes $ax + b\left(\frac{a}{b}\right)x + c\left(\frac{ax}{c}\right) = d$

$ax + ax + ax = d$

$3ax = d \Rightarrow \boxed{x = \frac{d}{3a}}$

so $y = \frac{a}{b}\left(\frac{d}{3a}\right) = \boxed{\frac{d}{3b}}$

$z = \frac{a}{c}\frac{d}{3a} = \boxed{\frac{d}{3c}}$

so max $V = xyz = \frac{d}{3a}\times\frac{d}{3b}\times\frac{d}{3c} = \boxed{\frac{d^3}{27abc}}$

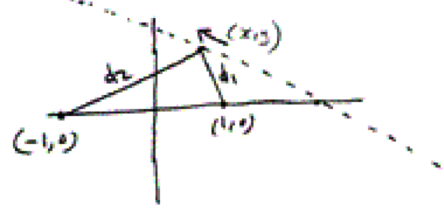
To verify: solve 9.6 using this formula.

in 9.6, $a=2$, $b=3$, $c=4$, $d=6$.

so max $V = \frac{6\times6\times6}{27(2\times3\times4)} = \frac{6\times6\times6}{27\times4} = \frac{6^2}{9\times2} = \boxed{\frac{1}{3}}$ which agrees with my solution for 9.6

Ch 4

[9.8] a point moves in the (x,y) plane on the line $2x+3y-4=0$. Where will it be when the sum of the squares of its distance from (1,0) and (-1,0) is smallest?



when $x=0$ $3y=4 \Rightarrow y=\frac{4}{3}$

when $y=0$ $2x=4 \Rightarrow x=2$

So line is as shown.

let d_1 be distance to (1,0)

d_2 be distance to (-1,0).

let point be (x,y). so $d_1^2=(x-1)^2+(y-0)^2$

and $d_2^2=(x+1)^2+(y-0)^2$

i.e. $d_1^2+d_2^2=(x-1)^2+y^2+(x+1)^2+y^2=x^2-2x+1+x^2+2x+1+2y^2$

$f(x,y)=d_1^2+d_2^2=2x^2+2y^2+2$

$\phi(x,y)=2x+3y-4=0$ or $\phi(x,y)=2x+3y=4$ —— (1)

so $\boxed{F=f+\lambda\phi}$

$F=2x^2+2y^2+2+\lambda(2x+3y)$

$\frac{\partial F}{\partial x}=4x+2\lambda=0$ —— (2)

$\frac{\partial F}{\partial y}=4y+3\lambda=0$ —— (3)

eliminate λ. from (3) $\lambda=-\frac{4}{3}y$. plug in (2) $\Rightarrow$ $4x+2\left(-\frac{4}{3}y\right)=0$

$\Rightarrow 4x-\frac{8}{3}y=0 \Rightarrow x=\frac{2}{3}y$. plug into (1) $\Rightarrow 2\left(\frac{2}{3}y\right)+3y=4$

$\Rightarrow \frac{4}{3}y+3y=4 \Rightarrow \frac{4+9}{3}y=4 \Rightarrow \frac{13}{3}y=4 \Rightarrow \boxed{y=\frac{12}{13}}$

so $x=\frac{2}{3}\frac{12}{13}=\frac{24}{3\times13}=\boxed{\frac{8}{13}}$

So point will be at $\boxed{\left(\frac{8}{13},\frac{12}{13}\right)}$ when sum of squares is smallest.

ch 4 **10.2** Find largest and smallest distance from origin to the conic whose equation is $5x^2-6xy+5y^2-32=0$ and hence determine the length of the semiaxis of conic.

Let $d^2 = x^2+y^2$. This is our $f(x,y)$ function we want to minimize.

$$\phi(x,y) = 5x^2 - 6xy + 5y^2 = 32 \qquad (1)$$

so $F = f + \lambda\phi$

$$F = x^2+y^2 + \lambda(5x^2-6xy+5y^2)$$

$$\frac{\partial F}{\partial x} = 0 = 2x + 10\lambda x - 6\lambda y \qquad (2)$$

$$\frac{\partial F}{\partial y} = 0 = 2y + 10\lambda y - 6\lambda x \qquad (3)$$

or

$$\frac{\partial F}{\partial x} = 0 = x(2+10\lambda) - 6\lambda y \qquad (2)$$

$$\frac{\partial F}{\partial y} = 0 = y(2+10\lambda) - 6\lambda x \qquad (3)$$

solve for λ from (1) and (2).

from (3) $y = \frac{6\lambda x}{2+10\lambda}$ —— (4)

Plug (4) into (2) $\Rightarrow$

$$0 = x(2+10\lambda) - 6\lambda\left(\frac{6\lambda x}{2+10\lambda}\right)$$

i.e. $0 = x(2+10\lambda) - \frac{36\lambda^2 x}{2+10\lambda}$

$$0 = x(2+10\lambda)(2+10\lambda) - 36\lambda^2 x$$

$\longrightarrow$

(13

$0 = x\,[\,4 + 100\lambda^2 + 40\lambda - 36\lambda^2\,]$

either $x = 0$ or $x \neq 0$.

if $x \neq 0$ then $64\lambda^2 + 40\lambda + 4 = 0$

$\Rightarrow \boxed{\lambda = -\frac{1}{8} \text{ or } -\frac{1}{2}}$

when $\lambda = -\frac{1}{8}$

from (3) $\Rightarrow$ $y\left(2 + 10\left(-\frac{1}{8}\right)\right) - 6\left(-\frac{1}{8}\right)x = 0$

$y\left(2 - \frac{10}{8}\right) + \frac{6}{8}x = 0$

$y\left(\frac{16-10}{8}\right) + \frac{6}{8}x = 0$

$\frac{6}{8}y + \frac{6}{8}x = 0 \Rightarrow \boxed{y = -x}$

from (1) $5x^2 - 6x(-x) + 5(-x)^2 = 32$

$5x^2 + 6x^2 + 5x^2 = 32$

$16x^2 = 32$

$x^2 = 2 \Rightarrow \boxed{x = \pm\sqrt{2}}$

so $\boxed{y = \mp\sqrt{2}}$

so for $\lambda = -\frac{1}{8}$, position points are

$\boxed{(\sqrt{2}, -\sqrt{2}) \text{ and } (-\sqrt{2}, \sqrt{2})}$

for $\lambda = -\frac{1}{2}$ from (3) $\Rightarrow$ $y\left(2 + 10\left(-\frac{1}{2}\right)\right) - 6\left(-\frac{1}{2}\right)x = 0$

i.e. $y(2 - 5) + 3x = 0$

i.e. $-3y + 3x = 0 \Rightarrow \boxed{y = x}$

so from (1) $5x^2 - 6x(x) + 5(x)^2 = 32$

$5x^2 - 6x^2 + 5x^2 = 32$

$4x^2 = 32 \Rightarrow \boxed{x = \pm 2\sqrt{2}}$

so $y = \pm 2\sqrt{2}$

so points are $(2\sqrt{2}, 2\sqrt{2})$ or $(-2\sqrt{2}, -2\sqrt{2}) \Longrightarrow$

now, all above points were found by assuming $x \neq 0$.

now for the case if $x = 0$.

from (1) $\Rightarrow$ $5y^2 = 32$

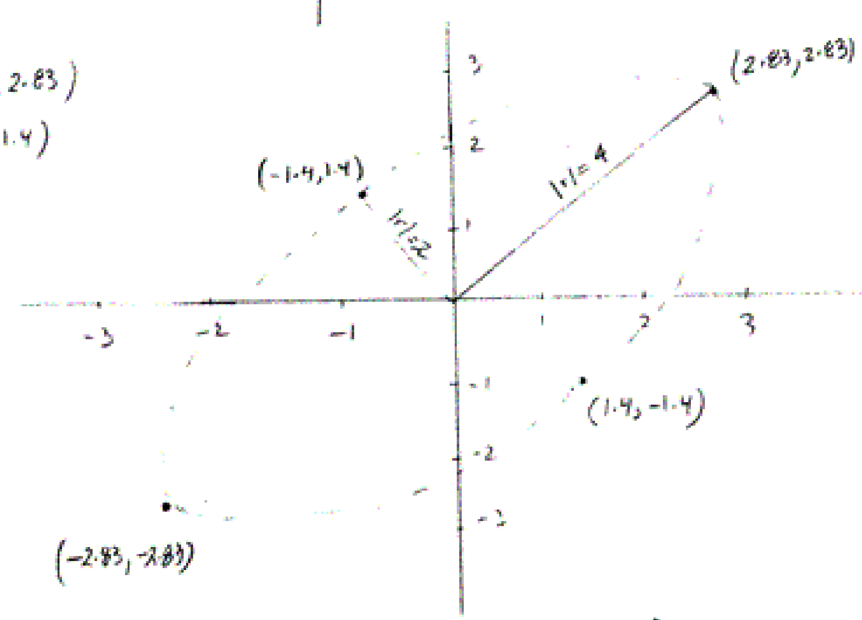
or $y^2 = \frac{32}{5}$ $\Rightarrow$ $y = \pm 2.53$

so points are $(0, 2.53)$ or $(0, -2.53)$.

So in summary I have found 6 points where $f(x,y)$ is either min or max. now for each point need to find the distance from origin to know which is max and which is min.

point	d^2	d	
$(0, \pm 2.53)$	6.4	2.53	
$(\pm 2\sqrt{2}, \pm 2\sqrt{2})$	8+8=16	4	→ largest distance
$(\pm\sqrt{2}, \pm\sqrt{2})$	2+2=4	2	→ smallest distance

note $(2\sqrt{2}, 2\sqrt{2}) = (2.83, 2.83)$
$(\sqrt{2}, \sqrt{2}) = (1.4, 1.4)$



here major axes length = 8   minor axis length = 4

ch 4

10.7

Find the Largest z for which $2x+4y=5$ and $x^2+z^2=2y$.

here the constraint $\Phi(x,y) = 2x+4y=5$ ---- (1)

and $z^2 = 2y - x^2$.

so Largest z is the that will make z^2 Largest as well.

Let $f(x,y) = 2y - x^2$.

hence $F = f + \lambda\Phi$

$= 2y - x^2 + \lambda(2x+4y)$

$\frac{\partial F}{\partial x} = 0 = -2x + 2\lambda$ ---- (2)

$\frac{\partial F}{\partial y} = 0 = 2 + 4\lambda$ ---- (3)

from (3), $\lambda = -\frac{1}{2}$

from (2) $0 = -2x + 2(-\frac{1}{2})$

i.e. $0 = -2x - 1$ i.e $\boxed{x = -\frac{1}{2}}$

so from $2x+4y=5 \Rightarrow 2(-\frac{1}{2})+4y=5 \Rightarrow y = \frac{5+1}{4} = \frac{6}{4} = \boxed{1.5}$

since $z^2 = 2y - x^2$

Then $z^2 = 2(\frac{6}{4}) - (-\frac{1}{2})^2 = 3 - \frac{1}{4} = \frac{12-1}{4} = \frac{11}{4}$

so $z = \pm\sqrt{\frac{11}{4}} = \pm\frac{1}{2}\sqrt{11}$

so Largest z is $\boxed{\frac{1}{2}\sqrt{11}}$

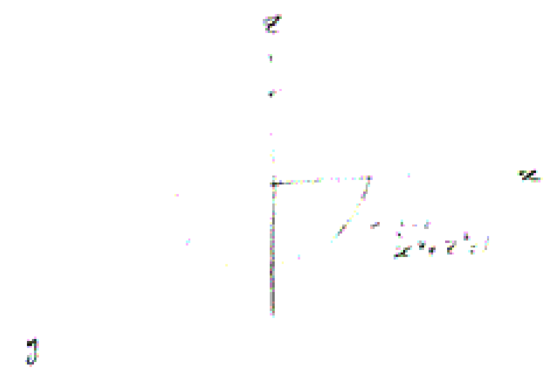
ch 4

10.10 the temp at point (x,y,z) in the sphere $x^2+y^2+z^2\leq 1$ is given by $T=y^2+xz$. Find largest and smallest values which T takes.

(a) on the circle $y=0$, $x^2+z^2=1$

(b) on the surface $x^2+y^2+z^2=1$

(c) in the whole sphere



(a). here the constraints are $y=0$ and $x^2+z^2=1$ while $f=y^2+xz$. but $y=0$, hence $f=xz$

so $\phi = x^2+z^2=1$ ...... (1)

$f=xz$

$F=f+\lambda\phi$

$F=xz+\lambda(x^2+z^2)$

$\frac{\partial F}{\partial x}=0=z+2\lambda x$ —— (2)

$\frac{\partial F}{\partial z}=0=x+2\lambda z$ —— (3)

from (3), $\lambda=-\frac{x}{2z}$. sub into (2) $\Rightarrow$ $0=z+2\cdot\left(\frac{-x}{2z}\right)x$

ie $0=\frac{2z^2-2x^2}{2z}$

z can't be zero, since if $z=0$ then (3) implies $x=0$ also. but then $x^2+z^2=1$ would be a contradiction. so we can divide by z to get $0=z^2-x^2$

ie $z^2=x^2$ ...... (4)

from (1) and (4) $\Rightarrow$ $2x^2=1$ or $x=\pm\frac{1}{\sqrt{2}}$ and also $z^2=\frac{1}{2}\Rightarrow z=\pm\frac{1}{\sqrt{2}}$ $\longrightarrow$

so temp $= y^2 + xz$

$= 0 + xz$

$= \frac{1}{\sqrt{2}} \cdot -\frac{1}{\sqrt{2}}$ or $\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}} \cdot -\frac{1}{\sqrt{2}}$

$$\boxed{T = -\frac{1}{2} \text{ or } \frac{1}{2}}$$

min at $\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$

max at $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ and at $\left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$

(b) on the surface $x^2+y^2+z^2=1$

here $\phi = x^2+y^2+z^2=1$ ——— (1)

$F = y^2 + xz + \lambda\left(x^2+y^2+z^2\right)$

$\frac{\partial F}{\partial x} = z + 2\lambda x = 0$ — (2)

$\frac{\partial F}{\partial y} = 2y + 2\lambda y = 0$ (3)

$\frac{\partial F}{\partial z} = x + 2\lambda z = 0$ (4)

Remove λ from 2,3,4. this will result in 2 new equations in x,y,z. which with equation 1, we get 3 equations in x,y,z to solve

from (4) $\lambda = \frac{-x}{2z}$. (5)

From (5) and (3) $\Rightarrow$ $0 = 2y + 2\left(-\frac{x}{2z}y\right)$

$0 = 2y - \frac{xy}{z}$ ——— (6)

From (5) and (2) $\Rightarrow$ $0 = z + 2\left(\frac{-x}{2z}\right)x$

$0 = z - \frac{x^2}{z}$ ——— (7)

so now we have

$x^2+y^2+z^2 = 1$ ——— (1)

$2yz - xy = 0$ ——— (6)

$z^2 - x^2 = 0$ ——— (7)

(1×)

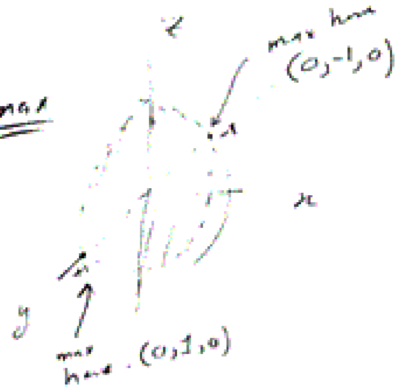
from (7), $x^2 = z^2$
from (6), $2z = x$ $\Big\}$ $\Longrightarrow$ $\boxed{x = 0,\ z = 0}$ only possible solution.

so from (1) $y^2 = 1$ or $\boxed{y = \pm 1}$

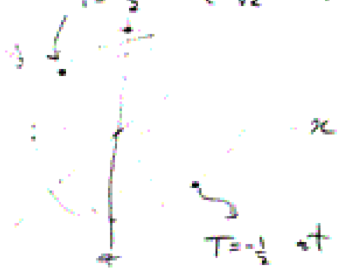
so $\boxed{T = y^2 + xz = 1}$

so $\boxed{T = 1 \text{ at } (0, \pm 1, 0)}$.

max



for the minimum, I am not sure how to find it. all what I see is that when $y = 0$, in this case which is part (a) solution.



min

(c) in the whole sphere:

this means to find T (max, min) inside sphere. i.e. $x^2+y^2+z^2<1$ —— (1) is the new constraint, in addition to constraint $x^2+y^2+z^2=1$ which I solved for in part (b). So only look at $x^2+y^2+z^2<1$ and see what min/max T I get and to compare to min/max T found in part (b) to decide.

$$\frac{\partial F}{\partial x} = 0 = z + 2\lambda x \quad \text{—— (2)}$$

$$\frac{\partial F}{\partial y} = 0 = 2y + 2\lambda y \quad \text{—— (3)}$$

$$\frac{\partial F}{\partial z} = 0 = x + 2\lambda z \quad \text{—— (4)} \quad \rightarrow$$

so from part (b), I set

$$x^2+y^2+z^2<1 \quad \text{—— ①}$$
$$2yz - xy = 0 \quad \text{—— ⑥}$$
$$z^2 - x^2 = 0 \quad \text{—— ⑦}$$

so $z^2 = x^2$ from ⑦ and from ⑥ $2z - x = 0$

So as in part (b), $x=0$, $z=0$.

so $y^2 < 1$

so $T = y^2 + xz$

$\Rightarrow$ $T<1$ inside sphere.

so at $y=0$ $T=0$, which is at origin

ie at $\boxed{(0,0,0) \quad T=0}$ which is the min.

for the max, max occurs on surface of sphere as per part (b).

(20

ch 4

11.1 in partial diff eq $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = 0$

put $s = y + 2x$, $t = y + 3x$ and show that eq becomes $\frac{\partial^2 z}{\partial s \partial t} = 0$. following the method of solving 11.6, solve the equation.

we can think of z as function $z(s,t)$. where $s(x,y)$, $t(x,y)$

So $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial x}$

So $\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)}_{\frac{\partial^2 z}{\partial x^2}} = \frac{\partial z}{\partial s}\frac{\partial^2 s}{\partial x^2} + \frac{\partial s}{\partial x}\frac{\partial^2 z}{\partial s \partial x} + \frac{\partial z}{\partial t}\frac{\partial^2 t}{\partial x^2} + \frac{\partial t}{\partial x}\frac{\partial^2 z}{\partial t \partial x}$

now find $\frac{\partial^2 z}{\partial x \partial y}$

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial z}{\partial s}\frac{\partial^2 s}{\partial x \partial y} + \frac{\partial s}{\partial x}\frac{\partial^2 z}{\partial s \partial y} + \frac{\partial z}{\partial t}\frac{\partial^2 t}{\partial x \partial y} + \frac{\partial t}{\partial x}\frac{\partial^2 z}{\partial t \partial y}$

now find $\frac{\partial^2 z}{\partial y^2}$:

$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial y}$

$\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial z}{\partial s}\frac{\partial^2 s}{\partial y^2} + \frac{\partial s}{\partial y}\frac{\partial^2 z}{\partial s \partial y}\right) + \left(\frac{\partial z}{\partial t}\frac{\partial^2 t}{\partial y^2} + \frac{\partial t}{\partial y}\frac{\partial^2 z}{\partial t \partial y}\right)$

So now plug all these in our PDE ⇒

$\frac{\partial z}{\partial s}\frac{\partial^2 s}{\partial x^2} + \frac{\partial s}{\partial x}\frac{\partial^2 z}{\partial s \partial x} + \frac{\partial z}{\partial t}\frac{\partial^2 t}{\partial x^2} + \frac{\partial t}{\partial x}\frac{\partial^2 z}{\partial t \partial x} - 5\frac{\partial z}{\partial s}\frac{\partial^2 s}{\partial x \partial y} - 5\frac{\partial s}{\partial x}\frac{\partial^2 z}{\partial s \partial y} - 5\frac{\partial z}{\partial t}\frac{\partial^2 t}{\partial x \partial y} - 5\frac{\partial t}{\partial x}\frac{\partial^2 z}{\partial t \partial y}$
$+ 6\frac{\partial z}{\partial s}\frac{\partial^2 s}{\partial y^2} + 6\frac{\partial s}{\partial y}\frac{\partial^2 z}{\partial s \partial y} + 6\frac{\partial z}{\partial t}\frac{\partial^2 t}{\partial y^2} + 6\frac{\partial t}{\partial y}\frac{\partial^2 z}{\partial t \partial y} = 0$ (1)

now, $\frac{\partial s}{\partial x} = 2$, $\frac{\partial^2 s}{\partial x^2} = 0$, $\frac{\partial t}{\partial x} = 3$, $\frac{\partial^2 t}{\partial x^2} = 0$, $\frac{\partial s}{\partial y} = 1$, $\frac{\partial^2 s}{\partial y^2} = 0$

$\frac{\partial t}{\partial y} = 1$, $\frac{\partial^2 t}{\partial y^2} = 0$, $\frac{\partial^2 s}{\partial x \partial y} = 0$, $\frac{\partial^2 t}{\partial x \partial y} = 0$ sub into (1) ⟶

(2)

$$0+2\frac{\partial^2 z}{\partial s\partial x}+0+3\frac{\partial^2 z}{\partial t\partial x}-0-5\times 2\frac{\partial^2 z}{\partial s\partial y}-0-5\times 3\frac{\partial^2 z}{\partial t\partial y}$$

$$+0+6\frac{\partial^2 z}{\partial s\partial y}+0+6\frac{\partial^2 z}{\partial t\partial y}=0$$

or $$2\frac{\partial^2 z}{\partial s\partial x}+3\frac{\partial^2 z}{\partial t\partial x}-10\frac{\partial^2 z}{\partial s\partial y}-15\frac{\partial^2 z}{\partial t\partial y}+6\frac{\partial^2 z}{\partial s\partial y}+6\frac{\partial^2 z}{\partial t\partial y}=0$$

$$2\frac{\partial^2 z}{\partial s\partial x}+3\frac{\partial^2 z}{\partial t\partial x}-4\frac{\partial^2 z}{\partial s\partial y}-9\frac{\partial^2 z}{\partial t\partial y}=0 \quad (1A)$$

now $$\frac{\partial^2 z}{\partial s\partial x}=\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial z}{\partial t}\frac{\partial t}{\partial x}\right)=\frac{\partial}{\partial s}\left(2\frac{\partial z}{\partial s}+3\frac{\partial z}{\partial t}\right)$$

$$=2\frac{\partial^2 z}{\partial s^2}+3\frac{\partial^2 z}{\partial s\partial t} \quad (2)$$

and $$\frac{\partial^2 z}{\partial t\partial x}=\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial t}\left(2\frac{\partial z}{\partial s}+3\frac{\partial z}{\partial t}\right)=2\frac{\partial^2 z}{\partial s\partial t}+3\frac{\partial^2 z}{\partial t^2} \quad (3)$$

and $$\frac{\partial^2 z}{\partial s\partial y}=\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial z}{\partial t}\frac{\partial t}{\partial y}\right)=\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial s}\cdot 1+\frac{\partial z}{\partial t}\cdot 1\right)$$

$$=\frac{\partial^2 z}{\partial s^2}+\frac{\partial^2 z}{\partial s\partial t} \quad (4)$$

and $$\frac{\partial^2 z}{\partial t\partial y}=\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial s}+\frac{\partial z}{\partial t}\right)=\frac{\partial^2 z}{\partial t\partial s}+\frac{\partial^2 z}{\partial t^2} \quad (5)$$

Plug (2),(3),(4),(5) into (1A) $\Rightarrow$

$$2\left(2\frac{\partial^2 z}{\partial s^2}+3\frac{\partial^2 z}{\partial s\partial t}\right)+3\left(2\frac{\partial^2 z}{\partial s\partial t}+3\frac{\partial^2 z}{\partial t^2}\right)-4\left(\frac{\partial^2 z}{\partial s^2}+\frac{\partial^2 z}{\partial s\partial t}\right)-9\left(\frac{\partial^2 z}{\partial t\partial s}+\frac{\partial^2 z}{\partial t^2}\right)=0$$

$$4\frac{\partial^2 z}{\partial s^2}+6\frac{\partial^2 z}{\partial s\partial t}+6\frac{\partial^2 z}{\partial s\partial t}+9\frac{\partial^2 z}{\partial t^2}-4\frac{\partial^2 z}{\partial s^2}-4\frac{\partial^2 z}{\partial s\partial t}-9\frac{\partial^2 z}{\partial t\partial s}-9\frac{\partial^2 z}{\partial t^2}=0$$

$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial s\partial t}=0}$$

Ch 4

11.3 Suppose $\omega = f(x,y)$ satisfies $\frac{\partial^2 \omega}{\partial x^2} - \frac{\partial^2 \omega}{\partial y^2} = 1$

Put $x = u + v$, $y = u - v$ and show that ω satisfies $\frac{\partial^2 \omega}{\partial u \partial v} = 1$. hence solve the equation.

$$\left.\begin{array}{l} x(u,v) = u + v \\ y(u,v) = u - v \end{array}\right\} \Rightarrow \begin{array}{l} \frac{\partial x}{\partial u} = 1, \frac{\partial x}{\partial v} = 1, \frac{\partial^2 x}{\partial u^2} = 0, \frac{\partial^2 x}{\partial v^2} = 0 \\ \frac{\partial y}{\partial u} = 1, \frac{\partial y}{\partial v} = -1, \frac{\partial^2 y}{\partial u^2} = 0, \frac{\partial^2 y}{\partial v^2} = 0 \end{array}$$

$$\frac{\partial \omega}{\partial u} = \frac{\partial \omega}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \omega}{\partial y}\frac{\partial y}{\partial u}$$

so $\frac{\partial \omega}{\partial u} = \frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y}$ ——— (1)

$$\frac{\partial \omega}{\partial v} = \frac{\partial \omega}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \omega}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial \omega}{\partial x} - \frac{\partial \omega}{\partial y} \quad \text{——— (2)}$$

From (1) and (2) solve for $\frac{\partial \omega}{\partial x}$ and $\frac{\partial \omega}{\partial y}$.

From (1), $\frac{\partial \omega}{\partial x} = \left(\frac{\partial \omega}{\partial u} - \frac{\partial \omega}{\partial y}\right)$

sub into (2) $\Rightarrow$ $\frac{\partial \omega}{\partial v} = \left(\frac{\partial \omega}{\partial u} - \frac{\partial \omega}{\partial y}\right) - \frac{\partial \omega}{\partial y}$

Now from above find $\frac{\partial \omega}{\partial y}$:

$$\frac{\partial \omega}{\partial v} = \frac{\partial \omega}{\partial u} - 2\frac{\partial \omega}{\partial y} \Rightarrow \boxed{\frac{\partial \omega}{\partial y} = \frac{1}{2}\left(\frac{\partial \omega}{\partial u} - \frac{\partial \omega}{\partial v}\right)} \text{——— (3)}$$

So from (1) $\frac{\partial \omega}{\partial u} = \frac{\partial \omega}{\partial x} + \frac{1}{2}\left(\frac{\partial \omega}{\partial u} - \frac{\partial \omega}{\partial v}\right)$

$$\frac{\partial \omega}{\partial x} = \frac{\partial \omega}{\partial u} - \frac{1}{2}\left(\frac{\partial \omega}{\partial u} - \frac{\partial \omega}{\partial v}\right)$$

$$\boxed{\frac{\partial \omega}{\partial x} = \frac{1}{2}\frac{\partial \omega}{\partial u} + \frac{1}{2}\frac{\partial \omega}{\partial v}} \text{——— (4)}$$

now need to find second derivatives $\frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}$

introduce $G = \frac{\partial w}{\partial x}$

$H = \frac{\partial w}{\partial y}$

so (3),(4) can be rewritten as

$$H = \frac{1}{2}\frac{\partial w}{\partial u} - \frac{1}{2}\frac{\partial w}{\partial v} \quad \text{—— (3A)}$$

$$G = \frac{1}{2}\frac{\partial w}{\partial u} + \frac{1}{2}\frac{\partial w}{\partial v} \quad \text{—— (4A)}$$

so $\frac{\partial^2 w}{\partial x^2} = \frac{\partial G}{\partial x}$

$\frac{\partial^2 w}{\partial y^2} = \frac{\partial H}{\partial y}$

so $1 = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \Rightarrow 1 = \frac{\partial G}{\partial x} - \frac{\partial H}{\partial y}$ —— (5)

now replace w by H in eq (3) and replace w by G in equation (4) $\Rightarrow$

$$\frac{\partial H}{\partial y} = \frac{1}{2}\frac{\partial H}{\partial u} - \frac{1}{2}\frac{\partial H}{\partial v} \quad \text{—— (3B)}$$

$$\frac{\partial G}{\partial x} = \frac{1}{2}\frac{\partial G}{\partial u} + \frac{1}{2}\frac{\partial G}{\partial v} \quad \text{—— (4B)}$$

sub 3B, 4B into equation (5)

$$1 = \frac{1}{2}\frac{\partial G}{\partial u} + \frac{1}{2}\frac{\partial G}{\partial v} - \frac{1}{2}\frac{\partial H}{\partial u} + \frac{1}{2}\frac{\partial H}{\partial v} \quad \text{—— (7)}$$

now need to find $\frac{\partial G}{\partial u}, \frac{\partial G}{\partial v}, \frac{\partial H}{\partial u}, \frac{\partial H}{\partial v}$. i.e. this from equations (3A) and (4A)

$\Rightarrow$

$$\frac{\partial G}{\partial u} = \frac{1}{2}\frac{\partial^2 w}{\partial u^2} + \frac{1}{2}\frac{\partial^2 w}{\partial v \partial u}$$

$$\frac{\partial G}{\partial v} = \frac{1}{2}\frac{\partial^2 w}{\partial u \partial v} + \frac{1}{2}\frac{\partial^2 w}{\partial v^2}$$

$$\frac{\partial H}{\partial u} = \frac{1}{2}\frac{\partial^2 w}{\partial u^2} - \frac{1}{2}\frac{\partial^2 w}{\partial v \partial u}$$

$$\frac{\partial H}{\partial v} = \frac{1}{2}\frac{\partial^2 w}{\partial u \partial v} - \frac{1}{2}\frac{\partial^2 w}{\partial v^2}$$

Plug these into (7) to get

$$1 = \frac{1}{2}\left(\frac{1}{2}\frac{\partial^2 w}{\partial u^2} + \frac{1}{2}\frac{\partial^2 w}{\partial v \partial u}\right) + \frac{1}{2}\left(\frac{1}{2}\frac{\partial^2 w}{\partial u \partial v} + \frac{1}{2}\frac{\partial^2 w}{\partial v^2}\right)$$
$$- \frac{1}{2}\left(\frac{1}{2}\frac{\partial^2 w}{\partial u^2} - \frac{1}{2}\frac{\partial^2 w}{\partial v \partial u}\right) - \frac{1}{2}\left(\frac{1}{2}\frac{\partial^2 w}{\partial u \partial v} - \frac{1}{2}\frac{\partial^2 w}{\partial v^2}\right)$$

$$1 = \frac{1}{4}w_{uu} + \frac{1}{4}w_{uv} + \frac{1}{4}w_{uv} + \frac{1}{4}w_{vv} - \frac{1}{4}w_{uu} + \frac{1}{4}w_{vu} + \frac{1}{4}w_{uv} - \frac{1}{4}w_{vv}$$

$$1 = \frac{1}{4}w_{vu} + \frac{1}{4}w_{uv} + \frac{1}{4}w_{vu} + \frac{1}{4}w_{vu}$$

and since $w_{uv} = w_{vu}$

then

$$\boxed{1 = w_{uv} = \frac{\partial^2 w}{\partial u \partial v}}$$

so solution is found from

$$\frac{\partial^2 w}{\partial u \partial v} = 1 \qquad \text{i.e. } \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial u}\right) = 1$$

or $\frac{\partial w}{\partial u} = v + A \implies \boxed{w = vu + Au + B}$

where A, B are constants.

Ch 4
11.6 reduce the equation $x^2\left(\frac{d^2y}{dx^2}\right)+2x\left(\frac{dy}{dx}\right)-5y=0$ to a differential equation with constant coefficients in $\frac{d^2y}{dz^2}, \frac{dy}{dz}$ and y by the change of variable $x=e^z$.

we are given $y(x)$. we need to rewrite the equation so that y is now a function of z instead.

$$\frac{dy}{dz}=\frac{dy}{dx}\frac{dx}{dz}$$

$$\frac{dy}{dz}=\frac{dy}{dx}e^z \Rightarrow \boxed{\frac{dy}{dx}=\frac{1}{e^z}\frac{dy}{dz}} \quad (2)$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(e^{-z}\frac{dy}{dz}\right)$$

$$=e^{-z}\frac{d^2y}{dz\,dx}+\frac{dy}{dz}\left(\frac{d}{dx}e^{-z}\right)$$

since $e^{-z}=x^{-1}$ given.

$$=e^{-z}\frac{d}{dz}\left(\frac{dy}{dx}\right)+\frac{dy}{dz}\left(\frac{d}{dx}\left(\frac{1}{x}\right)\right)$$

$$=e^{-z}\frac{d}{dz}\left(\frac{1}{e^z}\frac{dy}{dz}\right)+\frac{dy}{dz}\left(-\frac{1}{x^2}\right)$$

$$\frac{d^2y}{dx^2}=e^{-z}\left(e^{-z}\frac{d^2y}{dz^2}+\frac{dy}{dz}(-e^{-z})\right)-\frac{dy}{dz}\frac{1}{x^2}, \text{ but } \frac{1}{x^2}=e^{-2z}$$

so
$$\frac{d^2y}{dx^2}=e^{-2z}\frac{d^2y}{dz^2}-e^{-2z}\frac{dy}{dz}-\frac{dy}{dz}e^{-2z} \quad (3)$$

Plug (2) and (3) into (1) $\Rightarrow$ and replace x^2 by e^{2z}

$$e^{2z}\left(e^{-2z}\frac{d^2y}{dz^2}-e^{-2z}\frac{dy}{dz}-\frac{dy}{dz}e^{-2z}\right)+2e^{z}\left(e^{-z}\frac{dy}{dz}\right)-5y=0$$

$$\frac{d^2y}{dz^2}-\frac{dy}{dz}-\frac{dy}{dz}+2\frac{dy}{dz}-5y=0 \Rightarrow \boxed{\frac{d^2y}{dz^2}-5y=0.}$$

Ch 4 [11.7] transfer $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$

to $\frac{d^2y}{d\theta^2} + \cot\theta\frac{dy}{d\theta} + 2y = 0$ by using $x = \cos\theta$.

Solution:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

$$\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta}$$

$$= \frac{dy}{dx}(-\sin\theta)$$

so $\frac{dy}{dx} = \frac{-1}{\sin\theta}\frac{dy}{d\theta}$ ——— (1)

let $G_1 = \frac{dy}{dx}$

so $G_1 = \frac{dy}{dx} = -\frac{1}{\sin\theta}\frac{dy}{d\theta}$ ——— (1a)

so our DE is

$$(1-\cos^2\theta)\frac{dG_1}{dx} - 2\cos\theta\, G_1 + 2y = 0 \quad \text{——— (2)}$$

(1) is correct for any function. replacing y by G_1 in (1)

$$\frac{dG_1}{dx} = -\frac{1}{\sin\theta}\frac{dG_1}{d\theta} \quad \text{——— (3)}$$

sub (3) into (2)

$$(1-\cos^2\theta)\left(-\frac{1}{\sin\theta}\frac{dG_1}{d\theta}\right) - 2\cos\theta\, G_1 + 2y = 0 \quad \text{——— (4)}$$

need to find $\frac{dG_1}{d\theta}$. From (1a)

$$\frac{dG_1}{d\theta} = \frac{d}{d\theta}\left(-\frac{1}{\sin\theta}\frac{dy}{d\theta}\right) = -\left(-\frac{\cos\theta}{\sin^2\theta}\frac{dy}{d\theta} + \frac{1}{\sin\theta}\frac{d^2y}{d\theta^2}\right)$$

Sub the above into (4) to get the solution needed →

$$(1-\cos^2\theta)\left(-\frac{1}{\sin\theta}\left(\frac{\cos\theta}{\sin^2\theta}\frac{dy}{d\theta}-\frac{1}{\sin\theta}\frac{d^2y}{d\theta^2}\right)\right)-2\cos\theta\left(-\frac{1}{\sin\theta}\frac{dy}{d\theta}\right)+2y=0$$

So above becomes

$$(1-\cos^2\theta)\left(-\frac{\cos\theta}{\sin^3\theta}\frac{dy}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2y}{d\theta^2}\right)+2\frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta}+2y=0$$

$$-\frac{\cos\theta}{\sin^3\theta}\frac{dy}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2y}{d\theta^2}+\frac{\cos^3\theta}{\sin^3\theta}\frac{dy}{d\theta}-\frac{\cos^2\theta}{\sin^2\theta}\frac{d^2y}{d\theta^2}+2\frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta}+2y=0$$

$$\frac{d^2y}{d\theta^2}\left(\frac{1}{\sin^2\theta}-\frac{\cos^2\theta}{\sin^2\theta}\right)+\frac{dy}{d\theta}\left(\frac{-\cos\theta}{\sin^3\theta}+\frac{\cos^3\theta}{\sin^3\theta}+2\frac{\cos\theta}{\sin\theta}\right)+2y=0$$

$$\frac{d^2y}{d\theta^2}\left(\frac{1-\cos^2\theta}{\sin^2\theta}\right)+\frac{dy}{d\theta}\left(\frac{-\cos\theta+\cos^3\theta+2\cos\theta\sin^2\theta}{\sin^3\theta}\right)+2y=0$$

$$\frac{d^2y}{d\theta^2}\left(\frac{\sin^2\theta}{\sin^2\theta}\right)+\frac{dy}{d\theta}\left(\frac{-\cos\theta+\cos\theta(1-\sin^2\theta)+2\cos\theta\sin^2\theta}{\sin^3\theta}\right)+2y=0$$

$$\frac{d^2y}{d\theta^2}(1)+\frac{dy}{d\theta}\left(\frac{-\cos\theta+\cos\theta-\cos\theta\sin^2\theta+2\cos\theta\sin^2\theta}{\sin^3\theta}\right)+2y=0$$

$$\frac{d^2y}{d\theta^2}+\frac{dy}{d\theta}\left(\frac{\cos\theta\sin^2\theta}{\sin^3\theta}\right)+2y=0$$

$$\boxed{\frac{d^2y}{d\theta^2}+\cot\theta\frac{dy}{d\theta}+2y=0}$$

Ch 4

11.8 Change x to $u = 2\sqrt{x}$ in $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (1-x)y = 0$

and show eqn becomes $u^2 \frac{d^2y}{du^2} + u\frac{dy}{du} + (u^2-4)y = 0$

$u = 2\sqrt{x} \Rightarrow x = \left(\frac{u}{2}\right)^2, \quad x^2 = \left(\frac{u}{2}\right)^4$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{du}\frac{du}{dx}\right) = \frac{dy}{du}\frac{d^2u}{dx^2} + \frac{du}{dx}\frac{d^2y}{dudx}$$

So our DE becomes

$$\left(\frac{u}{2}\right)^4\left[\frac{dy}{du}\frac{d^2u}{dx^2} + \frac{du}{dx}\frac{d^2y}{dudx}\right] + \left(\frac{u}{2}\right)^2\left[\frac{dy}{du}\frac{du}{dx}\right] - \left(1-\left(\frac{u}{2}\right)^2\right)y = 0 \quad \text{—— (1)}$$

now $\frac{du}{dx} = 2\left(\frac{1}{2}x^{-\frac{1}{2}}\right) = \frac{1}{\sqrt{x}}$

$$\frac{d^2u}{dx^2} = -\frac{1}{2}\left(x^{-3/2}\right) = -\frac{1}{2x^{3/2}}$$

i.e. in terms of u, $\frac{du}{dx} = \frac{2}{u}$

and $\frac{d^2u}{dx^2} = -\frac{1}{2}\frac{1}{\left(\frac{u}{2}\right)^3} = -\frac{1}{2}\frac{8}{u^3} = -\frac{4}{u^3}$

So now (1) becomes

$$\frac{u^4}{16}\left[\frac{dy}{du}\left(-\frac{4}{u^3}\right) + \left(\frac{2}{u}\right)\frac{d^2y}{dudx}\right] + \frac{u^2}{4}\left[\frac{dy}{du}\left(\frac{2}{u}\right)\right] - \left(1-\frac{u^2}{4}\right)y = 0$$

$$-\frac{u}{4}\frac{dy}{du} + \frac{u^3}{8}\frac{d^2y}{dudx} + \frac{u}{2}\frac{dy}{du} - \left(1-\frac{u^2}{4}\right)y = 0 \quad \text{—— (2)}$$

now what is left is to find $\frac{d^2y}{dudx}$.

$$\frac{d^2y}{dudx} = \frac{d}{du}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{du}\frac{du}{dx}\right) = \frac{d}{du}\left(\frac{dy}{du}\frac{1}{\sqrt{x}}\right) = \frac{d}{du}\left(\frac{dy}{du}\frac{2}{u}\right) = 2\frac{d}{du}\left(\frac{1}{u}\frac{dy}{du}\right)$$

note: y is function of x here not u.

$$\frac{d^2 u}{du\,dx} = 2\left(\frac{1}{u}\frac{d^2 y}{dx^2} + \frac{dy}{dx}\left(-\frac{1}{u^2}\right)\right) = \boxed{\frac{2}{u}\frac{d^2 y}{du^2}}$$

= 0 since u is held constant here when diff. y w.r.t. x. i.e. y(x) only, so $\frac{du}{dx}$ = 0 here only.

So now (2) becomes

$$-\frac{u}{4}\frac{dy}{du} + \frac{u^2}{8}\left[\frac{2}{u}\frac{d^2y}{du^2}\right] + \frac{u}{2}\frac{dy}{du} - \left(1 - \frac{u^2}{4}\right)y = 0$$

$$-\frac{u}{4}\frac{dy}{du} + \frac{u^2}{4}\frac{d^2y}{du^2} + \frac{u}{2}\frac{dy}{du} - \left(1 - \frac{u^2}{4}\right)y = 0$$

$$\frac{d^2y}{du^2}\left(\frac{u^2}{4}\right) + \frac{dy}{du}\left(-\frac{u}{4} + \frac{u}{2}\right) + \left(\frac{u^2}{4} - 1\right)y = 0$$

$\frac{u}{4}$

×4 →

$$\boxed{u^2\frac{d^2y}{du^2} + u\frac{dy}{du} + (u^2 - 4)y = 0}$$

QED

Ch 4

11.9 if $x = e^s \cos t$, $y = e^s \sin t$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s}\left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right)$$

in $x = e^s \cos t$, here x is function of s and t.

i.e. $x(s,t) = e^s \cos t$.

and $y(s,t) = e^s \sin t$.

$$\frac{\partial x}{\partial s} = e^s \cos t$$

$$\frac{\partial x}{\partial t} = -e^s \sin t$$

$$\frac{\partial^2 x}{\partial s^2} = e^s \cos t$$

$$\frac{\partial^2 x}{\partial t^2} = -e^s \cos t$$

$$\frac{\partial y}{\partial s} = e^s \sin t$$

$$\frac{\partial y}{\partial t} = e^s \cos t$$

$$\frac{\partial^2 y}{\partial s^2} = e^s \sin t$$

$$\frac{\partial^2 y}{\partial t^2} = -e^s \sin t$$

$$\frac{\partial u(x,y)}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = \boxed{\frac{\partial u}{\partial x} e^s \cos t + \frac{\partial u}{\partial y} e^s \sin t} \quad (1)$$

$$\frac{\partial u(x,y)}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = \boxed{-\frac{\partial u}{\partial x} e^s \sin t + \frac{\partial u}{\partial y} e^s \cos t} \quad (2)$$

now, solve (1)(2) for $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$

(3)

From (1) $\frac{\partial u}{\partial y} = \frac{\frac{\partial u}{\partial s} - \frac{\partial u}{\partial x} e^s \cos t}{e^s \sin t}$

Sub. into (2)

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} e^s \sin t + \left(\frac{\frac{\partial u}{\partial s} - \frac{\partial u}{\partial x} e^s \cos t}{e^s \sin t} \right) e^s \cos t$$

$$\frac{\partial u}{\partial t} = \frac{-\frac{\partial u}{\partial x} e^{2s} \sin^2 t + \left(\frac{\partial u}{\partial s} - \frac{\partial u}{\partial x} e^s \cos t \right) e^s \cos t}{e^s \sin t}$$

$$e^s \sin t \frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} e^{2s} \sin^2 t + e^s \cos t \frac{\partial u}{\partial s} - e^{2s} \cos^2 t \frac{\partial u}{\partial x}$$

$$e^s \sin t \frac{\partial u}{\partial t} - e^s \cos t \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \left(-e^{2s} \sin^2 t - e^{2s} \cos^2 t \right)$$

so
$$\frac{\partial u}{\partial x} = \frac{e^s \sin t \frac{\partial u}{\partial t} - e^s \cos t \frac{\partial u}{\partial s}}{-e^{2s} \sin^2 t - e^{2s} \cos^2 t} = \frac{\sin t \frac{\partial u}{\partial t} - \cos t \frac{\partial u}{\partial s}}{-e^s \sin^2 t - e^s \cos^2 t}$$

$$= \frac{\sin t \frac{\partial u}{\partial t} - \cos t \frac{\partial u}{\partial s}}{-e^s \left(\sin^2 t + \cos^2 t \right)} = \frac{\sin t \frac{\partial u}{\partial t} - \cos t \frac{\partial u}{\partial s}}{-e^s}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{1}{e^s} \left(\cos t \frac{\partial u}{\partial s} - \sin t \frac{\partial u}{\partial t} \right)} \quad \text{------ (3)}$$

So, From (2), plug above into (2) to find $\frac{\partial u}{\partial y}$:

$$\frac{\partial u}{\partial t} = -\frac{1}{e^s} \left(\cos t \frac{\partial u}{\partial s} - \sin t \frac{\partial u}{\partial t} \right) e^s \sin t + \frac{\partial u}{\partial y} e^s \cos t$$

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$$\frac{\partial u}{\partial y} = \frac{\frac{\partial u}{\partial t} + \frac{1}{e^s}\left(\cos t \frac{\partial u}{\partial s} - \sin t \frac{\partial u}{\partial t}\right) e^s \sin t}{e^s \cos t}$$

$$= \frac{1}{e^s \cos t}\frac{\partial u}{\partial t} + \frac{1}{e^{2s}\cos t}\left(\cos t \frac{\partial u}{\partial s} - \sin t \frac{\partial u}{\partial t}\right) e^s \sin t$$

$$= \frac{1}{e^s \cos t}\frac{\partial u}{\partial t} + \frac{1}{e^{2s}}\frac{\partial u}{\partial s} e^s \sin t - \frac{e^s \sin^2 t}{e^{2s}\cos t}\frac{\partial u}{\partial t}$$

$$= \frac{1}{e^s \cos t}\frac{\partial u}{\partial t} + \frac{1}{e^s}\frac{\partial u}{\partial s}\sin t - \frac{\sin^2 t}{e^s \cos t}\frac{\partial u}{\partial t}$$

$$= \frac{\partial u}{\partial t}\left(\frac{1}{e^s \cos t} - \frac{\sin^2 t}{e^s \cos t}\right) + \frac{\partial u}{\partial s}\left(\frac{\sin t}{e^s}\right)$$

$$= \frac{\partial u}{\partial t}\left(\frac{1-\sin^2 t}{e^s \cos t}\right) + \frac{\partial u}{\partial s}\left(\frac{\sin t}{e^s}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t}\left(\frac{\cos t}{e^s}\right) + \frac{\partial u}{\partial s}\left(\frac{\sin t}{e^s}\right)$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{1}{e^s}\left(\cos t \frac{\partial u}{\partial t} + \sin t \frac{\partial u}{\partial s}\right)} \quad \text{——— (4)}$$

to find second derivatives, let $G = \frac{\partial u}{\partial x}$

$H = \frac{\partial u}{\partial y}$

So (3) becomes $G = \frac{1}{e^s}\left(\cos t \frac{\partial u}{\partial s} - \sin t \frac{\partial u}{\partial t}\right)$ ——— (5)

and (4) becomes $H = \frac{1}{e^s}\left(\cos t \frac{\partial u}{\partial t} + \sin t \frac{\partial u}{\partial s}\right)$ ——— (6)

now equations 3) and (4) are true for any Function, so replace u by G in (3) and u by H in (4) $\Rightarrow$

$$\frac{\partial G}{\partial x} = \frac{1}{e^s}\left(\cos t \frac{\partial G}{\partial s} - \sin t \frac{\partial G}{\partial t}\right) \quad \text{--- (7)}$$

From (3) by replacing u by G

$$\frac{\partial H}{\partial y} = \frac{1}{e^s}\left(\cos t \frac{\partial H}{\partial t} + \sin t \frac{\partial H}{\partial s}\right) \quad \text{--- (8)}$$

from (4) by replacing u by H.

so $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{\partial G}{\partial x} + \frac{\partial H}{\partial y}$

so, given (7) and (8), we get

$$\Rightarrow \frac{1}{e^s}\left(\cos t \frac{\partial G}{\partial s} - \sin t \frac{\partial G}{\partial t}\right) + \frac{1}{e^s}\left(\cos t \frac{\partial H}{\partial t} + \sin t \frac{\partial H}{\partial s}\right) \quad \text{--- (9)}$$

to find $\frac{\partial G}{\partial s}$, $\frac{\partial G}{\partial t}$, $\frac{\partial H}{\partial t}$, $\frac{\partial H}{\partial s}$, differentiate eqn (5), (6)

From (5)

$$\frac{\partial G}{\partial s} = \frac{1}{e^s}\left(\frac{\partial^2 u}{\partial s^2}\cos t - \sin t \frac{\partial^2 u}{\partial t \partial s}\right) + \left(\cos t \frac{\partial u}{\partial s} - \sin t \frac{\partial u}{\partial t}\right)\left(-\frac{1}{e^s}\right)$$

$$\frac{\partial G}{\partial s} = \frac{1}{e^s}\left(\frac{\partial^2 u}{\partial s^2}\cos t - \sin t \frac{\partial^2 u}{\partial t \partial s} + \sin t \frac{\partial u}{\partial t} - \cos t \frac{\partial u}{\partial s}\right) \quad (10)$$

$$\frac{\partial G}{\partial t} = \frac{1}{e^s}\cos t \frac{\partial^2 u}{\partial s \partial t} + \frac{1}{e^s}\frac{\partial u}{\partial s}(-\sin t) - \frac{1}{e^s}\sin t \frac{\partial^2 u}{\partial t^2} - \frac{1}{e^s}\frac{\partial u}{\partial t}\cos t$$

$$\frac{\partial G}{\partial t} = \frac{1}{e^s}\left(\frac{\partial^2 u}{\partial s \partial t}\cos t - \frac{\partial u}{\partial s}\sin t - \frac{\partial^2 u}{\partial t^2}\sin t - \frac{\partial u}{\partial t}\cos t\right) \text{--- (11)}$$

From (6)

$$\frac{\partial H}{\partial s} = \frac{1}{e^s}\left(\cos t \frac{\partial^2 u}{\partial t \partial s} + \sin t \frac{\partial^2 u}{\partial s^2}\right) + \left(\cos t \frac{\partial u}{\partial t} + \sin t \frac{\partial u}{\partial s}\right)\left(-\frac{1}{e^s}\right)$$

$$= \frac{1}{e^s}\left(\frac{\partial^2 u}{\partial t \partial s}\cos t + \frac{\partial^2 u}{\partial s^2}\sin t - \frac{\partial u}{\partial t}\cos t - \frac{\partial u}{\partial s}\sin t\right) \text{--- (12)}$$

$$\frac{\partial H}{\partial t} = \frac{1}{e^s}\cos t \frac{\partial^2 u}{\partial t^2} + \frac{1}{e^s}\frac{\partial u}{\partial t}(-\sin t) + \frac{1}{e^s}\sin t \frac{\partial^2 u}{\partial s \partial t} + \frac{1}{e^s}\frac{\partial u}{\partial s}\cos t$$

$\rightarrow$

$$\frac{\partial H}{\partial t} = \frac{1}{e^s}\left(\frac{\partial^2 u}{\partial t^2}\cos t - \frac{\partial u}{\partial t}\sin t + \frac{\partial^2 u}{\partial s\partial t}\sin t + \frac{\partial u}{\partial s}\cos t\right) \quad (13)$$

now sub above equations (10),(11),(12),(13) for $\frac{\partial G}{\partial s}, \frac{\partial G}{\partial t}, \frac{\partial H}{\partial s}, \frac{\partial H}{\partial t}$ into equation (9) $\Rightarrow$ (write C to mean $\cos t$, S to mean $\sin t$ write $u_s, u_{ss}, u_t, u_{tt}, u_{ts}$ to make it easier to do)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{e^s}\left(\cos t\frac{\partial G}{\partial s} - \sin t\frac{\partial G}{\partial t}\right) + \frac{1}{e^s}\left(\cos t\frac{\partial H}{\partial t} + \sin t\frac{\partial H}{\partial s}\right)$$

$$= \frac{1}{e^s}\left[\cos t\left(\frac{1}{e^s}(--)\right) - \sin t\left(\frac{1}{e^s}(--)\right)\right] + \frac{1}{e^s}\left[\cos t\left(\frac{1}{e^s}(--)\right) + \sin t\left(\frac{1}{e^s}(--)\right)\right]$$

$$= \frac{1}{e^{2s}}\Big[C(u_{ss}C - u_{st}S + u_t S - u_s C)$$
$$- S(u_{st}C - u_s S - u_{tt}S - u_t C)$$
$$+ C(u_{tt}C - u_t S + u_{ts}S + u_s C)$$
$$+ S(u_{ts}C + u_{ss}S - u_t C - u_s S)\Big]$$

$$= \frac{1}{e^{2s}}\Big[u_{ss}C^2 - u_{st}SC + u_t CS - u_s C^2$$
$$- u_{st}SC + u_s S^2 + u_{tt}S^2 + u_t SC$$
$$+ u_{tt}C^2 - u_t SC + u_{ts}SC + u_s C^2$$
$$+ u_{ts}SC + u_{ss}S^2 - u_t CS - u_s S^2\Big]$$

$$= \frac{1}{e^{2s}}\left[u_{ss}(C^2+S^2) + u_{tt}(C^2+S^2)\right]$$

but $C^2 + S^2 \equiv \cos^2 + \sin^2 = 1$

so $$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{e^{2s}}\left[u_{ss} + u_{tt}\right]}$$

QED

ch 4

12.1 Find $\frac{dy}{dx}$ given $y = \int_0^{\sqrt{x}} \sin t^2 \, dt$

$$\frac{dy}{dx} = \sin(\sqrt{x}^2)\frac{d}{dx}(\sqrt{x}) - \sin(0^2)$$

$$= \sin(x)\frac{d}{dx}\sqrt{x} = \sin(x)\left(\frac{1}{2}\frac{1}{\sqrt{x}}\right) = \boxed{\frac{\sin(x)}{2\sqrt{x}}}$$

12.2 if $S = \int_u^v \frac{1-e^t}{t} dt$ Find $\frac{\partial s}{\partial v}, \frac{\partial s}{\partial u}$ and also their limits as u and v tend to zero.

using Leibniz rule, take differential on the integral:

$$ds = d\left(\int_u^v \frac{1-e^t}{t} dt\right)$$

$$ds = \frac{1-e^{(v)}}{v} dv - \left(\frac{1-e^u}{u}\right) du$$

$$\frac{ds}{dv} = \frac{1-e^v}{v} - \left(\frac{1-e^u}{u}\right)\frac{du}{dv}$$ so assuming u is not a function of v.

so $$\boxed{\frac{\partial s}{\partial v} = \frac{1-e^v}{v}}$$

similarly $$\frac{ds}{du} = \frac{1-e^v}{v}\frac{dv}{du} - \left(\frac{1-e^u}{u}\right)$$

so $$\boxed{\frac{\partial s}{\partial u} = -\left(\frac{1-e^u}{u}\right)}$$

$$\lim_{v\to 0}\frac{\partial s}{\partial v} = \lim_{v \to 0}\frac{1-e^v}{v}$$

use L'Hopital Rule $\lim_{v\to 0}\frac{f(v)}{g(v)} = \lim_{v\to 0}\frac{f'(v)}{g'(v)}$

so $$\lim_{v\to 0}\frac{1-e^v}{v} = \lim_{v\to 0}\frac{-e^v}{1} = -e^0 = \boxed{-1}$$

and $$\lim_{u\to 0}\frac{e^u-1}{u} = \lim_{u\to 0} e^u = e^0 = \boxed{1}$$

Ch 4

12.4 $\lim_{x\to 2} \frac{1}{x-2}\int_2^x \frac{\sin t}{t}\,dt$.

$$\lim_{x\to 2} \frac{\frac{d}{dx}\int_2^x \frac{\sin t}{t}\,dt}{\frac{d}{dx}(x-2)} = \lim_{x\to 2} \frac{\frac{\sin x}{x}\frac{d}{dx}x - \frac{\sin 2}{2}\frac{d}{dx}(2)}{1}$$

$$= \lim_{x\to 2} \frac{\sin x}{x} = \boxed{\frac{1}{2}\sin 2}$$

12.7 if $\int_u^v e^{-t^2}\,dt = x$ and $u^v = y$, find $\left(\frac{\partial u}{\partial x}\right)_y$, $\left(\frac{\partial u}{\partial y}\right)_x$ and $\left(\frac{\partial v}{\partial x}\right)_u$ at $u=2, v=0$.

$\frac{\partial u}{\partial x}_y$ means $u(x,y)$ ⸺ next

take differential $d\int_{u(x,y)}^{v(x,y)} e^{-t^2}\,dt$

$$d(x) = d(v(x,y))\,e^{-(v^2)} - d(u(x,y))\,e^{-u^2}$$

so $$1 = \left(\frac{dv}{dx}+\frac{dv}{dy}\right)e^{-v^2} - \left(\frac{du}{dx}+\frac{du}{dy}\right)e^{-u^2}$$

$$1 = \frac{dv}{dx}e^{-v^2} + \frac{dv}{dy}e^{-v^2} - \frac{du}{dx}e^{-u^2} - \frac{du}{dy}e^{-u^2}$$

so $$\frac{du}{dx} = \left(-1 + \frac{dv}{dx}e^{-v^2} + \frac{dv}{dy}e^{-v^2} - \frac{du}{dy}e^{-u^2}\right)\frac{1}{e^{-u^2}}$$

$$\frac{du}{dx} = -1 + \frac{dv}{dx}e^{-v^2+u^2} + \frac{dv}{dy}e^{-v^2+u^2} - \frac{du}{dy}$$

since $u(x,y)$, rewrite above as →

$$\left(\frac{\partial u}{\partial x}\right)_y = -1 + \frac{\partial v}{\partial x}_y\, e^{u^2-v^2} + \frac{dv}{dy}\, e^{u^2-v^2} - \frac{du}{dy}$$

Ch 4

12.7

if $\int_u^v e^{-t}\,dt = x$, $u^v = y$, find $\left(\frac{\partial u}{\partial x}\right)_y$, $\frac{\partial u}{\partial y}$, $\frac{\partial y}{\partial x}$ at $u=2$, $v=0$.

we have $u(x,y)$, but v is constant. (since problem does not mention it is function of x or y)

take derivative, using Lebniz Rule, we get

$$\frac{d}{dx}(x) = \frac{d}{dx}\int_{u(x,y)}^{v} e^{-t}\,dt$$

$$1 = e^{-v}\frac{dv}{dx} - e^{-u(x,y)}\left(\frac{\partial u}{\partial x}\right)_y$$

(with $\frac{dv}{dx} = 0$)

so $\boxed{\left(\frac{\partial u}{\partial x}\right)_y = -\frac{1}{e^{-u(x,y)}}}$ at $u=2 \Rightarrow \left(\frac{\partial u}{\partial x}\right)_y = -\frac{1}{e^{-2}} = -e^{2}$ $= \boxed{-7.38}$

to find $\left(\frac{\partial u}{\partial y}\right)_x$, from Lebniz Rule:

$$\frac{d}{dy}(x) = \frac{d}{dy}\int_{u(x,y)}^{v} e^{-t}\,dt = e^{-v}\frac{dv}{dy} - e^{-u(x,y)}\left(\frac{\partial u}{\partial y}\right)_x$$

(with $\frac{dv}{dy} = 0$)

$$\frac{dx}{dy} = -e^{-u(x,y)}\left(\frac{\partial u}{\partial y}\right)_x$$

$$\left(\frac{\partial u}{\partial y}\right)_x = \boxed{-\frac{dx}{dy}\frac{1}{e^{-u(x,y)}}}$$ at $u=2 \Rightarrow +\frac{dx}{dy}\cdot 7.38$

from $u^v = y \Rightarrow v\log u = \log y \Rightarrow \log u = \frac{1}{v}\log y$.

so $d(\log u) = \frac{1}{v}d(\log y) \Rightarrow \frac{1}{u}du = \frac{1}{v}\frac{1}{y}dy$

so $\frac{du}{dy} = \frac{u}{v}\frac{1}{y} = \frac{u}{v}\frac{1}{u^v}$

Ch 4 2.8 if $\int_0^x e^{-s^2}\,ds = u$ find $\frac{dx}{du}$

$$\frac{du}{dx} = \frac{d}{dx}\int_0^x e^{-s^2}\,ds$$

$$= e^{-x^2}\frac{d}{dx}x - e^{-0^2}\frac{d}{dx}(0)$$

$$\frac{du}{dx} = e^{-x^2}\,1$$

so $\frac{du}{dx} = e^{-x^2}$ $\Rightarrow$ $\boxed{\frac{dx}{du} = e^{x^2}}$

Chapter 14: #1.6: 0

3.8 HW 7

Local contents

3.8.1 chapter 14, problem 1.6

Problem Find real and imaginary parts u, v of e^z

Solution

Let $z = x + iy$, then

$$
\begin{aligned}
f(z) &= e^z \\
&= e^{x+iy} \\
&= e^x e^{iy} \\
&= e^x\left(\cos y + i \sin y\right) \\
&= e^x \cos y + ie^x \sin y
\end{aligned}
$$

Hence $u\left(x,y\right) = e^x \cos y$ and $v\left(x,y\right) = e^x \sin y$

3.8.2 chapter 14, problem 1.12

Problem Find real and imaginary parts u, v of $f(z) = \frac{z}{z^2+1}$

Solution

Let $z = x + iy$ then

$$
\begin{aligned}
z^2 + 1 &= \left(x + iy\right)^2 + 1 \\
&= \left(x^2 - y^2 + 1\right) + i\left(2xy\right)
\end{aligned}
$$

Hence

$$
f(z) = \frac{x + iy}{\left(x^2 - y^2 + 1\right) + i\left(2xy\right)}
$$

Multiplying numerator and denominator by conjugate of denominator gives

$$\begin{aligned}
f(z) &= \frac{\left(x+iy\right)\left(\left(x^2-y^2+1\right)-i\left(2xy\right)\right)}{\left(\left(x^2-y^2+1\right)+i\left(2xy\right)\right)\left(\left(x^2-y^2+1\right)-i\left(2xy\right)\right)} \\
&= \frac{\left(x\left(x^2-y^2+1\right)+y\left(2xy\right)\right)+i\left(y\left(x^2-y^2+1\right)\left(y\left(2xy\right)\right)\right)}{\left(x^2-y^2+1\right)^2+\left(2xy\right)^2} \\
&= \frac{x\left(x^2-y^2+1\right)+2xy^2}{\left(x^2-y^2+1\right)^2+\left(2xy\right)^2}+i\frac{y\left(x^2-y^2+1\right)-2x^2y}{\left(x^2-y^2+1\right)^2+\left(2xy\right)^2}
\end{aligned}$$

Hence

$$\begin{aligned}
u\left(x,y\right) &= \frac{x\left(x^2-y^2+1\right)+2xy^2}{\left(x^2-y^2+1\right)^2+2xy} \\
v\left(x,y\right) &= \frac{y\left(x^2-y^2+1\right)-2x^2y}{\left(x^2-y^2+1\right)^2+\left(2xy\right)^2}
\end{aligned}$$

3.8.3 chapter 14, problem 2.22

Problem Use Cauchy-Riemann conditions to find if $f(z)=y+ix$ is analytic.

Solution

CR says a complex function $f(z)=u+iv$ is analytic if

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x} \tag{2}$$

Here $u=y$ and $v=x$, since $f(z)=z=x+iy$. Therefore $\frac{\partial u}{\partial x}=0, \frac{\partial v}{\partial y}=0$ and (1) is satisfied. And $\frac{\partial u}{\partial y}=1$ and $\frac{\partial v}{\partial x}=1$, hence (2) is NOT satisfied. Therefore not analytic.

3.8.4 chapter 14, problem 2.23

Problem Use Cauchy-Riemann conditions to find if $f(z)=\frac{x-iy}{x^2+y^2}$ is analytic.

Solution

CR says a complex function $f(z)=u+iv$ is analytic if

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x} \tag{2}$$

Here $f(z)=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}$, hence

$$\begin{aligned}
u &= \frac{x}{x^2+y^2} \\
v &= \frac{-y}{x^2+y^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1}{x^2+y^2}-\frac{x}{\left(x^2+y^2\right)^2}(2x) \\
&= \frac{x^2+y^2-2x^2}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}
\end{aligned}$$

And

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{-1}{x^2+y^2} + \frac{y}{\left(x^2+y^2\right)^2}\left(2y\right)\\ &= \frac{-\left(x^2+y^2\right)+2y^2}{\left(x^2+y^2\right)^2}\\ &= \frac{y^2-x^2}{\left(x^2+y^2\right)^2}\end{aligned}$$

Hence (1) is satisfied. And

$$\frac{\partial u}{\partial y} = \frac{-2xy}{\left(x^2+y^2\right)^2}$$

And

$$\frac{\partial v}{\partial x} = \frac{2xy}{\left(x^2+y^2\right)^2}$$

Hence (2) is satisfied also. Therefore $f(z)$ is analytic.

3.8.5 chapter 14, problem 2.34

Problem Write power series about origin for $f(z) = \ln(1-z)$. Use theorem 3 to find circle of convergence for each series.

Solution

From page 34, for $-1 < x \leq 1$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Hence

$$\begin{aligned}\ln(1-z) &= (-z) - \frac{(-z)^2}{2} + \frac{(-z)^3}{3} - \frac{(-z)^4}{4} + \cdots\\ &= -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \cdots\\ &= -\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots\right)\\ &= -\sum_{n=1} \frac{1}{n} z^n\end{aligned}$$

To find radius of convergence, use ratio test.

$$\begin{aligned}L &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}\\ &= \lim_{n\to\infty} \frac{\left|\frac{1}{n+1}\right|}{\left|\frac{1}{n}\right|}\\ &= \lim_{n\to\infty} \frac{n}{n+1}\\ &= 1\end{aligned}$$

Hence $R = \frac{1}{L} = 1$. Therefore converges for $|z| < 1$.

3.8.6 chapter 14, problem 2.37

Problem Find circle of convergence for tanh(z)

Solution

$$\tanh(z) = -i\tan(iz)$$

But $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{325}x^7 + \cdots$, therefore

$$\begin{aligned}\tanh(z) &= -i\left(iz + \frac{(iz)^3}{3} + \frac{2}{15}(iz)^5 + \frac{17}{325}(iz)^7 + \cdots\right)\\ &= -i\left(iz - \frac{iz^3}{3} + \frac{2}{15}iz^5 + \cdots\right)\\ &= z - \frac{z^3}{3} + \frac{2}{15}z^5 + \cdots\end{aligned}$$

This is the power series of tanh(z) about $z = 0$. Since $\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{\sinh(z)}{\cos(iz)}$ and $\cos(iz) = 0$ at $iz = \pm\frac{\pi}{2}$ then $|z| < \frac{\pi}{2}$ to avoid hitting a singularity. So radius of convergence is $R = \frac{\pi}{2}$.

3.8.7 chapter 14, problem 2.40

Problem Find series and circle of convergence for $\frac{1}{1-z}$

Solution

From Binomial expansion

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

For $|z| < 1$. Hence $R = 1$.

3.8.8 chapter 14, problem 2.55

Problem Show that $3x^2y - y^3$ is harmonic, that is, it satisfies Laplace equation, and find a function $f(z)$ of which this function is the real part. Show that the function $v(x,y)$ which you also find also satisfies Laplace equation.

Solution

The given function is the real part of $f(z)$. Hence $u(x,y) = 3x^2y - y^3$. To show this is harmonic, means it satisfies $\nabla^2 u = 0$ or $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. But

$$\begin{aligned}\frac{\partial u}{\partial x} &= 6xy\\ \frac{\partial^2 u}{\partial x^2} &= 6y\\ \frac{\partial u}{\partial y} &= 3x^2 - 3y^2\\ \frac{\partial^2 u}{\partial y^2} &= -6y\end{aligned}$$

Therefore $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, hence $u(x,y)$ is harmonic. Now, we want to find $f(z) = u(x,y) + iv(x,y)$ and analytic function, where its real part is what we are given above. So we need to find $v(x,y)$. Since $f(z)$ is analytic, then we apply Cauchy-Riemann equations to find $v(x,y)$ CR says a complex function $f(z) = u + iv$ is analytic if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

But $\frac{\partial u}{\partial x} = 6xy$, so (1) gives

$$\begin{aligned} 6xy &= \frac{\partial v}{\partial y} \\ v(x,y) &= \int 6xy dy \\ &= 3xy^2 + g(x) \end{aligned} \tag{3}$$

From (2) we obtain

$$-3x^2 + 3y^2 = \frac{\partial v}{\partial x}$$

But from (3), we see that $\frac{\partial v}{\partial x} = 3y^2 + g'(x)$, hence the above becomes

$$\begin{aligned} -3x^2 + 3y^2 &= 3y^2 + g'(x) \\ g'(x) &= -3x^2 \\ g(x) &= \int -3x^2 dx \\ &= -x^3 + C \end{aligned}$$

Therefore from (3), we find that

$$v(x,y) = 3xy^2 - x^3 + C$$

We can set any value to C. Let $C = 0$ to simplify things. Hence

$$\begin{aligned} f(z) &= u + iv \\ &= (3x^2y - y^3) + i(3xy^2 - x^3) \end{aligned}$$

Now we show that $v(x,y)$ is also harmonic. i.e. it satisfies Laplace.

$$\begin{aligned} \frac{\partial v}{\partial x} &= 3y^2 - 3x^2 \\ \frac{\partial^2 v}{\partial x^2} &= -6x \\ \frac{\partial v}{\partial y} &= 6xy \\ \frac{\partial^2 v}{\partial y^2} &= 6x \end{aligned}$$

Hence we see that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$. QED.

3.8.9 chapter 14, problem 2.55

Problem Show that xy is harmonic, that is, it satisfies Laplace equation, and find a function $f(z)$ of which this function is the real part. Show that the function $v(x,y)$ which you also find also satisfies Laplace equation.

Solution

The given function is the real part of $f(z)$. Hence $u(x,y) = xy$. To show this is harmonic, means it satisfies $\nabla^2 u = 0$ or $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. But

$$\begin{aligned}
\frac{\partial u}{\partial x} &= y\\
\frac{\partial^2 u}{\partial x^2} &= 0\\
\frac{\partial u}{\partial y} &= x\\
\frac{\partial^2 u}{\partial y^2} &= 0
\end{aligned}$$

Therefore $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, hence $u(x,y)$ is harmonic. Now, we want to find $f(z) = u(x,y) + iv(x,y)$ and analytic function, where its real part is what we are given above. So we need to find $v(x,y)$. Since $f(z)$ is analytic, then we apply Cauchy-Riemann equations to find $v(x,y)$ CR says a complex function $f(z) = u + iv$ is analytic if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

But $\frac{\partial u}{\partial x} = y$, so (1) gives

$$\begin{aligned}
y &= \frac{\partial v}{\partial y}\\
v(x,y) &= \int y dy\\
&= \frac{y^2}{2} + g(x) \qquad (3)
\end{aligned}$$

From (2) we obtain

$$-x = \frac{\partial v}{\partial x}$$

But from (3), we see that $\frac{\partial v}{\partial x} = g'(x)$, hence the above becomes

$$\begin{aligned}
-x &= g'(x)\\
g(x) &= \int -x dx\\
&= -\frac{x^2}{2} + C
\end{aligned}$$

Therefore from (3), we find that

$$v(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + C$$

We can set any value to C. Let $C = 0$ to simplify things. Hence

$$\begin{aligned}
f(z) &= u + iv\\
&= (xy) + i\left(\frac{y^2 - x^2}{2}\right)
\end{aligned}$$

Now we show that $v(x,y)$ is also harmonic. i.e. it satisfies Laplace.

$$\begin{aligned}
\frac{\partial v}{\partial x} &= -x \\
\frac{\partial^2 v}{\partial x^2} &= -1 \\
\frac{\partial v}{\partial y} &= y \\
\frac{\partial^2 v}{\partial y^2} &= 1
\end{aligned}$$

Hence we see that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$. QED.

3.8.10 chapter 14, problem 2.60

Problem Show that $\ln(x^2 + y^2)$ is harmonic, that is, it satisfies Laplace equation, and find a function $f(z)$ of which this function is the real part. Show that the function $v(x,y)$ which you also find also satisfies Laplace equation.

Solution

The given function is the real part of $f(z)$. Hence $u(x,y) = xy$. To show this is harmonic, means it satisfies $\nabla^2 u = 0$ or $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. But

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{2x}{x^2+y^2} \\
\frac{\partial^2 u}{\partial x^2} &= 2\left(\frac{1}{x^2+y^2}\right) + 2x\left(\frac{-1}{\left(x^2+y^2\right)^2}(2x)\right) \\
&= \frac{2}{x^2+y^2} - \frac{4x^2}{\left(x^2+y^2\right)^2} \\
&= \frac{2\left(x^2+y^2\right) - 4x^2}{\left(x^2+y\right)^2} \\
&= \frac{-2x^2+2y^2}{\left(x^2+y\right)^2} \\
\frac{\partial u}{\partial y} &= \frac{2y}{x^2+y^2} \\
\frac{\partial^2 u}{\partial y^2} &= 2\left(\frac{1}{x^2+y^2}\right) + 2y\left(\frac{-1}{\left(x^2+y^2\right)^2}(2y)\right) \\
&= \frac{2}{x^2+y^2} - \frac{4y^2}{\left(x^2+y^2\right)^2} \\
&= \frac{2\left(x^2+y^2\right) - 4y^2}{\left(x^2+y^2\right)^2} \\
&= \frac{2x^2-2y^2}{\left(x^2+y^2\right)^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{-2x^2+2y^2}{\left(x^2+y\right)^2} + \frac{2x^2-2y^2}{\left(x^2+y^2\right)^2} \\
&= 0
\end{aligned}$$

Hence $u(x,y)$ is harmonic. Now, we want to find $f(z) = u(x,y) + iv(x,y)$ and analytic function, where its real part is what we are given above. So we need to find $v(x,y)$. Since $f(z)$ is analytic, then we apply Cauchy-Riemann equations to find $v(x,y)$ CR says a complex function $f(z) = u + iv$ is analytic if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

But $\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}$, so (1) gives

$$\begin{aligned}\frac{2x}{x^2+y^2} &= \frac{\partial v}{\partial y} \\ v(x,y) &= \int \frac{2x}{x^2+y^2} dy \\ &= 2\arctan\left(\frac{y}{x}\right) + g(x) \qquad (3)\end{aligned}$$

From (2) we obtain

$$-\frac{2y}{x^2+y^2} = \frac{\partial v}{\partial x}$$

But from (3), we see that $\frac{\partial v}{\partial x} = -\frac{2y}{y^2+x^2} + g'(x)$, hence the above becomes

$$\begin{aligned}-\frac{2y}{x^2+y^2} &= -\frac{2y}{y^2+x^2} + g'(x) \\ g'(x) &= 0 \\ g(x) &= C\end{aligned}$$

Therefore from (3), we find that

$$v(x,y) = 2\arctan\left(\frac{y}{x}\right) + C$$

We can set any value to C. Let $C = 0$ to simplify things. Hence

$$v(x,y) = 2\arctan\left(\frac{y}{x}\right)$$

And therefore

$$\begin{aligned}f(z) &= u + iv \\ &= \ln\left(x^2+y^2\right) + i\left(2\arctan\left(\frac{y}{x}\right)\right)\end{aligned}$$

Now we show that $v(x,y)$ is also harmonic. i.e. it satisfies Laplace. We find that

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2} &= \frac{4xy}{\left(x^2+y^2\right)^2} \\ \frac{\partial^2 v}{\partial y^2} &= -\frac{4xy}{\left(x^2+y^2\right)^2}\end{aligned}$$

Hence we see that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$. QED.

3.8.11 chapter 14, problem 3.3(b)

Problem Find $\oint_C z^2 dz$ over the half unit circle arc shown.

Solution

Since $f(z) = z^2$ is clearly analytic on and inside C and no poles are inside, then by Cauchy's theorem $\oint_C z^2 dz = 0$

3.8.12 chapter 14, problem 3.5

Problem Find $\int e^{-z}dz$ along positive part of the line $y=\pi$. This is frequently written as $\int_{i\pi}^{\infty+i\pi} e^{-z}dz$

Solution

Let $z=x+iy$, then

$$\begin{aligned} I &= \int_{i\pi}^{\infty+i\pi} e^{-z}dz \\ &= \int_{i\pi}^{\infty+i\pi} e^{-x}e^{-iy}dz \end{aligned}$$

But $dz=dx+idy$, the above becomes

$$\begin{aligned} I &= \int_{i\pi}^{\infty+i\pi} e^{-x}e^{-iy}\left(dx+idy\right) \\ &= \int_{0}^{\infty} e^{-x}e^{-iy}dx + i\int_{i\pi}^{i\pi} e^{-x}e^{-iy}dy \\ &= \int_{0}^{\infty} e^{-x}e^{-iy}dx \end{aligned}$$

But $y=\pi$ over the whole integration. The above simplifies to

$$\begin{aligned} I &= e^{-i\pi}\int_{0}^{\infty} e^{-x}dx \\ &= e^{-i\pi}\left(\frac{e^{-x}}{-1}\right)_{0}^{\infty} \\ &= -e^{-i\pi}(0-1) \\ &= e^{i\pi} \\ &= -1 \end{aligned}$$

3.8.13 chapter 14, problem 3.17

Problem Using Cauchy integral formula to evaluate $\oint_C \frac{\sin z}{2z-\pi}dz$ where (a) C is circle $|z|=1$ and (b) C is circle $|z|=2$

Solution

For part (a), since the pole is at $z=\frac{\pi}{2}$, it is outside the circle $|z|=1$ and $f(z)$ is analytic inside and on C, then by Cauchy theorem $\oint_C \frac{\sin z}{2z-\pi}dz = 0$.

For part(b), since now the pole is inside, then

$$\oint_C \frac{\sin z}{2z-\pi}dz = 2\pi i \operatorname{Residue}\left(\frac{\pi}{2}\right)$$

But

$$\begin{aligned} \operatorname{Residue}\left(\frac{\pi}{2}\right) &= \lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)f(z) \\ &= \lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)\frac{\sin z}{2z-\pi} \\ &= \sin\left(\frac{\pi}{2}\right)\lim_{z\to\frac{\pi}{2}}\frac{\left(z-\frac{\pi}{2}\right)}{2z-\pi} \end{aligned}$$

Applying L'Hopital

$$\begin{aligned}\text{Residue}\left(\frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{2}\right)\lim_{z\to\frac{\pi}{2}}\frac{1}{2}\\ &= \frac{1}{2}\end{aligned}$$

Hence

$$\oint_C \frac{\sin z}{2z-\pi}dz = \pi i$$

3.8.14 chapter 14, problem 3.18

Problem Integrate $\oint_C \frac{\sin 2z}{6z-\pi}dz$ over circle $|z| = 3$

Solution

The pole is at $z = \frac{\pi}{6}$. This is inside $|z| = 3$. Hence

$$\oint_C \frac{\sin 2z}{6z-\pi}dz = 2\pi i\,\text{Residue}\left(\frac{\pi}{6}\right)$$

But

$$\begin{aligned}\text{Residue}\left(\frac{\pi}{6}\right) &= \lim_{z\to\frac{\pi}{6}}\left(z-\frac{\pi}{6}\right)\frac{\sin 2z}{6z-\pi}\\ &= \sin\left(\frac{\pi}{3}\right)\lim_{z\to\frac{\pi}{6}}\frac{\left(z-\frac{\pi}{6}\right)}{6z-\pi}\end{aligned}$$

Applying L'Hopitals

$$\begin{aligned}\text{Residue}\left(\frac{\pi}{6}\right) &= \sin\left(\frac{\pi}{3}\right)\lim_{z\to\frac{\pi}{6}}\frac{1}{6}\\ &= \frac{1}{6}\sin\left(\frac{\pi}{3}\right)\end{aligned}$$

Hence

$$\oint_C \frac{\sin 2z}{6z-\pi}dz = 2\pi i\left(\frac{1}{6}\sin\left(\frac{\pi}{3}\right)\right)$$

But $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and the above simplifies to

$$\begin{aligned}\oint_C \frac{\sin 2z}{6z-\pi}dz &= 2\pi i\left(\frac{1}{6}\frac{\sqrt{3}}{2}\right)\\ &= \frac{\pi i}{2\sqrt{3}}\end{aligned}$$

3.8.15 chapter 14, problem 3.19

Problem Integrate $\oint_C \frac{e^{3z}}{z-\ln 2}dz$ if C is square with vertices $\pm 1, \pm i$

Solution

The pole is at $z = \ln 2 = 0.693$ so inside C. Hence

$$\oint_C \frac{e^{3z}}{z-\ln 2}dz = 2\pi i\,\text{Residue}(\ln 2)$$

But

$$\begin{aligned}\text{Residue}(\ln 2) &= \lim_{z\to\ln 2}(z-\ln 2)f(z)\\ &= e^{3\ln 2}\lim_{z\to\ln 2}\frac{z-\ln 2}{z-\ln 2}\\ &= e^{3\ln 2}\end{aligned}$$

Hence

$$\begin{aligned}\oint_C \frac{e^{3z}}{z-\ln 2}dz &= 2\pi i e^{3\ln 2}\\ &= 2\pi i(2)^3\\ &= 16\pi i\end{aligned}$$

3.8.16 chapter 14, problem 3.20

Problem Integrate $\oint_C \frac{\cosh z}{2\ln 2 - z}dz$ if C is (a) circle with $|z| = 1$ and (b) Circle with $|z| = 2$

Solution

Part (a). Pole is at $z = 2\ln 2 = 1.38$. Hence pole is outside C. Therefore $\oint_C \frac{\cosh z}{2\ln 2 - z}dz = 0$ since $f(z)$ is analytic on C

Part(b). Now pole is inside. Hence

$$\oint_C \frac{\cosh z}{2\ln 2 - z}dz = 2\pi i\,\text{Residue}(2\ln 2)$$

But

$$\begin{aligned}\text{Residue}(2\ln 2) &= \lim_{z\to 2\ln 2}(z-2\ln 2)f(z)\\ &= \lim_{z\to 2\ln 2}(z-2\ln 2)\frac{\cosh z}{2\ln 2 - z}\\ &= \cosh(2\ln 2)\lim_{z\to\ln 2}\frac{z-2\ln 2}{2\ln 2 - z}\\ &= -\cosh(2\ln 2)\end{aligned}$$

Therefore

$$\begin{aligned}\oint_C \frac{\cosh z}{2\ln 2 - z}dz &= -2\pi i\cosh(2\ln 2)\\ &= -4.25\pi i\end{aligned}$$

3.8.17 chapter 14, problem 3.23

Problem Integrate $\oint_C \frac{e^{3z}}{(z-\ln 2)^4}dz$ if C is square between $\pm 1, \pm i$

Solution

The pole is at $z = \ln 2 = 0.69$ which is inside the square. The order is 4. Hence

$$\oint_C \frac{e^{3z}}{(z-\ln 2)^4}dz = 2\pi i\,\text{Residue}(\ln 2)$$

To find Residue(ln 2) we now use different method from earlier, since this is not a simple pole.

$$\begin{aligned}
\text{Residue}(\ln 2) &= \lim_{z\to\ln 2} \frac{1}{3!}\frac{d^3}{dz^3}(z-\ln 2)^4 f(z) \\
&= \lim_{z\to\ln 2} \frac{1}{3!}\frac{d^3}{dz^3}(z-\ln 2)^4\left(\frac{e^{3z}}{(z-\ln 2)^4}\right) \\
&= \lim_{z\to\ln 2} \frac{1}{3!}\frac{d^3}{dz^3}\left(e^{3z}\right) \\
&= \lim_{z\to\ln 2} \frac{1}{3!}\frac{d^2}{dz^2}\left(3e^{3z}\right) \\
&= \lim_{z\to\ln 2} \frac{1}{3!}9\frac{d}{dz}e^{3z} \\
&= \lim_{z\to\ln 2} \frac{1}{3!}27e^{3z} \\
&= \lim_{z\to\ln 2} \frac{27}{6}e^{3z} \\
&= \frac{27}{6}e^{3\ln 2} \\
&= (27)\left(\frac{8}{6}\right) \\
&= 36
\end{aligned}$$

Hence

$$\begin{aligned}
\oint_C \frac{e^{3z}}{(z-\ln 2)^4}dz &= 2\pi i 36 \\
&= 72\pi i
\end{aligned}$$

3.8.18 chapter 14, problem 4.6

Problem Find Laurent series and residue at origin for $f(z) = \frac{1}{z^2(1+z)^2}$

Solution

There is a pole at $z=0$ and at $z=-1$. We expand around a disk of radius 1 centered at $z=0$ to find Laurent series around $z=0$. Hence

$$f(z) = \frac{1}{z^2}\frac{1}{(1+z)^2}$$

For $|z|<1$ we can now expand $\frac{1}{(1+z)^2}$ using Binomial expansion

$$\begin{aligned}
f(z) &= \frac{1}{z^2}\left(1+(-2)z+(-2)(-3)\frac{z^2}{2!}+(-2)(-3)(-4)\frac{z^3}{3!}+\cdots\right) \\
&= \frac{1}{z^2}\left(1-2z+3z^2-4z^3+\cdots\right) \\
&= \frac{1}{z^2}-\frac{2}{z}+3-4z+\cdots
\end{aligned}$$

Hence residue is -2. To find Laurent series outside this disk, we write

$$\begin{aligned}
f(z) &= \frac{1}{z^2}\frac{1}{(1+z)^2} \\
&= \frac{1}{z^2}\frac{1}{\left(z\left(1+\frac{1}{z}\right)\right)^2} \\
&= \frac{1}{z^4}\frac{1}{\left(1+\frac{1}{z}\right)^2}
\end{aligned}$$

And now we can expand $\frac{1}{\left(1+\frac{1}{z}\right)^2}$ for $\left|\frac{1}{z}\right| < 1$ or $|z| > 1$ using Binomial and obtain

$$
\begin{aligned}
f(z) &= \frac{1}{z^4}\left(1 + (-2)\frac{1}{z} + \frac{(-2)(-3)}{2!}\left(\frac{1}{z}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{1}{z}\right)^3 + \cdots\right) \\
&= \frac{1}{z^4}\left(1 - \frac{2}{z} + 3\left(\frac{1}{z}\right)^2 - 4\left(\frac{1}{z}\right)^3 + \cdots\right) \\
&= \frac{1}{z^4} - \frac{2}{z^5} + \frac{3}{z^6} - \frac{4}{z^7} + \cdots
\end{aligned}
$$

We see that outside the disk, the Laurent series contains only the principal part and no analytical part as the case was in the Laurent series inside the disk.

3.8.19 chapter 14, problem 4.7

Problem Find Laurent series and residue at origin for $f(z) = \frac{2-z}{1-z^2}$

Solution

There is a pole at $z = \pm 1$. So we need to expand $f(z)$ for $|z| < 1$ around origin. Here there is no pole at origin, hence the series expansion should contain only an analytical part

$$
\begin{aligned}
f(z) &= \frac{2-z}{1-z^2} \\
&= \frac{2-z}{(1-z)(1+z)} \\
&= \frac{A}{(1-z)} + \frac{B}{(1+z)} \\
&= \frac{1}{2}\frac{1}{(1-z)} + \frac{3}{2}\frac{1}{(1+z)} \\
&= \frac{1}{2}\left(1 + z + z^2 + z^3 + \cdots\right) + \frac{3}{2}\left(1 - z + z^2 - z^3 + z^4 - \cdots\right) \\
&= 2 - z + 2z^2 - z^3 + 2z^4 - z^5 + \cdots
\end{aligned}
$$

No principal part. Only analytical part, since $f(z)$ is analytical everywhere inside the region. For $|z| > 1$ we write

$$
\begin{aligned}
f(z) &= \frac{1}{2}\frac{1}{(1-z)} + \frac{3}{2}\frac{1}{(1+z)} \\
&= \frac{1}{2z}\frac{1}{\left(\frac{1}{z} - 1\right)} + \frac{3}{2z}\frac{1}{\left(\frac{1}{z} + 1\right)} \\
&= \frac{-1}{2z}\frac{1}{\left(1 - \frac{1}{z}\right)} + \frac{3}{2z}\frac{1}{\left(\frac{1}{z} + 1\right)} \\
&= \frac{-1}{2z}\left(1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots\right) + \frac{3}{2z}\left(1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4 - \cdots\right) \\
&= \frac{1}{z} - \frac{2}{z^2} + \frac{1}{z^3} - \frac{2}{z^4} + \frac{1}{z^5} - \frac{2}{z^6} + \cdots
\end{aligned}
$$

We see that outside the disk, the Laurent series contains only the principal part and no analytical part.

3.8.20 chapter 14, problem 4.9

Problem Determine the type of singularity at the point given. If it is regular, essential, or pole (and indicate the order if so). (a) $f(z) = \frac{\sin z}{z}, z = 0$ (b) $f(z) = \frac{\cos z}{z^3}, z = 0$, (c) $f(z) = \frac{z^3-1}{(z-1)^3}, z = 1$, (d) $f(z) = \frac{e^z}{z-1}, z = 1$

Solution

(a) There is a singularity at $z = 0$, but we will check if it removable

$$\begin{aligned} f(z) &= \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z} \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \end{aligned}$$

So the series contain no principal part (since all powers are positive). Hence we have pole of order 1 which is removable. Therefore $z = 0$ is a regular point.

(b) There is a singularity at $z = 0$, but we will check if it removable

$$\begin{aligned} f(z) &= \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots}{z^3} \\ &= \frac{1}{z^3} - \frac{1}{2z} + \frac{z}{4!} - \cdots \end{aligned}$$

Hence we could not remove the pole. So the the point is a pole of order 3.

(c) There is a singularity at $z = 1$,

$$\begin{aligned} f(z) &= \frac{z^3 - 1}{(z-1)^3} \\ &= \frac{(z-1)\left(z^2 + 1 + z\right)}{(z-1)^3} \\ &= \frac{\left(z^2 + 1 + z\right)}{(z-1)^2} \end{aligned}$$

Hence a pole of order 2.

(d)

$$f(z) = \frac{e^z}{z-1}$$

There is no cancellation here. Hence $z = 1$ is a pole or order 1.

3.8.21 chapter 14, problem 4.10

Problem Determine the type of singularity at the point given. If it is regular, essential, or pole (and indicate the order if so). (a) $f(z) = \frac{e^z - 1}{z^2 + 4}, z = 2i$ (b) $f(z) = \tan^2 z, z = \frac{\pi}{2}$. (c) $f(z) = \frac{1 - \cos(z)}{z^4}, z = 0$, (d) $f(z) = \cos\left(\frac{\pi}{z - \pi}\right), z = \pi$

Solution

(a) To find if the point is essential or pole or regular, we expand $f(z)$ around the point, and look at the Laurent series. If the number of b_n terms is infinite, then it is essential singularity. If the number of b_n is finite, then it is a pole of order that equal the largest order of the b_n term. If the series contains only analytical part and no principal part (the part which has the b_n terms), then the point is regular.

So we need to expand $\frac{e^z - 1}{z^2 + 4}$ around $z = 2i$. For the numerator, this gives

$$e^z = e^{2i} + (z - 2i)e^{2i} + (z - 2i)^2 \frac{e^{2i}}{2!} + \cdots$$

For

$$\begin{aligned} \frac{1}{z^2 + 4} &= \frac{1}{(z - 2i)(z + 2i)} \\ &= -\frac{i}{4} \frac{1}{(z - 2i)} + \frac{1}{16} + \frac{i}{64}(z - 2i) - \frac{1}{256}(z - 2i)^2 - \cdots \end{aligned}$$

Hence

$$f(z)=\left(1-e^{2i}+(z-2i)e^{2i}+(z-2i)^2\frac{e^{2i}}{2!}+\cdots\right)\left(-\frac{i}{4}\frac{1}{(z-2i)}+\frac{1}{16}+\frac{i}{64}(z-2i)-\frac{1}{256}(z-2i)^2-\cdots\right)$$

We see that the resulting series will contain infinite number of b_n terms. These are the terms with $\frac{1}{(z-2i)^n}$. Hence the point $z=2i$ is essential singularity.

(b) We need to find the series of $\tan^2 z$ around $z=\frac{\pi}{2}$.

$$\begin{aligned}
\tan^2\left(z-\frac{\pi}{2}\right) &= \frac{\sin^2\left(z-\frac{\pi}{2}\right)}{\cos^2\left(z-\frac{\pi}{2}\right)}\\
&= \frac{\left(\left(z-\frac{\pi}{2}\right)-\frac{\left(z-\frac{\pi}{2}\right)^3}{3!}+\frac{\left(z-\frac{\pi}{2}\right)^5}{5!}-\cdots\right)^2}{\left(1-\frac{\left(z-\frac{\pi}{2}\right)^2}{2!}+\frac{\left(z-\frac{\pi}{2}\right)^4}{4!}-\cdots\right)^2}\\
&= \frac{\left(z-\frac{\pi}{2}\right)^2\left(1-\frac{\left(z-\frac{\pi}{2}\right)^2}{3!}+\frac{\left(z-\frac{\pi}{2}\right)^4}{5!}-\cdots\right)^2}{\left(1-\frac{\left(z-\frac{\pi}{2}\right)^2}{2!}+\frac{\left(z-\frac{\pi}{2}\right)^4}{4!}-\cdots\right)^2}\\
&= \frac{\left(z-\frac{\pi}{2}\right)^2\left(1-\frac{\left(z-\frac{\pi}{2}\right)^2}{3!}+\frac{\left(z-\frac{\pi}{2}\right)^4}{5!}-\cdots\right)^2}{\left(\left(z-\frac{\pi}{2}\right)\left(\frac{1}{z-\frac{\pi}{2}}-\frac{\left(z-\frac{\pi}{2}\right)}{2!}+\frac{\left(z-\frac{\pi}{2}\right)^3}{4!}-\cdots\right)\right)^2}\\
&= \frac{\left(z-\frac{\pi}{2}\right)^2\left(1-\frac{\left(z-\frac{\pi}{2}\right)^2}{3!}+\frac{\left(z-\frac{\pi}{2}\right)^4}{5!}-\cdots\right)^2}{\left(z-\frac{\pi}{2}\right)^2\left(\frac{1}{z-\frac{\pi}{2}}-\frac{\left(z-\frac{\pi}{2}\right)}{2!}+\frac{\left(z-\frac{\pi}{2}\right)^3}{4!}-\cdots\right)^2}\\
&= \frac{\left(1-\frac{\left(z-\frac{\pi}{2}\right)^2}{3!}+\frac{\left(z-\frac{\pi}{2}\right)^4}{5!}-\cdots\right)^2}{\left(\frac{1}{z-\frac{\pi}{2}}-\frac{\left(z-\frac{\pi}{2}\right)}{2!}+\frac{\left(z-\frac{\pi}{2}\right)^3}{4!}-\cdots\right)^2}
\end{aligned}$$

So we see that the number of b_n terms will be 2 if we simplify the above. We only need to look at the first 2 terms, which will come out as

$$f(z)=\frac{1}{\left(z-\frac{\pi}{2}\right)^2}-\frac{2}{3}+\frac{1}{15}\left(z-\frac{\pi}{2}\right)^2+\cdots$$

Since the order of the b_n is 2, from $\frac{1}{\left(z-\frac{\pi}{2}\right)^2}$, then this is a pole of order 2. If the number of b_n was infinite, this would have been essential singularity.

(c) $f(z)=\frac{1-\cos(z)}{z^4}$, Hence expanding around $z=0$ gives

$$\begin{aligned}
f(z) &= \frac{1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right)}{z^4}\\
&= \frac{\frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}+\cdots}{z^4}\\
&= \frac{1}{2}\frac{1}{z^2}-\frac{1}{4!}+\frac{z^2}{6!}+\cdots
\end{aligned}$$

Since $b_n = \frac{1}{2}\frac{1}{z^2}$ and highest power is 2, then this is pole of order 2.

(d) $f(z) = \cos\left(\frac{\pi}{z-\pi}\right)$. We need to expand $f(z)$ around $z=\pi$ and look at the series. Since $\cos(x)$ expanded around π is

$$\cos(x) = -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{24}(x-\pi)^4 + \cdots$$

Replacing $x = \frac{\pi}{z-\pi}$, the above becomes

$$\cos\left(\frac{\pi}{z-\pi}\right) = -1 + \frac{1}{2}\left(\left(\frac{\pi}{z-\pi}\right)-\pi\right)^2 - \frac{1}{24}\left(\left(\frac{\pi}{z-\pi}\right)-\pi\right)^4 + \cdots$$

The series diverges at $z=\pi$ so it is essential singularity at $z=\pi$. One can also see there are infinite number of b_n terms of the form $\frac{1}{(z-\pi)^n}$

3.8.22 chapter 14, problem 5.1

Problem If C is circle of radius R about z_0, show that

$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n=1 \\ 0 & \text{otherwise} \end{cases}$$

Solution

Since $z = z_0 + Re^{i\theta}$ then $dz = Rie^{i\theta}$ and the integral becomes

$$\begin{aligned} \int_0^{2\pi} \frac{Rie^{i\theta}}{\left(Re^{i\theta}\right)^n} d\theta &= \int_0^{2\pi} \left(Rie^{i\theta}\right)^{1-n} d\theta \\ &= (R)^{1-n} \int_0^{2\pi} ie^{i\theta(1-n)} d\theta \qquad (1) \end{aligned}$$

When $n=1$ the above becomes

$$\begin{aligned} \int_0^{2\pi} \frac{Rie^{i\theta}}{\left(Rie^{i\theta}\right)^n} d\theta &= \int_0^{2\pi} i d\theta \\ &= 2\pi i \end{aligned}$$

And when $n \neq 1$, then (1) becomes

$$\begin{aligned} \int_0^{2\pi} \frac{Rie^{i\theta}}{\left(Re^{i\theta}\right)^n} d\theta &= i(R)^{1-n} \left[\frac{e^{i\theta(1-n)}}{i(1-n)}\right]_0^{2\pi} \\ &= \frac{R^{1-n}}{1-n}\left[e^{i\theta(1-n)}\right]_0^{2\pi} \\ &= \frac{R^{1-n}}{1-n}\left(e^{i2\pi(1-n)} - 1\right) \end{aligned}$$

But $e^{i2\pi(1-n)} = 1$ since $1-n$ is integer. Hence the above becomes

$$\begin{aligned} \int_0^{2\pi} \frac{Rie^{i\theta}}{\left(Re^{i\theta}\right)^n} d\theta &= \frac{R^{1-n}}{1-n}(1-1) \\ &= 0 \end{aligned}$$

QED.

3.9 HW 8

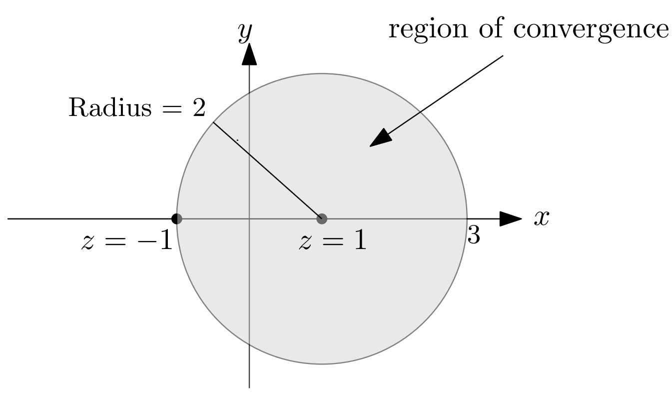
Local contents

3.9.1 chapter 14, problem 6.5

Problem Find Laurent series for $f(z) = \frac{e^z}{z^2-1}$ around $z = 1$

Solution

There are two poles $z = \pm 1$. Hence the expansion around $z = 1$ will extend around $z = 1$ up to the next pole, at $z = -1$. So it will make a circle centered at $z = 1$ and radius 2.



Method one

Let

$$
\begin{aligned}
g(z) &= (z-1)f(z) \\
&= (z-1)\frac{e^z}{(z-1)(z+1)} \\
&= \frac{e^z}{(z+1)}
\end{aligned}
$$

Now $g(z)$ is can be expanded around $z = 1$ since it is analytic at $z = 1$. Using Taylor series.

$$g'(z)=\frac{e^z(z+1)-e^z}{(z+1)^2}$$
$$g''(z)=\frac{(e^z(z+1)+e^z-e^z)(z+1)^2-(e^z(z+1)-e^z)2(z+1)}{(z+1)^4}$$
$$\vdots$$

Hence

$$g'(1)=\frac{2e-e}{4}=\frac{e}{4}$$
$$g''(1)=\frac{(2e+e-e)(4)-(2e-e)2(2)}{(2)^4}=\frac{4e}{16}=\frac{e}{4}$$
$$\vdots$$

Therefore, Taylor series for $g(z)$ around $z=1$ is

$$\begin{aligned}g(z)&=g(1)+g'(1)(z-1)+\frac{g''(1)(z-1)^2}{2!}+\cdots\\&=\frac{e}{2}+\frac{e}{4}(z-1)+\frac{e}{8}(z-1)^2+\cdots\end{aligned}$$

But $f(z)=\frac{g(z)}{(z-1)}$, hence

$$\begin{aligned}f(z)&=\frac{1}{(z-1)}\left(\frac{e}{2}+\frac{e}{4}(z-1)+\frac{e}{8}(z-1)^2+\cdots\right)\\&=\frac{e}{2(z-1)}+\frac{e}{4}-\frac{e}{8}(z-1)+\cdots\end{aligned}$$

Method two

In this method, and when the expansion is about a point z_0 which is not zero, it is easiest to use the substitution $u=z-z_0$ first. Hence $u=z-1$ or $z=u+1$ and now $f(z)$ becomes

$$\begin{aligned}f(z)&=\frac{e^{u+1}}{(u+1)^2-1}\\&=\frac{e^{u+1}}{u^2+2u}\\&=\frac{e^{u+1}}{u(u+2)}\\&=\frac{e^{u+1}}{u}\frac{1}{u+2}\\&=\frac{e^{u+1}}{2u}\frac{1}{\left(1+\frac{u}{2}\right)}\end{aligned}\tag{1}$$

And now we can expand $\frac{1}{\left(1+\frac{u}{2}\right)}$ using Binomial series for $\left|\frac{u}{2}\right|<1$ or $-1<\frac{u}{2}<1$ or $-2<u<2$ or $-2<z-1<2$ or $-1<z<3$. Hence (1) becomes

$$f(z)=\frac{e^{u+1}}{2u}\left(1-\frac{u}{2}+\left(\frac{u}{2}\right)^2-\left(\frac{u}{2}\right)^3+\cdots\right)$$

But $e^u=1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots$, hence the above can be written as

$$\begin{aligned}f(z)&=\frac{e}{2u}\left(1-\frac{u}{2}+\left(\frac{u}{2}\right)^2-\left(\frac{u}{2}\right)^3+\cdots\right)\left(1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots\right)\\&=\frac{e}{2u}\left[\left(1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots\right)-\frac{u}{2}\left(1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots\right)+\left(\frac{u}{2}\right)^2\left(1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots\right)+\cdots\right]\\&=\frac{e}{2u}\left[\left(1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots\right)-\left(\frac{u}{2}+\frac{u^2}{2}+\frac{u^3}{4}+\frac{u^4}{(2)3!}+\cdots\right)+\left(\frac{u^2}{4}+\frac{u^3}{4}+\frac{u^4}{8}+\frac{u^5}{24}+\cdots\right)+\cdots\right]\end{aligned}$$

The above simplifies to

$$\begin{aligned} f(z) &= \frac{e}{2u}\left(1+\frac{u}{2}+\frac{u^2}{4}+\frac{u^3}{24}+\cdots\right) \\ &= e\left(\frac{1}{2u}+\frac{1}{4}+\frac{u}{8}+\frac{u^2}{48}+\cdots\right) \end{aligned}$$

Replacing u back by $z-1$ gives

$$f(z) = \frac{e}{2(z-1)}+\frac{e}{4}+\frac{e(z-1)}{8}+\frac{e(z-1)^2}{48}+\cdots$$

Hence residue is $\frac{e}{2}$. The above is valid for $-1 < z < 3$. Or $|z-1| < 2$. The above is the same answer found using method one. Method one is more direct, but requires lots of differentiations to find the Taylor series for $g(z)$.

3.9.2 chapter 14, problem 6.6

Problem Find Laurent series for $f(z) = \sin\left(\frac{1}{z}\right)$ around $z = 0$

Solution

Since expansion about zero is

$$\sin(x) = x - \frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

Then

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z}-\frac{1}{(3!)z^3}+\frac{1}{(5!)z^5}-\frac{1}{(7!)z^7}+\cdots$$

Residue is 1. Since the series contains only the principal part and no analytical part and then number of terms with negative powers is infinite, then $z = 0$ is an essential singularity.

3.9.3 chapter 14, problem 6.7

Problem Find Laurent series for $f(z) = \frac{\sin \pi z}{4z^2-1}$ around $z = \frac{1}{2}$

Solution

$$\begin{aligned} f(z) &= \frac{1}{4}\frac{\sin \pi z}{z^2-\frac{1}{4}} \\ &= \frac{1}{4}\frac{\sin \pi z}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{2}\right)} \end{aligned} \tag{1}$$

Let us first consider $g(z) = \frac{1}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{2}\right)}$. When we have the expansion about point z_0 not zero, it is easiest to use the substitution $u = z - z_0$ first. Hence $u = z - \frac{1}{2}$ or $z = u + \frac{1}{2}$ and now $g(z)$ becomes

$$\begin{aligned} g(z) &= \frac{1}{u(1+u)} \\ &= \frac{1}{u}\left(1-u+u^2-u^3+\cdots\right) \\ &= \frac{1}{u}-1+u-u^2+\cdots \end{aligned}$$

Replacing back to z the above becomes

$$g(z) = \frac{1}{z-\frac{1}{2}}-1+\left(z-\frac{1}{2}\right)-\left(z-\frac{1}{2}\right)^2+\cdots$$

Hence (1) becomes

$$\begin{aligned} f(z) &= \frac{1}{4}\sin(\pi z) g(z) \\ &= \frac{1}{4}\sin(\pi z)\left(\frac{1}{z-\frac{1}{2}} - 1 + \left(z-\frac{1}{2}\right) - \left(z-\frac{1}{2}\right)^2 + \cdots\right) \end{aligned}$$

Now we expand $\sin(\pi z)$ but remember to expand it around $z=\frac{1}{2}$. The above becomes

$$\begin{aligned} f(z) &= \frac{1}{4}\left(1 - \frac{1}{2}\pi^2\left(z-\frac{1}{2}\right)^2 + \frac{1}{24}\pi^4\left(z-\frac{1}{2}\right)^4 - \cdots\right)\left(\frac{1}{z-\frac{1}{2}} - 1 + \left(z-\frac{1}{2}\right) - \left(z-\frac{1}{2}\right)^2 + \cdots\right) \\ &= \frac{1}{4}\frac{1}{z-\frac{1}{2}} - \frac{1}{4} + \left(\frac{1}{4} - \frac{\pi^2}{8}\right)\left(z-\frac{1}{2}\right) + \cdots \end{aligned}$$

Hence residue is $\frac{1}{4}$.

3.9.4 chapter 14, problem 6.9

Problem Find Laurent series for $f(z) = \frac{1+\cos z}{(z-\pi)^2}$ around $z=\pi$

Solution

We just need to expand $\cos z$ around $z=\pi$ here. This gives

$$\cos z = -1 + \frac{1}{2}(z-\pi)^2 - \frac{1}{2}(z-\pi)^4 + \cdots$$

Hence $f(z)$ becomes

$$\begin{aligned} f(z) &= \frac{1 + \left(-1 + \frac{1}{2}(z-\pi)^2 - \frac{1}{2}(z-\pi)^4 + \cdots\right)}{(z-\pi)^2} \\ &= \frac{\frac{1}{2}(z-\pi)^2 - \frac{1}{2}(z-\pi)^4 + \cdots}{(z-\pi)^2} \\ &= \frac{1}{2} - \frac{1}{2}(z-\pi)^4 + \cdots \end{aligned}$$

Residue is zero.

3.9.5 chapter 14, problem 6.8

Problem Find Laurent series for $f(z) = \frac{1}{z^2-5z+6}$ around $z=2$

Solution

$$f(z) = \frac{1}{(z-3)(z-2)}$$

Let $u = z-2$ or $z = u+2$ and the above becomes

$$\begin{aligned} f(z) &= \frac{1}{(u-1)u} \\ &= \frac{-1}{u}\left(\frac{1}{1-u}\right) \\ &= \frac{-1}{u}\left(1 + u + u^2 + u^3 + \cdots\right) \\ &= -\left(\frac{1}{u} + 1 + u + u^2 + \cdots\right) \end{aligned}$$

But $u = z - 2$ hence the above becomes

$$\begin{aligned} f(z) &= -\left(\frac{1}{z-2} + 1 + (z-2) + (z-2)^2 + \cdots\right) \\ &= \frac{-1}{z-2} - 1 - (z-2) - (z-2)^2 - \cdots \end{aligned}$$

Reside is -1.

3.9.6 chapter 14, problem 6.15

Problem Find residue at $z = \frac{1}{2}$ and $z = \frac{4}{5}$for $f(z) = \frac{1}{(1-2z)(5z-4)}$

Solution

$$\begin{aligned} \text{Residue}\left(\frac{1}{2}\right) &= \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)f(z) \\ &= \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{1}{(1-2z)(5z-4)} \\ &= \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{1}{\left(2\left(\frac{1}{2}-z\right)\right)(5z-4)} \\ &= \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{-1}{\left(2\left(z-\frac{1}{2}\right)\right)(5z-4)} \\ &= \lim_{z\to\frac{1}{2}}\frac{-1}{2(5z-4)} \\ &= \frac{-1}{2\left(5\left(\frac{1}{2}\right)-4\right)} \\ &= \frac{1}{3} \end{aligned}$$

And

$$\begin{aligned} \text{Residue}\left(\frac{4}{5}\right) &= \lim_{z\to\frac{4}{5}}\left(z-\frac{4}{5}\right)f(z) \\ &= \lim_{z\to\frac{4}{5}}\left(z-\frac{4}{5}\right)\frac{1}{(1-2z)(5z-4)} \\ &= \lim_{z\to\frac{4}{5}}\left(z-\frac{4}{5}\right)\frac{1}{(1-2z)5\left(z-\frac{4}{5}\right)} \\ &= \lim_{z\to\frac{4}{5}}\frac{1}{5(1-2z)} \\ &= \frac{1}{5\left(1-2\left(\frac{4}{5}\right)\right)} \\ &= \frac{-1}{2\left(5\left(\frac{1}{2}\right)-4\right)} \\ &= -\frac{1}{3} \end{aligned}$$

3.9.7 chapter 14, problem 6.19

Problem Find residue at $z = \frac{\pi}{2}$ for $f(z) = \frac{\sin^2 z}{2z-\pi}$

Solution

$$\begin{aligned}
\text{Residue}\left(\frac{\pi}{2}\right) &= \lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)\frac{\sin^2 z}{2z-\pi} \\
&= \lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)\frac{\sin^2 z}{2\left(z-\frac{\pi}{2}\right)} \\
&= \lim_{z\to\frac{\pi}{2}}\frac{\sin^2 z}{2} \\
&= \frac{\sin^2\left(\frac{\pi}{2}\right)}{2} \\
&= \frac{1}{2}
\end{aligned}$$

3.9.8 chapter 14, problem 6.23

Problem Find residue at $z=\frac{2i}{3}$ for $f(z)=\frac{e^{iz}}{9z^2+4}$

Solution

$$\begin{aligned}
\text{Residue}\left(\frac{2i}{3}\right) &= \lim_{z\to\frac{2i}{3}}\left(z-\frac{2i}{3}\right)\frac{e^{iz}}{9z^2+4} \\
&= \lim_{z\to\frac{2i}{3}}\left(z-\frac{2i}{3}\right)\frac{e^{iz}}{9\left(z^2+\frac{4}{9}\right)} \\
&= \frac{1}{9}\lim_{z\to\frac{2i}{3}}\left(z-\frac{2i}{3}\right)\frac{e^{iz}}{\left(z-\frac{2i}{3}\right)\left(z+\frac{2i}{3}\right)} \\
&= \frac{1}{9}\lim_{z\to\frac{2i}{3}}\frac{e^{iz}}{\left(z+\frac{2i}{3}\right)} \\
&= \frac{1}{9}\frac{e^{i\frac{2i}{3}}}{\left(\frac{2i}{3}+\frac{2i}{3}\right)} \\
&= \frac{1}{9}\frac{e^{\frac{-2}{3}}}{\frac{4i}{3}} \\
&= \frac{1}{3}\frac{e^{\frac{-2}{3}}}{4i} \\
&= -i\frac{e^{-\frac{2}{3}}}{12}
\end{aligned}$$

3.9.9 chapter 14, problem 6.31

Problem Find residue at $z=0$ for $f(z)=\frac{e^{3z}-3z-1}{z^4}$

Solution

Pole is of order $m=4$, so we use the formula.

$$
\begin{aligned}
\text{Residue}(0) &= \lim_{z\to 0} \frac{1}{(m-1)!}\left(\frac{d^{m-1}}{z^m}(z-0)^m f(z)\right)\\
&= \lim_{z\to 0} \frac{1}{3!}\left(\frac{d^3}{z^3}\left(z^4 \frac{e^{3z}-3z-1}{z^4}\right)\right)\\
&= \lim_{z\to 0} \frac{1}{3!}\left(\frac{d^3}{z^3}\left(e^{3z}-3z-1\right)\right)\\
&= \lim_{z\to 0} \frac{1}{3!}\left(\frac{d^2}{z^2}\left(3e^{3z}-3\right)\right)\\
&= \lim_{z\to 0} \frac{1}{3!}\left(\frac{d}{z}9e^{3z}\right)\\
&= \lim_{z\to 0} \frac{1}{3!}\left(27e^{3z}\right)\\
&= \frac{27}{6}\\
&= \frac{9}{2}
\end{aligned}
$$

3.9.10 chapter 14, problem 7.4

Problem Evaluate $I = \int_0^{2\pi} \frac{\sin^2\theta}{5+3\cos\theta} d\theta$

Solution let $z = e^{i\theta}$ then

$$
\begin{aligned}
dz &= ie^{i\theta} d\theta\\
d\theta &= \frac{dz}{ie^{i\theta}}\\
&= -i\frac{dz}{z}
\end{aligned}
$$

Hence $d\theta = -i\frac{dz}{z}$. Now using

$$
\begin{aligned}
\sin^2\theta &= \frac{1-\cos(2\theta)}{2}\\
&= \frac{1-\left(\frac{e^{i2\theta}+e^{-i2\theta}}{2}\right)}{2}\\
&= \frac{1-\left(\frac{z^2+z^{-2}}{2}\right)}{2}\\
&= \frac{\frac{2-z^2-z^{-2}}{2}}{2}\\
&= \frac{2-z^2-\frac{1}{z^2}}{4}\\
&= \frac{2z^2-z^4-1}{4z^2}
\end{aligned}
$$

And

$$
\cos\theta = \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right) = \left(\frac{z+z^{-1}}{2}\right)
$$

Then the integral becomes

$$
\begin{aligned}
I &= \int_0^{2\pi} \frac{\sin^2\theta}{5+3\cos\theta} d\theta \\
&= \oint \frac{\frac{2z^2-z^4-1}{4z^2}}{5+3\left(\frac{z+z^{-1}}{2}\right)} \left(-i\frac{dz}{z}\right) \\
&= -i \oint \frac{\frac{2z^2-z^4-1}{4z^2}}{z\left(5+3\left(\frac{z+z^{-1}}{2}\right)\right)} dz \\
&= -i \oint \frac{2\left(2z^2-z^4-1\right)}{4z^3\left(10+3\left(z+z^{-1}\right)\right)} \\
&= \frac{-i}{2} \oint \frac{2z^2-z^4-1}{z^3\left(\frac{10z+3\left(z^2+1\right)}{z}\right)} dz \\
&= \frac{-i}{2} \oint \frac{2z^2-z^4-1}{z^2\left(10z+3z^2+3\right)} dz
\end{aligned}
$$

Using residue theorem,

$$
\begin{aligned}
\oint \frac{2z^2-z^4-1}{z^2\left(10z+3z^2+3\right)} dz &= \oint f(z)\, dz \\
&= 2\pi i \sum \text{residues of } f(z) \text{ inside}
\end{aligned}
$$

Hence

$$
\begin{aligned}
I &= \frac{-i}{2} \oint \frac{2z^2-z^4-1}{z^2\left(10z+3z^2+3\right)} dz \\
&= \frac{-i}{2}\left[2\pi i \sum R(f(z))\right] \\
&= -i^2\pi \sum R(f(z)) \\
&= \pi \sum R(f(z))
\end{aligned}
$$

Now we need to find residues of $f(z)$

$$
\frac{2z^2-z^4-1}{z^2\left(10z+3z^2+3\right)} = f(z) = \frac{g(z)}{h(z)}
$$

factoring $h(z) = z^2(3+z)(1+3z)$ gives

$$
f(z) = \frac{g(z)}{h(z)} = \frac{2z^2-z^4-1}{z^2(3+z)(1+3z)}
$$

Need to find residue inside a unit circle. $3z = -1$ or $z_0 = -1/3$ is inside the unit circle, also $z_1 = 0$ is inside the circle

$$\begin{aligned}
\lim_{z\to z_0}(z-z_0)\frac{g(z)}{h(z)} &= \lim_{z\to -1/3}\left(z+\frac{1}{3}\right)\frac{2z^2-z^4-1}{z^2(3+z)(1+3z)}\\
&= \lim_{z\to -1/3}\frac{2z^2-z^4-1}{3z^2(3+z)}\\
&= \frac{2\left(-\frac{1}{3}\right)^2-\left(-\frac{1}{3}\right)^4-1}{3\left(-\frac{1}{3}\right)^2\left(3-\frac{1}{3}\right)}\\
&= \frac{\frac{2}{9}-\frac{1}{81}-1}{3\left(\frac{1}{9}\right)\left(3-\frac{1}{3}\right)}\\
&= \frac{\frac{18-1-81}{81}}{\left(\frac{1}{3}\right)\left(\frac{9-1}{3}\right)}\\
&= \frac{\frac{-64}{81}}{\left(\frac{8}{9}\right)}\\
&= \frac{-8}{9}
\end{aligned}$$

To find residue at zero, since it is order $m = 2$

$$\begin{aligned}
\lim_{z\to z_1}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-z_1)^2\frac{g(z)}{h(z)} &= \lim_{z\to 0}\frac{1}{(2-1)!}\frac{d}{dz}(z)^2\frac{2z^2-z^4-1}{z^2(3+z)3\left(\frac{1}{3}+z\right)}\\
&= \lim_{z\to 0}\frac{d}{dz}\frac{2z^2-z^4-1}{(3+z)3\left(\frac{1}{3}+z\right)}\\
&= \lim_{z\to 0}\frac{d}{dz}\frac{2z^2-z^4-1}{3+10z+3z^2}\\
&= \lim_{z\to 0}\frac{1}{3+10z+3z^2}\left(4z-4z^3\right)+\left[\left(2z^2-z^4-1\right)\left(\frac{-1}{\left(3+10z+3z^2\right)^2}\right)(10+6z)\right]\\
&= \frac{1}{3}(0)+(-1)\left(\frac{-1}{(3)^2}\right)(10)\\
&= \frac{10}{9}
\end{aligned}$$

Hence

$$\begin{aligned}
I &= \frac{-i}{2}\oint\frac{2z^2-z^4-1}{z^2\left(10z+3z^2+3\right)}dz\\
&= \pi\sum R(f(z))\\
&= \pi\left(\frac{10}{9}-\frac{8}{9}\right)
\end{aligned}$$

Therefore

$$\begin{aligned}
I &= \int_0^{2\pi}\frac{\sin^2\theta}{5+3\cos\theta}d\theta\\
&= \frac{2}{9}\pi
\end{aligned}$$

3.9.11 chapter 14, problem 7.5

Problem Evaluate $\int_0^{\pi}\frac{d\theta}{1-2r\cos\theta+r^2}$ for $0\le r<1$

Solution

Since even function then,

$$\int_0^{\pi}\frac{d\theta}{1-2r\cos\theta+r^2}=\frac{1}{2}\int_0^{2\pi}\frac{d\theta}{1-2r\cos\theta+r^2}$$

Using $\cos\theta=\frac{z+z^{-1}}{2}$ and $z=e^{i\theta}$ then $dz=ie^{i\theta}d\theta=izd\theta$. The integral becomes

$$\begin{aligned}
I &= \frac{1}{2}\oint_C \frac{1}{1-2r\left(\frac{z+z^{-1}}{2}\right)+r^2}\frac{dz}{iz}\\
&= -\frac{1}{2}i\oint_C \frac{1}{1-r\left(z+\frac{1}{z}\right)+r^2}\frac{dz}{z}\\
&= -\frac{1}{2}i\oint_C \frac{1}{z-r\left(z^2+1\right)+zr^2}dz\\
&= -\frac{1}{2}i\oint_C \frac{1}{z-rz^2-r+zr^2}dz\\
&= -\frac{1}{2}i\oint_C \frac{1}{z^2(-r)+z\left(1+r^2\right)-r}dz\\
&= \frac{1}{2}\frac{i}{r}\oint_C \frac{1}{z^2+z\left(\frac{-1}{r}-r\right)+1}dz\\
&= \frac{1}{2}\frac{i}{r}\oint_C \frac{1}{(z-r)\left(z-\frac{1}{r}\right)}dz
\end{aligned}$$

Since $|r|<1$ then only pole inside the unit circle is $z=r$. We do not need to find residue for the pole at $z=\frac{1}{r}$ since it is outside. Hence

$$\begin{aligned}
I &= \frac{i}{2r}2\pi i\,\text{Residue}(r)\\
&= \frac{-\pi}{r}\text{Residue}(r)
\end{aligned}$$

Where $f(z)=\frac{1}{(z-r)\left(z-\frac{1}{r}\right)}$ for purpose of finding residue.

$$\begin{aligned}
\text{Residue}(r) &= \lim_{z\to r}(z-r)\frac{1}{(z-r)\left(z-\frac{1}{r}\right)}\\
&= \lim_{z\to r}\frac{1}{\left(z-\frac{1}{r}\right)}\\
&= \frac{1}{\left(r-\frac{1}{r}\right)}\\
&= \frac{r}{r^2-1}
\end{aligned}$$

Hence

$$\begin{aligned}
I &= \frac{-\pi}{r}\left(\frac{r}{r^2-1}\right)\\
&= \frac{-\pi}{r^2-1}\\
&= \frac{\pi}{1-r^2}
\end{aligned}$$

3.9.12 chapter 14, problem 7.12

Problem Evaluate $\int_0^\infty \frac{x^2}{x^4+16}dx$

Solution

The integrand is even. Hence $I = \int_0^\infty \frac{x^2}{x^4+16}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{x^4+16}dx$. First we find location of poles. These are roots of

$$\begin{aligned} x^4 + 16 &= 0 \\ x^4 &= -16 \\ x &= (-16)^{\frac{1}{4}} \\ &= (16)^{\frac{1}{4}}(-1)^{\frac{1}{4}} \\ &= 2(-1)^{\frac{1}{4}} \end{aligned}$$

To find $(-1)^{\frac{1}{4}}$, we write it as

$$\begin{aligned} (-1)^{\frac{1}{4}} &= \left(e^{i\pi}\right)^{\frac{1}{4}} \\ &= e^{i\frac{(\pi+2n\pi)}{4}} \qquad n{=}0{,}1{,}2{,}3 \end{aligned}$$

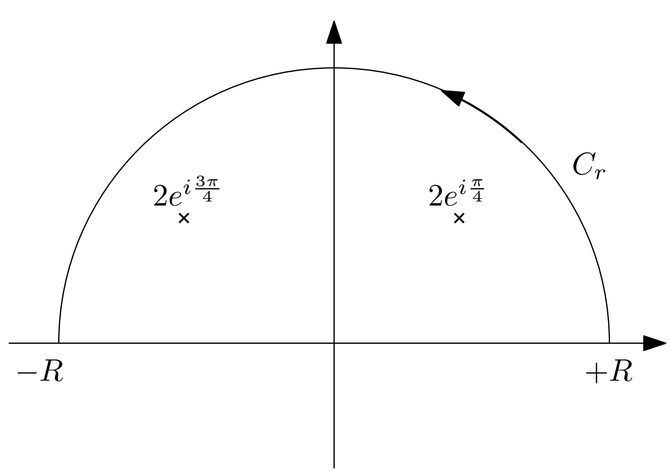
Therefore the four roots are

$$2e^{i\frac{\pi}{4}}, 2e^{i\frac{3}{4}\pi}, 2e^{i\frac{5}{4}\pi}, 2e^{i\frac{7}{4}\pi}$$

Or

$$2e^{\pm i\frac{\pi}{4}}, 2e^{\pm i\frac{3}{4}\pi}$$

Now we use the following contour to do the integration, such that the upper half circle include inside it the first two roots above, since these are the only ones in the upper half plane.



$$\oint_C f(z)dz = 2\pi i \sum \text{ sum of residues inside}$$

Where C above is the contour from $-R$ to $+R$ and around C_r as shown. So all what we need to do now if found the residues. There are two poles inside C. These are $z_1 = 2e^{i\frac{\pi}{4}}$ and $z_2 = 2e^{i\frac{3}{4}\pi}$. Therefore using

$$\begin{aligned}
\operatorname{Residue}(z_1) &= \lim_{z\to z_1}(z-z_1)f(z)\\
\operatorname{Residue}\left(2e^{i\frac{\pi}{4}}\right) &= \lim_{z\to 2e^{i\frac{\pi}{4}}}\left(z-2e^{i\frac{\pi}{4}}\right)\frac{z^2}{z^4+16}\\
&= \lim_{z\to 2e^{i\frac{\pi}{4}}}\frac{z^3-2z^2e^{\frac{i\pi}{4}}}{z^4+16}
\end{aligned}$$

Applying L'Hopitals rules gives

$$\begin{aligned}
\operatorname{Residue}(z_1) &= \lim_{z\to 2e^{i\frac{\pi}{4}}}\frac{3z^2-4ze^{\frac{i\pi}{4}}}{4z^3}\\
&= \frac{3\left(2e^{i\frac{\pi}{4}}\right)^2-4\left(2e^{i\frac{\pi}{4}}\right)e^{\frac{i\pi}{4}}}{4\left(2e^{i\frac{\pi}{4}}\right)^3}\\
&= \frac{12e^{i\frac{\pi}{2}}-8e^{i\frac{\pi}{2}}}{32e^{i\frac{3\pi}{4}}}\\
&= \frac{4e^{i\frac{\pi}{2}}}{32e^{i\frac{3\pi}{4}}}\\
&= \frac{1}{8}e^{-i\frac{\pi}{4}} \qquad (1)
\end{aligned}$$

Similarly

$$\begin{aligned}
\operatorname{Residue}(z_2) &= \lim_{z\to z_2}(z-z_2)f(z)\\
\operatorname{Residue}\left(2e^{i\frac{\pi}{4}}\right) &= \lim_{z\to 2e^{i\frac{3}{4}\pi}}\left(z-2e^{i\frac{3}{4}\pi}\right)\frac{z^2}{z^4+16}\\
&= \lim_{z\to 2e^{i\frac{3}{4}\pi}}\frac{z^3-2z^2e^{i\frac{3}{4}\pi}}{z^4+16}
\end{aligned}$$

Applying L'Hopitals rules gives

$$\begin{aligned}
\operatorname{Residue}(z_2) &= \lim_{z\to 2e^{i\frac{3}{4}\pi}}\frac{3z^2-4ze^{i\frac{3}{4}\pi}}{4z^3}\\
&= \frac{3\left(2e^{i\frac{3}{4}\pi}\right)^2-4\left(2e^{i\frac{3}{4}\pi}\right)e^{i\frac{3}{4}\pi}}{4\left(2e^{i\frac{3}{4}\pi}\right)^3}\\
&= \frac{12e^{i\frac{3}{2}\pi}-8e^{i\frac{3}{2}\pi}}{32e^{i\frac{9}{4}\pi}}\\
&= \frac{4e^{i\frac{3}{2}\pi}}{32e^{i\frac{9}{4}\pi}}\\
&= \frac{1}{8}e^{i\left(\frac{3\pi}{2}-\frac{9}{4}\pi\right)}\\
&= \frac{1}{8}e^{-i\frac{3}{4}\pi} \qquad (2)
\end{aligned}$$

From (1) and (2) then

$$\begin{aligned}\oint_C f(z)dz &= 2\pi i\left(\frac{1}{8}e^{-i\frac{\pi}{4}} + \frac{1}{8}e^{-i\frac{3}{4}\pi}\right)\\ &= 2\pi i\left(-\frac{1}{8}i\sqrt{2}\right)\\ &= \frac{1}{4}\sqrt{2}\,\pi\end{aligned}$$

Therefore, since

$$\oint_C f(z)dz = \lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{x^4+16}dx + \lim_{R\to\infty}\int_{CR}\frac{z^2}{z^4+16}dz$$

Then

$$\frac{1}{4}\sqrt{2}\,\pi = \lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{x^4+16}dx + \lim_{R\to\infty}\int_{CR}\frac{z^2}{z^4+16}dz \tag{3}$$

The only thing left is to show that $\lim_{R\to\infty}\int_{CR}\frac{z^2}{z^4+16}dz = 0$. Let $z = R\, ie^{\theta}$, then $\frac{dz}{d\theta} = Rie^{i\theta}$, therefore the second integral above can be written as

$$\begin{aligned}\lim_{R\to\infty}\int_{CR}\frac{z^2}{z^4+16}dz &= \lim_{R\to\infty}\int_0^{\pi}\frac{R^2\left(ie^{i\theta}\right)^2}{\left(Rie^{i\theta}\right)^4+16}Rie^{i\theta}d\theta\\ &= \lim_{R\to\infty}\int_0^{\pi}\frac{R^3\left(ie^{i\theta}\right)^2}{R^4e^{i4\theta}+16}ie^{i\theta}d\theta\end{aligned}$$

As $R\to\infty$ the integrand goes to zero, since the numerator has R^3 and the denominator has R^4. In other words, $\frac{R^3}{R^4+16}$ or $\frac{\frac{1}{R^4}}{1+\frac{16}{R^4}}$ which goes to zero as $R\to\infty$. Hence (3) simplifies to

$$\begin{aligned}\frac{1}{4}\sqrt{2}\,\pi &= \lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{x^4+16}dx\\ \int_{-\infty}^{\infty}\frac{x^2}{x^4+16}dx &= \frac{1}{4}\sqrt{2}\,\pi\end{aligned}$$

Therefore

$$\int_0^{\infty}\frac{x^2}{x^4+16}dx = \frac{1}{8}\sqrt{2}\,\pi$$

3.9.13 chapter 14, problem 7.18

Problem Evaluate

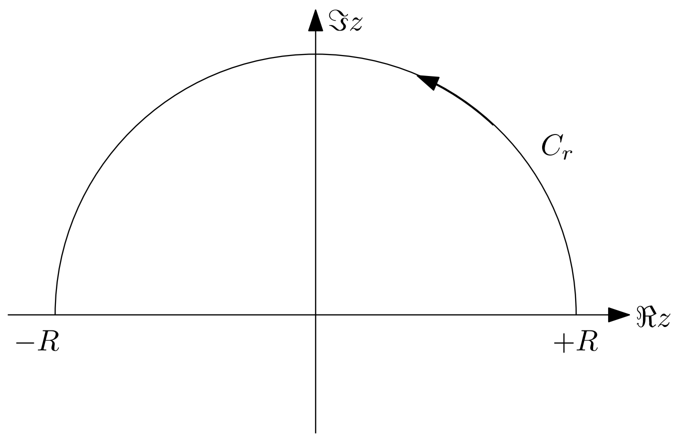
$$I = \int_0^{\infty}\frac{\cos\pi x}{1+x^2+x^4}dx$$

Solution

Since this is an even function then

$$\begin{aligned}\int_0^{\infty}\frac{\cos\pi x}{1+x^2+x^4}dx &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos\pi x}{1+x^2+x^4}dx\\ &= \frac{1}{2}\operatorname{Re}\int_{-\infty}^{\infty}\frac{e^{i\pi x}}{1+x^2+x^4}dx \end{aligned}\tag{1}$$

Now consider the contour shown



Then considering the integral

$$\begin{aligned} I &= \oint f(z)dz \\ &= \lim_{R\to\infty}\left(\int_{-R}^{R}\frac{e^{i\pi x}}{1+x^2+x^4}dx + \int_C \frac{e^{i\pi z}}{1+z^2+z^4}dz\right) \end{aligned} \tag{2}$$

But by Cauchy theorem,

$$2\pi i \sum \text{residues of } f(z) \text{ inside contour} = \oint f(z)dz$$

The second integral to the right in (1) can be shown to go to zero as $R\to\infty$ (See below). Hence the above simplifies to

$$2\pi i \sum \text{residues of } f(z) \text{ inside contour} = \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{i\pi x}}{1+x^2+x^4}dx \tag{3}$$

Now we need to find residues of

$$f(z) = \frac{e^{i\pi z}}{1+z^2+z^4}$$

Looking at $1+z^2+z^4$, let $z^2=\beta$, then find root of $1+\beta+\beta^2=0$, the roots are

$$\begin{aligned} \frac{-b\pm\sqrt{b^2-4ac}}{2a} &= \frac{-1\pm\sqrt{1-4}}{2} \\ &= \frac{-1\pm i\sqrt{3}}{2} \end{aligned}$$

Hence

$$\left(\beta-\frac{-1+i\sqrt{3}}{2}\right)\left(\beta-\frac{-1-i\sqrt{3}}{2}\right)=0$$

Replacing $z^2=\beta$ gives

$$\left(z^2-\frac{-1+i\sqrt{3}}{2}\right)\left(z^2-\frac{-1-i\sqrt{3}}{2}\right)=0$$

looking at each term. $z^2-\frac{-1+i\sqrt{3}}{2}=0$ results in

$$\begin{aligned} z &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{0\pm\sqrt{0-4\left(-\frac{-1+i\sqrt{3}}{2}\right)}}{2} \\ &= \pm\sqrt{\frac{-1+i\sqrt{3}}{2}} \\ &= \pm\sqrt{\frac{-1}{2}+\frac{i\sqrt{3}}{2}} \end{aligned}$$

Hence the first 2 roots are

$$\left(\frac{-1}{2}-\frac{i\sqrt{3}}{2}\right),\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$$

Now to find the second 2 roots, looking at second $\left(z^2-\frac{-1-i\sqrt{3}}{2}\right)=0$ results in

$$\begin{aligned}
z &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
&= \frac{0\pm\sqrt{-4\left(-\frac{-1-i\sqrt{3}}{2}\right)}}{2}\\
&= \frac{\pm\sqrt{-2-2i\sqrt{3}}}{2}\\
&= \pm\sqrt{-\frac{1}{2}-\frac{i}{2}\sqrt{3}}
\end{aligned}$$

Hence

$$=\pm\sqrt{\left(\frac{1}{2}-\frac{i}{2}\sqrt{3}\right)\left(\frac{1}{2}-\frac{i}{2}\sqrt{3}\right)}=\pm\sqrt{\left(\frac{1}{2}-\frac{i}{2}\sqrt{3}\right)^2}=\pm\left(\frac{1}{2}-\frac{i}{2}\sqrt{3}\right)$$

And the second two roots are

$$\left(\frac{1}{2}-\frac{i}{2}\sqrt{3}\right),\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)$$

So the poles of $f(z)$ are

$$\begin{aligned}
z_1 &= \left(\frac{-1}{2}-\frac{i\sqrt{3}}{2}\right)\\
z_2 &= \left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\\
z_3 &= \left(\frac{1}{2}-\frac{i}{2}\sqrt{3}\right)\\
z_4 &= \left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)
\end{aligned}$$

Now need to find which ones are inside the contour. Need the poles with a positive imaginary parts. Hence looking at above, these are z_2 and z_4. To find the residue of $f(z)$ at z_2

$$\begin{aligned}
\lim_{z\to z_2}(z-z_2)f(z) &= \lim_{z\to z_2}\left(z-\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\right)\frac{e^{i\pi z}}{1+z^2+z^4}\\
&= \lim_{z\to z_2}\frac{ze^{i\pi z}-\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)e^{i\pi z}}{1+z^2+z^4}
\end{aligned}$$

Applying L'Hopitals

$$
\begin{aligned}
\text{Residue}(z_2) &= \lim_{z\to z_2} \frac{e^{i\pi z} + i\pi z e^{i\pi z} - i\pi\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)e^{i\pi z}}{2z+4z^3}\\
&= \frac{e^{i\pi\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} + i\pi\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)e^{i\pi\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - i\pi\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)e^{i\pi\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}}{2\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)+4\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^3}\\
&= \frac{e^{\left(\frac{i\pi}{2}-\frac{\sqrt{3}\pi}{2}\right)} + i\pi\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)e^{i\pi\left(\frac{i\pi}{2}-\frac{\sqrt{3}\pi}{2}\right)} - i\pi\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)e^{i\pi\left(\frac{i\pi}{2}-\frac{\sqrt{3}}{2}\pi\right)}}{i\sqrt{3}-3}\\
&= \frac{ie^{-\frac{1}{2}\sqrt{3}\pi}}{i\sqrt{3}-3}
\end{aligned}
$$

Now to find the second residue

$$
\begin{aligned}
\lim_{z\to z_2}(z-z_4)f(z) &= \lim_{z\to z_4}\left(z-\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)\right)\frac{e^{i\pi z}}{1+z^2+z^4}\\
&= \lim_{z\to z_4}\frac{ze^{i\pi z}-\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)e^{i\pi z}}{1+z^2+z^4}
\end{aligned}
$$

Applying L'Hopitals

$$
\begin{aligned}
\text{Residue}(z_2) &= \lim_{z\to z_2} \frac{e^{i\pi z} + i\pi z e^{i\pi z} - i\pi\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)e^{i\pi z}}{2z+4z^3}\\
&= \frac{e^{i\pi\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)} + i\pi\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)e^{i\pi\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)} - i\pi\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)e^{i\pi\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)}}{2\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)+4\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)^3}\\
&= \frac{e^{-\frac{1}{2}\pi\left(\sqrt{3}+i\right)}}{i\sqrt{3}+3}
\end{aligned}
$$

The above is the second residue. Hence Sum of residues is

$$
\begin{aligned}
\frac{i\exp\left(-\frac{\pi\sqrt{3}}{2}\right)}{i\sqrt{3}-3} - \frac{i\ \exp\left(-\frac{\pi\sqrt{3}}{2}\right)}{i\sqrt{3}+3} &= i\exp\left(-\frac{\pi\sqrt{3}}{2}\right)\left(\frac{1}{i\sqrt{3}-3}-\frac{1}{i\sqrt{3}+3}\right)\\
&= i\exp\left(-\frac{\pi\sqrt{3}}{2}\right)\left(\frac{\left(i\sqrt{3}+3\right)-\left(i\sqrt{3}-3\right)}{\left(i\sqrt{3}-3\right)\left(i\sqrt{3}+3\right)}\right)\\
&= i\exp\left(-\frac{\pi\sqrt{3}}{2}\right)\left(\frac{6}{-3-9}\right)\\
&= i\exp\left(-\frac{\pi\sqrt{3}}{2}\right)\left(\frac{-1}{2}\right)\\
&= -\frac{i}{2}\exp\left(-\frac{\pi\sqrt{3}}{2}\right)
\end{aligned}
$$

Now, from (3) we obtain

$$
\begin{aligned}
\lim_{R\to\infty}\int_{-R}^{+R}\frac{e^{i\pi x}}{1+x^2+x^4}dx &= 2\pi i\left(-\frac{i}{2}\exp\left(-\frac{\pi\sqrt{3}}{2}\right)\right)\\
&= \pi\ \exp\left(-\frac{\pi\sqrt{3}}{2}\right)
\end{aligned}
$$

Taking the real part of both sides above gives

$$\int_{-\infty}^{+\infty} \frac{\cos \pi x}{1+x^2+x^4} dx = \pi \exp\left(-\frac{\pi\sqrt{3}}{2}\right)$$

Using the above in (1) gives

$$\begin{aligned}\int_0^{\infty} \frac{\cos \pi x}{1+x^2+x^4} dx &= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\cos \pi x}{1+x^2+x^4} dx \\ &= \frac{\pi}{2} \exp\left(-\frac{\pi\sqrt{3}}{2}\right)\end{aligned}$$

What is left is to show that $\lim_{R\to\infty} \int_C e^{imz} f(z) dz = 0$. To do this, we use Jordan Lemma, which says that integral $\lim_{R\to\infty} \int_C e^{imz} f(z) dz \to 0$ if $\left|f(z)\right|_{\max} \to 0$ as $|z| \to \infty$. But

$$\begin{aligned}\lim_{R\to\infty} \left|f(z)\right|_{\max} &= \lim_{R\to\infty} \frac{1}{\left|1+z^2+z^4\right|_{\min}} \\ &= \lim_{R\to\infty} \frac{1}{\left|1+\left(R\,e^{i\theta}\right)^2+\left(R\,e^{i\theta}\right)^4\right|_{\min}} \\ &= \lim_{R\to\infty} \frac{1}{1+R^2+R^4} \\ &= \lim_{R\to\infty} \frac{\frac{1}{R^4}}{\frac{1}{R^4}+\frac{1}{R^2}+1} \\ &= \frac{0}{1} \\ &= 0\end{aligned}$$

Therefore $\lim_{R\to\infty} \int_C e^{imz} f(z) dz = 0$, which is all what we needed to show to complete the solution.

3.9.14 chapter 14, problem 7.20

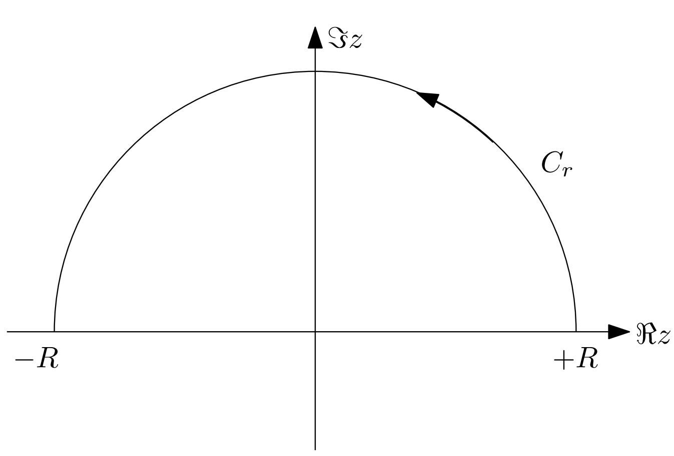
Problem Evaluate

$$I = \int_0^{\infty} \frac{\cos x}{\left(1+9x^2\right)^2} dx$$

Solution consider

$$\oint \frac{e^{iz}}{\left(1+9z^2\right)^2} dz$$

over the contour shown.



By Cauchy theorem,

$$\oint \frac{e^{iz}}{\left(1+9z^2\right)^2} dz = 2\pi i \sum \text{residues of } f(z) \text{ inside contour}$$

But

$$\oint \frac{e^{iz}}{\left(1+9z^2\right)^2} dz = \int_{-R}^{+R} \frac{e^{ix}}{\left(1+9z^2\right)^2} dx + \int_C \frac{e^{iz}}{\left(1+9z^2\right)^2} dz$$

second integral to the right above can be shown to go to zero as $R \to \infty$ as was done in earlier problem. Hence

$$\oint \frac{e^{iz}}{\left(1+9z^2\right)^2} dz = \int_{-\infty}^{+\infty} \frac{e^{ix}}{\left(1+9x^2\right)^2} dx = 2\pi i \sum \text{residues } f(z)$$

Or

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{\left(1+9x^2\right)^2} dx = 2\pi i \sum \text{residues } \ f(z)$$

Now we need to find residues of

$$f(z) = \frac{e^{iz}}{\left(1+9z^2\right)^2}$$

Looking at

$$\begin{aligned} 1+9z^2 &= 0 \\ z^2 &= -\frac{1}{9} \\ z &= \pm\frac{i}{3} \end{aligned}$$

Hence the poles of $f(z)$ are

$$\begin{aligned} z_1 &= -\frac{i}{3} \\ z_2 &= +\frac{i}{3} \end{aligned}$$

Each is of order $m = 2$. Hence, $\left(1+9z^2\right)^2$ can be written as $\left(9\left(z-\frac{i}{3}\right)\left(z+\frac{i}{3}\right)\right)^2$ or $81\left(z-\frac{i}{3}\right)^2\left(z+\frac{i}{3}\right)^2$. Therefore

$$f(z) = \frac{1}{81} \frac{e^{iz}}{\left(z-\frac{i}{3}\right)^2\left(z+\frac{i}{3}\right)^2}$$

Now need to find which ones are inside the contour. Need the poles with a positive imaginary parts. Looking at above, z_2 is the pole we need to find residue for. Hence, to find the residue R of $f(z)$ at z_2, using $m = 2$

$$
\begin{aligned}
R &= \lim_{z\to z_2} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-z_2)^m f(z)\\
&= \lim_{z\to z_2} \frac{d}{dz}(z-z_2)^2 \frac{1}{81}\frac{e^{iz}}{(z-z_1)^2(z-z_2)^2}\\
&= \frac{1}{81}\lim_{z\to z_2}\frac{d}{dz}\frac{e^{iz}}{(z-z_1)^2}\\
&= \frac{1}{81}\lim_{z\to z_2}\frac{1}{(z-z_1)^2}\left(ie^{iz}\right) + e^{iz}\left(-2\frac{1}{(z-z_1)^3}\right)\\
&= \frac{1}{81}\left(\frac{1}{(z_2-z_1)^2}\left(ie^{iz_2}\right) + e^{iz_2}\left[-2\frac{1}{(z_2-z_1)^3}\right]\right)\\
&= \frac{1}{81}e^{iz_2}\left(\frac{i}{(z_2-z_1)^2} - \frac{2}{(z_2-z_1)^3}\right)\\
&= \frac{1}{81}e^{i\frac{i}{3}}\left(\frac{i}{\left(\frac{i}{3}-\left(-\frac{i}{3}\right)\right)^2} - \frac{2}{\left(\frac{i}{3}-\left(-\frac{i}{3}\right)\right)^3}\right)\\
&= \frac{1}{81}e^{\frac{-1}{3}}\left(\frac{i}{\left(\frac{2i}{3}\right)^2} - \frac{2}{\left(\frac{2i}{3}\right)^3}\right)\\
&= \frac{1}{81}e^{\frac{-1}{3}}\left(\frac{-9i}{4} + \frac{54}{8i}\right)\\
&= \frac{1}{81}e^{\frac{-1}{3}}\left(\frac{-18i-54i}{8}\right)\\
&= \frac{1}{81}e^{\frac{-1}{3}}(-9i)
\end{aligned}
$$

Hence

$$R = -\frac{1}{9}ie^{\frac{-1}{3}}$$

Since

$$\int_{-\infty}^{+\infty}\frac{e^{ix}}{\left(1+9x^2\right)^2}dx = 2\pi i\sum \text{residues } \ f(z)$$

Then

$$
\begin{aligned}
\int_{-\infty}^{+\infty}\frac{e^{ix}}{\left(1+9x^2\right)^2}dx &= 2\pi i\left(-\frac{1}{9}ie^{\frac{-1}{3}}\right)\\
&= \left(\frac{1}{9}\right)2\pi\left(e^{\frac{-1}{3}}\right)\\
&= \frac{2}{9}\pi\left(e^{\frac{-1}{3}}\right)
\end{aligned}
$$

Taking the real part of both sides above gives

$$\int_{-\infty}^{+\infty}\frac{\cos x}{\left(1+9x^2\right)^2}dx = \frac{2}{9}\pi\left(e^{\frac{-1}{3}}\right)$$

Since $\frac{\cos x}{\left(1+9x^2\right)^2}$ is an even function, then

$$
\begin{aligned}
\int_{-\infty}^{+\infty}\frac{\cos x}{\left(1+9x^2\right)^2}dx &= 2\int_{0}^{+\infty}\frac{\cos x}{\left(1+9x^2\right)^2}dx\\
&= \frac{\pi}{9}\left(e^{\frac{-1}{3}}\right)
\end{aligned}
$$

3.9.15 chapter 14, problem 7.29

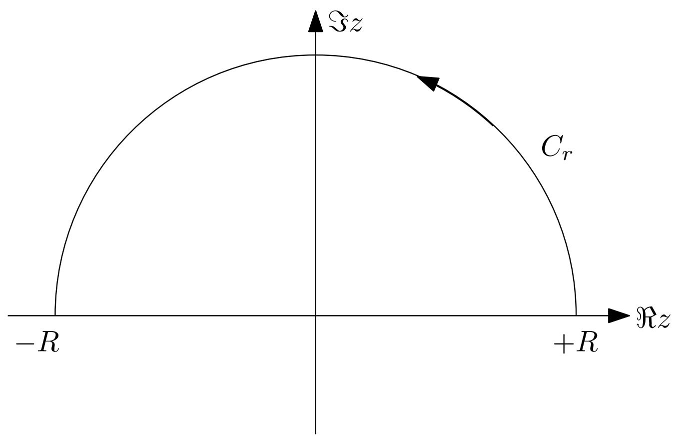
Evaluate the following integral. Find principal value if necessary

$$I=\int_0^\infty \frac{\sin\ ax}{x}dx$$

consider

$$\oint \frac{e^{iaz}}{z}dz$$

over the contour shown.



The contour avoids the singularity at $z=0$, hence by Cauchy theorem,

$$\oint \frac{e^{iaz}}{z}dz=0$$

since $f(z)$ is analytic on and everywhere inside the contour. Then

$$\oint \frac{e^{iaz}}{z}dz=0=\int_{-R}^{-r}\frac{e^{iax}}{x}dx+\int_{C'}\frac{e^{iaz}}{z}dz+\int_{+r}^{+R}\frac{e^{iax}}{x}dx+\int_{C}\frac{e^{iaz}}{z}dz$$

Looking at $\int_{C'}\frac{e^{iaz}}{z}dz$ let $z=re^{i\theta}, dz=ire^{i\theta}d\theta, \frac{dz}{z}=id\theta$, then

$$\int_{C'}\frac{e^{iaz}}{z}dz=\int_{\pi}^{0}e^{i\ a\ re^{i\theta}}\ id\theta$$

As $r\rightarrow 0$, $e^{i\ a\ re^{i\theta}}\rightarrow 1$, hence

$$\int_{C'}\frac{e^{iaz}}{z}dz\rightarrow\int_{\pi}^{0}id\theta=[i\theta]_{\pi}^{0}=-i\pi$$

Hence, as $r\rightarrow 0$ and $R\rightarrow\infty$

$$0=\int_{-\infty}^{-0}\frac{e^{iax}}{x}dx-i\pi+\int_{+0}^{+\infty}\frac{e^{iax}}{x}dx+\int_{C}\frac{e^{iaz}}{z}dz$$

But $\int_{C}\frac{e^{iaz}}{z}dz\rightarrow 0$ as $R\rightarrow\infty$ as from before and as shown in book page 603. Hence

$$\begin{aligned}0&=\int_{-\infty}^{-0}\frac{e^{iax}}{x}dx-i\pi+\int_{+0}^{+\infty}\frac{e^{iax}}{x}dx\\ i\pi&=\int_{-\infty}^{-0}\frac{e^{iax}}{x}dx+\int_{+0}^{+\infty}\frac{e^{iax}}{x}dx\\ i\pi&=\int_{-\infty}^{+\infty}\frac{e^{iax}}{x}dx\end{aligned}$$

Equating imaginary and real parts part of the above equation

$$i\pi = \int_{-\infty}^{+\infty} \frac{\cos ax + i \sin ax}{x} dx$$

Hence

$$\pi = \int_{-\infty}^{+\infty} \frac{\sin ax}{x} dx$$
$$0 = \int_{-\infty}^{+\infty} \frac{\cos ax}{x} dx$$

Now, $\frac{\sin ax}{x}$ is an even function, since $f(-x) = \frac{\sin(-ax)}{-x} = \frac{-\sin ax}{-x} = \frac{\sin ax}{x} = f(x)$. Therefore

$$\pi = \int_{-\infty}^{+\infty} \frac{\sin ax}{x} dx = 2\int_{0}^{+\infty} \frac{\sin ax}{x} dx$$

And

$$\int_{0}^{+\infty} \frac{\sin ax}{x} dx = \frac{\pi}{2}$$

3.9.16 Chapter 14, problem 7.30 part (a)

Evaluate the following integral by the method of example 2

$$I = \int_0^{\infty} \frac{1}{1+x^4} dx$$

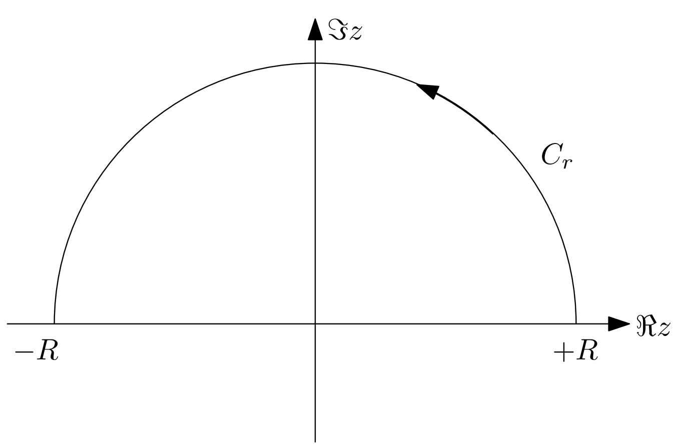
First, let me find what function we have.

$$\begin{aligned} f(-x) &= \frac{1}{1+(-x)^4} \\ &= \frac{1}{1+(x)^4} \\ &= f(x) \end{aligned}$$

Hence an even function. Consider

$$\oint \frac{1}{1+z^4} dz$$

over the contour shown.



Hence

$$\begin{aligned} \oint \frac{1}{1+z^4} dz &= \int_{-R}^{+R} \frac{1}{1+x^4} dx + \int_C \frac{1}{1+z^4} dz \\ &= 2\pi i \sum \text{residues } f(z) \text{ inside} \end{aligned}$$

As $R \to \infty$, $\int\limits_C \frac{1}{1+z^4} dz \to 0$ as shown before and as shown in book page 603. Hence

$$\oint \frac{1}{1+z^4} dz = \int_{-\infty}^{+\infty} \frac{1}{1+x^4} dx = 2\pi i \sum \text{residues } f(z) \text{ inside}$$
$$\int_{-\infty}^{+\infty} \frac{1}{1+x^4} dx = 2\pi i \sum \text{residues } f(z) \text{ inside}$$

Now to find residues of

$$f(z) = \frac{1}{1+z^4}$$

At poles inside C. Finding roots of polynomial $1 + z^4 = 0$

$$z^4 = -1$$
$$z = -1^{\frac{1}{4}} = \left(e^{i\pi}\right)^{\frac{1}{4}} = e^{i\frac{(\pi+2\pi n)}{4}} \qquad n = 0, 1, 2, 3$$

Hence the roots are

$$\left(e^{i(\pi)}\right)^{\frac{1}{4}}, \left(e^{-i(\pi+2\pi)}\right)^{\frac{1}{4}}, \left(e^{-i(\pi+4\pi)}\right)^{\frac{1}{4}}, \left(e^{-i(\pi+6\pi)}\right)^{\frac{1}{4}}$$
$$e^{i\left(\frac{\pi}{4}\right)}, e^{i\left(\frac{3}{4}\pi\right)}, e^{i\left(\frac{5}{4}\pi\right)}, e^{i\left(\frac{7}{4}\pi\right)}$$

The poles are

$$z_1 = e^{i\left(\frac{\pi}{4}\right)}$$
$$z_2 = e^{i\left(\frac{3}{4}\pi\right)}$$
$$z_3 = e^{-i\left(\frac{\pi}{4}\right)}$$
$$z_4 = e^{-i\left(\frac{3}{4}\pi\right)}$$

Out of these zeros, we want the ones with positive imaginary parts since those are the ones inside the contour. From above, those are z_1 and z_2. Therefore to find residue at z_1

$$\lim_{z\to z_1} (z - z_1) f(z) = \lim_{z\to z_1} \left(z - e^{i\frac{\pi}{4}}\right) \frac{1}{1+z^4}$$
$$= \lim_{z\to z_1} \frac{z - e^{i\frac{\pi}{4}}}{1+z^4}$$

Applying L'Hopitals

$$\text{Residue}(z_1) = \lim_{z\to z_1} \frac{1}{4z^3}$$
$$= \frac{1}{4\left(e^{i\frac{\pi}{4}}\right)^3}$$
$$= \frac{1}{4} e^{-i\frac{3\pi}{4}}$$

Now we find residue at the other pole, at z_2

$$\lim_{z\to z_2} (z - z_2) f(z) = \lim_{z\to z_2} \left(z - e^{i\frac{3}{4}\pi}\right) \frac{1}{1+z^4}$$
$$= \lim_{z\to z_1} \frac{z - e^{i\frac{3}{4}\pi}}{1+z^4}$$

Applying L'Hopitals

$$\begin{aligned}
\text{Residue}(z_1) &= \lim_{z\to z_1} \frac{1}{4z^3} \\
&= \frac{1}{4\left(e^{i\frac{3}{4}\pi}\right)^3} \\
&= \frac{1}{4} e^{-i\frac{9\pi}{4}} \\
&= \frac{1}{4} e^{-i\frac{\pi}{4}}
\end{aligned}$$

Hence sum of residues is

$$\begin{aligned}
\sum \text{residues} &= \frac{1}{4} e^{-i\frac{3\pi}{4}} + \frac{1}{4} e^{-i\frac{\pi}{4}} \\
&= \frac{-\sqrt{2}}{4} i
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{1}{1+x^4} dx &= 2\pi i \sum \text{residues } f(z) \text{ inside} \\
&= 2\pi i \left(\frac{-\sqrt{2}}{4} i \right) \\
&= \pi \frac{\sqrt{2}}{2}
\end{aligned}$$

But $\frac{1}{1+x^4}$ is an even function, hence

$$\int_{0}^{+\infty} \frac{1}{1+x^4} dx = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{1}{1+x^4} dx = \frac{1}{2} \pi \frac{\sqrt{2}}{2} = \pi \frac{\sqrt{2}}{4}$$

Therefore

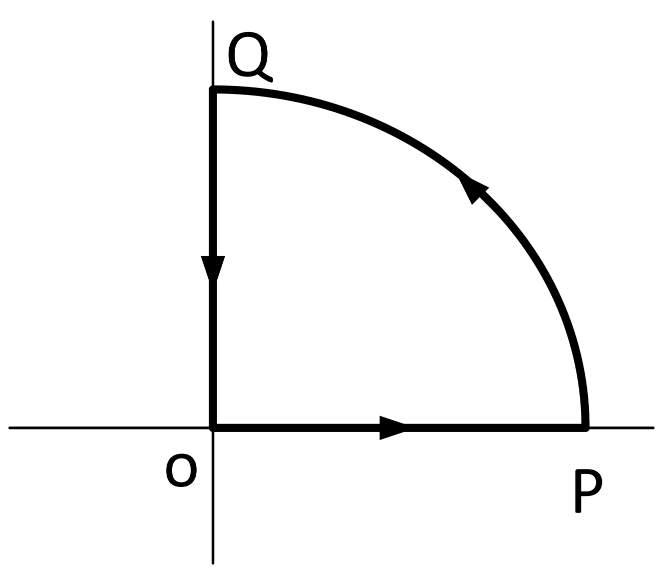
$$\int_{0}^{+\infty} \frac{1}{1+x^4} dx = \pi \frac{\sqrt{2}}{4}$$

3.9.17 chapter 14, problem 7.45

Problem determine in which quadrants the roots of the following equation lie

$$f(z) = z^3 + z^2 + z + 4 = 0$$

Look at $f(z)$ in the first quadrant, put a contour as shown in figure,



And using theorem 7.8, which says

$$N - P = \frac{1}{2\pi}\Big(\text{change of angle of } f(z) \text{ around contour}\Big)$$

Where N is number of zeros of $f(z)$ INSIDE contour, and P is number of poles of $f(z)$ INSIDE contour. The main idea is that we want to see how much does the argument changes as the complex number is mapped by $f(z)$ from $z \to z^3 + z^2 + z + 4$, then using 7.8 theorem this will tell us the number of zeros in the first quadrant. From this was can find where the other 2 roots are.

First, along path OP, $f(z)$ is real and equals $x^3 + x^2 + x + 4$ which is always > 0. On path QO, i.e. on imaginary y axis,

$$\begin{aligned} f(z) &= \left(iy\right)^3 + \left(iy\right)^2 + iy + 4 = -iy^3 - y^2 + iy + 4 \\ &= i\left(-y^3 + y\right) + \left(4 - y^2\right) \end{aligned}$$

If this to be zero, then $\left(4 - y^2\right) = 0$, or $y^2 = 4$, or $y = \pm 2$.Then looking at the imaginary part of $f(z)$ which is $\left(-y^3 + y\right)$ and substitute these y values, we get , when $y = +2$, $\left(-2^3 + 2\right) = -6 \neq 0$, and when $y = -2$, we get $\left(-(-2)^3 - 2\right) = 6 \neq 0$, hence $f(z)$ is not zero on OP and not zero on QO.

Now on the arc PQ, we can make the radius as large as we want to contains all zeros of $f(z)$ inside, so to make no zeros on PQ or outside. So, now we need to find the angle change of $f(z)$. on OP, z is real x, so any point on x is mapped to a point on x , hence no argument is changed, i.e. angle change is zero by the function. On arc PQ write $z = re^{i\theta}$, so mapping of $f(z)$ results in

$$\begin{aligned} f(z) &= \left(re^{i\theta}\right)^3 + \left(re^{i\theta}\right)^2 + re^{i\theta} + 4 \\ &= \left(re^{i\theta}\right)^3 + \left(re^{i\theta}\right)^2 + re^{i\theta} + 4 \\ &= r^3 e^{3i\theta} + r^2 e^{2i\theta} + re^{i\theta} + 4 \end{aligned}$$

For very large radius r, r^3term dominates, and so $f(z) \approx r^3 e^{3i\theta}$, so as z moves from 0 to $\frac{\pi}{2}$, the change in angle cause by $f(z)$ will be 3θ or $3 \times \frac{\pi}{2} = \frac{3\pi}{2}$. On QO, i.e. the imaginary axis, $z = iy$ then

$$f(z) = \left(iy\right)^3 + \left(iy\right)^2 + iy + 4 = i\left(-y^3 + y\right) + \left(4 - y^2\right)$$

Hence in the w domain, i.e. looking at the output of $w = f(z)$, the angle that the complex number w makes is $\tan\Theta = \frac{\left(y - y^3\right)}{\left(4 - y^2\right)}$.

For very large y, $\tan\Theta \approxeq -y = -\infty$ and from above we see that the angle start at $\frac{3\pi}{2}$ when y is very large. now we decrease y as we move down to the origin and see how the angle changes. Looking at $\tan\Theta = \frac{\left(y - y^3\right)}{\left(4 - y^2\right)}$, as y get smaller $y - y^3$ becomes smaller but remains negative, this is until $y - y^3 = 0$, or $y = 0$ at which time the angle is 2π (note the final angle is 2π, for $\arctan 0$ and not angle of zero, since the tangent was negative when we started and continued to be negative but smaller and smaller, hence the final angle is 2π).

Since we got to $y = 0$, this completes the contour. So angle change is 2π. From

$$N - P = \frac{1}{2\pi}\Big(\text{change of angle of } f(z) \text{ around contour}\Big)$$

And since poles of $f(z)$ do not exist (it has no denominator), then

$$N = \frac{1}{2\pi}(2\pi) = 1$$

Hence $f(z)$ has ONE zero in the first quadrant. Now using the same argument as on page 611 of text book, we know that a polynomial of real coefficients, when it has a complex root, then they come in conjugate pairs, hence the second complex root will be in the 4th quadrant (since when take a conjugate of a complex number in the first quadrant, we get a complex number in the fourth quadrant).

Now since a 3rd order polynomial must have number of zeros as its order, then the 3rd zero must be real (it can't be complex since complex roots come in pairs). In addition the third root(the real root) must be on the negative x-axis to make x^3+x^2+x+4 a zero quantity.

3.9.18 chapter 14, problem 8.15

Evaluate the following integral by computing residue at ∞ check answer by computing residues at all finite poles.

$$\oint \frac{z^2\,dz}{(2z+1)\left(z^2+9\right)}$$

around $|z| = 5$, where in the above, the circle is going in the positive direction, i.e. anticlockwise. We know that

$$\oint_{\substack{\text{clockwise}\\ \text{around zero}}} f(z) = -\oint_{\substack{\text{anticlockwise}\\ \text{around } \infty}} f(z)$$

but $\oint_{\text{clockwise}} f(z) = 2\pi i \sum$residues of $f(z)$ inside circle, therefore $\oint_{\text{clockwise}} f(z) = -2\pi i\left(\text{residues } \; f(z) \;\; at \;\; \infty\right)$

Hence we need to find residue of $f(z)$ at ∞. We know that residue of $f(z)$ at ∞ is residue of $-\frac{1}{z^2}f\left(\frac{1}{z}\right)$ at zero. Therefore

$$\begin{aligned}
-\frac{1}{z^2}f\left(\frac{1}{z}\right) &= -\frac{1}{z^2}\frac{(1/z)^2}{(2(1/z)+1)\left((1/z)^2+9\right)}\\
&= -\frac{1}{z^4}\frac{1}{\left(\frac{2}{z}+1\right)\left(\frac{1}{z^2}+9\right)}\\
&= -\frac{1}{z^4}\frac{1}{\left(\frac{2+z}{z}\right)\left(\frac{1+9z^2}{z^2}\right)}\\
&= -\frac{1}{z^4}\frac{z^3}{(2+z)\left(1+9z^2\right)}\\
&= \frac{-1}{z(2+z)\left(1+9z^2\right)}
\end{aligned}$$

So we have a simple pole at $z=0$, hence residue of the above function at zero is

$$\lim_{z\to 0} z\,\frac{-1}{z(2+z)\left(1+9z^2\right)} = \frac{-1}{2}$$

Hence

$$\oint \frac{z^2\,dz}{(2z+1)\left(z^2+9\right)} = -2\pi i\left(\frac{-1}{2}\right) = \pi i$$

To find the same integral by standard method, we write

$$\oint \frac{z^2\,dz}{(2z+1)\left(z^2+9\right)} = 2\pi i \sum \text{residues } \; f(z)$$

Poles are at $\frac{-1}{2}$ and at $z = \pm 3i$. Notice that all poles are inside $|z| = 5$ so must add residue of $f(z)$ for each pole. Residue at $\frac{-1}{2}$ is

$$\begin{aligned}
&\lim_{z\to-\frac{1}{2}}\left(z+\frac{1}{2}\right)\frac{z^2}{(2z+1)\left(z^2+9\right)}\\
&=\lim_{z\to-\frac{1}{2}}\frac{z^2}{2\left(z^2+9\right)}\\
&=\frac{\left(-\frac{1}{2}\right)^2}{2\left(\left(-\frac{1}{2}\right)^2+9\right)}\\
&=\frac{\frac{1}{4}}{2\left(\frac{1}{4}+9\right)}\\
&=\frac{\frac{1}{4}}{2\left(\frac{1}{4}+9\right)}\\
&=\frac{1}{2(37)}\\
&=\frac{1}{74}
\end{aligned}$$

Residue at $3i$ is

$$\begin{aligned}
\lim_{z\to3i}(z-3i)\frac{z^2}{(2z+1)\left(z^2+9\right)} &= \lim_{z\to3i}(z-3i)\frac{z^2}{(2z+1)(z-3i)(z+3i)}\\
&=\lim_{z\to3i}\frac{z^2}{(2z+1)\,(z+3i)}\\
&=\frac{(3i)^2}{(2(3i)+1)\,(3i+3i)}\\
&=\frac{-9}{(6i+1)(6i)}\\
&=\frac{-9}{-36+6i}\\
&=\frac{-3}{-12+2i}\\
&=\frac{-3}{-12+2i}\left(\frac{-12-2i}{-12-2i}\right)\\
&=\frac{36+6i}{144+4}\\
&=\frac{36+6i}{148}\\
&=\frac{18+3i}{74}
\end{aligned}$$

Hence the residue at the last pole will be $\frac{18-3i}{74}$ symmetric. Hence sum of residues is

$$\begin{aligned}
\frac{18-3i}{74}+\frac{18+3i}{74}+\frac{1}{74} &= \frac{18+18+1}{74}\\
&=\frac{37}{74}
\end{aligned}$$

Hence

$$\begin{aligned}
\oint\frac{z^2\,dz}{(2z+1)\left(z^2+9\right)} &= 2\pi i\frac{37}{74}\\
&=\pi i
\end{aligned}$$

which agrees with my answer earlier using the residue around ∞ approach. I think the approach using residue around ∞ required less effort to do.

3.9.19 chapter 14, problem 8.4

Problem Determine if ∞ is a regular point, essential singularity or a pole (and of what order) and find the residue

$$f(z) = \frac{2z+3}{(z+2)^2}$$

Solution We start by doing the transformation $w = \frac{1}{z}$ and examine the function $f(w)$ at zero.

$$\begin{aligned} f(w) &= f\left(\frac{1}{z}\right) \\ &= \frac{2\frac{1}{z}+3}{\left(\frac{1}{z}+2\right)^2} \\ &= \frac{\frac{2+3z}{z}}{\left(\frac{1+2z}{z}\right)^2} \\ &= \frac{\frac{2+3z}{z}}{\frac{(1+2z)^2}{z^2}} \\ &= \frac{z(2+3z)}{(1+2z)^2} \end{aligned}$$

At $z = 0$, $f\left(\frac{1}{z}\right) = 0$, hence $f(z)$ is a *regular function* at ∞. To find the residue at ∞, we want to find the residue of

$$\left(-\frac{1}{(z-z_0)^2}\right) f\left(\frac{1}{z}\right)$$

at $z_0 = 0$. The term $-\frac{1}{(z-z_0)^2}$ comes from doing $Z = 1/z$, $dZ = -\frac{1}{z^2}dz$. Hence, we want to the residue of

$$\begin{aligned} \left(-\frac{1}{(z-z_0)^2}\right) f\left(\frac{1}{z}\right) &= \left(-\frac{1}{(z-z_0)^2}\right)\frac{z(2+3z)}{(1+2z)^2} \\ &= \left(-\frac{1}{z^2}\right)\frac{z(2+3z)}{(1+2z)^2} \\ &= \frac{-(2+3z)}{z(1+2z)^2} \end{aligned}$$

To find the residue of $\frac{-(2+3z)}{z(1+2z)^2}$ at 0, we see the function has a pole of order 2 at $-1/2$ and a simple pole of at $z = 0$, therefore

$$\begin{aligned} \text{residue} &= \lim_{z\to 0}(z)\frac{-(2+3z)}{z(1+2z)^2} \\ &= \frac{-(2+3(0))}{(1+2(0))^2} \\ &= \frac{-2}{(1)^2} \\ &= -2 \end{aligned}$$

Or

$$\text{residue of} \frac{2z+3}{(z+2)^2} \text{at } \infty \text{ is } -2$$

3.9.20 chapter 14, problem 8.5

Problem Determine if ∞ is a regular point, essential singularity or a pole (and of what order) and find the residue

$$f(z) = \sin\frac{1}{z}$$

We start by doing the transformation $w = \frac{1}{z}$ and examine the function $f(w)$ at zero.

$$f(w) = f\left(\frac{1}{z}\right) = \sin z$$

at $z = 0$, $f\left(\frac{1}{z}\right) = 0$, hence $f(z) = \sin\frac{1}{z}$ is a *regular function* at ∞. To find the residue at ∞, we want to find the residue of

$$\left(-\frac{1}{(z-z_0)^2}\right) f\left(\frac{1}{z}\right)$$

at $z_0 = 0$. The $-\frac{1}{(z-z_0)^2}$ term above comes from doing $Z = 1/z$, $dZ = -\frac{1}{z^2}dz$. Hence, we want to the residue of

$$\begin{aligned}
\left(-\frac{1}{(z-z_0)^2}\right) f\left(\frac{1}{z}\right) &= \left(-\frac{1}{(z-z_0)^2}\right)\sin z \\
&= -\frac{1}{z^2}\sin z
\end{aligned}$$

To find the residue of $-\frac{1}{z^2}\sin z$ at 0, we see the function has a pole of order $m = 2$ at $z = 0$, hence

$$\begin{aligned}
\text{residue} &= \frac{1}{m-1}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z) \\
&= \lim_{z\to z_0}\frac{d}{dz}(z-z_0)^2\left(-\frac{1}{z^2}\right)\sin z \\
&= \lim_{z\to z_0}\frac{d}{dz} - \sin z = \lim_{z\to z_0} - \cos z \\
&= -\cos z_0 \\
&= -\cos 0 \\
&= -1
\end{aligned}$$

Hence

$$\text{residue of } \sin\frac{1}{z} \text{at } \infty \text{ is } -1$$

3.10 HW 9

Local contents

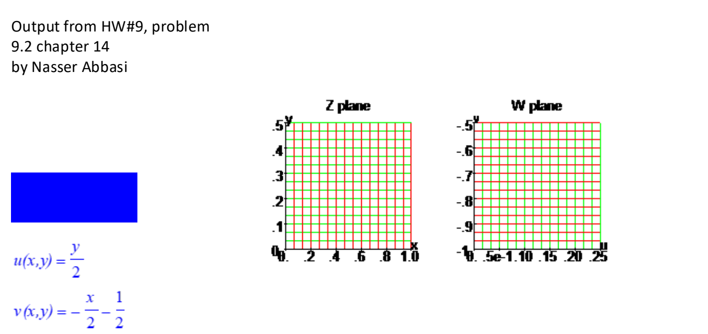
3.10.1 chapter 14, problem 9.2

For following function $w = f(z) = u + iv$ find u and v as functions of x and y. Sketch the graphs in the (x, y) plane of the images of $u = const$ and $v = const$ for several values of u and several values of v where $w = \frac{z+1}{2i}$

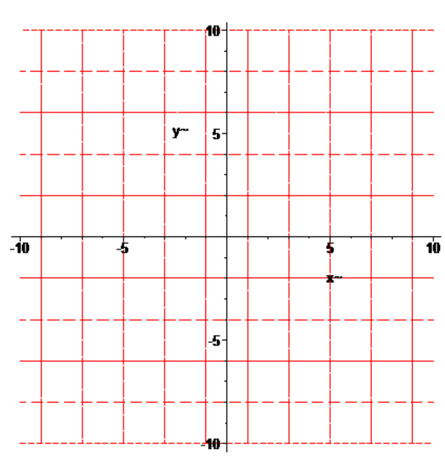
Answer let $z = x + iy$, hence $w = \frac{z+1}{2i} = \frac{x+iy+1}{2i} = -i\frac{x+iy+1}{2} = \frac{-ix+y-i}{2} = \frac{y}{2} + i\left(\frac{-1-x}{2}\right)$. So, since $w = u + iv$ then $u = \frac{y}{2}$ and $v = \left(\frac{-1-x}{2}\right)$ Then $u = C$, where C is a constant, gives the equation $\frac{y}{2} = C$. Which is the equation of a straight line $y = C$.

v =constant, gives the equation $\left(\frac{-1-x}{2}\right) = C$, gives the equation of the straight line $x = C - 1$.

These two equations are plotted for few points. The following shows the plots generated for the mapping from the z-plane to the w-plane, and then the image of u=const and the image of v=const back into the xy plane.



Mapping of f(z) from the Z plane to the W plane.



Showing the images of U and V on the Z plane (the X,Y plane). The DASHED lines are the image of the equation U=const, and the solid lines are the image of the V=const equation.

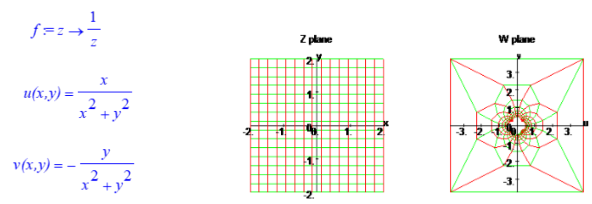
3.10.2 chapter 14, problem 9.3

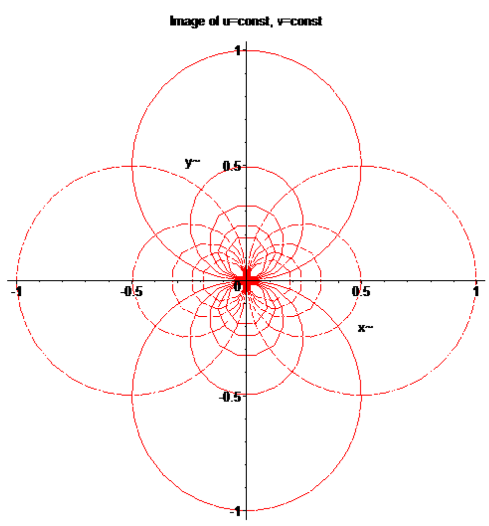
For following function $w = f(z) = u + iv$ find u and v as functions of x and y. Sketch the graphs in the (x, y) plane of the images of $u = C$ and $v = C$, where C is constant, for several values of u and several values of v. $w = \frac{1}{z}$

Answer let $z = x + iy$, hence $w = \frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy}\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + \frac{-iy}{x^2+y^2}$. Hence, since $w = u + iv$ then $u = \frac{x}{x^2+y^2}$ and $v = \frac{-y}{x^2+y^2}$. Then $u = C$ gives the equation $\frac{x}{x^2+y^2} = C$. $v = C$ gives the equation $\frac{-y}{x^2+y^2} = C$.

These 2 equations were plotted for few points. The following shows the plots generated for the mapping from the z-plane to the w-plane, and then the image of $u = C$ and the image of $v = C$ back into the xy plane.

Analysis for problem 9.3 chapter 14
Mary Boas text book.
by Nasser Abbasi





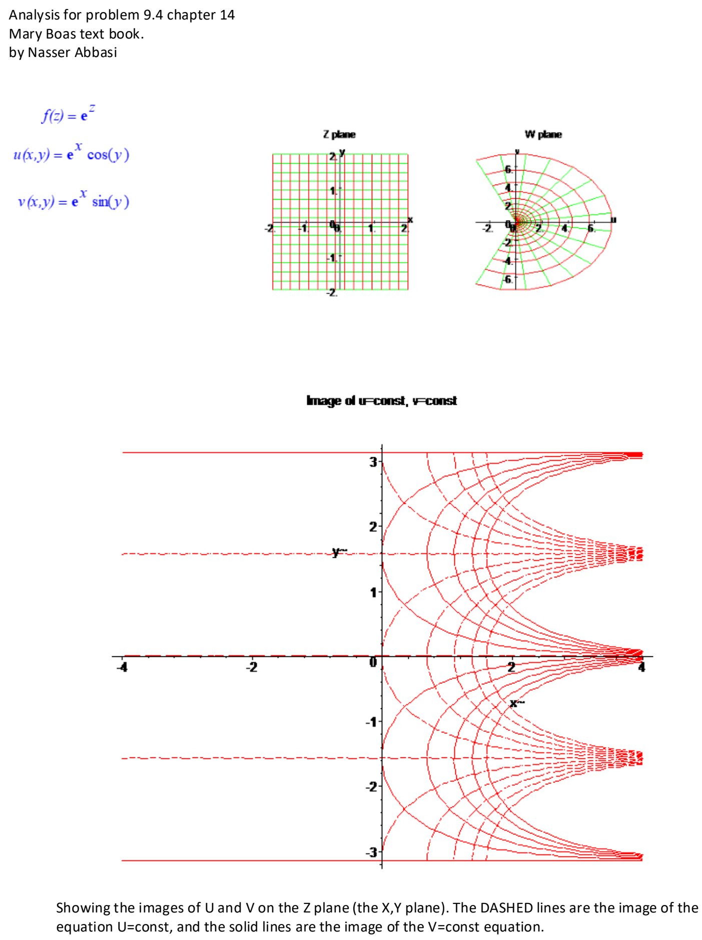
Showing the images of U and V on the Z plane (the X,Y plane). The DASHED lines are the image of the equation U=const, and the solid lines are the image of the V=const equation.

3.10.3 chapter 14, problem 9.4

For the function $w = f(z) = u + iv$ shown below, find u and v as functions of x and y. Sketch the graphs in the (x, y) plane of the images of $u = const$ and $v = const$ for several values of u and several values of v. $w = e^z$

Answer let $z = x + iy$, hence $w = e^z = e^{x+iy} = e^x e^{iy} = e^x\left(\cos y + i\sin y\right) = e^x\cos(y) + i\,e^x\sin(y)$. Therefore, since $w = u + iv$ then $u = e^x\cos(y)$ and $v = e^x\sin(y)$. Then $u = C$ gives the equation $e^x\cos(y) = C$ and $v = C$ gives the equation $e^x\sin(y) = C$

These 2 equations are plotted for few points. The following shows the plots generated for the mapping from the z-plane to the w-plane, and then the image of u=const and the image of v=const back into the xy plane.



3.10.4 chapter 14, problem 9.7

For the function $w = f(z) = u + iv$ shown below, find u and v as functions of x and y. Sketch the graphs in the (x, y) plane of the images of $u = C$ and $v = C$ for several values of u and several values of v. use $w = \sin(z)$

Answer let $z = x + iy$, hence

$$\begin{aligned} w &= \sin(z) = \sin\left(x + iy\right) \\ &= \sin(x)\cos\left(iy\right) + \cos(x)\sin\left(iy\right) \end{aligned}$$

But $\cos\left(iy\right) = \cosh(y)$ and $\sin\left(iy\right) = i\sinh(y)$, therefore

$$w = \sin(x)\cosh(y) + \cos(x)i\sinh(y)$$

Since $w = u + iv$ then $u = \sin(x)\cosh(y)$ and $v = \cos(x)\sinh(y)$. Then $u = C$ gives the equation

$$\sin(x)\cosh(y) = C$$

And $v = C$ gives the equation

$$\cos(x)\sinh(y) = C$$

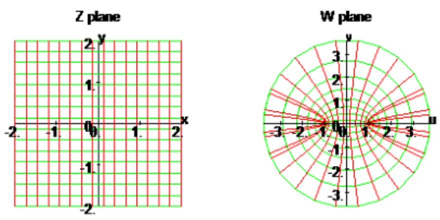
These 2 equations are plotted for few points. The following shows the plots generated for the mapping from the z-plane to the w-plane, and then the image of u=const and the image of v=const back into the xy plane.

Analysis for problem 9.7 chapter 14
Mary Boas text book.
by Nasser Abbasi

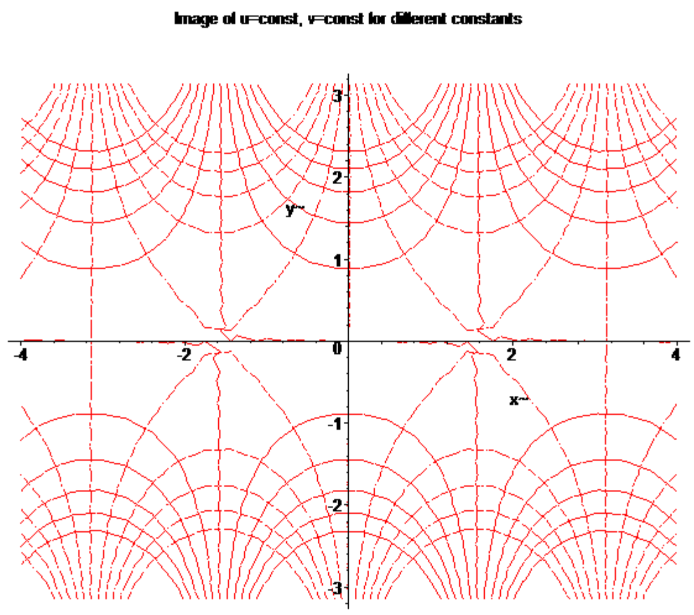
$f(z) = \sin(z)$

$u(x,y) = \sin(x)\cosh(y)$

$v(x,y) = \cos(x)\sinh(y)$



Mapping of f(z) from the Z plane to the W plane.



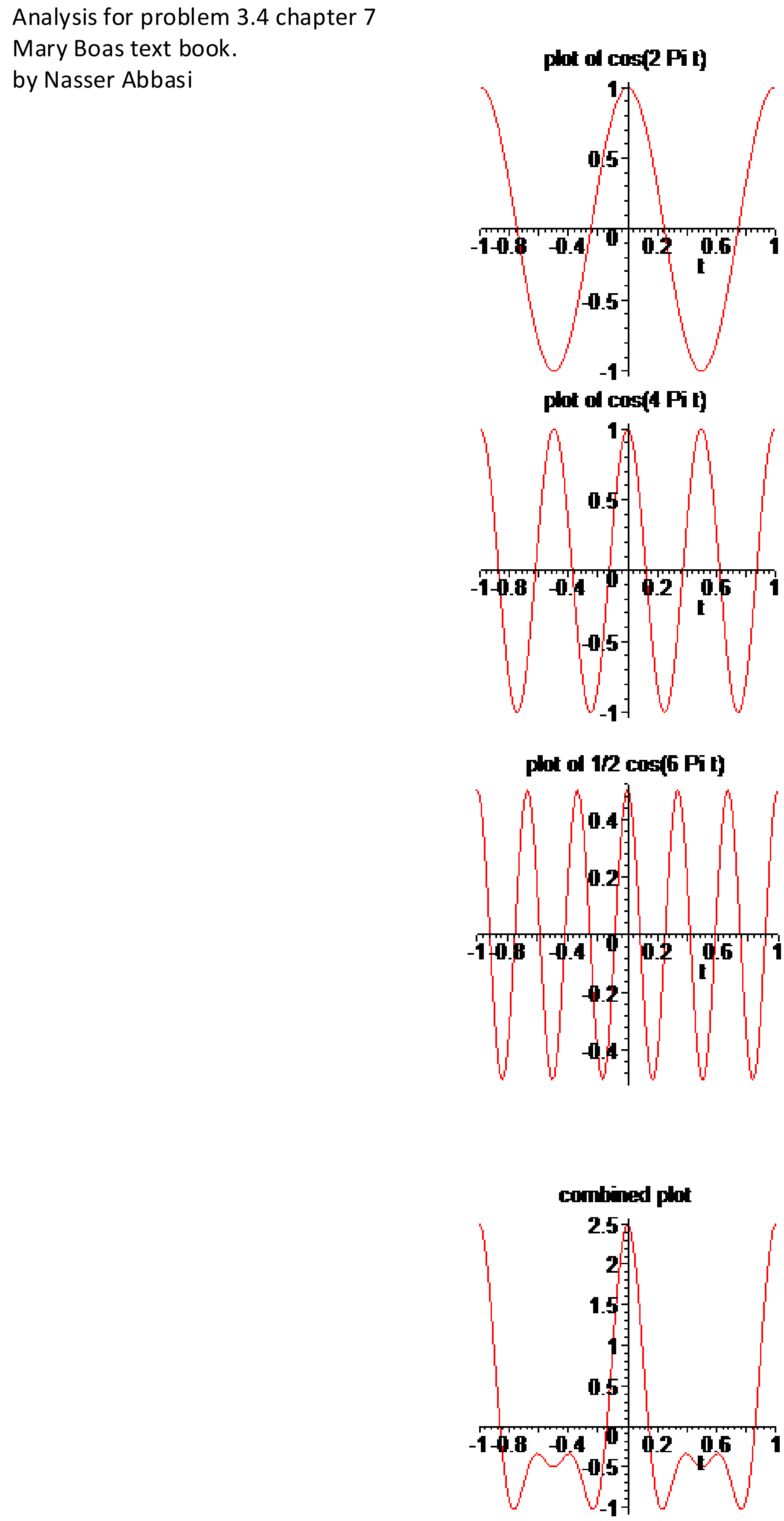
Showing the images of U and V on the Z plane (the X,Y plane). The DASHED lines are the image of the equation U=const, and the solid lines are the image of the V=const equation.

3.10.5 chapter 7, problem 3.4

Draw a graph over a whole period for $f(t) = \cos(2\pi t) + \cos(4\pi t) + \frac{1}{2}\cos(6\pi t)$

Answer First, find the period of the above function. A function is periodic with period p if $f\left(t+p\right) = f(t)$ for all t. We know that $\cos(nt)$ has the same period as $\cos(nt + 2n\pi)$ for n integer, since the function $\cos(x)$ has a period of 2π. So a period of $f(t)$ is 2π since it is the sum of $\cos(x)$ functions. To plot this function, we plot each of its components over the same period of 2π and then sum them together.

This plot below shows the result



3.10.6 chapter 7, problem 3.6

Draw a graph of $f(x) = \sin(2x) + \sin 2\left(x + \frac{\pi}{3}\right)$. what are the period and amplitude? Write as a single harmonic.

<u>Answer</u>

Since $\sin\left(x + y\right) = \sin(x)\cos(y) + \cos(x)\sin(y)$ then

$$\sin 2\left(x + \frac{\pi}{3}\right) = \sin\left(2x + \frac{2}{3}\pi\right) = \sin(2x)\cos\left(\frac{2}{3}\pi\right) + \cos(2x)\sin\left(\frac{2}{3}\pi\right)$$

Hence

$$f(x) = \sin(2x) + \sin 2\left(x + \frac{\pi}{3}\right) = \sin(2x) + \sin(2x)\cos\left(\frac{2}{3}\pi\right) + \cos(2x)\sin\left(\frac{2}{3}\pi\right)$$

Now $\cos\left(\frac{2}{3}\pi\right) = -\frac{1}{2}$ and $\sin\left(\frac{2}{3}\pi\right) = \frac{\sqrt{3}}{2}$ so above can be written as

$$f(x) = \sin(2x) - \frac{1}{2}\sin(2x) + \frac{\sqrt{3}}{2}\cos(2x)$$
$$f(x) = \frac{1}{2}\sin(2x) + \frac{\sqrt{3}}{2}\cos(2x)$$

But $\sin(2x) = \cos(\frac{\pi}{2} - 2x)$

$$f(x) = \frac{1}{2}\cos(\frac{\pi}{2} - 2x) + \frac{\sqrt{3}}{2}\cos(2x)$$

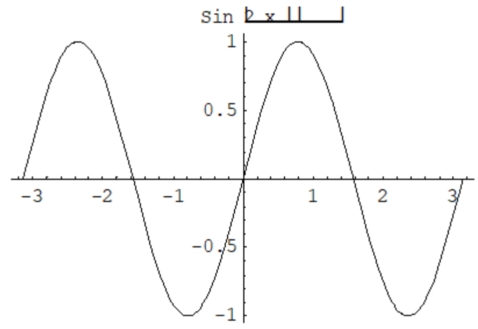
Now this is in term of a single harmonic function. Hence, we see that $f(x)$ is the sum of harmonics of the same periods (the cos function have period of 2π).hence the period of $f(x)$ is 2π. To find Max amplitude, this is a problem of finding a maximum of a function.

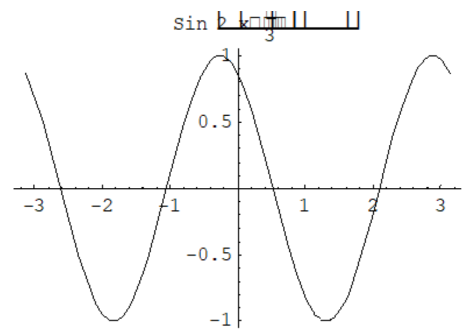
$$\begin{aligned}\frac{d}{dx}f(x) &= -\frac{1}{2}\sin(\frac{\pi}{2} - 2x)(-2) - \frac{\sqrt{3}}{2}\sin(2x)(2)\\ &= \sin(\frac{\pi}{2} - 2x) - \sqrt{3}\sin(2x)\\ &= \cos(2x) - \sqrt{3}\sin(2x)\end{aligned}$$

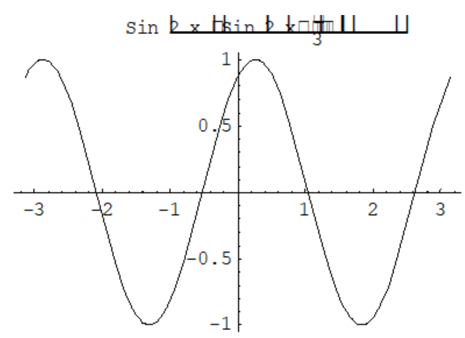
Hence for a maximum, $\cos(2x) - \sqrt{3}\sin(2x) = 0$. A root for this equation is found at $x = 0.261799$ so I use this value in $f(x)$ to find the amplitude.

$f(0.261799) = 1$. This is the maximum value, or the amplitude.The following is a plot of this function

Analysis for problem 3.6 chapter 7
Mary Boas text book.
by Nasser Abbasi



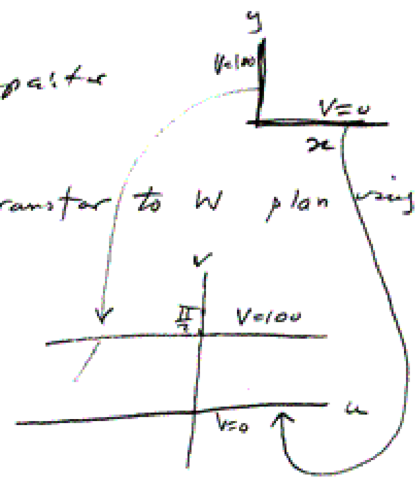




3.10.7 First part of HW9 was scanned

problem 5, section 10, chapter 14
Nasser Abbasi

this is the capacitor



to solve, transfer to W plan using $w = \ln(z)$ transformation.
this gives

now we solve the problem in the w plane. need to solve for V in shaded area.

the solution is linear. so at $v=0$, $V=0$
at $v=\frac{\pi}{2}$, $V=100$.

so need $V = \frac{100}{\pi/2} v$

now convert back to x-y plane.

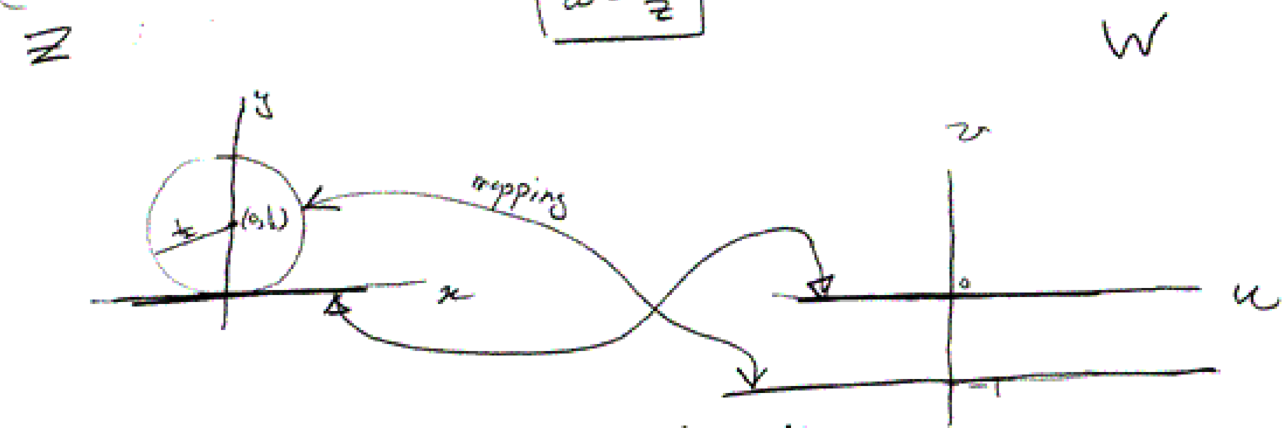
since $v = \arg(z)$, then $v = \arctan \frac{y}{x}$

so solution is $V = \frac{100}{\pi/2} \arctan \frac{y}{x}$

Section 10, problem 6, chapter 14.

Nasser Abbasi

$$\boxed{w = \frac{1}{z}}$$



the mapping $w=\frac{1}{z}$ is found as this:

let $z = x+iy$.

so $\frac{1}{z} = \frac{1}{x+iy}\,\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$.

so $w = \frac{1}{z} = u+iv$.

hence $u = \frac{x}{x^2+y^2}$, $v = \frac{-y}{x^2+y^2}$.

for $v=-1$, we get $-1 = \frac{-y}{x^2+y^2}$ or $x^2+y^2-y=0$. (1)

a circle equation is $(x-a)^2+(y-b)^2 = r^2$ (2)

to convert (1) to (2)

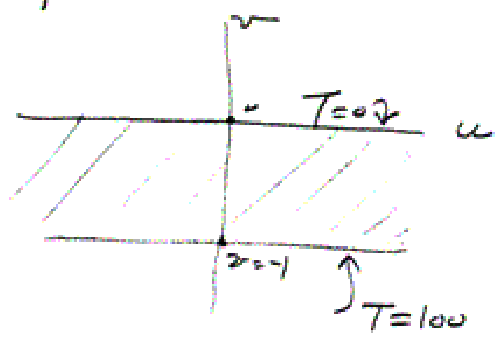
$$(x-0)^2+\left(y-\tfrac{1}{2}\right)^2-\tfrac{1}{4}=0 \Rightarrow \sqrt{(x-0)^2+\left(y-\tfrac{1}{2}\right)^2} = \tfrac{1}{4}.$$

so the line **[$v=-1$ maps to circle with center $(0,\frac{1}{2})$ and radius $=\frac{1}{2}$ in Z plane]**

and line $v=0$ (u axis) maps to $y=0$ or x-axis

hence area outside cylinder $\Longrightarrow$ area between lines $v=-1$ and $v=0$ in W plane $\longrightarrow$

here in W plane we have this



So $T = 100$ at $v = -1$

$T = 0$ at $v = 0$. ✓

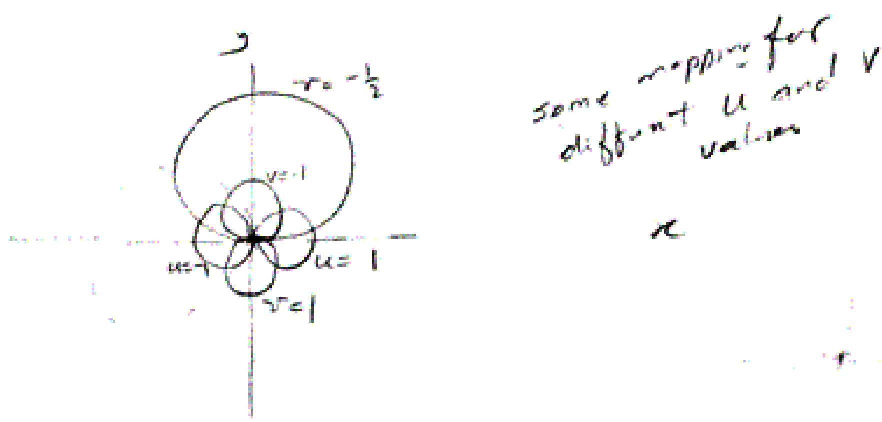
so since T is linear $\Rightarrow$ $\boxed{T = -100\,v}$

now to get the solution in the x-y, replace v by

$$\frac{-y}{x^2+y^2}.$$

hence $T = -100\left(\frac{-y}{x^2+y^2}\right)$

$$\boxed{T = 100\,\frac{y}{x^2+y^2}}$$ ✓



problem 11, section 10, chapter 14

Nasser Abbasi

for $w = \ln[(z+1)/(z-1)]$ show that the images of $u=$ const and $v=$ const are two orthogonal sets of circles. Find centers and radii of 5 or six circles of each set and sketch them.

let $z = x+iy$.

then $w = \ln\left(\frac{x+iy+1}{x+iy-1}\right) = \ln\left(\frac{(x+1)+iy}{(x-1)+iy}\frac{(x-1)-iy}{(x-1)-iy}\right)$

$$w = \ln\left(\frac{x^2+y^2-1-2yi}{(x-1)^2+y^2}\right)$$

So $e^w = \dfrac{(x^2+y^2-1)+i(-2y)}{(x-1)^2+y^2}$.

but $w = u+iv$.

so $e^u e^{iv} = \dfrac{(x^2+y^2-1)+i(-2y)}{(x-1)^2+y^2}$ (0)

$e^u(\cos v + i\sin v) =$ " (1)

$e^u\cos v + i e^u \sin v =$ " (2)

so $e^u\cos v = \dfrac{(x^2+y^2-1)}{(x-1)^2+y^2}$; $e^u \sin v = \dfrac{-2y}{(x-1)^2+y^2}$

$\longrightarrow$

so divide one equation by another, we set

$$\frac{e^u \sin v}{e^u \cos v} = \tan v = \frac{-2y}{x^2+y^2-1}$$

for $v=$ const, this mean $\tan v =$ const. i.e.

$$\frac{-2y}{x^2+y^2-1} = \text{constant} = \frac{1}{k} \text{ (say)}$$

so equation is $-2yK = x^2+y^2-1$

or. $x^2+y^2-1+2yk=0$

or $x^2+y^2+2yk=1$

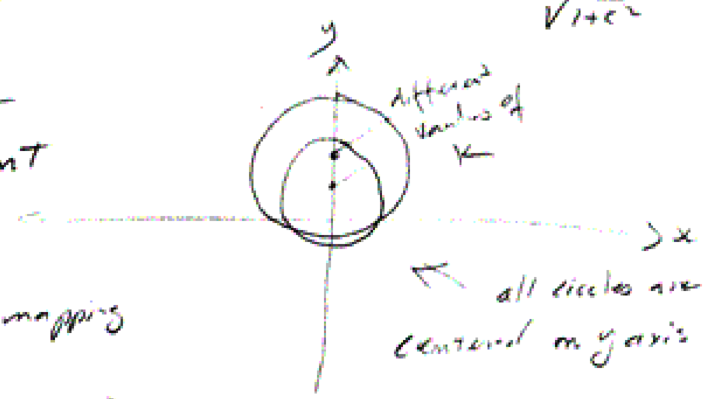
this is equation of circle

$$(x-0)^2+(y+K)^2-K^2=1$$

$$(x-0)^2+(y+K)^2 = 1+K^2$$

so this is a circle with radius at $(0,K)$ and radius $\sqrt{1+K^2}$

mapping of $v=$ constant



now I need to find mapping of $u=$ constant →

for u = constant, we get, by taking Abs of

$$e^u e^{iv} \Rightarrow |e^u e^{iv}| = e^u$$

$$= \left| \frac{(x^2+y^2-1) + i(-2y)}{(x-1)^2+y^2} \right|$$

so $$e^{2u} = \frac{(x^2+y^2-1)^2 + (-2y)^2}{\left((x-1)^2+y^2\right)^2} = \frac{(x+1)^2+y^2}{(x-1)^2+y^2}$$

for u = constant, then e^{2u} = const = k say.

so $k\left((x-1)^2+y^2\right) = (x+1)^2+y^2$

or $k(x^2-2x+1+y^2) = x^2+2x+1+y^2$

or $kx^2-2kx+k+ky^2 = x^2+2x+1+y^2$

$$x^2(k-1)+y^2(k-1)+x(-2k-2)+k-1=0$$

let $k-1=A$, a new constant.

so $Ax^2+Ay^2+2Ax=-A$ ——— (3)

or $x^2+y^2+2x = M$ (new constant)

this is equation of circle.

$$(x-a)^2+(y-b)^2=r^2$$

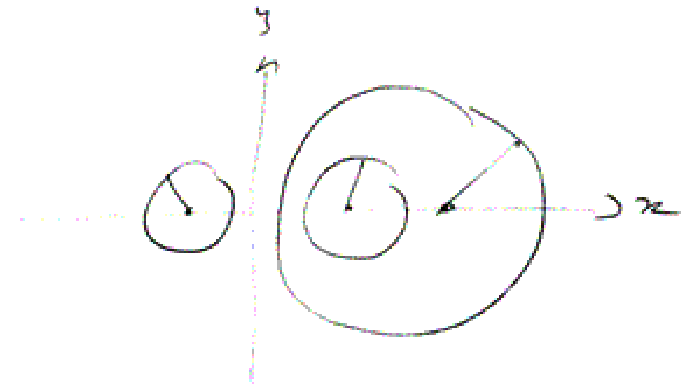
i.e. $x^2-2ax+a^2+y^2-2yb+b^2=r^2$ ——— (4)

Compare (3),(4) we get $\boxed{(x+1)^2+(y-0)^2 = \text{Constant}}$ ⟶

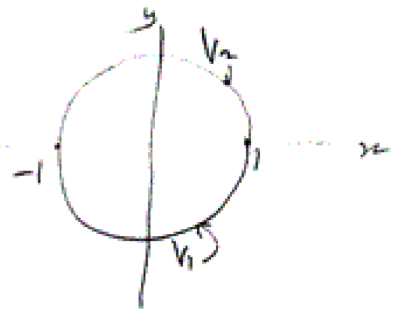
so mappings of $u=$ constant are

circles with center on x-axis.

different circles
$u=$ constant
for different
constants.



problem 13, section 10, chapter 14
Nasser Abbasi



Find $V(x,y)$ between plates in inside circle.

let $z = re^{i\theta}$.
use mapping $w = \ln\left(\frac{z+1}{z-1}\right)$ ✓

from result of problem 11, is mapping we want.

hence V at $\frac{3\pi}{2} = V_1$
V at $\frac{\pi}{2} = V_2$. ✓

so solution in W plane is

$$V = V_2 + \left(v - \frac{\pi}{2}\right)\cdot\left(\frac{V_1 - V_2}{\pi}\right)$$

$$= V_2 + \frac{vV_1}{\pi} - \frac{vV_2}{\pi} - \frac{V_1}{2} + \frac{V_2}{2} = \frac{3}{2}V_2 - \frac{V_1}{2} + \frac{v}{\pi}(V_1 - V_2)$$

$$\boxed{V = V_2\left(\frac{3}{2} - \frac{v}{\pi}\right) + V_1\left(\frac{v}{\pi} - \frac{1}{2}\right)}$$

now change back to x-y plane →

$v =$ angle of $\dfrac{x^2+y^2-1-2iy}{(x-1)^2+y^2}$ (from problem 11)

so this substitute in (1) below gives

V in x-y plane

✓

3.11 HW 10

Local contents

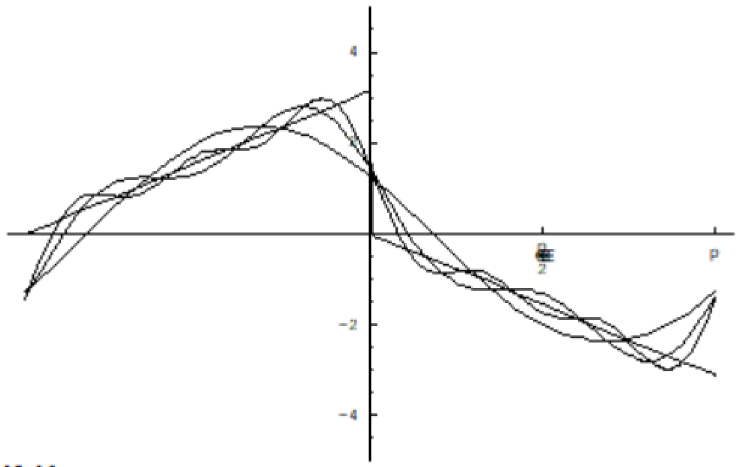
3.11.1 chapter 7, problem 4.10

I wrote a Mathematica program to help me understand the Fourier problems. This below is the output showing how series converges to the function for a number of n-values as n increases. Problem 4.10, chapter 7. Mary Boas second edition.

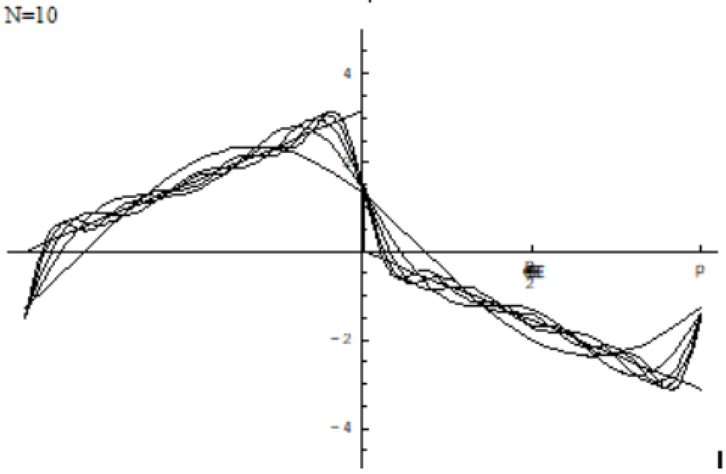
This is fourier series for

f[x_]:=x+π /; -π<=x<0
f[x_]:=-x /; 0<=x≤π

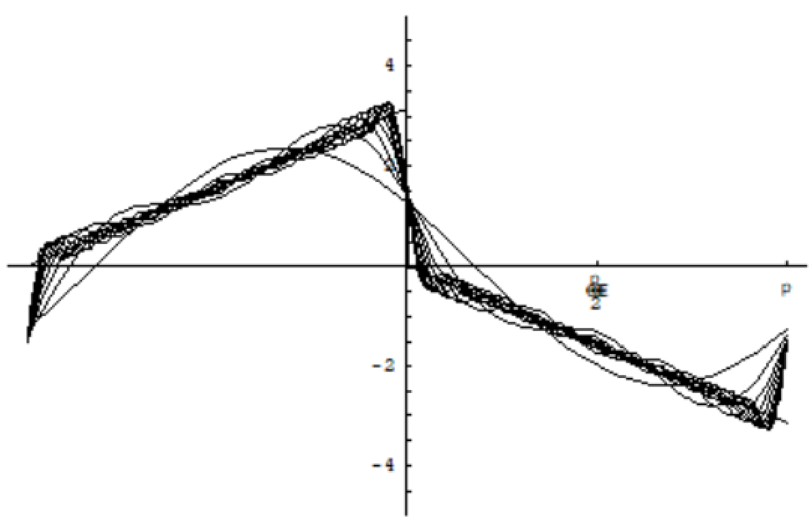
N=5



N=10



N=20

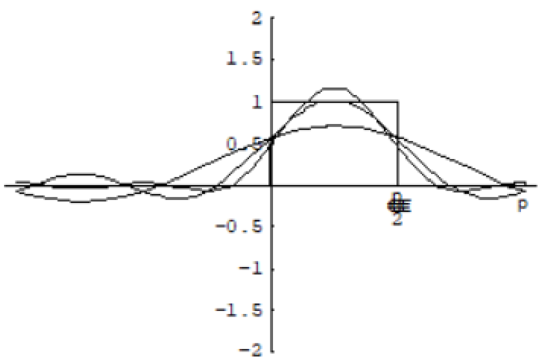


3.11.2 chapter 7, problem 4.2

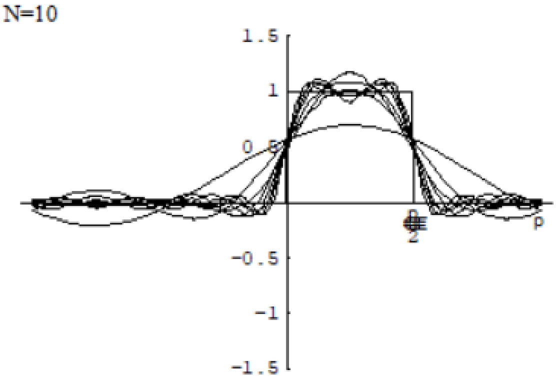
I wrote a Mathematica program to help me understand the Fourier problems. This below is the output showing how series converges to the function for a number of n-values as n increases.

I wrote a mathematica program to help me understand the Fourier problems.
This below is the output showing how series converges to the function for a number of n-values as n increases.

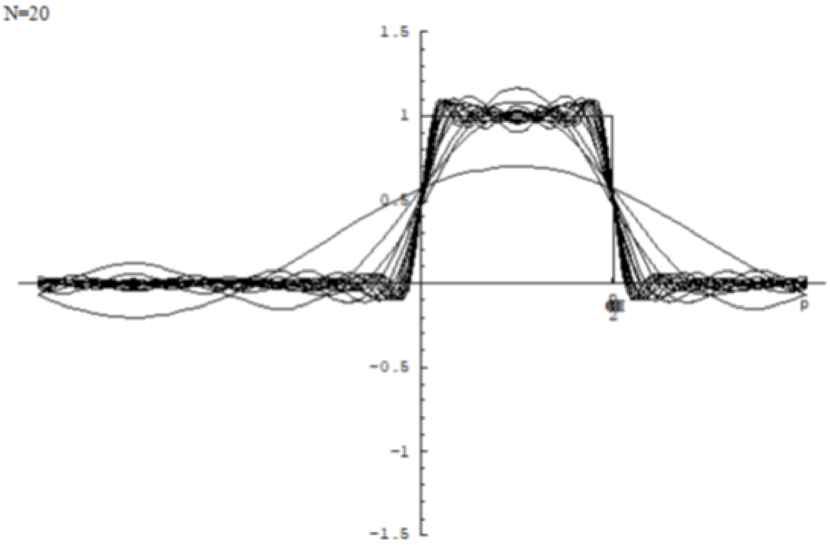
N=5



N=10



N=20

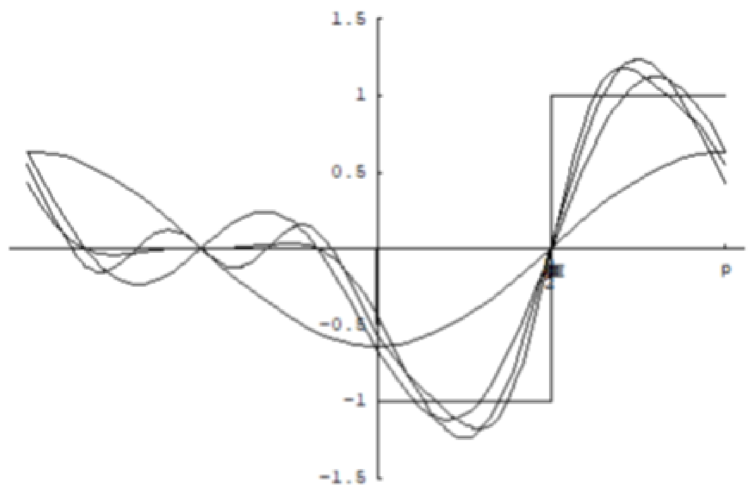


3.11.3 chapter 7, problem 4.5

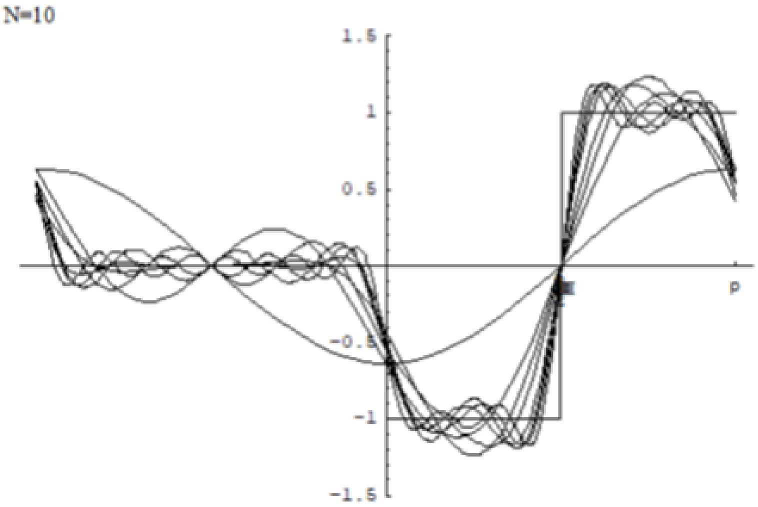
I wrote a Mathematica program to help me understand the Fourier problems. This below is the output showing how series converges to the function for a number of n-values as n increases.

I wrote a mathematica program to help me understand the Fourier problems.
This below is the output showing how series converges to the function for a number of n-values as n increases. Problem 4.5, chapter 7, Mary Boas second edition.

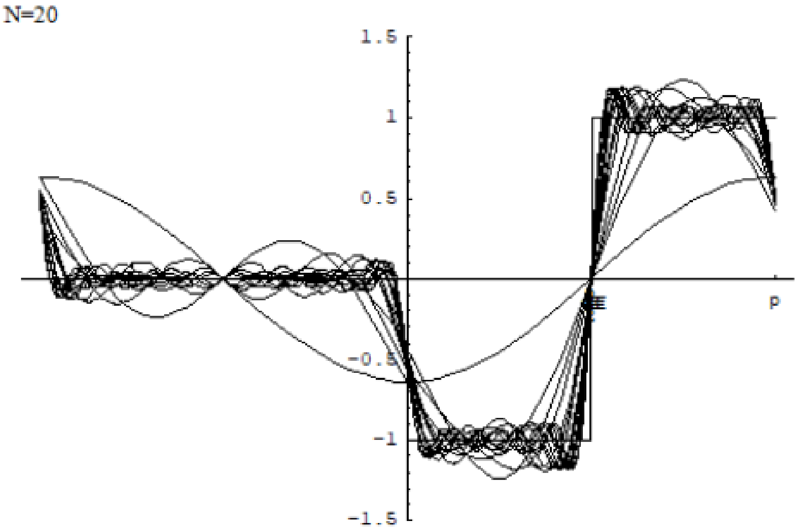
N=5



N=10



N=20



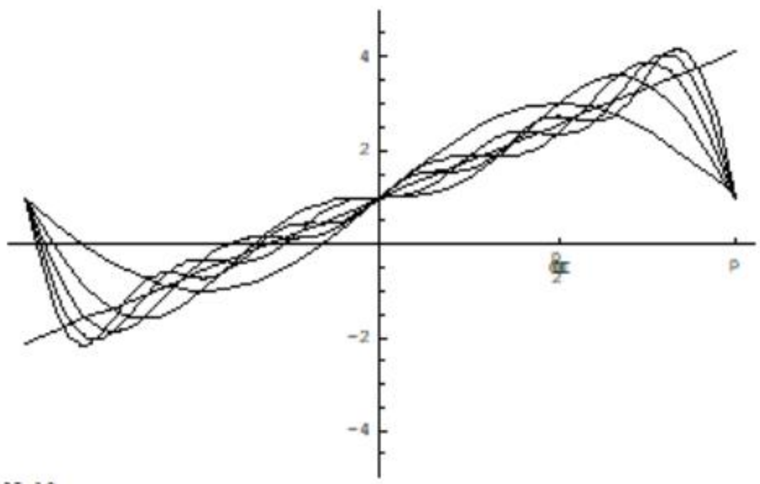
3.11.4 chapter 7, problem 4.8

I wrote a Mathematica program to help me understand the Fourier problems. This below is the output showing how series converges to the function for a number of n-values as n increases.

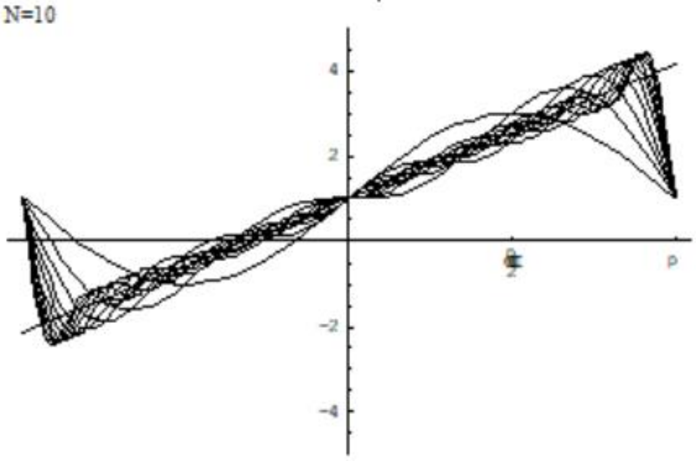
I wrote a mathematica program to help me understand the Fourier problems.
This below is the output showing how series converges to the function for a number of n-values as n increases. Problem 4.8, chapter 7, Mary Boas second edition

This is fourier series for
F(x)=1+x

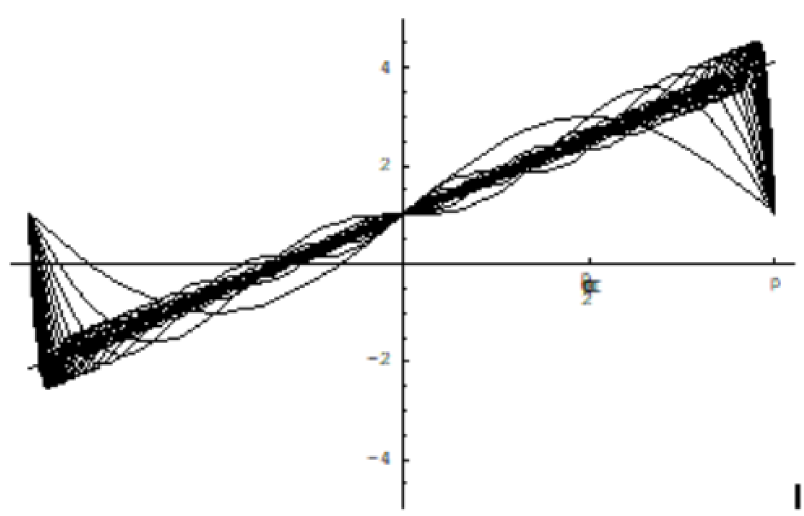
N=5



N=10



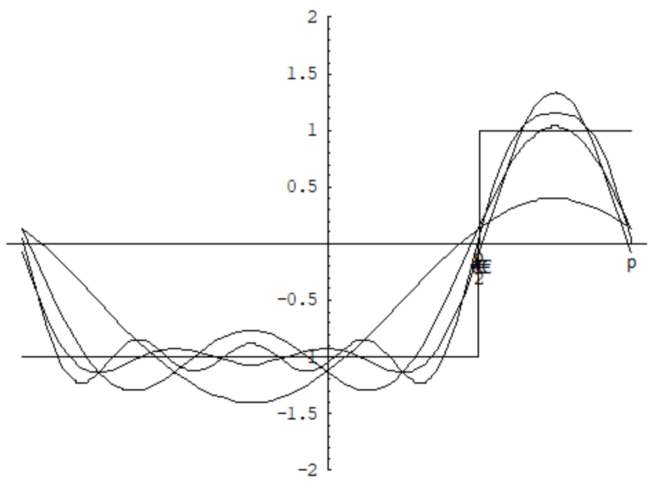
N=20



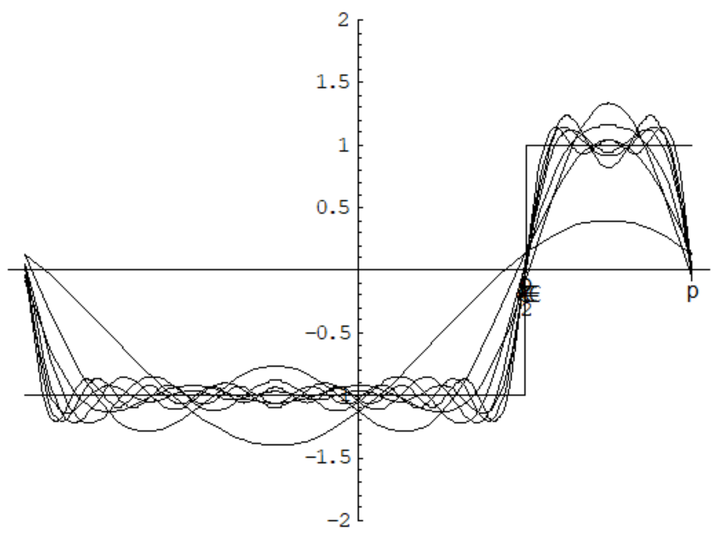
3.11.5 chapter 7, problem 5.4

I wrote a Mathematica program to help me understand the Fourier problems. This below is the output showing how series converges to the function for a number of n-values as n increases.

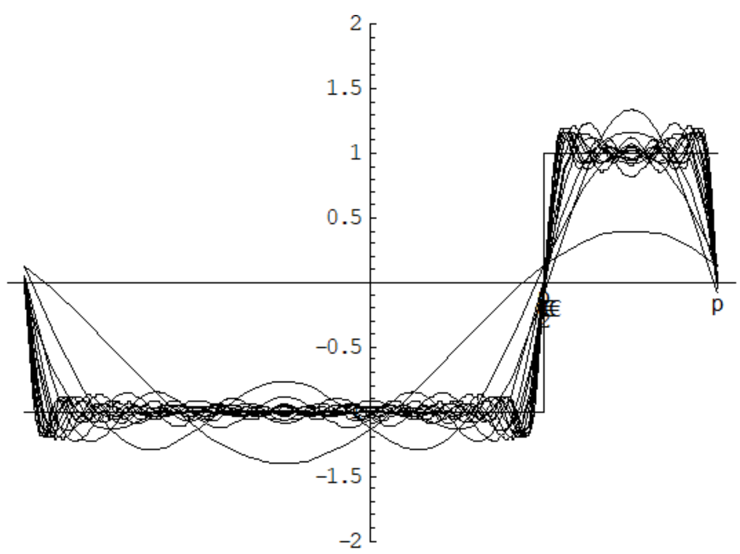
N=5



N=10



N=20



3.11.6 First part of HW 10 was scanned

Math 121A

HW # 10

2/2

Nasser Abbasi

UCB extension

Ch 7, problem 4.12.

Find average value of $\cos^2 \frac{7\pi}{2}x$ over $(0, \frac{8}{7})$.

by definition, this is $I = \frac{1}{8/7} \int_0^{8/7} \cos^2 \frac{7\pi}{2}x \, dx$

$\cos^2 \alpha = \frac{1 + \cos(2\alpha)}{2}$

so $\cos^2\left(\frac{7\pi}{2}x\right) = \frac{1 + \cos(7\pi x)}{2}$ ✓

so $I = \frac{7}{8} \int_0^{8/7} \frac{1+\cos(7\pi x)}{2} dx = \frac{7}{16} \int_0^{8/7} 1 + \cos(7\pi x) dx$

$= \frac{7}{16}\left[x + \frac{1}{7\pi}\sin(7\pi x)\right]_0^{8/7}$ ✓

$= \frac{7}{16}\left[\frac{8}{7} + \frac{1}{7\pi}\sin\left(7\pi \frac{8}{7}\right) - \left(0 + \frac{1}{7\pi}\sin(0)\right)\right]$

$= \frac{7}{16}\left[\frac{8}{7} + \frac{1}{7\pi}\sin(8\pi) - 0\right] = \frac{7}{16}\left[\frac{8}{7} + 0\right] = \left(\frac{7}{16}\right)\left(\frac{8}{7}\right) = \boxed{\frac{1}{2}}$ ✓

Chapter 7
Problem 4.13.

Using 4.3 and argument similar to 4.5 to 4.7 show that

$$\int_a^b \sin^2 kx\, dx = \int_a^b \cos^2 kx\, dx = \frac{1}{2}(b-a)$$

if $k(b-a)$ is multiple of π.

eq. 4.3 is Av. $f(x)$ over $(a,b) = \frac{1}{b-a}\int_a^b f(x)\,dx$

$$\sin^2(kx) = \frac{1-\cos(2kx)}{2}$$ ✓

$$\text{so } \int_a^b \sin^2(kx)dx = \frac{1}{2}\int_a^b 1-\cos(2kx)\,dx = \frac{1}{2}\left[x - \frac{1}{2k}\sin(2kx)\right]_a^b$$

$$= \frac{1}{2}\left[b - \frac{1}{2k}\sin(2kb) - a + \frac{1}{2k}\sin(2ka)\right]$$

$$= \frac{1}{2}\left[(b-a) + \frac{1}{2k}\left(\sin(2ka) - \sin(2kb)\right)\right]$$ ✓

but $\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$.

$$\text{so } \int_a^b \sin^2(kx)dx = \frac{1}{2}\left[(b-a) + \frac{1}{2k}\left(2\sin\left(\frac{2ka-2kb}{2}\right)\cos\left(\frac{2ka+2kb}{2}\right)\right)\right]$$

$$\int_a^b \sin^2(kx)dx = \frac{1}{2}\left[(b-a) + \frac{1}{k}\left(\sin(k(a-b))\cos(k(a+b))\right)\right] \quad (1)$$ ✓

similarly

$$\int_a^b \cos^2(kx)\,dx = \frac{1}{2}\int_a^b 1+\cos(2kx)\,dx$$

$$= \frac{1}{2}\left[x + \frac{1}{2k}\sin(2kx)\right]_a^b \longrightarrow$$

$$= \frac{1}{2}\left[b + \frac{1}{2k}\sin(2kb) - a - \frac{1}{2k}\sin(2ka) \right]$$

$$= \frac{1}{2}\left[(b-a) + \frac{1}{2k}\left(\sin(2kb) - \sin(2ka) \right) \right]$$

but $\sin(\alpha) - \sin(\beta) = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$ ✓

so

$$= \frac{1}{2}\left[(b-a) + \frac{1}{2k}\left(2\sin\left(\frac{2kb-2ka}{2}\right)\cos\left(\frac{2kb+2ka}{2}\right)\right)\right]$$

$$= \frac{1}{2}\left[(b-a) + \frac{1}{k}\left(\sin(k(b-a))\cos(k(a+b)) \right) \right] \rightarrow (2)$$

looking at (1) and (2) above ✓ we see that

if $k(b-a)$ is multiple of π, we get

$\sin k(b-a) \rightarrow 0$, hence we get simplification to:

$$\int_a^b \sin^2 kx \, dx = \frac{1}{2}(b-a)$$

and $\int_a^b \cos^2 kx \, dx = \frac{1}{2}(b-a)$ ✓

if $k(b-a) = n\pi$

or $k = \frac{n\pi}{(b-a)}$

n integer.

chapter 7
problem 4.15

evaluate $\int_{-\frac{1}{4}}^{\frac{11}{4}} \cos^2 \pi x \, dx$ (upper limit $\frac{11}{4}$ marked b, lower limit $-\frac{1}{4}$ marked a)

the limit is $\left[\frac{11}{4} + \frac{1}{4}\right] = \frac{12}{4} = 3$

but from problem 13 we found that

$$\int_a^b \cos^2 kx = \frac{1}{2}(b-a)$$

if $k(b-a) = n\pi$ for n integer.

here $(b-a) = 3$

and $k = \pi$.

hence $k(b-a)$ is a multiple of π. so we

can use result from problem 13.

i.e.

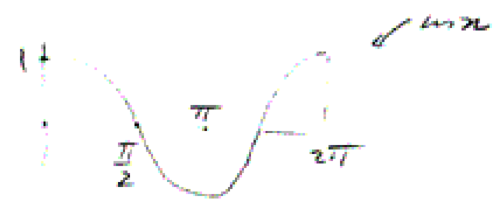
$$\int_{-\frac{1}{4}}^{\frac{11}{4}} \cos^2 \pi x \, dx = \frac{1}{2}(b-a) = \frac{1}{2}\left(\frac{11}{4} + \frac{1}{4}\right)$$

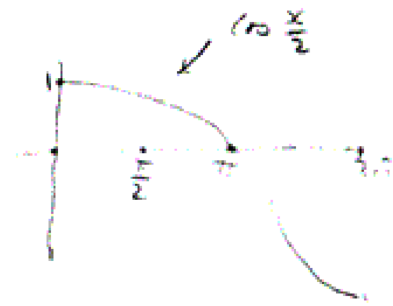
$$= \boxed{\frac{3}{2}}$$

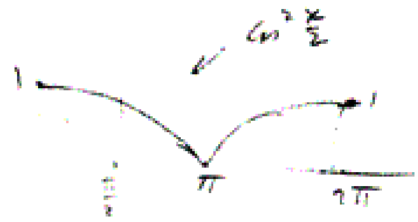
HW#10

ch 7, 45 Find average value of

$\cos^2 \frac{x}{2}$ on $(0, \frac{\pi}{2})$







by definition, average value of $f(x)$ over (a, b) is

$$\frac{1}{(b-a)} \int_a^b f(x)\, dx = \frac{1}{\pi/2} \int_0^{\pi/2} \cos^2 \frac{x}{2}\, dx .$$

but $\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2}$. so $\cos^2\left(\frac{x}{2}\right) = \frac{1+\cos(x)}{2}$.

$$\text{so } I = \frac{2}{\pi} \int_0^{\pi/2} \frac{1+\cos(x)}{2}\, dx = \frac{1}{\pi} \int_0^{\pi/2} 1+\cos(x)\, dx$$

$$= \frac{1}{\pi}\left[x + \sin(x)\right]_0^{\pi/2} = \frac{1}{\pi}\left[\left(\frac{\pi}{2}+1\right) - (0+0)\right] = \frac{1}{\pi}\left[\frac{\pi}{2}+1\right]$$

$$= \boxed{\frac{1}{2} + \frac{1}{\pi}}$$

chapter 7

problem 5.2

Given function over interval $-\pi < x < \pi$, sketch it over several periods. Expand the periodic function in a sin-cosine Fourier series

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

expand over several periods:

now to expand in Fourier series

$$a_n = \frac{\int_{-\pi}^{\pi} f(x) \cos nx \, dx}{\int_{-\pi}^{\pi} \cos^2 nx \, dx} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \qquad n \geq 0$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \cos x + \int_{0}^{\pi/2} 1 \cdot \cos nx \, dx + \int_{\pi/2}^{\pi} 0 \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \int_{0}^{\pi/2} \cos nx \, dx \quad \begin{cases} \frac{1}{\pi} \left[\frac{1}{n} \sin nx \right]_0^{\pi/2} = \frac{1}{\pi} \left[\frac{1}{n} \sin \frac{n\pi}{2} \right] = \frac{1}{\pi n} \sin \frac{n\pi}{2} & (n \neq 0) \\ \frac{1}{\pi} \left[x \right]_0^{\pi/2} = \frac{1}{\pi} \left[\frac{\pi}{2} \right] = \frac{1}{2} & (n = 0) \end{cases}$$

$$a_n = \begin{cases} \frac{1}{\pi n} \sin \frac{n\pi}{2} & n \neq 0 \\ \frac{1}{2} & n = 0 \end{cases}$$

so the cosin series is:

$$= \frac{1}{4} + \left(\frac{1}{\pi}\sin\frac{\pi}{2}\right)\cos x + \left(\frac{1}{2\pi}\sin\frac{2\pi}{2}\right)\cos 2x + \left(\frac{1}{3\pi}\sin\frac{3\pi}{2}\right)\cos 3x + \left(\frac{1}{4\pi}\sin\frac{4\pi}{2}\right)\cos 4x$$

$$+ \left(\frac{1}{5\pi}\sin\frac{5\pi}{2}\right)\cos 5\pi + \cdots$$

$$\boxed{= \frac{1}{4} + \frac{1}{\pi}\cos x - \frac{1}{3\pi}\cos 3x + \frac{1}{5\pi}\cos 5\pi - \cdots} \quad (1)$$

now to find the 'sin' series part

$$b_n = \frac{\int_{-\pi}^{\pi} f(x)\sin nx\,dx}{\int_{-\pi}^{\pi}\sin^2 nx\,dx} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad n > 0.$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \sin nx + \int_{0}^{\pi/2} 1 \sin nx\,dx + \int_{\pi/2}^{\pi} 0 \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\cos nx\Big|_0^{\pi/2}\right] = -\frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cos 0\right] =$$

$$= -\frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - 1\right] = \frac{1}{n\pi} - \frac{1}{n\pi}\cos\frac{n\pi}{2}$$

so sin series is

$$b_n = \left(\frac{1}{\pi} - \frac{1}{\pi}\cos\frac{\pi}{2}\right)\sin x + \left(\frac{1}{2\pi} - \frac{1}{2\pi}\cos\pi\right)\sin 2x + \left(\frac{1}{3\pi} - \frac{1}{3\pi}\cos\frac{3\pi}{2}\right)\sin 3x +$$

$$\left(\frac{1}{4\pi} - \frac{1}{4\pi}\cos\frac{4\pi}{2}\right)\sin 4x + \left(\frac{1}{5\pi} - \frac{1}{5\pi}\cos\frac{5\pi}{2}\right)\sin 5x + \cdots \longrightarrow$$

$$b_n = \frac{1}{\pi}\sin x + \left(\frac{1}{2\pi}+\frac{1}{2\pi}\right)\sin 2x + \left(\frac{1}{3\pi}\right)\sin 3x + \left(\frac{1}{4\pi}-\frac{1}{4\pi}\right)\sin 4x$$

$$+ \left(\frac{1}{5\pi}\right)\sin 5x + \cdots$$

$$= \frac{1}{\pi}\sin x + \frac{2\sin 2x}{2\pi} + \frac{\sin 3x}{3\pi} + \frac{\sin 5x}{5\pi} + \cdots$$

$$\boxed{= \frac{1}{\pi}\left(\sin x + \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)} \quad (2)$$

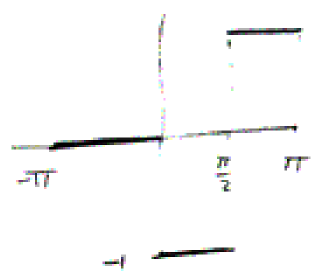
So $f(x) = (1) + (2)$

as shown above.

problem 5, chapter 7, section 5.

find Fourier Series for

$$f(x)=\begin{cases}0 & -\pi<x<0\\ -1 & 0<x<\frac{\pi}{2}\\ 1 & \frac{\pi}{2}<x<\pi\end{cases}$$



for extend to 2π period:

to find fourier series.

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\left[\int_0^{\pi/2}(-1)\cos nx\,dx+\int_{\pi/2}^{\pi}\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left[-\int_0^{\pi/2}\cos nx\,dx+\int_{\pi/2}^{\pi}\cos nx\,dx\right]$$

for $n=0$ $\quad a_0=\frac{1}{\pi}\left[\left(-\frac{\pi}{2}\right)+\left(\frac{\pi}{2}\right)\right]=0$

for $n\neq 0$

$$a_n=\frac{1}{\pi}\left[-\left[\frac{1}{n}\sin nx\right]_0^{\pi/2}+\left[\frac{1}{n}\sin nx\right]_{\pi/2}^{\pi}\right]$$

$$=\frac{1}{\pi}\left[-\frac{1}{n}\left[\sin\frac{n\pi}{2}-0\right]+\frac{1}{n}\left[\sin n\pi-\sin\frac{n\pi}{2}\right]\right]$$

$$=\frac{1}{\pi}\left[-\frac{1}{n}\left(\sin\frac{n\pi}{2}\right)+\frac{1}{n}\left(-\sin\frac{n\pi}{2}\right)\right]=\frac{-2}{n\pi}\sin\frac{n\pi}{2}$$

so looking at few terms:

$a_1 = \frac{-2}{\pi}$

$a_2 = \frac{-2}{2\pi}(0) = 0$

$a_3 = \frac{-2}{3\pi}\sin\left(\frac{3}{2}\pi\right) = +\frac{2}{3\pi}$

so a series is

$$\boxed{\frac{-2}{\pi}\cos x + \frac{2}{3\pi}\cos 3x - \frac{2}{5\pi}\cos 5x + \cdots}$$

notice, no a_0 term.

now to Find the sin series:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left(\int_0^{\pi/2} -\sin nx\,dx + \int_{\pi/2}^{\pi}\sin nx\,dx\right)$$

$$= \frac{1}{\pi}\left(-\left(-\frac{\cos nx}{n}\right)_0^{\pi/2} + \left(-\frac{\cos nx}{n}\right)_{\pi/2}^{\pi}\right)$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\left[\cos nx\right]_0^{\pi/2} - \frac{1}{n}\left[\cos nx\right]_{\pi/2}^{\pi}\right)$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\left[\cos\frac{n\pi}{2} - 1\right] - \frac{1}{n}\left[\cos n\pi - \cos\frac{n\pi}{2}\right]\right)$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\left[\cos\frac{n\pi}{2} - 1 - \cos n\pi + \cos\frac{n\pi}{2}\right)\right.$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\left(2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right)\right] = \frac{1}{n\pi}\left(2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right)$$

looking at few terms;

$\longrightarrow$

$n=1 \quad \frac{1}{\pi}\left[2\cos\frac{\pi}{2} - 1 - \cos\pi\right] = \frac{1}{\pi}\left[-1+1\right] = 0$

$n=2 \quad \frac{1}{2\pi}\left[2\cos\pi - 1 - \cos 2\pi\right] = \frac{1}{2\pi}\left[-2-1-1\right] = \frac{1}{2\pi}\left[-5\right] = -\frac{5}{2\pi}$

$n=3 \quad \frac{1}{3\pi}\left[2\cos\frac{3\pi}{2} - 1 - \cos 3\pi\right] = \frac{1}{3\pi}\left[0-1+1\right] = 0$

$n=4 \quad \frac{1}{4\pi}\left[2\cos 2\pi - 1 - \cos 4\pi\right] = \frac{1}{4\pi}\left[2-1-1\right] = 0$

$n=5 \quad \frac{1}{5\pi}\left[2\cos\frac{5\pi}{2} - 1 - \cos 5\pi\right] = \frac{1}{5\pi}\left[0-1+1\right] = 0$

$n=6 \quad \frac{1}{6\pi}\left[2\cos\frac{6\pi}{2} - 1 - \cos 6\pi\right] = \frac{1}{6\pi}\left[-2-1-1\right] = \frac{-5}{6\pi}$

So sine series is

$$-\frac{5}{2\pi}\sin 2x - \frac{5}{6\pi}\sin 6x - \frac{5}{10\pi}\sin 10x - \cdots$$

hence fourier series is

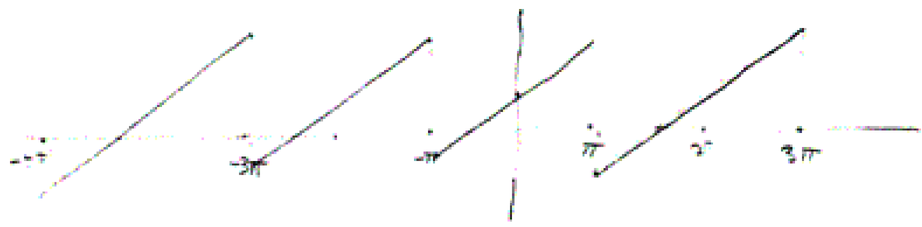
$$\left(\frac{-2}{\pi}\cos x + \frac{2}{3\pi}\cos 3x - \frac{2}{5\pi}\cos 5x + \cdots\right)$$

$$+\left(\frac{-5}{2\pi}\sin 2x - \frac{5}{6\pi}\sin 6x - \frac{5}{10\pi}\sin 10x - \cdots\right)$$

problem 8 chapter 7, section 5

find fourier series for $f(x) = 1 + x \quad -\pi < x < \pi$

extend to make it periodic over 2π:



to find a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (1+x)\cos nx \, dx.$$

$$n=0 \Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (1+x)\, dx = \frac{1}{\pi}\left[2\pi + \frac{1}{2}\left[x^2\right]_{-\pi}^{\pi}\right]$$

$$a_0 = \frac{1}{\pi}\left[2\pi + \frac{1}{2}\left(\pi^2 - (-\pi)^2\right)\right] = \frac{1}{\pi}\left[2\pi + \frac{1}{2}\left(\pi^2 - \pi^2\right)\right] = 2$$

now, for a_n, $n > 1$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)\cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi}\cos nx + \int_{-\pi}^{\pi} x \cos nx \, dx\right]$$

($=0$ since over one period) ($=0$ since odd over period)

(x odd, $\cos nx$ even $\rightarrow$ odd)

hence $a_n = 0 \quad n > 1$. $\longrightarrow$

to find b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (1+x) \sin nx \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx + \int_{-\pi}^{\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

integrate by parts. let $u = x$, $dv = \sin nx$.

$du = 1$, $V = -\frac{1}{n}\cos nx$

$$\Rightarrow \int_{-\pi}^{\pi} x \sin nx \, dx = \left[-x \frac{\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} 1 \cdot \frac{1}{n}\cos nx \, dx$$

$$= -\frac{1}{n}\left[x\cos nx\right]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} \cos nx \, dx \quad = 0$$

$$= -\frac{1}{n}\left[\pi \cos n\pi - (-\pi)\cos(-n\pi)\right] = -\frac{1}{n}\left[\pi \cos n\pi + \pi \cos(n\pi)\right]$$

$$= -\frac{1}{n}\left[2\pi \cos n\pi\right] = -\frac{2\pi}{n}\cos n\pi$$

for few values of n.

$n=1 \Rightarrow -2\pi \cos \pi = 2\pi$

$n=2 \Rightarrow -\pi \cos 2\pi = -\pi$

$n=3 \Rightarrow -\frac{2}{3}\pi \cos 3\pi = \frac{2}{3}\pi$

$n=4 \Rightarrow -\frac{1}{2}\pi \cos 4\pi = -\frac{1}{2}\pi$

so Fourier Series

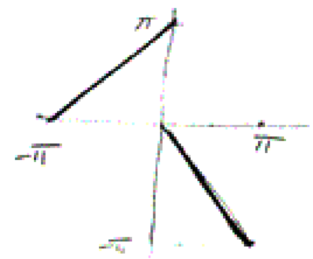
$$f(x) = 1 + \frac{1}{\pi}\left(2\pi \sin x - \pi \sin 2x + \frac{2}{3}\pi \sin 3x - \frac{1}{2}\pi \sin 4x \cdots\right)$$

$$\boxed{f(x) = 1 + 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x \cdots}$$

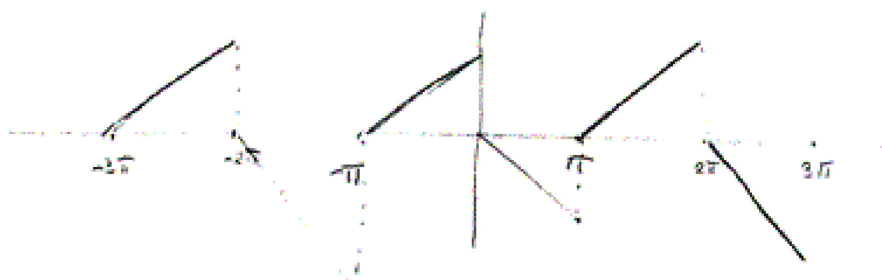
problem 10, section 5, chapter 7

Find fourier series for

$$f(x) = \begin{cases} x+\pi & -\pi < x < 0 \\ -x & 0 < x < \pi \end{cases}$$



extend it



to find a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

for $n=0$, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} x+\pi \, dx + \int_{0}^{\pi} -x \, dx\right]$

$$= \frac{1}{\pi}\left[\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \pi\left[x\right]_{-\pi}^{0} - \left[\frac{x^2}{2}\right]_{0}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\left(0 - (-\pi)^2\right) + \pi\left[0 + \pi\right] - \frac{1}{2}\left[\pi^2\right]\right] = \frac{1}{\pi}\left(\frac{1}{2}\left(-\pi^2\right) + \pi^2 - \frac{\pi^2}{2}\right)$$

$$= \frac{1}{\pi}\left(-\frac{\pi^2}{2} + \pi^2 - \frac{\pi^2}{2}\right) = 0$$

for $n>0$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(x+\pi)\cos nx\,dx - \int_0^{\pi} x\cos nx\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} x\cos nx\,dx + \int_{-\pi}^{0}\pi\cos nx\,dx - \int_0^{\pi} x\cos nx\,dx\right]$$

$$\int_{-\pi}^{0} x\cos nx\,dx \Rightarrow \quad u=x, \quad dv=\cos nx$$

$$\Rightarrow \left[x\frac{\sin nx}{n}\right]_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\sin nx}{n}\,dx = \frac{1}{n}\left[-\pi\sin(-n\pi)\right] - \frac{1}{n}\int_{-\pi}^{0}\sin nx\,dx$$

$$= -\frac{\pi}{n}\left[\sin(-n\pi)\right] + \frac{1}{n}\left[\frac{1}{n}\cos nx\right]_{-\pi}^{0}$$

$$= \frac{\pi}{n}\sin(n\pi) + \frac{1}{n^2}\left[\cos 0 - \cos(n\pi)\right] = \frac{\pi}{n}\sin n\pi + \frac{1}{n^2}\left[1-\cos n\pi\right]$$

$$\int_0^{\pi} x\cos nx\,dx = \left[x\frac{\sin nx}{n}\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx$$

$$= \frac{1}{n}\left[\pi\sin n\pi\right] + \frac{1}{n}\frac{1}{n}\left(\cos nx\right)_0^{\pi} = \frac{1}{n^2}\left(\cos n\pi - 1\right)$$

$$= \frac{1}{n^2}\left(\cos n\pi - 1\right)$$

$$\int_{-\pi}^{0}\pi\cos nx\,dx = \pi\left[\frac{1}{n}\sin nx\right]_{-\pi}^{0} = \frac{\pi}{n}\left[\sin 0 - \sin \pi n\right]$$

so $$a_n = \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{1}{n^2}\cos n\pi - \left(\frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{2}{n^2} - \frac{2}{n^2}\cos n\pi\right]$$

so for $n=1$, $\quad a_1 = \frac{1}{\pi}\left[2+2\right] = \frac{4}{\pi}$

$n=2 \quad a_2 = \frac{1}{\pi}\left[\frac{2}{4} - \frac{2}{4}\right] = 0$ $\longrightarrow$

$n=3 \quad a_3 = \frac{1}{\pi}\left[\frac{2}{9} - \frac{2}{9}(-1)\right] = \frac{1}{\pi}\left[\frac{4}{9}\right]$

$n=4 \quad a_4 = \frac{1}{\pi}\left[\frac{2}{16} - \frac{2}{16}\right] = 0$

so The cosine series is

$$\frac{4}{\pi}\cos x + \frac{4}{9\pi}\cos 3x + \frac{4}{25}\cos 5x + \dots$$

to find the b_n terms.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} x\sin nx\,dx + \int_{-\pi}^{0}\pi\sin nx\,dx - \int_{0}^{\pi} x\sin nx\,dx\right]$$

$$\int_{-\pi}^{0} x\sin nx\,dx = -\left[\frac{x\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{n}\int_{-\pi}^{0}\cos nx\,dx$$

$$= -\frac{1}{n}\left[0 - (-\pi)\cos n(-\pi)\right] + \frac{1}{n}\frac{1}{n}\left[\sin nx\right]_{-\pi}^{0}$$

$$= -\frac{1}{n}\left[\pi\cos n\pi\right] + \frac{1}{n^2}\left[0 - \sin(-n\pi)\right]$$

$$= -\frac{1}{n}\left[\pi\cos n\pi\right] + \frac{1}{n^2}\left[\sin n\pi\right]$$

$$\int_{-\pi}^{0}\pi\sin nx\,dx = -\frac{\pi}{n}\left[\cos n\pi\right]_{-\pi}^{0} = -\frac{\pi}{n}\left[1 - \cos n\pi\right]$$

→

$$\int_0^{\pi} x \sin nx\, dx = \left[-\frac{x\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos nx\, dx$$

$$= -\frac{1}{n}\left[\pi \cos n\pi\right] + \frac{1}{n}\frac{1}{n}\left[\sin nx\right]_0^{\pi}$$

$$= -\frac{\pi}{n}\left[\cos n\pi\right] + \frac{1}{n^2}\left[0\right] = -\frac{\pi}{n}\cos n\pi$$

so $b_n = -\frac{\pi}{n}\cos n\pi + \frac{1}{n^2}\sin n\pi - \frac{\pi}{n} + \frac{\pi}{n}\cos n\pi + \frac{\pi}{n}\cos n\pi$

$$= \frac{1}{n^2}\sin n\pi - \frac{\pi}{n} + \frac{\pi}{n}\cos n\pi.$$

look at few n values:

$b_1 = -\pi + \pi\cos\pi = -\pi - \pi = -2\pi$

$b_2 = -\frac{\pi}{2} + \frac{\pi}{2}\cos 2\pi = -\frac{\pi}{2} + \frac{\pi}{2} = 0$

$b_3 = -\frac{\pi}{3} + \frac{\pi}{3}\cos 3\pi = -\frac{\pi}{3} - \frac{\pi}{3} = -\frac{2}{3}\pi$

so sin series is

$$-2\sin x - \frac{2}{3}\sin 3x - \frac{2}{5}\sin 5x - \cdots$$

hence Fourier series is

$$\left(\frac{4}{\pi}\cos x + \frac{4}{9\pi}\cos 3x + \frac{4}{25}\cos 5x + \cdots\right)$$

$$-\left(2\sin x + \frac{2}{3}\sin 3x + \frac{2}{5}\sin 5x + \cdots\right)$$

problems 5, section 6, chapter 7

use Dirichlet theorem to find the value to which Fourier series converges at $x = 0, \pm\frac{\pi}{2}, \pm\pi, \pm 2\pi$ for

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ -1 & 0 < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$

at $x = 0$, it will converge to $\frac{f(0_-) + f(0_+)}{2} = \frac{0-1}{2} = \boxed{-\frac{1}{2}}$

at $x = -\frac{\pi}{2}$, $\frac{f(-\frac{\pi}{2}_-) + f(-\frac{\pi}{2}_+)}{2} = \frac{-1+1}{2} = \boxed{0}$

at $x = \frac{\pi}{2}$, $\frac{f(\frac{\pi}{2}_-) + f(\frac{\pi}{2}_+)}{2} = \frac{-1+1}{2} = \boxed{0}$

at $x = -\pi$, $\frac{f(-\pi_-) + f(-\pi_+)}{2} = \frac{1+0}{2} = \boxed{\frac{1}{2}}$

at $x = \pi$, $\boxed{\frac{1}{2}}$

$x = 2\pi$, $\boxed{-\frac{1}{2}}$

$x = -2\pi$, $\boxed{-\frac{1}{2}}$

problem 4, section 7, chapter 7.

expand the following function in Fourier series of complex exponentials e^{inx} in interval $(-\pi,\pi)$ and verify using Euler formula that answer is equal to one found in section 5.

$$f(x) = \begin{cases} -1 & -\pi < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$

$$f(x) = c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + c_2 e^{2ix} + c_{-2} e^{-2ix} + \cdots$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

Find c_n.

$$c_n = \frac{1}{2\pi}\left[\int_{-\pi}^{\frac{\pi}{2}} -e^{-inx}\, dx + \int_{\frac{\pi}{2}}^{\pi} e^{-inx}\, dx \right]$$

$$= \frac{1}{2\pi}\left[-\frac{1}{-in}\left[e^{-inx} \right]_{-\pi}^{\frac{\pi}{2}} + \frac{1}{-in}\left[e^{-inx} \right]_{\frac{\pi}{2}}^{\pi} \right]$$

$$= \frac{1}{2\pi}\frac{1}{in}\left[\left(e^{-in\frac{\pi}{2}} - e^{+in\pi} \right) - \left(e^{-in\pi} - e^{-in\frac{\pi}{2}} \right) \right]$$

$$= \frac{1}{2\pi i n}\left[e^{-in\frac{\pi}{2}} + e^{-in\frac{\pi}{2}} - e^{-in\pi} - e^{+in\pi} \right] \longrightarrow$$

notice few relationships on $e^{in\pi}$

$e^{-in\pi} = \cos n\pi - i\sin n\pi = \cos n\pi = -1$ for odd $\pm n$
$= 1$ for even $\pm n$, 0

$e^{in\pi} = \cos n\pi + i\sin n\pi = \cos n\pi = -1$ for odd $\pm n$
$= 1$ for even $\pm n$, 0

$e^{-in\frac{\pi}{2}} = \cos n\frac{\pi}{2} - i\sin n\frac{\pi}{2} = -i$ for odd n 1, 5, 9, … −3, −7, −11, …
$= i$ for odd n 3, 7, 11, … −1, −5, −9, …
$= -1$ for even $\pm n$ 2, 6, 10, …
$= 1$ for even $\pm n$ 4, 8, 12, …, 0

$e^{in\frac{\pi}{2}} = \cos n\frac{\pi}{2} + i\sin n\frac{\pi}{2} = i$ for odd n 1, 5, 9, … −3, −7, −9, …
$= -i$ for odd n 3, 7, 11, … −1, −5, −9, …
$= -1$ for even n 2, 6, 10, …
$= 1$ for even n 0, 4, 8, 12, …

So C_n becomes

$$C_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{\frac{\pi}{2}} -dx + \int_{\frac{\pi}{2}}^{\pi} dx\right] = \frac{1}{2\pi}\left[-\left[x\right]_{-\pi}^{\frac{\pi}{2}} + \left[x\right]_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[-\left(\frac{\pi}{2}+\pi\right) + \left(\pi - \frac{\pi}{2}\right)\right] = \frac{1}{2\pi}\left[-\pi\right] = \boxed{-\frac{1}{2}}$$

→

So $f(x)$ is

positive n

$$+\frac{1}{2\pi i}\left(\frac{1}{1}\left(e^{-i\frac{\pi}{2}}+e^{-i\frac{\pi}{2}}-e^{-i\pi}-e^{i\pi}\right)e^{ix}+\frac{1}{2}\left(e^{-i\pi}+e^{-i\pi}-e^{-2i\pi}-e^{i\pi}\right)e^{i2x}+\cdots\right)$$

$$+\frac{1}{2\pi i}\left(-1\left(e^{i\frac{\pi}{2}}+e^{i\frac{\pi}{2}}-e^{i\pi}-e^{-i\pi}\right)e^{-ix}+\left(-\frac{1}{2}\right)\left(e^{i\pi}+e^{i\pi}-e^{2i\pi}-e^{-2i\pi}\right)e^{-i2x}+\cdots\right)$$

negative n

$$=\frac{1}{2\pi i}\left((-i-i+1+1)e^{ix}+\left(\frac{1}{2}\right)(-1-1-1-1)e^{i2x}+\cdots\right)$$

$$+\frac{1}{2\pi i}\left(-1(-i-i+1+1)e^{-ix}+\left(-\frac{1}{2}\right)(-1-1-1-1)e^{-i2x}+\cdots\right)$$

$$=\frac{1}{2\pi i}\left((-2i+2)e^{ix}-2e^{i2x}+\cdots\right)$$

$$+\frac{1}{2\pi i}\left((2i-2)e^{-ix}+2e^{-2ix}+\cdots\right)$$

$$=\frac{1}{2\pi i}\left(\left(-2ie^{ix}+2e^{ix}-2e^{i2x}+\cdots\right)+\left(2ie^{-ix}-2e^{-ix}+2e^{-i2x}+\cdots\right)\right)$$

$\longrightarrow$

Collect Terms

$$= \frac{1}{2\pi i}\left(\left(2ie^{-ix}-2ie^{ix}\right)+\left(2e^{ix}-2e^{-ix}\right)+\left(2e^{-2ix}-2e^{2ix}\right)\cdots\right.$$

$$=\frac{1}{\pi}\left(\underbrace{\left(e^{-ix}-e^{ix}\right)}_{-2\cos x}+\underbrace{\left(\frac{e^{ix}-e^{-ix}}{i}\right)}_{2\sin x}+\underbrace{\left(\frac{e^{-2ix}-e^{2ix}}{i}\right)}_{2\sin 2x}+\cdots\right.$$

$$=\frac{1}{\pi}\left(-2\cos x+2\sin x+2\sin 2x\ \cdots\right)$$

So

$$f(x)=\frac{1}{2}+\frac{2}{\pi}\left(-\cos x+\sin x+\sin 2x\right)$$

$\swarrow$ c_0

$\nearrow$ this matches result of #5.4 by the Fourier series. Verified using mathematica.

7.7 chapter 7 (2/2) Name? Nasser Abbasi

$$f(x) = \frac{1}{2}a_0 + \sum_1^\infty a_n \cos nx + \sum_1^\infty b_n \sin nx = \sum_{-\infty}^{\infty} c_n e^{inx}$$

$$= \frac{1}{2}a_0 + (a_1 \cos nx + a_2 \cos 2nx + \cdots) + (b_1 \sin x + b_2 \sin 2x + \cdots)$$

$$= \cdots + c_{-1} e^{-inx} + c_0 + c_1 e^{inx} + \cdots$$

so $c_0 = \frac{1}{2} a_0$

expand $e^{-inx} = \cos nx - i \sin nx$

$e^{inx} = \cos nx + i \sin nx$

so $= \frac{1}{2}a_0 + (a_1 \cos nx + \cdots \surd) + (b_1 \sin x + b_2 \sin 2x + \cdots)$

$= \cdots + c_{-1}(\cos x - i \sin nx) + c_0 + c_1(\cos x + i \sin nx) + \cdots$

$+ \cos x (c_{-1} + c_1) + \sin x(-i c_{-1} + i c_1)$

so $a_1 = c_{-1} + c_1$

$b_1 = i(-c_{-1} + c_1)$ ✓

$a_n = c_{-n} + c_n$

$b_n = i(-c_{-n} + c_n)$

problem 12, section 7, Chapter 7.

Show that if a real $f(x)$ is expanded in complex exp. fourier series $\sum_{-\infty}^{\infty} c_n e^{inx}$, then $c_{-n} = \bar{c}_n$ where $\bar{c}_n$ means complex conjugate of c_n.

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \quad (1)$$

$$\text{so } c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{inx}\, dx \quad (2)$$

from (1), $\bar{c}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)}\;\overline{e^{-inx}}\, dx$, but $f(x)$ is real, so $\overline{f(x)} = f(x)$

so $\bar{c}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, \overline{e^{-inx}}\, dx$.

but $\overline{e^{-inx}} = e^{inx}$

$$\text{so } \bar{c}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{inx}\, dx \quad (3)$$

compare (2) and (3)

$$\Rightarrow \boxed{c_{-n} = \bar{c}_n}$$

(1

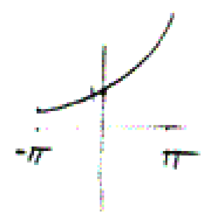
Problem 12, Section 8, Chpt 7

sketch several periods of function and find fourier series:

(a) $f(x) = e^x \quad -\pi < x < \pi$

(b) $f(x) = e^x \quad 0 < x < 2\pi$

(a)



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$

$$n=0 \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi} = \boxed{\frac{1}{\pi}\left[e^{\pi}-e^{-\pi}\right]} \checkmark$$

$$n>0 \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx.$$

integration by parts:

let $u = \cos nx \quad du = -n\sin nx$

$dv = e^x$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\left[\cos nx\, e^x\right]_{-\pi}^{\pi} - n\int \sin nx\, e^x\,dx\right]$$

$$I = \frac{1}{\pi}\left[\left(\cos nx\, e^x\right)_{-\pi}^{\pi} - n\left[\left(\sin nx\; e^x\right)_{-\pi}^{\pi} - n\int \cos nx\, e^x\,dx\right]\right]$$

$$I = \frac{1}{\pi}\left[\left(\cos nx\, e^x\right)_{-\pi}^{\pi} - n\left(\sin nx\, e^x\right]_{-\pi}^{\pi} + n^2 I\right]$$

$$I - \frac{n^2}{\pi} I = \frac{1}{\pi}\left(\cos nx\, e^x\right)_{-\pi}^{\pi} - \frac{n}{\pi}\left(\sin x\; e^x\right)_{-\pi}^{\pi} \checkmark \longrightarrow$$

(2

$$a_n\left(1-\frac{n^2}{\pi}\right) = \frac{1}{\pi}\left(\cos n\pi e^{\pi} - \cos n\pi e^{-\pi}\right) - \frac{n}{\pi}\left(\sin \pi e^{\pi} + \sin\pi e^{-\pi}\right)$$

$$a_n = \frac{1}{\left(\frac{\pi-n^2}{\pi}\right)}\frac{1}{\pi}\left[\left(\cos n\pi e^{\pi} - \cos n\pi e^{-\pi}\right) - n\left(\sin \pi e^{\pi} + \sin\pi e^{-\pi}\right)\right]$$

$$a_n = \left(\frac{1}{\pi-n^2}\right)\left[\cos n\pi\left(e^{\pi}-e^{-\pi}\right) - n\left(\overset{=0}{\sin n\pi}\left(e^{\pi}+e^{-\pi}\right)\right)\right]$$

$$= \frac{1}{\pi-n^2}\cos n\pi\left(e^{\pi}-e^{-\pi}\right) = \boxed{\frac{e^{\pi}-e^{-\pi}}{\pi-n^2}\cos n\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin nx\, e^{x}\, dx = \frac{1}{\pi}\left[\left(\sin nx\, e^{x}\right)_{-\pi}^{\pi} - n\int \cos nx\, e^{x}\right]$$

$$= \frac{1}{\pi}\left[\left(\sin n\pi e^{\pi} + \sin n\pi e^{-\pi}\right) - n\left[\left(\cos nx\, e^{x}\right)_{-\pi}^{\pi} + n\int \sin nx\, e^{x}\, dx\right]\right]$$

$$= \frac{1}{\pi}\left[\sin n\pi\left(e^{\pi}-e^{-\pi}\right) - n\left[\left(\cos n\pi e^{\pi} - \cos n\pi e^{-\pi}\right) + n\, I\right]\right]$$

$$I = \sin n\pi\left(\frac{e^{\pi}-e^{-\pi}}{\pi}\right) - \frac{n}{\pi}\left(\cos n\pi\left(e^{\pi}-e^{-\pi}\right)\right) - \frac{n^2}{\pi} I$$

$$I + \frac{n^2}{\pi} I = \frac{e^{\pi}-e^{-\pi}}{\pi}\left(\sin n\pi - n\cos n\pi\right)$$

$$b_n = \frac{1}{\left(1+\frac{n^2}{\pi}\right)}\frac{e^{\pi}-e^{-\pi}}{\pi}\left(\overset{=0}{\sin n\pi} - n\cos n\pi\right)$$

$$b_n = \left(\frac{e^{\pi}-e^{-\pi}}{\pi+n^2}\right)(-n\cos n\pi)$$

so $a_n =$

$n=1 \quad \left(\frac{e^{\pi}-e^{-\pi}}{\pi-1}\right)(-1)$

$n=2 \quad \frac{e^{\pi}-e^{-\pi}}{\pi-4}$

$n=3 \quad \frac{e^{\pi}-e^{-\pi}}{\pi-9}(-1)$ etc..

b_n:

$b_1 = \frac{e^{\pi}-e^{-\pi}}{\pi+1}(1)$

$b_2 = \frac{e^{\pi}-e^{-\pi}}{\pi+4}(-2)$

$$\text{so } f(x) = \frac{1}{2\pi}\left(e^{\pi}-e^{-\pi}\right) + \left(-\frac{(e^{\pi}-e^{-\pi})}{\pi-1}\cos x + \frac{e^{\pi}-e^{-\pi}}{\pi-4}\cos 2x + \cdots\right)$$

$$+\left(\frac{e^{\pi}-e^{-\pi}}{\pi+1}\sin x - 2\,\frac{e^{\pi}-e^{-\pi}}{\pi+4}\sin 2x + \cdots\right)$$

$$f(x) = e^{\pi}-e^{-\pi}\left[\left(\frac{1}{2\pi} - \frac{1}{\pi-1}\cos x + \frac{1}{\pi-4}\cos 2x + \cdots\right) + \left(\frac{1}{\pi+1}\sin x - \frac{2}{\pi+4}\sin 2x + \cdots\right)\right]$$

3.12 HW 11

Local contents

3.12.1 chapter 15, problem 2.11

Problem Find the inverse transform of the function $F(p) = \frac{3p+2}{3p^2+5p-2}$

Solution

Need to simplify the above expression to some expressions which are shown in the table on page 636.

$$\begin{aligned} F(p) &= \frac{3p+2}{3p^2+5p-2} \\ &= \frac{2}{(3p-1)(p+2)} + \frac{3p}{(3p-1)(p+2)} \end{aligned} \tag{1}$$

Expanding in partial fractions. For the first term in (1):

$$\begin{aligned} \frac{1}{(3p-1)(p+2)} &= \frac{A}{(3p-1)} + \frac{B}{(p+2)} = \frac{A(p+2)+B(3p-1)}{(3p-1)(p+2)} \\ A(p+2)+B(3p-1) &= 1 \\ Ap+2A+3Bp-B &= 1 \end{aligned}$$

Hence $2A-B=1$ and $(A+3B)=0$ which gives $A=\frac{1}{2}+\frac{B}{2}$. Therefore $\frac{1}{2}+\frac{B}{2}+3B=0$ or $\frac{1+B+6B}{2}=0$ or $1+7B=0$ or $B=\frac{-1}{7}$. Hence $A=\frac{1}{2}+\frac{\frac{-1}{7}}{2}=\frac{1}{2}-\frac{1}{14}=\frac{7-1}{14}=\frac{6}{14}$.

Now the first term in (1) can be written as $2\left(\frac{A}{(3p-1)}+\frac{B}{(p+2)}\right)$ or $2\left(\frac{\frac{6}{14}}{(3p-1)}+\frac{\frac{-1}{7}}{(p+2)}\right)$ or

$$\frac{6}{7(3p-1)} - \frac{2}{7(p+2)} \tag{2}$$

Doing partial fraction on the second term in (1) which is $\frac{3p}{(3p-1)(p+2)}$ gives

$$
\begin{aligned}
\frac{p}{(3p-1)(p+2)} &= \frac{A}{(3p-1)} + \frac{B}{(p+2)} = \frac{A(p+2)+B(3p-1)}{(3p-1)(p+2)} \\
A(p+2)+B(3p-1) &= p \\
Ap+2A+3Bp-B &= p
\end{aligned}
$$

Hence $2A-B=0$ and $(A+3B)=1$, therefore $A=\frac{B}{2}$. Hence $\left(\frac{B}{2}+3B\right)=1$ or $\frac{B+6B}{2}=1$ or $7B=2$ or $B=\frac{2}{7}$. Therefore $A=\frac{B}{2}=\frac{2}{14}$. Hence $\frac{3p}{(3p-1)(p+2)}=3\left(\frac{A}{(3p-1)}+\frac{B}{(p+2)}\right)=3\left(\frac{\frac{2}{14}}{(3p-1)}+\frac{\frac{2}{7}}{(p+2)}\right)$ or

$$
\frac{3}{7(3p-1)} + \frac{6}{7(p+2)} \tag{3}
$$

Combining (2) and (3) gives

$$
\begin{aligned}
F(p) &= \frac{6}{7(3p-1)} - \frac{2}{7(p+2)} + \frac{3}{7(3p-1)} + \frac{6}{7(p+2)} \\
&= \frac{6}{7}\frac{1}{(3p-1)} - \frac{2}{7}\frac{1}{(p+2)} + \frac{2}{7}\frac{1}{(3p-1)} + \frac{6}{7}\frac{1}{(p+2)} \\
&= \frac{4}{7}\frac{1}{(p+2)} + \frac{9}{7}\frac{1}{(3p-1)} \\
&= \frac{4}{7}\frac{1}{(p+2)} + \frac{9}{21}\frac{1}{\left(p-\frac{1}{3}\right)}
\end{aligned}
$$

Now we can use the table to find the inverse transform. Use property L2, which says

$$
\mathscr{L}\left(e^{-at}\right) = \frac{1}{p+a}
$$

Hence , setting $a=2$, gives $\mathscr{L}\left(e^{-2t}\right)=\frac{1}{p+2}$ and setting $a=-\frac{1}{3}$ gives $\mathscr{L}\left(e^{-\frac{1}{3}t}\right)^{\frac{1}{3}t}=\frac{1}{p-\frac{1}{3}}$. Hence $f(t)=\frac{4}{7}e^{-2t}+\frac{9}{21}e^{\frac{1}{3}t}$ and the inverse Laplace transform is

$$
f(t) = \frac{4}{7}e^{-2t} + \frac{3}{7}e^{\frac{1}{3}t}
$$

3.12.2 chapter 15, problem 2.17

Problem Use L32 and L11 to obtain $\mathscr{L}\left(t^2 \sin at\right)$

Solution

$$
\mathscr{L}\left(t^n f(t)\right) = (-1)^n \frac{d^n F(p)}{dp^n} \tag{L32}
$$

$$
\mathscr{L}(t \sin at) = \frac{2ap}{\left(p^2+a^2\right)^2} \tag{L11}
$$

we set $f(t) = t\ \sin at$ then we can write using L32

$$\mathscr{L}\left(t\ f(t)\right) = (-1)\frac{d\mathscr{L}\left(f(t)\right)}{dp} \tag{1}$$

But $\mathscr{L}\left(f(t)\right) = \mathscr{L}(t\ \sin at) = \frac{2ap}{\left(p^2+a^2\right)^2}$ from table L11 (1) becomes

$$\begin{aligned}
\mathscr{L}\left(t\ f(t)\right) &= -\frac{d}{dp}\left(\frac{2ap}{\left(p^2+a^2\right)^2}\right)\\
\mathscr{L}(t\times t\ \sin at) &= -\left(p\frac{-2\times 2a}{\left(p^2+a^2\right)^3}\times 2p + \frac{2a}{\left(p^2+a^2\right)^2}\times 1\right)\\
\mathscr{L}\left(t^2\ \sin at\right) &= \frac{8ap^2}{\left(p^2+a^2\right)^3} - \frac{2a}{\left(p^2+a^2\right)^2}\\
&= \frac{8ap^2 - 2a\left(p^2+a^2\right)}{\left(p^2+a^2\right)^3}\\
&= \frac{a\left(8p^2 - 2p^2 - 2a^2\right)}{\left(p^2+a^2\right)^3}\\
&= \frac{a\left(6p^2 - 2a^2\right)}{\left(p^2+a^2\right)^3}
\end{aligned}$$

Or

$$\mathscr{L}\left(t^2\ \sin at\right) = \frac{6ap^2 - 2a^3}{\left(p^2+a^2\right)^3}$$

3.12.3 chapter 15, problem 2.18

Problem Use L31 to derive L21

Solution

$$\mathscr{L}\left(\frac{f(t)}{t}\right) = \int_{u=p}^{\infty} F(u)du \tag{L31}$$

$$\mathscr{L}\left(\frac{e^{-at} - e^{-bt}}{t}\right) = \ln\frac{p+b}{p+a} \tag{L21}$$

we set $f(t) = e^{-at} - e^{-bt}$ then we can write using L31

$$\begin{aligned}
\mathscr{L}\left(\frac{f(t)}{t}\right) &= \int_{u=p}^{\infty} F(u)du\\
&= \int_{u=p}^{\infty} \mathscr{L}\left[f(t)\right]\ du
\end{aligned}$$

but $\mathscr{L}\left[f(t)\right] = \mathscr{L}\left(e^{-at} - e^{-bt}\right) = \mathscr{L}\left(e^{-at}\right) - \mathscr{L}\left(e^{-bt}\right) = \frac{1}{p+a} - \frac{1}{p+b}$ By using L2. But since we are using u in place of p in integral, we need to call $p = u$. Hence

$$
\begin{aligned}
\mathscr{L}\left(\frac{f(t)}{t}\right) &= \int_{u=p}^{\infty}\left(\frac{1}{u+a}-\frac{1}{u+b}\right)\,du \\
&= \int_{u=p}^{\infty}\frac{1}{u+a}\,du-\int_{u=p}^{\infty}\frac{1}{u+b}\,du \\
&= [\ln(u+a)]_p^{\infty}-[\ln(u+b)]_p^{\infty} \\
&= \left[\ln(\infty+a)-\ln\left(p+a\right)\right]-\left[\ln(\infty+b)-\ln\left(p+b\right)\right] \\
&= \ln(\infty)-\ln\left(p+a\right)-\ln(\infty)+\ln\left(p+b\right) \\
&= \ln\left(p+b\right)-\ln\left(p+a\right)
\end{aligned}
$$

but $\ln A - \ln B = \ln \frac{A}{B}$, therefore

$$
\begin{aligned}
\mathscr{L}\left(\frac{f(t)}{t}\right) &= \ln\left(p+b\right)-\ln\left(p+a\right) \\
\mathscr{L}\left(\frac{e^{-at}-e^{-bt}}{t}\right) &= \ln\left(\frac{p+b}{p+a}\right)
\end{aligned}
$$

3.12.4 chapter 15, problem 2.2

Problem Use relation L2 to find L7 and L8 in laplace table.

Solution

$$
\mathscr{L}\left(e^{-at}\right)=\frac{1}{p+a} \tag{L2}
$$

for $\Re\left(p+a\right)>0$

$$
\mathscr{L}\frac{e^{-at}-e^{-bt}}{b-a}=\frac{1}{\left(p+a\right)\left(p+b\right)} \tag{L7}
$$

$$
\mathscr{L}\frac{ae^{-at}-be^{-bt}}{a-b}=\frac{p}{\left(p+a\right)\left(p+b\right)} \tag{L8}
$$

For $\Re\left(p+a\right)>0$ and $\Re\left(p+b\right)>0$. Where $\mathscr{L}f(t)$ is the laplace transform of $f(t)$ defined as $\mathscr{L}f(t)=F(p)=\int_0^{\infty}e^{-pt}f(t)dt$.

From the linearity property of the $\mathscr{L}$ operator, expand the LHS of L7, we get

$$
\mathscr{L}\frac{e^{-at}-e^{-bt}}{b-a}=\frac{1}{b-a}\mathscr{L}\left(e^{-at}-e^{-bt}\right)=\frac{1}{b-a}\left(\mathscr{L}\left(e^{-at}\right)-\mathscr{L}\left(e^{-bt}\right)\right)
$$

Now applying L2 gives for $\Re\left(p+a\right)>0$ and $\Re\left(p+b\right)>0$

$$
\begin{aligned}
\mathscr{L}\frac{e^{-at}-e^{-bt}}{b-a} &= \frac{1}{b-a}\left(\frac{1}{p+a}-\frac{1}{p+b}\right)\\
&= \frac{1}{b-a}\left(\frac{p+b-\left(p+a\right)}{\left(p+a\right)\left(p+b\right)}\right)\\
&= \frac{1}{b-a}\left(\frac{b-a}{\left(p+a\right)\left(p+b\right)}\right)\\
&= \frac{1}{\left(p+a\right)\left(p+b\right)}
\end{aligned}
$$

For Which is L7 as required to show. Similarly for L8, expand the LHS of L8 we get for $\Re\left(p+a\right)>0$ and $\Re\left(p+b\right)>0$

$$
\begin{aligned}
\mathscr{L}\frac{ae^{-at}-be^{-bt}}{a-b} &= \frac{1}{a-b}\mathscr{L}\left(ae^{-at}-be^{-bt}\right)\\
&= \frac{1}{a-b}\left(\mathscr{L}\left(e^{-at}\right)-\mathscr{L}\left(e^{-bt}\right)\right)\\
&= \frac{1}{a-b}\left(a\mathscr{L}\left(e^{-at}\right)-b\mathscr{L}\left(e^{-bt}\right)\right)\\
&= \frac{1}{a-b}\left(a\frac{1}{p+a}-b\frac{1}{p+b}\right)\\
&= \frac{1}{a-b}\left(\frac{a\left(p+b\right)-b\left(p+a\right)}{\left(p+a\right)\left(p+b\right)}\right)\\
&= \frac{1}{a-b}\left(\frac{ap+ab-bp-ba}{\left(p+a\right)\left(p+b\right)}\right)\\
&= \frac{1}{a-b}\left(\frac{ap-bp}{\left(p+a\right)\left(p+b\right)}\right)\\
&= \frac{1}{a-b}\left(\frac{p(a-b)}{\left(p+a\right)\left(p+b\right)}\right)\\
&= \frac{p}{\left(p+a\right)\left(p+b\right)}
\end{aligned}
$$

Which is L8.

3.12.5 chapter 15, problem 2.21

Problem Use L29 and L11 to obtain $\mathscr{L}\left(te^{-at}\ \sin bt\right)$

Solution

$$
\mathscr{L}\left(e^{-at}\ f(t)\right) = F\left(p+a\right) \tag{L29}
$$

$$
\mathscr{L}(t\ \sin at) = \frac{2ap}{\left(p^2+a^2\right)^2} \tag{L11}
$$

Then from L11 we get

$$
\mathscr{L}(t\ \sin bt) = \frac{2bp}{\left(p^2+b^2\right)^2}
$$

Now, let $p = \left(p+a\right)$ then from L29, the above becomes

$$\mathscr{L}\left(e^{-at}\ t\ \sin bt\right) = \frac{2b\left(p+a\right)}{\left(\left(p+a\right)^2+b^2\right)^2}$$

3.12.6 chapter 15, problem 2.22

Problem similar to problem 2.21, Use L29 and L12 to obtain $\mathscr{L}\left(te^{-at}\ \cos bt\right)$

Solution

$$\mathscr{L}\left(e^{-at}\ f(t)\right) = F\left(p+a\right) \tag{L29}$$

$$\mathscr{L}(t\ \cos at) = \frac{p^2-a^2}{\left(p^2+a^2\right)^2} \tag{L12}$$

then from L12 we get

$$\mathscr{L}(t\ \cos bt) = \frac{p^2-b^2}{\left(p^2+b^2\right)^2}$$

Now, let $p = \left(p+a\right)$ then from L29, the above becomes

$$\mathscr{L}\left(e^{-at}\ t\ \cos bt\right) = \frac{\left(p+a\right)^2-b^2}{\left(\left(p+a\right)^2+b^2\right)^2}$$

3.12.7 chapter 15, problem 2.23

Problem use result obtained in problem 2.21 and 2.22 to find inverse transform for $\frac{p^2+2p-1}{\left(p^2+4p+5\right)^2}$

Solution

Recall, from 2.21 we showed that $\mathscr{L}\left(te^{-at}\ \sin bt\right) = \frac{2b\left(p+a\right)}{\left(\left(p+a\right)^2+b^2\right)^2}$ and from 2.22 $\mathscr{L}\left(te^{-at}\ \cos bt\right) = \frac{\left(p+a\right)^2-b^2}{\left(\left(p+a\right)^2+b^2\right)^2}$ Hence

$$\begin{aligned}\mathscr{L}\left(te^{-at}\ \cos bt\right) - \mathscr{L}\left(te^{-at}\ \sin bt\right) &= \frac{\left(p+a\right)^2-b^2}{\left(\left(p+a\right)^2+b^2\right)^2} - \frac{2b\left(p+a\right)}{\left(\left(p+a\right)^2+b^2\right)^2}\\ &= \frac{\left(p+a\right)^2+b^2\ -2b\left(p+a\right)}{\left(\left(p+a\right)^2+b^2\right)^2}\end{aligned}$$

Now Let $a = 2$ and let $b = 1$ we get

$$\mathscr{L}\left(te^{-2t}\ \cos t\right) - \mathscr{L}\left(te^{-2t}\ \sin t\right) = \frac{\left(p+2\right)^2 - 1^2 - 2\left(p+2\right)}{\left(\left(p+2\right)^2 + 1^2\right)^2}$$

$$\mathscr{L}\left(te^{-2t}\ \cos t - te^{-2t}\ \sin t\right) = \frac{p^2 + 4 + 4p - 1 - 2p - 4}{\left(\left(p+2\right)^2 + 1^2\right)^2}$$

$$= \frac{p^2 + 2p - 1}{\left(p^2 + 4p + 5\right)^2}$$

Hence inverse transform of $\frac{p^2+2p-1}{\left(p^2+2p+5\right)^2}$ is $te^{-2t}\ \cos t - te^{-2t}\ \sin t$ or $te^{-2t}\ (\cos t - \sin t)$

3.12.8 chapter 15, problem 2.3

Problem Using either relation L2 or L3 and L4, verify L9 and L10 in laplace table.

Solution

$$\mathscr{L}\left(e^{-at}\right) = \frac{1}{p+a} \qquad \mathrm{Re}\left(p+a\right) > 0 \tag{L2}$$

$$\mathscr{L}\sin at = \frac{a}{p^2+a^2} \qquad \mathrm{Re}(p) > |\mathrm{Im}\, a| \tag{L3}$$

$$\mathscr{L}\cos at = \frac{p}{p^2+a^2} \qquad \mathrm{Re}(p) > |\mathrm{Im}\, a| \tag{L4}$$

$$\mathscr{L}\sinh at = \frac{a}{p^2-a^2} \qquad \mathrm{Re}(p) > |\ \mathrm{Re}(a)| \tag{L9}$$

$$\mathscr{L}\cosh at = \frac{p}{p^2-a^2} \qquad \mathrm{Re}(p) > |\ \mathrm{Re}(a)| \tag{L10}$$

Where $\mathscr{L}\left(f(t)\right)$ is the laplace transform of $f(t)$ defined as $\mathscr{L}\left(f(t)\right) = Y(s) = \int_0^\infty e^{-pt} f(t)dt$. To derive L9, use the relation that

$$i \sinh(x) = \sin(ix)$$

Hence, using L3, we get

$$\mathscr{L}(i \sinh at) = \mathscr{L}(\sin iat\ )$$

$$i\mathscr{L}(\sinh at) = \frac{ia}{p^2 + (ia)^2} \qquad \mathrm{Re}(p) > |\mathrm{Im}\, a|$$

$$\mathscr{L}(\sinh at) = \frac{a}{p^2 - a^2} \qquad \mathrm{Re}(p) > |\ \mathrm{Re}(a)|$$

which is L9. To find L10, use the relation

$$\cosh(x) = \cos(ix)$$

And using L4, we get

$$\begin{aligned}
\mathscr{L}(\cosh(at)) &= \mathscr{L}(\cos(iat)\,) \\
\mathscr{L}(\cosh at) &= \frac{p}{p^2 + (ia)^2} && \operatorname{Re}(p) > |\operatorname{Im} a| \\
\mathscr{L}(\sinh at) &= \frac{p}{p^2 - a^2} && \operatorname{Re}(p) > |\operatorname{Re}(a)|
\end{aligned}$$

Which is L10.

3.12.9 chapter 15, problem 2.4

Problem by differentiating the appropriate formulas w.r.t. 'a', verify L12

Solution

L12 is

$$\mathscr{L}(t\ \cos t) = \frac{p^2 - a^2}{\left(p^2 + a^2\right)^2} \qquad \operatorname{Re}\left(p\right) > |\operatorname{Im} a| \tag{L12}$$

Where $\mathscr{L}\left(f(t)\right)$ is the laplace transform of $f(t)$ defined as $\mathscr{L}\left(f(t)\right) = F(p) = \int_0^\infty e^{-pt} f(t) dt$. To derive this, I start with L3, which says

$$\mathscr{L}(\sin at) = \frac{a}{p^2 + a^2} \qquad \operatorname{Re}(p) > |\operatorname{Im} a|$$

The above can be rewritten in the full definition of the transform to make it easier to see

$$\int_0^\infty e^{-pt}\ \sin at\ \ dt = \frac{a}{p^2 + a^2}$$

Taking derivative of both sides w.r.t. a gives

$$\begin{aligned}
\frac{d}{da}\left[\int_0^\infty e^{-pt}\ \sin at\ \ dt\right] &= \frac{d}{da}\left[\frac{a}{p^2 + a^2}\right] \\
\int_0^\infty \frac{d}{da}\left[e^{-pt}\ \sin at\right]\ dt &= a\left(\frac{-2a}{\left(p^2 + a^2\right)^2}\right) + \frac{1}{p^2 + a^2} \times 1 \\
\int_0^\infty e^{-pt}\ t \cos at\ \ dt &= \frac{-2a^2}{\left(p^2 + a^2\right)^2} + \frac{1}{p^2 + a^2} \\
\mathscr{L}(t\ \cos at) &= \frac{-2a^2 + \left(p^2 + a^2\right)}{\left(p^2 + a^2\right)^2} \\
&= \frac{p^2 - a^2}{\left(p^2 + a^2\right)^2}
\end{aligned}$$

Which is L12

3.12.10 chapter 15, problem 2.5

Problem by integrating the appropriate formulas w.r.t. 'a', verify L19

Solution L19 is

$$\mathscr{L}\left(\frac{\sin at}{t}\right) = \arctan\frac{a}{p} \qquad \operatorname{Re}(p) > |\operatorname{Im} a| \tag{L19}$$

Where $\mathscr{L}\left(f(t)\right)$ is the laplace transform of $f(t)$ defined as $\mathscr{L}\left(f(t)\right) = F(p) = \int_0^\infty e^{-pt} f(t) dt$. To derive this, we start with L4, which says

$$\mathscr{L}(\cos at) = \frac{p}{p^2 + a^2} \qquad \operatorname{Re}(p) > |\operatorname{Im} a|$$

The above can be rewritten in the full definition of the transform to make it easier to see

$$\int_0^\infty e^{-pt} \cos at \; dt = \frac{p}{p^2 + a^2}$$

Integrating both sides w.r.t. a gives

$$\begin{aligned}
\int\left(\int_{t=0}^{t=\infty} e^{-pt} \cos at \; dt\right)da &= \int \frac{p}{p^2 + a^2} da \\
\int_{t=0}^{t=\infty}\left(\int e^{-pt} \cos at \, da\right) dt &= \arctan\frac{a}{p} + K \\
\int_{t=0}^{t=\infty} e^{-pt}\left(\int \cos at \, da\right) dt &= \arctan\frac{a}{p} + K
\end{aligned}$$

Now, since $\int \cos at \, da = \frac{\sin at}{t} + k$ and choose zero for values of the K's, the constants of integration gives

$$\int_0^\infty e^{-pt}\left(\frac{\sin at}{t}\right) dt = \arctan\frac{a}{p}$$

Hence

$$\mathscr{L}\left(\frac{\sin at}{t}\right) = \arctan\frac{a}{p}$$

Which is L19

3.12.11 chapter 15, problem 2.9

Problem Find the inverse transform of the function $F\left(p\right) = \frac{5-2p}{p^2+p-2}$

Solution

Need to simplify the above expression to some expressions which are shown in the table on page 636.

$$\begin{aligned}
F(p) &= \frac{5 - 2p}{p^2 + p - 2} \\
&= \frac{5}{p^2 + p - 2} - \frac{2p}{p^2 + p - 2} \\
&= \frac{5}{\left(p - 1\right)\left(p + 2\right)} - \frac{2p}{\left(p - 1\right)\left(p + 2\right)}
\end{aligned} \tag{1}$$

From table, L7, we see that $\mathscr{L}\left(\frac{e^{-at}-e^{-bt}}{b-a}\right) = F(p) = \frac{1}{(p+a)(p+b)}$. Setting $a = -1, b = 2$ we get the first expression in (1), that is

$$\mathscr{L}\left(\frac{e^{t} - e^{-2t}}{3}\right) = \frac{1}{(p-1)(p+2)} \tag{2}$$

also from table, we see that L8 is $\mathscr{L}\left(\frac{ae^{-at}-be^{-bt}}{a-b}\right) = F(p) = \frac{p}{(p+a)(p+b)}$. Hence, letting $a = -1, b = 2$ we get the second expression in (1), that is

$$\mathscr{L}\left(\frac{-e^{+t} - 2e^{-2t}}{-3}\right) = \frac{p}{(p-1)(p+2)} \tag{3}$$

So combine (2) and (3) we get

$$5\mathscr{L}\left(\frac{e^{+t} - e^{-2t}}{2+1}\right) - 2\mathscr{L}\left(\frac{-e^{+t} - 2e^{-2t}}{-1-2}\right) = \frac{5}{(p-1)(p+2)} - \frac{2p}{(p-1)(p+2)}$$

which is (1). Hence

$$\begin{aligned} f(t) &= 5\frac{e^{+t} - e^{-2t}}{3} - 2\frac{-e^{+t} - 2e^{-2t}}{-3} \\ &= 5\frac{e^{+t} - e^{-2t}}{3} + 2\frac{-e^{+t} - 2e^{-2t}}{3} \\ &= \frac{5e^{+t} - 5e^{-2t} - 2e^{+t} - 4e^{-2t}}{3} \\ &= \frac{3e^{t} - 9e^{-2t}}{3} \\ &= e^{t} - 3e^{-2t} \end{aligned}$$

Hence the inverse transform of $F(p)$ is $e^t - 3e^{-2t}$

3.12.12 chapter 15, problem 3.11

Problem Use laplace transform to solve $y'' - 4y = 4e^{2t}$,$y_0 = 0, y'_0 = 1$

Solution

Let $\mathscr{L}(y(t)) = Y(p)$, and taking the laplace transform of both sides, noting first that $\mathscr{L}(y'') = p^2Y - py_0 - y'_0$ then we get

$$\begin{aligned} \mathscr{L}(y'') - 4\mathscr{L}(y(t)) &= \mathscr{L}(4e^{2t}) \\ (p^2Y - py_0 - y'_0) - 4Y &= 4\mathscr{L}(e^{2t}) \end{aligned}$$

L2 from table on page 636 : $\mathscr{L}(e^{-at}) = \frac{1}{p+a}$, hence $\mathscr{L}(e^{2t}) = \frac{1}{p-2}$, hence, after applying boundary conditions, we get

$$
\begin{aligned}
\left(p^2Y - 1\right) - 4Y &= 4\frac{1}{p-2} \\
Y\left(p^2-4\right) - 1 &= 4\frac{1}{p-2} \\
Y &= 4\frac{1}{\left(p^2-4\right)\left(p-2\right)} + \frac{1}{\left(p^2-4\right)} \\
Y &= 4\frac{1}{\left(p-2\right)\left(p+2\right)\left(p-2\right)} + \frac{1}{\left(p^2-4\right)} \\
Y &= 4\frac{1}{\left(p+2\right)\left(p-2\right)^2} + \frac{1}{\left(p^2-4\right)}
\end{aligned}
$$

Doing partial fractions, repeated roots, gives

$$
\begin{aligned}
\frac{1}{\left(p+2\right)\left(p-2\right)^2} &= \frac{A}{\left(p+2\right)} + \frac{B}{\left(p-2\right)} + \frac{C}{\left(p-2\right)^2} \\
1 &= A\left(p-2\right)^2 + B\left(p+2\right)\left(p-2\right) + C\left(p+2\right) \\
1 &= A\left(p^2-4p+4\right) + B\left(p^2-4\right) + C\left(p+2\right) \\
1 &= p^2(A+B) + p(-4A+C) + 4A - 4B + 2C
\end{aligned}
$$

Hence

$$
\begin{aligned}
A + B &= 0 \\
-4A + C &= 0 \\
4A - 4B + 2C &= 1
\end{aligned}
$$

hence $A = -B$, then $-4B - 4B + 2C = 1$ or $-8B + 2C = 1$ or $C = \frac{1+8B}{2}$, therefore $4B + \frac{1+8B}{2} = 0$ then $B = -\frac{1}{16}$, then $A = \frac{1}{16}, C = \frac{1+8\left(-\frac{1}{16}\right)}{2} = \frac{1-\left(\frac{1}{2}\right)}{2} = \frac{1}{4}$. Hence

$$
\begin{aligned}
Y &= 4\frac{1}{\left(p+2\right)\left(p-2\right)^2} + \frac{1}{\left(p^2-4\right)} \\
&= 4\left(\frac{A}{\left(p+2\right)} + \frac{B}{\left(p-2\right)} + \frac{C}{\left(p-2\right)^2}\right) + \frac{1}{\left(p^2-4\right)} \\
&= 4\left(\frac{\frac{1}{16}}{\left(p+2\right)} + \frac{-\frac{1}{16}}{\left(p-2\right)} + \frac{\frac{1}{4}}{\left(p-2\right)^2}\right) + \frac{1}{\left(p^2-4\right)} \\
&= \frac{1}{4}\frac{1}{\left(p+2\right)} - \frac{1}{4}\frac{1}{\left(p-2\right)} + \frac{1}{\left(p-2\right)^2} + \frac{1}{\left(p^2-4\right)} \\
&= \frac{1}{4}\frac{1}{\left(p+2\right)} - \frac{1}{4}\frac{1}{\left(p-2\right)} + \frac{1}{\left(p-2\right)^2} + \left(\frac{1}{4}\frac{1}{\left(p-2\right)} - \frac{1}{4}\frac{1}{\left(p+2\right)}\right) \\
&= \frac{1}{\left(p-2\right)^2}
\end{aligned}
$$

$\frac{1}{\left(p-2\right)^2} \rightarrow$ using L6, we have $\mathscr{L}\left(t^k e^{-at}\right) = \frac{k!}{\left(p+a\right)^{k+1}}$, here $a = -2, k = 1$ and $\frac{1}{\left(p-2\right)^2} \rightarrow te^{2t}$.
Hence ,

$$f(t) = te^{2t}$$

3.12.13 chapter 15, problem 3.24

Problem Use laplace transform to solve $y'' - 2y' + y = 2\cos t$,$y_0 = 5, y_0' = -2$

Solution

Let $\mathscr{L}\left(y(t)\right) = Y(p)$, Take the laplace transform of both sides, noting first that $\mathscr{L}\left(y''\right) = p^2Y - py_0 - y_0'$, $\mathscr{L}\left(y'\right) = pY - y_0$then we get

$$\begin{aligned}
\mathscr{L}\left(y''\right) - 2\mathscr{L}\left(y'\right) + \mathscr{L}\left(y(t)\right) &= \mathscr{L}(2\cos t) \\
\left(p^2Y - py_0 - y_0'\right) - 2\left(pY - y_0\right) + Y &= 2\frac{p}{p^2+1}
\end{aligned}$$

$$\begin{aligned}
\left(p^2Y - 5p + 2\right) - 2\left(pY - 5\right) + Y &= 2\frac{p}{p^2+1} \\
Y\left(p^2 - 2p + 1\right) + 12 - 5p &= 2\frac{p}{p^2+1} \\
Y\left(p^2 - 2p + 1\right) &= \frac{2p}{p^2+1} - 12 + 5p \\
Y &= \frac{\frac{2p}{p^2+1} - 12 + 5p}{\left(p^2 - 2p + 1\right)} \\
Y &= \frac{2p}{\left(p^2+1\right)\left(p^2 - 2p + 1\right)} + \frac{5p - 12}{\left(p^2 - 2p + 1\right)} \\
Y &= \frac{2p}{\left(p^2+1\right)\left(p-1\right)^2} + \frac{5p - 12}{\left(p-1\right)^2}
\end{aligned}$$

Doing partial fractions $\frac{2p}{\left(p-1\right)^2\left(p^2+1\right)} = \frac{A}{\left(p-1\right)} + \frac{B}{\left(p-1\right)^2} + \frac{Cp+D}{\left(p^2+1\right)}$. Solving, we get $A = 0, B = 1, C = 0, D = -1$ Hence

$$\begin{aligned}
Y &= \frac{1}{\left(p-1\right)^2} - \frac{1}{\left(p^2+1\right)} + \frac{5p - 12}{\left(p-1\right)^2} \\
&= \frac{1}{\left(p-1\right)^2} - \frac{1}{\left(p^2+1\right)} + \frac{5p}{\left(p-1\right)^2} - 12\frac{1}{\left(p-1\right)^2} \\
&= \frac{-11}{\left(p-1\right)^2} - \frac{1}{\left(p^2+1\right)} + \frac{5p}{\left(p-1\right)^2}
\end{aligned}$$

Hence , using table, we get inverse laplace transform $\frac{1}{\left(p-1\right)^2} \rightarrow te^t$ and $\frac{1}{\left(p^2+1\right)} \rightarrow \sin t$ and $\frac{p}{\left(p-1\right)^2} \rightarrow e^t + te^t$, hence

$$\begin{aligned}
f(t) &= -11te^t - \sin t + 5\left(e^t + te^t\right) \\
&= -11te^t - \sin t + 5e^t + 5te^t \\
&= 5e^t - 6te^t - \sin t
\end{aligned}$$

3.12.14 chapter 15, problem 3.25

Problem Use laplace transform to solve $y'' \ + 4y' + 5y = 2e^{-2t}\cos t$, $,y_0 = 0, y'_0 = 3$

Solution

Let $\mathscr{L}\big(y(t)\big) = Y(p)$, Taking the laplace transform of both sides, noting first that $\mathscr{L}\big(y''(t)\big) = p^2Y - py_0 - y'_0$, $\mathscr{L}\big(y'(t)\big) = pY - y_0$then we get

$$\begin{aligned}\mathscr{L}\big(y''(t)\big) + 4\mathscr{L}\big(y'(t)\big) + 5\mathscr{L}\big(y(t)\big) &= \mathscr{L}\big(2e^{-2t}\cos t\big)\\ \big(p^2Y - py_0 - y'_0\big) \ + 4\big(pY - y_0\big) + 5Y &= 2\mathscr{L}\big(e^{-2t}\cos t\big)\end{aligned}$$

Applying initial conditions

$$\big(p^2Y \ - 3\big) \ + 4\big(pY\big) + 5Y = 2\mathscr{L}\big(e^{-2t}\cos t\big)$$

From table using L14 $\mathscr{L}\big(e^{-2t}\cos t\big) = \frac{p+2}{(p+2)^2+1}$. Hence

$$\begin{aligned}\big(p^2Y \ - 3\big) \ + 4\big(pY\big) + 5Y &= 2\frac{p+2}{\big(p+2\big)^2 + 1}\\ Y\big(p^2 + 4p + 5\big) - 3 &= 2\frac{p+2}{\big(p+2\big)^2 + 1}\\ Y &= \frac{2\frac{p+2}{(p+2)^2+1} + 3}{\big(p^2 + 4p + 5\big)}\\ Y &= 2\frac{p+2}{\Big(\big(p+2\big)^2 + 1\Big)\big(p^2 + 4p + 5\big)} + 3\frac{1}{\big(p^2 + 4p + 5\big)}\end{aligned}$$

but $\big(p^2 + 4p + 5\big) = \big(p+2\big)^2 + 1$, therefore

$$\begin{aligned}Y &= 2\frac{p+2}{\Big(\big(p+2\big)^2 + 1\Big)^2} + \frac{3}{\big(p+2\big)^2 + 1}\\ &= 2\frac{p+2}{\Big(\big(p+2\big)^2 + 1\Big)\Big(\big(p+2\big)^2 + 1\Big)} + \frac{3}{\big(p+2\big)^2 + 1}\end{aligned}$$

But inverse transform of $\frac{p+2}{\left((p+2)^2+1\right)^2} = \frac{1}{2}te^{-2t}\sin t$ and inverse transform of $\frac{1}{(p+2)^2+1} = e^{-2t}\sin t$. Hence

$$\begin{aligned}f(t) &= 2\left(\frac{1}{2}te^{-2t}\sin t\right) + 3\big(e^{-2t}\sin t\big)\\ &= te^{-2t}\sin t + 3e^{-2t}\sin t\\ &= (t+3)e^{-2t} \ \sin t\end{aligned}$$

3.12.15 chapter 15, problem 3.29

Problem Use laplace transform to solve $y' \ + z' - 2y = 1, y_0 = z_0 = 1, z - y' = t$

Solution

Taking laplace transform of both equations, then we get 2 equations in Y and Z, then solve for them. Let $\mathscr{L}\left(y(t)\right) = Y(p)$, $\mathscr{L}(z(t)) = Z(p)$

$$\begin{aligned} \mathscr{L}\left(y'(t)\right) \ + \mathscr{L}(z'(t)) - 2\mathscr{L}\left(y(t)\right) &= \mathscr{L}(1) \\ \mathscr{L}(z(t)) \ - \mathscr{L}\left(y'(t)\right) &= \mathscr{L}(t) \end{aligned}$$

Then we get

$$\begin{aligned} \left(pY - y_0\ \right) + \left(pZ - z_0\ \right) - 2Y &= \frac{1}{p} \\ Z\ - \left(pY - y_0\ \right) \ &= \frac{1}{p^2} \end{aligned}$$

Putting initial conditions gives

$$\begin{aligned} \left(pY - 1\ \right) + \left(pZ - 1\ \right) - 2Y &= \frac{1}{p} \\ Z\ - \left(pY - 1\ \right) \ &= \frac{1}{p^2} \end{aligned}$$

$$Y\left(p-2\right) + \ pZ - 2 \ = \frac{1}{p} \tag{1}$$

$$Z\ - pY + 1 \ = \frac{1}{p^2} \tag{2}$$

Solving for Y,From (1), $Z = \frac{\frac{1}{p}+2-Y\left(p-2\right)}{p}$, and substituting into (2) gives

$$\begin{aligned} \frac{\frac{1}{p} + 2 - Y\left(p-2\right)}{p} - pY + 1 \ &= \frac{1}{p^2} \\ \frac{1}{p} + 2 - Y\left(p-2\right) - p^2 Y + p &= \frac{1}{p} \\ 2 - Y\left(p-2\right) - p^2 Y \ &= -p \\ Y\left(-p + 2 - p^2\right) \ &= \ -p - 2 \\ Y \ &= \ \frac{-p-2}{\left(-p+2-p^2\right)} \\ Y \ &= \ -\frac{p+2}{\left(-p+1\right)\left(p+2\right)} \\ Y &= -\frac{1}{\left(-p+1\right)} \end{aligned}$$

Hence $Y = \frac{1}{p-1}$ so from L2

$$y(t) = e^t$$

Now, that we have Y, we solve for Z. From (2)

$$
\begin{aligned}
Z - pY + 1 &= \frac{1}{p^2} \\
Z - p\left(\frac{1}{p-1}\right) + 1 &= \frac{1}{p^2} \\
Z &= \frac{1}{p^2} - 1 + \frac{p}{p-1} \\
Z &= \frac{p - 1 - p^2\left(p-1\right) + p^3}{p^2\left(p-1\right)} \\
Z &= \frac{p - 1 + p^2}{p^2\left(p-1\right)} \\
Z &= \frac{p - 1 + p^2}{p^2\left(p-1\right)}
\end{aligned}
$$

Doing partial fraction on the above, we get Z = $\frac{1}{p^2} + \frac{1}{p-1}$, Hence

$$z(t) = t + e^t$$

3.12.16 chapter 15, problem 3.30

Problem Use laplace transform to solve $y' + 2z = 1, y_0 = 0,\ 2y - z' = 2t, z_0 = 1$

Solution

Take laplace transform of both equations, then we get 2 equations in Y and Z, then solve for them.

Let $\mathscr{L}\left(y(t)\right) = Y(p)$, $\mathscr{L}(z(t)) = Z(p)$

$$
\begin{aligned}
\mathscr{L}\left(y'(t)\right) + 2Z &= \mathscr{L}(1) \\
2Y - \mathscr{L}(z'(t)) &= \mathscr{L}(2t)
\end{aligned}
$$

$$
\begin{aligned}
pY - y_0 + 2Z &= \mathscr{L}(1) \\
2Y - \left(pZ - z_0\right) &= \mathscr{L}(2t)
\end{aligned}
$$

Then we get, by putting $z_0 = 1, y_0 = 0$

$$pY + 2Z = \frac{1}{p} \tag{1}$$

$$2Y - \left(pZ - 1\right) = \frac{2}{p^2} \tag{2}$$

Obtain Z from first equation and sub into the second to solve for Y, $Z = \frac{\frac{1}{p} - pY}{2}$, Hence

$$
\begin{aligned}
2Y - \left(p\left(\frac{\frac{1}{p} - pY}{2}\right) - 1\right) &= \frac{2}{p^2}\\
2Y - \frac{p}{2}\left(\frac{1}{p} - pY\right) + 1 &= \frac{2}{p^2}\\
2Y - \frac{p}{2}\left(\frac{1 - p^2 Y}{p}\right) &= \frac{2}{p^2} - 1\\
2Y - \frac{1}{2}\left(1 - p^2 Y\right) &= \frac{2 - p^2}{p^2}\\
2Y - \frac{1}{2} + \frac{1}{2}p^2 Y &= \frac{2 - p^2}{p^2}\\
Y\left(2 + \frac{p^2}{2}\right) &= \frac{2 - p^2}{p^2} + \frac{1}{2}\\
Y &= \frac{\frac{4-p^2}{2p^2}}{\left(\frac{4+p^2}{2}\right)}\\
Y &= \frac{4 - p^2}{p^2\left(4 + p^2\right)}
\end{aligned}
$$

Hence , using partial fraction gives

$$
\begin{aligned}
Y &= \frac{4 + p^2}{p^2\left(4 + p^2\right)}\\
&= \frac{Ap + B}{p^2} + \frac{Cp + D}{\left(4 + p^2\right)}
\end{aligned}
$$

Then

$$
\begin{aligned}
\left(Ap + B\right)\left(4 + p^2\right) + \left(Cp + D\right)p^2 &= 4 - p^2\\
4Ap + Ap^3 + 4B + Bp^2 + Cp^3 + Dp^2 &= 4 - p^2\\
p^3(A + C) + p^2(B + D) + p(4A) + 4B &= 4 - p^2
\end{aligned}
$$

Hence, $4B = 4 \to B = 1$ and $4A = 0 \to A = 0$ and $B + D = -1$ and $A + C = 0 \to C = 0$, therefore $D = -1 - B \to D = -2$. Hence

$$
\begin{aligned}
Y &= \frac{Ap + B}{p^2} + \frac{Cp + D}{\left(4 + p^2\right)}\\
&= \frac{1}{p^2} - \frac{2}{\left(4 + p^2\right)}
\end{aligned}
$$

Using tables for inverse transform gives

$$
y(t) = t - \sin 2t
$$

Now, to find $z(t)$, subtituting value we found for Y into equation (1) above.

$$
\begin{aligned}
pY \quad + 2Z &= \frac{1}{p} \\
p\left(\frac{1}{p^2} - \frac{2}{\left(4+p^2\right)}\right) \quad + 2Z &= \frac{1}{p} \\
\frac{1}{p} - \frac{2p}{\left(4+p^2\right)} \quad + 2Z &= \frac{1}{p} \\
Z &= \frac{p}{\left(4+p^2\right)} \\
Z &= \frac{p}{\left(4+p^2\right)}
\end{aligned}
$$

From tables, using L4

$$z(t) = \cos 2t$$

3.12.17 chapter 15, problem 3.4

Problem Use laplace transform to solve $y'' + y = \sin t$,$y_0 = 1, y_0' = 0$

Solution

Let $\mathscr{L}\big(y(t)\big) = Y(p)$, Take the laplace transform of both sides, noting first that $\mathscr{L}\big(y''(t)\big) = p^2 Y - p y_0 - y_0'$ then we get

$$
\begin{aligned}
\mathscr{L}\big(y''(t)\big) + \mathscr{L}\big(y(t)\big) &= \mathscr{L}(\sin t) \\
\left(p^2 Y - p y_0 - y_0'\right) + Y &= \frac{1}{p^2+1}
\end{aligned}
$$

Where I used L3 from table on page 636 which says that $\mathscr{L}(\sin at) = \frac{a}{p^2+a^2}$. Now solving for Y, noting that $y_0 = 1$ and $y_0' = 0$ gives

$$
\begin{aligned}
\left(p^2 Y - p\right) + Y &= \frac{1}{p^2+1} \\
Y\left(p^2+1\right) \; - p &= \frac{1}{p^2+1} \\
Y\left(p^2+1\right) &= \frac{1}{p^2+1} + p \\
Y &= \frac{1}{\left(p^2+1\right)^2} + \frac{p}{\left(p^2+1\right)} \qquad (1)
\end{aligned}
$$

From table using L12, $\frac{p}{\left(p^2+1\right)} \to \cos t$ and using L17, $\frac{1}{\left(p^2+1\right)^2} \to \frac{1}{2}(\sin t - t\cos t)$. Hence, putting these together into (1) gives

$$f(t) = \cos t + \frac{1}{2}(\sin t - t\cos t)$$

This is the particular solution to the ODE.

3.12.18 chapter 15, problem 3.6

Problem Use laplace transform to solve $y'' - 6y' + 9y = te^{3t}$,$y_0 = 0, y_0' = 5$

Solution

Let $\mathscr{L}\big(y(t)\big) = Y(p)$, Take the laplace transform of both sides, noting first that $\mathscr{L}\big(y''(t)\big) = p^2Y - py_0 - y_0'$ and $\mathscr{L}\big(y'(t)\big) = pY - y_0$ then we get

$$\mathscr{L}\big(y''(t)\big) - 6\mathscr{L}\big(y'(t)\big) + 9\mathscr{L}\big(y(t)\big) = \mathscr{L}\big(t\, e^{3t}\big)$$
$$\big(p^2Y - py_0 - y_0'\big) - 6\big(pY - y_0\big) + 9Y = \mathscr{L}\big(t\, e^{3t}\big)$$

I use L6 from table on page 636 which says that $\mathscr{L}\big(t^k\, e^{-at}\big) = \frac{k!}{(p+a)^{k+1}}$, hence for $k = 1, a = -3$, we get $\mathscr{L}\big(t^k\, e^{-at}\big) = \frac{1}{(p-3)^2}$

$$\big(p^2Y - py_0 - y_0'\big) - 6\big(pY - y_0\big) + 9Y = \frac{1}{(p-3)^2}$$

Applying boundary conditions gives

$$\begin{aligned}
\big(p^2Y - 5\big) - 6\big(pY\big) + 9Y &= \frac{1}{(p-3)^2}\\
Y\big(p^2 - 6p + 9\big) - 5 &= \frac{1}{(p-3)^2}\\
Y\big(p^2 - 6p + 9\big) &= \frac{1}{(p-3)^2} + 5\\
Y &= \frac{1}{\big(p^2 - 6p + 9\big)(p-3)^2} + \frac{5}{\big(p^2 - 6p + 9\big)}\\
Y &= \frac{1}{(p-3)^2(p-3)^2} + \frac{5}{(p-3)^2}\\
Y &= \frac{1}{(p-3)^4} + \frac{5}{(p-3)^2} \qquad (1)
\end{aligned}$$

Now using table, from L6, $\frac{1}{(p-3)^2}$, let $a = -3, k = 1$ hence $\frac{1}{(p-3)^2} \to te^{3t}$ And using L6 again, $\frac{1}{(p-3)^4}$, let $k = 3, a = -3$ then $\frac{6}{(p-3)^4} \to t^3e^{3t}$, therefore (1) becomes

$$\begin{aligned}
f(t) &= \frac{1}{6}t^3e^{3t} + 5\, te^{3t}\\
&= e^{3t}\left(\frac{1}{6}t^3 + 5t\right)
\end{aligned}$$

3.12.19 chapter 15, problem 3.8

Problem Use laplace transform to solve $y'' + 16y = 8\cos 4t$,$y_0 = 0, y_0' = 0$

Solution

Let $\mathscr{L}\big(y(t)\big) = Y(p)$, Taking the laplace transform of both sides, noting first that $\mathscr{L}\big(y''(t)\big) = p^2Y - py_0 - y_0'$ results in

$$\begin{aligned}\mathscr{L}\left(y''(t)\right) + 16\mathscr{L}\left(y(t)\right) &= \mathscr{L}(8\cos 4t)\\ \left(p^2Y - py_0 - y_0'\right) + 16Y &= 8\mathscr{L}(\cos 4t)\\ Y\left(p^2+16\right) &= 8\frac{p}{p^2+16}\end{aligned}$$

I used L6 from table on page 636 : $\mathscr{L}(\cos at) = \frac{p}{p^2+a^2}$

$$Y = 8\frac{p}{\left(p^2+16\right)^2}$$

Now looking at L11, which says $\mathscr{L}(t\ \sin at) = \frac{2ap}{\left(p^2+a^2\right)^2}$, hence letting $a = 4$ gives the solution

$$f(t) = t\ \sin 4t$$

3.13 HW 12

Local contents

3.13.1 chapter 15, problem 4.12

Problem Find the exponential Fourier transform of the given $f(x)$ and write $f(x)$ as a fourier integral.

$$f(x)=\begin{cases}\sin x & |x|<\frac{\pi}{2}\\ \\ 0, & |x|>\frac{\pi}{2}\end{cases}$$

Solution

Let $\mathsf{F}(\alpha)$ be the Fourier transform of $f(x)$ defined as $\mathsf{F}(\alpha)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}f(x)\,e^{-i\alpha x}\,dx$

$$\mathsf{F}(\alpha)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}f(x)\,e^{-i\alpha x}\,dx$$
$$2\pi\ \mathsf{F}(\alpha)=\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin x\ \ e^{-i\alpha x}\,dx$$

Integration by parts, $u=\sin x, du=\cos x, v=\frac{e^{-i\alpha x}}{-i\alpha}$, hence $\int u\,dv\ =uv-\int du\ v$

$$
\begin{aligned}
I &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \; e^{-i\alpha x}\,dx \\
&= \left[\sin x \, \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, \frac{e^{-i\alpha x}}{-i\alpha}\,dx \\
&= \left[\sin x \, \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\alpha}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \; e^{-i\alpha x}\,dx
\end{aligned}
$$

Integration by parts the second integral again. $u = \cos x, du = -\sin x$

$$
\begin{aligned}
I &= \left[\sin x \, \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\alpha}\left\{\left[\cos x \, \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-\sin x) \, \frac{e^{-i\alpha x}}{-i\alpha}\,dx\right\} \\
I &= \left[\sin x \, \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\alpha}\left\{\left[\cos x \, \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \left(\frac{1}{i\alpha}\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \; e^{-i\alpha x}\,dx\right\} \\
I &= \left[\sin x \, \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\alpha}\left[\cos x \, \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{1}{i^2\alpha^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \; e^{-i\alpha x}\,dx \\
I &= \left[\sin x \, \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\alpha}\left[\cos x \, \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{\alpha^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \; e^{-i\alpha x}\,dx
\end{aligned}
$$

But the last integral on the right above is the same as the integral we start with, so

$$
\begin{aligned}
I &= \left[\sin x \, \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\alpha}\left[\cos x \, \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{\alpha^2} I \\
I - \frac{1}{\alpha^2} I &= \left[\sin x \, \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{i}{\alpha}\left[\cos x \, \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
I\left(1 - \frac{1}{\alpha^2}\right) &= \left[\sin\left(\frac{\pi}{2}\right) \frac{e^{-i\alpha\frac{\pi}{2}}}{-i\alpha} - \sin\left(-\frac{\pi}{2}\right) \frac{e^{-i\alpha\left(-\frac{\pi}{2}\right)}}{-i\alpha}\right] - \frac{i}{\alpha}\left[\cos\left(\frac{\pi}{2}\right) \frac{e^{-i\alpha\frac{\pi}{2}}}{-i\alpha} - \cos\left(-\frac{\pi}{2}\right) \frac{e^{-i\alpha\left(-\frac{\pi}{2}\right)}}{-i\alpha}\right] \\
I\left(\frac{\alpha^2 - 1}{\alpha^2}\right) &= \left[\frac{e^{-i\alpha\frac{\pi}{2}}}{-i\alpha} + \frac{e^{i\alpha\left(\frac{\pi}{2}\right)}}{-i\alpha}\right] - \frac{i}{\alpha}(0) \\
I &= \left(\frac{\alpha^2}{\alpha^2 - 1}\right)\left[\frac{-e^{-i\alpha\frac{\pi}{2}}}{i\alpha} - \frac{e^{i\alpha\frac{\pi}{2}}}{i\alpha}\right] \\
I &= \left(\frac{\alpha}{\alpha^2 - 1}\right)\frac{-1}{i}\left[e^{i\alpha\frac{\pi}{2}} + e^{-i\alpha\frac{\pi}{2}}\right] \\
I &= \left(\frac{\alpha i}{\alpha^2 - 1}\right)\left[e^{i\alpha\frac{\pi}{2}} + e^{-i\alpha\frac{\pi}{2}}\right] \\
I &= \left(\frac{\alpha i}{\alpha^2 - 1}\right) 2\cos\left(\alpha\frac{\pi}{2}\right)
\end{aligned}
$$

Hence the Fourier transform of $f(x)$ is

$$\begin{aligned}2\pi\ \mathsf{F}(\alpha) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\ e^{-i\alpha x}\,dx\\ &= \left(\frac{\alpha i}{\alpha^2-1}\right) 2\cos\left(\alpha\frac{\pi}{2}\right)\end{aligned}$$

Therefore

$$\begin{aligned}g(\alpha) &= \frac{1}{\pi}\left(\frac{\alpha i}{\alpha^2-1}\right)\cos\left(\alpha\frac{\pi}{2}\right)\\ &= \frac{\alpha\ i\ \cos\left(\alpha\frac{\pi}{2}\right)}{\left(\alpha^2-1\right)\pi}\end{aligned}$$

To obtain $f(x)$ given its fourier transform $\mathsf{F}(\alpha)$, then we apply the inverse fourier transform

$$\begin{aligned}f(x) &= \int_{-\infty}^{\infty} \mathsf{F}(\alpha)\ e^{i\alpha x}\ d\alpha\\ &= \int_{-\infty}^{\infty} \frac{\alpha\ i\ \cos\left(\alpha\frac{\pi}{2}\right)}{\left(\alpha^2-1\right)\pi}\ e^{i\alpha x}\ d\alpha\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\alpha\ i\ \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2-1}\ e^{i\alpha x}\ d\alpha\end{aligned}$$

3.13.2 chapter 15, problem 4.18

Problem Find the fourier sin transform of the given $f(x)$ and write $f(x)$ as a fourier integral. Verify the answer is the same as the exponential fourier transform.

$$f(x) = \begin{cases} x & |x| < 1 \\ \\ 0, & |x| > 1 \end{cases}$$

Solution

Let $\mathscr{F}_s f(x)$ be the Fourier sin transform of $f(x)$ defined as $g_s(\alpha) = \mathscr{F}_s f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\ \sin(\alpha x)\,dx$.

Hence, for the function above we get

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^1 x\ \sin(\alpha x)\ dx$$

Notice, we integrate from zero, not from -1, since the sin transform is defined only for positive x. Integrating by parts, $u = x, v = \frac{-\cos(\alpha x)}{\alpha}$, hence $\int u\ dv\ = uv - \int du\ v$

$$
\begin{aligned}
g_s(\alpha) &= \sqrt{\frac{2}{\pi}}\left\{\left[x\left(\frac{-\cos(\alpha x)}{\alpha}\right)\right]_0^1 - \int_0^1 \frac{-\cos(\alpha x)}{\alpha}\,dx\right\}\\
&= \sqrt{\frac{2}{\pi}}\left\{\frac{-1}{\alpha}[x\ \cos(\alpha x)]_0^1 + \frac{1}{\alpha}\int_0^1 \cos(\alpha x)\,dx\right\}\\
&= \sqrt{\frac{2}{\pi}}\left\{\frac{-1}{\alpha}[x\ \cos(\alpha x)]_0^1 + \frac{1}{\alpha}\left[\frac{\sin(\alpha x)}{\alpha}\right]_0^1\right\}\\
&= \sqrt{\frac{2}{\pi}}\frac{1}{\alpha}\left\{-[\cos(\alpha) - 0] + \frac{1}{\alpha}\,[\sin(\alpha x)]_0^1\right\}\\
&= \sqrt{\frac{2}{\pi}}\frac{1}{\alpha}\left\{-\cos(\alpha) + \frac{1}{\alpha}\,[\sin(\alpha) - 0]\right\}\\
&= \sqrt{\frac{2}{\pi}}\frac{1}{\alpha}\left(-\cos(\alpha) + \frac{1}{\alpha}\,\sin(\alpha)\right)\\
&= \sqrt{\frac{2}{\pi}}\frac{1}{\alpha}\left(\frac{1}{\alpha}\,\sin(\alpha) - \cos(\alpha)\right)
\end{aligned}
$$

Hence the Sin Fourier transform of $f(x)$ is

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}}\frac{1}{\alpha}\left(\frac{1}{\alpha}\,\sin(\alpha) - \cos(\alpha)\right)$$

To obtain $f(x)$ given its sin fourier transform $g_s(\alpha)$, then we apply the inverse sin fourier transform

$$
\begin{aligned}
f_s(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty g_s(\alpha)\ \sin\alpha x\,d\alpha\\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{1}{\alpha}\left(\frac{1}{\alpha}\,\sin\alpha - \cos\alpha\right)\sin\alpha x\,d\alpha\\
&= \frac{2}{\pi}\int_0^\infty\left(\frac{1}{\alpha^2}\,\sin\alpha - \frac{1}{\alpha}\cos\alpha\right)\sin\alpha x\,d\alpha\\
&= \frac{2}{\pi}\int_0^\infty\left(\frac{\sin\alpha - \alpha\cos\alpha}{\alpha^2}\right)\sin\alpha x\ \ d\alpha \qquad \text{(A0)}
\end{aligned}
$$

Now we need to show that the above is the same as the inverse fourier transform found for problem 6. From back of the book, the IFT for problem 6 is given as

$$f(x) = \int_{-\infty}^{\infty}\frac{\sin\alpha - \alpha\cos\alpha}{i\pi\alpha^2}e^{i\alpha x}d\alpha$$

Need to convert the above to $f_s(x)$. Since $e^{i\alpha x} = \cos\alpha x + i\sin\alpha x$

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty}\frac{\sin\alpha - \alpha\cos\alpha}{i\pi\alpha^2}(\cos\alpha x + i\sin\alpha x)d\alpha\\
&= \int_{-\infty}^{\infty}\frac{(\sin\alpha - \alpha\cos\alpha)\cos\alpha x}{i\pi\alpha^2} + \frac{(\sin\alpha - \alpha\cos\alpha)i\sin\alpha x}{i\pi\alpha^2}d\alpha\\
&= \int_{-\infty}^{\infty}\frac{(\sin\alpha - \alpha\cos\alpha)\cos\alpha x}{i\pi\alpha^2}d\alpha + \int_{-\infty}^{\infty}\frac{(\sin\alpha - \alpha\cos\alpha)i\sin\alpha x}{i\pi\alpha^2}d\alpha \qquad (1)
\end{aligned}
$$

Looking at the first integral,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\left(\overbrace{\sin\alpha}^{\text{odd}} - \overbrace{\alpha}^{\text{odd}} \overbrace{\cos\alpha}^{\text{even}}\right)\overbrace{\cos\alpha x}^{\text{even}}}{i\pi\,\overbrace{\alpha^2}^{\text{even}}}\, d\alpha \\
&= \int_{-\infty}^{\infty} \frac{(\text{odd} - \text{odd}\times\text{even})\times\text{even}}{\text{even}}\, d\alpha \\
&= \int_{-\infty}^{\infty} \frac{(\text{odd} - \text{odd}\ )\times\text{even}}{\text{even}}\, d\alpha \\
&= \int_{-\infty}^{\infty} \frac{\text{odd}\ \times\text{even}}{\text{even}}\, d\alpha \\
&= \int_{-\infty}^{\infty} \frac{\text{odd}}{\text{even}}\, d\alpha \\
&= \int_{-\infty}^{\infty} \text{odd}\ d\alpha
\end{aligned}$$

Hence the integral vanishes. Hence (1) becomes

$$f(x) = \int_{-\infty}^{\infty} \frac{(\sin\alpha - \alpha\cos\alpha) i \sin\alpha x}{i\pi\alpha^2} d\alpha \tag{2}$$

Looking at the above

$$\begin{aligned}
\frac{\left(\overbrace{\sin\alpha}^{\text{odd}} - \overbrace{\alpha}^{\text{odd}} \overbrace{\cos\alpha}^{\text{even}}\right)\overbrace{i\sin\alpha x}^{\text{odd}}}{i\pi\,\overbrace{\alpha^2}^{\text{even}}} &= \frac{(\text{odd} - \text{odd}\times\text{even})\times\text{odd}}{\text{even}} \\
&= \frac{\text{odd}\ \ \times\text{odd}}{\text{even}} \\
&= \frac{\text{even}}{\text{even}} \\
&= \text{even}
\end{aligned}$$

Since the integrand is even, then $\int_{-\infty}^{\infty} = 2\int_0^{\infty}$ Hence (2) becomes

$$\begin{aligned}
f(x) &= 2\int_0^{\infty} \frac{(\sin\alpha - \alpha\cos\alpha) i \sin\alpha x}{i\pi\alpha^2}\, d\alpha \\
&= \frac{2}{\pi}\int_0^{\infty} \frac{(\sin\alpha - \alpha\cos\alpha)}{\alpha^2} \sin\alpha x\ d\alpha
\end{aligned} \tag{3}$$

comparing this to equation (A1) above, we see that

$$f_s(x) = f(x)$$

Which is what we are asked to show.

3.13.3 chapter 15, problem 4.20

Problem Find the fourier sin transform of the given $f(x)$ and write $f(x)$ as a fourier integral. Verify the answer is the same as the exponential fourier transform.

$$f(x) = \begin{cases} \sin x & |x| < \frac{\pi}{2} \\ \\ 0, & |x| > \frac{\pi}{2} \end{cases}$$

Solution

Let $\mathscr{F}_s f(x)$ be the Fourier sin transform of $f(x)$ defined as $g_s(\alpha) = \mathscr{F}_s f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\, \sin(\alpha x)\, dx$. Hence, for the function above we get

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^1 \sin x\ \sin(\alpha x)\, dx$$

Notice, we integrate from zero, not from -1, since the sin transform is defined only for positive x. Since $\sin\beta \sin\gamma = \frac{1}{2}\cos(\beta-\gamma) - \frac{1}{2}\cos(\beta+\gamma)$ Then $\sin x\ \sin(\alpha x) = \frac{1}{2}\cos(x-\alpha x) - \frac{1}{2}\cos(x+\alpha x)$. Hence

$$\begin{aligned} g_s(\alpha) &= \sqrt{\frac{2}{\pi}} \int_0^1 \frac{1}{2}\cos(x-\alpha x) - \frac{1}{2}\cos(x+\alpha x)\, dx \\ &= \frac{1}{2}\sqrt{\frac{2}{\pi}} \left\{ \left[\frac{\sin(x-\alpha x)}{1-\alpha}\right]_0^1 - \left[\frac{\sin(x+\alpha x)}{1+\alpha}\right]_0^1 \right\} \\ &= \frac{1}{2}\sqrt{\frac{2}{\pi}} \left\{ \left[\frac{\sin(1-\alpha)}{1-\alpha} - 0\right] - \left[\frac{\sin(1+\alpha)}{1+\alpha} - 0\right]_0^1 \right\} \\ &= \frac{1}{2}\sqrt{\frac{2}{\pi}} \left(\frac{\sin(1-\alpha)}{1-\alpha} - \frac{\sin(1+\alpha)}{1+\alpha}\right) \end{aligned}$$

Hence the Sin Fourier transform of $f(x)$ is

$$g_s(\alpha) = \frac{1}{2}\sqrt{\frac{2}{\pi}} \left(\frac{\sin(1-\alpha)}{1-\alpha} - \frac{\sin(1+\alpha)}{1+\alpha}\right)$$

Therefore, to obtain $f(x)$ given its sin fourier transform $g_s(\alpha)$, we apply the inverse sin fourier transform

$$\begin{aligned} f_s(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty g_s(\alpha)\ \sin\alpha x\, d\alpha \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{2}\sqrt{\frac{2}{\pi}} \left(\frac{\sin(1-\alpha)}{1-\alpha} - \frac{\sin(1+\alpha)}{1+\alpha}\right) \sin\alpha x\, d\alpha \\ &= \frac{1}{\pi} \int_0^\infty \left(\frac{\sin(1-\alpha)}{1-\alpha} - \frac{\sin(1+\alpha)}{1+\alpha}\right) \sin\alpha x\, d\alpha \qquad (1) \end{aligned}$$

Now we need to show that the above is the same as the exponential inverse fourier transform found for problem 12. The exponential IFT for problem 12 is

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\alpha\, i\, \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2 - 1}\ e^{i\alpha x}\, d\alpha \qquad (2)$$

So Need to show that (1) and (2) are the same. Need to convert the above (1) to $f_s(x)$ in (2).Since $e^{i\alpha x} = \cos\alpha x + i\sin\alpha x$, (2) can be written as

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\alpha\, i\, \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2-1}(\cos\alpha x + i\sin\alpha x)d\alpha \\ &= \frac{1}{\pi}\left[\int_{-\infty}^{\infty}\frac{\alpha\, i\, \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2-1}\cos\alpha x\;\, d\alpha + \int_{-\infty}^{\infty}\frac{\alpha\, i\, \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2-1}\, i\sin\alpha x\;\, d\alpha\right] \end{aligned} \tag{3}$$

Looking at the first integral,

$$\begin{aligned} &\int_{-\infty}^{\infty}\frac{\overbrace{\alpha}^{\text{odd}}\, i\, \overbrace{\cos\left(\alpha\frac{\pi}{2}\right)}^{\text{even}}}{\underbrace{\overbrace{\alpha^2}}_{\text{even}}-1}\overbrace{\cos\alpha x}^{\text{even}}\; d\alpha \\ &= \int_{-\infty}^{\infty}\frac{(\text{odd}\times\ \text{even})\times\text{even}}{\text{even}}\; d\alpha \\ &= \int_{-\infty}^{\infty}\frac{\text{odd}\ \times\text{even}}{\text{even}}\; d\alpha \\ &= \int_{-\infty}^{\infty}\frac{\text{odd}}{\text{even}}\; d\alpha \\ &= \int_{-\infty}^{\infty}\text{odd}\;\, d\alpha \end{aligned}$$

Hence the integral vanishes. So (3) becomes

$$\begin{aligned} f(x) &= 0 + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\alpha\, i\, \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2-1}\, i\sin\alpha x\;\, d\alpha \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\alpha\, i\, \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2-1}\, i\sin\alpha x\;\, d\alpha \end{aligned}$$

Looking at the above integrand,

$$\begin{aligned} \frac{\overbrace{\alpha}^{\text{odd}}\, i\, \overbrace{\cos\left(\alpha\frac{\pi}{2}\right)}^{\text{even}}}{\underbrace{\overbrace{\alpha^2}}_{\text{even}}-1}\overbrace{i\sin\alpha x}^{\text{odd}} &= \frac{(\text{odd}\ \times\text{even})\times\text{odd}}{\text{even}} \\ &= \frac{\text{odd}\ \ \times\text{odd}}{\text{even}} \\ &= \frac{\text{even}}{\text{even}} \\ &= \text{even} \end{aligned}$$

Since the integrand is even, then $\int_{-\infty}^{\infty} = 2\int_{0}^{\infty}$. Hence (2) becomes

$$\begin{aligned} f(x) &= \frac{2}{\pi}\int_{0}^{\infty}\frac{\alpha\, i\, \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2-1}\, i\sin\alpha x\;\, d\alpha \\ &= \frac{-2}{\pi}\int_{0}^{\infty}\frac{\alpha\;\, \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2-1}\sin\alpha x\;\, d\alpha \\ &= \frac{2}{\pi}\int_{0}^{\infty}\frac{\alpha\;\, \cos\left(\alpha\frac{\pi}{2}\right)}{1-\alpha^2}\sin\alpha x\;\, d\alpha \end{aligned}$$

But $1-\alpha^2=(1+\alpha)(1-\alpha)$, therefore

$$f(x)=\frac{2}{\pi}\int_0^\infty \frac{\alpha\ \cos\left(\alpha\frac{\pi}{2}\right)}{(1+\alpha)(1-\alpha)}\ \sin\alpha x\ d\alpha$$

But $\cos\left(\alpha\frac{\pi}{2}\right)\sin\alpha x=\frac{1}{2}\left(-\sin\left(\alpha\frac{\pi}{2}-\alpha x\right)+\sin\left(\alpha\frac{\pi}{2}+\alpha x\right)\right)$, hence the above becomes

$$\begin{aligned}
f(x)&=\frac{2}{\pi}\int_0^\infty \frac{\alpha\ \frac{1}{2}\left(-\sin\left(\alpha\frac{\pi}{2}-\alpha x\right)+\sin\left(\alpha\frac{\pi}{2}+\alpha x\right)\right)}{(1+\alpha)(1-\alpha)}\ d\alpha\\
f(x)&=\frac{1}{\pi}\int_0^\infty \frac{\alpha\ \left(-\sin\left(\alpha\frac{\pi}{2}-\alpha x\right)+\sin\left(\alpha\frac{\pi}{2}+\alpha x\right)\right)}{(1+\alpha)(1-\alpha)}\ d\alpha\\
f(x)&=\frac{1}{\pi}\int_0^\infty \frac{\alpha\sin\left(\alpha\frac{\pi}{2}+\alpha x\right)}{(1+\alpha)(1-\alpha)}-\frac{\alpha\sin\left(\alpha\frac{\pi}{2}-\alpha x\right)}{(1+\alpha)(1-\alpha)}\ d\alpha
\end{aligned}$$

3.13.4 chapter 15, problem 4.21

Problem Find the fourier transform of the given $f(x)=e^{\frac{-x^2}{2\sigma^2}}$

Solution

Let $\mathsf{F}(\alpha)$ be the Fourier sin transform of $f(x)$ defined as

$$\mathsf{F}(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\ e^{-i\alpha x}\ dx$$

So, for the function above we get

$$\begin{aligned}
\mathsf{F}(\alpha)&=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}}\ e^{-i\alpha x}\ dx\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}-i\alpha x}\ dx\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-x^2-i\alpha x\left(2\sigma^2\right)}{2\sigma^2}}\ dx\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-x^2-2i\alpha\sigma^2 x}{2\sigma^2}}\ dx \qquad (1)
\end{aligned}$$

looking at the exponent $\frac{-x^2-2i\alpha\sigma^2 x}{2\sigma^2}$. completing the square in x gives

$$x^2+2i\alpha\sigma^2 x\ =(x+Z\ )^2-Y$$

Solving for Z,Y gives

$$x^2+2i\alpha\sigma^2 x\ =x^2+2xZ+Z^2-Y$$

Therefore $Z=i\alpha\sigma^2, Z^2-Y=0$, and $Y=-\alpha^2\sigma^4$. Hence Exponent can be written as

$$
\begin{aligned}
\frac{x^2+2i\alpha\sigma^2 x}{-2\sigma^2} &= \frac{\left(x+i\alpha\sigma^2\right)^2-\left(-\alpha^2\sigma^4\right)}{-2\sigma^2}\\
&= \frac{\left(x+i\alpha\sigma^2\right)^2+\alpha^2\sigma^4}{-2\sigma^2}\\
&= \frac{\left(x+i\alpha\sigma^2\right)^2}{-2\sigma^2}-\frac{\alpha^2\sigma^4}{2\sigma^2}\\
&= \frac{\left(x+i\alpha\sigma^2\right)^2}{-2\sigma^2}-\frac{\alpha^2\sigma^2}{2}
\end{aligned}
$$

The integral (1) becomes

$$
\begin{aligned}
F(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{\left(x+i\alpha\sigma^2\right)^2}{-2\sigma^2}-\frac{\alpha^2\sigma^2}{2}}\,dx\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{\left(x+i\alpha\sigma^2\right)^2}{-2\sigma^2}}\,e^{-\frac{\alpha^2\sigma^2}{2}}\,dx
\end{aligned}
$$

Moving $e^{-\frac{\alpha^2\sigma^2}{2}}$ outside the integral because it does not depend on x gives

$$
F(\alpha) = \frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{2\pi}\int_{-\infty}^{\infty} e^{\frac{\left(x+i\alpha\sigma^2\right)^2}{-2\sigma^2}}\,dx
$$

Let $y = x + i\alpha\sigma^2$, $dy = dx$ and the limits do not change. Hence we get

$$
F(\alpha) = \frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{2\pi}\int_{-\infty}^{\infty} e^{\frac{y^2}{-2\sigma^2}}\,dy
$$

Since the exponential function is raised to a square power, then we can write $\int_{-\infty}^{\infty} e^{y^2} = 2\int_{0}^{\infty} e^{y^2}$ (since even function). Hence above integral becomes

$$
F(\alpha) = \frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{\pi}\int_{0}^{\infty} e^{\frac{y^2}{-2\sigma^2}}\,dy
$$

Let $\zeta = \frac{y}{\sqrt{2}\sigma}$, then $y = \sqrt{2}\sigma\zeta$, and $y^2 = 2\sigma^2\zeta^2$. Hence $d\zeta = \frac{1}{\sqrt{2}\sigma}dy$ and the above integral becomes

$$
\begin{aligned}
F(\alpha) &= \frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{\pi}\int_{0}^{\infty} e^{-\zeta^2}\,\sqrt{2}\sigma\,d\zeta\\
F(\alpha) &= \sqrt{2}\sigma\frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{\pi}\int_{0}^{\infty} e^{-\zeta^2}\,d\zeta \qquad (2)
\end{aligned}
$$

Now from equation 9.5 on page 468

$$\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\zeta^2}\, d\zeta = 1$$
$$\int_0^\infty e^{-\zeta^2}\, d\zeta = \frac{\sqrt{\pi}}{2}$$

Hence (2) becomes

$$F(\alpha) = \sqrt{2}\,\sigma\frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{\pi}\left(\frac{\sqrt{\pi}}{2}\right)$$
$$F(\alpha) = \frac{\sigma}{\sqrt{2\pi}}\, e^{-\frac{\alpha^2\sigma^2}{2}}$$

Which is what we are asked to show.

3.13.5 chapter 15, problem 4.23

Problem Show that

$\int_0^\infty \frac{1-\cos\pi\alpha}{\alpha}\sin\alpha\, d\alpha = \frac{\pi}{2}$ and $\int_0^\infty \frac{1-\cos\pi\alpha}{\alpha}\sin\pi\alpha\, d\alpha = \frac{\pi}{4}$

Solution

From problem 17, the Fourier sin transform for $f(x)$ shown in problem 3 is

$$g_s(\alpha) = \frac{\sqrt{2}\,(1-\cos\pi\alpha)}{\sqrt{\pi}\,\alpha}$$

From equation 4.14 page 651, $f(x)$ can be obtained from inverse sin transform is

$$\begin{aligned}
f_s(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty g_s(\alpha)\sin\alpha x\, d\alpha \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty\left(\frac{\sqrt{2}\,(1-\cos\pi\alpha)}{\sqrt{\pi}\,\alpha}\right)\sin\alpha x\, d\alpha \\
&= \frac{2}{\pi}\int_0^\infty \frac{(1-\cos\pi\alpha)}{\alpha}\sin\alpha x\, d\alpha \qquad (1)
\end{aligned}$$

Now, from the definition of $f(x)$, which is

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \\ 0, & |x| > \pi \end{cases}$$

We see that for $x = 1$, $f(x) = 1$, hence substitute in (1) we get

$$1 = -\frac{2}{\pi}\int_0^\infty \frac{(1-\cos\pi\alpha)}{\alpha}\sin\alpha\, d\alpha$$
$$\frac{\pi}{2} = \int_0^\infty \frac{(1-\cos\pi\alpha)}{\alpha}\sin\alpha\, d\alpha$$

Which is the first result we required to show. For the second result, let $x = \pi$ hence $f(\pi)$ = average value of $f(x)$ at $x = \pi$. Which is given by $\frac{f(\pi_{-})+f(\pi_{+})}{2} = \frac{1+0}{2} = \frac{1}{2}$. Hence substitute in (1) we get

$$
\begin{aligned}
f_s(x=\pi) = \frac{1}{2} &= \frac{2}{\pi}\int_0^{\infty} \frac{(1-\cos\pi\alpha)}{\alpha}\sin\alpha\pi\; d\alpha \\
\frac{\pi}{4} &= \int_0^{\infty} \frac{(1-\cos\pi\alpha)}{\alpha}\sin\alpha\pi\; d\alpha
\end{aligned}
$$

Which is the second result we are asked to show.

3.13.6 chapter 15, problem 4.25

Problem Show that

(a) represent as an exponential fourier transform the function

$$
f(x) = \begin{cases} \sin x & 0 < x < \pi \\ \\ 0, & \text{otherwise} \end{cases}
$$

(b) Show that the result can be written as

$$
f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{\cos\alpha x + \cos\alpha(x-\pi)}{1-\alpha^2} d\alpha
$$

Solution

The exponential Fourier transform is defined as

$$
g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-i\alpha x}\, dx
$$

Applying the function $f(x)$ gives

$$
g(\alpha) = \frac{1}{2\pi}\int_0^{\pi} \sin x\; e^{-i\alpha x}\, dx
$$

But

$$
\sin x = \frac{e^{ix}-e^{-ix}}{2i}
$$

Hence the transform can be written as

$$
\begin{aligned}
g(\alpha) &= \frac{1}{2\pi}\int_0^{\pi} \frac{e^{ix}-e^{-ix}}{2i}\, e^{-i\alpha x}\, dx \\
&= \frac{1}{4i\pi}\int_0^{\pi} e^{ix-i\alpha x} - e^{-ix-i\alpha x}\ dx \\
&= \frac{1}{4i\pi}\int_0^{\pi} e^{x(i-i\alpha)} - e^{x(-i-i\alpha)}\ dx \\
&= \frac{1}{4i\pi}\left(\left[\frac{e^{x(i-i\alpha)}}{i-i\alpha}\right]_0^{\pi} - \left[\frac{e^{x(-i-i\alpha)}}{-i-i\alpha}\right]_0^{\pi}\right) \\
&= \frac{1}{4i\pi}\left(\left[\frac{e^{\pi(i-i\alpha)}}{i-i\alpha} - \frac{1}{i-i\alpha}\right] - \left[\frac{e^{\pi(-i-i\alpha)}}{-i-i\alpha} - \frac{1}{-i-i\alpha}\right]\right) \\
&= \frac{1}{4i\pi}\left(\frac{1}{i-i\alpha}\left[e^{\pi(i-i\alpha)}-1\right] - \frac{1}{-i-i\alpha}\left[e^{\pi(-i-i\alpha)}-1\right]\right) \\
&= \frac{1}{4i\pi}\left(\frac{1}{i-i\alpha}\left[e^{\pi(i-i\alpha)}-1\right] + \frac{1}{i+i\alpha}\left[e^{\pi(-i-i\alpha)}-1\right]\right) \\
&= \frac{1}{i}\frac{1}{4i\pi}\left(\frac{1}{1-\alpha}\left[e^{\pi(i-i\alpha)}-1\right] + \frac{1}{1+\alpha}\left[e^{\pi(-i-i\alpha)}-1\right]\right) \\
&= \frac{-1}{4\pi}\left(\frac{e^{\pi(i-i\alpha)}}{1-\alpha} - \frac{1}{1-\alpha} + \frac{e^{\pi(-i-i\alpha)}}{1+\alpha} - \frac{1}{1+\alpha}\right) \\
&= \frac{-1}{4\pi}\left(\frac{(1+\alpha)e^{\pi(i-i\alpha)} + (1-\alpha)e^{\pi(-i-i\alpha)}}{(1-\alpha)(1+\alpha)} - \frac{(1+\alpha)+(1-\alpha)}{(1-\alpha)(1+\alpha)}\right) \\
&= \frac{-1}{4\pi}\left(\frac{(1+\alpha)e^{\pi(i-i\alpha)} + (1-\alpha)e^{\pi(-i-i\alpha)}}{\left(1-\alpha^2\right)} - \frac{2}{\left(1-\alpha^2\right)}\right) \\
&= \frac{-1}{\left(1-\alpha^2\right)4\pi}\left(e^{\pi(i-i\alpha)} + \alpha e^{\pi(i-i\alpha)} + e^{\pi(-i-i\alpha)} - \alpha e^{\pi(-i-i\alpha)} - 2\right) \\
&= \frac{-1}{\left(1-\alpha^2\right)4\pi}\left(e^{\pi i}\, e^{-i\pi\alpha} + \alpha e^{\pi i}\, e^{-i\pi\alpha} + e^{-\pi i}\, e^{-i\pi\alpha} - \alpha e^{-\pi i}\, e^{-i\pi\alpha} - 2\right)
\end{aligned}
$$

But $e^{\pi i} = -1$ and $e^{-\pi i} = -1$

$$
\begin{aligned}
g(\alpha) &= \frac{-1}{\left(1-\alpha^2\right)4\pi}\left(-e^{-i\pi\alpha} - \overbrace{\alpha e^{-i\pi\alpha}} - e^{-i\pi\alpha} + \overbrace{\alpha e^{-i\pi\alpha}} - 2\right) \\
g(\alpha) &= \frac{-1}{\left(1-\alpha^2\right)4\pi}\left(-e^{-i\pi\alpha} - e^{-i\pi\alpha} - 2\right) \\
g(\alpha) &= \frac{1}{\left(1-\alpha^2\right)4\pi}\left(e^{-i\pi\alpha} + e^{-i\pi\alpha} + 2\right) \\
g(\alpha) &= \frac{2e^{-i\pi\alpha} + 2}{\left(1-\alpha^2\right)4\pi}
\end{aligned}
$$

Hence the exponential fourier transform is

$$
g(\alpha) = \frac{e^{-i\pi\alpha} + 1}{\left(1-\alpha^2\right)2\pi}
$$

Therefore $f(x)$ can be rewritten as

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} g(\alpha)\, e^{i\alpha x}\, d\alpha \\ &= \int_{-\infty}^{\infty} \frac{e^{-i\pi\alpha} + 1}{\left(1-\alpha^2\right)2\pi} e^{i\alpha x}\, d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1+e^{-i\pi\alpha}}{\left(1-\alpha^2\right)} e^{i\alpha x}\, d\alpha \end{aligned} \tag{1}$$

Which is the answer required to show.

Part(b)

Now need to show that the above can be written as

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{\cos\alpha x + \cos\alpha(x-\pi)}{1-\alpha^2} d\alpha$$

From (1)

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1+e^{-i\pi\alpha}}{\left(1-\alpha^2\right)} e^{i\alpha x}\, d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\alpha x} + e^{i\alpha x}e^{-i\pi\alpha}}{\left(1-\alpha^2\right)} d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\alpha x} + e^{i\alpha x - i\pi\alpha}}{\left(1-\alpha^2\right)} d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\alpha x} + e^{i\alpha(x-\pi)}}{\left(1-\alpha^2\right)} d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2(\cos\alpha x + \cos\alpha(x-\pi))}{\left(1-\alpha^2\right)} d\alpha \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos\alpha x + \cos\alpha(x-\pi)}{\left(1-\alpha^2\right)} d\alpha \end{aligned}$$

Which is what is required to show.

3.13.7 chapter 15, problem 4.3

Problem Find the exponential fourier transform of the given $f(x)$ and write $f(x)$ as a fourier integral.

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \\ 0, & |x| > \pi \end{cases}$$

Solution

Let $\mathsf{F}(\alpha)$ be the Fourier transform of $f(x)$ defined as $\mathsf{F}(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$, hence

$$
\begin{aligned}
\mathsf{F}(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx \\
2\pi\, \mathsf{F}(\alpha) &= \int_{-\pi}^{0} -\ e^{-i\alpha x}\, dx + \int_{0}^{\pi} e^{-i\alpha x}\, dx \\
&= -\left[\frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\pi}^{0} + \left[\frac{e^{-i\alpha x}}{-i\alpha}\right]_{0}^{\pi} \\
&= \frac{1}{i\alpha}\left[e^{-i\alpha x}\right]_{-\pi}^{0} - \frac{1}{i\alpha}\left[e^{-i\alpha x}\right]_{0}^{\pi} \\
&= \frac{1}{i\alpha}\left[e^{0} - e^{i\alpha\pi}\right] - \frac{1}{i\alpha}\left[e^{-i\alpha\pi} - e^{0}\right]_{0}^{\pi} \\
&= \frac{1}{i\alpha}\left[1 - e^{i\alpha\pi}\right] - \frac{1}{i\alpha}\left[e^{-i\alpha\pi} - 1\right]_{0}^{\pi} \\
&= \frac{1}{i\alpha} - \frac{e^{i\alpha\pi}}{i\alpha} - \frac{e^{-i\alpha\pi}}{i\alpha} + \frac{1}{i\alpha} \\
&= \frac{2}{i\alpha} - \frac{1}{i\alpha}\left(e^{i\alpha\pi} + e^{-i\alpha\pi}\right)
\end{aligned}
$$

But $e^{i\alpha\pi} + e^{-i\alpha\pi} = 2\cos\alpha\pi$. Hence

$$
\begin{aligned}
2\pi\, \mathsf{F}(\alpha) &= \frac{2}{i\alpha} - \frac{1}{i\alpha}(2\cos\alpha\pi) \\
&= \frac{2}{i\alpha}(1 - \cos\alpha\pi)
\end{aligned}
$$

Hence the Fourier transform of $f(x)$ is

$$
\begin{aligned}
\mathsf{F}(\alpha) &= \frac{1}{2\pi}\left[\frac{2}{i\alpha}(1 - \cos\alpha\pi)\right] \\
&= \frac{1}{\pi\alpha i}\,(1 - \cos\alpha\pi)
\end{aligned}
$$

To obtain $f(x)$ given its fourier transform $\mathsf{F}(\alpha)$, then we apply the inverse fourier transform

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} \mathsf{F}(\alpha)\, e^{i\alpha x}\, d\alpha \\
&= \int_{-\infty}^{\infty} \frac{1}{\pi\alpha i}\,(1 - \cos\alpha\pi)\, e^{i\alpha x}\, d\alpha \\
&= \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1}{\alpha}(1 - \cos\alpha\pi)\, e^{i\alpha x}\, d\alpha
\end{aligned}
$$

3.13.8 chapter 15, problem 4.5

Problem Find the exponential fourier transform of the given $f(x)$ and write $f(x)$ as a fourier integral.

$$
f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}
$$

Solution

Let $\mathsf{F}(s)$ be the Fourier transform of $f(x)$ defined as $\mathsf{F}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$

$$
\begin{aligned}
\mathsf{F}(s) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx \\
2\pi\ \mathsf{F}(s) &= \int_0^1 e^{-isx}\, dx \\
&= \left[\frac{e^{-isx}}{-is}\right]_0^1 \\
&= \frac{-1}{is}\left[e^{-isx}\right]_0^1 \\
&= \frac{-1}{is}\left[e^{-is} - e^0\right] \\
&= \frac{-1}{is}\left[e^{-is} - 1\right]
\end{aligned}
$$

Hence the Fourier transform of $f(x)$ is

$$
\begin{aligned}
\mathsf{F}(s) &= \frac{1}{2\pi}\left[\frac{-1}{is}\left[e^{-is} - 1\right]\right] \\
&= \frac{i}{2\pi s}\left(e^{-is} - 1\right)
\end{aligned}
$$

To obtain $f(x)$ given its fourier transform $\mathsf{F}(s)$, then we apply the inverse fourier transform

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} \mathsf{F}(s)\, e^{isx}\, ds \\
&= \int_{-\infty}^{\infty} \frac{i}{2\pi s}\left(e^{-is} - 1\right) e^{isx}\, ds \\
&= \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{1}{s}\left(e^{-is} - 1\right) e^{isx}\, ds
\end{aligned}
$$

3.13.9 chapter 15, problem 4.7

Problem Find the exponential fourier transform of the given $f(x)$ and write $f(x)$ as a fourier integral.

$$
f(x) = \begin{cases} |x| & |x| < 1 \\ 0, & |x| > 1 \end{cases}
$$

Solution

Let $\mathsf{F}(\alpha)$ be the Fourier transform of $f(x)$ defined as $\mathsf{F}(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$

$$
\begin{aligned}
\mathsf{F}(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx \\
2\pi\ \mathsf{F}(\alpha) &= \int_{-1}^{0} -x\ e^{-i\alpha x}\, dx + \int_0^1 x\ e^{-i\alpha x}\, dx
\end{aligned}
$$

Integrating by parts, $u = x, v = \frac{e^{-i\alpha x}}{-i\alpha}$, hence $\int u\, dv = uv - \int du\, v$. The first integral is

$$
\begin{aligned}
\int_{-1}^{0} -x\ e^{-i\alpha x}\,dx &= \left[(-x)\ \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-1}^{0} + \int_{-1}^{0} \frac{e^{-i\alpha x}}{-i\alpha}\,dx \\
&= \left[(x)\ \frac{e^{-i\alpha x}}{i\alpha}\right]_{-1}^{0} + \int_{-1}^{0} \frac{e^{-i\alpha x}}{-i\alpha}\,dx \\
&= \frac{1}{i\alpha}\ \left[0-(-1)\times e^{i\alpha}\right] - \frac{1}{i\alpha}\int_{-1}^{0} e^{-i\alpha x}\,dx \\
&= \frac{1}{i\alpha}\ \left[e^{i\alpha}\right] - \frac{1}{i\alpha}\left[\frac{e^{-i\alpha x}}{-i\alpha}\right]_{-1}^{0} \\
&= \frac{1}{i\alpha}\ \left[e^{i\alpha}\right] + \frac{1}{i^2\alpha^2}\left[e^{-i\alpha x}\right]_{-1}^{0} \\
&= \frac{1}{i\alpha}\ \left[e^{i\alpha}\right] - \frac{1}{\alpha^2}\left[1-\ e^{i\alpha}\right] \\
&= \frac{e^{i\alpha}}{i\alpha} - \frac{1}{\alpha^2} + \frac{e^{i\alpha}}{\alpha^2}
\end{aligned}
$$

And the second integral

$$
\begin{aligned}
\int_{0}^{1} x\ e^{-i\alpha x}\,dx &= \left[x\ \frac{e^{-i\alpha x}}{-i\alpha}\right]_{0}^{1} - \int_{0}^{1} \frac{e^{-i\alpha x}}{-i\alpha}\,dx \\
&= \frac{1}{-i\alpha}\ \left[1\times e^{-i\alpha}-0\right] + \frac{1}{i\alpha}\int_{0}^{1} e^{-i\alpha x}\,dx \\
&= \frac{1}{-i\alpha}\ \left[e^{-i\alpha}\right] + \frac{1}{i\alpha}\left[\frac{e^{-i\alpha x}}{-i\alpha}\right]_{0}^{1} \\
&= \frac{1}{-i\alpha}\ \left[e^{-i\alpha}\right] - \frac{1}{i^2\alpha^2}\left[e^{-i\alpha x}\right]_{0}^{1} \\
&= \frac{1}{-i\alpha}\ \left[e^{-i\alpha}\right] + \frac{1}{\alpha^2}\left[e^{-i\alpha}-1\right] \\
&= \frac{e^{-i\alpha}}{-i\alpha} + \frac{e^{-i\alpha}}{\alpha^2} - \frac{1}{\alpha^2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
2\pi\ \mathsf{F}(\alpha) &= \left(\frac{e^{i\alpha}}{i\alpha} - \frac{1}{\alpha^2} + \frac{e^{i\alpha}}{\alpha^2}\right) + \left(\frac{e^{-i\alpha}}{-i\alpha} + \frac{e^{-i\alpha}}{\alpha^2} - \frac{1}{\alpha^2}\right) \\
2\pi\ \mathsf{F}(\alpha) &= \frac{e^{i\alpha}}{i\alpha} - \frac{1}{\alpha^2} + \frac{e^{i\alpha}}{\alpha^2} - \frac{e^{-i\alpha}}{i\alpha} + \frac{e^{-i\alpha}}{\alpha^2} - \frac{1}{\alpha^2} \\
&= \frac{e^{i\alpha}}{i\alpha} + \frac{e^{i\alpha}}{\alpha^2} - \frac{e^{-i\alpha}}{i\alpha} + \frac{e^{-i\alpha}}{\alpha^2} - \frac{2}{\alpha^2} \\
&= \frac{1}{\alpha}\left(\frac{e^{i\alpha}}{i} - \frac{e^{-i\alpha}}{i}\right) + \frac{1}{\alpha^2}\left(e^{i\alpha}+e^{-i\alpha}\right) - \frac{2}{\alpha^2} \\
&= \frac{1}{\alpha}\left(\frac{e^{i\alpha}-e^{-i\alpha}}{i}\right) + \frac{1}{\alpha^2}\left(e^{i\alpha}+e^{-i\alpha}\right) - \frac{2}{\alpha^2}
\end{aligned}
$$

But $e^{i\alpha}+e^{-i\alpha}=2\cos\alpha$ and $\frac{e^{i\alpha}-e^{-i\alpha}}{i}=2\sin\alpha$, Hence the above becomes

$$
\begin{aligned}
2\pi\ \mathrm{F}(\alpha) &= \frac{1}{\alpha}(2\sin\alpha) + \frac{1}{\alpha^2}(2\cos\alpha\ ) - \frac{2}{\alpha^2} \\
&= \frac{2}{\alpha^2}[\alpha\sin\alpha + \cos\alpha\ - 1] \\
\mathrm{F}(\alpha) &= \frac{1}{\pi\alpha^2}[\alpha\sin\alpha + \cos\alpha\ - 1]
\end{aligned}
$$

Hence the Fourier transform of $f(x)$ is

$$
\mathrm{F}(\alpha) = \frac{1}{\pi\alpha^2}[\alpha\sin\alpha + \cos\alpha\ - 1]
$$

To obtain $f(x)$ given its fourier transform $\mathrm{F}(\alpha)$, then we apply the inverse fourier transform

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} \mathrm{F}(\alpha)\ e^{i\alpha x}\ d\alpha \\
&= \int_{-\infty}^{\infty} \frac{1}{\pi\alpha^2}[\alpha\sin\alpha + \cos\alpha\ - 1]\ \ e^{i\alpha x}\ d\alpha \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{\alpha^2}[\alpha\sin\alpha + \cos\alpha\ - 1]\ \ e^{i\alpha x}\ d\alpha
\end{aligned}
$$

3.13.10 chapter 15, problem 5.1

Problem Show that $g(t) \circledast h(t) = h(t) \circledast g(t)$

Solution

By definition,

$$
g(t) \circledast h(t) = \int_0^t g(t-\tau)\ h(\tau)\ d\tau \tag{1}
$$

Let $u = t - \tau$, $du = -d\tau$, when $\tau = 0$, $u = t$, when $\tau = t$, $u = 0$, Hence The RHS becomes

$$
\begin{aligned}
\int_0^t g(t-\tau)\ h(\tau)\ d\tau &= \int_{u=t}^{u=0} g(u)\ h(t-u)\ (-du) \\
&= -\int_{u=t}^{u=0} g(u)\ h(t-u)\ du \\
&= \int_{u=0}^{u=t} g(u)\ h(t-u)\ du
\end{aligned}
$$

Since u is a dummy variable of integration, call it anything we want, say τ so above integral becomes

$$
\begin{aligned}
\int_0^t g(t-\tau)\ h(\tau)\ d\tau &= \int_0^t g(\tau)\ h(t-\tau)\ d\tau \\
\int_0^t g(t-\tau)\ h(\tau)\ d\tau &= \int_0^t h(t-\tau)\ g(\tau)\ \ d\tau \qquad (2)
\end{aligned}
$$

Hence from (2) $g(t) \circledast h(t) = h(t) \circledast g(t)$

3.13.11 chapter 15, problem 5.10

Problem

Use convolution integral to find the inverse transform of $\frac{1}{p\left(p^2+a^2\right)^2}$

Solution

$$\frac{1}{p\left(p^2+a^2\right)^2}=\frac{1}{p}\frac{1}{\left(p^2+a^2\right)^2}=GH$$

From Tables using L1 and L17 $g(t)=1$ and $h(t)=\frac{\sin at - at\ \cos at}{2a^3}$. Hence the inverse transform of $GH=\ g(t)\circledast h(t)$. Using L34

$$g(t)\circledast h(t)=\int_0^t g(t-\tau)\,h(\tau)\,d\tau \tag{L34}$$

Hence

$$\begin{aligned} g(t)\circledast h(t) &= \int_0^t 1\times\frac{\sin a(t-\tau)-a(t-\tau)\ \cos a(t-\tau)}{2a^3}\ d\tau \\ &= \frac{1}{2a^3}\int_0^t \sin a(t-\tau)-a(t-\tau)\ \cos a(t-\tau)\ d\tau \\ &= \frac{1}{2a^3}\left[\int_0^t \sin a(t-\tau)\ d\tau - at\int_0^t \cos a(t-\tau)d\tau + a\int_0^t \tau\cos a(t-\tau)\ d\tau\right] \end{aligned} \tag{1}$$

The last integral can be integrated by parts. $u=\tau, v=\frac{\sin a(t-\tau)}{-a}$

$$\begin{aligned} \int_0^t \tau\cos a(t-\tau)\ d\tau &= \left[\tau\frac{\sin a(t-\tau)}{-a}\right]_0^t - \int_0^t \frac{\sin a(t-\tau)}{-a}\ d\tau \\ &= \frac{-1}{a}\left[\tau\sin a(t-\tau)\right]_0^t + \frac{1}{a}\int_0^t \sin a(t-\tau)\ d\tau \\ &= \frac{-1}{a}\left[\tau\sin a(t-\tau)\right]_0^t + \frac{1}{a}\left[\frac{-\cos a(t-\tau)}{-a}\right]_0^t \\ &= \frac{-1}{a}\left[\tau\sin a(t-\tau)\right]_0^t + \frac{1}{a^2}\left[\ \cos a(t-\tau)\right]_0^t \\ &= \frac{-1}{a}\left[t\sin a(t-t)-0\right] + \frac{1}{a^2}\left[\ \cos a(t-t)-\cos a(t-0)\right] \\ &= \frac{-1}{a}[0] + \frac{1}{a^2}\left[\ \cos a(0)-\cos a(t)\right] \\ &= \frac{1}{a^2}\left[\,1-\cos at\right] \end{aligned}$$

Hence (1) becomes

$$
\begin{aligned}
g(t) \circledast h(t) &= \frac{1}{2a^3}\left[\int_0^t \sin a(t-\tau)\, d\tau - at\int_0^t \cos a(t-\tau) d\tau + a\frac{1}{a^2}[1-\cos at]\right] \\
g(t) \circledast h(t) &= \frac{1}{2a^3}\left[\left[\frac{-\cos a(t-\tau)}{-a\tau}\right]_0^t - at\left[\frac{\sin a(t-\tau)}{-a\tau}\right] + \frac{1}{a}[1-\cos at]\right] \\
&= \frac{1}{2a^3}\left[\frac{1}{a}[\cos a(t-\tau)]_0^t + t\,[\sin a(t-\tau)]_0^t + \frac{1}{a}[1-\cos at]\right] \\
&= \frac{1}{2a^3}\left[\frac{1}{a}[\cos a(t-t) - \cos a(t-0)] + t\,[\sin a(t-t) - \sin a(t-0)]_0^t + \frac{1}{a}[1-\cos at]\right] \\
&= \frac{1}{2a^3}\left[\frac{1}{a}[1-\cos at] + t\,[-\sin at] + \frac{1}{a}[1-\cos at]\right] \\
&= \frac{1}{2a^3}\left[\frac{1}{a}[1-\cos at] - t\;\;\sin at + \frac{1}{a}[1-\cos at]\right] \\
&= \frac{1}{2a^4}(2 - 2\cos at - at\;\;\sin at)
\end{aligned}
$$

So the inverse Laplace transform of $\frac{1}{p\left(p^2+a^2\right)^2}$ is

$$\frac{1}{2a^4}(2 - 2\cos at - at\;\;\sin at)$$

3.13.12 chapter 15, problem 5.2

Problem Use L34 and L2 to find the inverse transform of $G\left(p\right)H(p)$ when $G(p) = \frac{1}{\left(p+a\right)}$ and $H(p) = \frac{1}{\left(p+b\right)}$ your result should be L7

Solution

$$\mathscr{L}\left(e^{-at}\right) = \frac{1}{p+a} \tag{L2}$$

$$g(t) \circledast h(t) = \int_0^t g(t-\tau)\, h(\tau)\, d\tau \tag{L34}$$

Using L2, $g(t) = \mathscr{L}^{-1}\frac{1}{\left(p+a\right)} = e^{-at}$, and $h(t) = \mathscr{L}^{-1}\frac{1}{\left(p+b\right)} = e^{-bt}$. Now Let $Y(p) = G(p)H(p)$, But

$$G(p)H(p) = \mathscr{L}\left\{\, g(t) \circledast h(t)\, \right\}$$

Then

$$
\begin{aligned}
y(t) = g(t) \circledast h(t) &= \int_0^t g(t-\tau)\, h(\tau)\, d\tau \\
&= \int_0^t e^{-a(t-\tau)}\;\; e^{-b\tau}\, d\tau \\
&= \int_0^t e^{-at+a\tau}\;\; e^{-b\tau}\, d\tau \\
&= \int_0^t e^{-at}\, e^{a\tau}\;\; e^{-b\tau}\, d\tau
\end{aligned}
$$

e^{-at} can be moved outside the integral

$$
\begin{aligned}
y(t) &= e^{-at} \int_0^t e^{a\tau} \; e^{-b\tau} \, d\tau \\
y(t) &= e^{-at} \int_0^t e^{a\tau - b\tau} \; d\tau \\
y(t) &= e^{-at} \int_0^t e^{\tau(a-b)} \; d\tau \\
y(t) &= e^{-at} \left[\frac{e^{\tau(a-b)}}{a-b} \right]_0^t \\
y(t) &= \frac{e^{-at}}{a-b} \left[e^{t(a-b)} - 1 \right] \\
y(t) &= \frac{e^{-at+t(a-b)} - e^{-at}}{a-b} \\
y(t) &= \frac{e^{-bt} - e^{-at}}{a-b} \\
y(t) &= \frac{e^{-at} - e^{-bt}}{b-a}
\end{aligned}
$$

Which is L7 as required to show.

3.13.13 chapter 15, problem 5.22

Problem

Verify Parseval's theorem for $f(x) = e^{-|x|}$ and $g(\alpha)$ =Fourier transform of $f(x)$

Solution

Parseval theorem says that total energy in a signal equal to the sum of the energies in the harmonics that make up the signal. i.e.

$$
\int_{-\infty}^{\infty} \left| g(\alpha) \right|^2 d\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| f(x) \right|^2 dx
$$

$$
\begin{aligned}
\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| f(x) \right|^2 dx &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| e^{-|x|} \right|^2 dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2|x|} \, dx \\
&= \frac{1}{2\pi} \left\{ \int_{-\infty}^{0} e^{2x} \, dx + \int_{0}^{\infty} e^{-2x} \, dx \right\} \\
&= \frac{1}{2\pi} \left\{ \left[\frac{e^{2x}}{2} \right]_{-\infty}^{0} + \left[\frac{e^{-2x}}{-2} \right]_{0}^{\infty} \right\} \\
&= \frac{1}{4\pi} \left\{ \left[e^{2x} \right]_{-\infty}^{0} - \left[e^{-2x} \right]_{0}^{\infty} \right\} \\
&= \frac{1}{4\pi} \left\{ \left[e^{0} - 0 \right] - \left[0 - e^{0} \right] \right\} \\
&= \frac{1}{4\pi} \{1 + 1\} \\
&= \frac{1}{2\pi} \qquad (1)
\end{aligned}
$$

Now we find the Fourier transform for $f(x)$

$$
\begin{aligned}
g(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|x|}e^{-ix\alpha}dx \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{x}e^{-ix\alpha}dx + \int_{0}^{\infty} e^{-x}e^{-ix\alpha}dx\right] \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{x(1-i\alpha)}\,dx + \int_{0}^{\infty} e^{\,x(-1-i\alpha)}\,dx\right] \\
&= \frac{1}{2\pi}\left(\left[\frac{e^{x(1-i\alpha)}}{1-i\alpha}\right]_{-\infty}^{0} + \left[\frac{e^{x(-1-i\alpha)}}{-1-i\alpha}\right]_{0}^{\infty}\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{1-i\alpha}\left[e^{x(1-i\alpha)}\right]_{-\infty}^{0} - \frac{1}{1+i\alpha}\left[e^{x(-1-i\alpha)}\right]_{0}^{\infty}\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{1-i\alpha}\left[e^{x(1-i\alpha)}\right]_{-\infty}^{0} - \frac{1}{1+i\alpha}\left[e^{x(-1-i\alpha)}\right]_{0}^{\infty}\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{1-i\alpha}\left[1-e^{-\infty(1-i\alpha)}\right] - \frac{1}{1+i\alpha}\left[e^{\infty(-1-i\alpha)}-1\right]\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{1-i\alpha}[1] - \frac{1}{1+i\alpha}[-1]\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{1-i\alpha} + \frac{1}{1+i\alpha}\right) \\
&= \frac{1}{2\pi}\left(\frac{1+i\alpha+1-i\alpha}{(1-i\alpha)(1+i\alpha)}\right) \\
&= \frac{1}{2\pi}\left(\frac{2}{1+\alpha^2}\right) \\
&= \frac{1}{\pi\left(1+\alpha^2\right)}
\end{aligned}
$$

So

$$
\begin{aligned}
\int_{-\infty}^{\infty} \left|g(\alpha)\right|^2 d\alpha &= \int_{-\infty}^{\infty}\left|\frac{1}{\pi\left(1+\alpha^2\right)}\right|^2 d\alpha \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{1}{\left(1+\alpha^2\right)^2}d\alpha
\end{aligned}
$$

But $\int_{-\infty}^{\infty}\frac{1}{\left(1+\alpha^2\right)^2}d\alpha = \frac{\pi}{2}$, Hence

$$
\begin{aligned}
\int_{-\infty}^{\infty} \left|g(\alpha)\right|^2 d\alpha &= \frac{1}{\pi^2}\left(\frac{\pi}{2}\right) \\
&= \frac{1}{2\pi} \qquad (2)
\end{aligned}
$$

Comparing (1) and (2). They are the same. Hence

$$
\int_{-\infty}^{\infty} \left|g(\alpha)\right|^2 d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|f(x)\right|^2 dx
$$

was verified for this problem as required.

3.13.14 chapter 15, problem 5.4

Problem

Use convolution integral to find the inverse transform of $\frac{1}{(p+a)(p+b)^2}$

Solution

$$\frac{1}{(p+a)(p+b)^2} = \frac{1}{(p+a)}\frac{1}{(p+b)^2}$$

From but from L6

$$\mathscr{L}\left(t^k e^{-at}\right) = \frac{k!}{(p+a)^{k+1}} \tag{L2}$$

Hence $\frac{1}{(p+a)} = \mathscr{L}\left(e^{-at}\right)$ and $\frac{1}{(p+b)^2} = \mathscr{L}\left(e^{-bt}\right)$. Hence the inverse transform of $\frac{1}{(p+a)}\frac{1}{(p+b)^2} =$ $e^{-at} \circledast te^{-bt}$. Using L34

$$g(t) \circledast h(t) = \int_0^t g(t-\tau)\, h(\tau)\, d\tau \tag{L34}$$

Hence

$$\begin{aligned}
e^{-at} \circledast te^{-bt} &= \int_0^t e^{-a(t-\tau)}\ \tau e^{-b\tau}\ d\tau\\
&= \int_0^t e^{-at+a\tau}\ \tau e^{-b\tau}\ d\tau\\
&= \int_0^t e^{-at}e^{a\tau}\ \tau e^{-b\tau}\ d\tau\\
&= e^{-at}\int_0^t e^{a\tau}\ \tau e^{-b\tau}\ d\tau\\
&= e^{-at}\int_0^t \tau e^{\tau(a-b)}\ \ d\tau
\end{aligned}$$

Integrate by parts. $u = \tau, v = \frac{e^{\tau(a-b)}}{a-b}$

$$
\begin{aligned}
e^{-at} \circledast te^{-bt} &= e^{-at}\left\{ \left[\tau \frac{e^{\tau(a-b)}}{a-b} \right]_0^t - \int_0^t \frac{e^{\tau(a-b)}}{a-b}\ d\tau \right\} \\
&= e^{-at}\left\{ \left[\tau \frac{e^{\tau(a-b)}}{a-b} \right]_0^t - \frac{1}{a-b}\left[\frac{e^{\tau(a-b)}}{a-b} \right]_0^t \right\} \\
&= e^{-at}\left(\frac{1}{a-b}\left[\tau e^{\tau(a-b)} \right]_0^t - \frac{1}{(a-b)^2}\left[e^{\tau(a-b)} \right]_0^t \right) \\
&= e^{-at}\left(\frac{1}{a-b}\left[te^{t(a-b)} \right] - \frac{1}{(a-b)^2}\left[e^{t(a-b)} - 1 \right] \right) \\
&= e^{-at}\left(\frac{te^{ta-tb}}{a-b} - \frac{e^{ta-tb}-1}{(a-b)^2} \right) \\
&= \frac{te^{-tb}}{a-b} - \frac{e^{-tb}-e^{-at}}{(a-b)^2} \\
&= \frac{(a-b)te^{-tb} - e^{-tb} + e^{-at}}{(a-b)^2} \\
&= \frac{((a-b)t-1)e^{-tb} + e^{-at}}{(a-b)^2}
\end{aligned}
$$

So the inverse laplace transform of $\frac{1}{(p+a)(p+b)^2}$ is

$$
\frac{((a-b)t-1)\, e^{-tb} + e^{-at}}{(a-b)^2}
$$

3.13.15 chapter 15, problem 6.2

Problem

Find the inverse laplace transform using 6.6 of the function $\frac{1}{p^4-1}$

Solution

6.6 states that $f(t)$ = sum of all residues of $F(z)e^{zt}$ at all poles. Poles of $F(z) = \frac{1}{z^4-1}$are at $\pm 1, \pm i$. Hence

$$
F(z) = \frac{e^{zt}}{(z-1)(z+1)(z-i)(z+i)}
$$

Since each pole is of order 1, we use equation 6.1 page 599 which says

$$
\text{Residue of } F(z) \text{ at } z = z_0 \text{ is } \lim_{z \to z_0} (z - z_0)F(z)
$$

Hence sum of residue is

$$\begin{aligned}
R &= \lim_{z\to+1}\frac{e^{zt}}{(z+1)(z-i)(z+i)} + \lim_{z\to-1}\frac{e^{zt}}{(z-1)(z-i)(z+i)}\\
&+ \lim_{z\to+i}\frac{e^{zt}}{(z-1)(z+1)(z+i)} + \lim_{z\to-i}\frac{e^{zt}}{(z-1)(z+1)(z-i)}\\
&= \frac{e^{t}}{(1+1)(1-i)(1+i)} + \frac{e^{-t}}{(-1-1)(-1-i)(-1+i)}\\
&+ \frac{e^{it}}{(i-1)(i+1)(i+i)} + \frac{e^{-it}}{(-i-1)(-i+1)(-i-i)}\\
&= \frac{e^{t}}{4} + \frac{e^{-t}}{-4} + \frac{e^{it}}{-4i} + \frac{e^{-it}}{4i}\\
&= \left(\frac{e^{t}-e^{-t}}{4}\right) - \frac{1}{2}\left(\frac{e^{it}-e^{-it}}{2i}\right)\\
&= \left(\frac{e^{t}-e^{-t}}{4}\right) - \frac{1}{2}(\sin t)
\end{aligned}$$

3.13.16 chapter 15, problem 6.4

Problem

Find the inverse laplace transform using 6.6 of the function $\frac{p^3}{p^4-16}$

Solution

6.6 states that $f(t)$ = sum of all residues of $F(z)e^{zt}$ at all poles. Poles of $F(z) = \frac{z^3}{z^4-16}$ are at $\pm 2, \pm 2i$, Hence

$$F(z) = \frac{z^3 e^{zt}}{(z-2)(z+2)(z-2i)(z+2i)}$$

Since each pole is of order 1, we use equation 6.1 page 599 which says

$$\text{Residue of } F(z) \text{ at } z = z_0 \text{ is } \lim_{z\to z_0}(z-z_0)F(z)$$

Hence sum of residue is

$$R = \lim_{z\to+2}\frac{z^3e^{zt}}{(z+2)(z-2i)(z+2i)} + \lim_{z\to-2}\frac{z^3e^{zt}}{(z-2)(z-2i)(z+2i)}$$
$$+ \lim_{z\to+2i}\frac{z^3e^{zt}}{(z-2)(z+2)(z+2i)} + \lim_{z\to-2i}\frac{z^3e^{zt}}{(z-2)(z+2)(z-2i)}$$

$$= \frac{8e^{2t}}{(2+2)(2-2i)(2+2i)} + \frac{-8e^{-2t}}{(-2-2)(-2-2i)(-2+2i)}$$
$$+ \frac{(2i)^3e^{2it}}{(2i-2)(2i+2)(2i+2i)} + \frac{(-2i)^3e^{-2it}}{(-2i-2)(-2i+2)(-2i-2i)}$$

$$= \frac{8e^{2t}}{(4)\,8} + \frac{-8e^{-2t}}{(-4)8} + \frac{(-8i)e^{2it}}{-8(4i)} + \frac{(8i)e^{-2it}}{-8(-4i)}$$
$$= \frac{e^{2t}}{4} + \frac{e^{-2t}}{4} + \frac{e^{2it}}{4} + \frac{e^{-2it}}{4}$$
$$= \frac{e^{2t}+e^{-2t}}{4} + \frac{1}{2}(\cos 2t)$$

So inverse Laplace transform of $\frac{p^3}{p^4-16}$ is

$$\frac{e^{2t}+e^{-2t}}{4} + \frac{1}{2}(\cos 2t)$$

3.13.17 chapter 15, problem 6.5

Problem

Find the inverse laplace transform using 6.6 of the function $\frac{3p^2}{p^3+8}$

Solution

6.6 states that $f(t)$ = sum of all residues of $F(z)e^{zt}$ at all poles. To find poles, look at $p^3+8=0$, hence $p^3=-8$, $p=-8^{\frac{1}{3}}$. Let

$$8^{\frac{1}{3}} = re^{i\theta}$$

Then the roots are

$$= 8^{\frac{1}{3}}e^{\frac{i0}{3}}, 8^{\frac{1}{3}}e^{\frac{i(0+2\pi)}{3}}, 8^{\frac{1}{3}}e^{\frac{i(0+4\pi)}{3}}$$
$$= 8^{\frac{1}{3}}\,, 8^{\frac{1}{3}}(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}), 8^{\frac{1}{3}}(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3})$$
$$= 2\,, 2(\cos 120^0 + i\sin 120^0), 2(\cos 240^0 + i\sin 240^0)$$
$$= 2\,, 2(-\frac{1}{2} + i\frac{\sqrt{3}}{2}), 2(-\frac{1}{2} + i\frac{-\sqrt{3}}{2})$$
$$= 2, -1+i\sqrt{3}, -1-i\sqrt{3}$$

Hence

$$p = -2, 1-i\sqrt{3}, 1+i\sqrt{3}$$

And

$$\begin{aligned} F(z) &= \frac{3z^2}{z^3+8}e^{zt} \\ &= \frac{3z^2}{(z+2)\left(z-\left(1-i\sqrt{3}\right)\right)\left(z-\left(1+i\sqrt{3}\right)\right)}e^{zt} \end{aligned}$$

Since each pole is of order 1, we use equation 6.1 page 599 which says

$$\text{Residue of } F(z) \text{ at } z = z_0 \text{ is } \lim_{z\to z_0} (z-z_0)F(z)$$

Hence sum of residue is

$$\begin{aligned} R &= \lim_{z\to -2} \frac{3z^2}{\left(z-\left(1-i\sqrt{3}\right)\right)\left(z-\left(1+i\sqrt{3}\right)\right)}e^{zt} + \lim_{z\to\left(1-i\sqrt{3}\right)} \frac{3z^2}{(z+2)\left(z-\left(1+i\sqrt{3}\right)\right)}e^{zt} \\ &\quad + \lim_{z\to\left(1+i\sqrt{3}\right)} \frac{3z^2}{(z+2)\left(z-\left(1-i\sqrt{3}\right)\right)}e^{zt} \\ &= \frac{3(-2)^2}{\left(-2-\left(1-i\sqrt{3}\right)\right)\left(-2-\left(1+i\sqrt{3}\right)\right)}e^{-2t} + \frac{3\left(1-i\sqrt{3}\right)^2}{\left(\left(1-i\sqrt{3}\right)+2\right)\left(\left(1-i\sqrt{3}\right)-\left(1+i\sqrt{3}\right)\right)}e^{\left(1-i\sqrt{3}\right)t} \\ &\quad + \frac{3\left(1+i\sqrt{3}\right)^2}{\left(\left(1+i\sqrt{3}\right)+2\right)\left(\left(1+i\sqrt{3}\right)-\left(1-i\sqrt{3}\right)\right)}e^{\left(1+i\sqrt{3}\right)t} \\ &= \frac{12}{\left(-3+i\sqrt{3}\right)\left(-3-i\sqrt{3}\right)}e^{-2t} + \frac{3\left(-2-2i\sqrt{3}\right)}{\left(3-i\sqrt{3}\right)\left(-2i\sqrt{3}\right)}e^{\left(1-i\sqrt{3}\right)t} + \frac{3\left(-2+2i\sqrt{3}\right)}{\left(3+i\sqrt{3}\right)\left(2i\sqrt{3}\right)}e^{\left(1+i\sqrt{3}\right)t} \\ &= \frac{12}{12}e^{-2t} + \frac{-6-6i\sqrt{3}}{-6i\sqrt{3}-6}e^{\left(1-i\sqrt{3}\right)t} + \frac{-6+6i\sqrt{3}}{6i\sqrt{3}-6}e^{\left(1+i\sqrt{3}\right)t} \\ &= e^{-2t} + e^{\left(1-i\sqrt{3}\right)t} + e^{\left(1+i\sqrt{3}\right)t} \\ &= e^{-2t} + e^t e^{-i\sqrt{3}t} + e^t e^{i\sqrt{3}t} \\ &= e^{-2t} + e^t\left(e^{i\sqrt{3}t} + e^{-i\sqrt{3}t}\right) \\ &= e^{-2t} + e^t\left(2\cos\sqrt{3}t\right) \end{aligned}$$

So inverse Laplace transform of $\frac{3p^2}{p^3+8}$ is

$$e^{-2t} + e^t\left(2\cos\sqrt{3}t\right)$$

3.13.18 chapter 15, problem 6.9

Problem

Find the inverse laplace transform using 6.6 of the function $\frac{p}{p^4-1}$

Solution

6.6 states that $f(t) =$ sum of all residues of $F(z)e^{zt}$ at all poles. To find poles, look at $p^4-1=0$, hence $p^4=1$, $p=1^{\frac{1}{4}}$. Let

$$1^{\frac{1}{4}} = re^{i\theta}$$

Then roots are

$$
\begin{aligned}
&= 1^{\frac{1}{4}} e^{\frac{i0}{4}}, 1^{\frac{1}{4}} e^{\frac{i(0+2\pi)}{4}}, 1^{\frac{1}{4}} e^{\frac{i(0+4\pi)}{4}}, ^{\frac{1}{4}} e^{\frac{i(0+6\pi)}{4}} \\
&= 1\,, 1\left(\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right), 1\left(\cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4}\right), 1\left(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right) \\
&= 1\,, (0+i), (-1+i\,0), (0-i\,1) \\
&= \ 1, i, -1, -i
\end{aligned}
$$

Therefore

$$
p = \ 1, i, -1, -i
$$

Hence

$$
\begin{aligned}
F(z) &= \frac{z}{z^4 - 1} e^{zt} \\
&= \frac{z}{(z-1)(z-i)(z+1)(z+i)} e^{zt}
\end{aligned}
$$

Since each pole is of order 1, we use equation 6.1 page 599 which says

$$
\text{Residue of } F(z) \text{ at } z = z_0 \ \text{ is } \ \lim_{z \to z_0} \ (z - z_0) F(z)
$$

Hence sum of residue is

$$
\begin{aligned}
R = &\lim_{z\to 1} \frac{z}{(z-i)(z+1)(z+i)} e^{zt} + \lim_{z \to i} \frac{z}{(z-1)(z+1)(z+i)} e^{zt} \\
&+ \lim_{z\to -1} \frac{z}{(z-1)(z-i)(z+i)} e^{zt} + \lim_{z\to -i} \frac{z}{(z-1)(z-i)(z+1)} e^{zt} \\
= &\ \frac{1}{(1-i)(1+1)(1+i)} e^{t} + \frac{i}{(i-1)(i+1)(i+i)} e^{it} \\
&+ \frac{-1}{(-1-1)(-1-i)(-1+i)} e^{-t} + \frac{-i}{(-i-1)(-i-i)(-i+1)} e^{-it} \\
= &\ \frac{1}{4} e^{t} + \frac{i}{-4i} e^{it} + \frac{1}{4} e^{-t} + \frac{-i}{4i} e^{-it} \\
= &\ \frac{1}{4}\left(e^{t} + e^{-t}\right) - \frac{1}{2}(\cos t)
\end{aligned}
$$

So inverse Laplace transform of $\frac{p}{p^4-1}$ is

$$
\frac{1}{4}\left(e^{t} + e^{-t}\right) - \frac{1}{2}(\cos t)
$$

3.13.19 chapter 15, problem 7.11

Problem

Using the δ function method, Find the response of the following system to a unit impulse. $\frac{d^4 y}{dy^4} - y = \delta(t - t_0)$

Solution

Taking the laplace transform of each side gives (assuming initial conditions for the system are at rest)

$$
\begin{aligned}
Yp^4 - Y &= e^{-pt_0}\\
Y &= \frac{e^{-pt_0}}{p^4-1}\\
Y &= \frac{e^{-pt_0}}{\left(p^2-1\right)\left(p^2+1\right)}
\end{aligned}
$$

Finding the inverse laplace of $\frac{1}{(p-1)(p+1)(p^2+1)} = \frac{1}{(p-1)(p+1)}\frac{1}{(p^2+1)} = GH$. Then $g(t) = \frac{e^t - e^{-t}}{2}$ using L7 and, $h(t) = \sin t$ using L3. Hence the inverse transform is

$$
\begin{aligned}
g(t) \circledast h(t) &= \int_0^t \frac{e^{\tau} - e^{-\tau}}{2} \sin(t-\tau)\, d\tau\\
&= \frac{1}{2}(\sinh t - \sin t)
\end{aligned}
$$

Using L28 with the result above we get

$$
y(t) = \begin{cases} \frac{1}{2}(\sinh(t-t_0) - \sin(t-t_0)) & t > t_0\\ 0 & t < t_0 \end{cases}
$$

Or by expressing sinh using exp, the above becomes

$$
y(t) = \begin{cases} \frac{1}{4}\left(e^{t-t_0} - e^{-t+t_0} - 2\sin(t-t_0)\right) & t > t_0\\ 0 & t < t_0 \end{cases}
$$

3.13.20 chapter 15, problem 7.7

Problem

Using the δ function method, Find the response of the following system to a unit impulse. $y'' + 2y' + y = \delta(t-t_0)$

Solution

Take the laplace transform of each side we get (assume initial conditions for a system at rest)

$$
\begin{aligned}
Yp^2 + 2Yp + Y &= e^{-pt_0}\\
Y &= \frac{e^{-pt_0}}{p^2+2p+1}\\
Y &= \frac{e^{-pt_0}}{\left(p+1\right)^2}
\end{aligned}
$$

Using L28 and L6 (for $k=1$)

$$
y(t) = \begin{cases} (t-t_0)e^{-(t-t_0)} & t > t_0\\ 0 & t < t_0 \end{cases}
$$

3.13.21 chapter 15, problem 7.9

Problem

Using the δ function method, Find the response of the following system to a unit impulse.
$y'' + 2y' + 10y = \delta(t - t_0)$

Solution

Taking the laplace transform of each side we get (assume initial conditions for a system at rest)

$$
\begin{aligned}
Yp^2 + 2Yp + 10Y &= e^{-pt_0} \\
Y &= \frac{e^{-pt_0}}{p^2 + 2p + 10} \\
Y &= \frac{e^{-pt_0}}{\left(p-a\right)\left(p-b\right)}
\end{aligned}
$$

Where $a = -1 + 3i,\ b = -1 - 3i$ the roots of $p^2 + 2p + 10$. Using L28 and L7

$$
y(t) = \begin{cases} \frac{e^{a\left(t-t_0\right)} - e^{b\left(t-t_0\right)}}{(-b)-(-a)} & t > t_0 \\ 0 & t < t_0 \end{cases}
$$

Replacing values for a, b gives

$$
\begin{aligned}
y(t) &= \frac{e^{a(t-t_0)} - e^{b(t-t_0)}}{a-b} \\
&= \frac{e^{(-1+3i)(t-t_0)} - e^{(-1-3i)(t-t_0)}}{(-1+3i) - (-1-3i)} \\
&= \frac{e^{(-1+3i)(t-t_0)} - e^{(-1-3i)(t-t_0)}}{6i} \\
&= \frac{e^{-t+t_0+3it-3it_0} - e^{-t+t_0-3it+3it_0}}{6i} \\
&= e^{-t+t_0}\frac{e^{3i(t-t_0)} - e^{-3i(t-t_0)}}{6i} \\
&= e^{-t+t_0}\left(\frac{1}{3}\right)\left(\frac{e^{3i(t-t_0)} - e^{-3i(t-t_0)}}{2i}\right) \\
&= \frac{e^{-t+t_0}}{3}\sin 3(t-t_0)
\end{aligned}
$$

Therefore

$$
y(t) = \begin{cases} \frac{e^{-t+t_0}}{3}\sin 3(t-t_0) & t > t_0 \\ 0 & t < t_0 \end{cases}
$$

3.14 HW 13

Local contents

3.14.1 chapter 9, problem 2.1

Problem

Write and solve the Euler equation to make the following integral stationary

$\int_{x_1}^{x_2} \sqrt{x}\sqrt{1+y'^2}\,dx$

Solution

Let $F = \left(x,y,y'\right) = \sqrt{x}\sqrt{1+y'^2}$

The Euler equation is

$$\begin{aligned}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} &= 0\\ \frac{\partial F}{\partial y} &= \frac{\partial}{\partial y}\left(\sqrt{x}\sqrt{1+y'^2}\right) = 0\end{aligned}$$

Hence the Euler equation becomes

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

This means that $\frac{\partial F}{\partial y'} = C$ for some constant C.

$$\frac{\partial F}{\partial y'} = \sqrt{x}\,\frac{y'}{\sqrt{1+y'^2}}$$

Hence

$$\begin{aligned}
\sqrt{x}\frac{y'}{\sqrt{1+y'^2}} &= C\\
y'^2 &= \frac{C^2\left(1+y'^2\right)}{x}\\
x &= \frac{C^2+C^2y'^2}{y'^2}\\
x &= \frac{C^2}{y'^2}+k\\
y'^2 &= \frac{C^2}{x-C^2}\\
y' &= \frac{C}{\sqrt{x-C^2}}\\
y(x) &= \frac{2C}{\sqrt{x-C^2}}+C_1\\
\frac{y(x)}{2C}-\frac{C_1}{C} &= \frac{1}{\sqrt{x-C^2}}
\end{aligned}$$

Let $\frac{C_1}{C}=-b$ (some constant), and Let $\frac{1}{2C}=a$ (constant), Hence above becomes

$$\begin{aligned}
a\,y+b &= \frac{1}{\sqrt{x-\frac{1}{4a^2}}}\\
a\,y+b &= \frac{2a}{\sqrt{4a^2\,x-1}}
\end{aligned}$$

This is equation of a parabola.

3.14.2 chapter 9, problem 2.3

Problem

Write and solve the Euler equation to make the following integral stationary $\int_{x_1}^{x_2} x\sqrt{1-y'^2}\,dx$

Solution

Let $F=\left(x,y,y'\right)=x\sqrt{1-y'^2}$. The Euler equation is

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y} &= 0\\
\frac{\partial F}{\partial y} &= \frac{\partial}{\partial y}\left(x\sqrt{1-y'^2}\right)=0
\end{aligned}$$

Hence Euler equation becomes

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$$

This means that $\frac{\partial F}{\partial y'}=C$ for some constant C.

$$\frac{\partial F}{\partial y'}=\frac{-x\,y'}{\sqrt{1-y'^2}}$$

Hence

$$\begin{aligned}
\frac{-x\,y'}{\sqrt{1-y'^2}} &= C\\
y'^2 &= \frac{C^2\left(1-y'^2\right)}{x^2}\\
x^2 &= \frac{C^2-C^2y'^2}{y'^2}\\
x^2 &= \frac{C^2}{y'^2}-C^2\\
y'^2 &= \frac{C^2}{x^2+C^2}\\
y' &= \frac{C}{\sqrt{x^2+C^2}}\\
y(x) &= C\operatorname{arcsinh}\left(\frac{x}{C}\right)+C_1\\
\frac{y-C_1}{C} &= \operatorname{arcsinh}\left(\frac{x}{C}\right)\\
\frac{x}{C} &= \sinh\left(\frac{y}{C}-\frac{C_1}{C}\right)
\end{aligned}$$

Let $\frac{C_1}{C}=-b$ (some constant). Let $\frac{1}{C}=a$ (some constant). Hence the above becomes

$$a\,x=\sinh\left(a\,y+b\right)$$

3.14.3 chapter 9, problem 2.6

Problem

Write and solve the Euler equation to make the following integral stationary $\int_{x_1}^{x_2}\left(y'^2+\sqrt{y}\right)dx$

Solution

Let $F\left(x,y,y'\right)=y'^2+\sqrt{y}$. Since F does not depend on x, we change the integration variable to y. Let $y'=\frac{1}{x'}$, then $dx=\frac{dx}{dy}dy$. Hence the integral becomes

$$\int_{y_1}^{y_2}\left(\frac{1}{x'^2}+\sqrt{y}\right)x'\,dy=\int_{y_1}^{y_2}\left(\frac{1}{x'}+x'\sqrt{y}\right)dy$$

Now $F\left(y,x'\right)=\left(\frac{1}{x'}+x'\sqrt{y}\right)$. The Euler equation changes from $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0$ to $\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right)-\frac{\partial F}{\partial x}=0$. Now, $\frac{\partial F}{\partial x}=0$ since F does not depend on x,Hence the Euler equation reduces to

$$\begin{aligned}
\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) &= 0\\
\frac{d}{dy}\left(-\frac{1}{x'^2}+\sqrt{y}\right) &= 0
\end{aligned}$$

Hence $-\frac{1}{x'^2} + \sqrt{y} = C$ where C is some constant

$$
\begin{aligned}
-\frac{1}{x'^2} &= C - \sqrt{y} \\
-\frac{1}{C - \sqrt{y}} &= x'^2 \\
\frac{1}{b + \sqrt{y}} &= x'^2 \qquad \text{where } b \text{ is a new constant } = -C \\
\frac{1}{\sqrt{k + \sqrt{y}}} &= \frac{dx}{dy} \\
\int \frac{dy}{\sqrt{b + \sqrt{y}}} &= \int dx \\
\frac{4}{3}\left(-2b + \sqrt{y}\right)\left(\sqrt{b + \sqrt{y}}\right) &= x + a \qquad \text{Where } a \text{ is constant of integration}
\end{aligned}
$$

Hence the solution is

$$
\frac{4}{3}\left(\sqrt{y} - 2b\right)\left(\sqrt{b + \sqrt{y}}\right) = x + a
$$

3.14.4 chapter 9, problem 3.2

Problem

Write and solve the Euler equation to make the following integral stationary $\int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{y^2} dx$

Solution

Let $F(x, y, y') = \frac{\sqrt{1+y'^2}}{y^2}$. Since F does not depend on x, we change the integration variable to y. Let $y' = \frac{1}{x'}$, hence $dx = \frac{dx}{dy} dy$. The integral becomes

$$
\int_{y_1}^{y_2} \left(\frac{\sqrt{1 + \frac{1}{x'^2}}}{y^2} \right) x' \, dy = \int_{y_1}^{y_2} \frac{\sqrt{x'^2 + 1}}{y^2} \, dy
$$

Now $F(y, x') = \frac{\sqrt{x'^2+1}}{y^2}$. The Euler equation changes from $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ to $\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0$. But $\frac{\partial F}{\partial x} = 0$ since F does not depend on x,Hence the Euler equation reduces to

$$
\begin{aligned}
\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) &= 0 \\
\frac{\partial F}{\partial x'} &= \frac{\partial}{\partial x'}\left(\frac{\sqrt{x'^2 + 1}}{y^2}\right) \\
&= \frac{x'}{y^2 \sqrt{x'^2 + 1}}
\end{aligned}
$$

Hence

$$
\frac{d}{dy}\left(\frac{x'}{y^2 \sqrt{x'^2 + 1}}\right) = 0
$$

Hence $\frac{x'}{y^2\sqrt{x'^2+1}} = C$ where C is some constant

$$\begin{aligned}
\frac{x'^2}{x'^2+1} &= C\,y^4\\
\frac{x'^2+1}{x'^2} &= \frac{1}{C\,y^4}\\
1+\frac{1}{x'^2} &= \frac{1}{C\,y^4}\\
\frac{1}{x'^2} &= \frac{1-Cy^4}{C\,y^4}\\
\frac{\sqrt{C}\;y^2}{\sqrt{1-Cy^4}} &= x'\\
\frac{\sqrt{C}\;y^2}{\sqrt{1-Cy^4}} &= \frac{dx}{dy}\\
\int\frac{\sqrt{C}\;y^2}{\sqrt{1-Cy^4}} &= \int dx
\end{aligned}$$

The solution is

$$\frac{\sqrt{C}\;y^3}{3\sqrt{1-Cy^4}} = x + C_1$$

Where C_1 is constant of integration. Let $C_1 = a, C = b$ hence solution can be written as

$$\frac{\sqrt{b}\;y^3}{3\sqrt{1-by^4}} = x + a$$

3.14.5 chapter 9, problem 3.4

Problem

Write and solve the Euler equation to make the following integral stationary $\int_{x_1}^{x_2} y\sqrt{y'^2+y^2}\,dx$

Solution

Let $F\left(x,y,y'\right) = y\sqrt{y'^2+y^2}$. Since F does not depend on x, we change the integration variable to y. Let $y' = \frac{1}{x'}$ and $dx = \frac{dx}{dy}dy$. Hence the integral becomes

$$\int_{y_1}^{y_2}\left(y\sqrt{\frac{1}{x'^2}+y^2}\right)x'\,dy = \int_{y_1}^{y_2} y\sqrt{1+x'^2\,y^2}\,dy$$

Now $F\left(y,x'\right) = y\sqrt{1+x'^2\,y^2}$. The Euler equation changes from $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y} = 0$ to $\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right)-\frac{\partial F}{\partial x} = 0$. But $\frac{\partial F}{\partial x} = 0$ since F does not depend on x,Hence the Euler equation reduces to

$$\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) = 0$$

$$\begin{aligned}
\frac{\partial F}{\partial x'} &= \frac{\partial}{\partial x'}\left(y\sqrt{1+x'^2\,y^2}\right)\\
&= y\left(\frac{x'\,y^2}{\sqrt{1+x'^2\,y^2}}\right)\\
&= \frac{x'\,y^3}{\sqrt{1+x'^2\,y^2}}
\end{aligned}$$

Hence

$$\frac{d}{dy}\left(\frac{x'\,y^3}{\sqrt{1+x'^2\,y^2}}\right)=0$$

Hence $\frac{x'\,y^3}{\sqrt{1+x'^2\,y^2}}=C$ where C is some constant

$$\begin{aligned}
x'\,y^3 &= C\sqrt{1+x'^2\,y^2}\\
x'^2\,y^6 &= C^2\left(1+x'^2\,y^2\right)\\
x'^2\,y^6 &= C^2+C^2\,x'^2\;y^2\\
x'^2\left(y^6-C^2\,y^2\right) &= C^2\\
x'^2 &= \frac{C^2}{\left(y^6-C^2\,y^2\right)}\\
x' &= \frac{C}{y\sqrt{y^4-C^2}}\\
\int dx &= C\int\frac{1}{y\sqrt{y^4-C^2}}dy
\end{aligned}$$

The solution is (using Mathematica)

$$a\,x=-\frac{1}{2}i\log\left(\frac{-2iC+2\sqrt{-C^2+y^4}}{y^2}\right)$$

3.14.6 chapter 9, problem 3.6

Problem

Write and solve the Euler equation to make the following integral stationary $\int_{x_1}^{x_2}\frac{y\,y'^2}{1+y\,y'}\,dx$

Solution

Let $F\left(x,y,y'\right)=\frac{y\,y'^2}{1+y\,y'}$. Since F does not depend on x, we change the integration variable to y. Let $y'=\frac{1}{x'},dx=\frac{dx}{dy}dy$. Hence the integral becomes

$$\begin{aligned}
\int_{y_1}^{y_2}\left(\frac{y\,\frac{1}{x'^2}}{1+y\,\frac{1}{x'}}\right)x'\,dy &= \int_{y_1}^{y_2}\left(\frac{y\,\frac{1}{x'^2}}{\frac{x'+\,y}{x'}}\right)x'\,dy\\
&= \int_{y_1}^{y_2}\left(\frac{y\,\frac{1}{x'}}{x'\,+\,y}\right)x'\,dy\\
&= \int_{y_1}^{y_2}\left(\frac{y}{x'+y}\right)dy
\end{aligned}$$

Now $F\left(y,x'\right)=\left(\frac{y}{x'+y}\right)$. The Euler equation changes from $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0$ to $\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right)-\frac{\partial F}{\partial x}=0$. $\frac{\partial F}{\partial x}=0$ since F does not depend on x,Hence the Euler equation reduces to

$$\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right)=0$$

$$\begin{aligned}
\frac{\partial F}{\partial x'} &= \frac{\partial}{\partial x'}\left(\frac{y}{x'+y}\right)\\
&= y\left(-\frac{1}{\left(x'+y\right)^2}\right)\\
&= \frac{-y}{\left(x'+y\right)^2}
\end{aligned}$$

Hence

$$\frac{d}{dy}\left(\frac{-y}{\left(x'+y\right)^2}\right)=0$$

Hence $\frac{-y}{\left(x'+y\right)^2}=C$ where C is some constant

$$-y=C\left(x'+y\right)^2$$

Let $C=-k$

$$\begin{aligned}
y &= k\ \left(x'+y\right)^2\\
\sqrt{\frac{y}{k}} &= x'+y\\
\sqrt{\frac{y}{k}}-y &= x'\\
\sqrt{\frac{y}{k}}-y &= \frac{dx}{dy}\\
\int\sqrt{\frac{y}{k}}-y\,dy &= \int dx\\
-\frac{y^2}{2}+\frac{2}{3}y\sqrt{\frac{y}{k}} &= x+\beta
\end{aligned}$$

Where β is the integration constant. Let $\frac{1}{\sqrt{k}}=\alpha$ a new constant

$$x=-\frac{1}{2}y^2+\frac{2}{3}\alpha\,y^{\frac{3}{2}}-\beta$$

Let $\frac{2}{3}\alpha=a$ a new integration constant, let $-\beta=b$ a new constant, we get

$$x=a\,y^{\frac{3}{2}}-\frac{1}{2}y^2+b$$

3.14.7 chapter 9, problem 3.9

Problem

Write and solve the Euler equation to make the following integral stationary $\int_{\phi_1}^{\phi_2}\sqrt{\theta'^2+\sin^2\theta}\,d\phi$, $\theta'=\frac{d\theta}{d\phi}$

Solution

Here $F\left(x,y(x),y'(x)\right)$ becomes $F\left(\phi,\theta\left(\phi\right),\theta'\left(\phi\right)\right)$. So now $x\to\phi, y\to\theta, y'\to\theta'$. Since $F(\theta',\theta)$ does not depend on ϕ, we change the integration variable to θ, so we want to change from $\theta'=\frac{d\theta}{d\phi}$ to $\phi'=\frac{d\phi}{d\theta}$. Let $\theta'=\frac{1}{\phi'}, d\phi=\frac{d\phi}{d\theta}d\theta$. Hence the integral becomes

$$\int_{\theta_1}^{\theta_2}\left(\sqrt{\frac{1}{\phi'^2}+\sin^2\theta}\right)\phi'\,d\theta=\int_{\theta_1}^{\theta_2}\sqrt{1+\phi'^2\sin^2\theta}\,d\theta$$

So now

$$F\left(\phi',\theta\right)=\sqrt{1+\phi'^2\sin^2\theta}$$

The Euler equation changes from $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0$ to $\frac{d}{d\theta}\left(\frac{\partial F}{\partial\phi'}\right)-\frac{\partial F}{\partial\phi}=0$. $\frac{\partial F}{\partial\phi}=0$ since F does not depend on ϕ,Hence the Euler equation reduces to

$$\frac{d}{d\theta}\left(\frac{\partial F}{\partial\phi'}\right)=0$$

$$\begin{aligned}\frac{\partial F}{\partial \phi'} &= \frac{\partial}{\partial \phi'}\left(\sqrt{1+\phi'^2\sin^2\theta}\right)\\ &= \frac{\phi'\sin^2\theta}{\sqrt{1+\phi'^2\sin^2\theta}}\end{aligned}$$

Hence

$$\frac{d}{d\theta}\left(\frac{\phi'\sin^2\theta}{\sqrt{1+\phi'^2\sin^2\theta}}\right)=0$$

Hence $\frac{\phi'\sin^2\theta}{\sqrt{1+\phi'^2\sin^2\theta}}=C$ where C is some constant

$$\begin{aligned}\phi'\sin^2\theta &= C\sqrt{1+\phi'^2\sin^2\theta}\\ \phi'^2\sin^4\theta &= C^2\left(1+\phi'^2\sin^2\theta\right)\\ \phi'^2\sin^4\theta &= C^2+C^2\phi'^2\sin^2\theta\\ \phi'^2 &= \frac{C^2}{\sin^4\theta-C^2\sin^2\theta}\\ \phi' &= \frac{C}{\sin\theta\sqrt{\sin^2\theta-C^2}}\\ \int d\phi &= \int\frac{C}{\sin\theta\sqrt{\sin^2\theta-C^2}}d\theta\\ \phi+\alpha &= -\frac{C\ \tanh^{-1}\left(\frac{\sqrt{2}\sqrt{C^2}\cos(\theta)}{\sqrt{1-2C^2-\cos(2\theta)}}\right)}{\sqrt{-C^2}}\end{aligned}$$

The last integral value was found using mathematica. Hence

$$\frac{\sqrt{-C^2}\left(\phi+\alpha\right)}{-C}=\operatorname{arctanh}\left(\frac{\sqrt{2}\sqrt{C^2}\cos(\theta)}{\sqrt{1-2C^2-\cos(2\theta)}}\right)$$

Let $\frac{\sqrt{-C^2}}{-C}=A$, let $\sqrt{2}\sqrt{C^2}=B, 1-2C^2=D$, then

$$\begin{aligned}A\left(\phi+\alpha\right) &= \operatorname{arctanh}\left(\frac{B\cos(\theta)}{\sqrt{D-\cos(2\theta)}}\right)\\ \tanh\left(A\left(\phi+\alpha\right)\right) &= \frac{B\cos(\theta)}{\sqrt{D-\cos(2\theta)}}\end{aligned}$$

3.14.8 chapter 9, problem 5.2

Problem

Set up Lagrange equations in cylindrical coordinates for a particle of mass m in a potential field $V(r,\theta,z)$

Solution

$L=T-V$ where T is the K.E. and V the potential energy. $T=\frac{1}{2}mv^2$, But

$$ds^2=dr^2+r^2d\theta^2+dz^2$$

As shown on page 219 equation 4.4 , now differentiate both sides w.r.t. time

$$\begin{aligned}2\,ds\,\frac{ds}{dt} &= 2dr\,\dot{r}+\left(r^2\,2\,d\theta\,\dot{\theta}+2r\,\dot{r}\,d\theta^2\right)+2dz\,\dot{z}\\ \frac{ds}{dt} &= \frac{dr\,\dot{r}+r^2\ d\theta\,\dot{\theta}+r\,\dot{r}\,d\theta^2+dz\,\dot{z}}{\sqrt{dr^2+r^2d\theta^2+dz^2}}\end{aligned}$$

Hence

$$v^2 = \frac{\left(dr\,\dot{r} + r^2 d\theta\,\dot{\theta} + r\,\dot{r}\,d\theta^2 + dz\,\dot{z}\right)^2}{dr^2 + r^2 d\theta^2 + dz^2}$$

I used Mathematica to simplify this getting

$$v^2 = \dot{r}^2 + r^2\,\dot{\theta}^2 + \dot{z}^2$$

Hence,

$$L = \overbrace{\frac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + \dot{z}^2\right)}^{\text{K.E.}} - \overbrace{V(r,\theta,z)}^{\text{P.E.}}$$

The Lagrange equations are

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} &= 0\\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0\\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} &= 0
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\frac{d}{dt}(m\,\dot{r}) - \left(mr\dot{\theta}^2 - \frac{\partial V}{\partial r}\right) &= 0\\
\frac{d}{dt}\left(mr^2\dot{\theta}\right) + \frac{\partial V}{\partial \theta} &= 0\\
\frac{d}{dt}(m\dot{z}) + \frac{\partial V}{\partial z} &= 0
\end{aligned}$$

Now differentiating w.r.t. time, and remembering that $r(t)$ also changes with time.

$$\begin{aligned}
m\,\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} &= 0\\
m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) + \frac{\partial V}{\partial \theta} &= 0\\
m\ddot{z} + \frac{\partial V}{\partial z} &= 0
\end{aligned}$$

Hence finally we get

$$\begin{aligned}
m\left(\ddot{r} - r\dot{\theta}^2\right) &= -\frac{\partial V}{\partial r}\\
m\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) &= -\frac{1}{r}\frac{\partial V}{\partial \theta}\\
m\ddot{z} &= -\frac{\partial V}{\partial z}
\end{aligned}$$

3.14.9 chapter 9, problem 5.6

Problem

A particle moves on the surface of a sphere of radius a under the action of the earth gravitational field. Find the θ,ϕ equations of motion. (this is called the spherical pendulum).

Solution

$L = T - V$ where T is the K.E. and V the potential energy. Using spherical coordinates.

$$x = a\sin\theta\cos\phi,\ y = a\sin\theta\sin\phi,\ z = a\cos\theta$$

Hence a position vector

$$\mathbf{r} = \mathbf{i}\, a\sin\theta\cos\phi + \mathbf{j}\, a\sin\theta\sin\phi + \mathbf{k}\, a\cos\theta$$

So velocity is

$$\begin{aligned}\dot{\mathbf{r}} &= \mathbf{i}\,\frac{d}{dt}\left(a\sin\theta\cos\phi\right) + \mathbf{j}\,\frac{d}{dt}\left(a\sin\theta\sin\phi\right) + \mathbf{k}\,\frac{d}{dt}(a\cos\theta)\\ &= \mathbf{i}\,\left(-a\sin\theta\sin\phi\,\dot{\phi} + a\cos\theta\,\dot{\theta}\,\cos\phi\right) + \mathbf{j}\,\left(a\sin\theta\cos\phi\dot{\phi} + a\cos\theta\,\dot{\theta}\,\sin\phi\right) + \mathbf{k}\,\left(-a\sin\theta\,\dot{\theta}\right)\end{aligned}$$

Hence

$$\dot{r} = \|\dot{\mathbf{r}}\| = \sqrt{\left(-a\sin\theta\sin\phi\,\dot{\phi} + a\cos\theta\,\dot{\theta}\cos\phi\right)^2 + \left(a\sin\theta\cos\phi\,\dot{\phi} + a\cos\theta\,\dot{\theta}\,\sin\phi\right)^2 + \left(-a\sin\theta\,\dot{\theta}\right)^2}$$

Then

$$\begin{aligned}v^2 = \dot{r}^2 &= \left(-a\sin\theta\sin\phi\,\dot{\phi} + a\cos\theta\,\dot{\theta}\cos\phi\right)^2 + \left(a\sin\theta\cos\phi\dot{\phi} + a\cos\theta\,\dot{\theta}\,\sin\phi\right)^2 + \left(-a\sin\theta\,\dot{\theta}\right)^2\\
&= \left(a^2\sin^2\theta\sin^2\phi\,\dot{\phi}^2 + a^2\cos^2\theta\,\dot{\theta}^2\cos^2\phi - \overbrace{2a^2\sin\theta\sin\phi\,\dot{\phi}\cos\theta\,\dot{\theta}\cos\phi}\right)\\
&+ \left(a^2\sin^2\theta\cos^2\phi\,\dot{\phi}^2 + a^2\cos^2\theta\,\dot{\theta}^2\,\sin^2\phi\,\overbrace{+\,2a^2\sin\theta\cos\phi\dot{\phi}\,\cos\theta\,\dot{\theta}\,\sin\phi}\right) + \left(a^2\sin^2\theta\,\dot{\theta}^2\right)\\
&= \overbrace{a^2\sin^2\theta\sin^2\phi\,\dot{\phi}^2} + a^2\cos^2\theta\,\dot{\theta}^2\cos^2\phi + \overbrace{a^2\sin^2\theta\cos^2\phi\,\dot{\phi}^2} + a^2\cos^2\theta\,\dot{\theta}^2\,\sin^2\phi + a^2\sin^2\theta\,\dot{\theta}^2\\
&= a^2\dot{\phi}^2\sin^2\theta\,\overbrace{\left(\sin^2\phi + \cos^2\phi\right)}^{=1} + a^2\dot{\theta}^2\cos^2\theta\,\overbrace{\left(\cos^2\phi + \sin^2\phi\right)}^{=1} + a^2\sin^2\theta\,\dot{\theta}^2\\
&= a^2\dot{\phi}^2\sin^2\theta + a^2\dot{\theta}^2\overbrace{\left(\cos^2\theta + \sin^2\theta\right)}^{=1}\\
&= a^2\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right)\end{aligned}$$

Hence $T = \frac{1}{2}mv^2$. For a particle, taking mass as one unit. Hence

$$T = \frac{1}{2}a^2\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right)$$

The P.E. is $mga\ \cos\theta$. Hence the Lagrangian is

$$\begin{aligned}L &= T - V\\ L &= \frac{1}{2}a^2\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) - ga\ \cos\theta\end{aligned}$$

We have 2 independent variables, hence we need 2 Lagrangian equations

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} &= 0\\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial L}{\partial\dot{\theta}} &= a^2\dot{\theta}\\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) &= a^2\ddot{\theta}\\ \frac{\partial L}{\partial\theta} &= a^2\left(\dot{\phi}^2\sin\theta\ \cos\theta\right) + ga\ \sin\theta\end{aligned}$$

Hence the first equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0$$
$$a^2\ddot{\theta}-a^2\left(\dot{\phi}^2\sin\theta\ \cos\theta\right)-ga\ \sin\theta=0$$
$$a\ \ddot{\theta}-a\ \left(\dot{\phi}^2\sin\theta\ \cos\theta\right)-g\ \sin\theta=0$$

To find the second equation

$$\frac{\partial L}{\partial \dot{\phi}}=a^2\left(2\dot{\phi}\sin^2\theta\right)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)=\frac{d}{dt}\left(a^2\left(2\dot{\phi}\sin^2\theta\right)\right)$$
$$\frac{\partial L}{\partial \phi}=0$$

Hence the second equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)-\frac{\partial L}{\partial \phi}=0$$
$$\frac{d}{dt}\left(a^2\left(2\dot{\phi}\sin^2\theta\right)\right)=0$$
$$\frac{d}{dt}\left(\ 2\dot{\phi}\sin^2\theta\right)=0$$
$$\frac{d}{dt}\left(\ 2\dot{\phi}\sin^2\theta\right)=0$$

3.14.10 chapter 9, problem 6.1

Problem

Find surface of revolution formed by rotating the curve around the x-axis that has a minimum area subject to a curve of give length l joining 2 points.

Solution

Area is

$$I=\int_{x_1}^{x_2}2\pi y\sqrt{1+y'^2}\,dx \tag{1}$$

Since integrand does not depend on x we change the independent variable to y. $dx=\frac{dx}{dy}dy, y'=\frac{1}{x'}$. Hence (1) becomes

$$I=\int_{y_1}^{y_2}2\pi y\sqrt{1+\frac{1}{x'^2}}\ x'dy \tag{1}$$
$$=\int_{y_1}^{y_2}2\pi y\sqrt{x'^2+1}\ dy$$

Hence $F\left(y,x',x\right)=2\pi y\sqrt{x'^2+1}$. Now finding the constraint

$$g=\int ds=l$$
$$=\int_{x_1}^{x_2}\sqrt{1+y'^2}\,dx$$

Since integrand does not depend on x we change the independent variable to y. $dx=\frac{dx}{dy}dy, y'=\frac{1}{x'}$. Hence

$$g=\int_{y_1}^{y_2}\sqrt{1+\frac{1}{x'^2}}\ x'dy$$
$$=\int_{y_1}^{y_2}\sqrt{x'^2+1}\ \ dy$$

So $G = \sqrt{x'^2+1}$. Hence we get

$$F+\lambda G = \left(2\pi y\sqrt{x'^2+1}\right)+\lambda\sqrt{x'^2+1}$$

As the new Euler equation (with constrains). Solving

$$\begin{aligned}\frac{d}{dy}\left(\frac{\partial}{\partial x'}(F+\lambda G)\right)-\overbrace{\frac{\partial}{\partial x}(F+\lambda G)}^{0\text{ since does not depend on }x} &= 0\\ \frac{d}{dy}\left(\frac{\partial}{\partial x'}\left(2\pi y\sqrt{x'^2+1}+\lambda\sqrt{x'^2+1}\right)\right) &= 0\\ \frac{d}{dy}\left(\frac{2\pi y x'}{\sqrt{x'^2+1}}+\frac{\lambda x'}{\sqrt{x'^2+1}}\right) &= 0\end{aligned}$$

Hence

$$\begin{aligned}\frac{2\pi y x'}{\sqrt{x'^2+1}}+\frac{\lambda x'}{\sqrt{x'^2+1}} &= c\\ \frac{2\pi y x'+\lambda x'}{\sqrt{x'^2+1}} &= c\\ x'\left(2\pi y+\lambda\right) &= c\sqrt{x'^2+1}\\ x'^2\left(2\pi y+\lambda\right)^2 &= c^2\left(x'^2+1\right)\\ \frac{x'^2}{\left(x'^2+1\right)} &= \frac{c^2}{\left(2\pi y+\lambda\right)^2}\\ \frac{\left(x'^2+1\right)}{x'^2} &= \frac{\left(2\pi y+\lambda\right)^2}{c^2}\\ 1+\frac{1}{x'^2} &= \frac{\left(2\pi y+\lambda\right)^2}{c^2}\\ \frac{1}{x'^2} &= \frac{\left(2\pi y+\lambda\right)^2-c^2}{c^2}\\ \frac{c^2}{\left(2\pi y+\lambda\right)^2-c^2} &= x'^2\\ \frac{c}{\sqrt{\left(2\pi y+\lambda\right)^2-c^2}} &= x'\\ \frac{dx}{dy} &= \frac{c}{\sqrt{\left(2\pi y+\lambda\right)^2-c^2}}\\ \int dx &= \int\frac{c}{\sqrt{\left(2\pi y+\lambda\right)^2-c^2}}dy\\ \int dx &= \int\frac{1}{\sqrt{\left(\frac{2\pi y+\lambda}{c}\right)^2-1}}dy\\ x &= \frac{c}{2\pi}\operatorname{arccosh}\left(\frac{2\pi y+\lambda}{c}\right)+c_1\end{aligned}$$

To express this as y a function of x we get

$$\begin{aligned}\frac{2\pi}{c}(x-c_1) &= \operatorname{arccosh}\left(\frac{2\pi y+\lambda}{c}\right)\\ \cosh\left(\frac{2\pi}{c}(x-c_1)\right) &= \frac{2\pi y+\lambda}{c}\\ \frac{c\cosh\left(\frac{2\pi}{c}(x-c_1)\right)-\lambda}{2\pi} &= y\end{aligned}$$

We have 3 unknowns, c, c_1, λ that we can use boundary conditions, and length l to determine.

3.14.11 chapter 9, problem 6.2

Problem

Find the equation of the curve subject to a curve of give length l joining 2 points so that the plane area between the curve and straight line joining the points is a maximum.

Solution

Area is $\int y\, dx$. Hence area is $I = \int_{x_1}^{x_2} y\, dx$ subject to constraint that $\int ds = l$ or $g = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx = l$. Hence the Euler equation with constrains now becomes

$$F + \lambda G = y + \lambda \sqrt{y'^2 + 1}$$

Therefore

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\partial}{\partial y'}(F+\lambda G)\right) - \frac{d}{dy}(F+\lambda G) &= 0 \\
\frac{d}{dy}\left(\frac{\lambda\, y'}{\sqrt{y'^2+1}}\right) - 1 &= 0 \\
\frac{\lambda\, y'}{\sqrt{y'^2+1}} &= x + c
\end{aligned}$$

This simplifies to

$$\begin{aligned}
\int dy &= \int \frac{(x+c)}{\sqrt{\lambda^2 - (x+c)^2}} dx \\
y + c_1 &= -\sqrt{\lambda^2 - (x+c)^2} \\
\left(y + c_1\right)^2 &= \lambda^2 - (x+c)^2 \\
\left(y + c_1\right)^2 + (x+c)^2 &= \lambda^2
\end{aligned}$$

This is the equation of a circle.

3.14.12 chapter 9, problem 6.5

Problem

Given surface area of solid of revolution, finds its shape to make its volume a maximum.

Solution

Volume is $\int \pi y^2 ds$ where ds is a small segment of the curve length. Hence

$$I = \int_{x_1}^{x_2} \pi y^2 \sqrt{1+y'^2}\, dx \tag{1}$$

Constraint is that area is given, say A. Hence

$$g = \int_{x_1}^{x_2} 2\pi y \sqrt{1+y'^2}\, dx = A \tag{2}$$

Since both integrands do not depend on x we change the independent variable to y. $dx = \frac{dx}{dy}dy, y' = \frac{1}{x'}$. Hence (1) becomes

$$\begin{aligned}
I &= \int_{x_1}^{x_2} \pi y^2 \sqrt{1 + \frac{1}{x'^2}}\, x' dy \\
&= \int_{x_1}^{x_2} \pi y^2 \sqrt{x'^2 + 1}\, dy
\end{aligned}$$

And (2) becomes

$$\begin{aligned} g &= \int_{y_1}^{y_2} 2\pi y \sqrt{1+\frac{1}{x'^2}}\ x' dy \\ &= \int_{y_1}^{y_2} 2\pi y \sqrt{x'^2+1}\ dy \end{aligned}$$

Hence we get

$$F+\lambda G = \left(\pi y^2 \sqrt{x'^2+1}\right) + 2\lambda\pi y\sqrt{x'^2+1}$$

as the new Euler equation (with constrains) to solve.

$$\begin{aligned} \frac{d}{dy}\left(\frac{\partial}{\partial x'}(F+\lambda G)\right) - \overbrace{\frac{\partial}{\partial x}(F+\lambda G)}^{0 \text{ since does not depend on } x} &= 0 \\ \frac{d}{dy}\left(\frac{\partial}{\partial x'}\left(\pi y^2\sqrt{x'^2+1} + 2\lambda\pi y\sqrt{x'^2+1}\right)\right) &= 0 \\ \frac{d}{dy}\left(\frac{\pi y^2 x'}{\sqrt{x'^2+1}} + \frac{2\lambda\pi y x'}{\sqrt{x'^2+1}}\right) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} \frac{\pi y^2 x'}{\sqrt{x'^2+1}} + \frac{2\lambda\pi y x'}{\sqrt{x'^2+1}} &= c \\ \frac{\pi y^2 x' + 2\lambda\pi y x'}{\sqrt{x'^2+1}} &= c \\ \pi y^2 x' + 2\lambda\pi y x' &= c\sqrt{x'^2+1} \\ x'^2\left(\pi y^2 + 2\lambda\pi y\right)^2 &= c^2\left(x'^2+1\right) \\ \frac{x'^2}{\left(x'^2+1\right)} &= \frac{c^2}{\left(\pi y^2+2\lambda\pi y\right)^2} \\ \frac{\left(x'^2+1\right)}{x'^2} &= \frac{\left(\pi y^2+2\lambda\pi y\right)^2}{c^2} \\ 1+\frac{1}{x'^2} &= \frac{\left(\pi y^2+2\lambda\pi y\right)^2}{c^2} \\ \frac{1}{x'^2} &= \frac{\left(\pi y^2+2\lambda\pi y\right)^2 - c^2}{c^2} \\ \frac{c^2}{\left(\pi y^2+2\lambda\pi y\right)^2 - c^2} &= x'^2 \\ \frac{c}{\sqrt{\left(\pi y^2+2\lambda\pi y\right)^2 - c^2}} &= x' \\ \frac{dx}{dy} &= \frac{c}{\sqrt{\left(\pi y^2+2\lambda\pi y\right)^2 - c^2}} \\ \int dx &= \int \frac{c}{\sqrt{\left(\pi y^2+2\lambda\pi y\right)^2 - c^2}} dy \\ x &= \int \frac{c}{\sqrt{\left(\pi y^2+2\lambda\pi y\right)^2 - c^2}} dy \\ x &= \int \frac{1}{\sqrt{\left(\frac{\pi y^2+2\lambda\pi y}{c}\right)^2 - 1}} dy \end{aligned}$$

Hence

$$x = \left(\frac{c}{2y\pi + 2\lambda\pi}\right)\cosh^{-1}\left(\frac{\pi y^2 + 2\lambda\pi y}{c}\right)$$

3.14.13 chapter 15, problem 8.12

Problem

Solve $y'' + y = f(x)$ with $y(0) = y\left(\frac{\pi}{2}\right) = 0$ using 8.17:

$$y(x) = -\cos x \int_0^x \sin(x')f(x')\,dx' - \sin x \int_x^{\frac{\pi}{2}} \cos(x')f(x')\,dx'$$

when $f(x) = \sec x$

Solution

$$y(x) = -\cos x \int_0^x \sin(x')\sec x'\,dx' - \sin x \int_x^{\frac{\pi}{2}} \cos(x')\sec x'\,dx'$$

Since $\sec x' = \frac{1}{\cos x'}$ we get

$$y(x) = -\cos x \int_0^x \tan x'\,dx' - \sin x \int_x^{\frac{\pi}{2}} dx'$$

But $\int_0^x \tan x'\,dx' = -\log(\cos(x))$, Hence

$$\begin{aligned} y(x) &= \cos(x)\log(\cos(x)) - \sin x\left(\frac{1}{2}\pi - x\right) \\ &= \cos(x)\log(\cos(x)) - \frac{1}{2}\pi\sin x + x\sin x \end{aligned}$$

3.14.14 chapter 15, problem 8.15

Problem

Use Green function method and the given solutions of the homogeneous equation to find a particular solution to $y'' - y = \sec h(x)$, where $y_1(x) = \sinh(x)$, $y_2(x) = \cosh(x)$

Solution

$$y_p = y_2 \int \frac{y_1 f}{W} dx - y_1 \int \frac{y_2 f}{W} dx \tag{1}$$

Where $f = \sec h(x)$

$$\begin{aligned} W &= \begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix} \\ &= \begin{vmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{vmatrix} \\ &= \cosh^2 x - \sinh^2 x \\ &= 1 \end{aligned}$$

So from (1) we get

$$y_p = \cosh(x)\int \sinh(x)\ \sec h(x)\,dx - \sinh(x)\int \cosh(x)\sec h(x)\,dx$$

But $\sec h(x) = \frac{1}{\cosh x}$, Hence

$$\begin{aligned} y_p &= \cosh(x) \int \sinh(x) \frac{1}{\cosh x}\, dx - \sinh(x) \int \cosh(x) \frac{1}{\cosh x}\, dx \\ &= \cosh(x) \int \tan sh(x)\, dx - \sinh(x) \int dx \end{aligned}$$

But $\int \tan sh(x)\, dx = \log(\cosh(x))$, Hence

$$y_p = \cosh(x) \log(\cosh(x)) - x\ \sinh(x)$$

3.14.15 chapter 15, problem 8.17

Problem

Use Green function method and the given solutions of the homogeneous equation to find a particular solution to $y'' - 2\left(\csc^2(x)\right) y = \sin^2(x)$, where $y_1(x) = \cot x,\ \ y_2(x) = 1 - x\ \cot(x)$

Solution

Note $\cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}, \csc(x) = \frac{1}{\sin(x)}$

$$y_p = y_2 \int \frac{y_1 f}{W} dx - y_1 \int \frac{y_2 f}{W} dx \tag{1}$$

Where $f = \sin^2(x)$.

$$\begin{aligned} y_1' &= \frac{d}{dx}(\cot(x)) = -\cot^2 x - 1 \\ &= -\frac{1}{\sin^2(x)} \end{aligned}$$

And

$$\begin{aligned} y_2' &= \frac{d}{dx}(1 - x\ \cot(x)) \\ &= -\frac{\cos(x)}{\sin(x)} + \frac{x}{\sin^2(x)} \end{aligned}$$

Therefore

$$\begin{aligned} W &= \begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix} \\ &= \begin{vmatrix} -\frac{1}{\sin^2(x)} & -\frac{\cos(x)}{\sin(x)} + \frac{x}{\sin^2(x)} \\ \frac{\cos(x)}{\sin(x)} & 1 - \frac{x\ \cos(x)}{\sin(x)} \end{vmatrix} \\ &= \left(-\frac{1}{\sin^2(x)}\right)\left(1 - \frac{x\ \cos(x)}{\sin(x)}\right) - \left(-\frac{\cos(x)}{\sin(x)} + \frac{x}{\sin^2(x)}\right)\frac{\cos(x)}{\sin(x)} \\ &= -\frac{1}{\sin^2(x)} + \overbrace{\frac{x\ \cos(x)}{\sin^3(x)}} + \frac{\cos^2(x)}{\sin^2(x)} - \overbrace{\frac{x \cos(x)}{\sin^3(x)}} \\ &= -\frac{1}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)} \end{aligned}$$

So from (1) we get

$$\begin{aligned} y_p &= \left(1 - \frac{x \cos x}{\sin x}\right) \int \frac{\frac{\cos x}{\sin x} \sin^2(x)}{-\frac{1}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)}} dx - \frac{\cos x}{\sin x} \int \frac{\left(1 - \frac{x \cos x}{\sin(x)}\right) \sin^2(x)}{-\frac{1}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)}} dx \\ &= \left(1 - \frac{x \cos x}{\sin x}\right) \int \frac{\cos x\ \sin x}{\frac{-1+\cos^2 x}{\sin^2(x)}} dx - \frac{\cos x}{\sin x} \int \frac{\sin^2 x - x\ \cos x\ \sin x}{\frac{-1+\cos^2 x}{\sin^2(x)}} dx \\ &= \left(1 - \frac{x \cos x}{\sin x}\right) \int \frac{\cos x\ \sin^3 x}{-1 + \cos^2 x} dx - \frac{\cos x}{\sin x} \int \frac{\sin^4 x - x\ \cos x\ \sin^3 x}{-1 + \cos^2 x} dx \end{aligned}$$

but $I = \int \frac{\cos x \ \sin^3 x}{\cos^2 x - 1} = \int \frac{\cos x \ \sin^3 x}{-\sin^2 x} = \int -\cos x \ \sin x = \frac{1}{2}\cos^2 x$ And

$$\begin{aligned}
I &= \int \frac{\sin^4 x - x \ \cos x \ \sin^3 x}{-1+\cos^2 x} \\
&= \int \frac{\sin^4 x - x \ \cos x \ \sin^3 x}{-\sin^2 x} \\
&= \int -\sin^2 x + x \ \cos x \ \sin x \\
&= -\int \sin^2(x)\, dx + \int x \ \cos(x) \ \sin(x)\, dx
\end{aligned}$$

But $\int \sin^2(x)\, dx = \frac{x}{2} - \frac{1}{4}\sin(2x)$ and $\int x \ \cos(x) \ \sin(x)\, dx = -\frac{1}{4}x\cos(2x) + \frac{1}{8}\sin(2x)$, therefore

$$\begin{aligned}
-\int \sin^2(x)\, dx + \int x \ \cos(x) \ \sin(x)\, dx &= \left(-\frac{x}{2} + \frac{1}{4}\sin(2x)\right) + \left(-\frac{1}{4}x\cos(2x) + \frac{1}{8}\sin(2x)\right) \\
&= -\frac{x}{2} + \frac{1}{4}\sin(2x) - \frac{1}{4}x\cos(2x) + \frac{1}{8}\sin(2x) \\
&= \frac{3}{8}\sin 2x - \frac{1}{2}x - \frac{1}{4}x\cos 2x
\end{aligned}$$

Hence (2) becomes

$$\begin{aligned}
y_p(x) &= \left(1 - \frac{x\cos x}{\sin x}\right)\left(\frac{1}{2}\cos^2 x\right) - \frac{\cos x}{\sin x}\left(\frac{3}{8}\sin 2x - \frac{1}{2}x - \frac{1}{4}x\cos 2x\right) \\
&= \left(\frac{1}{2}\cos^2 x - \frac{1}{2}\frac{x\cos^3 x}{\sin x}\right) - \left(\frac{3}{8}\sin 2x\frac{\cos x}{\sin x} - \frac{1}{2}x\frac{\cos x}{\sin x} - \frac{1}{4}x\cos 2x\frac{\cos x}{\sin x}\right) \\
&= \frac{1}{2}\cos^2 x - \frac{1}{2}\frac{x\cos^3 x}{\sin x} - \frac{3}{8}\sin 2x\frac{\cos x}{\sin x} + \frac{1}{2}x\frac{\cos x}{\sin x} + \frac{1}{4}x\cos 2x\frac{\cos x}{\sin x} \\
&= \frac{1}{4}\cot x\,(x - \cos x \ \sin x)
\end{aligned}$$

3.14.16 chapter 15, problem 8.2

Problem

Solve $y'' + \omega^2 y = f(t)$ using $y(t) = \int_0^t \frac{1}{\omega}\sin\omega(t-t')\, f(t')\, dt'$ when $f(t) = \sin\omega t$

Solution

$$\begin{aligned}
y(t) &= \int_0^t \frac{1}{\omega}\sin\omega(t-t') f(t')\, dt' \\
&= \int_0^t \frac{1}{\omega}\sin\omega(t-t')\sin\omega t'\, dt' \qquad (1)
\end{aligned}$$

But $\sin\alpha\sin\beta = \frac{1}{2}\cos\left(\alpha-\beta\right) - \frac{1}{2}\cos\left(\alpha+\beta\right)$, hence

$$\begin{aligned}
\sin\omega(t-t')\sin\omega t' &= \frac{1}{2}\cos(\omega(t-t') - \omega t') - \frac{1}{2}\cos(\omega(t-t') + \omega t') \\
&= \frac{1}{2}\cos(t\omega - 2\omega t') - \frac{1}{2}\cos\omega t
\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}
y(t) &= \int_0^t \frac{1}{\omega}\frac{1}{2}\cos(\omega t - 2\omega t') - \frac{1}{2}\cos\omega t\, dt' \\
&= \frac{1}{2\omega}\int_0^t \cos(\omega t - 2\omega t')\, dt' - \frac{1}{2}\cos\omega t \int_0^t dt' \\
&= \frac{1}{2\omega}\left[\frac{\sin(\omega t - 2\omega t')}{-2\omega}\right]_0^t - \frac{1}{2} t\cos t\omega \\
&= \frac{-1}{4\omega^2}(\sin(\omega t - 2\omega t) - \sin(\omega t)) - \frac{1}{2} t\cos t\omega \\
&= \frac{1}{2\omega^2}\sin t\omega - \frac{1}{2} t\cos t\omega \\
&= \frac{1}{2\omega^2}(\sin t\omega - \omega t\cos t\omega)
\end{aligned}$$

$$y(t) = \frac{1}{2\omega^2}(\sin t\omega - \omega t\cos t\omega)$$

3.14.17 chapter 15, problem 8.3

Problem

Solve $y'' + \omega^2 y = f(t)$ using $y(t) = \int_0^t \frac{1}{\omega}\sin\omega(t-t')\, f(t')\, dt'$ when $f(t) = e^{-t}$

Solution

$$\begin{aligned}
y(t) &= \int_0^t \frac{1}{\omega}\sin\omega(t-t')\, f(t')\, dt' \\
&= \frac{1}{\omega}\int_0^t \sin\omega(t-t')\, e^{-t'}\, dt' \qquad (1)
\end{aligned}$$

$$\text{Let } I = \int_0^t \sin\omega(t-t')\, e^{-t'}\, dt'$$

Integrate by part, let $u = \sin(\omega t - \omega t')$, $v = -e^{-t'}$

$$\begin{aligned}
I &= [\sin\omega(t-t')\left(-e^{-t'}\right)]_0^t - \omega\int_0^t \cos(\omega t - \omega t')\, e^{-t'}\, dt' \\
&= \sin\omega t\ - \omega\int_0^t \cos(\omega t - \omega t')\, e^{-t'}\, dt'
\end{aligned}$$

Integrate by parts again. $u = \cos(\omega t - \omega t')$, $v = -e^{-t'}$

$$\begin{aligned}
I &= \sin\omega t - \omega\left([\cos(\omega t - \omega t')\left(-e^{-t'}\right)]_0^t + \omega\int_0^t \sin\omega(t-t') e^{-t'}\, dt'\right) \\
I &= \sin\omega t - \omega\left([-e^{-t} + \cos(\omega t)] + \omega I\right) \\
I &= \sin\omega t + \omega e^{-t} - \omega\cos(\omega t) - \omega^2 I \\
I + \omega^2 I &= \sin\omega t + \omega e^{-t} - \omega\cos(\omega t) \\
I &= \frac{\sin\omega t + \omega e^{-t} - \omega\cos(\omega t)}{1+\omega^2}
\end{aligned}$$

Hence from (1)

$$y(t) = \frac{1}{\omega}\frac{\omega e^{-t} - \omega\cos(\omega t) + \sin(\omega t)}{1+\omega^2}$$

3.14.18 chapter 9, problem 3.1

Problem

Change the independent variable to simplify the Euler equation and then find the first integral of it. $\int_{x_2}^{x_1} y^{\frac{3}{2}} ds$

Solution

$$ds = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\sqrt{1 + y'^2}$$

Hence

$$I = \int_{x_2}^{x_1} y^{\frac{3}{2}} ds = \int_{x_2}^{x_1} y^{\frac{3}{2}} \sqrt{1 + y'^2}\, dx$$

Since integrand does not depend on x, changing the independent variable to y in order to simplify solution. Using $dx = \frac{dx}{dy} dy \rightarrow y' = \frac{1}{x'}$. The integral now becomes

$$\begin{aligned} I &= \int_{x_2}^{x_1} y^{\frac{3}{2}} \sqrt{1 + \frac{1}{x'^2}}\, x'\, dy \\ &= \int_{x_2}^{x_1} y^{\frac{3}{2}} \sqrt{x'^2 + 1}\, dy \end{aligned}$$

$$F(y, x', x) = y^{\frac{3}{2}} \sqrt{x'^2 + 1}$$

The Euler equation is

$$\begin{aligned} \frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \overbrace{\frac{\partial F}{\partial x}}^{0} &= 0 \\ \frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) &= 0 \\ \frac{\partial F}{\partial x'} &= c \\ y^{\frac{3}{2}} \frac{x'}{\sqrt{x'^2 + 1}} &= c \end{aligned}$$

Simplifying gives

$$\begin{aligned} x' &= \frac{c}{\sqrt{y^3 - c^2}} \\ \frac{dx}{dy} &= \frac{c}{\sqrt{y^3 - c^2}} \\ x &= \int \frac{1}{\sqrt{\frac{y^3}{c^2} - 1}} dy \end{aligned}$$

We can stop here as the problem did not ask to fully solve the integral.

Chapter 4

Exams

Local contents

4.1 First midterm exam

Local contents

4.1.1 questions

Math 121a, Spring 2004, F.Rezakhanlou

1. (6 pts)Evaluate

(a)$\left(\frac{1-i}{1+\sqrt{3}i}\right)^{36}$,

(b)$(-2-2i)^{\frac{1}{5}}$,

(c)$\lim_{z\to 0}\left(\frac{1}{z^2}-\frac{\cos z}{\sin^2 z}\right)$.

2. (3 pts)Evaluate

$$\left(\sum_{n=0}^{\infty} r^{2n}\cos nt\right)^2+\left(\sum_{n=0}^{\infty} r^{2n}\sin nt\right)^2$$

for a real number r satisfying $|r|<1$.

3. (5 pts) (a)Find the circle of convergence for the following complex power series:

$$\sum_{n=1}\frac{(n!)^3\, \ln n}{(3n)!}(z-i)^n$$

(b) Let z be a nonzero complex number. For what value of z is the series $\sum_{n=1}^{\infty} z^{\ln n}$ absolutely convergent? Explain your answer.

4. (3 pts)(a) Find complex numbers z such that $|z-3i|=2-2zi$.

(b) Describe the set of points z such that $Im(e^{i\pi/2}z)<1$.

5. (4 pts)(a) Find two variable Maclaurin series for $\frac{\cos y}{1-x}$.

(b) About how much does a relative error of 2 percent in a and b effect the relative error of $\sqrt{\frac{a}{b^3}}$ in the worst case?

6. (3 pts)Use power series to evaluate $e^{-x^2/2} - 1 - ln\ \cos x$ at $x = .0011$.

4.1.2 Key solution

① (a) $1-i=\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)=\sqrt{2}\,e^{-i\frac{\pi}{4}}$, $1+\sqrt{3}i=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2e^{i\frac{\pi}{3}}$

$$\left(\frac{1-i}{1+\sqrt{3}i}\right)^{36}=\frac{1}{2^{18}}e^{-i(21)\pi}=-\frac{1}{2^{18}}.$$

why?

(b) $-2-2i=2\sqrt{2}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=2\sqrt{2}\,e^{-\frac{3\pi}{4}i}$,

$$(-2-2i)^{1/5}=2^{\frac{3}{10}}e^{-\frac{3\pi}{20}i+\frac{2k\pi}{5}}\qquad k=0,1,2,3,4.$$

(c) $$\frac{1}{x^2}-\frac{\cos x}{\sin^2 x}=\frac{\sin^2 x-x^2\cos x}{x^2\sin^2 x}=\frac{\left(x-\frac{x^3}{3!}+\cdots\right)^2-x^2\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-\cdots\right)}{x^2\left(x-\frac{x^3}{3!}+\cdots\right)^2}$$

$$=\frac{\left(1-\frac{x^2}{3!}+\cdots\right)^2-\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-\cdots\right)}{x^2\left(1-\frac{x^2}{3!}+\cdots\right)^2}=\frac{1-2\frac{x^2}{3!}+\cdots-\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-\cdots\right)}{x^2+\cdots}$$

Hence $\lim \frac{1}{x^2}-\frac{\cos x}{\sin^2 x}=\frac{1}{6}$.

② $r^{2n}e^{int}=r^{2n}\cos nt+i\,r^{2n}\sin nt$.

$$\sum_0^\infty r^{2n}e^{int}=\sum_0^\infty r^{2n}\cos nt+i\sum_0^\infty r^{2n}\sin nt$$

Hence $$\left(\sum_0^\infty r^{2n}\cos nt\right)^2+\left(\sum_0^\infty r^{2n}\sin nt\right)^2=\left|\sum_0^\infty r^{2n}e^{int}\right|^2$$

$$=\left|\sum_0^\infty (r^2e^{it})^n\right|^2=\left|\frac{1}{1-r^2e^{it}}\right|^2=\frac{1}{(1-r^2\cos t)^2+r^4\sin^2 t}$$

$$=\frac{1}{1-2r^2\cos t+r^4}.$$

③ (a) $a_n=\frac{(n!)^3\ln n}{(3n)!}$, $\frac{a_{n+1}}{a_n}=\left(\frac{(n+1)!}{n!}\right)^3\frac{\ln(n+1)}{\ln n}\frac{(3n)!}{(3(n+1))!}$

$$=\frac{(n+1)^3}{(3n+1)(3n+2)(3n+3)}\frac{\ln(n+1)}{\ln n}\to\frac{1}{27}\quad\text{as } n\to\infty$$

Circle of convergence : $\{z: |z-i|\le 27\}$.

(b) $|z^{\ln n}| = |e^{\ln n \ln z}| = e^{\ln n \operatorname{Re} \ln z} = e^{\ln n \ln|z|}$

$= n^{\ln|z|}$.

Now $\sum_{n=2}^{\infty} n^{\ln|z|}$ converges if and only if $\ln|z| < -1$

or $|z| < \frac{1}{e}$. ✓

(4) (a) If $z = x+iy$, then $2-2zi = (2+2y) - 2ix \geq 0$.

Hence $x=0$ and $2+2y \geq 0$. Now $|z-3i| = |iy-3i| = |y-3|$

and $|y-3| = 2+2y$ means either $y-3 = 2+2y$, $y \geq 3$

or $3-y = 2+2y$ and $y<3$, $2+2y>0$. The first case

means $y=-5$ which is inconsistent with $y \geq 3$. The second case

means $y = \frac{1}{3}$. The answer is $z = i/3$.

(b) $\operatorname{Im}(iz) < 1$ means $x < 1$.

(5) (a) $\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots$

$\frac{1}{1-x} = 1 + x + x^2 + \ldots$,

$\frac{\cos y}{1-x} = (1+x+x^2+\ldots)(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots)$

$= 1 + x + x^2 - \frac{y^2}{2!} - \frac{xy^2}{2!} - \frac{x^2y^2}{2!} + \frac{y^4}{4!} + \ldots$

(b) $f(a,b) = a^{1/2} b^{-3/2}$, $df = \frac{1}{2} a^{-1/2} b^{-3/2} da - \frac{3}{2} a^{1/2} b^{-5/2} db$,

$\frac{df}{f} = \frac{1}{2}\frac{da}{a} - \frac{3}{2}\frac{db}{b}$, Answer $1+3 = 4\%$.

(6) $e^{-x^2/2} - 1 - \ln \cos x$

$$= 1 - \frac{x^2}{2} + \frac{(x^2/2)^2}{2} + \cdots - 1 - \ln\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

$$= -\frac{x^2}{2} + \frac{x^4}{8} + \cdots - \left\{\left(-\frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right) - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right)^2 + \cdots\right\}$$

$$= \frac{x^4}{8} - \frac{x^4}{4!} + \frac{1}{8}x^4 \cdots = \frac{x^4}{4!} + \cdots,$$

Answer $\frac{5(.0011)^4}{4!}$

4.2 Second midterm exam

4.2.1 Key solution

Math 121a, Spring 2004, F.Rezakhanlou

1. (5 points) Define

$$f(z) = \frac{(z+2)^2(z+3)}{1-\cos \pi z}.$$

Find all the isolated singularitis of f and identify each as a removable, a pole (give the order) or an essential singularity. What is the radius of convergence of the Taylor expansion of f at $z = 1/2$? The denominator vanishes when $\cos \pi z = 1$ or $z = 2k$ for some integer k. Since for $\varphi(z) = 1 - \cos\pi z$ we have $\begin{cases}\varphi'(z) = -\pi \sin \pi z, \\ \varphi''(z) = -\pi^2 \cos \pi z\end{cases}$ we have $\varphi'(2k) = 0$, $\varphi''(2k) \neq 0$, so each $2k$ is a zero of order 2. This means $z = -2$ is a removable singularity but $2k$ for $k \neq -1$ is a pole of order 2.

The radius of convergence at $z = 1/2$ is $\frac{1}{2}$ because a disk of center $1/2$, radius $1/2$ is the largest to avoid singularities.

2. (4 point) Find the Laurent expansion of $f(z) = (1+z^2)^{-1} + (z+3)^{-1}$ in the set $\{z : 1 < |z| < 3\}$.

First $\frac{1}{1+z^2} = \frac{1}{z^2} \frac{1}{1+\frac{1}{z^2}} = \sum_0^\infty (-1)^n z^{-2n-2}$ is convergent for $|z| > 1$. Also

$$\frac{1}{z+3} = \frac{1}{3}\frac{1}{1+z/3} = \frac{1}{3}\sum_{n=0}^\infty \left(\frac{z}{3}\right)^n (-1)^n$$ is convergent for $|z| < 3$. Hence for $1 < |z| < 3$,

$$f(z) = \sum_0^\infty (-1)^n z^{-2-2n} + \sum_0^\infty (-1)^n z^n 3^{-n-1}.$$

3. (6 points) Evaluate $\int_0^{2\pi} \frac{1}{\sin\theta - 3i} d\theta$ and $\int_0^\infty \frac{\cos 2x}{x^2} dx$.

① $\int_0^{2\pi} \frac{d\theta}{\sin\theta - 3i} = \int_C \frac{1}{(z-\frac{1}{z})\frac{1}{2i} - 3i} \frac{dz}{iz} = \int_C \frac{2\,dz}{6z + z^2 - 1}$ C unit circle

$z^2 + 6z - 1 = 0 \Rightarrow z = -3 \pm \sqrt{10}$, $z_0 = -3 + \sqrt{10}$ inside C, so

$\int_0^{2\pi} \frac{d\theta}{\sin\theta - 3i} = 2\pi i \operatorname{Res}_{z_0} F$ with $F(z) = \frac{1}{z - z_0} \frac{2}{z - z_1}$ with $z_1 = -3 - \sqrt{10}$,

$\operatorname{Res}_{z_0} F = \frac{2}{z_0 - z_1} = \frac{1}{\sqrt{10}}$. Hence integral $= \frac{2\pi i}{\sqrt{10}}$

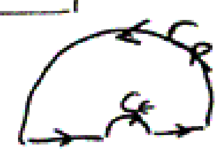
② $\int_0^\infty \frac{\cos 2x}{x^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos 2x}{x^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2} dx$

The function $F(z) = \frac{e^{i2z}}{z^2}$ has a simple pole at $z = 0$. So

4. (2 points) Find
$$\frac{d}{dx} \int_{x^2}^{\cos x} e^{xt^2} dt.$$

$$= \int_{x^2}^{\cos x} t^2 e^{xt^2} dt - e^{x\cos^2 x} \sin x - 2x e^{x^5}$$

$\int_{C_R} F + \int_{C_\epsilon} F + \left(\int_{-R}^{-\epsilon} + \int_\epsilon^R \right) F = 0.$



First $\int_{C_R} F = 0$ because $|e^{2iz}| = e^{-2\operatorname{Im} z} \le 1$, so

$\left| \int_{C_R} \frac{F(z)}{z^2} \right| \le \pi R \frac{1}{R^2} \to 0$ as $R \to \infty$. Also

$\int_{C_\epsilon} F = \int_{C_\epsilon} \frac{e^{2iz} - 2iz}{z^2} dz + \int_{C_\epsilon} \frac{2iz}{z^2}$, first integral goes to zero because the integrand is bounded. The second one is $2 \int_{C_\epsilon} \frac{i}{z} = -2\pi$.

Thus $\int^\infty \frac{\cos 2x}{x} = \pi$.

5. (4 points) Assume

$$xz_x + yz_y + z = 0 \qquad (1)$$

where $z = z(x,y)$ and z_x and z_y denote the partial derivatives of z with respect to x and y. Make the change of variables $r = x$, $t = y/x$. Derive an equation for z as a function of r and t. Use this equation to solve the equation (1).

$$z_x = z_r \frac{\partial r}{\partial x} + z_t \frac{\partial t}{\partial x} = z_r + z_t \left(\frac{-y}{x^2}\right) = z_r - z_t \frac{t}{r}$$

$$z_y = z_r \frac{\partial r}{\partial y} + z_t \frac{\partial t}{\partial y} = z_t \frac{1}{x} = z_t \frac{1}{r}$$

So $x z_x + y z_y + z = r(z_r - \frac{t}{r} z_t) + tr\, z_t \frac{1}{r} + z = r z_r + z = 0$.

Hence $\frac{\partial z}{\partial r} = -\frac{z}{r}$, $\frac{dz}{z} = -\frac{dr}{r} \Rightarrow \ln z + c = -\ln r$,

$z = \frac{c}{r}$. But C could depend on t. Thus

$$z = \frac{1}{r} C(t) = \frac{1}{x} C\left(\frac{y}{x}\right).$$

6. (5 points)Find the smallest and largest value of $F = yz + x^2 + z$ on the sphere $x^2 + y^2 + z^2 = 1$.

Using Lagrange multiplier, $\nabla F = \lambda \nabla G$, hence at a critical point $(2x, z, y+1) = \lambda(2x, 2y, 2z)$. If $x \neq 0$ we get $\lambda = 1$, $\begin{cases} z = 2y \\ y+1 = 2z \end{cases} \Rightarrow y+1 = 4y \Rightarrow y = \frac{1}{3}, z = 2/3, x = \sqrt{1-\frac{5}{9}} = \pm\frac{2}{3}$

The corresponding F is $\frac{2}{9} + \frac{4}{9} + \frac{6}{9} = \frac{12}{9} = \frac{4}{3}$. If $x = 0$ we get

$y+1 = 2\lambda z$, $z = 2\lambda y$; $\frac{y+1}{z} = \frac{z}{y}$, $y^2 + y = z^2$, also $y^2 + z^2 = 1$. So

$y^2 + y = 1 - y^2$, $2y^2 + y - 1 = 0$, $y = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = \begin{cases} -1 & \Rightarrow z = 0 \\ 1/2 & \Rightarrow z = \pm\frac{\sqrt{3}}{2} \end{cases}$

The corresponding F are $\begin{cases} 0 & x = 0, z = 0, y = -1 \\ \frac{3\sqrt{3}}{4} & x = 0, y = 1/2, z = \frac{\sqrt{3}}{2} \end{cases}$

Max $= 4/3$

4.3 Finals

4.3.1 questions

Math 121a Final, Spring 2004, F.Rezakhanlou

1. (3 points) Find the Lagrange's equation in polar coordinates for a particle moving in a plane if the potential energy is $V = r^{-1}$.

2. (3 points) Use the cylindrical coordinates to find the equation of the shortest path connecting two points on a circular cylinder.

$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$

3. (3 points) Solve the Euler equation corresponding to the action

$$\int_a^b \frac{\sqrt{1+y'^2}}{1+y}dx.$$

4. (3 points) Find the inverse Laplace transform of $\tilde{f}(z) = (z-1)^{-2}(z^2+4)^{-1}$.

5. (3 points)Find the exponential and sine Fourier transform of a function

$$f(x) = \begin{cases} 1 \text{ if } 0 < x \leq 2 \\ -1 \text{ if } -2 \leq x \leq 0 \\ 0 \text{ otherwise.} \end{cases}$$

Use this to evaluate

$$\int_0^\infty \frac{(\cos 2y - 1)\sin y}{y} dy.$$

✓ 6. (3 points) Solve $(x^2+1)y'' - 2xy' + 2y = (x^2+1)^2$ using the fact that x and $1-x^2$ are solutions to the homogeneous equations.

7. (3 points) Find $y = y(x)$ such that $y(0) = y'(0) = 0$ and $4y'' + 4y' + 10y = \delta(x - x_0)$.

8. (3 points) Given $f(x) = |x|$ on $(-\pi, \pi)$, expand f in an appropriate Fourier series of period 2π. To what value does the series converges at π?

9. (3 points) Evaluate $\int_0^\infty \frac{\cos x}{1+x^4} dx$.

10. (3 point) Use the transformation $f(z) = z^{-1}$ to find the temperature distribution T in the region

$$\{(x, y) : (x-1)^2 + y^2 > 1, x > 0\}$$

provided that $T(x, y) = 20$ if $(x-1)^2 + y^2 = 1$ and $T(0, y) = 10$.

11. (2 points) Evaluate

$$\frac{d^2}{dx^2}\int_0^x\int_0^x f(s,t)dsdt.$$

12. (3 points)If $z = xy$, $2x^3 + 2y^3 = 3t^2$ and $3x^2 + 3y^2 = 6t$, find $\frac{dz}{dt}$.