① (a) $1-i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \sqrt{2}\, e^{-i\pi/4}$, $\quad 1+\sqrt{3}\,i = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = 2\, e^{i\frac{\pi}{3}}$

why?

$$\left(\frac{1-i}{1+\sqrt{3}\,i}\right)^{36} = \frac{1}{2^{18}}\, e^{-i(21)\pi} = -\frac{1}{2^{18}}.$$

(b) $-2-2i = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right) = 2\sqrt{2}\, e^{-\frac{3\pi}{4} i}$,

$$(-2-2i)^{1/5} = 2^{\frac{3}{10}}\, e^{-\frac{3\pi}{20} i + \frac{2k\pi}{5}} \qquad k = 0,1,2,3,4.$$

(c) $$\frac{1}{x^2} - \frac{\cos x}{\sin^2 x} = \frac{\sin^2 x - x^2 \cos x}{x^2 \sin^2 x} = \frac{\left(x - \frac{x^3}{3!} + \cdots\right)^2 - x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right)}{x^2\left(x - \frac{x^3}{3!} + \cdots\right)^2}$$

$$= \frac{\left(1 - \frac{x^2}{3!} + \cdots\right)^2 - \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right)}{x^2\left(1 - \frac{x^2}{3!} + \cdots\right)^2} = \frac{1 - 2\frac{x^2}{3!} + \cdots - \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right)}{x^2 + \cdots}$$

Hence $\lim \frac{1}{x^2} - \frac{\cos x}{\sin^2 x} = \frac{1}{6}$.

② $r^{2n} e^{int} = r^{2n}\cos nt + i\, r^{2n} \sin nt$.

$$\sum_0^\infty r^{2n} e^{int} = \sum_0^\infty r^{2n}\cos nt + i \sum_0^\infty r^{2n} \sin nt$$

Hence $$\left(\sum_0^\infty r^{2n}\cos nt\right)^2 + \left(\sum_0^\infty r^{2n}\sin nt\right)^2 = \left|\sum_0^\infty r^{2n} e^{int}\right|^2$$

$$= \left|\sum_0^\infty (r^2 e^{it})^n\right|^2 = \left|\frac{1}{1 - r^2 e^{it}}\right|^2 = \frac{1}{(1 - r^2\cos t)^2 + r^4 \sin^2 t}$$

$$= \frac{1}{1 - 2r^2\cos t + r^4}.$$

③ (a) $a_n = \frac{(n!)^3 \ln n}{(3n)!}$, $\quad \frac{a_{n+1}}{a_n} = \left(\frac{(n+1)!}{n!}\right)^3 \frac{\ln(n+1)}{\ln n} \frac{(3n)!}{(3(n+1))!}$

$$= \frac{(n+1)^3}{(3n+1)(3n+2)(3n+3)} \frac{\ln(n+1)}{\ln n} \to \frac{1}{27} \quad \text{as } n \to \infty$$

Circle of convergence : $\{ z : |z i| \le 27 \}$.

(b) $|z^{\ln n}| = |e^{\ln n \ln z}| = e^{\ln n \operatorname{Re} \ln z} = e^{\ln n \ln|z|}$

$= n^{\ln|z|}$.

Now $\sum_{n=2}^{\infty} n^{\ln|z|}$ converges if and only if $\ln|z| < -1$

or $|z| < \frac{1}{e}$. ✓

(4) (a) If $z = x+iy$, then $2 - 2zi = (2+2y) - 2ix \geq 0$.

Hence $x = 0$ and $2+2y \geq 0$. Now $|z-3i| = |iy-3i| = |y-3|$

and $|y-3| = 2+2y$ means either $y-3 = 2+2y$, $y \geq 3$

or $3-y = 2+2y$ and $y<3$, $2+2y>0$. The first case

means $y = -5$ which is inconsistent with $y \geq 3$. The second case

means $y = \frac{1}{3}$. The answer is $z = i/3$.

(b) $\operatorname{Im}(iz) < 1$ means $x < 1$.

(5) (a) $\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots$

$\frac{1}{1-x} = 1 + x + x^2 + \cdots$,

$$\frac{\cos y}{1-x} = (1+x+x^2+\cdots)\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right)$$
$$= 1 + x + x^2 - \frac{y^2}{2!} - \frac{xy^2}{2!} - \frac{x^2y^2}{2!} + \frac{y^4}{4!} + \cdots$$

(b) $f(a,b) = a^{1/2} b^{-3/2}$, $df = \frac{1}{2} a^{-1/2} b^{-3/2} da - \frac{3}{2} a^{1/2} b^{-5/2} db$,

$\frac{df}{f} = \frac{1}{2}\frac{da}{a} - \frac{3}{2}\frac{db}{b}$, Answer $1+3 = 4\%$.

(6) $e^{-x^2/2} - 1 - \ln \cos x$

$$= 1 - \frac{x^2}{2} + \frac{(x^2/2)^2}{2} + \cdots - 1 - \ln\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

$$= -\frac{x^2}{2} + \frac{x^4}{8} + \cdots - \left\{\left(-\frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right) - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right)^2 + \cdots\right\}$$

$$= \frac{x^4}{8} - \frac{x^4}{4!} + \frac{1}{8}x^4 + \cdots = \frac{x^4}{4!} + \cdots ,$$

Answer $\quad \dfrac{5(.0011)^4}{4!}$