University Course

MAE 170
Introduction to control system

University Of California, Irvine (UCI)
Winter 2005

My Class Notes

Nasser M. Abbasi

Winter 2005

Contents

Chapter 1

Introduction

Local contents

1.1 Links

1. Course web page `http://eee.uci.edu/05w/16380/`

1.2 Instructor

Very good instructor, was helpful in office hrs.

```
Prof. James Bobrow
Email: jebobrow@uci.edu
Phone: (949) 824-4116
Office Location: S3220 Engineering Gateway
Office Hours: T, Th 1:30-2:30
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`http://gram.eng.uci.edu/~bobrow/`

Teaching assistants:

1. Zhi Liang
2. Benjamin Park

1.3 Course and Text book

Engineering, Mechanical and Aerospace

EngrMAE 170 INTRO CNTRL SYSTEMS

Code	Typ	Sec	Unt	Instructor	Time	Place	Max	Enr	WL	Req	Nor	Rstr	Ead	Web	Status
16380	Lec	A	4	BOBROW, J.E.	TuTh 9:30-10:50	MSTB 118	101	100	0	103	0	A	Ead	Web	OPEN

Figure 1.1: course schedule

MODERN
CONTROL
ENGINEERING
Fourth Edition

KATSUHIKO OGATA

Figure 1.2: textbook

1.4 syllabus

University of California, Irvine
Department of Mechanical and Aerospace Engineering
MAE 170 Course Outline, Winter Quarter, 2005

Instructor: J.E. Bobrow; Office, Engineering Gateway 3220; Phone (949) 824-4116, email: jebobrow@uci.edu, office hours: Tuesday and Thursday 1:30-2:30 pm.

Classroom: MSTB 118, Tuesday and Thursday, 9:30–10:50 am.

Discussion: Teaching assistants Zhi Liang and Benjamin Park will lead discussions on homework and on the use of Matlab on Wednesday 12:00-12:50p, room SST 220B; Thursday 11:00-11:50 and 12:00-12:50 in room IERF B011.

Text: **Modern Control Engineering,** Fourth Edition, by K. Ogata, Prentice Hall.

Web Reference: A nice control tutorial with Matlab is located at http://www.engin.umich.edu/group/ctm/ .

Software: Students need to use Matlab with the control systems toolbox. You can purchase the software, or use the Engineering PC lab in EG3151. You can get a key card with a $20 deposit from the SOE Dean's office, REC 305.

Course topics (and sections covered in text): This course presents the control theory that complements the experiments in MAE 106. Some demonstrations will be given class.

1. Review of Laplace Transforms (All of Chapter 2).
2. System Modelling (Chapter 3, pages 53-95, 112-113. Op-amps are covered in MAE106).
3. Thermal and Fluid systems (Selected material from Chapter 4, pages 169-191).
4. Response of systems to general inputs (Selected material from all of Chapter 5).
5. Root Locus analysis (All of Chapter 6).
6. Root Locus design (All of Chapter 7).
7. Frequency response analysis (Selected topics from Chapter 8).
8. Frequency response design (Selected topics from Chapter 9).

Grading Policy:

20% Homework
35% Midterm Exam (at approximately week 6)
45% Final Exam

Chapter 2

study note

Local contents

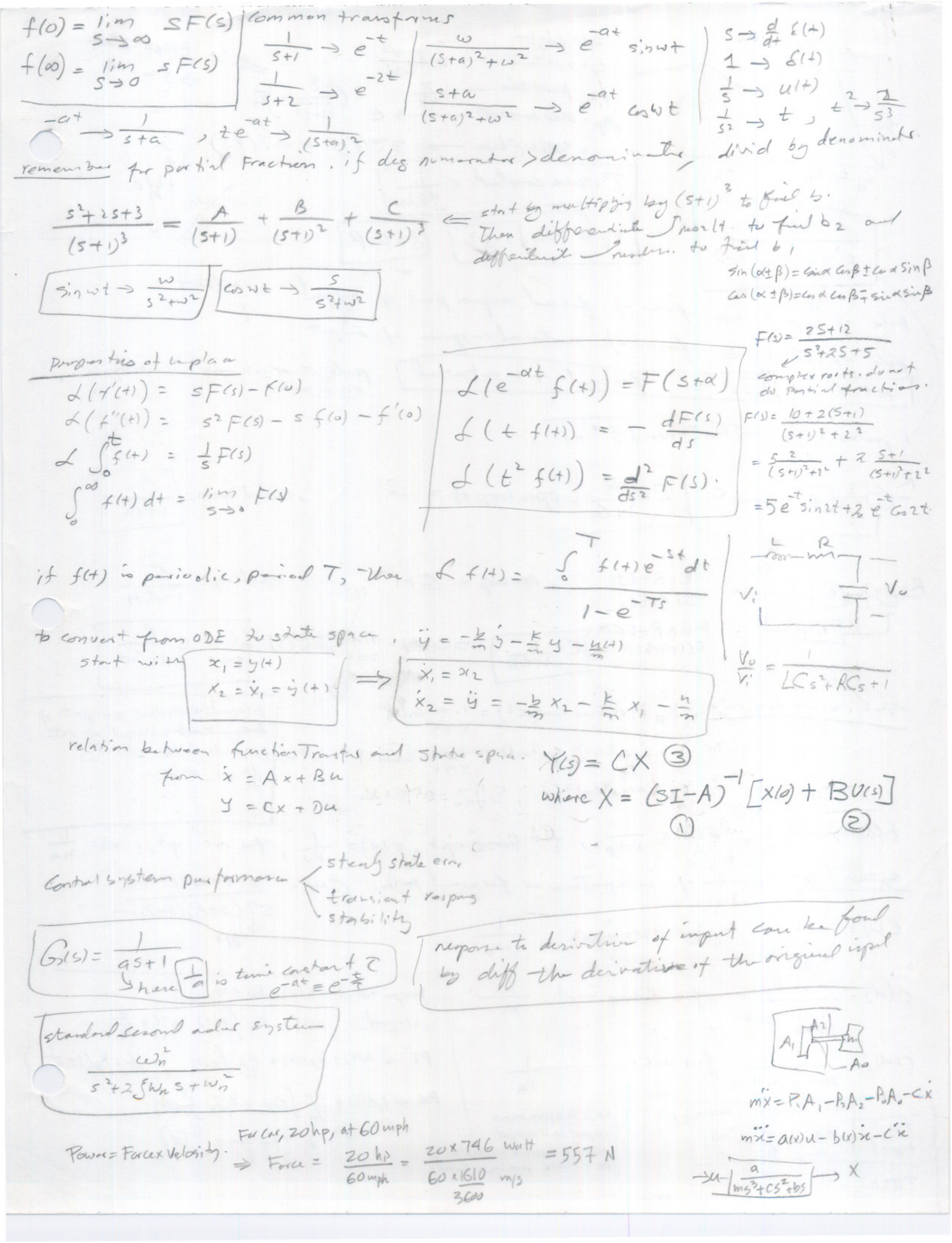
2.1 some Laplace properties

Laplace transform properties

$\mathscr{L}f(t) = F(s) = \int_0^\infty f(t)\, e^{-st} dt$

$\mathscr{L}f(t-a) = e^{-as}F(s)$	$\mathscr{L}\delta(t) = 1$ impulse	$\mathscr{L}u(t) = \frac{1}{s}$	$\mathscr{L}u(t-a) = e^{-as}\frac{1}{s}$	$\mathscr{L}e^{-at}f(t) = F(s+a)$
$\mathscr{L}f\left(\frac{t}{\alpha}\right) = \alpha F(\alpha\ s)$	$\mathscr{L}t = \frac{1}{s^2}$ ramp	$\mathscr{L}\cos\omega t = \frac{s}{s^2+\omega^2}$	$\mathscr{L}\sin\omega t = \frac{\omega}{s^2+\omega^2}$	

2.2 my cheat sheet



$f(0) = \lim_{s\to\infty} sF(s)$

$f(\infty) = \lim_{s\to 0} sF(s)$

Common transforms

$\frac{1}{s+1} \to e^{-t}$

$\frac{1}{s+2} \to e^{-2t}$

$\frac{\omega}{(s+a)^2+\omega^2} \to e^{-at} \sin\omega t$

$\frac{s+a}{(s+a)^2+\omega^2} \to e^{-at} \cos\omega t$

$s \to \frac{d}{dt}\delta(t)$

$1 \to \delta(t)$

$\frac{1}{s} \to u(t)$

$\frac{1}{s^2} \to t$, $t^2 \to \frac{1}{s^3}$

$e^{-at} \to \frac{1}{s+a}$, $te^{-at} \to \frac{1}{(s+a)^2}$

remember for partial Fraction. if deg numerator > denominator, divid by denominator.

$$\frac{s^2+2s+3}{(s+1)^3} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

⇐ start by multiplying by $(s+1)^3$ to find b_3. Then differentiate result to find b_2 and differentiate result to find b_1

$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\sin\omega t \to \frac{\omega}{s^2+\omega^2}$

$\cos\omega t \to \frac{s}{s^2+\omega^2}$

properties of laplace

$\mathcal{L}(f'(t)) = sF(s) - f(0)$

$\mathcal{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$

$\mathcal{L}\int_0^t f(t) = \frac{1}{s}F(s)$

$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s)$

$\mathcal{L}(e^{-\alpha t} f(t)) = F(s+\alpha)$

$\mathcal{L}(t\,f(t)) = -\frac{dF(s)}{ds}$

$\mathcal{L}(t^2 f(t)) = \frac{d^2}{ds^2}F(s).$

$F(s) = \frac{2s+12}{s^2+2s+5}$

complex roots. do not do partial fractions.

$F(s) = \frac{10+2(s+1)}{(s+1)^2+2^2}$

$= 5\frac{2}{(s+1)^2+2^2} + 2\frac{s+1}{(s+1)^2+2^2}$

$= 5e^{-t}\sin 2t + 2e^{-t}\cos 2t$

if f(t) is periodic, period T, then $\mathcal{L}f(t) = \frac{\int_0^T f(t)e^{-st}dt}{1-e^{-Ts}}$

$\frac{V_o}{V_i} = \frac{1}{LCs^2+RCs+1}$

to convert from ODE to state space, $\ddot{y} = -\frac{b}{m}\dot{y} - \frac{k}{m}y - \frac{u(t)}{m}$

start with $x_1 = y(t)$, $x_2 = \dot{x}_1 = \dot{y}(t)$ $\Longrightarrow$ $\dot{x}_1 = x_2$, $\dot{x}_2 = \ddot{y} = -\frac{b}{m}x_2 - \frac{k}{m}x_1 - \frac{u}{m}$

relation between function Transfer and state space.

from $\dot{x} = Ax + Bu$

$y = Cx + Du$

$Y(s) = CX$ ③

where $X = \underbrace{(sI-A)^{-1}}_{①}\underbrace{[x(0) + BU(s)]}_{②}$

Control system performance
- steady state error
- transient response
- stability

$G(s) = \frac{1}{as+1}$ here $\boxed{\frac{1}{a}}$ is time constant τ $e^{-at} \equiv e^{-\frac{t}{\tau}}$

response to derivative of input can be found by diff the derivative of the original input

standard second order system

$$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

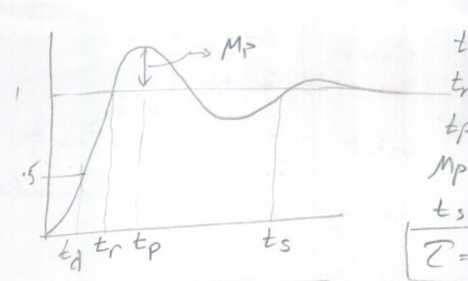
$m\ddot{x} = P_1A_1 - P_2A_2 - P_0A_0 - c\dot{x}$

$m\dddot{x} = a(x)u - b(x)\dot{x} - c\ddot{x}$

$u \to \boxed{\frac{a}{ms^3+cs^2+bs}} \to X$

Power = Force x Velocity. For car, 20hp, at 60 mph

$\Rightarrow$ Force $= \frac{20\text{ hp}}{60\text{ mph}} = \frac{20\times 746 \text{ watt}}{\frac{60\times 1610}{3600}\text{ m/s}} = 557$ N



t_d = delay time

t_r = rise time $\longrightarrow \frac{\pi-\beta}{\omega_d}$

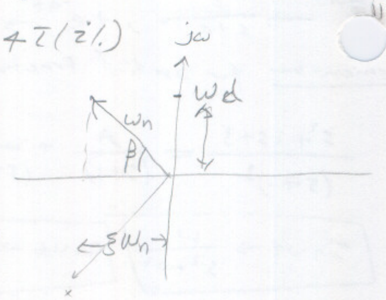
t_p = peak time $\longrightarrow \frac{\pi}{\omega_d}$

M_p = max overshoot $\longrightarrow e^{-\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\pi}$ $\Rightarrow$ i.e. if $M_p = .254$ $\rightarrow \xi = 0.4$

t_s = settling time $\longrightarrow 3\tau\,(5\%)$ or $4\tau\,(2\%)$

$\boxed{\tau = \text{time constant} = \frac{1}{\xi\omega_n}}$

$\boxed{\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}}$ $\boxed{\omega_d = \omega_n\sqrt{1-\xi^2}}$ $\boxed{\xi = \cos\beta}$

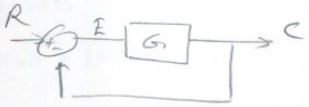


pole of input $\longrightarrow$ form of forced response part of output

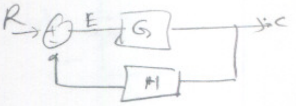
pole of $G(s)$ $\Rightarrow$ form of natural response of output.

$\boxed{\text{total response} = \text{forced} + \text{natural}}$ pole at origin of input $\longrightarrow$ 1 response.

} poles at $-\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$



$\boxed{E = \frac{R}{1+G}}$, open loop tf = G. , closed loop = $\boxed{\frac{G}{1+G}}$



$E = R - CH$, open loop = GH , closed loop = $\frac{G}{1+GH}$

but $C = EG$

so $E = R - EGH$

$E(1+GH) = R \Rightarrow \boxed{E = \frac{R}{1+GH}}$ $\Rightarrow e(\infty) = \lim_{s\to 0} sE(s) = \boxed{\lim_{s\to 0} s\frac{R}{1+GH}}$

Static error constants

K_p (position constant) $= \lim_{s\to 0} G(s)$

K_v (velocity constant) $= \lim_{s\to 0} sG(s)$

K_a (acc. constant) $= \lim_{s\to 0} s^2G(s)$

} K

integral: controller output is ~~[illegible]~~ changed at rate proportional to error i.e. $\frac{du}{dt} = K_i e(t)$

for step input $\frac{1}{s}$, $e(\infty) = \frac{1}{1+K_p}$, for Ramp input, $e(\infty) = \frac{1}{K_v}$, for acc. input, $e(\infty) = \frac{1}{K_a}$

system type number of integrators in forward path. $G(s) = \frac{K(s+z_1)(\cdots)}{s^{n}(s+p_1)(s+p_2)\cdots}$ (n → type.)

$e(\infty) = \frac{1}{1+\lim_{s\to 0} G(s)}$ for step input $\frac{1}{1+K_p}$

$e(\infty) = \frac{1}{\lim_{s\to 0} sG(s)}$ for Ramp input $\frac{1}{K_v}$

$e(\infty) = \frac{1}{s\to 0\ s^2G(s)}$ for acc. $\frac{1}{K_a}$

proportional $u(t) = K_p e(t)$

integral $u(t) = K_i \int^t e(t) \Rightarrow \frac{U}{E} = \frac{K_i}{s}$

PI $\Rightarrow u(t) = K_p e(t) + \frac{K_p}{T_i}\int^t e(t)$ or $\frac{U}{E} = K_p\left(1+\frac{1}{\ }\right)$

PID $\Rightarrow K_p e(t) + \frac{K_p}{T_i}\int e(t) + K_p T_d \frac{d}{dt} e(t)$

or $K_p\left(1+\frac{1}{T_i s}+sT_d\right)$

	step	ramp	Acc = $\frac{1}{2}t^2$
type 0	$\frac{1}{1+K}$	∞	∞
type 1	0	$\frac{1}{K}$	∞
type 2	0	0	$\frac{1}{K}$

ss(∞)

Velocity error means input and output move at same velocity but have position error.

Chapter 3

HWs

Local contents

3.1 HW 1

Local contents

3.1.1 Problem B 2-1

From Modern Control Engineering, 4th edition by Ogata

Question

1. Find Laplace transform for

$$\begin{aligned} f(t) &= 0 & t &< 0 \\ f(t) &= e^{-0.4t}\cos 12t & t &\geq 0 \end{aligned}$$

2. Find Laplace transform for

$$\begin{aligned} f(t) &= 0 & t &< 0 \\ f(t) &= \sin\left(4t+\frac{\pi}{3}\right) & t &\geq 0 \end{aligned}$$

Solution

3.1.1.1 Part a

This is of the form $e^{-at}f(t)$, hence use the property of Laplace transform

$$\mathscr{L}\left(e^{-at}f(t)\right) = F(s+a) \tag{1}$$

Where $F(s)$ is Laplace transform of $f(t)$. But $\mathscr{L}(\cos\omega t) = \frac{s}{s^2+\omega^2}$, therefore

$$F(s) = \mathscr{L}(\cos 12t) = \frac{s}{s^2+144}$$

Hence (1) becomes

$$\begin{aligned} \mathscr{L}\left(e^{-at}f(t)\right) &= \mathscr{L}\left(e^{-0.4t}\cos 12t\right) \\ &= F(s+a) \\ &= \frac{(s+0.4)}{(s+0.4)^2+144} \end{aligned}$$

3.1.1.2 Part b

I can not solve $\mathscr{L}\left(\sin\left(4t+\frac{\pi}{3}\right)\right)$ by using the property that

$$\mathscr{L}\left(f(t-a)\right)=e^{-as}F(s)$$

Because here delay $\frac{\pi}{3}>0$ where the above property is valid for $a<0$. Instead, writing

$$\begin{aligned}\sin(\omega t+\theta)&=\sin(\omega t)\cos\theta+\cos(\omega t)\sin\theta\\ \sin\left(4t+\frac{\pi}{3}\right)&=\sin(4t)\cos\frac{\pi}{3}+\cos(4t)\sin\frac{\pi}{3}\\ \mathscr{L}\left(\sin\left(4t+\frac{\pi}{3}\right)\right)&=\cos\frac{\pi}{3}\mathscr{L}(\sin 4t)+\sin\frac{\pi}{3}\mathscr{L}(\cos 4t)\end{aligned}\tag{2}$$

But $\cos\frac{\pi}{3}=\frac{1}{2}$ and $\sin\frac{\pi}{3}=\sqrt{3}$ and $\mathscr{L}(\sin\omega t)=\frac{\omega}{s^2+\omega^2}\implies\mathscr{L}(\sin 4t)=\frac{4}{s^2+16}$ and $\mathscr{L}(\cos\omega t)=\frac{s}{s^2+\omega^2}\implies\mathscr{L}(\cos 4t)=\frac{s}{s^2+16}$. Hence substituting into eq (2) gives

$$\begin{aligned}\mathscr{L}\left(\sin\left(4t+\frac{\pi}{3}\right)\right)&=\frac{1}{2}\frac{4}{s^2+16}+\sqrt{3}\frac{s}{s^2+16}\\&=\frac{1}{2}\left(\frac{4+\sqrt{3}s}{s^2+16}\right)\end{aligned}$$

3.1.2 Problem B 2-6

Question

(a) find Laplace transform for

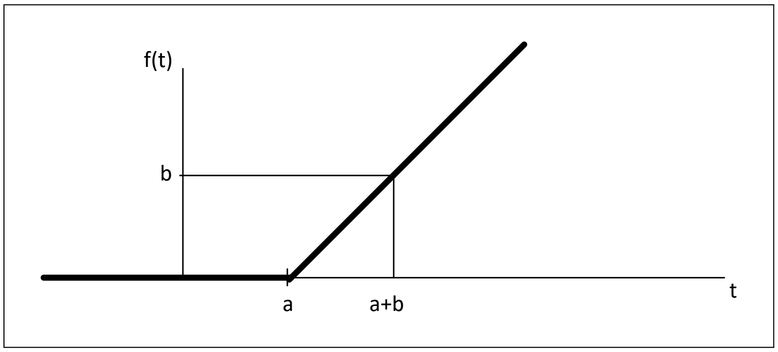


Figure 3.1: function

solution (a)This is a delayed ramp with slope=1. Hence ramp equation is $f(t)=t$. The amount of delay is a

Hence we want to find Laplace transform for $g(t)=f(t-a)$ which is, from Laplace properties, is $e^{-as}F(s)$

But $F(s)=\mathscr{L}(t)=\frac{1}{s^2}$ therefore the answer is

$$e^{-as}\frac{1}{s^2}$$

3.1.3 Problem B 2-7

From Modern Control Engineering, 4th edition by Ogata

Question

(a) find Laplace transform for

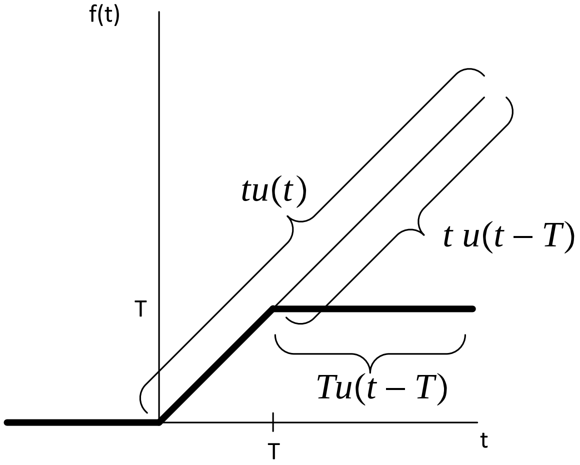


Figure 3.2: function

Solution

The above function can be constructed as follows

Let $f(t) = t$ (the ramp function)

$g(t) = f(t)\,u(t) - f(t)\,u(t-T) + Tu(t-T)$

Where $u(t)$ is the unit step function

This is ilustrated in this diagram

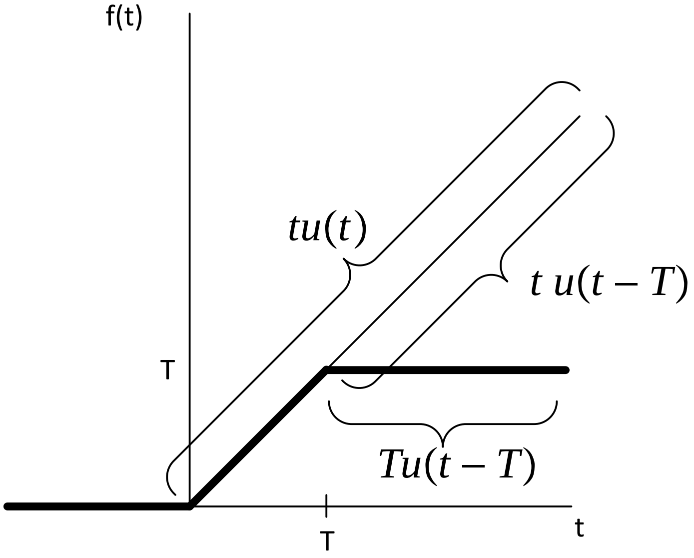


Figure 3.3: function

Now

$$\mathscr{L}\Big(f(t)\,u(t)\Big) = \frac{1}{s^2}$$

And

$$\begin{aligned}\mathscr{L}\Big(f(t)\,u(t-T)\Big) &= \int_T^\infty te^{-st}dt = \frac{-t}{s}e^{-st}\Big|_T^\infty - \int_T^\infty \frac{e^{-st}}{s}dt\\ &= \frac{-1}{s}\left(0 - Te^{-sT}\right) - \frac{1}{s}\left(\frac{-1}{s}\right)\ e^{-st}\Big|_T^\infty\\ &= \frac{T}{s}e^{-sT} + \frac{1}{s^2}\left(0 - e^{-sT}\right) = \frac{T}{s}e^{-sT} - \frac{e^{-sT}}{s^2}\end{aligned}$$

And

$$\mathscr{L}(Tu(t-T)) = \frac{T}{s}e^{-sT}$$

Hence

$$\begin{aligned}\mathscr{L}\Big(g(t)\Big) &= \frac{1}{s^2} - \left(\frac{T}{s}e^{-sT} - \frac{e^{-sT}}{s^2}\right) + \frac{T}{s}e^{-Ts}\\ &= \frac{1}{s^2} - \frac{T}{s}e^{-sT} + \frac{e^{-sT}}{s^2} + \ \frac{T}{s}e^{-Ts}\\ &= \frac{1}{s^2} + \frac{e^{-sT}}{s^2}\\ &= \frac{1 - e^{-sT}}{s^2}\end{aligned}$$

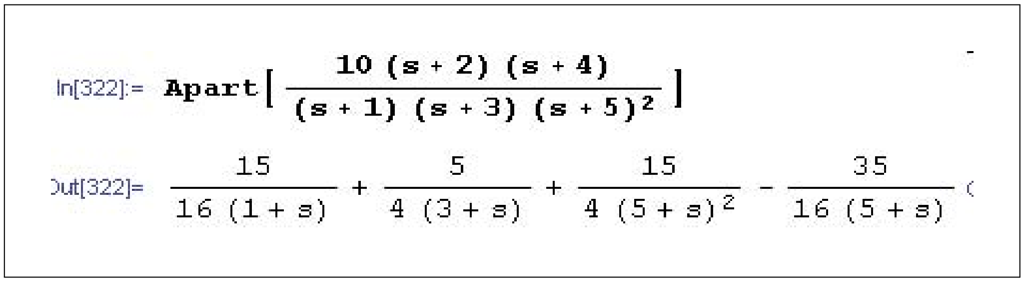
3.1.4 Problem B 2-15

From Modern Control Engineering, 4th edition by Ogata

Question Obtain partial-fraction using MATLAB for

$F(s) = \frac{10(s+2)(s+4)}{(s+1)(s+3)(s+5)^2}$ and then find inverse laplace transform

Solution I used Mathematica to find Partial-fraction



In[322]:= $\text{Apart}\left[\frac{10\ (s+2)\ (s+4)}{(s+1)\ (s+3)\ (s+5)^2}\right]$

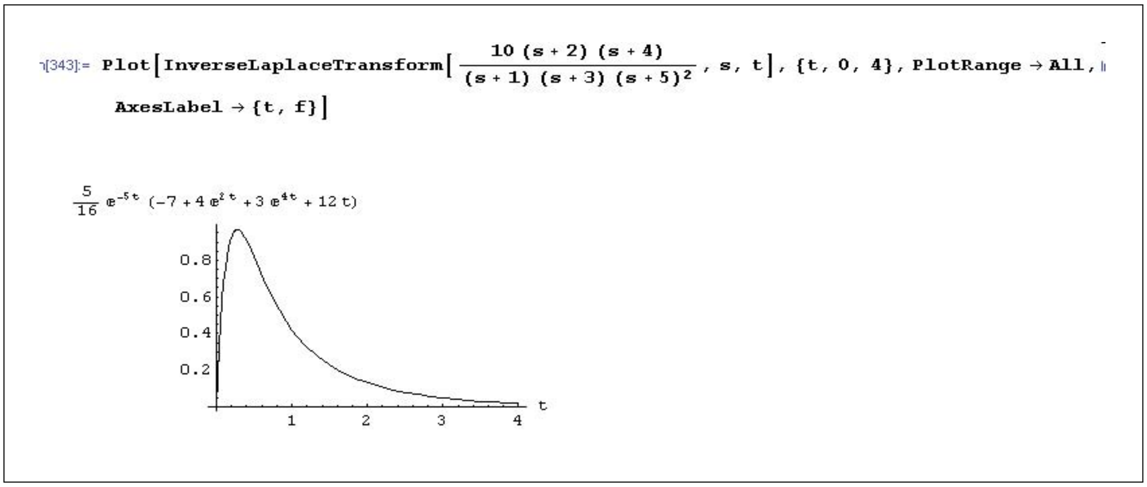
ut[322]= $\frac{15}{16\ (1+s)} + \frac{5}{4\ (3+s)} + \frac{15}{4\ (5+s)^2} - \frac{35}{16\ (5+s)}$ (

Figure 3.4: Code

Hence the inverse laplace tranform is

$$f(t) = \frac{15}{16}e^{-t} + \frac{5}{4}e^{-3t} + \frac{15}{4}te^{-5t} - \frac{35}{16}e^{-5t}$$

Here is a plot of the solution

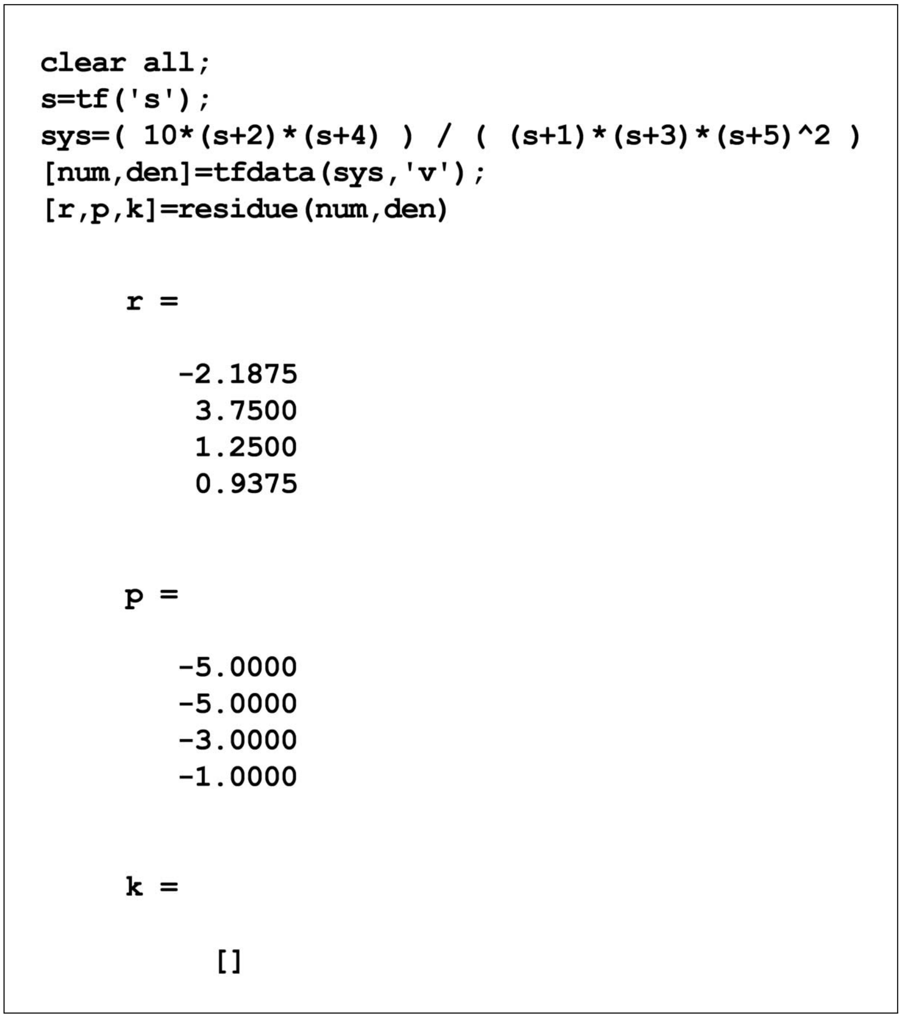


n[343]:= $\text{Plot}\left[\text{InverseLaplaceTransform}\left[\frac{10\ (s+2)\ (s+4)}{(s+1)\ (s+3)\ (s+5)^2}, s, t\right], \{t, 0, 4\}, \text{PlotRange} \to \text{All}, \right.$
$\left.\text{AxesLabel} \to \{t, f\}\right]$

$\frac{5}{16} e^{-5t} (-7 + 4 e^{2t} + 3 e^{4t} + 12 t)$

Figure 3.5: Plot

The Matlab code to find Partial-fraction for this problem is below.



```
clear all;
s=tf('s');
sys=( 10*(s+2)*(s+4) ) / ( (s+1)*(s+3)*(s+5)^2 )
[num,den]=tfdata(sys,'v');
[r,p,k]=residue(num,den)


    r =

       -2.1875
        3.7500
        1.2500
        0.9375


    p =

       -5.0000
       -5.0000
       -3.0000
       -1.0000


    k =

         []
```

Figure 3.6: Matlab code

3.1.5 Problem B 2-16

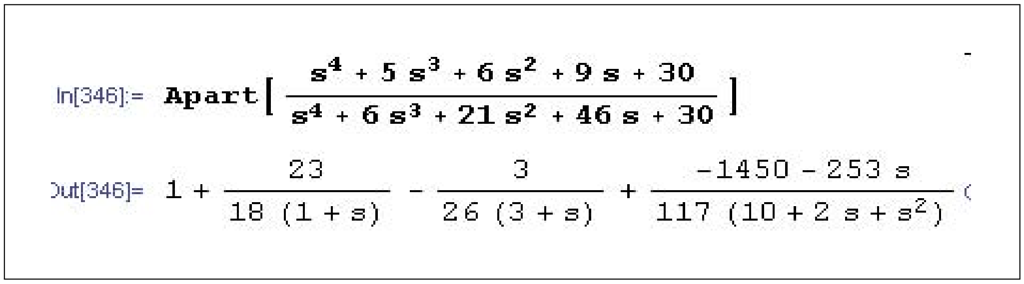
From Modern Control Engineering, 4th edition by Ogata

Question Obtain partial-fraction using MATLAB for

$$F(s) = \frac{s^4 + 5s^3 + 6s^2 + 9s + 30}{s^4 + 6s^3 + 21s^2 + 46s + 30}$$

And then find inverse laplace transform Solution

Using Mathematica to find Partial-fraction



```
                s⁴ + 5 s³ + 6 s² + 9 s + 30
In[346]:= Apart[ ---------------------------- ]
                s⁴ + 6 s³ + 21 s² + 46 s + 30

                    23             3          -1450 - 253 s
Out[346]= 1 + ----------- - ----------- + --------------------
              18 (1 + s)    26 (3 + s)    117 (10 + 2 s + s²)
```

Figure 3.7: Code

Now find the Inverse Laplace transform for each term in the above result as follows.

$$
\begin{aligned}
\mathscr{L}\delta(t) &= 1\\
\mathscr{L}\frac{23}{18}e^{-t} &= \frac{23}{18}\frac{1}{1+s}\\
\mathscr{L}\frac{3}{26}e^{-3t} &= \frac{3}{26}\frac{1}{1+s}
\end{aligned}
$$

The inverse laplace transform of the last term $\frac{-1450-253s}{117\left(10+2s+s^2\right)}$ is found by writing $10+2s+s^2=(s+1)^2+3^2$. Therefore this terms becomes

$$
\begin{aligned}
\frac{-1450-253s}{117\left(10+2s+s^2\right)} &= \frac{\frac{-1450}{117}-\frac{253}{117}s}{(s+1)^2+3^2}\\
&= \frac{\frac{-1450}{117}-\frac{253}{117}(s+1)+\frac{253}{117}}{(s+1)^2+3^2}\\
&= \frac{-\frac{1197}{117}-\frac{253}{117}(s+1)}{(s+1)^2+3^2}\\
&= -\frac{\frac{1197}{117}}{(s+1)^2+3^2}-\frac{\frac{253}{117}(s+1)}{(s+1)^2+3^2}\\
&= -\frac{1197}{(117)(3)}\frac{3}{(s+1)^2+3^2}-\frac{253}{117}\frac{(s+1)}{(s+1)^2+3^2}
\end{aligned}
$$

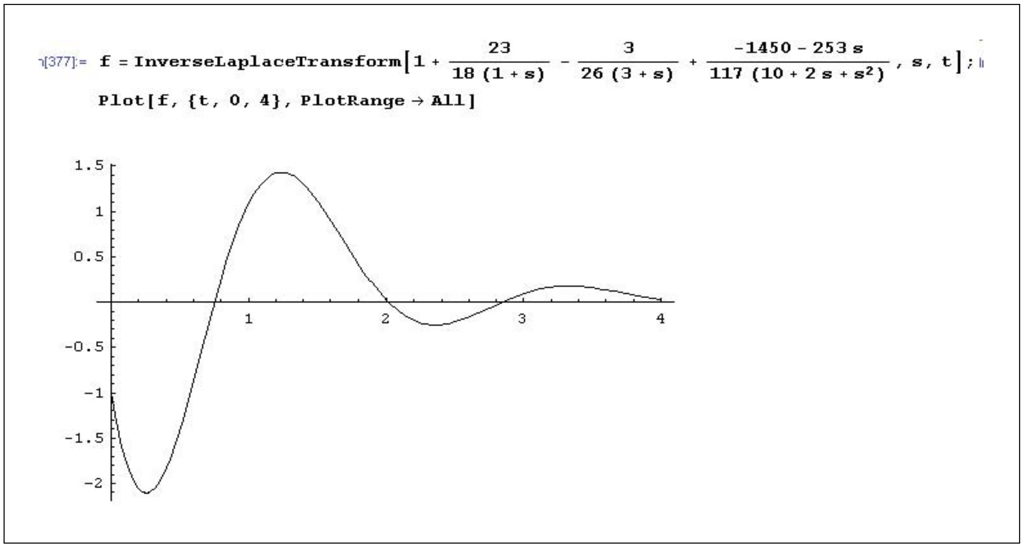
Hence

$$
\begin{aligned}
\mathscr{L}^{-1}\left(-\frac{1197}{117\times 3}\frac{3}{(s+1)^2+3^2}-\frac{253}{117}\frac{(s+1)}{(s+1)^2+3^2}\right) &= -\frac{1197}{(117)(3)}\mathscr{L}^{-1}\left(\frac{3}{(s+1)^2+3^2}\right)\\
&= -\frac{253}{117}\mathscr{L}^{-1}\left(\frac{(s+1)}{(s+1)^2+3^2}\right)\\
&= -\frac{1197}{(117)(3)}e^{-t}\sin 3t-\frac{253}{117}e^{-t}\cos 3t\\
&= -\frac{e^{-t}}{117}\left(399\sin 3t+253\cos 3t\right)
\end{aligned}
$$

Adding all of the above, gives the Inverse Laplace transform as

$$f(t) = \delta(t) + \frac{23}{18}e^{-t} - \frac{3}{26}e^{-3t} - \frac{e^{-t}}{117}(399 \sin 3t + 253 \cos 3t)$$

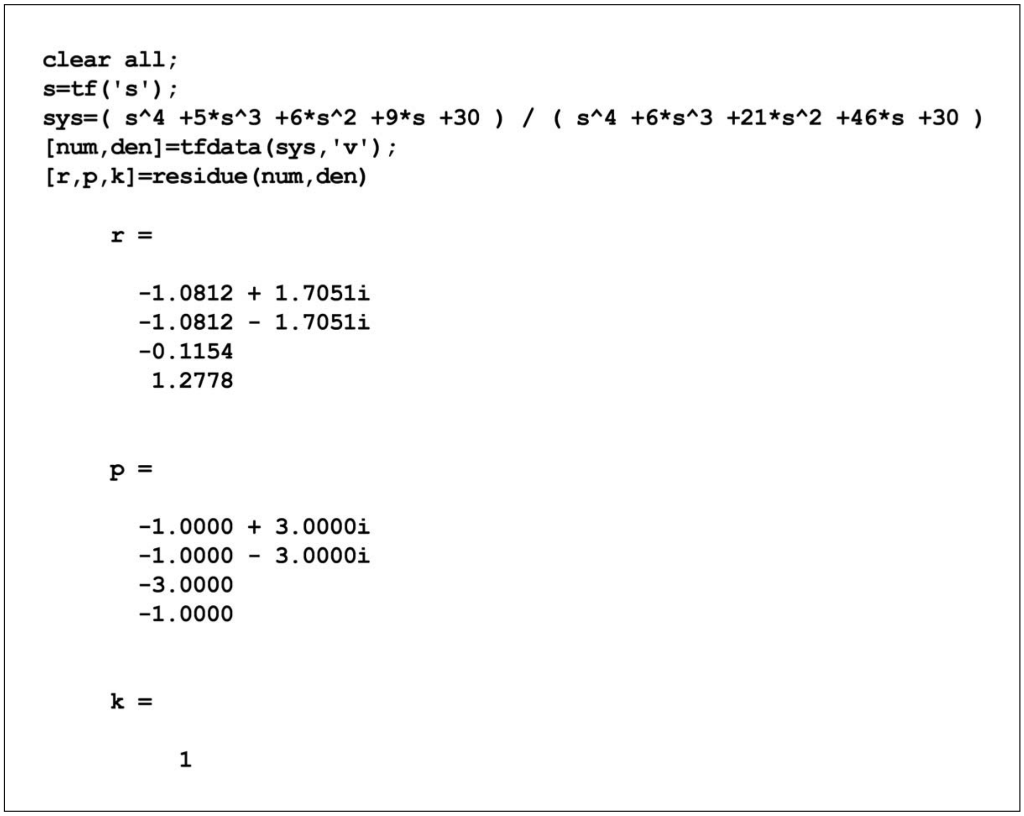
Here is a plot of the solution



```
[377]:= f = InverseLaplaceTransform[1 + 23/(18 (1 + s)) - 3/(26 (3 + s)) + (-1450 - 253 s)/(117 (10 + 2 s + s^2)), s, t];
        Plot[f, {t, 0, 4}, PlotRange → All]
```

Figure 3.8: Plot

The Matlab code to find Partial-fraction for this problem is below.



```
clear all;
s=tf('s');
sys=( s^4 +5*s^3 +6*s^2 +9*s +30 ) / ( s^4 +6*s^3 +21*s^2 +46*s +30 )
[num,den]=tfdata(sys,'v');
[r,p,k]=residue(num,den)

    r =

      -1.0812 + 1.7051i
      -1.0812 - 1.7051i
      -0.1154
       1.2778


    p =

      -1.0000 + 3.0000i
      -1.0000 - 3.0000i
      -3.0000
      -1.0000


    k =

         1
```

Figure 3.9: Matlab code

3.1.6 Problem B 2-17

From Modern Control Engineering, 4th edition by Ogata

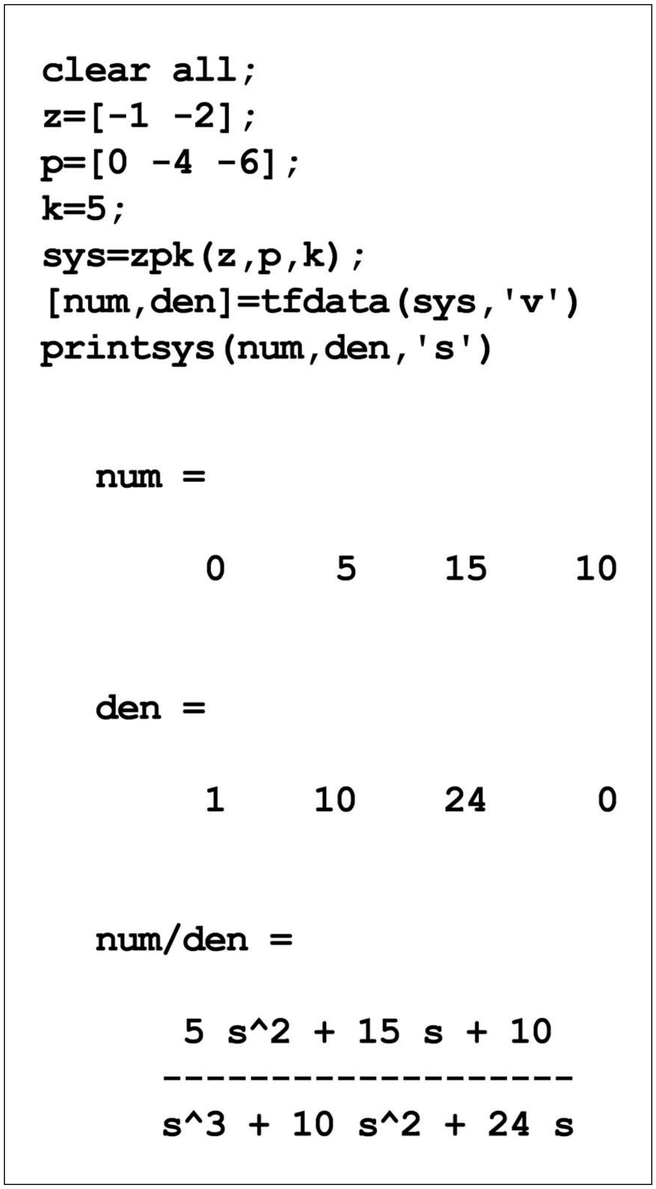
Question

A function $B(s)/A(s)$ consists of the following zeros, poles, and gain K. Zeros at $s=-1, s=-2$, poles at $s=0, s=-4, s=-6$, gain $k=5$.

Obtain an expression for $B(s)/A(s)$ using Matlab.

Solution

In Matlab



```
clear all;
z=[-1 -2];
p=[0 -4 -6];
k=5;
sys=zpk(z,p,k);
[num,den]=tfdata(sys,'v')
printsys(num,den,'s')


num =

     0     5    15    10


den =

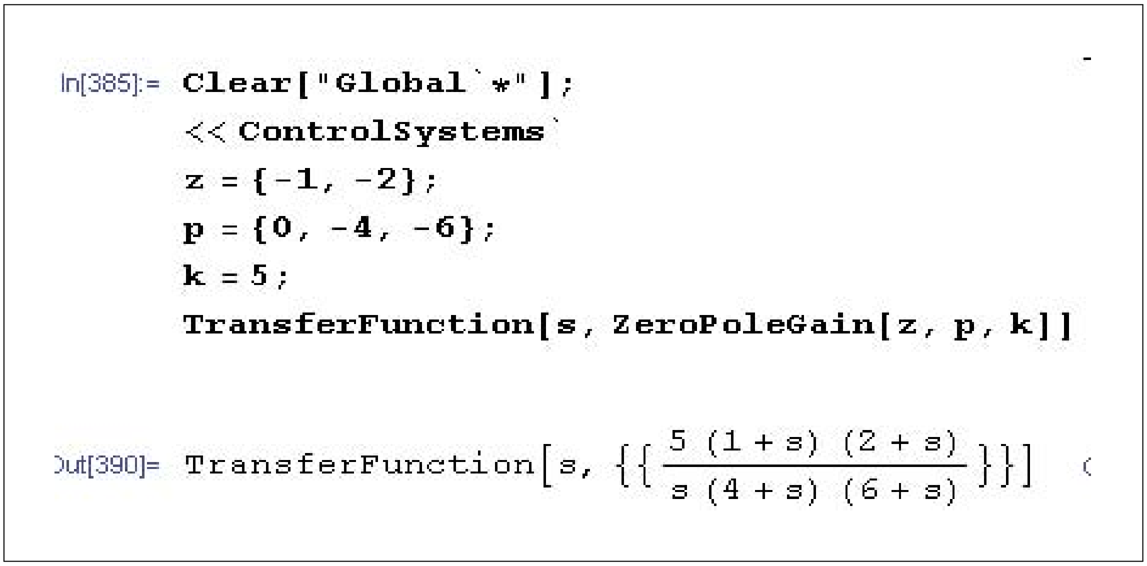
     1    10    24     0


num/den =

    5 s^2 + 15 s + 10
   -------------------
   s^3 + 10 s^2 + 24 s
```

Figure 3.10: Code

In Mathematica, the solution is as follows



```
In[385]:= Clear["Global`*"];
          << ControlSystems`
          z = {-1, -2};
          p = {0, -4, -6};
          k = 5;
          TransferFunction[s, ZeroPoleGain[z, p, k]]


Out[390]= TransferFunction[s, {{ 5 (1 + s) (2 + s)  }}]
                                 s (4 + s) (6 + s)
```

Figure 3.11: Code

3.1.7 Problem B 2-23

From Modern Control Engineering, 4th edition by Ogata

Question

Solve the following ODE

$x'' + 2x' + 10x = e^{-t}$

$x(0) = 0$

$x'(0) = 0$

The forcing function e^{-t} is given at $t = 0$ when the system is at rest.

Solution

Taking laplace transform of the differential equation gives

$$\begin{aligned} s(sX(s) - x(0)) - x'(0) + 2(sX(s) - x(0)) + 10X(s) &= \mathscr{L}\left(e^{-t}\right) \\ s^2X(s) - sx(0) - x'(0) + 2sX(s) - 2x(0) + 10X(s) &= \frac{1}{s+1} \end{aligned}$$

Applying the initial conditions results in

$$\begin{aligned} s^2X(s) + 2sX(s) + 10X(s) &= \frac{1}{s+1} \\ X(s)\left(s^2 + 2s + 10\right) &= \frac{1}{s+1} \\ X(s) &= \frac{1}{(s+1)}\frac{1}{\left(s^2 + 2s + 10\right)} \end{aligned}$$

Taking the inverse laplace transform of $X(s)$ and using partial fraction gives

$$\frac{1}{(s+1)}\frac{1}{\left(s^2+2s+10\right)}=\frac{A}{s+1}+\frac{B}{s^2+2s+10} \tag{1}$$

Multiplying (1) by $s+1$ gives

$$\frac{1}{s^2+2s+10}=A+\frac{B(s+1)}{s^2+2s+10}$$

Evaluating at $s=-1$ gives

$$\begin{aligned}\frac{1}{1-2+10}&=A\\ \frac{1}{9}&=A\end{aligned}$$

Multiplying eq (1) by $s^2+2s+10$ gives

$$\frac{1}{s+1}=\frac{A\left(s^2+2s+10\right)}{s+1}+B$$

Evaluating at $s=3i-1$ gives

$$\frac{1}{3i}=B$$

Therefore

$$\begin{aligned}\frac{1}{s+1}\frac{1}{s^2+2s+10}&=\frac{1}{9}\frac{1}{s+1}+\frac{1}{3i}\frac{1}{(s+1)^2+3^2}\\ &=\frac{1}{9}\frac{1}{s+1}-\frac{i}{9}\frac{3}{(s+1)^2+3^2}\end{aligned}$$

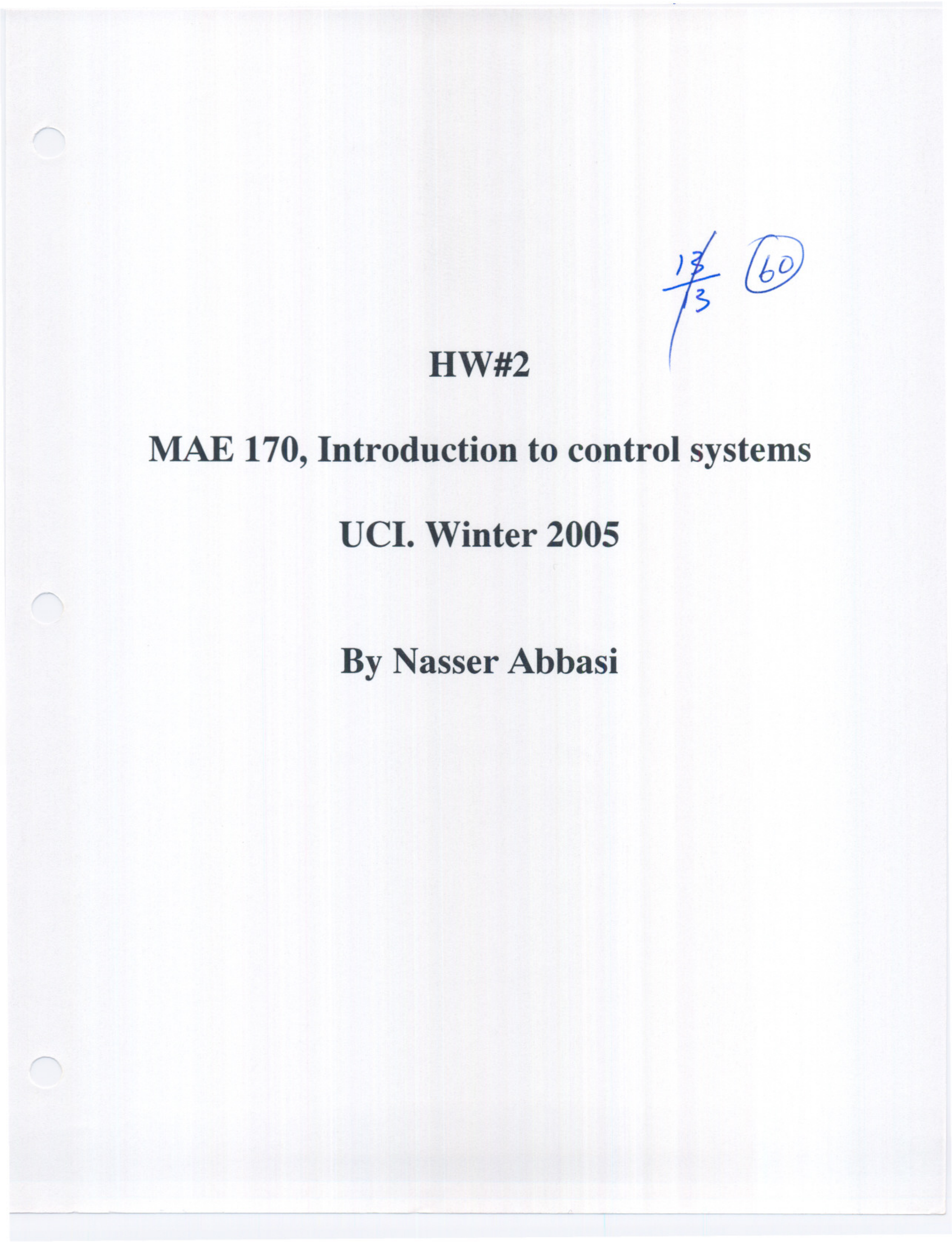
Using tables the inverse Laplace transform is

$$f(t)=\frac{1}{9}e^{-t}-\frac{i}{9}e^{-t}\sin 3t=\frac{1}{9}e^{-t}(1-i\sin 3t)$$

3.2 HW 2

Local contents

3.2.1 my solution



13/13 (60)

HW#2

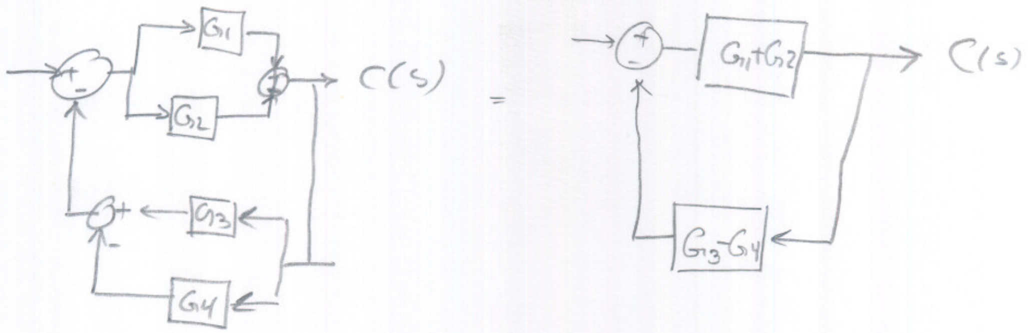
MAE 170, Introduction to control systems

UCI. Winter 2005

By Nasser Abbasi

Nasser Abbasi
HW#2

B-3-1



$$= \quad R(s) \longrightarrow \boxed{\frac{G_1+G_2}{1+(G_1+G_2)(G_3-G_4)}} \longrightarrow C(s)$$

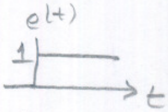
so Transfer function is $$\boxed{\frac{G_1+G_2}{1+(G_1+G_2)(G_3-G_4)}}$$

✓

HW# 2
problem B-3-4
Nasser Abbasi

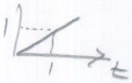
Please note I'll call $1(t)$ as $u(t)$ and the output (response) as $y(t)$ instead of $u(t)$ as book. I find $u(t)$ more clear than $1(t)$

$e(t)$ = unit step



$\Rightarrow \quad E(s) = \frac{1}{s}$

$e(t)$ = unit ramp

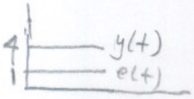


$\Rightarrow \quad E(s) = \frac{1}{s^2}$

For unit step $e(t)$:

proportional

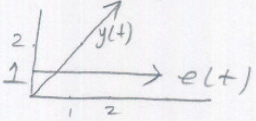
$\frac{Y(s)}{E(s)} = k_p = 4 \Rightarrow Y(s) = 4E(s) \Rightarrow \boxed{y(t) = 4\ 1(t)} \Rightarrow$



integral

$\frac{Y(s)}{E(s)} = \frac{k_i}{s} = \frac{2}{s} \Rightarrow sY(s) = 2\frac{1}{s} \Rightarrow \frac{dy}{dt} = 2\ u(t) \Rightarrow y(t) = 2\int_0^t dt = 2t$

so $\boxed{y(t) = 2t}$ i.e

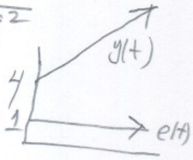


proportional + integral

$\frac{Y(s)}{E(s)} = K_p\left(1+\frac{1}{T_i s}\right) \Rightarrow \frac{Y(s)}{E(s)} = 4\left(1+\frac{1}{2s}\right) \Rightarrow Y(s) = \frac{4}{s} + \frac{4}{2s^2}$

so $\boxed{y(t) = 4u(t) + 2\ \text{unitRamp}(t)}$ i.e for $t>0$, $\boxed{y(t) = 4 + 2t} \Rightarrow$

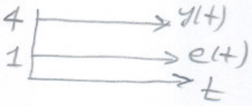
eq of line, slop=2, intersection=4



Propertional + derivative

$\frac{Y(s)}{E(s)} = K_p(1+T_d s) \Rightarrow \frac{Y(s)}{E(s)} = 4(1+0.8s) \rightarrow Y(s) = \frac{4}{s} + 3.2$

so $y(t) = 4u(t) + 3.2\delta(t)$ so for $t>0$ $\boxed{y(t) = 4}$



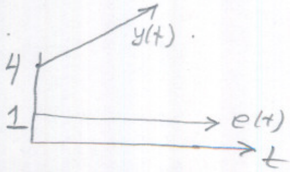
notice that P+D has same response as propertional since $e(t)$ is unit step, whose derivative is zero for $t>0$.

PID

$\frac{Y(s)}{E(s)} = K_p\left(1+\frac{1}{T_i s} + T_d s\right) = 4\left(1+\frac{1}{2s} + 0.8s\right) \Rightarrow Y(s) = \frac{4}{s} + \frac{1}{s^2} + 0.8$

so $y(t) = 4u(t) + \text{unitramp}(t) + 0.8\delta(t)$.

so for $t>0$, $\boxed{y(t) = 4 + t}$



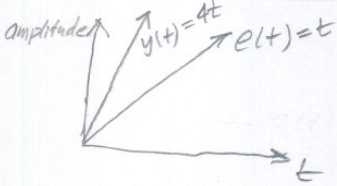
notice that PID response same as PI, sine the "D" effect is cancelled due to $e(t)$ having zero derivative for $t>0$. →

Problem B-3-4 (Cont.)

Now for $e(t) =$ unit ramp $\quad E(s)=\frac{1}{s^2}$

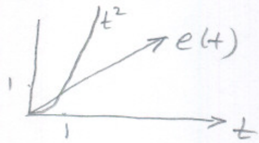
Proportional

$$\frac{Y(s)}{E(s)} = K_p \Rightarrow Y(s) = 4E(s) \Rightarrow Y(s)=\frac{4}{s^2} \Rightarrow \boxed{y(t)=4 \text{ unitramp}(t)}$$



integral

$$\frac{Y(s)}{E(s)}=\frac{K_i}{s} \Rightarrow Y(s)=\frac{2}{s}\frac{1}{s^2}=\frac{2}{s^3} \text{ From Tables } \Rightarrow \boxed{y(t)=t^2}$$



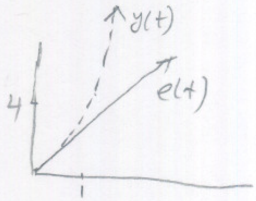
P+I

$$\frac{Y(s)}{E(s)} = K_p\left(1+\frac{1}{T_i s}\right) \Rightarrow Y(s)=4\left(1+\frac{1}{2s}\right)\frac{1}{s^2} \Rightarrow \frac{4}{s^2}+\frac{1}{2s^3}$$

$\frac{4}{s^2} \rightarrow 4 \text{ Unitramp}(t)$

$\frac{1}{2s^3} \rightarrow \frac{1}{4}\frac{2}{s^3} \rightarrow \frac{1}{4}t^2$

so $\boxed{y(t) = 4t+\frac{1}{4}t^2}$ for $t>0$.

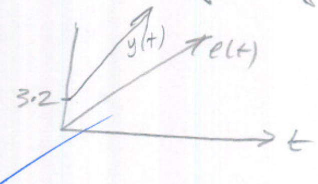


P+D

$$\frac{Y(s)}{E(s)}=K_p(1+T_d s)\Rightarrow Y(s)=(K_p+K_pT_d s)\frac{1}{s^2} \Rightarrow Y(s)=\frac{4}{s^2}+\frac{3.2}{s}$$

so $y(t) = 4 \text{ unitramp}(t) + 3.2\, u(t)$

so for $t>0$, $\boxed{y(t) = \underset{\text{slop}}{4}t + \underset{\text{intersection.}}{3.2}}$



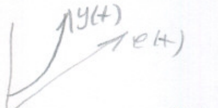
PID

$$\frac{Y(s)}{E(s)}=K_p\left(1+\frac{1}{T_i s}+T_d s\right)\Rightarrow Y(s)=\left(4+\frac{4}{2s}+(4)(0.8)\,s\right)\frac{1}{s^2}$$

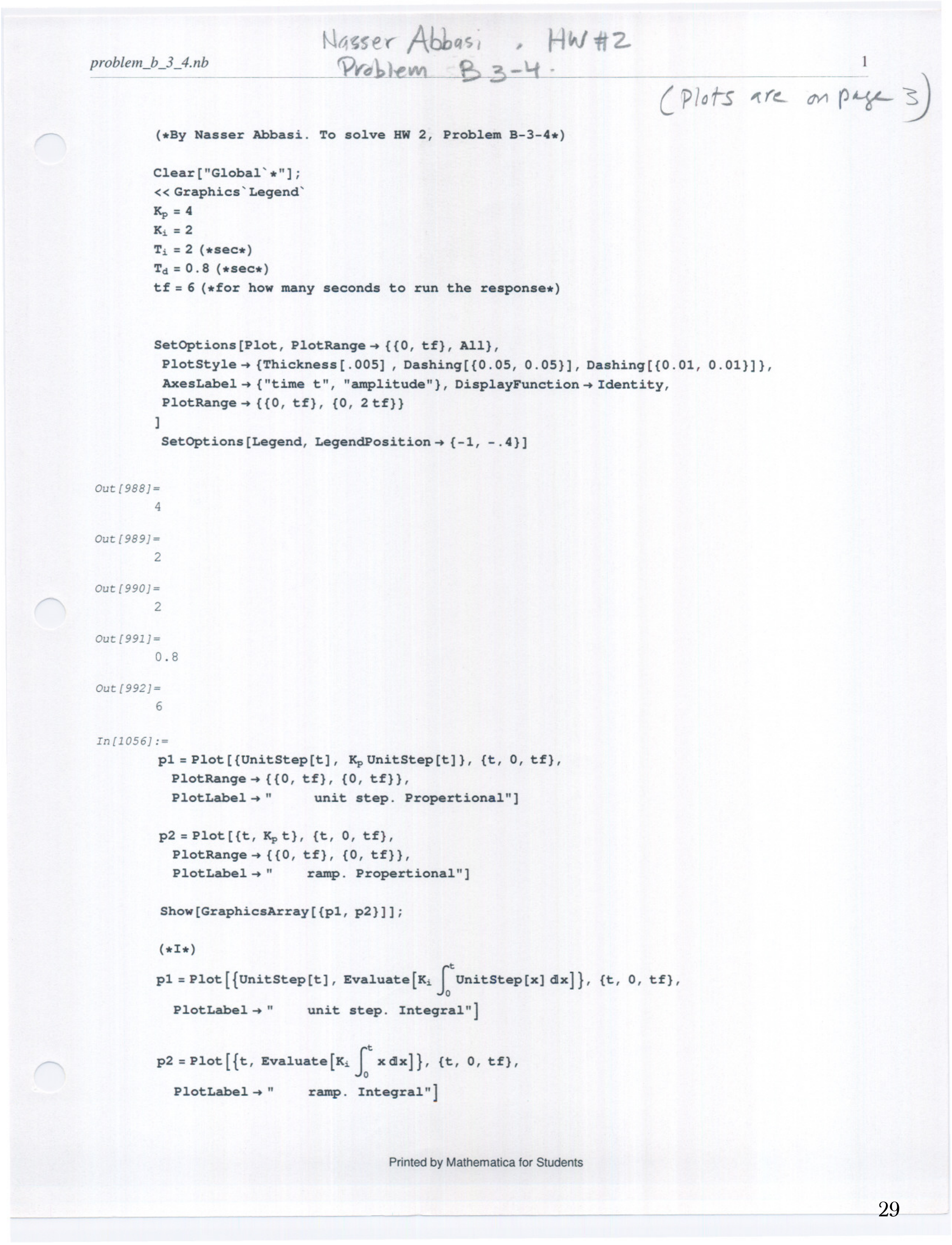
$$Y(s)=\frac{4}{s^2}+\frac{2}{s^3}+\frac{3.2}{s}\Rightarrow y(t)=4 \text{ unitramp}(t)+t^2+3.2 \text{ unitstep}(t).$$

So for $t>0$, $\underset{\text{quadratic equation}}{\boxed{y(t)=4t+t^2+3.2}}$

$t=0$, $y(t)=3.2$
$t=1$, $y(t)=8.2$



P.S. Just For Fun, I wrote the following small program which displays output of these controllers, which can be used for any $e(t)$ function.



problem_b_3_4.nb Nasser Abbasi , HW #2 1
Problem B 3-4.

(Plots are on page 3)

```
(*By Nasser Abbasi. To solve HW 2, Problem B-3-4*)

Clear["Global`*"];
<< Graphics`Legend`
Kp = 4
Ki = 2
Ti = 2 (*sec*)
Td = 0.8 (*sec*)
tf = 6 (*for how many seconds to run the response*)


SetOptions[Plot, PlotRange → {{0, tf}, All},
 PlotStyle → {Thickness[.005] , Dashing[{0.05, 0.05}], Dashing[{0.01, 0.01}]},
 AxesLabel → {"time t", "amplitude"}, DisplayFunction → Identity,
 PlotRange → {{0, tf}, {0, 2 tf}}
]
 SetOptions[Legend, LegendPosition → {-1, -.4}]


Out[988]=
       4

Out[989]=
       2

Out[990]=
       2

Out[991]=
       0.8

Out[992]=
       6

In[1056]:=
        p1 = Plot[{UnitStep[t], Kp UnitStep[t]}, {t, 0, tf},
          PlotRange → {{0, tf}, {0, tf}},
          PlotLabel → "     unit step. Propertional"]

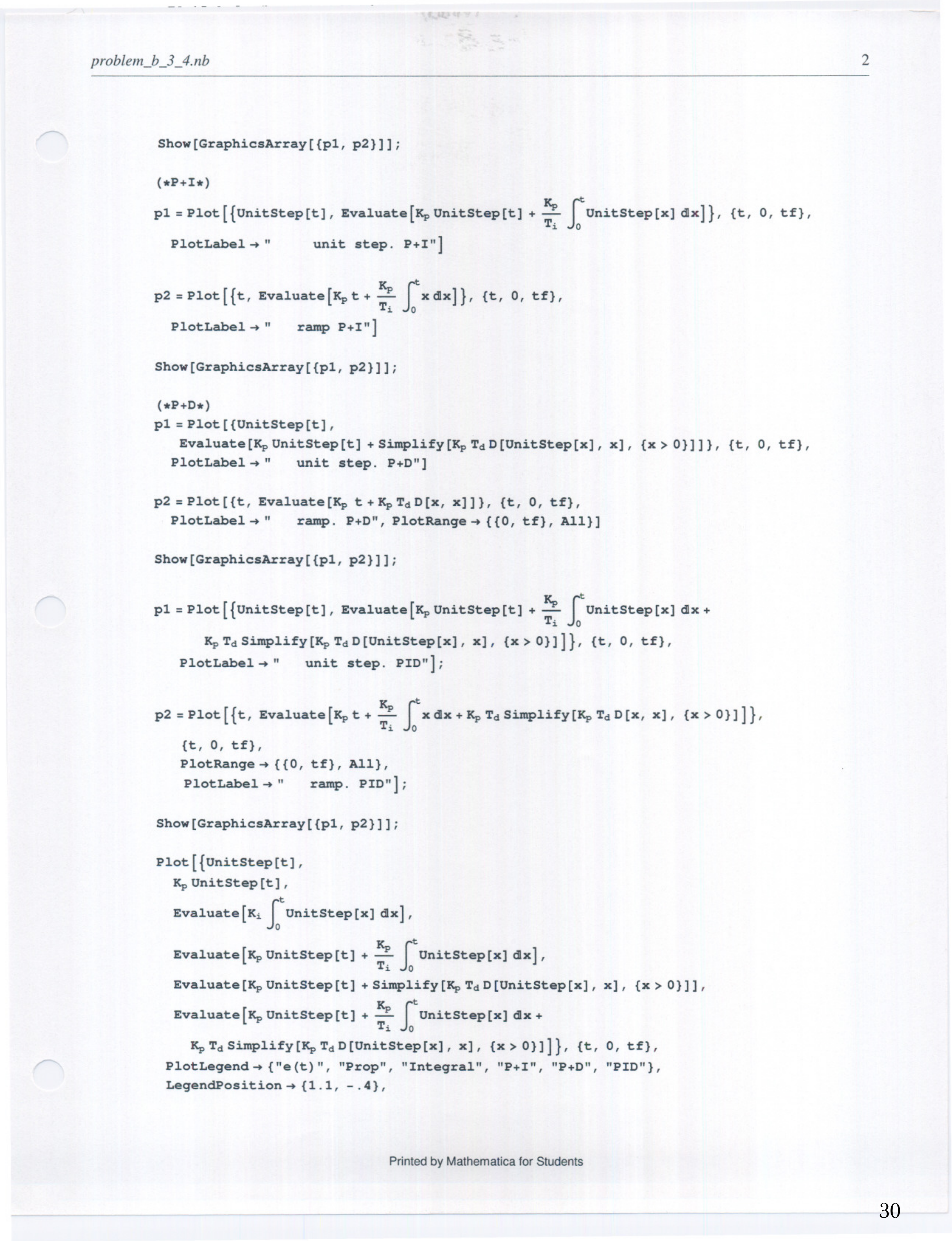
        p2 = Plot[{t, Kp t}, {t, 0, tf},
          PlotRange → {{0, tf}, {0, tf}},
          PlotLabel → "    ramp. Propertional"]

        Show[GraphicsArray[{p1, p2}]];

        (*I*)
        p1 = Plot[{UnitStep[t], Evaluate[Ki ∫_0^t UnitStep[x] ⅆx]}, {t, 0, tf},
          PlotLabel → "    unit step. Integral"]

        p2 = Plot[{t, Evaluate[Ki ∫_0^t x ⅆx]}, {t, 0, tf},
          PlotLabel → "    ramp. Integral"]
```

Printed by Mathematica for Students



```
Show[GraphicsArray[{p1, p2}]];

(*P+I*)
p1 = Plot[{UnitStep[t], Evaluate[K_p UnitStep[t] + K_p/T_i ∫_0^t UnitStep[x] ⅆx]}, {t, 0, tf},
  PlotLabel → "     unit step. P+I"]

p2 = Plot[{t, Evaluate[K_p t + K_p/T_i ∫_0^t x ⅆx]}, {t, 0, tf},
  PlotLabel → "   ramp P+I"]

Show[GraphicsArray[{p1, p2}]];

(*P+D*)
p1 = Plot[{UnitStep[t],
   Evaluate[K_p UnitStep[t] + Simplify[K_p T_d D[UnitStep[x], x], {x > 0}]]}, {t, 0, tf},
  PlotLabel → "   unit step. P+D"]

p2 = Plot[{t, Evaluate[K_p t + K_p T_d D[x, x]]}, {t, 0, tf},
  PlotLabel → "   ramp. P+D", PlotRange → {{0, tf}, All}]

Show[GraphicsArray[{p1, p2}]];

p1 = Plot[{UnitStep[t], Evaluate[K_p UnitStep[t] + K_p/T_i ∫_0^t UnitStep[x] ⅆx +
      K_p T_d Simplify[K_p T_d D[UnitStep[x], x], {x > 0}]]}, {t, 0, tf},
   PlotLabel → "   unit step. PID"];

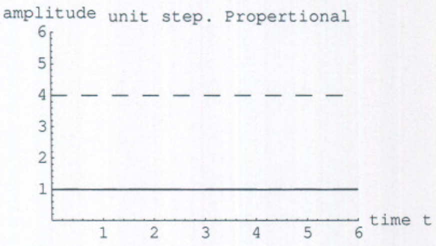
p2 = Plot[{t, Evaluate[K_p t + K_p/T_i ∫_0^t x ⅆx + K_p T_d Simplify[K_p T_d D[x, x], {x > 0}]]},
   {t, 0, tf},
   PlotRange → {{0, tf}, All},
    PlotLabel → "   ramp. PID"];

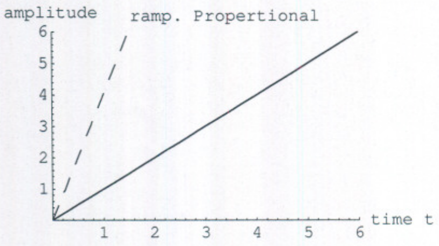
Show[GraphicsArray[{p1, p2}]];

Plot[{UnitStep[t],
  K_p UnitStep[t],
  Evaluate[K_i ∫_0^t UnitStep[x] ⅆx],
  Evaluate[K_p UnitStep[t] + K_p/T_i ∫_0^t UnitStep[x] ⅆx],
  Evaluate[K_p UnitStep[t] + Simplify[K_p T_d D[UnitStep[x], x], {x > 0}]],
  Evaluate[K_p UnitStep[t] + K_p/T_i ∫_0^t UnitStep[x] ⅆx +
    K_p T_d Simplify[K_p T_d D[UnitStep[x], x], {x > 0}]]}, {t, 0, tf},
 PlotLegend → {"e(t)", "Prop", "Integral", "P+I", "P+D", "PID"},
 LegendPosition → {1.1, -.4},
```

```
PlotRange → {{0, 3}, {0, 7}},
PlotLabel → "    different controllers response to unit step e(t)",
PlotStyle → {Dashing[{0.06, 0.06}], Dashing[{0.01, 0.01}], Dashing[{0.02, 0.02}],
  Dashing[{0.03, 0.03}], Dashing[{0.04, 0.04}]}, DisplayFunction → $DisplayFunction]

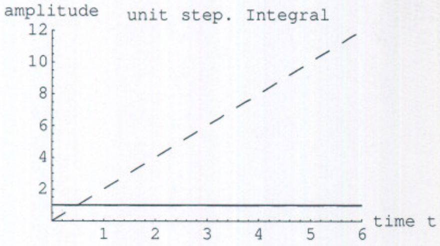
Plot[{t,
  Kp t,
  Evaluate[Ki ∫_0^t x ⅆx],
  Evaluate[Kp t + Kp/Ti ∫_0^t x ⅆx],
  Evaluate[Kp t + Simplify[Kp Td D[x, x], {x > 0}]],
  Evaluate[Kp t + Kp/Ti ∫_0^t x ⅆx + Kp Td Simplify[Kp Td D[ x, x], {x > 0}]]},
 {t, 0, tf},
 PlotLegend → {"e(t)", "Prop", "Integral", "P+I", "P+D", "PID"},
 LegendPosition → {1.1, -.4},
 PlotRange → {{0, tf}, {0, 30}},
 PlotLabel → "    different controllers response to ramp e(t)",
 PlotStyle →
  { Thickness[.008], Dashing[{0.06, 0.06}], Dashing[{0.01, 0.01}], Dashing[{0.02, 0.02}],
   Dashing[{0.03, 0.03}], Dashing[{0.04, 0.04}]}, DisplayFunction → $DisplayFunction
]
```

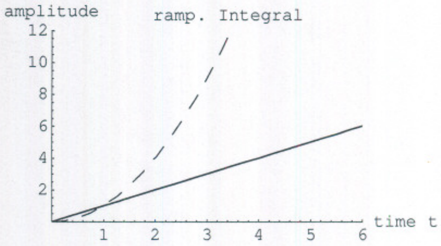


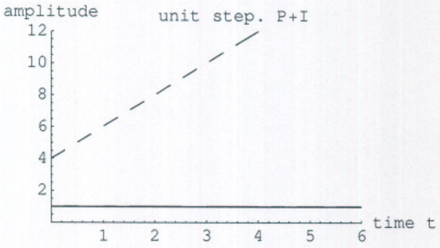


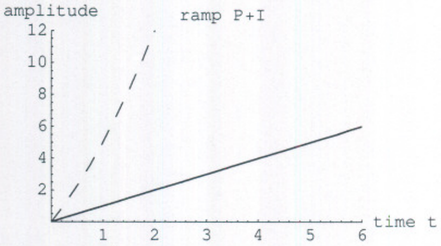
legend —— e(t)

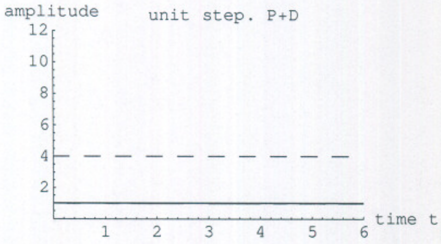
---- u(t)

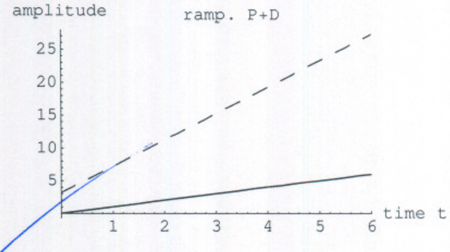

amplitude
unit step. Integral
12
10
8
6
4
2
1
2
3
4
5
6
time t

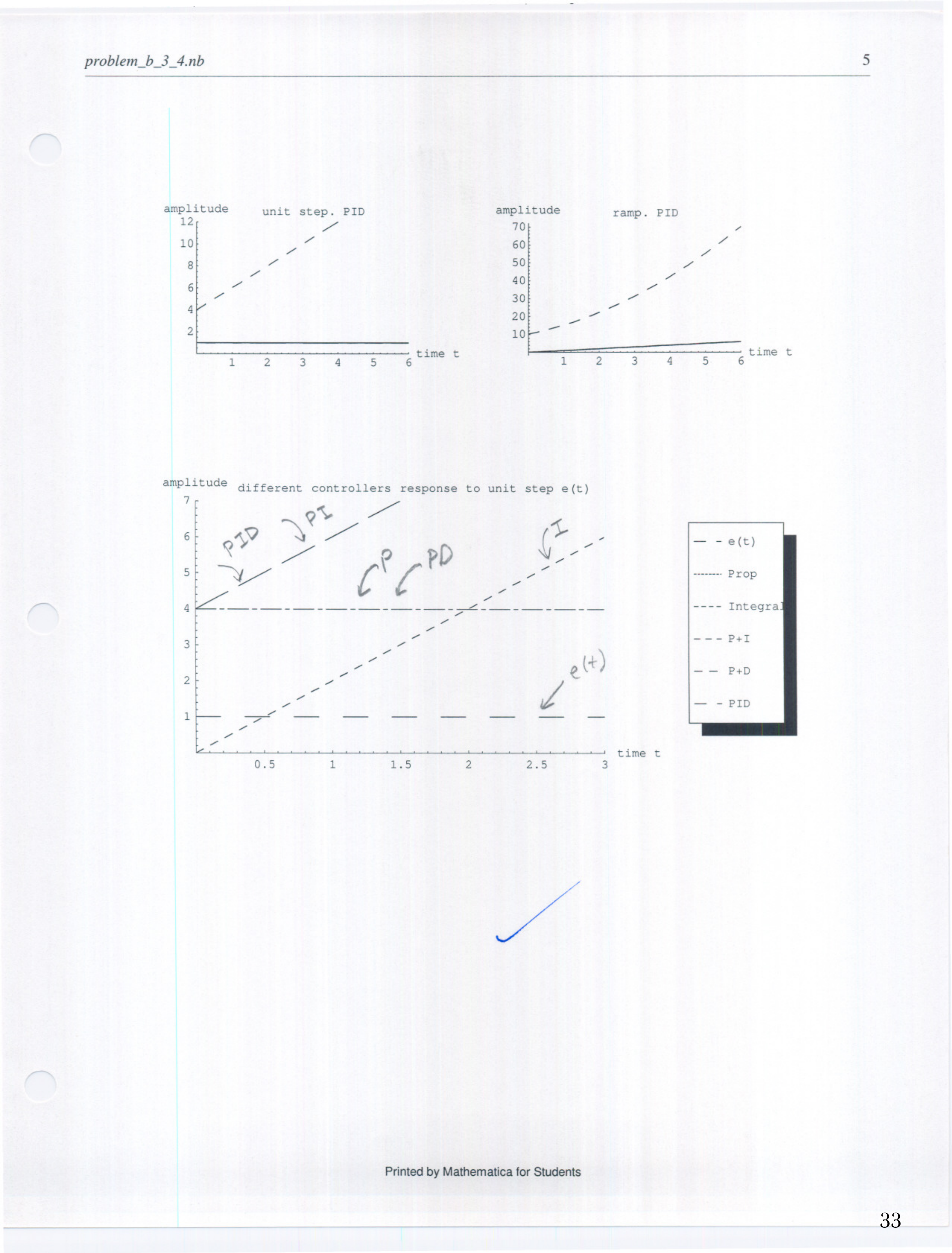

amplitude
ramp. Integral
12
10
8
6
4
2
1
2
3
4
5
6
time t


amplitude
unit step. P+I
12
10
8
6
4
2
1
2
3
4
5
6
time t


amplitude
ramp P+I
12
10
8
6
4
2
1
2
3
4
5
6
time t


amplitude
unit step. P+D
12
10
8
6
4
2
1
2
3
4
5
6
time t


amplitude
ramp. P+D
25
20
15
10
5
1
2
3
4
5
6
time t



problem_b_3_4.nb 5

✓

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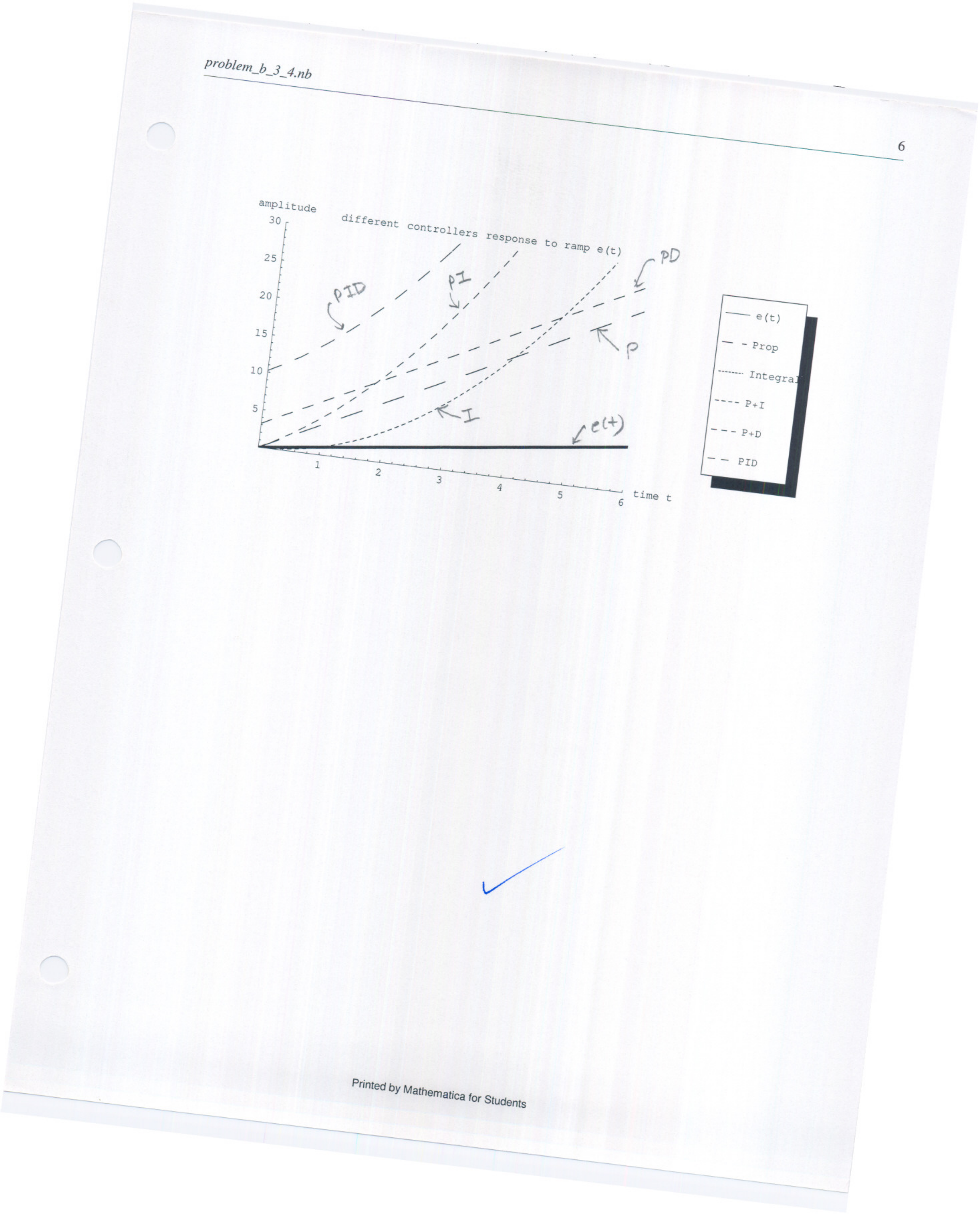

problem_b_3_4.nb

6

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HW#2
Problem B-3-9
Nasser Abbasi

Consider $\dddot{y} + 3\ddot{y} + 2\dot{y} = u$

derive state space representation.

solution here $n=3$ (order of D.E.)

let $\left\{\begin{array}{l} X_1 = y \\ x_2 = \dot{y} \\ x_3 = \ddot{y} \longrightarrow (n-1) \end{array}\right.$

state variables ←

so DE can be written as $\dot{x}_3 = -3x_3 - 2x_2 + u$

and $\dot{x}_1 = \dot{y} = x_2$

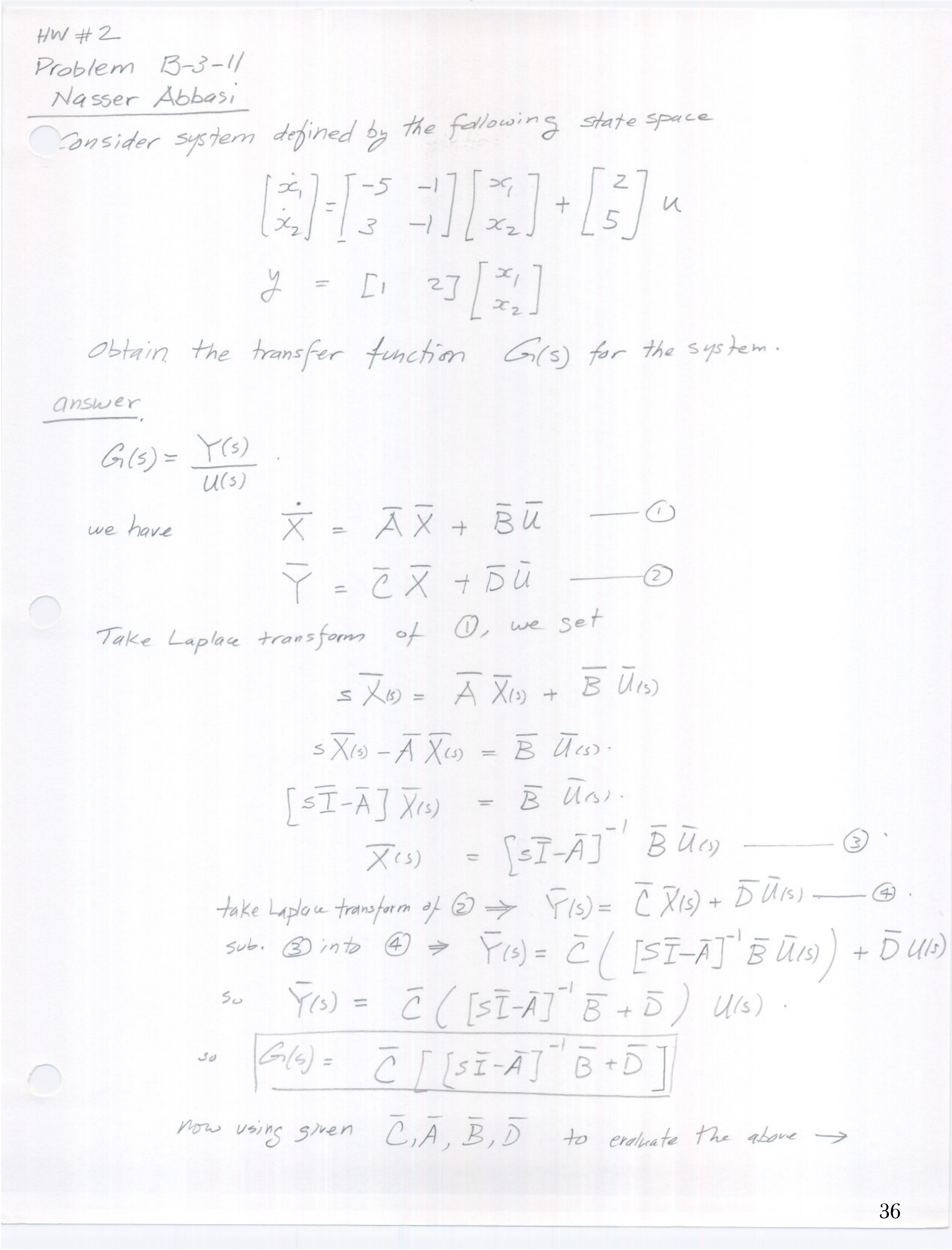
and $\dot{x}_2 = \ddot{y} = x_3$.

since $\underline{\dot{x}} = A\,\underline{X} + B\,\underline{u}$ Then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} [u]$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{\substack{C \\ 1\times3}} \underset{3\times1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} + \underbrace{[0]}_{\substack{D \\ 1\times1}} \underset{1\times1}{[u]}$$

✓



HW #2

Problem B-3-11

Nasser Abbasi

Consider system defined by the following state space

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Obtain the transfer function $G(s)$ for the system.

answer.

$$G(s) = \frac{Y(s)}{U(s)}.$$

we have
$$\dot{\bar{X}} = \bar{A}\bar{X} + \bar{B}\bar{u} \text{ ——— } ①$$
$$\bar{Y} = \bar{C}\bar{X} + \bar{D}\bar{u} \text{ ——— } ②$$

Take Laplace transform of ①, we set

$$s\bar{X}(s) = \bar{A}\bar{X}(s) + \bar{B}\bar{U}(s)$$
$$s\bar{X}(s) - \bar{A}\bar{X}(s) = \bar{B}\bar{U}(s).$$
$$[s\bar{I} - \bar{A}]\bar{X}(s) = \bar{B}\bar{U}(s).$$
$$\bar{X}(s) = [s\bar{I} - \bar{A}]^{-1}\bar{B}\bar{U}(s) \text{ ——— } ③.$$

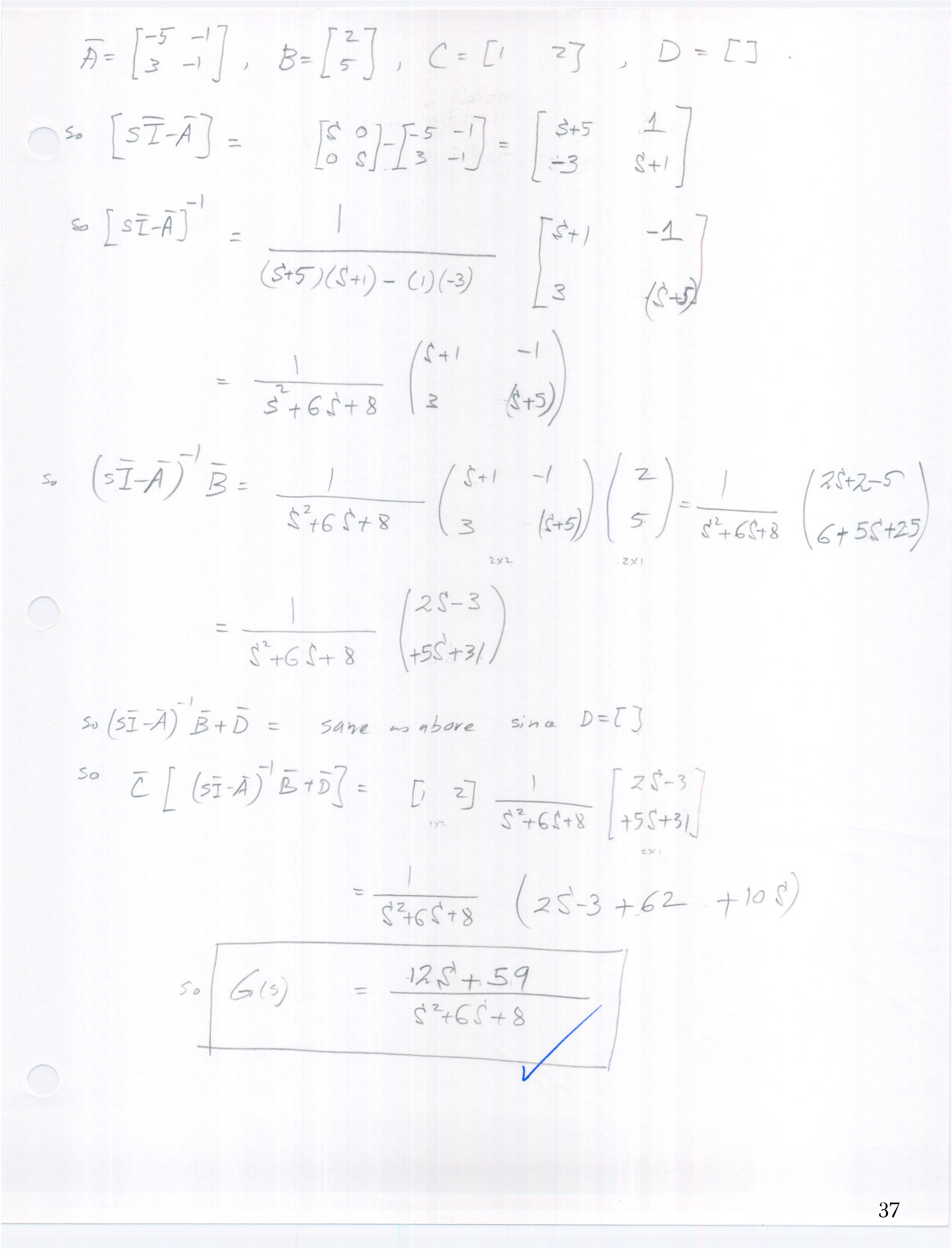
take Laplace transform of ② $\Rightarrow$ $\bar{Y}(s) = \bar{C}\bar{X}(s) + \bar{D}\bar{U}(s)$ ——— ④.

sub. ③ into ④ $\Rightarrow$ $\bar{Y}(s) = \bar{C}\left([s\bar{I} - \bar{A}]^{-1}\bar{B}\bar{U}(s)\right) + \bar{D}U(s)$

so $\bar{Y}(s) = \bar{C}\left([s\bar{I} - \bar{A}]^{-1}\bar{B} + \bar{D}\right)U(s)$.

so
$$\boxed{G(s) = \bar{C}\left[[s\bar{I} - \bar{A}]^{-1}\bar{B} + \bar{D}\right]}$$

now using given $\bar{C}, \bar{A}, \bar{B}, \bar{D}$ to evaluate the above →



$\bar{A}=\begin{bmatrix}-5 & -1\\ 3 & -1\end{bmatrix}$, $B=\begin{bmatrix}2\\ 5\end{bmatrix}$, $C=[1 \quad 2]$, $D=[\,]$.

so $[s\bar{I}-\bar{A}] = \begin{bmatrix}s & 0\\ 0 & s\end{bmatrix}-\begin{bmatrix}-5 & -1\\ 3 & -1\end{bmatrix} = \begin{bmatrix}s+5 & 1\\ -3 & s+1\end{bmatrix}$

so $[s\bar{I}-\bar{A}]^{-1} = \dfrac{1}{(s+5)(s+1)-(1)(-3)}\begin{bmatrix}s+1 & -1\\ 3 & (s+5)\end{bmatrix}$

$$= \frac{1}{s^2+6s+8}\begin{pmatrix}s+1 & -1\\ 3 & (s+5)\end{pmatrix}$$

so $(s\bar{I}-\bar{A})^{-1}\bar{B} = \dfrac{1}{s^2+6s+8}\underset{2\times2}{\begin{pmatrix}s+1 & -1\\ 3 & (s+5)\end{pmatrix}}\underset{2\times1}{\begin{pmatrix}2\\ 5\end{pmatrix}} = \dfrac{1}{s^2+6s+8}\begin{pmatrix}2s+2-5\\ 6+5s+25\end{pmatrix}$

$$= \frac{1}{s^2+6s+8}\begin{pmatrix}2s-3\\ +5s+31\end{pmatrix}$$

so $(s\bar{I}-\bar{A})^{-1}\bar{B}+\bar{D}$ = same as above since $D=[\,]$

so $\bar{C}\left[(s\bar{I}-\bar{A})^{-1}\bar{B}+\bar{D}\right] = \underset{1\times2}{[1 \quad 2]}\dfrac{1}{s^2+6s+8}\underset{2\times1}{\begin{bmatrix}2s-3\\ +5s+31\end{bmatrix}}$

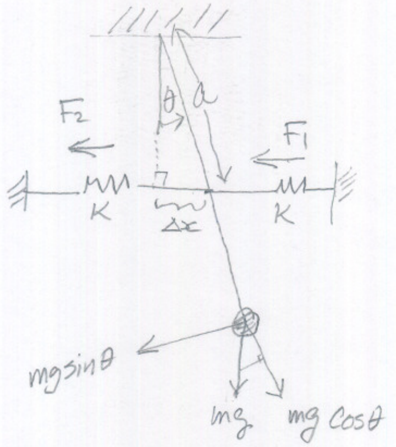
$$= \frac{1}{s^2+6s+8}\left(2s-3+62+10s\right)$$

so $\boxed{G(s) = \dfrac{12s+59}{s^2+6s+8}}$ ✓

HW# 2

Problem B-3-16

Nasser Abbasi

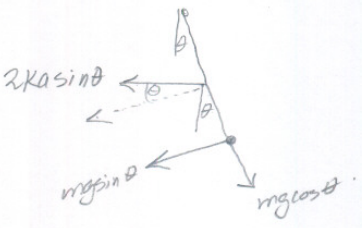


length of bar $= l$

$\sin\theta = \frac{\Delta x}{a} \Rightarrow \Delta x = a \sin\theta$. This is the amount of stretch of spring.

so total spring Force is $2(K\Delta x) = 2Ka\sin\theta$

so normal component of spring force will be $\boxed{2Ka\sin\theta\cos\theta}$



Torque produced by this force is $2Ka\sin\theta\cos\theta\, a$

$= \boxed{2Ka^2\sin\theta\cos\theta}$

Torque produced by tangential component of mg is $\boxed{mg\sin\theta\, l}$

so total torque is $-\left(2Ka^2\sin\theta\cos\theta + mg\sin\theta\, l\right)$

use equation of motion $\boxed{\text{Torque} = J\ddot{\theta}}$, where J is moment of inertia of bar around hinge.

$\boxed{J = ml^2}$ assume all mass in ball.

so $\boxed{ml^2\ddot{\theta} + 2Ka^2\sin\theta\cos\theta + mg\sin\theta\, l = 0}$ ✓

for small θ, since $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$, $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots$.

so $\sin\theta \rightarrow \theta$, $\cos\theta \rightarrow 1$ by ignoring term with powers ≥ 2 .

so $\boxed{ml^2\ddot{\theta} + 2Ka^2\theta + mg\theta l = 0}$ →

$$\ddot{\theta} + \theta\left(\frac{2Ka^2 + mg\ell}{m\ell^2}\right) = 0$$

let $\dfrac{2Ka^2+mg\ell}{m\ell^2} = B$.

so $\ddot{\theta} + B\theta = 0$. char. equation is $\overbrace{(D^2+B)}^{a=1,\, b=0,\, c=B}\theta = 0$

so roots of char eq. $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{\pm\sqrt{-4B}}{2} = \pm\sqrt{-B}$

$= \pm i\sqrt{B}$

so solution is $\boxed{\theta = \alpha e^{i\sqrt{B}} + \beta e^{-i\sqrt{B}}}$

or $\theta = \alpha e^{\sqrt{\frac{-2Ka^2-mg\ell}{m\ell^2}}} + \beta e^{-\sqrt{\frac{-2Ka^2-mg\ell}{m\ell^2}}}$

wher α, β are constants found from initial conditions.

we see that natural frequency is

$$\sqrt{\frac{2Ka^2+mg\ell}{m\ell^2}} = \sqrt{\frac{2Ka^2}{m\ell^2} + \frac{g}{\ell}}.$$

we see if there were no springs, i.e. $k=0$ this simplifies to $\sqrt{\frac{g}{\ell}}$ as expected.

✓

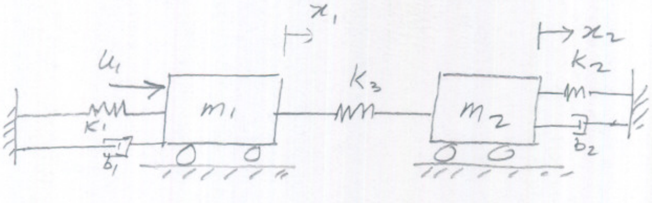
HW # 2

Problem B-3-18

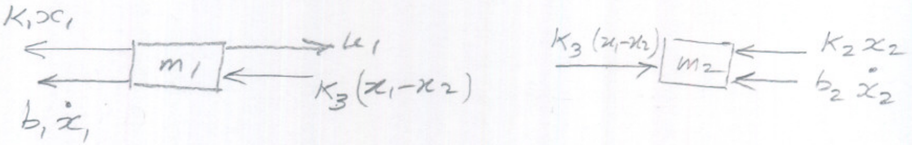
Nasser Abbasi

Obtain transfer functions $\frac{X_1(s)}{U(s)}$ and $\frac{X_2(s)}{U(s)}$ of mechanical system shown.

solution



assume m_1 is moving faster than m_2 to the right. draw free body diagrams



so, equation of motion $\boxed{F=ma}$ applied to each mass

$$u_1 - k_3(x_1-x_2) - b_1\dot{x}_1 - k_1 x_1 = m_1\ddot{x}_1$$

$$k_3(x_1-x_2) - k_2 x_2 - b_2\dot{x}_2 = m_2\ddot{x}_2$$

simplify, we get

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + k_1 x_1 + k_3 x_1 - k_3 x_2 = u \quad \text{———} \; ①$$

$$m_2\ddot{x}_2 + b_2\dot{x}_2 + k_2 x_2 - k_3 x_1 + k_3 x_2 = 0 \quad \text{———} \; ②$$

For eq ①. take Laplace transform, assume initial conditions.

$$m_1 s^2 X_1(s) + b_1 s X_1(s) + k_1 X_1(s) + k_3 X_1(s) - k_3 X_2(s) = U(s)$$

$$X_1(s)\left[m_1 s^2 + b_1 s + k_1 + k_3\right] - k_3 X_2(s) = U(s) \quad \text{———} \; ③.$$

For eq ②. take Laplace transform →

$$m_2 s^2 X_2(s) + b_2 s X_2(s) + k_2 X_2(s) - k_3 X_1(s) + k_3 X_2(s) = 0$$

→

so

$$X_2(s)\left[m_2 s^2 + b_2 s + K_2 + K_3\right] - K_3 X_1(s) = 0 \quad \text{———} \quad ④$$

i.e.

$$X_2(s) = \frac{K_3}{m_2 s^2 + b_2 s + K_2 + K_3} X_1(s) \quad \text{———} \quad ⑤$$

sub ⑤ into ③ we set

$$X_1(s)\left[m_1 s^2 + b_1 s + K_1 + K_3\right] - K_3 \frac{K_3}{m_2 s^2 + b_2 s + K_2 + K_3} X_1(s) = U(s).$$

So

$$X_1(s)\left[m_1 s^2 + b_1 s + K_1 + K_3 - \frac{K_3^2}{m_2 s^2 + b_2 s + K_2 + K_3}\right] = U(s).$$

So

$$G_1(s) = \boxed{\frac{X_1(s)}{U(s)} = \frac{m_2 s^2 + b_2 s + K_2 + K_3}{(m_1 s^2 + b_1 s + K_1 + K_3)(m_2 s^2 + b_2 s + K_2 + K_3) - K_3^2}}$$

✓

To find $G_2(s)$. sub ③ into ⑤ we set

$$\frac{X_2(s)}{U(s)} = \frac{K_3}{(m_2 s^2 + b_2 s + K_2 + K_3)} \frac{(m_2 s^2 + b_2 s + K_2 + K_3)}{(m_1 s^2 + b_1 s + K_1 + K_3)(m_2 s^2 + b_2 s + K_2 + K_3) - K_3^2}$$

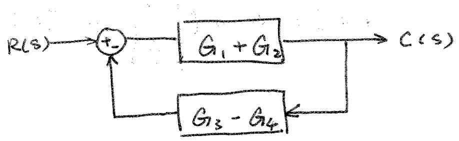
$$\boxed{\frac{X_2(s)}{U(s)} = \frac{K_3}{(m_1 s^2 + b_1 s + K_1 + K_3)(m_2 s^2 + b_2 s + K_2 + K_3) - K_3^2}}$$

✓

3.2.2 key solution

MAE 170 Homework 2 Solutions.

3.1



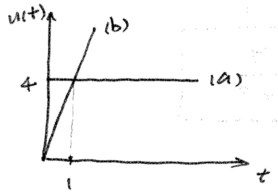
$$\frac{C}{R} = \frac{G_1+G_2}{1+(G_1+G_2)(G_3-G_4)}$$

3.4

$$\frac{U(s)}{E(s)} = K_p = 4$$

(a) $u(t) = 4 \cdot 1(t)$

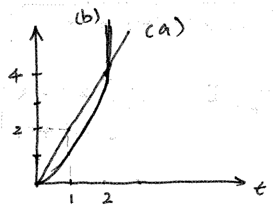
(b) $u(t) = 4t$



$$\frac{U(s)}{E(s)} = \frac{K_i}{s} = \frac{2}{s}$$

(a) $u(t) = 2t$

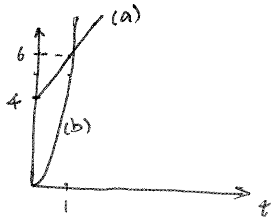
(b) $u(t) = t^2$



$$\frac{U(s)}{E(s)} = K_p\left(1+\frac{1}{T_i s}\right) = 4\left(1+\frac{1}{2s}\right)$$

(a) $u(t) = \mathcal{L}^{-1}\left(\frac{4}{s} + \frac{4}{2}\frac{1}{s^2}\right) = 4\cdot 1(t) + 2t$

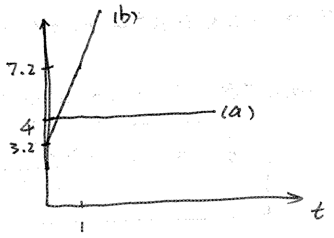
(b) $u(t) = \mathcal{L}^{-1}\left(\frac{4}{s^2} + 2\frac{1}{s^3}\right) = 4t + t^2$



$$\frac{U(s)}{E(s)} = 4(1+0.8s)$$

(a) $u(t) = \mathcal{L}^{-1}\left[4\frac{1}{s} + \underset{3.2}{\cancel{0.8}}\right] = 4\,1(t) + \overset{3.2}{\cancel{0.8}}\,\delta(t)$

(b) $u(t) = \mathcal{L}^{-1}\left[\frac{4}{s^2} + \frac{\overset{3.2}{\cancel{0.8}}}{s}\right] = 4t + \overset{3.2}{\cancel{0.8}}\cdot 1(t)$



3.9 $y''' + 3y'' + 2y' = u$

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad y = [1 \quad 0 \quad 0] \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix} + [0]\, u$$

3.11 $A = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad C = [1 \quad 2] \quad D = 0$

$$G(s) = C(sI - A)^{-1}B + D = C \frac{1}{\det(sI-A)} [\operatorname{adj}(sI-A)]\, B$$

$$= [1 \quad 2] \frac{1}{(s+5)(s+1) + 3} \begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$= \frac{1}{s^2+6s+8} [1 \quad 2] \begin{bmatrix} 2s-3 \\ 5s+31 \end{bmatrix} = \boxed{\frac{12s + 59}{s^2 + 6s + 8}}$$

using matlab

```
[num, den] = ss2tf (A, B, C, D) ;
```

3.16 assuming small θ, $T = J\ddot{\theta}$

$\Rightarrow J = ml^2 \qquad \sin\theta = \theta$

$$T = -2k\theta a^2 - mgl\sin\theta = J\ddot{\theta}$$

$$ml^2\ddot{\theta} = -2ka^2\theta - mgl\theta$$

$$\boxed{\ddot{\theta} = -\frac{2ka^2}{ml^2}\theta - \frac{g}{l}\theta}$$

3.18

Differential Equation of motion for the system

$$m_1\ddot{x}_1 = -k_1x_1 - b_1\dot{x}_1 - k_3(x_1 - x_2) + u$$
$$m_2\ddot{x}_2 = -k_2x_2 - b_2\dot{x}_2 - k_3(x_2 - x_1)$$

Hence

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + k_1x_1 + k_3x_1 = k_3x_2 + u$$
$$m_2\ddot{x}_2 + b_2\dot{x}_2 + k_2x_2 + k_3x_2 = k_3x_1$$

perform Laplace transform

$$(m_1s^2 + b_1s + k_1 + k_3)\,X_1(s) = K_3X_2(s) + U(s)$$
$$(m_2s^2 + b_2s + k_2 + k_3)\,X_2(s) = K_3X_1(s)$$

$$(m_1s^2 + b_1s + k_1 + k_3)X_1(s) = \frac{k_3\,k_3\,X_1(s)}{m_2s^2 + b_2s + k_2 + k_3} + U(s)$$

$$\Rightarrow \boxed{\frac{X_1(s)}{U(s)} = \frac{m_2s^2 + b_2s + k_2 + k_3}{(m_1s^2 + b_1s + k_1 + k_3)(m_2s^2 + b_2s + k_2 + k_3) - k_3^2}}$$

since $X_2(s) = \dfrac{k_3\,X_1(s)}{m_2s^2 + b_2s + k_2 + k_3}$

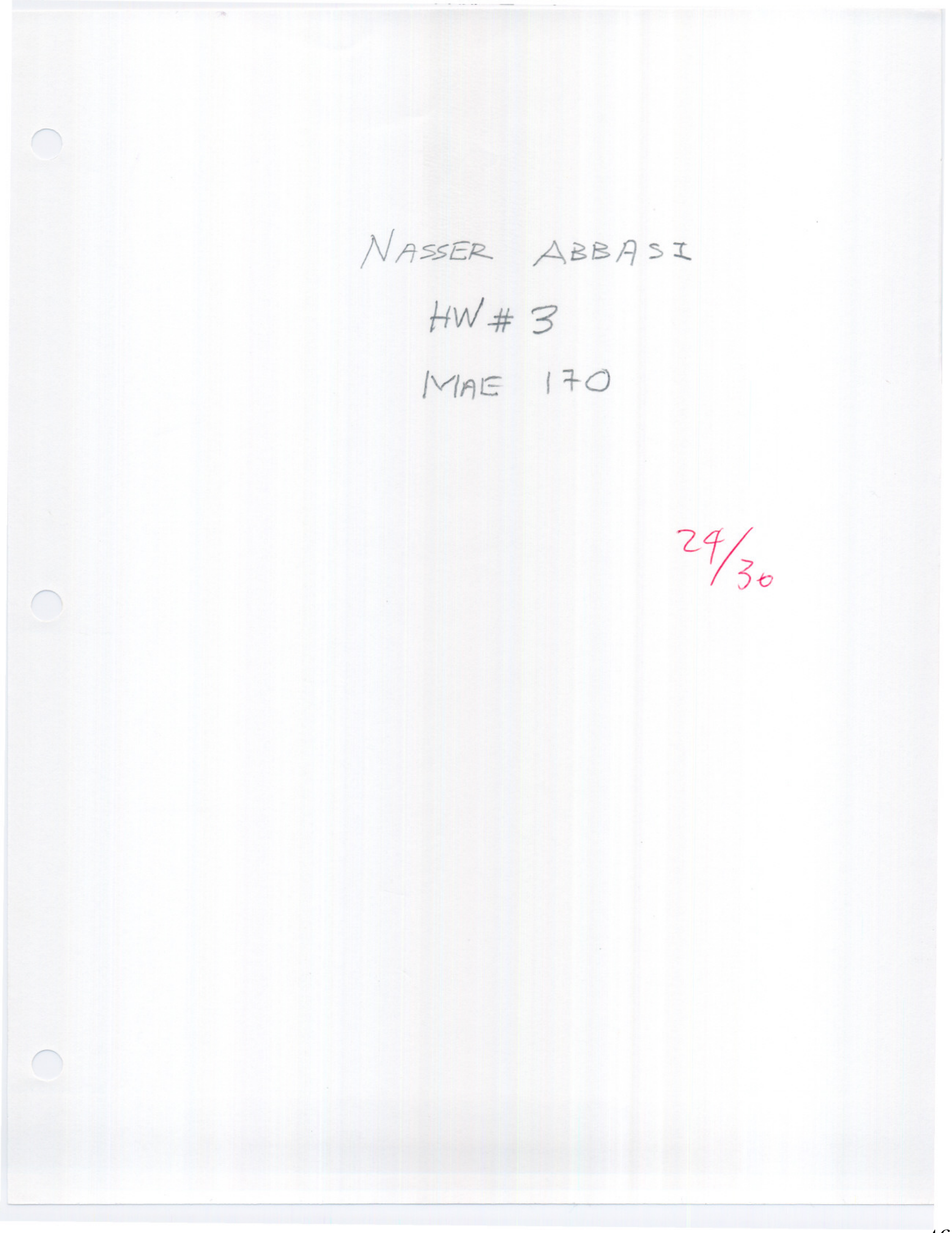
$$\Rightarrow \boxed{\frac{X_2(s)}{U(s)} = \frac{X_2(s)}{X_1(s)}\cdot\frac{X_1(s)}{U(s)} = \frac{k_3}{(m_1s^2 + b_1s + k_1 + k_3)(m_2s^2 + b_2s + k_2 + k_3) - k_3^2}}$$

3.1 (1 pt)
3.4 (4 pts)
3.9 (2 pts)
3.11 (2 pts)
3.16 (2 pts)
3.18 (2 pts)

Total $\underline{\underline{13}}$ pts

3.3 HW 3

3.3.1 my solution



NASSER ABBASI

HW # 3

MAE 170

24/30

HW 3, MAE 170.
Computer Problem **1**, cruise Control with no engine delay
by Nasser Abbasi
UCI, Winter 2005.

10

Solution

The first disturbance.

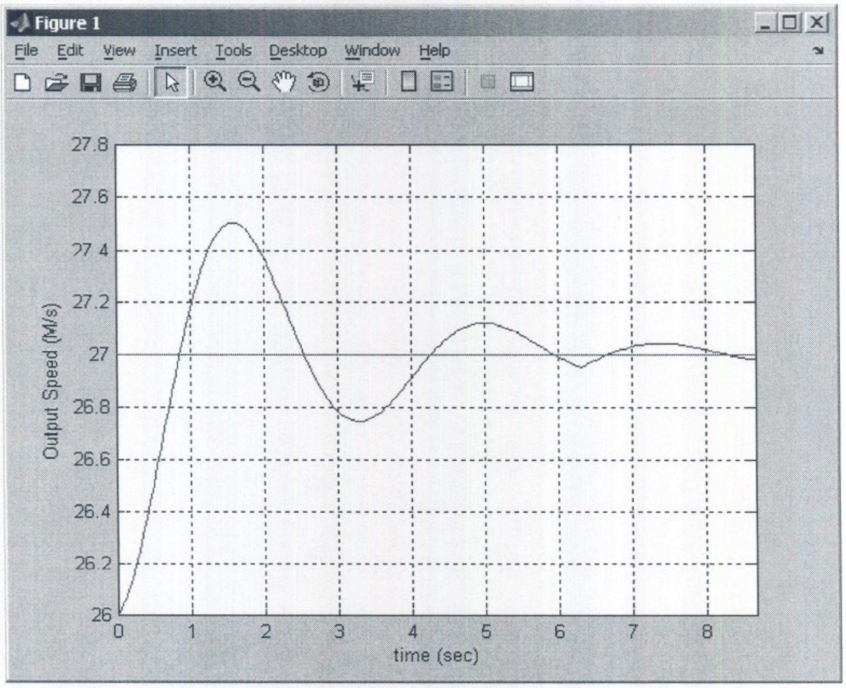
10

Using the input specified by $u = [27 \qquad fd + 200\ sign(\sin(0.5t))]$

By trial and error, I changed k_p and k_i to find a velocity output that remained as close as possible to the reference velocity without exceeding the maximum engine output of 3381 N. I addition, I was looking for the least amount of overshoot over the reference velocity. I was not able to eliminate completely the overshoot, but had to settle with an initial of only 0.5 m/s overshoot. After trying a number of different combinations, I found the following to give the best result.

$$\begin{aligned} k_p &= 1500 \\ k_i &= 6500 \end{aligned}$$

The max engine output was 3347 N
The transient values, read from this plot, are



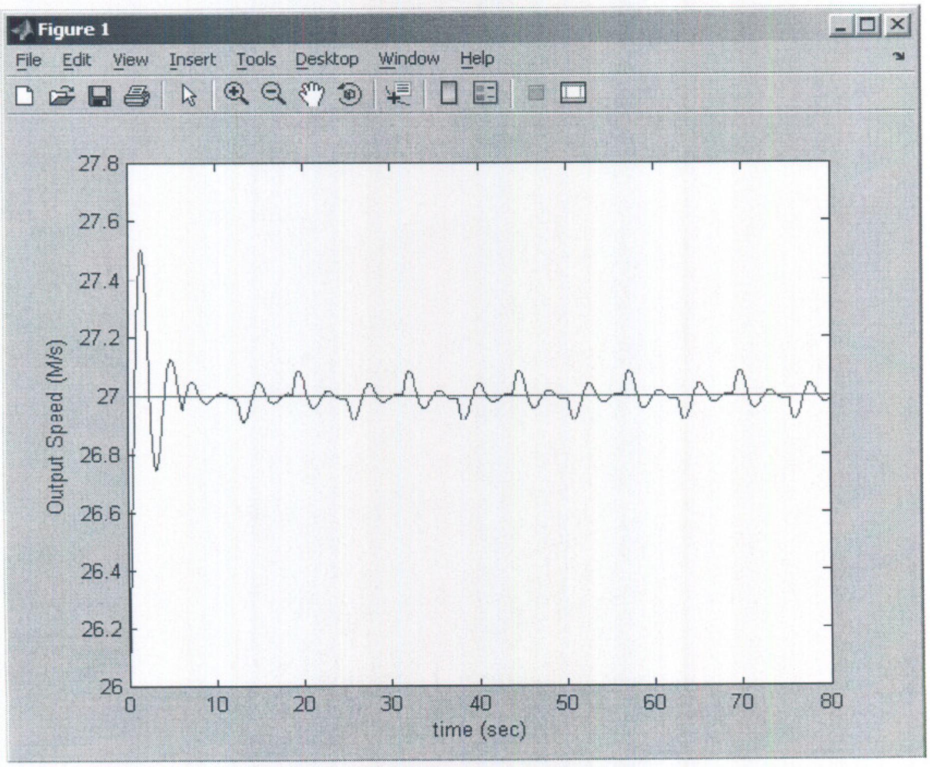
delay time=0.4 sec
rise time=0.8 sec
peak time=1.5 sec
settling time=6.2 sec
Max overshoot=0.5 m/s
(reference on page 230 of text for more explanation of these values).

The state space matrices are

$A = \begin{bmatrix} \frac{-k_p}{m} & \frac{k_i}{m} \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{k_p}{m} & \frac{-1}{m} \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ -k_p & k_i \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ k_p & 0 \end{bmatrix}$

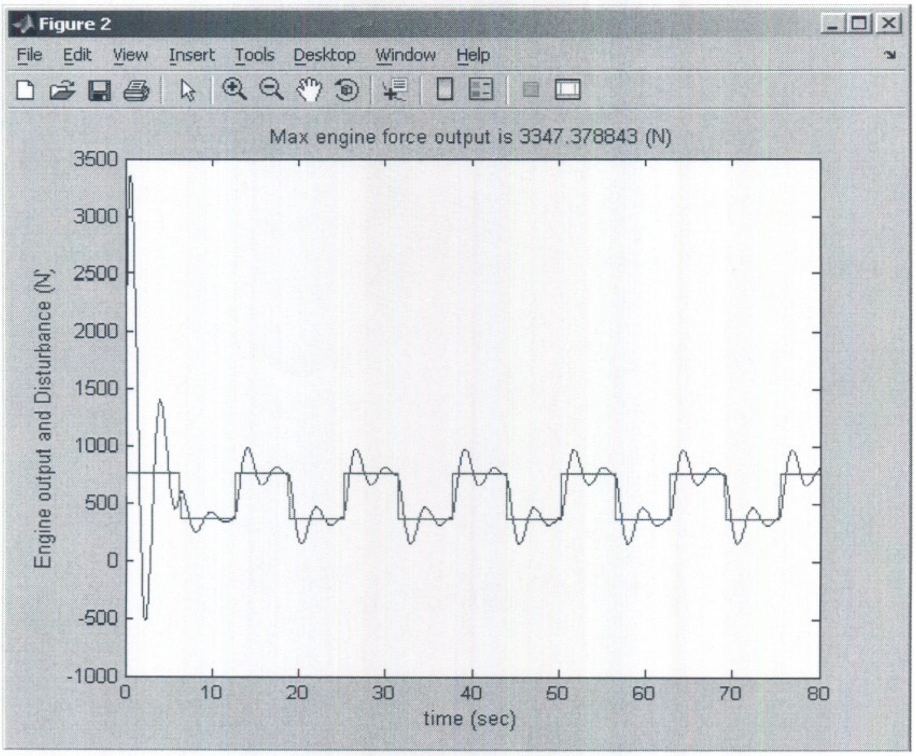
The output matrix is $\begin{bmatrix} V \\ Z \end{bmatrix}$

The output using the above values is shown in these plots



Transfer Function for 1^{st} disturbance from input velocity to output velocity is

$$\boxed{\frac{K_i + s K_p}{m s^2 + s K_p + K_i}}$$

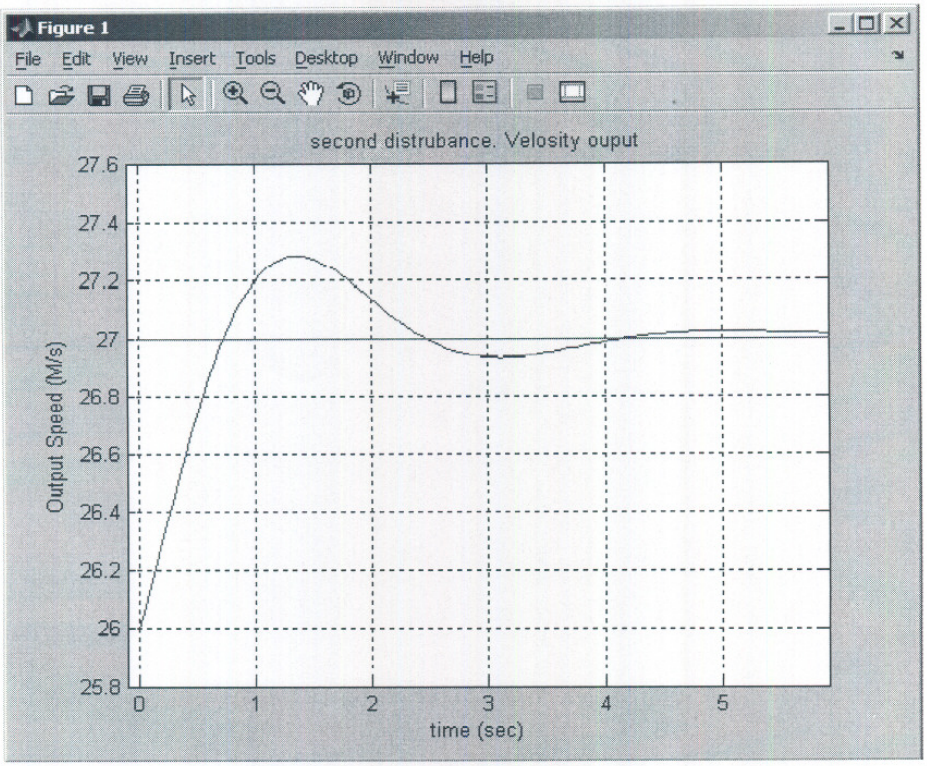


The second disturbance.

Using the input specified by $u = [27 \qquad fd + 200 \quad \sin(0.5t)]$
By trial and error, I changed k_p and k_i to find a velocity output that remained as close as possible to the reference velocity without exceeding the maximum engine output of 3381 N. I addition, I was looking for the least amount of overshoot over the reference velocity. I was not able to eliminate completely the overshoot. After trying a number of different I found the following to give the best result.

$$\begin{aligned} k_p &= 3100 \\ k_i &= 7000 \end{aligned}$$

The max engine output was 3371 N



delay time=0.3625 sec
rise time=0.75 sec
peak time=1.25 sec
settling time=3.8 sec
Max overshoot=0.25 m/s

Notice that the transient response for the second disturbance is better than the first. It has a faster rise time and settling time and less max overshoot. This is due to the fact that the second disturbance is smoother than the first. The first was a square wave, and the second is a sin wave.

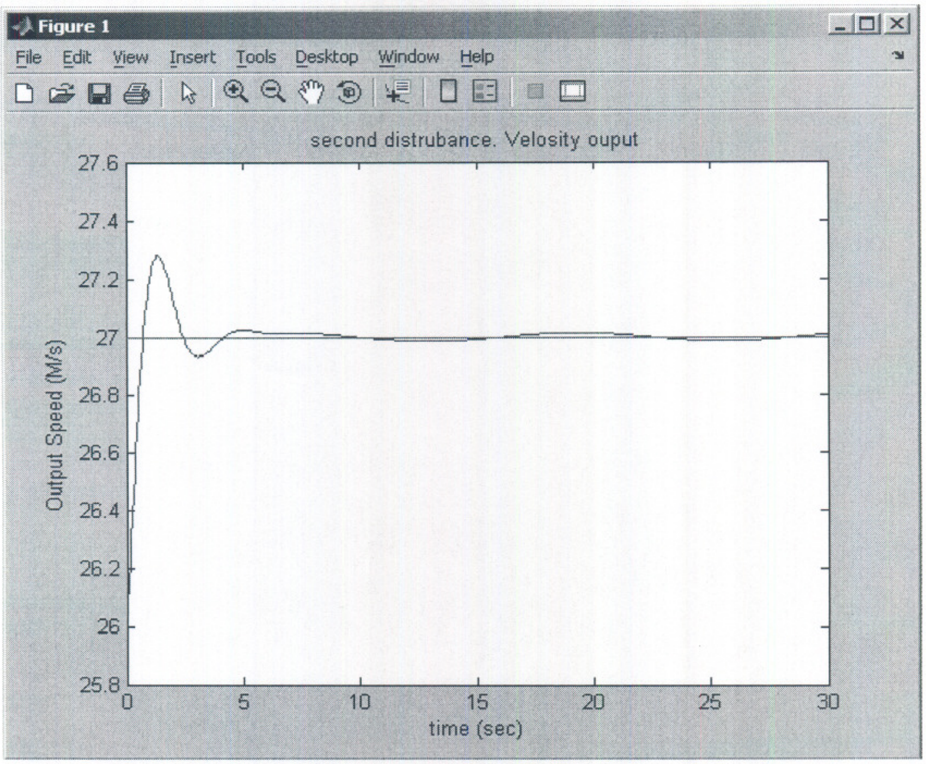


Figure 1
File
Edit
View
Insert
Tools
Desktop
Window
Help
second distrubance. Velosity ouput
Output Speed (M/s)
27.6
27.4
27.2
27
26.8
26.6
26.4
26.2
26
25.8
0
5
10
15
20
25
30
time (sec)

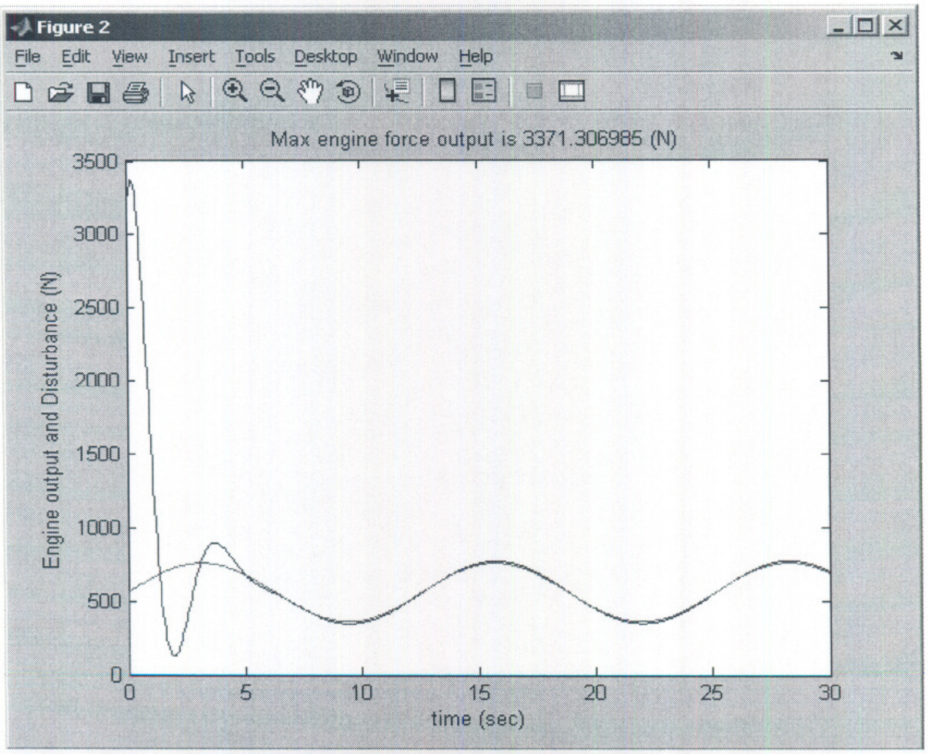


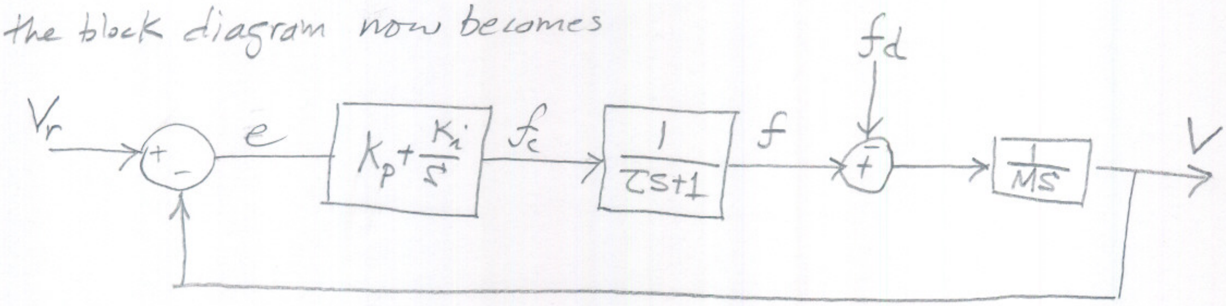
Figure 2
File
Edit
View
Insert
Tools
Desktop
Window
Help
Max engine force output is 3371.306985 (N)
Engine output and Disturbance (N)
3500
3000
2500
2000
1500
1000
500
0
0
5
10
15
20
25
30
time (sec)

MAE 170
HW#3
second computer problem
Nasser Abbasi

9

an ensine, 1^{st} order, time constan τ, gain is 1 is $\boxed{\frac{1}{\tau s+1}}$

so the block diagram now becomes



so now $f_c = K_p(V_r-V)+K_i\int(V_r-V)$ —— ①

$$F(s) = F_c(s)\frac{1}{\tau s+1}$$

so $\boxed{\tau\dot{f}+f = f_c}$ —— ②

from ①, ② $\Rightarrow$ $\tau\dot{f}+f = K_p(V_r-V)+K_i\int V_r-V$

so $\boxed{\dot{f} = \frac{K_p}{\tau}(V_r-V)+\frac{K_i}{\tau}\underbrace{\int V_r-V}_{z}-\frac{f}{\tau}}$ —— ③

so Force into plant is $f-f_d$.

so from F=ma, we set

$\boxed{f-f_d = m\dot{V}}$ —— ④

so $\boxed{\dot{V} = \frac{f}{M}-\frac{f_d}{M}}$ —— ⑤

Let $z = \int V_r-V$ $\longrightarrow$

now I need to set up the 3 DE with $\dot{V}, \dot{Z}, \dot{f}$ in LHS.

$$\boxed{\begin{aligned} \dot{V} &= \frac{f}{M} - \frac{f_d}{M} \\ \dot{Z} &= V_r - V \\ \dot{f} &= \frac{k_p}{\tau} V_r - \frac{k_p}{\tau} V + \frac{k_i}{\tau} Z - \frac{f}{\tau} \end{aligned}}$$

so

$$\begin{pmatrix} \dot{V} \\ \dot{Z} \\ \dot{f} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{M} \\ -1 & 0 & 0 \\ -\frac{k_p}{\tau} & \frac{k_i}{\tau} & -\frac{1}{\tau} \end{pmatrix} \begin{pmatrix} V \\ Z \\ f \end{pmatrix} + \underset{3\times 2}{\begin{pmatrix} 0 & -\frac{1}{M} \\ 1 & 0 \\ \frac{k_p}{\tau} & 0 \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} V_r \\ f_d \end{pmatrix}}$$

$$\begin{pmatrix} V \\ f_c \end{pmatrix} = \overset{2\times 3}{\begin{pmatrix} 1 & 0 & 0 \\ -k_p & k_i & 0 \end{pmatrix}} \overset{3\times 1}{\begin{pmatrix} V \\ Z \\ f \end{pmatrix}} + \overset{2\times 2}{\begin{pmatrix} 0 & 0 \\ k_p & 0 \end{pmatrix}} \overset{2\times 1}{\begin{pmatrix} V_r \\ f_d \end{pmatrix}}$$

→ you don't want to know fc, you want to know **f** -1

So now I can use the above A, B, C, D matrices to solve this. please see next.

Transfer Function from $V_{\text{reference}}$ to V_{out} is

$$\longleftarrow \boxed{\frac{k_i + s k_p}{k_i + s k_p + m s^2 (1 + s\tau)}}$$

HW 3, MAE 170.
Computer Problem **2**, cruise Control with no engine delay
by Nasser Abbasi
UCI, Winter 2005.

Solution

The state space matrices are

$$A=\begin{bmatrix}0&0&\frac{1}{m}\\-1&0&0\\\frac{-k_p}{\tau}&\frac{k_i}{\tau}&\frac{-1}{\tau}\end{bmatrix},\quad B=\begin{bmatrix}0&\frac{-1}{m}\\1&0\\\frac{k_p}{\tau}&0\end{bmatrix},\quad C=\begin{bmatrix}1&0&0\\-k_p&k_i&0\end{bmatrix},D=\begin{bmatrix}0&0\\k_p&0\end{bmatrix}$$

The output matrix is $\begin{bmatrix}V\\ \\f_c\end{bmatrix}$ state matrix $X=\begin{bmatrix}V\\Z\\f\end{bmatrix}$

The first disturbance.

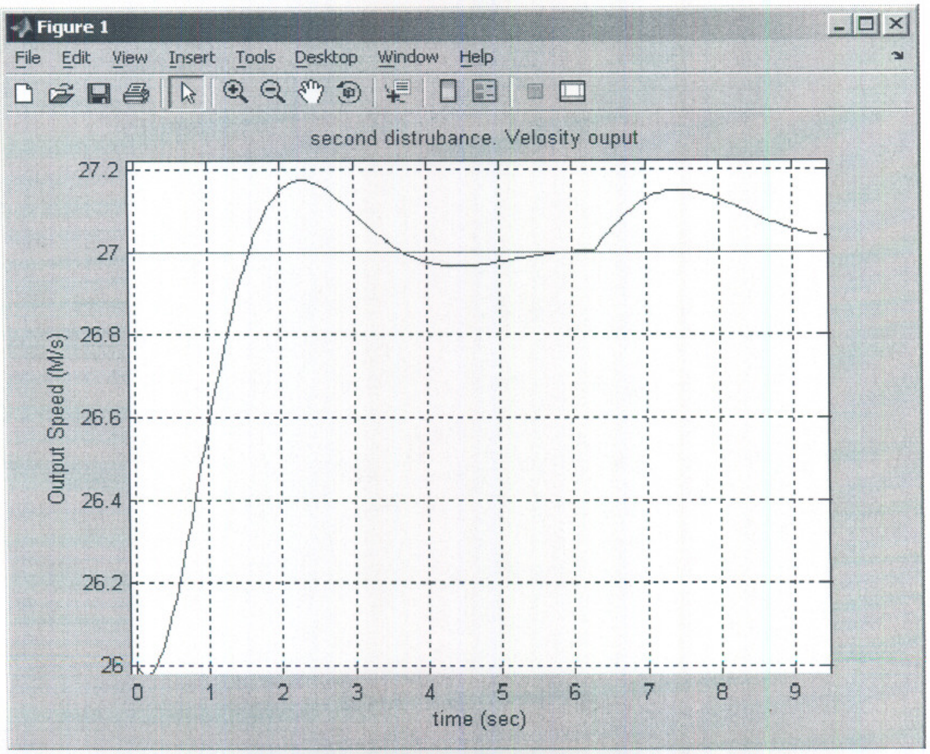
Using the input specified by $u=[27 \qquad fd+200\ sign(\sin(0.5t))]$

By trial and error, I changed k_p and k_i to find a velocity output that remained as close as possible to the reference velocity without exceeding the maximum engine output of 3381 N. I addition, I was looking for the least amount of overshoot over the reference velocity. I was not able to eliminate completely the overshoot, but had to settle with an initial of only 0.5 m/s overshoot. After trying a number of different combinations, I found the following to give the best result.

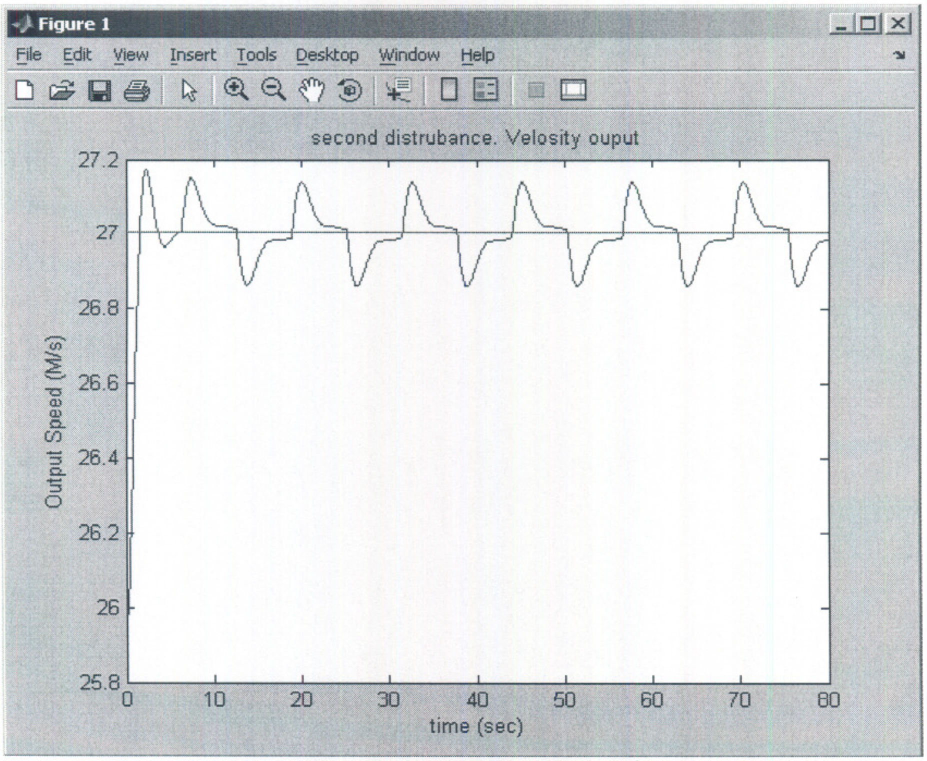
$$\begin{aligned}k_p&=3100\\k_i&=1000\end{aligned}$$

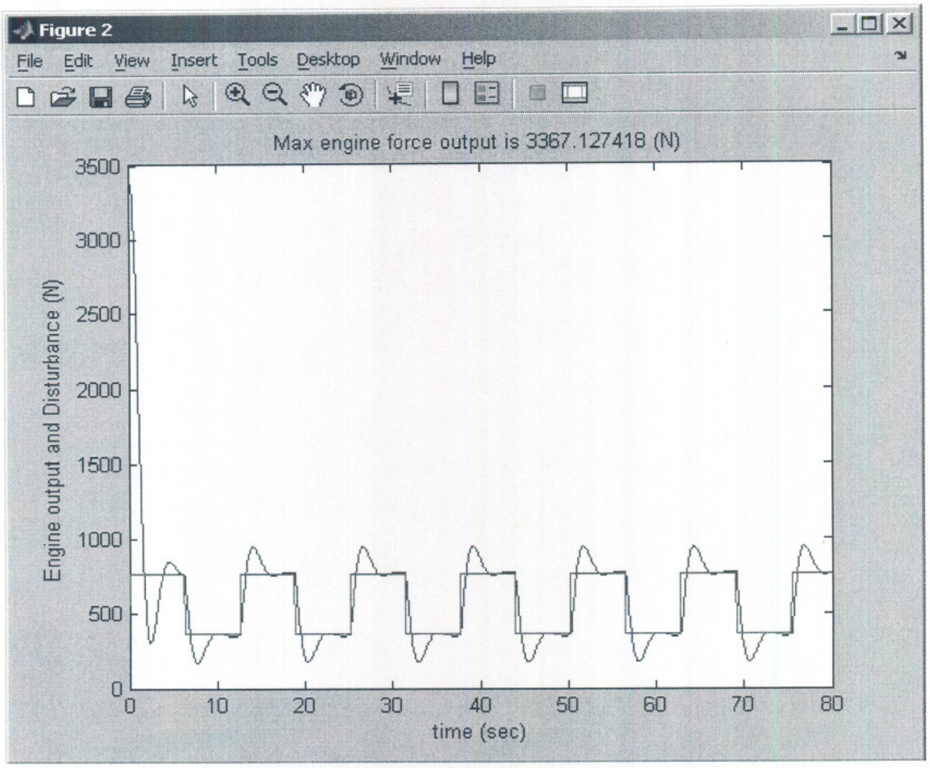
The max engine output was 3367 N

The transient values, read from this plot, are



delay time=0.8 sec
rise time=1.6 sec
peak time=2.2 sec
settling time=5 sec
Max overshoot=0.19 m/s
(reference on page 230 of text for more explanation of these values).
The output using the above values is shown in these plots



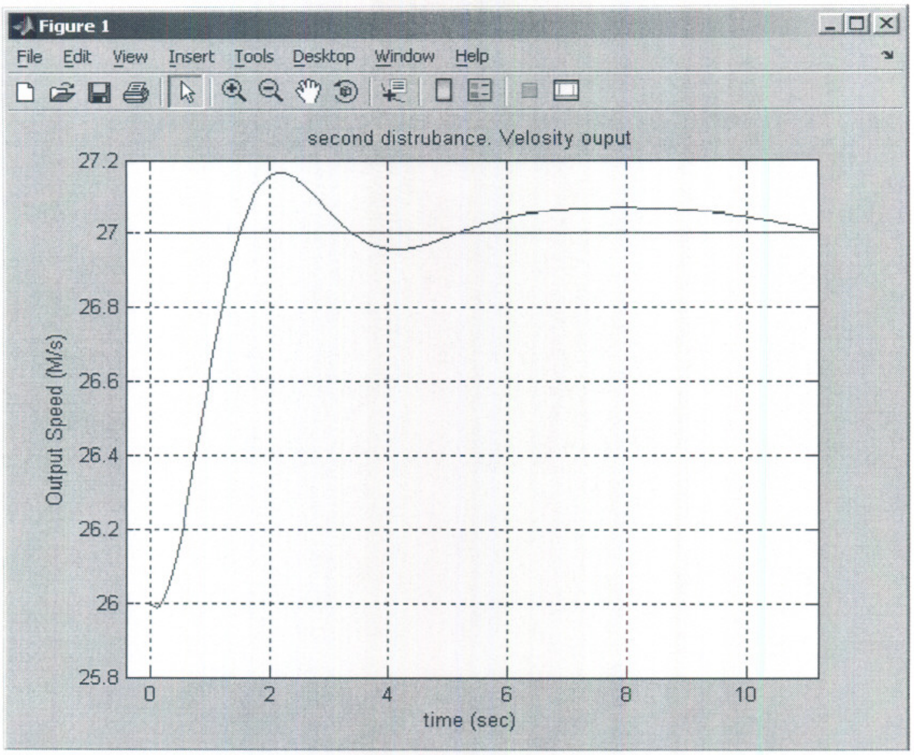


The second disturbance.

Using the input specified by $u = [27 \qquad fd + 200 \quad \sin(0.5t)]$
By trial and error, I changed k_p and k_i to find a velocity output that remained as close as possible to the reference velocity without exceeding the maximum engine output of 3381 N. I addition, I was looking for the least amount of overshoot over the reference velocity. After trying a number of different I found the following to give the best result.

$$\begin{aligned} k_p &= 3150 \\ k_i &= 1000 \end{aligned}$$

The max engine output was 3351 N



delay time=0.75 sec
rise time=1.5 sec
peak time=2.2 sec
settling time=5 sec
Max overshoot=0.18 m/s

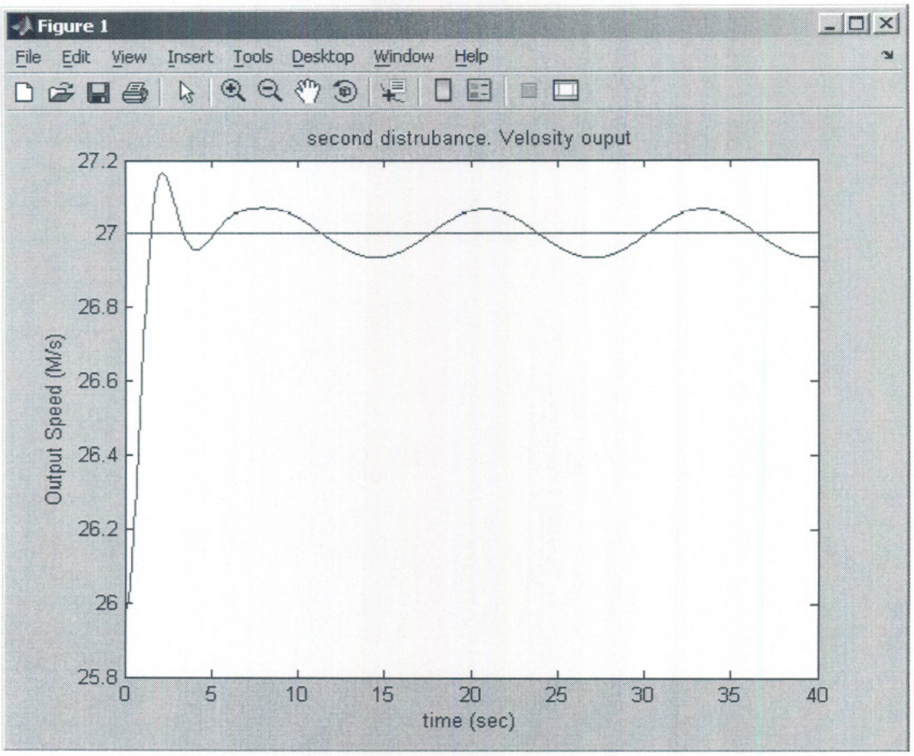


Figure 1
File Edit View Insert Tools Desktop Window Help
second distrubance. Velosity ouput
27.2
27
26.8
26.6
26.4
26.2
26
25.8
Output Speed (M/s)
0
5
10
15
20
25
30
35
40
time (sec)

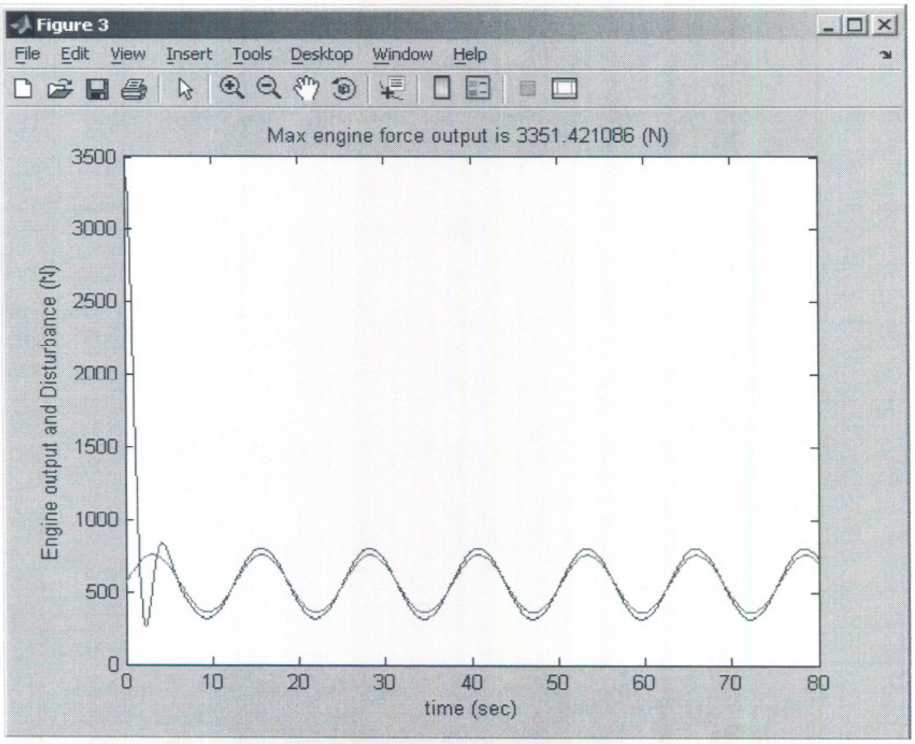


Figure 3
File Edit View Insert Tools Desktop Window Help
Max engine force output is 3351.421086 (N)
3500
3000
2500
2000
1500
1000
500
0
Engine output and Disturbance (N)
0
10
20
30
40
50
60
70
80
time (sec)

HW#3
problem 3
Nasser Abbasi

S

why start here? You should have used the given diff eqns.

−5

The Closed Loop System is $X_r \rightarrow \boxed{\frac{ka}{ms^3+cs^2+bs+ka}} \rightarrow X$

use $a=1$, $b=350$, $m=1$, $c=20$.

Find gain K that gives fast step response.

states: position, velocity, force.

$$\frac{X}{X_r} = \frac{ka}{ms^3+cs^2+bs+ka} \Rightarrow \boxed{m\dddot{x}+c\ddot{x}+b\dot{x}+kax = kax_r}$$

divide by m

$$\dddot{x} + \frac{c}{m}\ddot{x} + \frac{b}{m}\dot{x} + \frac{ka}{m}x = \frac{ka}{m}x_r$$

$\boxed{F=m\ddot{x}}$

$\boxed{V=\dot{x}}$

tion ← let $x_1 = x$

$V \leftarrow x_2 = \dot{x} = \dot{x}_1$

$F \leftarrow x_3 = m\ddot{x} = m x_2 \Rightarrow \ddot{x} = \frac{x_3}{m}$

?

Then the DE can be written as

$$\frac{\dot{x}_3}{m} + \frac{c}{m}\frac{x_3}{m} + \frac{b}{m}x_2 + \frac{ka}{m}x_1 = \frac{ka}{m}x_r$$

i.e

$$\frac{\dot{x}_3}{m} = \frac{ka}{m}x_r - \frac{c}{m}\frac{x_3}{m} - \frac{b}{m}x_2 - \frac{ka}{m}x_1$$

or $\boxed{\dot{F} = kax_r - \frac{c}{m}F - bV - kax}$ ← I replaced $\dot{x}_3$ by $\dot{F}$, x_3 by F, x_2 by V and x_1 by x →

$\boxed{\dot{x} = V}$

$\boxed{\dot{V} = \frac{F}{m}}$

hence.

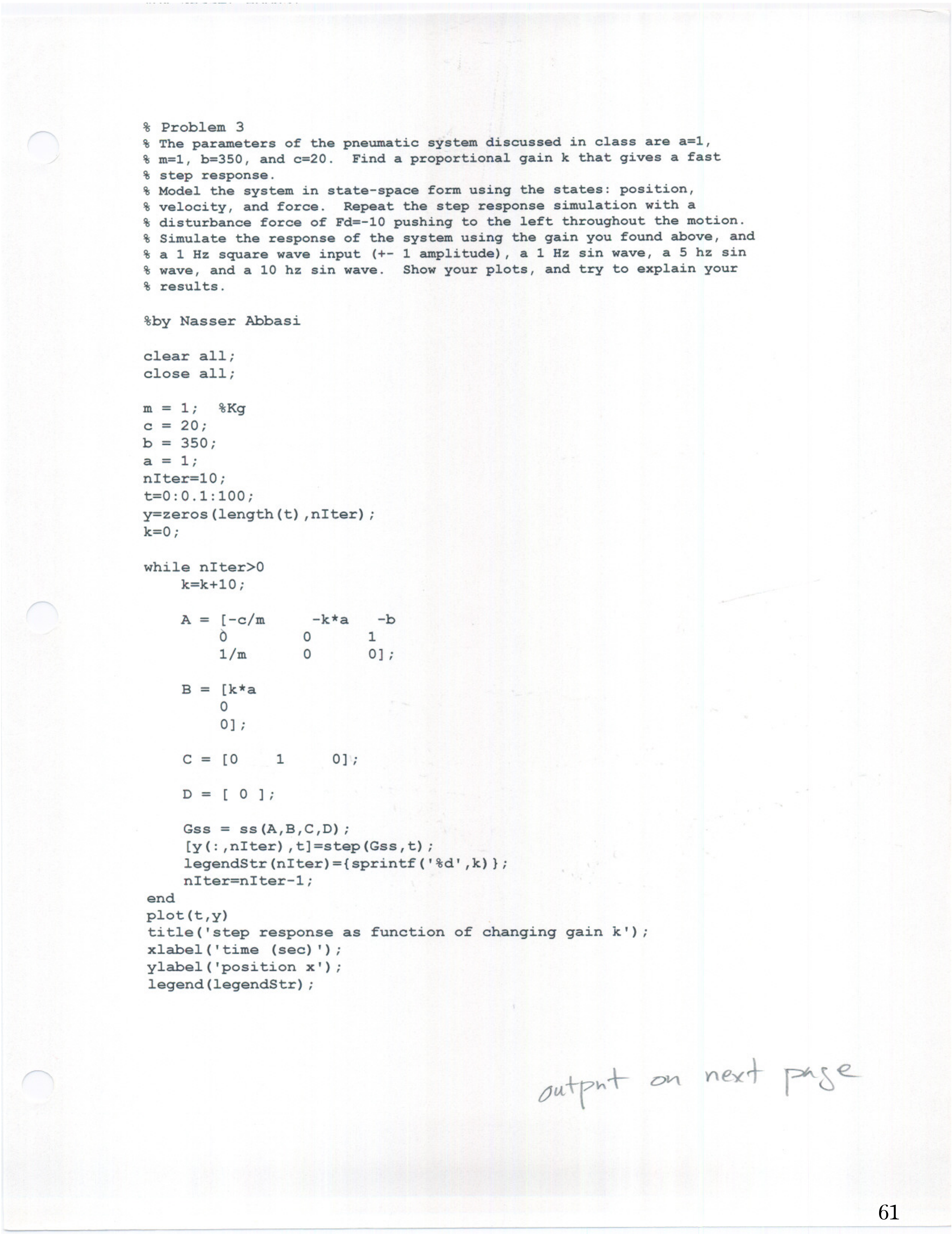
$$\begin{bmatrix}\dot{F}\\ \dot{x}\\ \dot{V}\end{bmatrix} = \overbrace{\begin{bmatrix}-\frac{c}{m} & -Ka & -b\\ 0 & 0 & 1\\ \frac{1}{m} & 0 & 0\end{bmatrix}}^{A}\begin{bmatrix}F\\ x\\ V\end{bmatrix} + \overbrace{\begin{bmatrix}Ka\\ 0\\ 0\end{bmatrix}}^{B}\begin{bmatrix}x_r\end{bmatrix}$$

For output, use X

so
$$\begin{bmatrix}X\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 & 0\end{bmatrix}}_{\substack{1\times 3\\ C}}\underbrace{\begin{bmatrix}F\\ X\\ V\end{bmatrix}}_{3\times 1} + \underbrace{\begin{bmatrix}\text{null}\end{bmatrix}}_{D}\begin{bmatrix}x_r\end{bmatrix}$$

The following is $\overset{\text{computed}}{\text{result}}$ to find best value of K to give fast step response →

?



```
% Problem 3
% The parameters of the pneumatic system discussed in class are a=1,
% m=1, b=350, and c=20.  Find a proportional gain k that gives a fast
% step response.
% Model the system in state-space form using the states: position,
% velocity, and force.  Repeat the step response simulation with a
% disturbance force of Fd=-10 pushing to the left throughout the motion.
% Simulate the response of the system using the gain you found above, and
% a 1 Hz square wave input (+- 1 amplitude), a 1 Hz sin wave, a 5 hz sin
% wave, and a 10 hz sin wave.  Show your plots, and try to explain your
% results.

%by Nasser Abbasi

clear all;
close all;

m = 1;  %Kg
c = 20;
b = 350;
a = 1;
nIter=10;
t=0:0.1:100;
y=zeros(length(t),nIter);
k=0;

while nIter>0
    k=k+10;

    A = [-c/m       -k*a    -b
         0          0       1
         1/m        0       0];

    B = [k*a
         0
         0];

    C = [0     1      0];

    D = [ 0 ];

    Gss = ss(A,B,C,D);
    [y(:,nIter),t]=step(Gss,t);
    legendStr(nIter)={sprintf('%d',k)};
    nIter=nIter-1;
end
plot(t,y)
title('step response as function of changing gain k');
xlabel('time (sec)');
ylabel('position x');
legend(legendStr);
```

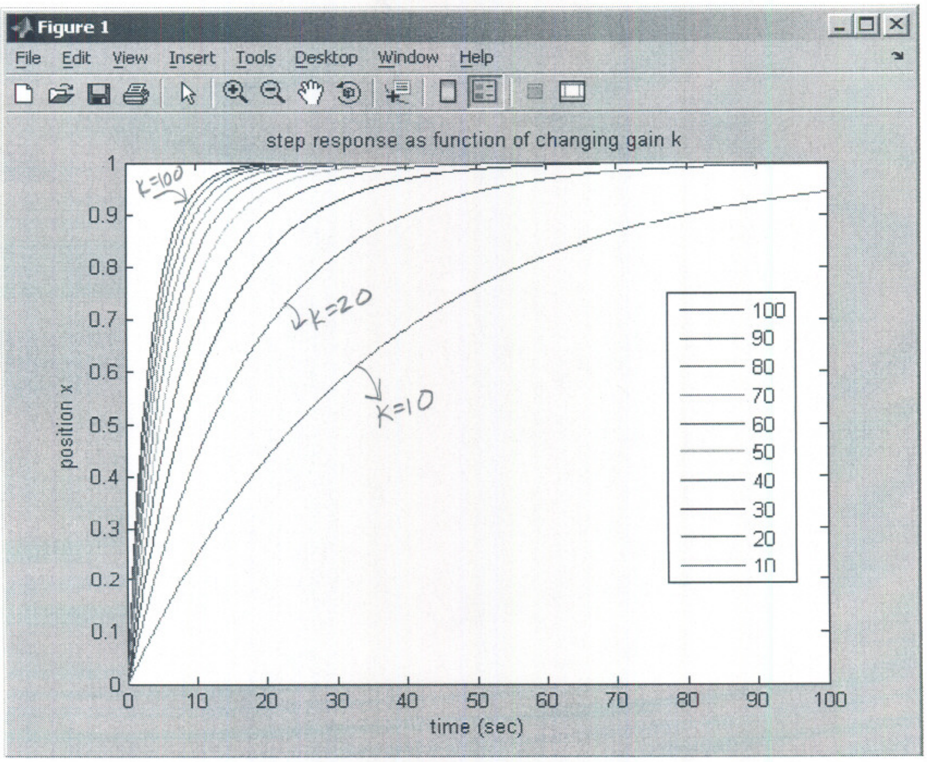
output on next page

HW 3, MAE 170.
Computer Problem **3**
by Nasser Abbasi
UCI, Winter 2005.

Solution

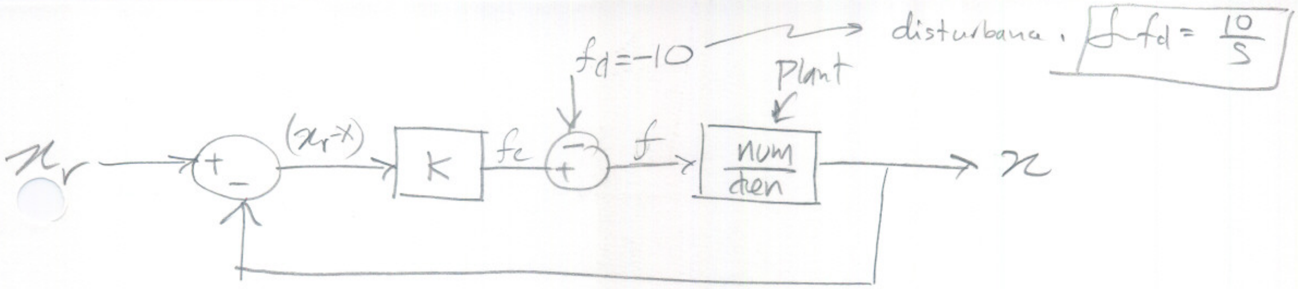
This is the result of running the step response as k is changed.

This shows that as k increases, the response is faster. at k=100 we have a rise time of about 20 seconds. at k=10 the rise time was more than 100 seconds.



(PS. Plot is in Color, but my printer is black/white)

next, I repeat with $f_d = -10$ →



so $G = \left[K - F_d\right]\frac{num}{den}$ where $\frac{num}{den} = \frac{a/m}{s^3 + \frac{c}{m}s^2 + \frac{bs}{m}}$

so closed Loop transfer function is

$$\frac{G}{1+G} = \frac{(K-f_d)\frac{a}{m}}{(K-f_d)\frac{a}{m} + s^3 + \frac{c}{m}s^2 + \frac{bs}{m}}$$

$$= \boxed{\frac{(K-F_d)\,a}{(K-F_d)a + ms^3 + cs^2 + bs}}$$

?

using $K=100$ from first part, and

$C=20$, $a=1$, $m=1$, $b=350$

now $$\frac{X}{X_r} = \frac{(K-F_d)a}{(K-F_d)a + ms^3 + cs^2 + bs}$$

ie $$(K-F_d)ax + \dddot{x}m + c\ddot{x} + b\dot{x} = (K-F_d)ax_r$$

$$Kax - aF_dx + \dddot{x}m + c\ddot{x} + b\dot{x} = Kax_r - F_d a x_r$$

$$\dddot{x}m + c\ddot{x} + b\dot{x} + Kax - aF_dx = Kax_r - F_d a x_r$$

$$\dddot{x} + \frac{c}{m}\ddot{x} + \frac{b}{m}\dot{x} + \frac{Kax}{m} - \frac{aF_d}{m}x = \frac{Ka}{m}x_r - \frac{F_d a}{m}x_r$$

let

position ← $x_1 = x$

Velocity ← $x_2 = \dot{x} = \dot{x}_1$

Force ← $x_3 = m\ddot{x} = m\dot{x}_2$

→

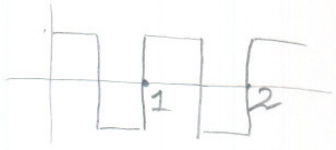
$$\frac{\dot{F}}{m} = -\frac{c}{m}\frac{F}{m} - \frac{b}{m}V - \frac{ka}{m}x + \frac{aF_d}{m}x + \frac{ka}{m}x_r - \frac{F_d a}{m}x_r$$

$$\boxed{\begin{aligned} \dot{F} &= -\frac{c}{m}F - bV - kax + aF_d x + kax_r - F_d a x_r \\ \dot{V} &= \frac{F}{m} \\ \dot{x} &= V \end{aligned}}$$

so
$$\begin{bmatrix} \dot{F} \\ \dot{X} \\ \dot{V} \end{bmatrix} = \underset{3\times3}{\begin{bmatrix} -\frac{c}{m} & -ka+aF_d & -b \\ 0 & 0 & 1 \\ \frac{1}{m} & 0 & 0 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} F \\ X \\ V \end{bmatrix}} + \underset{3\times1}{\begin{bmatrix} ka - F_d a \\ 0 \\ 0 \end{bmatrix}} \underset{1\times1}{\begin{bmatrix} x_r \end{bmatrix}}$$

$$\begin{bmatrix} X \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} F \\ X \\ V \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} x_r \end{bmatrix}$$

now apply 1 Hz square wave for X_r

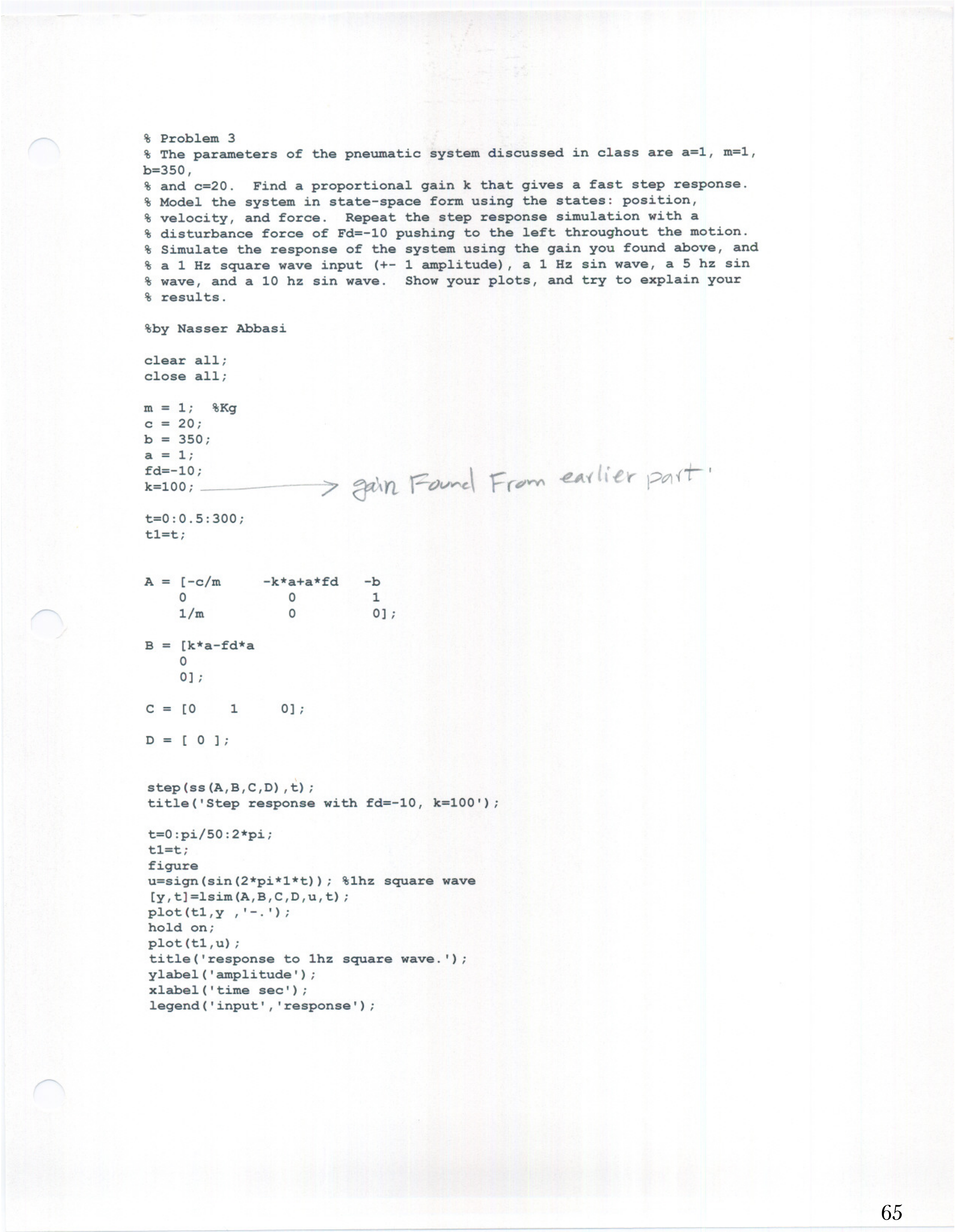


, Then 1 HZ sin wave for X_r

then 5 HZ sin wave.

then 10 HZ.

Computer solution ⟶



```
% Problem 3
% The parameters of the pneumatic system discussed in class are a=1, m=1,
b=350,
% and c=20.  Find a proportional gain k that gives a fast step response.
% Model the system in state-space form using the states: position,
% velocity, and force.  Repeat the step response simulation with a
% disturbance force of Fd=-10 pushing to the left throughout the motion.
% Simulate the response of the system using the gain you found above, and
% a 1 Hz square wave input (+- 1 amplitude), a 1 Hz sin wave, a 5 hz sin
% wave, and a 10 hz sin wave.  Show your plots, and try to explain your
% results.

%by Nasser Abbasi

clear all;
close all;

m = 1;  %Kg
c = 20;
b = 350;
a = 1;
fd=-10;
k=100;
```

→ gain Found From earlier part'

```
t=0:0.5:300;
t1=t;


A = [-c/m      -k*a+a*fd    -b
     0            0          1
     1/m          0          0];

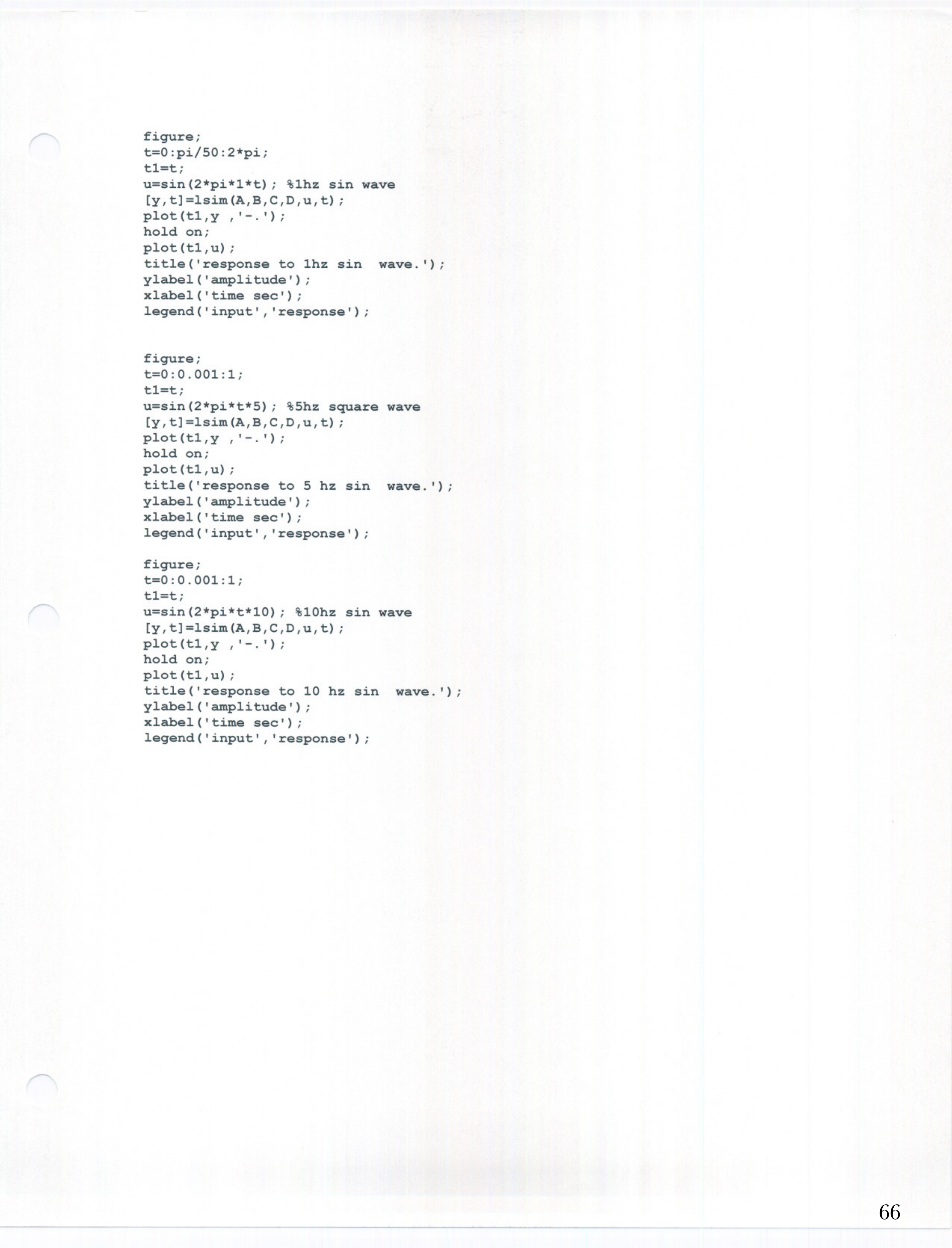
B = [k*a-fd*a
     0
     0];

C = [0    1     0];

D = [ 0 ];


step(ss(A,B,C,D),t);
title('Step response with fd=-10, k=100');

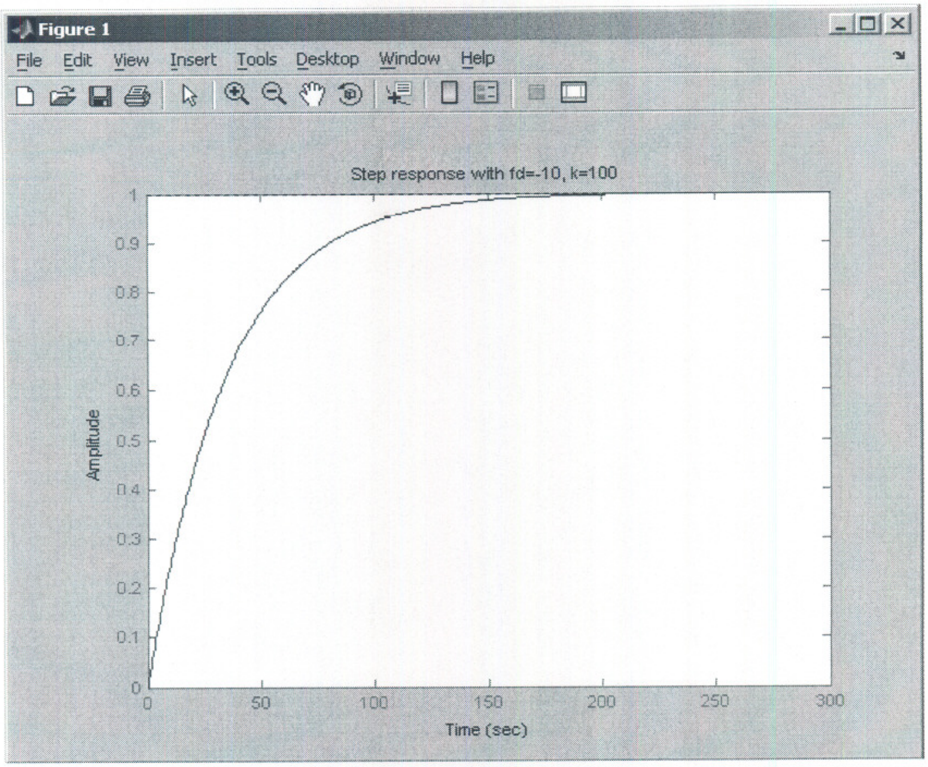
t=0:pi/50:2*pi;
t1=t;
figure
u=sign(sin(2*pi*1*t)); %1hz square wave
[y,t]=lsim(A,B,C,D,u,t);
plot(t1,y ,'-.');
hold on;
plot(t1,u);
title('response to 1hz square wave.');
ylabel('amplitude');
xlabel('time sec');
legend('input','response');
```



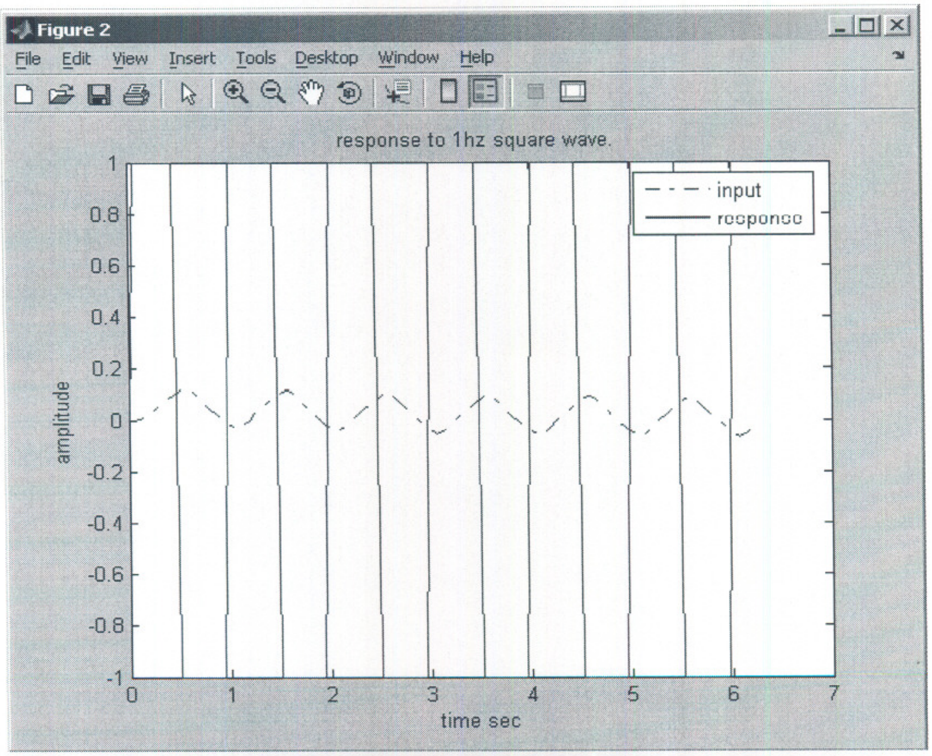
```
figure;
t=0:pi/50:2*pi;
t1=t;
u=sin(2*pi*1*t); %1hz sin wave
[y,t]=lsim(A,B,C,D,u,t);
plot(t1,y ,'-.');
hold on;
plot(t1,u);
title('response to 1hz sin  wave.');
ylabel('amplitude');
xlabel('time sec');
legend('input','response');

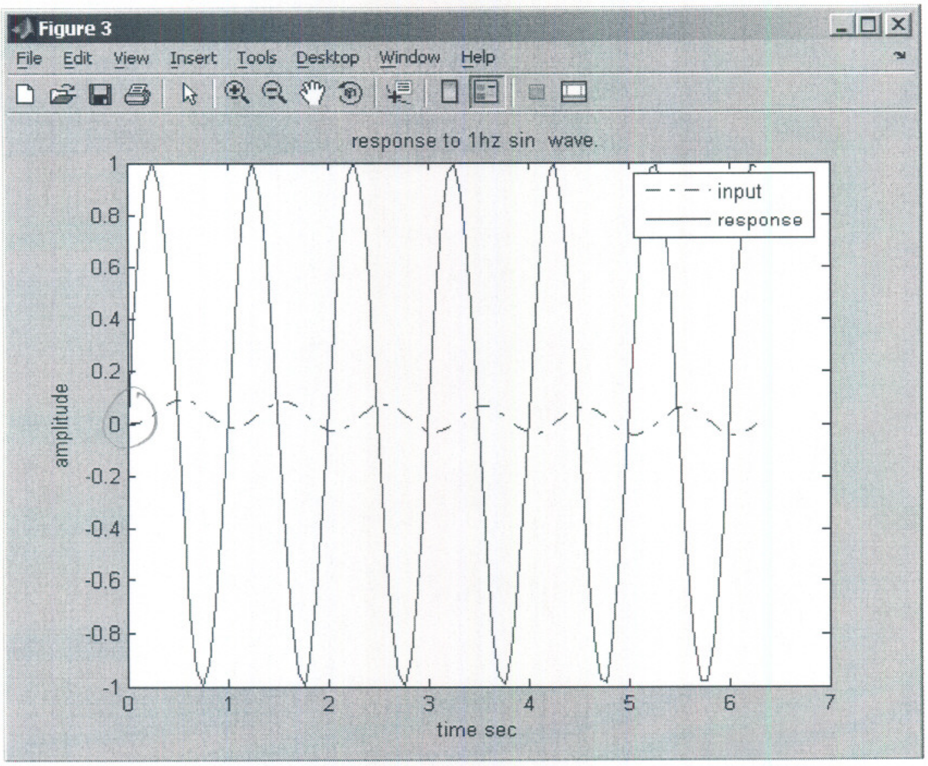

figure;
t=0:0.001:1;
t1=t;
u=sin(2*pi*t*5); %5hz square wave
[y,t]=lsim(A,B,C,D,u,t);
plot(t1,y ,'-.');
hold on;
plot(t1,u);
title('response to 5 hz sin  wave.');
ylabel('amplitude');
xlabel('time sec');
legend('input','response');

figure;
t=0:0.001:1;
t1=t;
u=sin(2*pi*t*10); %10hz sin wave
[y,t]=lsim(A,B,C,D,u,t);
plot(t1,y ,'-.');
hold on;
plot(t1,u);
title('response to 10 hz sin  wave.');
ylabel('amplitude');
xlabel('time sec');
legend('input','response');
```



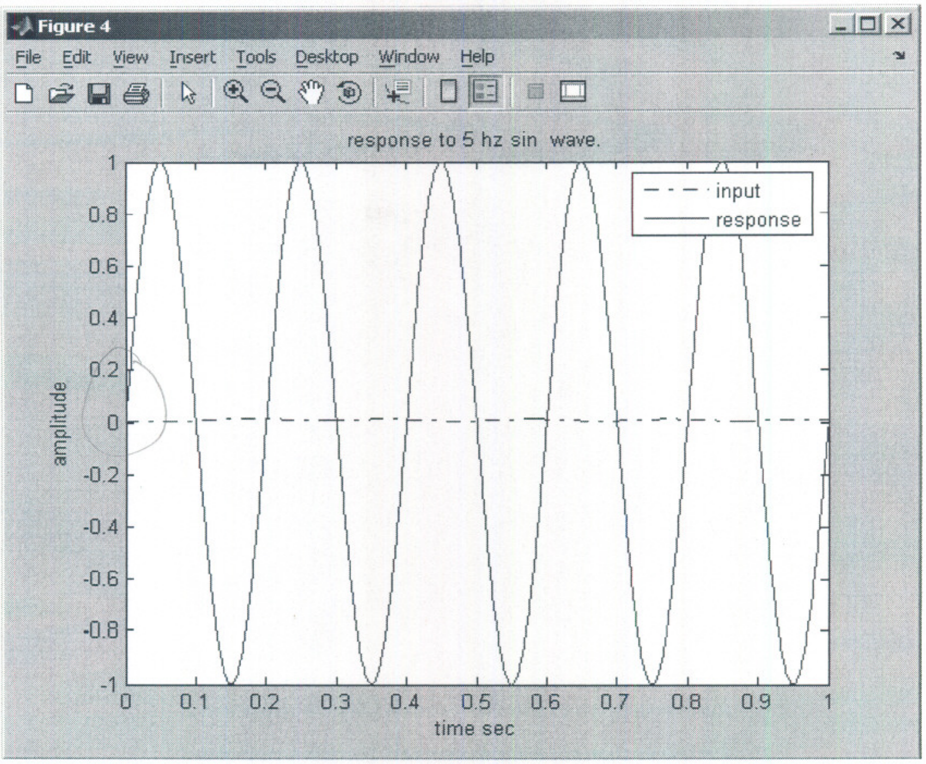
notice that now the rise time has increased when fd is present. This is expected.



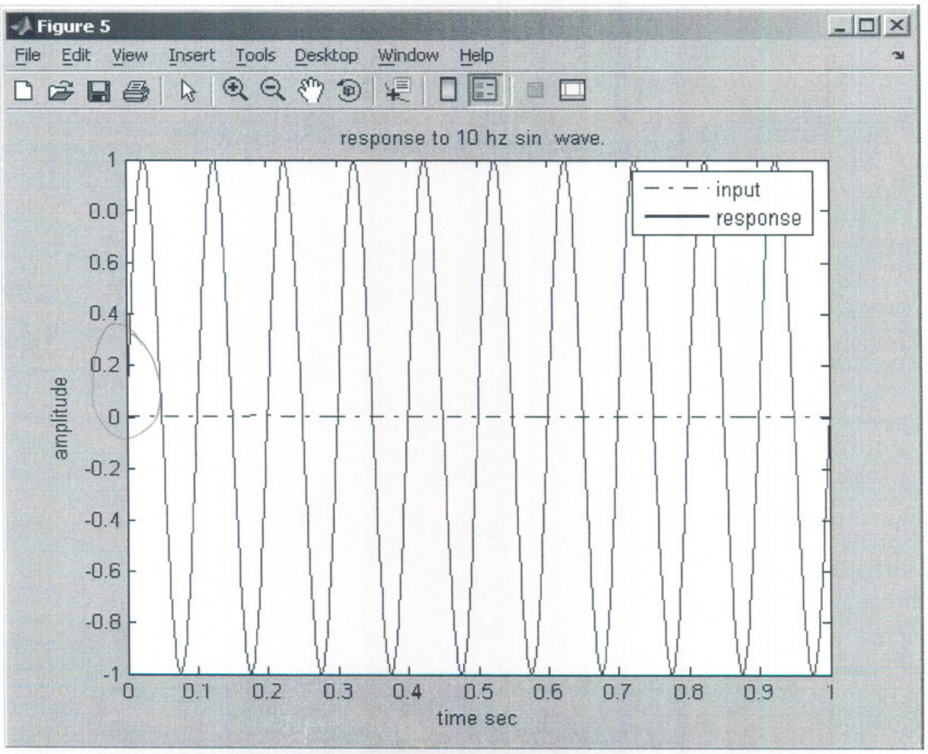


notice The response has the same frequency as the input, but different amplitude (scaling) and phase shift!

this is a property of linear systems.



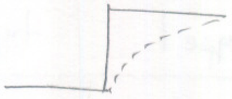
responce
same freq.
as input.
different
phase and
scaling.



MAE 170
Tuesday

①

$K_p : \lim_{s \to 0} GH$



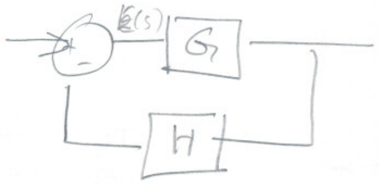
$e_{ss} = \frac{1}{1+K_p}$

$K_v = \lim_{s \to 0} sGH$

$e_{ss} = \frac{1}{K_v}$

$K_a = \lim_{s \to 0} s^2 GH$

$e_{ss} = \frac{1}{K_a}$



System type

$GH = \frac{N(s)}{D(s)}$ if no s in D(s) ⇒ type 0

$GH = \frac{N(s)}{s\,D(s)}$ Type 1

extra s

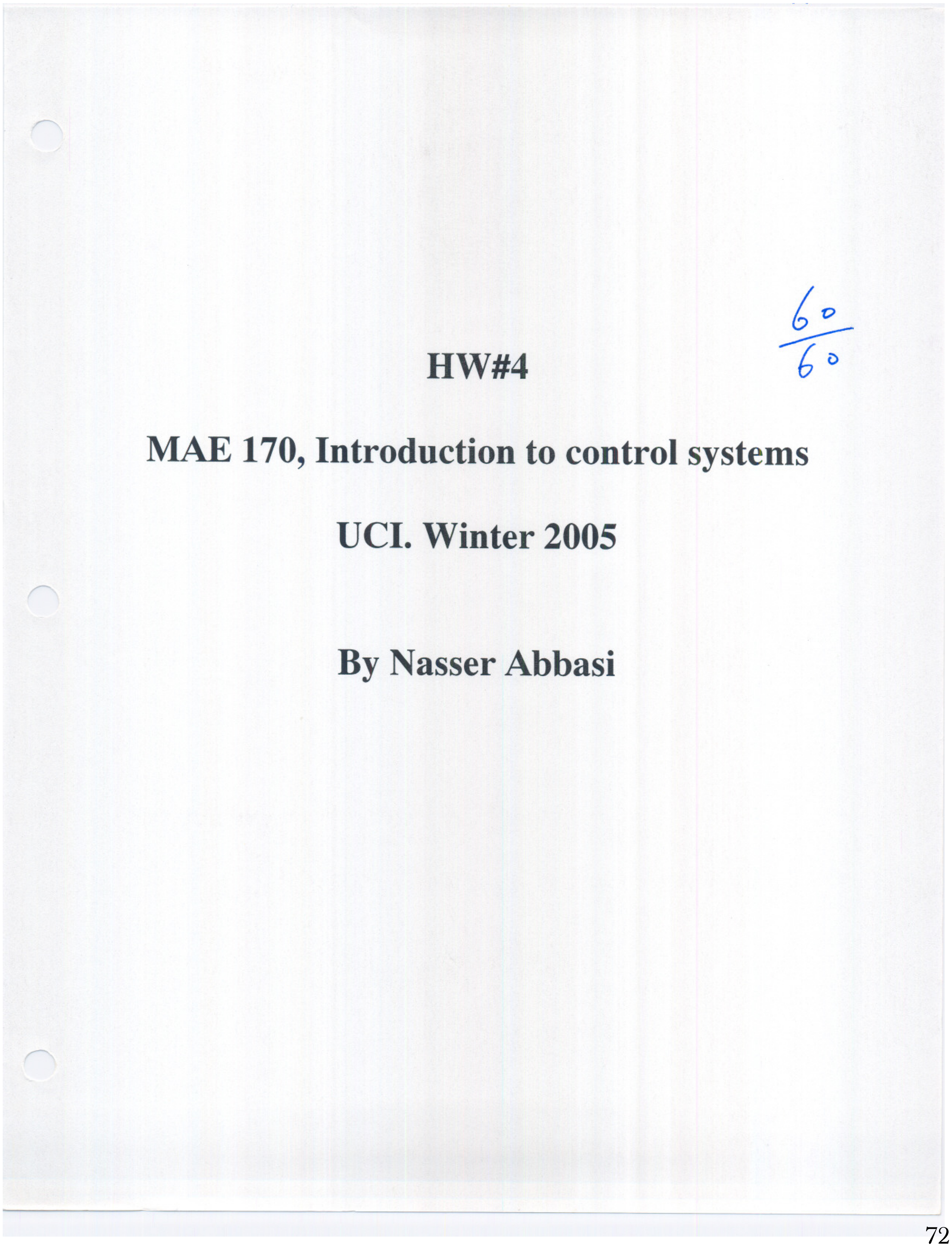
$GH = \frac{N}{s^2 D}$ type 2.

so for type 2, to find $K_a = \cancel{s^2} \frac{N}{\cancel{s^2} D} = \frac{N}{D}$

so for type 1, $K_a = 0$ (since s in Numerator. so $e_{ss} \to \infty$)

3.4 HW 4

3.4.1 my solution



$\frac{60}{60}$

HW#4

MAE 170, Introduction to control systems

UCI. Winter 2005

By Nasser Abbasi

Nasser Abbasi
HW#4
B-5-1

Thermo requires 60 seconds to indicate 98% of the response of the system to a step response. Assuming thermo to be a first order system, find τ (time constant). If thermo placed in path, temp. is changing linearly at rate 10°/min. how much error does thermo show?

Answer

since First Order System, then

$$\underset{\text{output}}{0.98} = \boxed{1 - e^{-\frac{t}{\underset{\text{time Constant}}{\tau}}}}$$

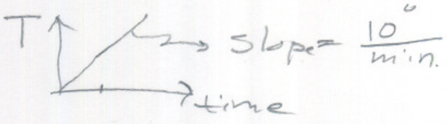
let $t = 60$ seconds. Solve for τ.

$$\text{so } 0.98 = 1 - e^{\frac{60}{\tau}} \Rightarrow 0.02 = e^{-\frac{60}{\tau}}.$$

$$\Rightarrow \ln 0.02 = -\frac{60}{\tau}$$

$$\tau = \frac{-60}{\ln 0.02} = \boxed{15.337 \text{ seconds}}$$

now the input is ramp



since 1st order system, then the transfer function is $\boxed{\frac{a}{s+a}}$

so $\frac{Y(s)}{X(s)} = \frac{a}{s+a}$. when input is ramp with slope $\underset{\text{secs in minute}}{\frac{10°}{60}}$ $\Rightarrow$

$$Y(s) = \left(\underset{\text{Laplace of input}}{\frac{10}{60}\frac{1}{s^2}}\right)\frac{a}{s+a} \equiv \frac{A}{s^2} + \frac{B}{s}$$

$$\Rightarrow A = \frac{10}{60}, \quad B = \frac{10}{60}\frac{1}{a}$$

$$\text{so } Y(s) = \frac{10}{60}\frac{1}{s^2} + \frac{10}{60}\frac{1}{a}\frac{1}{s}$$

$$\text{so } \boxed{y(t) = \frac{10}{60}t + \frac{10}{60}\frac{1}{a}u(t)} \longrightarrow$$

since we found $\tau = 15.337$ seconds.

and since $\boxed{\tau = \frac{1}{a}} \Longrightarrow \boxed{a = \frac{1}{15.337}}$ $\leftarrow$ this is Hz. (natural freq. of system).

so temp. at time t as indicated by Thermo is given by

$$y_{\text{measured}}(t) = \frac{10}{60} t + \frac{10}{60}(15.337)$$

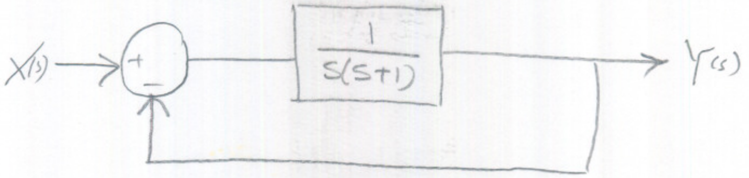
but <u>actual</u> temp is $y_a(t) = \frac{10}{60} t$.

so error is $\left[\frac{10}{60} t + \frac{10}{60}(15.337)\right] - \frac{10}{60} t$

$$= \boxed{2.56^\circ}$$ ✓

HW# 4
problem B-5-2
Nasser Abbasi

given



when X(s) is $\frac{1}{s}$ (i.e unit step), obtain rise time, peak time, maximum overshoot, settling time.

solution

Rise Time: time required for response to rise from 10% to 90% of its final value or 0% to 100% depending on system.

The closed Loop transfer function is $\frac{G(s)}{1+G(s)} = \frac{\frac{1}{s(s+1)}}{1+\frac{1}{(s+1)s}}$

$$= \frac{1}{1+(s+1)s} = \boxed{\frac{1}{s^2+s+1}}$$

this is a second order system.

$\omega_n^2 = 1 \Rightarrow \omega_n = 1$

$2\xi\omega_n = 1 \Rightarrow \boxed{\xi = \frac{1}{2}}$, so system is underdamped. hence according to text, page 230, use time from 0% to 100%.

From page 231, Rise time $t_r = \frac{1}{\omega_d} \tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \boxed{\frac{\pi-\beta}{\omega_d}}$

where $\omega_d = \omega_n\sqrt{1-\xi^2}$, $\sigma = \xi\omega_n$.

so $\omega_d = \sqrt{1-.5^2} = 0.866$, $\sigma = \frac{1}{2}$, $\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}\frac{0.866}{0.5}$

so $\beta = 1.04718$. so $t_r = \frac{\pi-1.04718}{0.866} = \boxed{2.41849 \text{ sec}}$

$\rightarrow t_r$

→

Peak time $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{0.866} = \boxed{3.6277 \text{ Sec}}$

Max. overshoot $M_p = e^{-(\sigma/\omega_n)\pi} = e^{-\left(\frac{0.5}{1}\right)\pi} = 0.2078.$

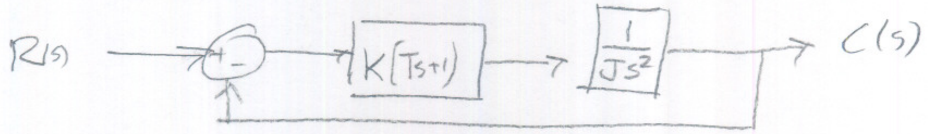
so max percent overshoot is $\boxed{20.78\%}$

Settling time using 2% criterion.

$$t_s = \frac{4}{\sigma} = \frac{4}{0.5} = \boxed{8 \text{ sec}}$$

HW#4
Problem B-5-4
Nasser Abbasi

This is a block diagram of space-vehicle attitude control system. assuming time constant T of the controller to be 3 sec and the ratio $\frac{K}{J}$ to be $\frac{2}{9}$ rad^2/sec^2, find damping ratio of system.



Solution

need to find ξ.

open loop transfer function $G(s) = \frac{K(Ts+1)}{Js^2} = \frac{2}{9}\frac{(3s+1)}{s^2}$

so closed loop transfer function $= \frac{G}{1+G} =$

$$\frac{C(s)}{R(s)} = \frac{\frac{2}{9}\frac{(3s+1)}{s^2}}{1+\frac{2(3s+1)}{9s^2}} = \boxed{\frac{6s+2}{9s^2+6s+2}}$$

so $9s^2+6s+2 \to s^2+\frac{2}{3}s+\frac{2}{9}$

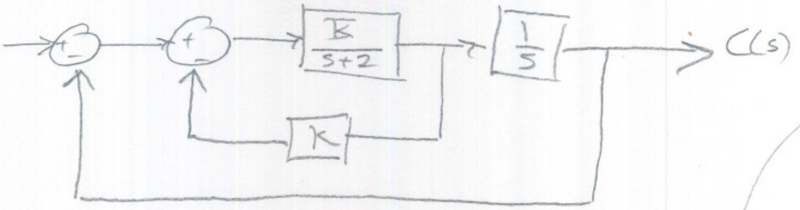
i.e. $\omega_n^2=\frac{2}{9}$, $2\xi\omega_n=\frac{2}{3}$

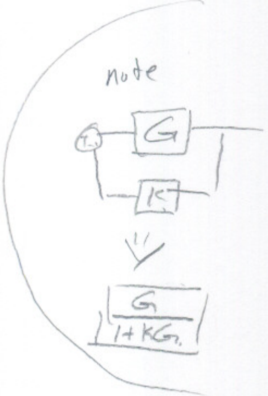
$\omega_n=\frac{\sqrt{2}}{3}$. so $\xi=\frac{1}{3}\frac{3}{\sqrt{2}}=\frac{1}{\sqrt{2}}$

so $\boxed{\xi=\frac{1}{\sqrt{2}}}=0.707$

HW#4
Problem B-5-10
Nasser Abbasi

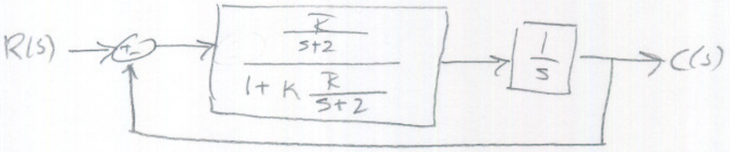
ref system in fig 5-84, determine $\bar{K}$ and K such that system has $\xi = 0.7$ and $\omega_n = 4$ rad/sec.

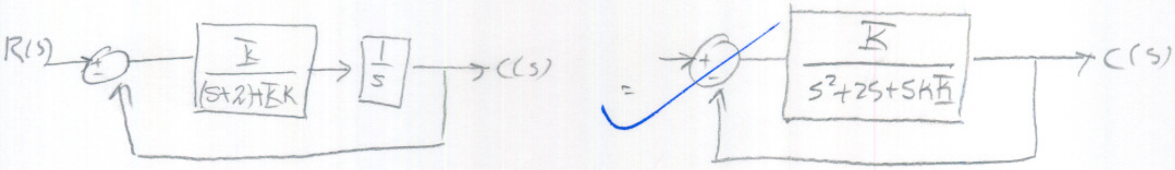


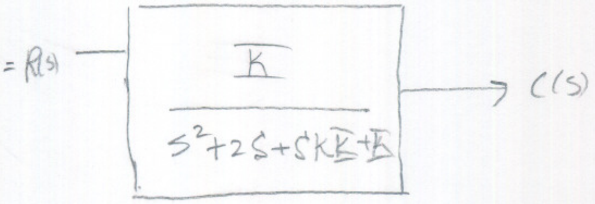


Solution

First simplify system using block diagram reduction.



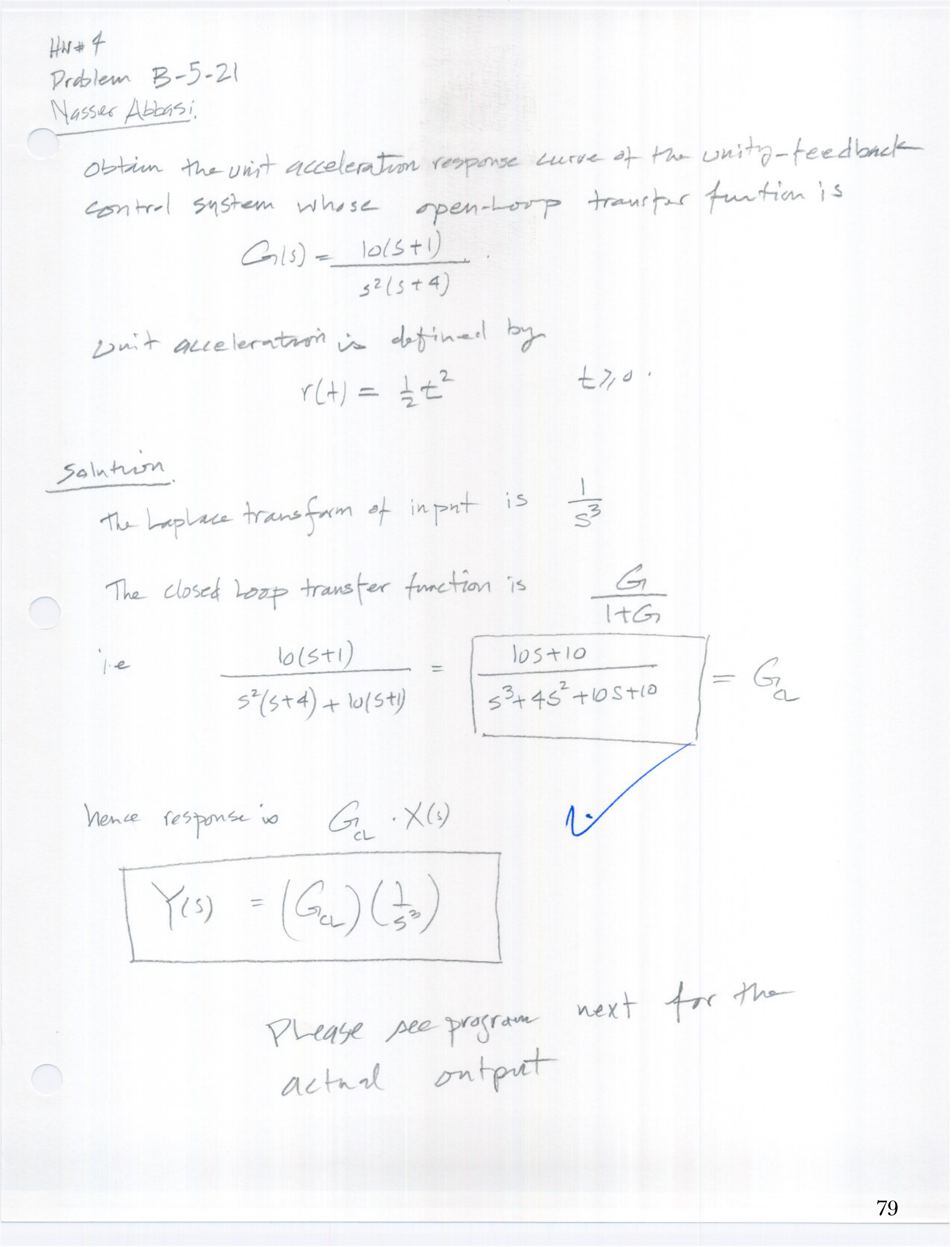




hence $G(s) = \dfrac{\bar{K}}{s^2+s(2+K\bar{K})+\bar{K}}$ $\Rightarrow$ $\boxed{\begin{aligned} 2\xi\omega_n &= 2+K\bar{K} \\ \omega_n^2 &= \bar{K} \end{aligned}}$

Closed loop

$\Rightarrow \boxed{\bar{K} = 16}$ so. $K = \dfrac{2\xi\omega_n - 2}{\bar{K}}$ $\Rightarrow$ $K = \dfrac{2(.7)(4)-2}{16} = \boxed{0.225}$



HW# 4
Problem B-5-21
Nasser Abbasi.

obtain the unit acceleration response curve of the unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{10(s+1)}{s^2(s+4)}.$$

unit acceleration is defined by

$$r(t) = \frac{1}{2}t^2 \qquad t \geq 0.$$

Solution

The Laplace transform of input is $\frac{1}{s^3}$

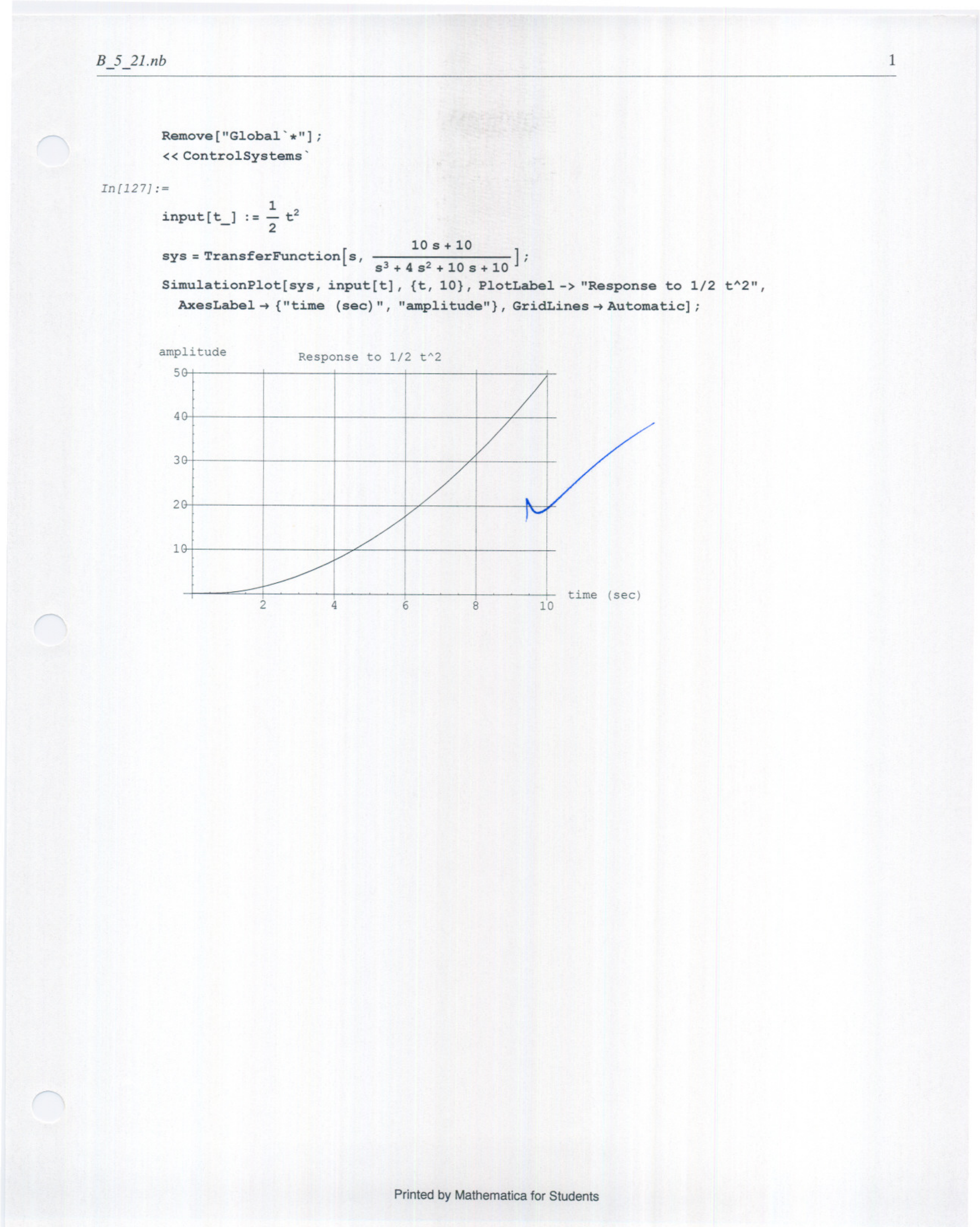
The closed loop transfer function is $\frac{G}{1+G}$

i.e $$\frac{10(s+1)}{s^2(s+4)+10(s+1)} = \boxed{\frac{10s+10}{s^3+4s^2+10s+10}} = G_{CL}$$

hence response is $G_{CL} \cdot X(s)$

$$\boxed{Y(s) = (G_{CL})\left(\frac{1}{s^3}\right)}$$

Please see program next for the actual output



B_5_21.nb 1

```
Remove["Global`*"];
<< ControlSystems`
```

In[127]:=

```
input[t_] := 1/2 t^2
sys = TransferFunction[s, (10 s + 10)/(s^3 + 4 s^2 + 10 s + 10)];
SimulationPlot[sys, input[t], {t, 10}, PlotLabel -> "Response to 1/2 t^2",
  AxesLabel → {"time (sec)", "amplitude"}, GridLines → Automatic];
```

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HW# 4
Problem B-5-31
Nasser Abbasi

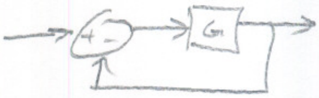
Consider unity feedback control system whose open Loop tf is

$$G(s) = \frac{K}{s(Js+B)}$$

discuss the effects of varying values of K and B has on the steady state error in unit-Ramp response.
sketch typical unit-ramp response curves for a small, medium, Large values of K, assuming B is constant.

Solution

First obtain the Closed Loop TF.



$$\text{so Closed Loop TF} = \frac{G}{1+G}$$

$$= \frac{K}{s(Js+B)+K} = \frac{K}{s^2J+Bs+K}$$

$$= \frac{K/J}{s^2+\frac{B}{J}s+K/J}.$$

hence the natural undamped frequency $\boxed{\omega_n = \sqrt{\frac{K}{J}}}$

and $\boxed{2\zeta\omega_n = \frac{B}{J}} \Rightarrow \zeta = \frac{B}{2J\omega_n} = \frac{B}{2J\sqrt{\frac{K}{J}}} = \boxed{\frac{B}{2\sqrt{JK}}}$

so, for fixed B, as K increases, ω_n will increase and ζ will decrease i.e more oscillation will result.

now to Look at steady state errors $\longrightarrow$

now $Y(s) = G(s)\,X(s)$ → this is $\frac{1}{s^2}$

this is G(s) Found earlier.

so $E(s) = X(s) - Y(s)$

$$E(s) = \frac{1}{s^2} - \left(\frac{1}{s^2}\,\frac{K/J}{s^2 + B/J\,s + K/J}\right) = \frac{1}{s^2} - \frac{1}{s^2}\,\frac{K}{Js^2+Bs+K}$$

using $\boxed{\text{Find value theorm}}$

$$e(\infty) = \lim_{s\to 0} s\,E(s)$$

$$= \lim_{s\to 0} \frac{1}{s} - \frac{1}{s}\,\frac{K}{Js^2+Bs+K} = \lim_{s\to 0} \frac{Js^2+Bs+K-K}{s\,(Js^2+Bs+K)}$$

$$= \lim_{s\to 0} \frac{Js^2+Bs}{s\,(Js^2+Bs+K)} = \lim_{s\to 0} \frac{Js+B}{Js^2+Bs+K} = \boxed{\frac{B}{K}}$$

so steady state error is $\frac{B}{K}$. (position error).

so for fixed B, $\boxed{\text{as K increases, the position error decrease}}$

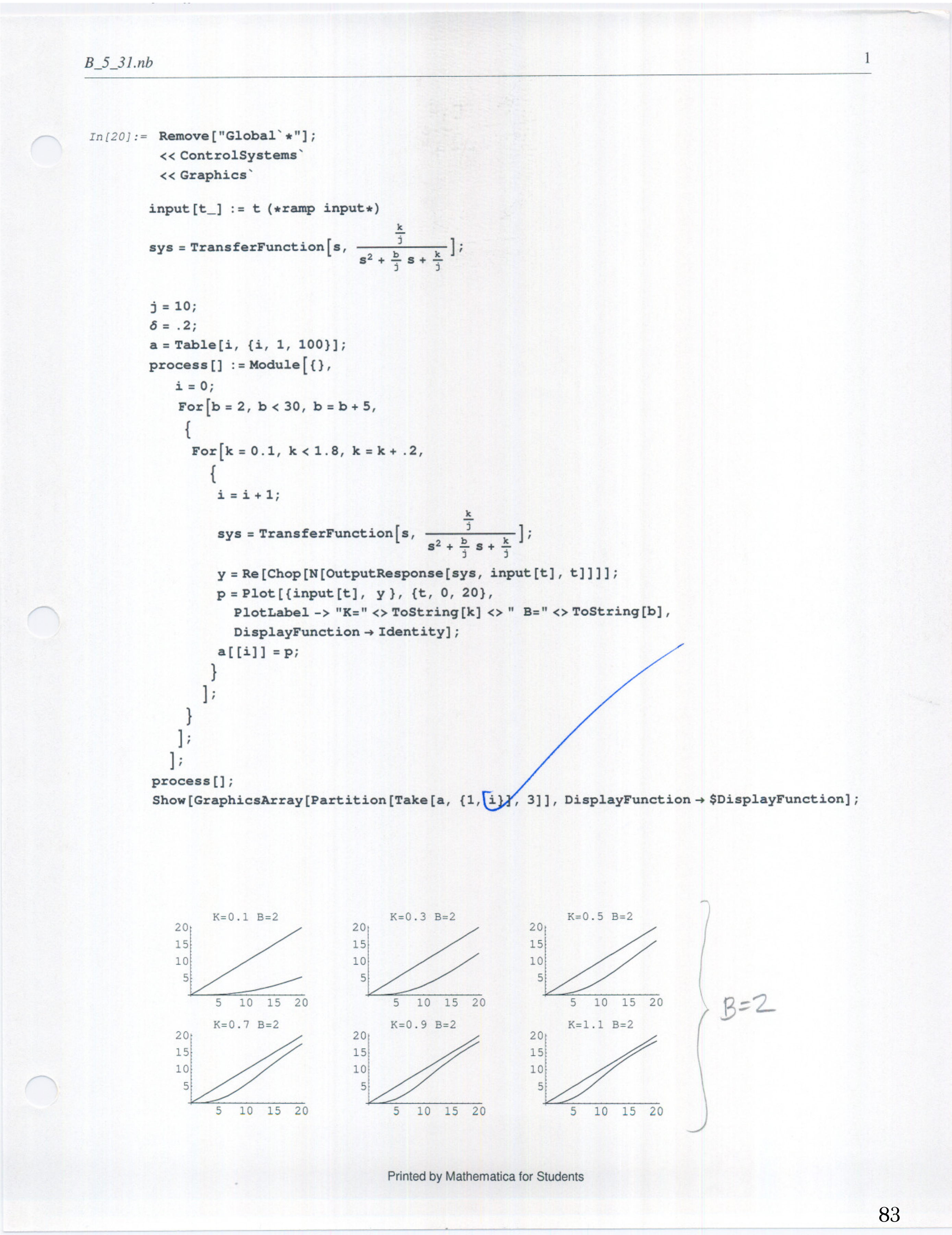
in the next program, I will plot the response for different values of K, for different Fixed values of B.

→

Conclusion

as a result of these plots, we see that as B increases, steady state error increases. For the same K. For Example, for K=1.7, when B was 2 there was much less ss error than when B was 27.

When B is fixed, as we increase K, steady state error decreased. so Best combination is to have large K and small B.



B_5_31.nb 1

```
In[20]:= Remove["Global`*"];
          << ControlSystems`
          << Graphics`

        input[t_] := t (*ramp input*)

        sys = TransferFunction[s, (k/j)/(s^2 + b/j s + k/j)];


        j = 10;
        δ = .2;
        a = Table[i, {i, 1, 100}];
        process[] := Module[{},
            i = 0;
            For[b = 2, b < 30, b = b + 5,
             {
              For[k = 0.1, k < 1.8, k = k + .2,
                {
                 i = i + 1;

                 sys = TransferFunction[s, (k/j)/(s^2 + b/j s + k/j)];

                 y = Re[Chop[N[OutputResponse[sys, input[t], t]]]];
                 p = Plot[{input[t],  y }, {t, 0, 20},
                   PlotLabel -> "K=" <> ToString[k] <> " B=" <> ToString[b],
                   DisplayFunction → Identity];
                 a[[i]] = p;
                }
               ];
             }
            ];
          ];
        process[];
        Show[GraphicsArray[Partition[Take[a, {1, i}], 3]], DisplayFunction → $DisplayFunction];
```

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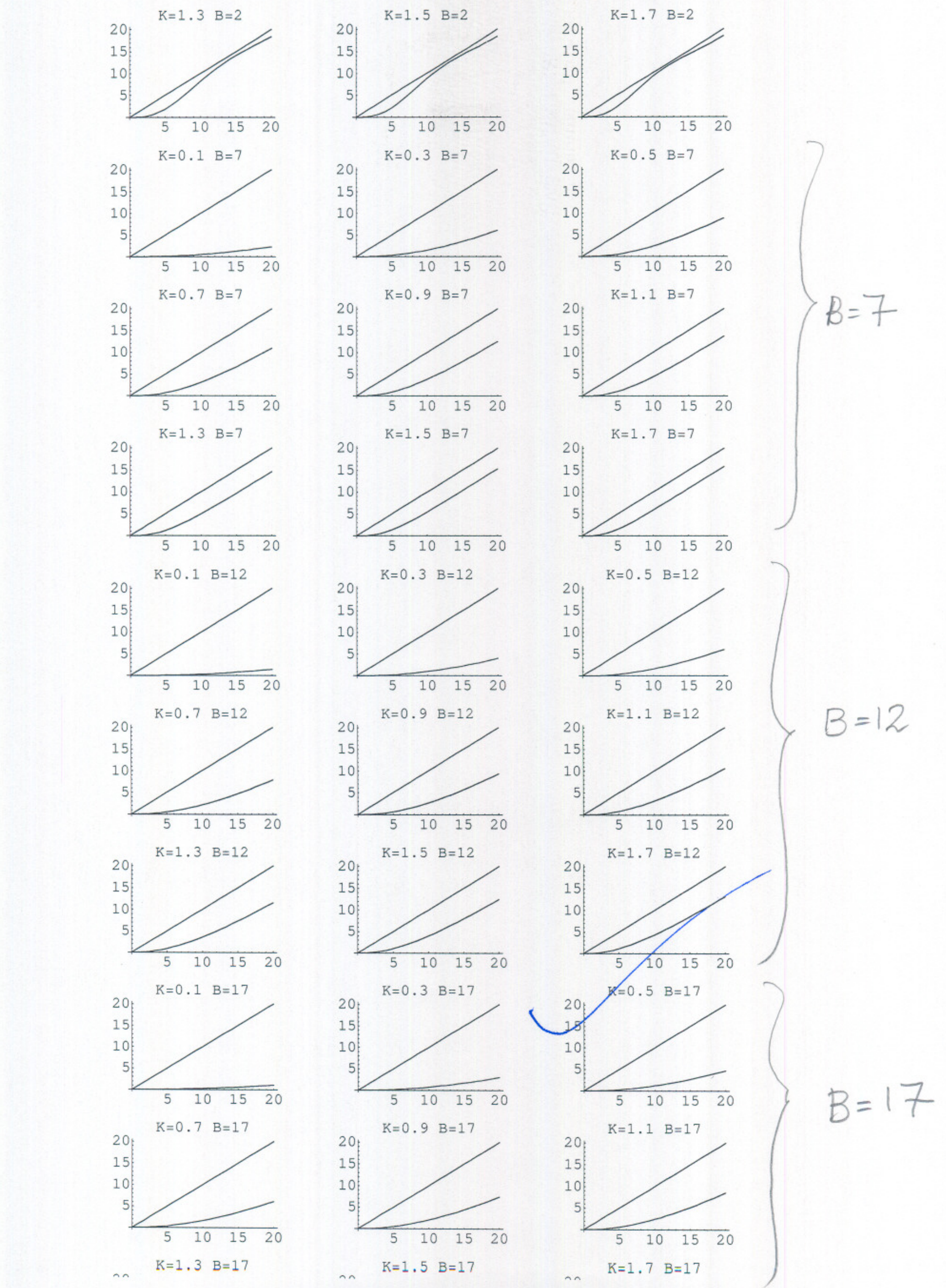
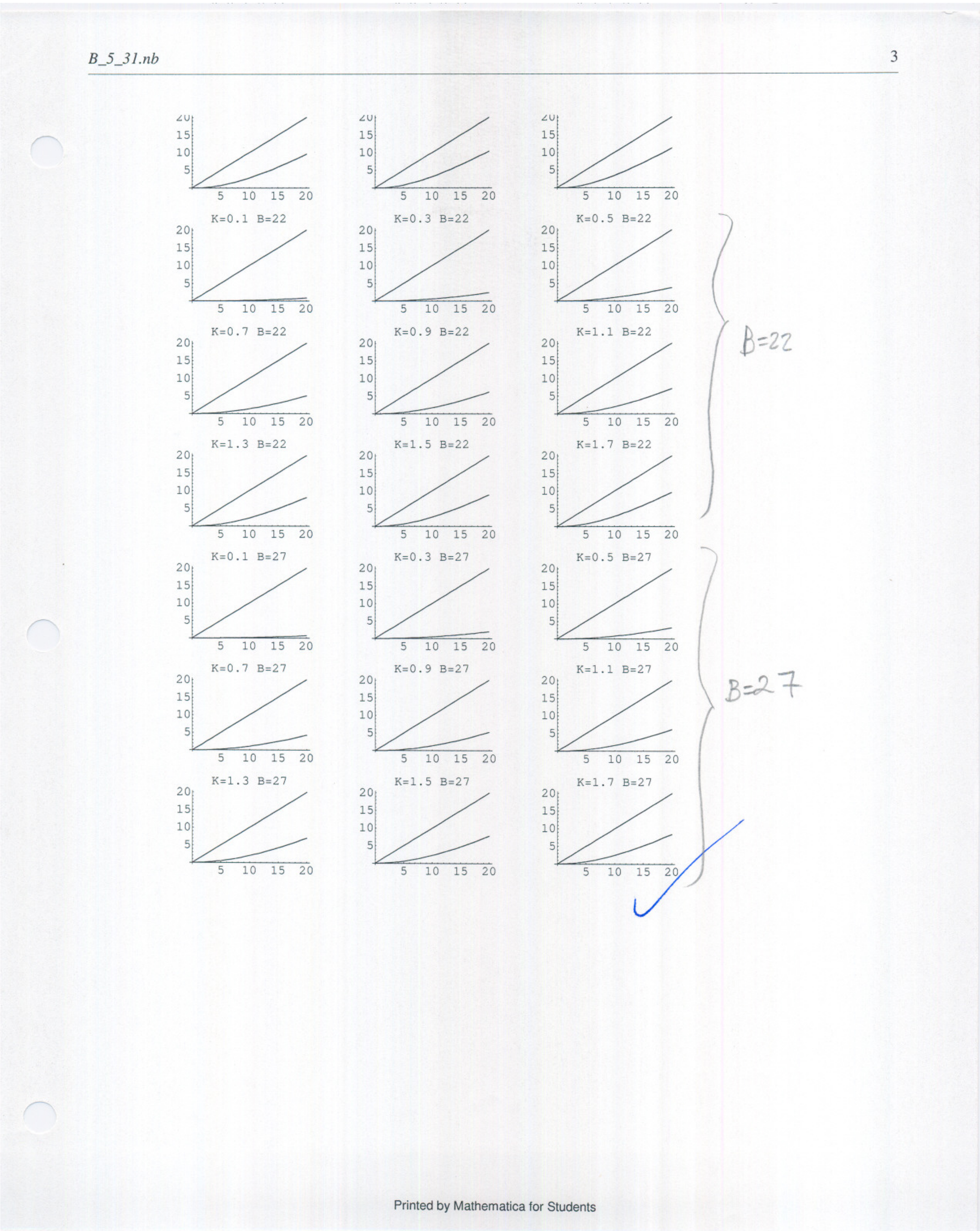

K=1.3 B=2
K=1.5 B=2
K=1.7 B=2
K=0.1 B=7
K=0.3 B=7
K=0.5 B=7
K=0.7 B=7
K=0.9 B=7
K=1.1 B=7
K=1.3 B=7
K=1.5 B=7
K=1.7 B=7
K=0.1 B=12
K=0.3 B=12
K=0.5 B=12
K=0.7 B=12
K=0.9 B=12
K=1.1 B=12
K=1.3 B=12
K=1.5 B=12
K=1.7 B=12
K=0.1 B=17
K=0.3 B=17
K=0.5 B=17
K=0.7 B=17
K=0.9 B=17
K=1.1 B=17
K=1.3 B=17
K=1.5 B=17
K=1.7 B=17
5 10 15 20
20 15 10 5
B=7
B=12
B=17



K=0.1 B=22 K=0.3 B=22 K=0.5 B=22

K=0.7 B=22 K=0.9 B=22 K=1.1 B=22

K=1.3 B=22 K=1.5 B=22 K=1.7 B=22

B=22

K=0.1 B=27 K=0.3 B=27 K=0.5 B=27

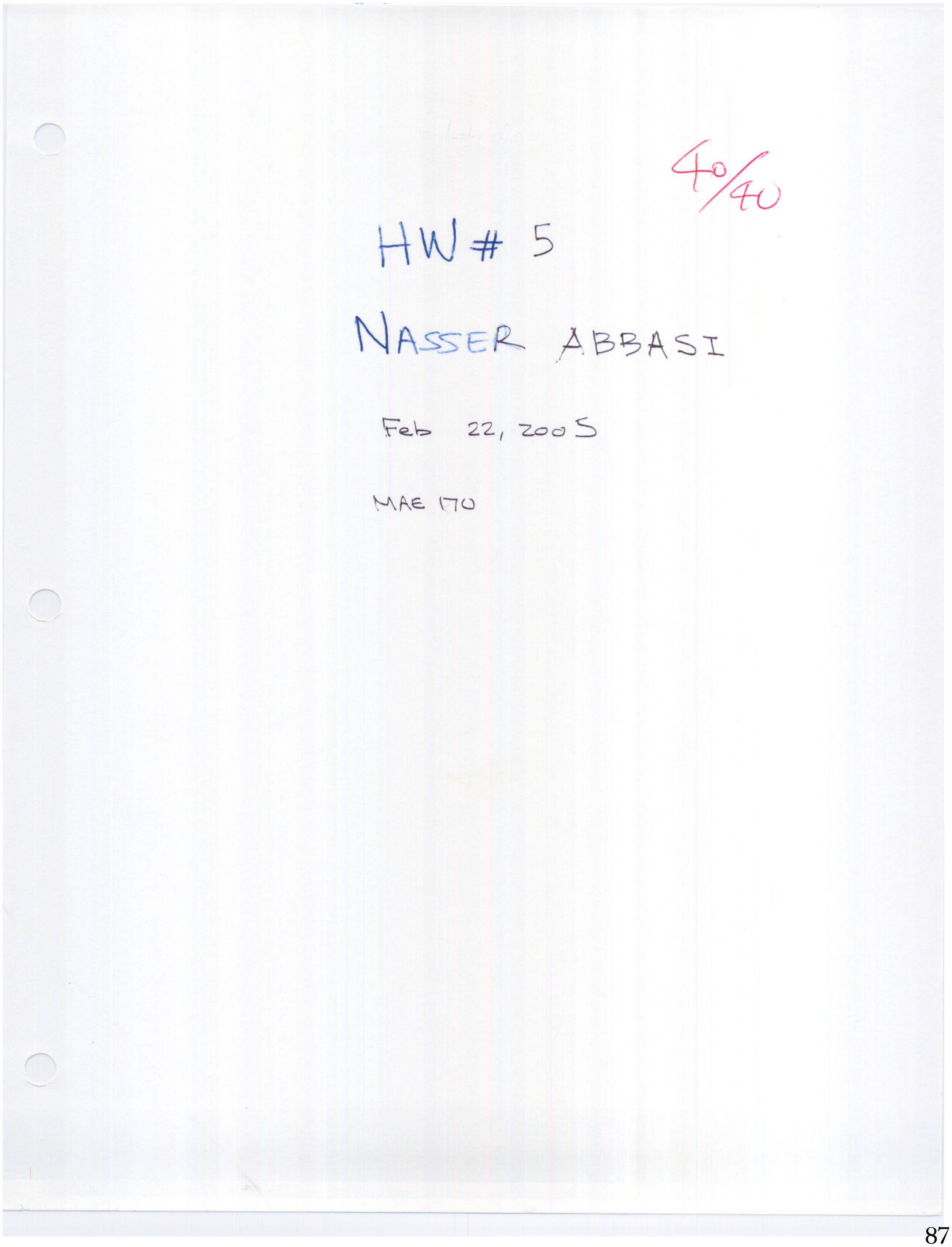
K=0.7 B=27 K=0.9 B=27 K=1.1 B=27

K=1.3 B=27 K=1.5 B=27 K=1.7 B=27

B=27

3.5 HW 5

3.5.1 my solution



40/40

HW # 5

NASSER ABBASI

Feb 22, 2005

MAE 170

HW 1, MAE 170.
Problem B 6-1, Modern Control Engineering, 4th edition by Ogata
by Nasser Abbasi
UCI, Winter 2005.

Question

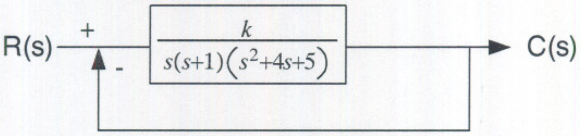
Plot the root loci for $G(s) = \frac{k}{s(s+1)(s^2+4s+5)}$

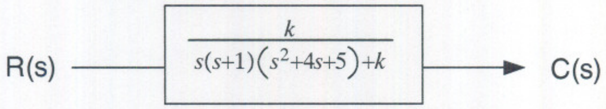
Solution

The closed loop transfer function is

$$G_{cl}(s) = \frac{G}{1+G} = \frac{\frac{k}{s(s+1)(s^2+4s+5)}}{1+\frac{k}{s(s+1)(s^2+4s+5)}} = \frac{k}{s(s+1)(s^2+4s+5)+k}$$

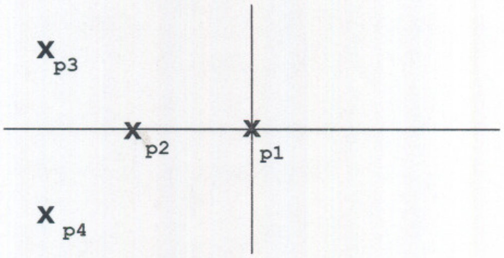
Hence the system is





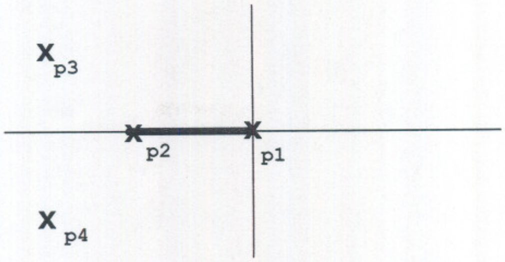
step 1

Plot the open loop poles and zeros:
Poles are at $s=0, s=-1$, and roots of (s^2+4s+5) which is $s=\frac{-4\pm\sqrt{16-4\times5}}{2}=\frac{-4\pm2i}{2}=-2\pm i$
There are no finite zeros (there are however 4 zeros at ∞)

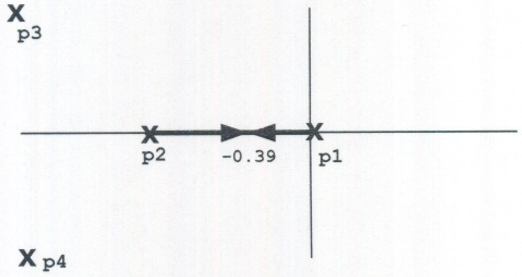


step 2

Apply angle condition to find initial segments on real axis that could be part of the root loci.
We see that on the RHP, it is not possible to have loci, since then the sum of angles to a text point will not add to $180 \pm n360$. For segments between p1 and p2, we will have a loci. for the segment to the right of p2, it is not possible to have a loci. Hence we now get this plot: (I use bold line to show where loci is)



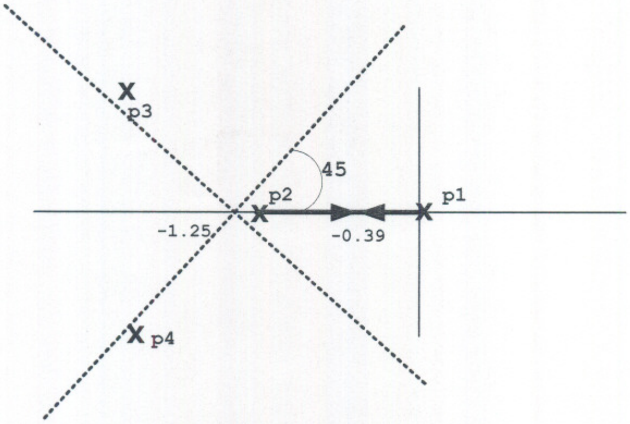
step 3
Since loci starts at a pole and ends at a zero of the open loop poles/zeros, then we know loci must start at p1 as well it must start at p2. Hence there must be a break away point between p1 and p2. Now we find this break away point. For this we use the condition that $\frac{dk}{ds}=0$
The characteristic equation for the close loop is $s\left(s+1\right)\left(s^2+4s+5\right)+k=0$ hence
$\frac{dk}{ds}=\frac{d}{ds}\left(s\left(s+1\right)\left(s^2+4s+5\right)\right)=\frac{d}{ds}\ \left(5s+9s^2+5s^3+s^4\right)=5+18s+15s^2+4s^3\ =0$, Solution is:
$\left[s=-1.6785-0.60278i\right],\left[s=-1.6785+0.60278i\right],\left[s=-0.39299\right]$
Since we are looking for a solution on the real axis, and one that is between p1 and p2, hence only possible solution is $s=-0.39299$, I add this point to the diagram, now it looks as follows



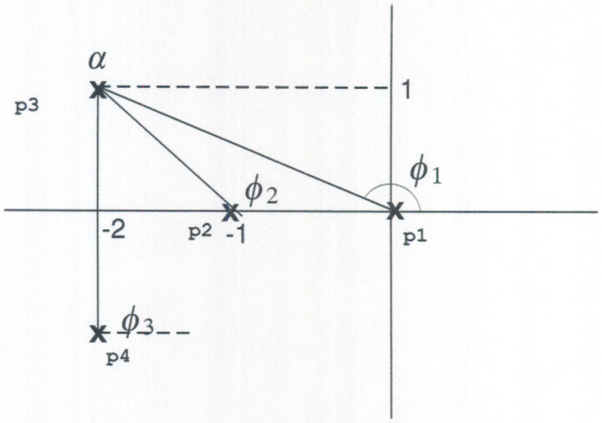
step 4
Now I need to find where the asymptotes lines cross at the real axis, and need to find angles that asymptotes leave the real axis at.
to find where asymptotes meet at the real axis:
$\sigma_a=\frac{\sum_m z_i-\sum_n p_i}{n-m}=\frac{0-(0+1+(2-i)+(2+i))}{4}=\frac{-(1+4)}{4}=-1.25$
To find angles, put a test point s very far away, and consider the sum of angles from the finite poles and zeros to that test point. we have here only 4 finite poles and no finite zeros. We know that sums of these angles must be 180 degrees, so we write

$$\begin{aligned} 4\theta &= 180\pm n360 \\ \theta &= 45\pm n90 \end{aligned}$$

Hence the angles are $45, 45+90, 45+180, 45+270$ or $45, 135, 225, 315$ so now I get this diagram, where I just added the asymptotes lines



step 5
Now I need to find angles of departures of loci from p3 and p4. To do this, put a test point s very close to p3 and solve for the angle conditions. a little bit of geometry is needed here. We get



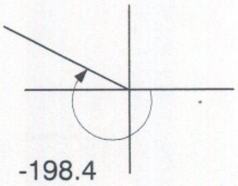
Hence, for a test point 's' near p3, we get from the angle condition, the following

$$\theta_1 + \theta_2 + \theta_3 + \alpha = 180 \pm n360$$

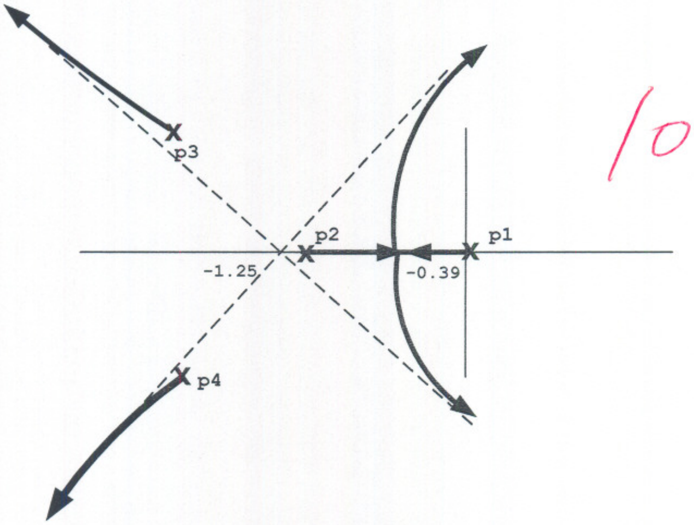
but
$\theta_3 = 90^0$
$\theta_2 = 90^0 + \tan^{-1}(1/1) = 135^0$
$\theta_1 = 90^0 + \tan^{-1}(2/1) = 90^0 + 63.4^0 = 153.4^0$
hence

$$\begin{aligned} \alpha + 90 + 135 + 153.4 &= 180^0 \pm n360 \\ \alpha + 378.4 &= 180^0 \pm n360 \\ \alpha &= 180^0 \pm n360 - 378.4 \\ &= -198.4 \pm n360 \end{aligned}$$

Since angles are positive anticlockwise, then this angle of -198.4 means this:



Hence, this is the angle of departures of p3. By symmetry we know the angle of departure of p4. Hence the plot now looks like this:



10

Note: at the break away point, -0.39, the loci is at 90^0 from the real axis.

step 6

Find where loci cross the j axis.

Looking at the charaterstic equation for the close loop $s(s+1)(s^2+4s+5)+k$, set $s=j\omega$ and solve

$$s(s+1)\left(s^2+4s+5\right)+k=0$$

Hence

$$\begin{aligned}\left(s^2+s\right)\left(s^2+4s+5\right)+k &= 0\\ 5s+9s^2+5s^3+s^4+k &= 0\end{aligned}$$

Let $s=j\omega$

$$5j\omega-9\omega^2-5j\omega^3+\omega^4+k=0$$

Hence equating real parts and imaginary parts, we get

$$\begin{aligned}5\omega-5\omega^3 &= 0\\ -9\omega^2+\omega^4+k &= 0\end{aligned}$$

From first equation, we get $1-\omega^2=0$ or $\omega=\ \pm1$

Hence the loci crosses the imaginary axis at $\pm i$
To find the gain k at these point, using the second equation, we get

$$\begin{aligned} -9i^2 + i^4 + k &= 0 \\ +9+1 &= -k \end{aligned}$$

hence gain is 10 (negative gain) where it cross the point $\boxed{(0,i)}$ by symmetry, the gain will be 10 (positive gain) where it cross at point $\boxed{(0,-i)}$

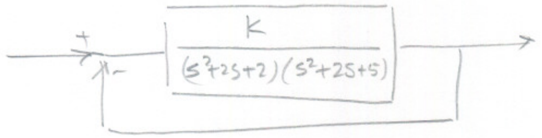
HW#5

Problem B-6-5

Plot Root loci For

$$G(s)=\frac{K}{(s^2+2s+2)(s^2+2s+5)}$$

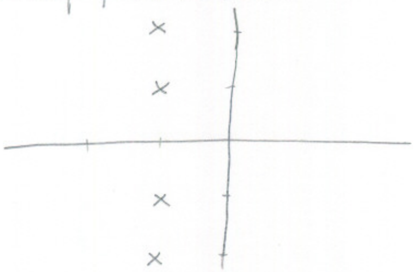
Solution



step 1 Plot open Loop Poles and Zeros.

For $s^2+2s+2=0 \rightarrow -1 \pm i$

For $s^2+2s+5=0 \rightarrow -1 \pm 2i$



Step 2 Find Loci on real axis. we see that using angle Conditions, there will be no loci on real axis. it is not possible to put have test point on real axis for loci.

Step 3 since Loci starts at poles and ends at zeros, then since there are no Finite zeros, then all loci will end at zero at ∞. since there are no break away or break in points here, nothing to do in this step.

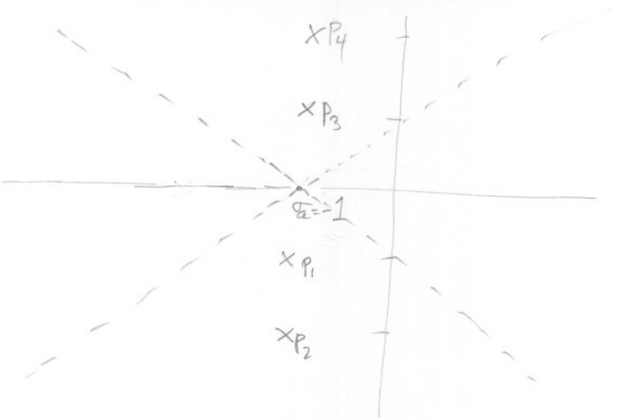
Step 4 determine number of asymptote lines and σ_a.

$$\sigma_a = \frac{\sum z_i - \sum P_i}{n-m} = \frac{0-(1+i+1-i+1-2i+1+2i)}{4} = -\left(\frac{4}{4}\right) = -1$$

now Find angles of asymptotes.

Put a test point very far. we see that $4\theta = 180 \pm n360$

or $\theta = 45^\circ \pm n90^\circ$, i.e 4 asymptotes $\longrightarrow$



$\boxed{\text{Step 5}}$ Find angles of departures for P_3, P_4 (by symmetry we find angle of departures for P_1, P_2).

<u>For P_3</u> Put test point 's' very close to P_3, we set

so $\alpha + 270 + 90 + 90 = 180 \pm n360$

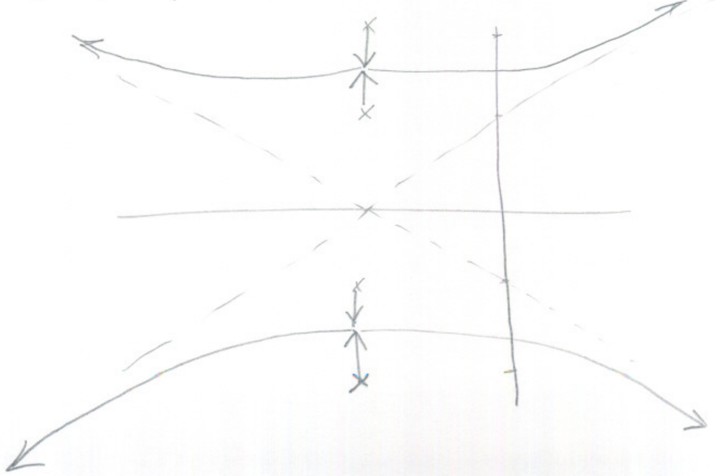
$\alpha + 450 = 180 \pm n360$

$\alpha = (180 - 450) \pm n360$

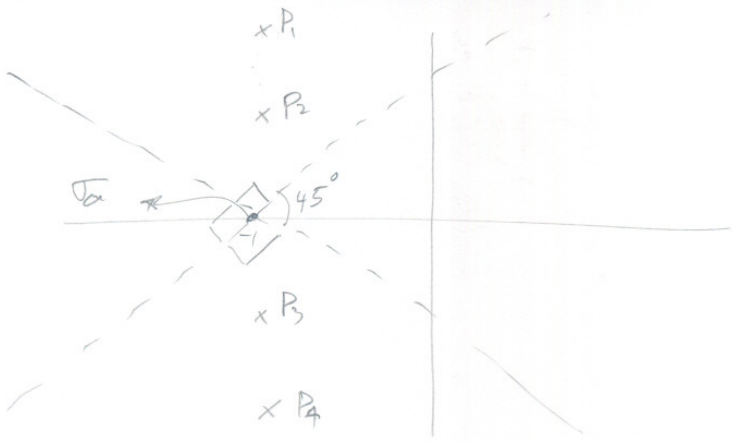
$\alpha = -270 \pm n360$

hence α is

hence angle of departures are -270° (or 90° From horizontal), by symmetry we set:



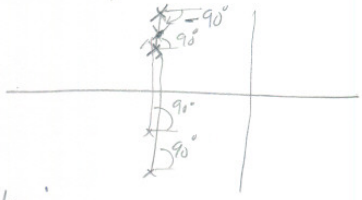
$\longrightarrow$



since we have 4 poles, we must have 4 branches.

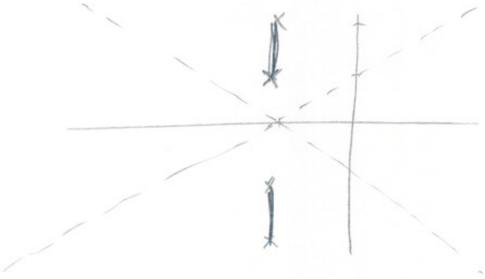
Put a test point 's' between P_1, P_2, then angle condition is

$$-90^\circ + 90^\circ + 90^\circ + 90^\circ = 180^\circ.$$



hence Path between $P_1 P_2$ is an loci.

Similarly path between P_3, P_4 is loci



$\boxed{\text{step6}}$ Find where loci cross $J\omega$ axis.

From char. equation of closed loop

10

$$(s^2+2s+2)(s^2+2s+5)+K=0.$$

$$s^4+2s^3+5s^2+2s^3+4s^2+10s+2s^2+4s+10+k=0$$

$$s^4+4s^3+11s^2+14s+10+K=0. \qquad \text{let } s=j\omega$$

$$\omega^4-4j\omega^3-11\omega^2+14j\omega+10+K=0$$

equate real and imaginary parts.

$$\omega^4-11\omega^2+10+K=0$$

$$-4\omega^3+14\omega=0 \Longrightarrow 14=4\omega^2 \rightarrow \omega=\pm\sqrt{7/2}=\pm 1.87$$

so loci crosses imaginary axis at $\boxed{\pm 1.87 i}$

HW#5

Problem B6-7

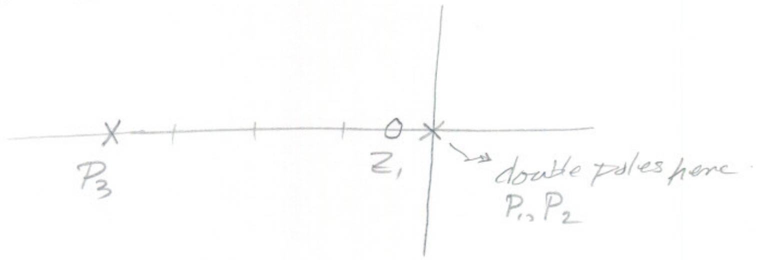
Plot loci for $G(s)=\frac{K(s+0.2)}{s^2(s+3.6)}$ $H(s)=1$

Solution

Step 1 Plot poles and zeros of open Loop

$P_1=0,\ P_2=0,\ P_3=-3.6,$

$Z_1=-0.2$



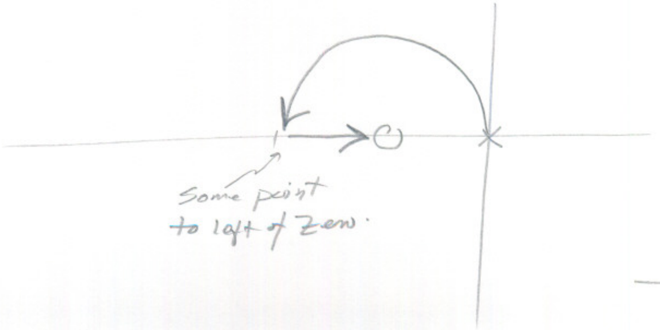
Step2 Determin Loci on real axis:

between P_1 and Z_1, angle conditions is (sum vectors from poles only to test point).

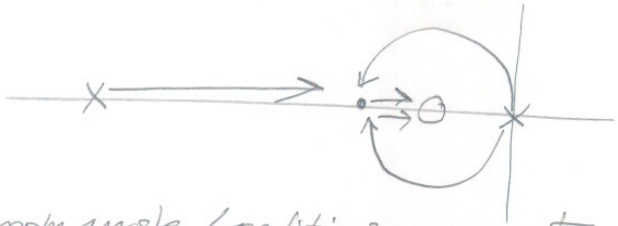
$180^\circ+180^\circ+0 \neq 180^\circ$

hence line between P_1, Z_1 is not on Loci. Line on +ve real axis also not on loci. since angle conditions there is $0+0+0=0$.

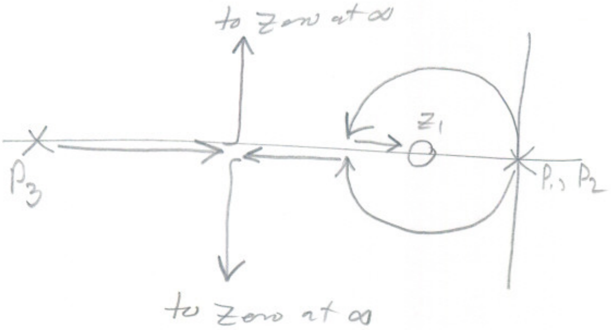
we know that loci must start at P_1 and end at a Zero Z_1. but since line between P_1, Z_1 is not on loci, how will occure then? it must be that loci jump over the imaginary plan and falls back to real axis and go back to the zero Z_1 as follows:



to verify, put a test point to left of zero as follows



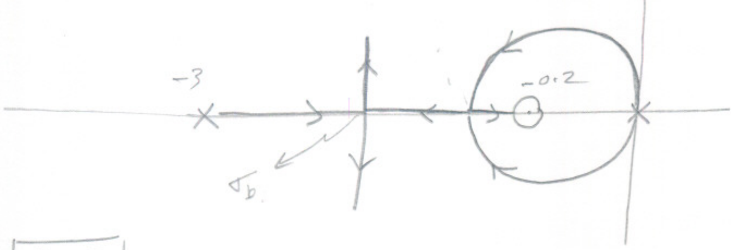
now apply angle conditions, we set $0+0+0 \neq 180^\circ$.
Then it must be that branch from P_2 will not go back to Z_1 but will go to P_3 and have a break point in between:



now try again with a test point to left of Z_1, we set

$0+180^\circ+0 = 180^\circ$. ok.

hence loci is:



$\boxed{\text{Step 3}}$
now I need to find point σ_b

$\frac{dK}{ds}=0$ at this points

char eq. for closed Loop is $s^2(s+3.6)+K(s+0.2)=0$

so $K = \frac{-s^2(s+3.6)}{s+0.2} \Rightarrow \frac{dK}{ds} = \frac{s^2(s+3.6)}{-(s+0.2)^2}+\frac{1}{(s+0.2)}\left[s^2+2s(s+3.6)\right]$

$$\frac{dK}{ds} = \frac{s^3+3.6s^2}{-(s+0.2)^2}+\frac{s^2+2s^2+7.2s}{s+0.2} = \frac{s^3+3.6s^2-(s+0.2)(s^2+2s^2+7.2s)}{-(s+0.2)^2}$$

so set $\frac{dK}{ds}=0$

Then $S^3+3.6S^2-[S^3+2S^3+7.2S^2+0.2S^2+0.4S^2+1.44S)=0$

i.e $S^3+3.6S^2-S^3-2S^3-7.4S^2-0.4S^2-1.44S=0$

i.e $-2S^3-4.2S^2-1.44S=0$

i.e $-2S^2-4.2S-1.44=0$

so $S^2+2.1S+.72=0$

i.e $S=\frac{-2.1\pm\sqrt{2.1^2-4\times.72}}{2} = \frac{-2.1\pm1.2369}{2} = -1.668$ or 0.018

so $\boxed{\sigma_b=-1.668}$

hence Final Loci is

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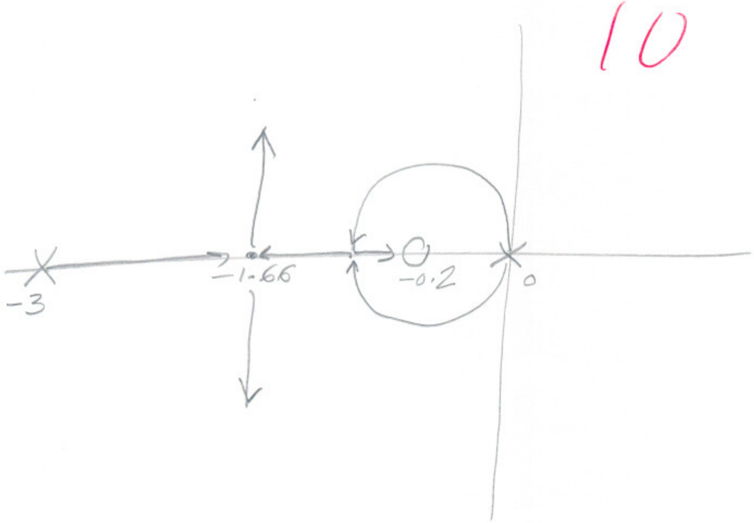


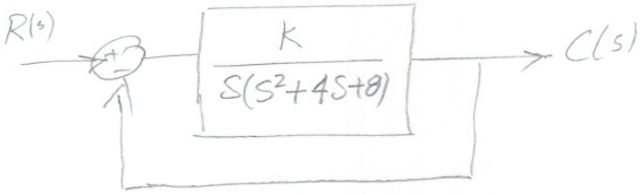
HW#5
Problem B 6-11

Consider Unity feedback Control systems with the Following feedforward transfer function.

$$G(s)=\frac{K}{s(s^2+4s+8)}$$

Plot the root loci for the system. if value of gain K is set equal to 2, where are the closed loop poles located?

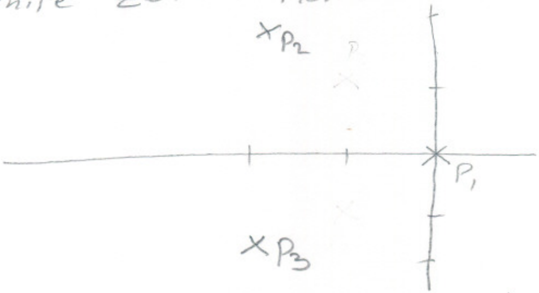
Answer



Step1 Plot the open Loop poles and zeros.

$P_1=0$, $P_2=$ roots $s^2+4s+8=0 \Rightarrow \frac{-4\pm\sqrt{16-4\times 8}}{2} = -2\pm 2i$

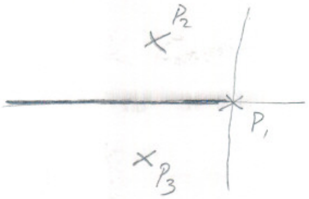
no finite zeros. hence



Step2 apply angle conditions to find loci on real axis.

no Loci to right of P_1.
put test point to left of P_1. angles = 180°. ok. so loci on all of Left real axis.

→



$\boxed{\text{Step3}}$ locate σ_b (break away points).
since no other pole on real axis. nothing to do.

$\boxed{\text{step 4}}$ Locate asymptotes:

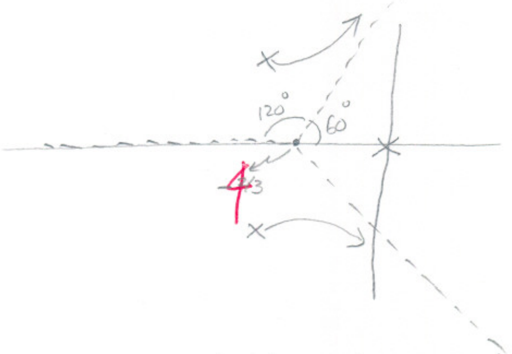
$$\sigma_a = \frac{\sum z_i - \sum P_i}{n-m} = \frac{0-(0+(2-2i)+(2+2i))}{3} = \frac{-(2)}{3} = -2/3$$

put test point very far.

Then we have $3\theta = 180° \pm n360$

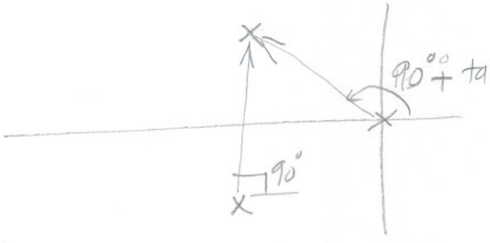
$\theta = 60° \pm 120n$

so

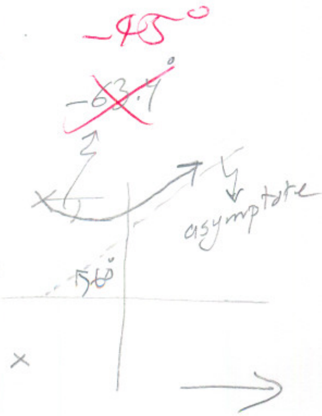


$\boxed{\text{step 5}}$ Find angles of departures of P_2, P_3 to asymptote.

put test point very close to P_2. we set



$90° + \tan^{-1}(1) = 1\cancel{53.43}°$.



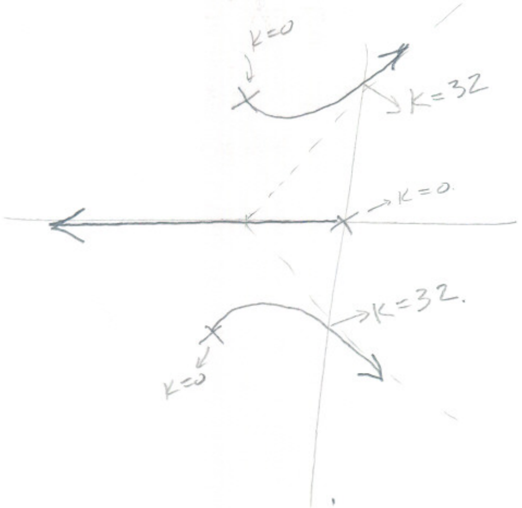
so $153.4° + 90° + \alpha = 180 \pm n360$

$\alpha = -63.4° \pm n360$, hence angle of departure is

→

by symmetry, angle of departure from P_2 is found.

so loci so far looks like



Step 6 Find where loci cross imaginary axis.

char equation for closed Loop is

$$s(s^2+4s+8)+k=0$$
$$s^3+4s^2+8s+k=0$$

let $s=j\omega$.

so $-j\omega^3-4\omega^2+8j\omega+k=0$

so $-\omega^3+8\omega=0 \longrightarrow \omega^2=8 \rightarrow \omega=\pm 2.828$

so loci cross imaginary axis at $\boxed{\pm 2.828i}$

and $-4\omega^2+k=0$

so $K=4(2.828)^2 = 32$.

now need to answer final part.

Put K=2. so need to find where on loci is K=2.

From loci diagram we see that system is stable for $\boxed{0<K<32}$

→

when $K=2$

Closed Loop TF = $\dfrac{K}{s(s^2+4s+8)+K}$

so Char eq when $K=2$ is

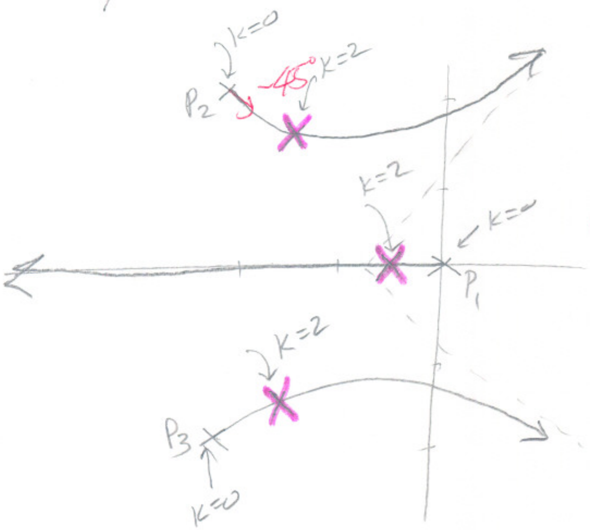
$$s(s^2+4s+8)+2=0$$

roots are $s=-0.288$

$s=-1.855+1.866i$ ✓

$s=-1.855-1.866i$

These ^closed loop^ poles are on loci as shown:



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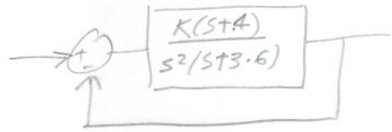
so when gain = 2, we see poles ^P2, P3^ have moved to the right to new location, and pole P_1 have moved to left.

HW#6

problem A-6-6

Sketch root loci for

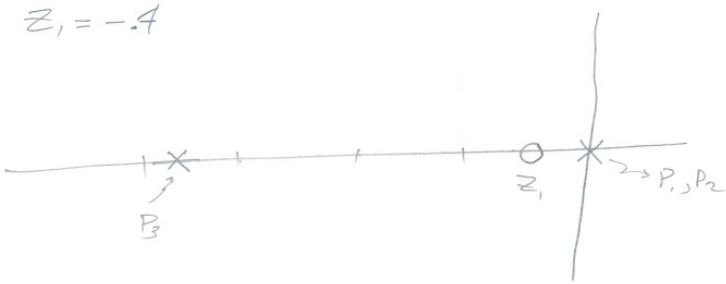


Solution

$\boxed{\text{Step 1}}$ Plot open Loop poles and zeros.

$P_1 = 0$, $P_2 = 0$, $P_3 = -3.6$

$Z_1 = -.4$



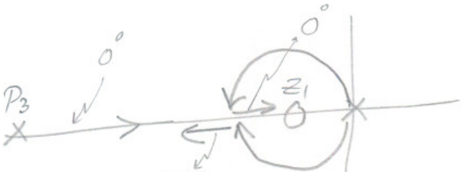
$\boxed{\text{Step 2}}$ use angle conditions to find loci on real axis

RHP: $0+0+0 \neq 180^\circ$

testpoint between P_1, Z_1: $180+180+0 \neq 180^\circ \pm n360^\circ$

so line between P_1 and Z_1 not part of Loci.

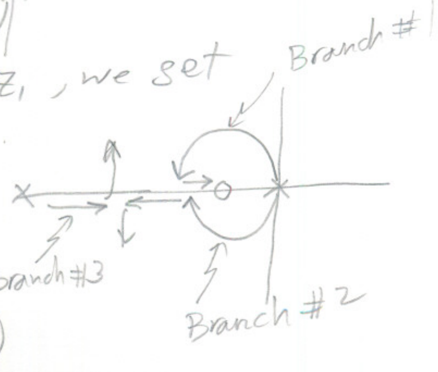
testpoint between Z_1, P_3: since line P_1Z_1 not on Loci, how will loci reach Line Z_1P_3 then? it must go up around Z_1 as follows:



so need for a test point on line P_3Z_1, we set

$0^\circ + 180^\circ + 0^\circ = 180$. OK.

So we have this loci so far



So now need to find breakaway point for branches 2,3 (σ_b)

→

$\boxed{\text{step3}}$ Find σ_b ; break away point for branchs # 2,3.

$\frac{ds}{dK}=0$. from closed loop char eq.

$$G_{CL}(s)=\frac{K(s+.4)}{s^2(s+3.6)+K(s+.4)}$$

so $s^2(s+3.6)+K(s+.4)=0$

$s^3+3.6s^2+K(s+.4)=0 \longrightarrow K=-\frac{s^3+3.6s^2}{s+.4}$

so $\frac{dK}{ds}=\frac{-(s^3+3.6s^2)}{(s+.4)^2}+\frac{1}{(s+.4)}(3s^2+7.2s)$

$\Rightarrow -(s^3+3.6s^2)+(s+.4)(3s^2+7.2s)=0$

i.e $-s^3-3.6s^2+3s^3+7.2s^2+1.2s^2+2.88s=0$

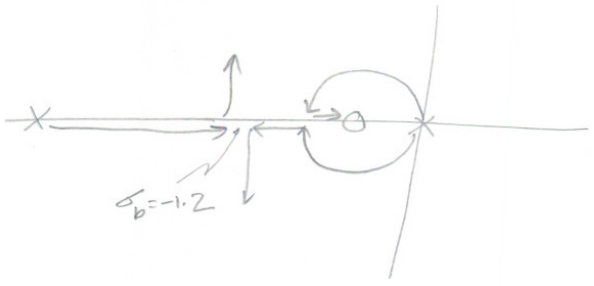
i.e $2s^3+4.8s^2+2.88s=0$

or $s^2+2.4s+1.44=0$ ($s=0$ is a solution).

$(s+1.2)^2=0 \Longrightarrow s=-1.2$ (double root) $\hookrightarrow$ double root means $\frac{d^2K}{ds^2}=0$.

so solutions are $\boxed{s=0 \text{ and } s=-1.2}$

so $\sigma_b=-1.2$ since that is to left of zero where we are looking for break away, so Loci looks like this so far



next find asymptotes

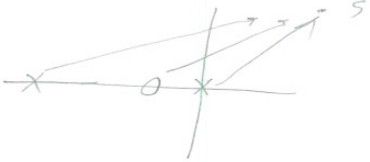
→

$\boxed{\text{Step 4}}$ find σ_a and number of asymptotes.

$$\frac{\sum P_i - \sum z_i}{3-1} = \frac{(0+0-3.6)-(-0.4)}{2} = \frac{-3.6+.4}{2} \boxed{=-1.6}$$

note to TA: here I write $\sum P_i - \sum z_i$ instead of book $\sum z_i - \sum P_i$, so I do not have to negate poles/zeros values. it is the same answer, but this way is more clear to me.

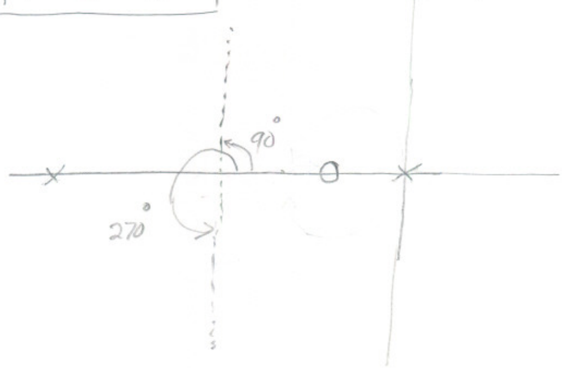
for angles of asymptote: put a test point 's' very far, we set



$$\underbrace{\theta_1+\theta_1+\theta_1}_{\text{from 3 poles}} \underset{\text{from the zero}}{-\theta_1} = 180^\circ \pm n360$$

so $2\theta = 180^\circ \pm n360 \Rightarrow \theta = 90^\circ \pm 180^\circ n$

so angles are $\boxed{90^\circ, 270^\circ}$, so 2 asymptotes



next find angle of departures.

Step 2

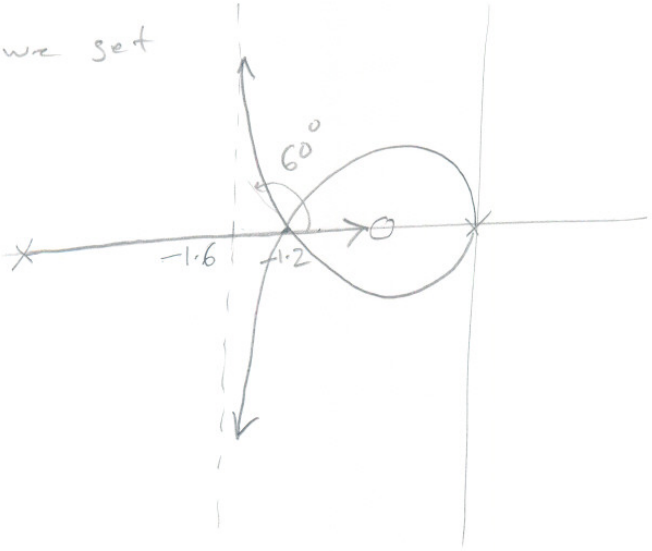
need to find this angle.
how?

I am not sure how to find this angle.

according to book, since there are triple root of the characteristic equation there (at $s=-1.2$, the σ_b point),

then $\frac{180^\circ}{\#\text{ of roots at }\sigma_b} = \frac{180^\circ}{3} = \boxed{60^\circ}$ (but I am not too clear on this point. can you please explain this to us more?)

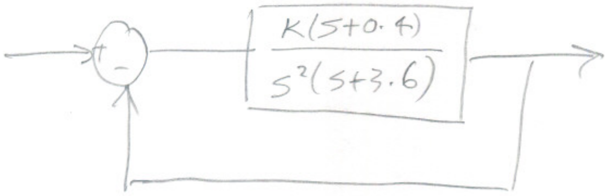
hence we set



HW# 6

Problem A-6-7

ref to



Obtain equation for root Locus branches for the system. Show that loci branches cross real axis at breakaway point at angles ±60°.

answer

since each point on loci must satisfy $\angle G(s) = 180^\circ \pm n360^\circ$

then let $s = \sigma + j\omega$, we set

$$\angle G(\sigma+j\omega) = 180^\circ \pm n360^\circ$$

$$\angle N(s) - \angle D(s) = 180^\circ \pm n360$$

$$\angle K(\sigma+j\omega+0.4) - \angle(\sigma+j\omega)^2(\sigma+j\omega+3.6 = 180^\circ \pm n360^\circ$$

since K is (+ve constant), it has no effect on phase, so

$$\angle \sigma+j\omega+0.4 - \left[\angle \sigma+j\omega + \angle \sigma+j\omega + \angle \sigma+j\omega+3.6\right] = 180 \pm n360$$

$$\tan^{-1}\frac{\omega}{\sigma+0.4} - \left[2\tan^{-1}\left(\frac{\omega}{\sigma}\right) + \tan^{-1}\left(\frac{\omega}{\sigma+3.6}\right)\right] = 180^\circ \pm n360$$

$$\tan^{-1}\frac{\omega}{\sigma+.4} - 2\tan^{-1}\left(\frac{\omega}{\sigma}\right) - \tan^{-1}\left(\frac{\omega}{\sigma+3.6}\right) = 180^\circ \pm n360$$

Let $\tan^{-1}\frac{\omega}{\sigma+.4} = A$

let $\tan^{-1}\frac{\omega}{\sigma} = B$

let $\tan^{-1}\frac{\omega}{\sigma+3.6} = C$

so This can be written as

$$A - 2B - C = 180^\circ \pm n360$$

$$A - B = C + B + 180^\circ \pm n360$$

Let $B + 180^\circ \pm n360 = D$

so $A - B = C + D$

take tan of both sides

$$\tan(A-B) = \tan(C+D)$$

but $\tan(A-B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$, $\tan(C+D) = \frac{\tan C - \tan D}{1-\tan C\tan D}$

hence

$$\frac{\tan A + \tan B}{1-\tan A\tan B} = \frac{\tan C - \tan D}{1-\tan C\tan D} \text{ ——— } ①$$

but $\tan D = \tan(B+180^\circ) = \tan B$ $\quad\because \tan(\alpha+180) = \tan\alpha.$

so ① becomes

$$\frac{\tan A + \tan B}{1-\tan A\tan B} = \frac{\tan C - \tan B}{1-\tan C\tan B} \text{ ——— } ②$$

but $\tan A = \frac{\omega}{\sigma+.4}$, $\tan B = \frac{\omega}{\sigma}$, $\tan C = \frac{\omega}{\sigma+3.6}$

so ② becomes

$$\frac{\frac{\omega}{\sigma+.4} + \frac{\omega}{\sigma}}{1-\frac{\omega}{\sigma+.4}\frac{\omega}{\sigma}} = \frac{\frac{\omega}{\sigma+3.6} - \frac{\omega}{\sigma}}{1-\frac{\omega}{\sigma+3.6}\frac{\omega}{\sigma}} \longrightarrow$$

This Leads to

$$\omega(\sigma^3+2.4\sigma^2+1.44\sigma+1.6\omega^2+\sigma\omega^2)=0$$

so $\boxed{\omega=0 \text{ is a solution.}}$

another solution from

$$\omega^2(\sigma+1.6)+\sigma(\sigma^2+2.4\sigma+1.44)=0$$

using quadratic equation Leads to

$$\boxed{\begin{array}{l} \omega=(\sigma+1.2)\sqrt{\dfrac{-\sigma}{\sigma+1.6}} \\ \text{and, } \omega=-(\sigma+1.2)\sqrt{\dfrac{-\sigma}{\sigma+1.6}} \end{array}}$$

so for $\omega=0$, means imaginary part $=0$. i.e the real axis part of Loci.

in complex plan, equation of Loci is given by (let me write $y=\omega$, $\sigma=x$ to make it more familiar)

$$y=\pm(x+1.2)\sqrt{\frac{-x}{x+1.6}}$$

square both sides, we get

$$y^2=(x+1.2)^2\left(\frac{-x}{x+1.6}\right)$$

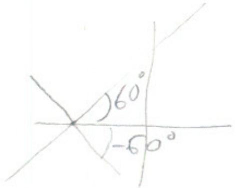
$$\boxed{y^2x+1.6y^2=x^2+2.4x+1.44}$$

this is the equation of Loci in complex plan.

at $x=-1.2$, slope is $\left.\frac{dy}{dx}\right|_{x=-1.2}$

but $\frac{dy}{dx}=\pm\sqrt{\frac{-x}{x+1.6}} \Rightarrow \left.\frac{dy}{dx}\right|_{x=-1.2}=\pm\sqrt{3}$

i.e $\boxed{\text{slope}=\pm 60^\circ}$ so at breakaway we have

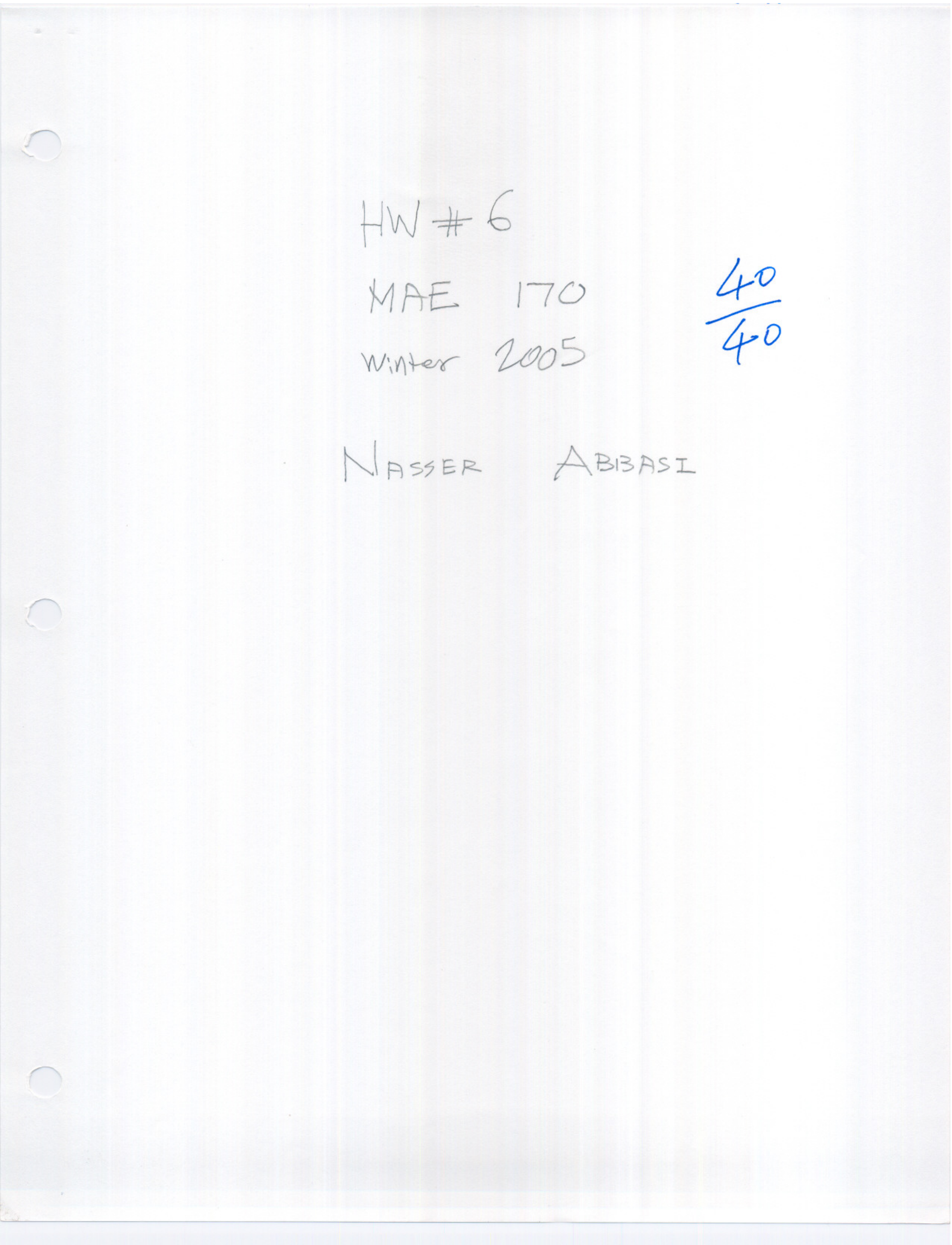


QED

3.6 HW 6

Local contents

3.6.1 my solution



HW # 6

MAE 170

Winter 2005

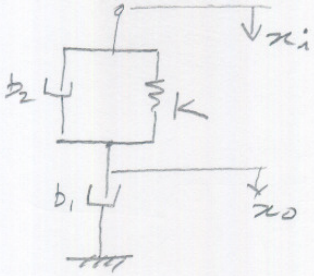
NASSER ABBASI

$\frac{40}{40}$

HW # 6

Problem B-7-1

Consider mechanical system below. obtain transfer function of system. x_i is input, x_o is output. is this a Lead or Lag network?



Solution

write the dynamic equations of system:

$$\boxed{b_2(\dot{x}_i-\dot{x}_o)+K(x_i-x_o)=b_1(\dot{x}_o)}\ ✓$$

take Laplace transform

$$b_2\,s(X_i-X_o)+K(X_i-X_o)=b_1\,sX_o$$

$$X_i\left[b_2s+K\right]=X_o\left[b_1s+b_2s+K\right]$$

so

$$\boxed{\frac{X_o}{X_i}=\frac{b_2s+K}{s(b_1+b_2)+K}}$$

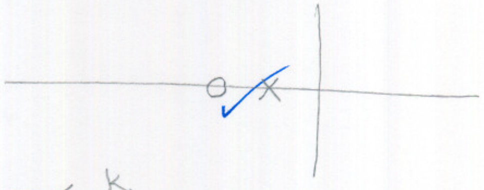
i.e.

$$X_i \longrightarrow \boxed{\frac{b_2s+K}{(b_1+b_2)s+K}} \longrightarrow X_o$$

so poles/zeros of the transfer function are.

Zero: $s=-\frac{K}{b_2}$

Pole: $s=-\frac{K}{(b_1+b_2)}$



Since $b_1+b_2>b_2$, then $\frac{K}{b_1+b_2}<\frac{K}{b_2}$

this means The zero is to the left of the pole $\Longrightarrow$ $\boxed{\text{Lag Network}}$ ✓

HW#6
Problem B-7-3

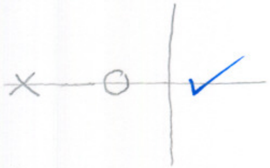
is the following $G_c(s)$ a lead or Lag ?

$$G_c(s) = \frac{3.5s+1.4}{s+2}$$

Answer

$$G_c(s) = \frac{s+\frac{1.4}{3.5}}{s+2} = \frac{s+0.4}{s+2}$$

since zero of $G_c(s)$ is at $s=-0.4$
and pole of $G_c(s)$ is at $s=-2$

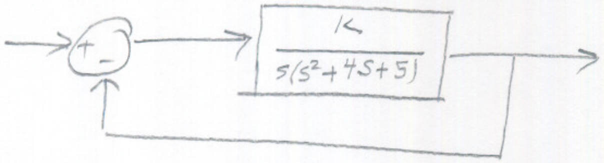


then this is a $\boxed{\text{Lead}}$ ✓ Compensator
(pole to left of zero).

HW#6

Problem B-7-6

Consider system shown, plot loci for system. Determine value for K such that ξ of dominant pole for closed loop is 0.5. Then determine all closed loop poles. plot unit step using Matlab.



Answer

step 1 put pole/zero of open loop

$s = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

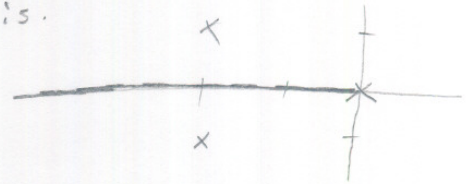
$= \frac{-4 \pm \sqrt{16-4\times5}}{2}$

$= -2 \pm i$

step2 determine loci on real axis. (apply angle condition).

No loci on positive real axis.

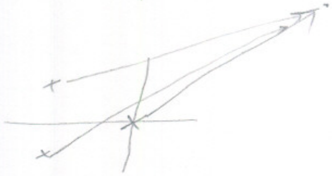
loci on all negative real axis.



✓

step 3 determine asymptotes.

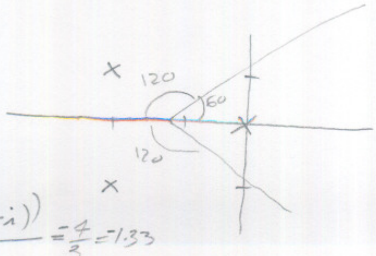
put a test point very far from origin



then $3\theta = 180 \pm n360$

so $\theta = 60^\circ \pm n\,120^\circ$

so



now find where asymptotes meet i.e. σ_a.

$\sigma_a = \frac{\Sigma P - \Sigma Z}{n-m} = \frac{(0+(-2+i)+(-2-i))}{3} = \frac{-4}{3} = -1.33$

→

so we have this:

Step 4 find σ_b (break away) use $\frac{dk}{ds}=0$.

The char. equation for closed loop is

$$K+s(s^2+4s+5)=0$$

so

$$\frac{dK}{ds}=\frac{d}{ds}\left(s^3+4s^2+5s\right)$$

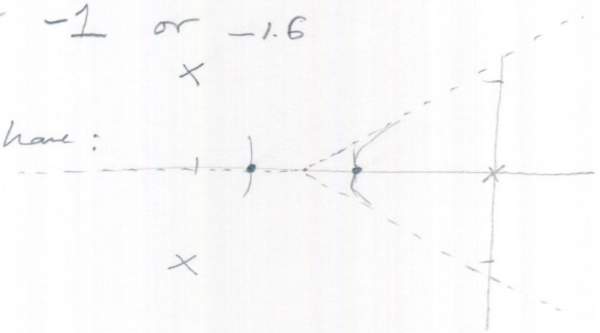
$$=3s^2+8s+5=0$$

so $s=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-8\pm\sqrt{64-4\times3\times5}}{6}=\frac{-8\pm\sqrt{4}}{6}$

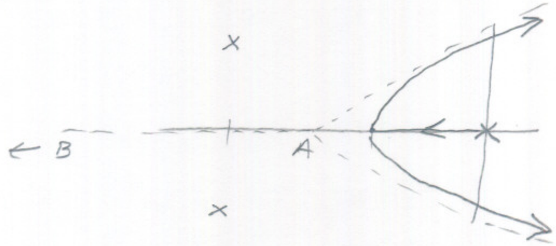
$$=\frac{-8\pm2}{6}=\frac{-8+2}{6} \text{ or } \frac{-8-2}{6}$$

i.e -1 or -1.6

i.e we now have:

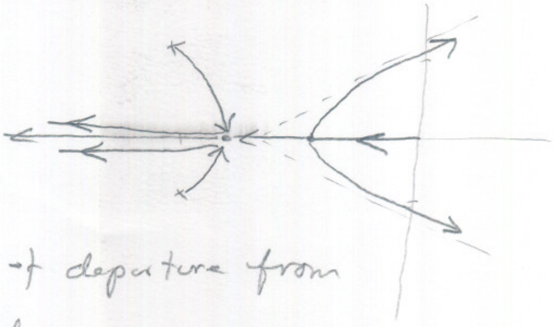


now since a loci starts at a pole ands at a zero, then loci must start at pole at origin. take point -1 as breakaway. we set



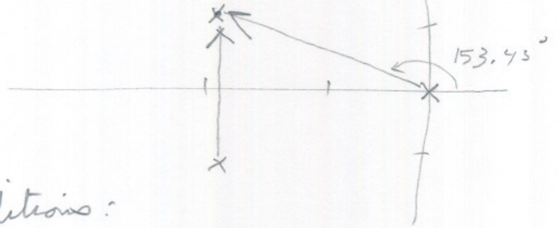
✓

and sine line AB is an asymptotes, then it mean point -1.6 is a break in for branches comming down from conjugate poles i.e →



Step 5 to find angle of departure from the conjugate poles:

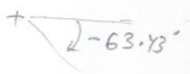
put a small test point close to (−2+i).



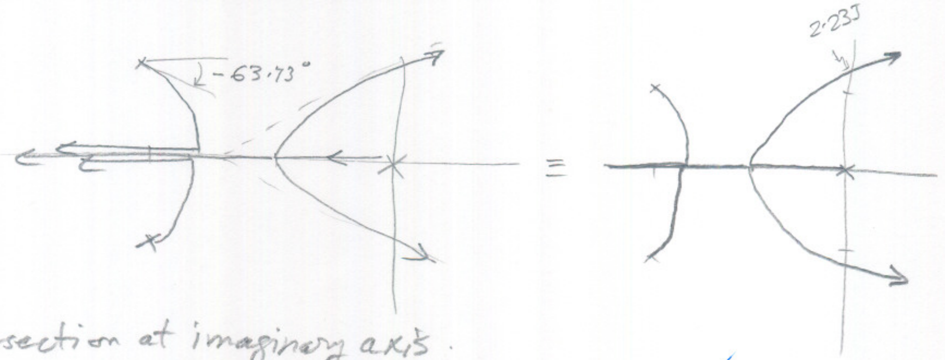
apply angle conditions:

$$153.43^\circ + 90^\circ + \alpha = 180 \pm n360$$

so $\alpha = 180 - 243.4^\circ = -63.43' \pm n360$ i.e



so departure angle is $\boxed{-63.43^\circ}$



Step 6 find intersection at imaginary axis.

characteristic equation is $K + s(s^2+4s+5) = 0$ ✓

so $K + s^3 + 4s^2 + 5s = 0$

or $k + -j\omega^3 - 4\omega^2 + 5j\omega = 0$

so $K - 4\omega^2 + j(-\omega^3 + 5\omega) = 0 \Rightarrow \left.\begin{array}{l} K-4\omega^2=0 \\ -\omega^3+5\omega=0 \end{array}\right\} \Rightarrow \begin{array}{l} K-4\omega^2=0 \\ -\omega^2=-5 \end{array}$

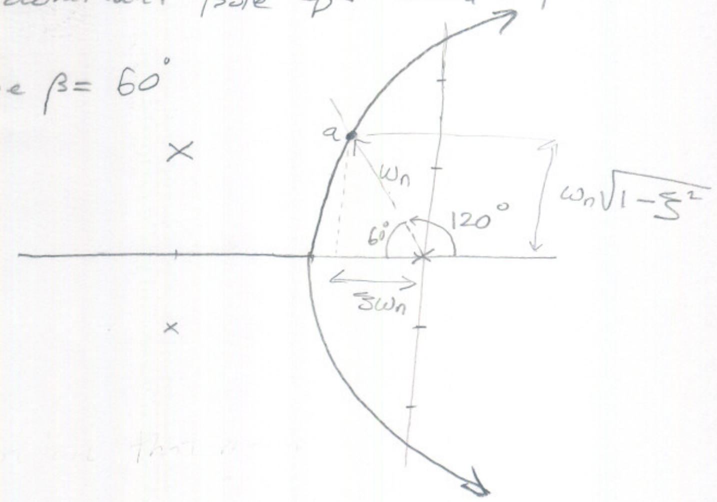
i.e $\omega = \sqrt{5} \Rightarrow$ intersection points $\boxed{\pm 2.236J}$

and K at that point is $\boxed{20}$

now that loci is plotted, complete solution of problem.

find K such that ξ of dominant pole for closed loop is 0.5.

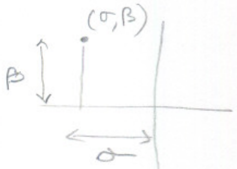
$\xi = 0.5 \Rightarrow \cos\beta = \xi \Rightarrow$ i.e $\beta = 60^\circ$



Closed Loop $\boxed{G(s) = \dfrac{K}{K + s(s^2+4s+5)}}$

The dominant Closed Loop pole by definition is on loci. assume it is at point 'a'. this point makes angle 60° as shown.

roots of $K + s(s^2+4s+5) = 0$ are the Closed Loop poles.



let point 'a' be $(\sigma + \beta i)$

$$K + (\sigma + i\beta)\left((\sigma+i\beta)^2 + 4(\sigma+i\beta) + 5\right) = 0$$

so $$(\sigma + i\beta)\left[\sigma^2 + 2i\sigma\beta - \beta^2 + 4\sigma + 4i\beta + 5\right] = -K$$

$$\sigma^3 + 2i\sigma^2\beta - \sigma\beta^2 + 4\sigma^2 + 4i\sigma\beta + 5\sigma$$
$$+ i\sigma\beta - 2\sigma\beta^2 - i\beta^3 + 4\sigma i\beta - 4\beta^2 + 5i\beta = K.$$

i.e $$\left(\sigma^3 - \sigma\beta^2 + 4\sigma^2 + 5\sigma - 2\sigma\beta^2 - 4\beta^2\right) + i\left(2\sigma^2\beta + 4\sigma\beta + \sigma\beta - \beta^3 + 4\sigma\beta + 5\beta\right) = -K.$$

equate real parts and imaginary parts.

so $$\boxed{\begin{aligned} &\sigma^3 - 3\sigma\beta^2 - 4\beta^2 + 4\sigma^2 + 5\sigma = -K \\ &2\sigma^2\beta + 9\sigma\beta - \beta^3 + 5\beta = 0 \end{aligned}}$$

This is 2 eqn, but 3 unknowns. now using $\tan 60^\circ = \frac{\beta}{\sigma}$ gives 3rd eq. →

so we have

$$\left.\begin{array}{l}\sigma^3-3\sigma\beta^2-4\beta^2+4\sigma^2+5\sigma=-K \\ 2\sigma^2\beta+9\sigma\beta-\beta^3+5\beta=0 \\ 1.732=\frac{\beta}{\sigma}\end{array}\right\}\begin{array}{l}\text{---} ① \\ \text{---} ② \\ \text{---} ③\end{array}$$

3 equations, 3 unknowns. now we can find K.

$\beta = 1.732\sigma$

sub in eq ②

$$2\sigma^2(1.732\sigma)+9\sigma(1.732\sigma)-(1.732\sigma)^3+5(1.732\sigma)=0$$

so $3.464\sigma^3+15.588\sigma^2-5.195\sigma^3+8.66\sigma=0$

or $-1.731\sigma^3+15.5886\sigma^2+8.66\sigma=0$

or $-1.731\sigma^2+15.5886\sigma+8.66=0$

so

$$\sigma=\frac{-15.588\pm\sqrt{15.588^2-4\times(-1.731)\times 8.66}}{2\times(-1.731)}=\frac{-15.588\pm 17.4}{-3.462}$$

$$=\frac{-15.588+17.4}{-3.462} \text{ or } \frac{-15.588-17.4}{-3.462}$$

$= -0.523$ or 9.5 ⇒ since pole is on negative plan ⇒ $\boxed{\sigma=-0.52}$

so $\beta = (1.732)(-0.523) = \boxed{-0.9058}$

from ① we find K:

$$(-0.523)^3-3(-0.523)(-0.9058)^2-4(-0.9058)^2+4(-0.523)^2+5(-0.523)=-K$$

$$\Longrightarrow \boxed{K=3.67} ✓$$

now to find all closed loop poles. i.e. solutions of Char eq

$$\boxed{3.67+s(s^2+4s+5)=0}$$

⟶

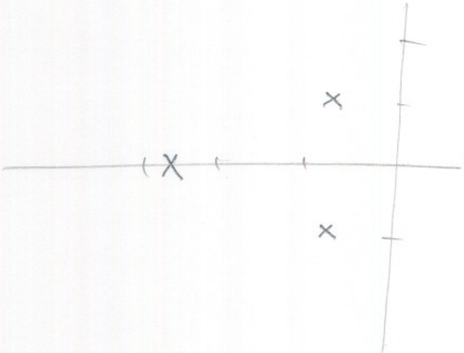
now to find poles of closed loop system, solved char eq.

$$3.67 + s(s^2+4s+5) = 0 .$$

solutions are

$$\boxed{\begin{aligned} s &= -2.629 \\ s &= -0.685 - 0.962\, i \ \checkmark \\ s &= -0.685 + 0.962\, i \ \checkmark \end{aligned}}$$

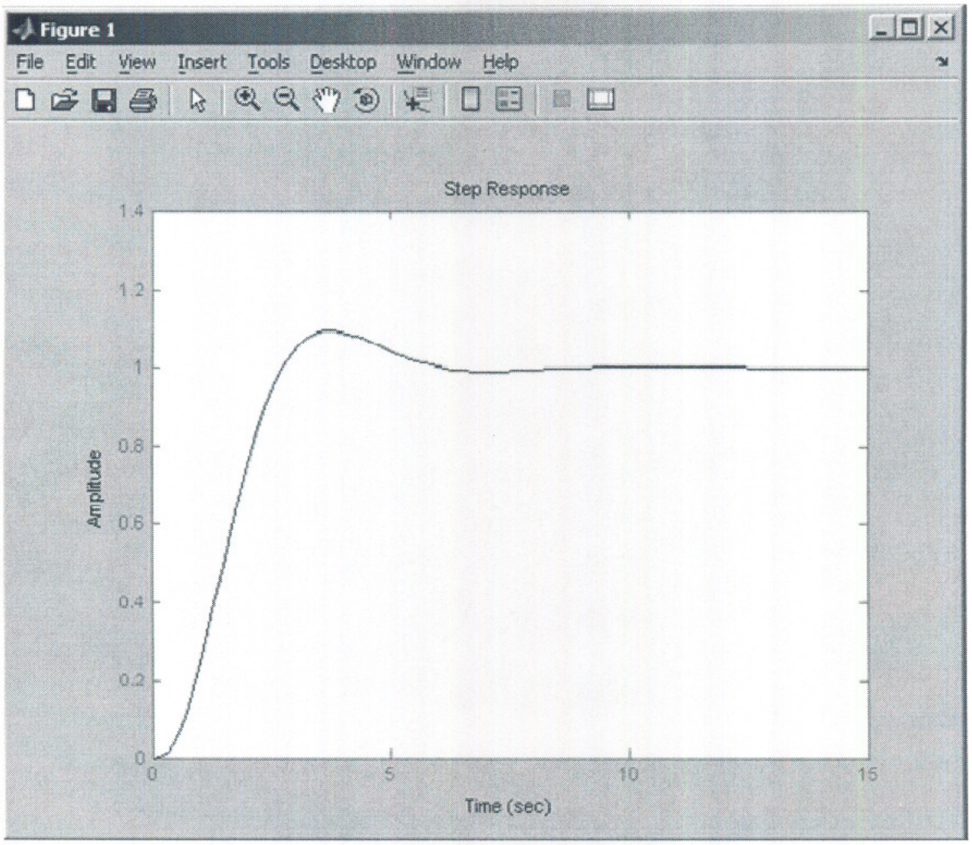
these are the closed loop poles



now I plot using matlab the step response:

✓

```
s=tf('s');
k=3.67
G=k/(s*(s^2+4*s+5));
sys=G/(1+G);
step(sys)
```

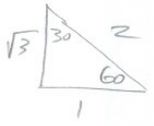


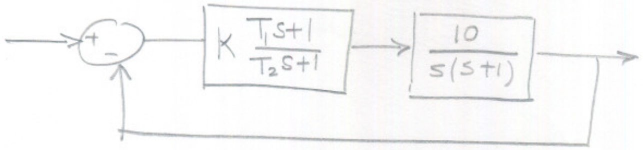
✓

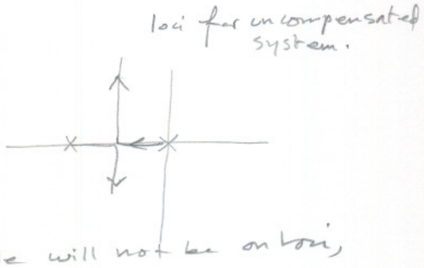
HW#6

Problem B-7-7

Determine the values of K, T_1, T_2 of system shown so that the dominant closed-loop poles have $\xi = 0.5$ and the undamped $\omega_n = 3$ rad/sec.







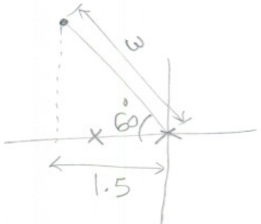
Solution

Poles/Zeros for uncompensated system is

$\sigma_b = \frac{-1}{2} = -.5$.

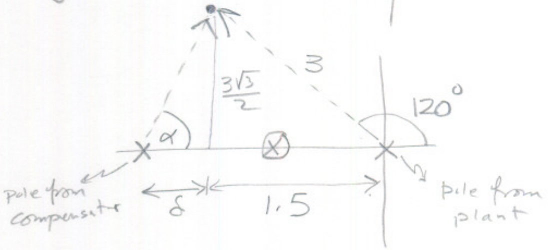
we see that for uncompensated system, closed pole will not be on loci,

we want closed loop pole to be at:



so need the compensator.

if we put the zero of the compensator on top of the pole at −1, we get



now apply angle conditions:

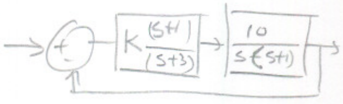
$120 + \alpha = 180 \pm n360$

$\Rightarrow \boxed{\alpha = 60^\circ}$

so $\tan 60 = \frac{3\sqrt{3}/2}{\delta} \Rightarrow \delta = \frac{3\sqrt{3}/2}{\sqrt{3}} = \frac{3}{2} = 1.5$

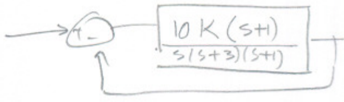
so pole from compensator at $\boxed{-3}$.

so $\boxed{G_c(s) = K\frac{(s+1)}{(s+3)}}$



Now apply Magnitude Conditions. need Char. equation

so



$\Rightarrow \left| K\frac{(s+1)}{(s+3)}\frac{10}{s(s+1)} \right| = 1$

let $s = \left(-1.5 + \frac{3\sqrt{3}}{2}j\right)$. The location of one dominant closed Loop pole.

now we find K ⟶

$$\left| \frac{10K\left(-1.5+j\frac{3\sqrt{3}}{2}+1\right)}{\left(-1.5+j\frac{3\sqrt{3}}{2}+3\right)\left(-1.5+j\frac{3\sqrt{3}}{2}\right)\left(-1.5+j\frac{3\sqrt{3}}{2}+1\right)} \right| = 1$$

$$\left| \frac{10K\cancel{\left(-0.5+j\frac{3\sqrt{3}}{2}\right)}}{\left(1.5+j\frac{3\sqrt{3}}{2}\right)\left(-1.5+j\frac{3\sqrt{3}}{2}\right)\cancel{\left(-.5+j\frac{3\sqrt{3}}{2}\right)}} \right| = 1$$

$$\left| \frac{10K}{-1.5^2 - \frac{9(3)}{4}} \right| = 1$$

$$\left| \frac{10K}{2.25 - 6.75} \right| = 1 \Rightarrow \left| \frac{10K}{-4.5} \right| = 1$$

so $\frac{10K}{4.5} = 1 \Rightarrow \boxed{K = \frac{4.5}{10} = 0.45}$

so $\boxed{G_c(s) = 0.45 \frac{(s+1)}{(s+3)}}$ compare to $\boxed{K \frac{T_1 s + 1}{T_2 s + 1}}$

so $\frac{0.45s + 0.45}{s+3} = \frac{KT_1 s + K}{T_2 s + 1}$

multiply, and equate coefficents of s^2, s, k, we get

$$\boxed{\begin{aligned} 0.45T_2 &= T_1 K \\ .45 + T_2 0.45 &= 3T_1 K + K \\ .45 &= 3K \end{aligned}}$$

so $\boxed{K = .15} \Rightarrow \boxed{\begin{aligned} .45 + .45T_2 &= .45T_1 + .15 \\ 0.45T_2 &= .15T_1 \end{aligned}}$

so $T_1 = 3T_2 \Rightarrow .45 + .45T_2 = .45(3T_2) + .15 \Rightarrow .45 + .45T_2 - 1.35T_2 - .15 = 0$

so $T_2(.45 - 1.35) = .15 - .45 \Rightarrow T_2 = \frac{-0.3}{-0.9} = \boxed{\frac{1}{3}}$

so $T_1 = 3\left(\frac{1}{3}\right) = \boxed{1} \Rightarrow \boxed{G_c(s) = 0.15 \frac{(1s+1)}{(\frac{1}{3}s+1)}} \longrightarrow \boxed{\begin{aligned} K &= 0.15 \\ T_1 &= 1 \\ T_2 &= \tfrac{1}{3} \end{aligned}}$ ✓

3.6.1.1 problem B 6.3

HW 1, MAE 170.
Problem B 6-1, Modern Control Engineering, 4th edition by Ogata
by Nasser Abbasi
UCI, Winter 2005.

Question

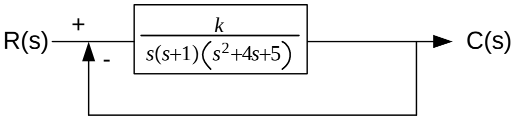
Plot the root loci for $G(s) = \frac{k}{s(s+1)(s^2+4s+5)}$

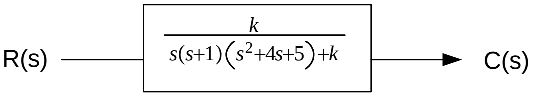
Solution

The closed loop transfer function is

$$G_{cl}(s) = \frac{G}{1+G} = \frac{\frac{k}{s(s+1)(s^2+4s+5)}}{1+\frac{k}{s(s+1)(s^2+4s+5)}} = \frac{k}{s(s+1)(s^2+4s+5)+k}$$

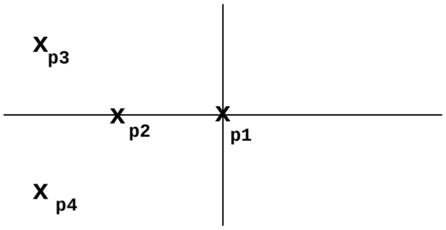
Hence the system is





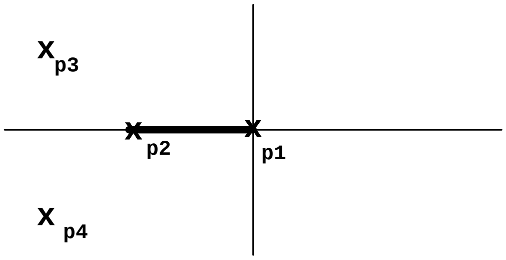
step 1

Plot the open loop poles and zeros:
Poles are at $s=0, s=-1$, and roots of (s^2+4s+5) which is $s=\frac{-4\pm\sqrt{16-4\times5}}{2}=\frac{-4\pm2i}{2}=-2\pm i$
There are no finite zeros (there are however 4 zeros at ∞)



step 2

Apply angle condition to find initial segments on real axis that could be part of the root loci.
We see that on the RHP, it is not possible to have loci, since then the sum of angles to a text point will not add to $180 \pm n360$. For segments between p1 and p2, we will have a loci. for the segment to the right of p2, it is not possible to have a loci. Hence we now get this plot: (I use bold line to show where loci is)

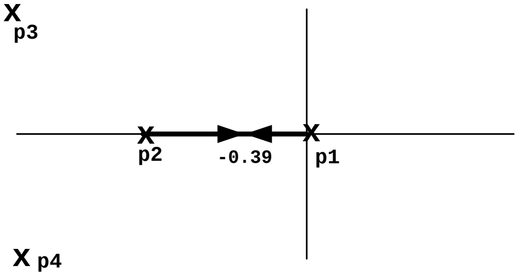


step 3

Since loci starts at a pole and ends at a zero of the open loop poles/zeros, then we know loci must start at p1 as well it must start at p2. Hence there must be a break away point between p1 and p2. Now we find this break away point. For this we use the condition that $\frac{dk}{ds}=0$

The characteristic equation for the close loop is $s\left(s+1\right)\left(s^{2}+4s+5\right)+k=0$ hence $\frac{dk}{ds}=\frac{d}{ds}\left(s\left(s+1\right)\left(s^{2}+4s+5\right)\right)=\frac{d}{ds}\ \left(5s+9s^{2}+5s^{3}+s^{4}\right)=5+18s+15s^{2}+4s^{3}\ =0$, Solution is: $\left[s=-1.\,678\,5-0.602\,78i\right],\left[s=-1.\,678\,5+0.602\,78i\right],\left[s=-0.392\,99\right]$

Since we are looking for a solution on the real axis, and one that is between p1 and p2, hence only possible solution is $s=-0.392\,99$, I add this point to the diagram, now it looks as follows



step 4

Now I need to find where the asymptotes lines cross at the real axis, and need to find angles that asymptotes leave the real axis at.

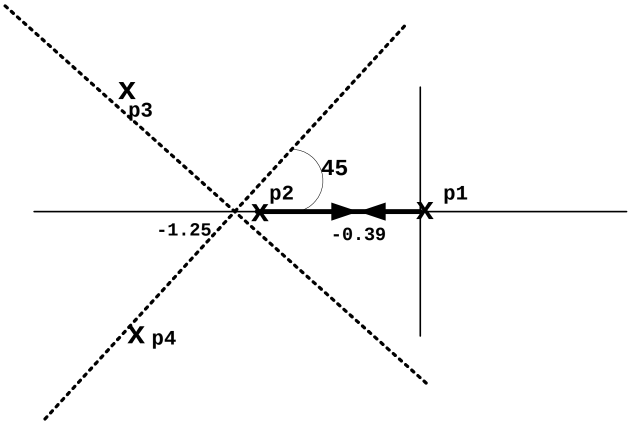
to find where asymptotes meet at the real axis:

$\sigma_{a}=\frac{\sum_{m}z_{i}-\sum_{n}p_{i}}{n-m}=\frac{0-(0+1+(2-i)+(2+i))}{4}=\frac{-(1+4)}{4}=-1.25$

To find angles, put a test point s very far away, and consider the sum of angles from the finite poles and zeros to that test point. we have here only 4 finite poles and no finite zeros. We know that sums of these angles must be 180 degrees, so we write

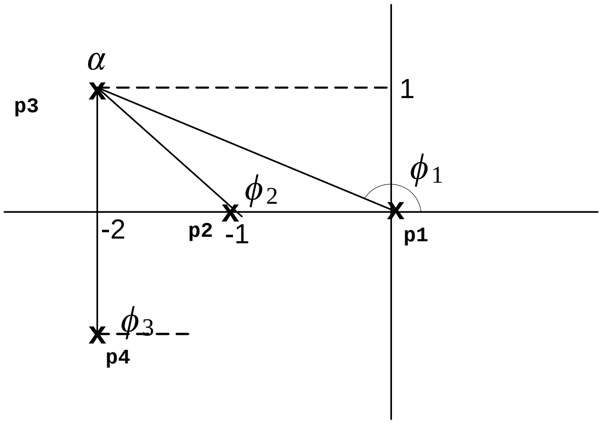
$$\begin{aligned} 4\theta &= 180\pm n360 \\ \theta &= 45\pm n90 \end{aligned}$$

Hence the angles are $45,45+90,45+180,45+270$ or $45,135,225,315$ so now I get this diagram, where I just added the asymptotes lines



step 5

Now I need to find angles of departures of loci from p3 and p4. To do this, put a test point s very close to p3 and solve for the angle conditions. a little bit of geometry is needed here. We get



Hence, for a test point 's' near p3, we get from the angle condition, the following

$$\theta_1 + \theta_2 + \theta_3 + \alpha = 180 \pm n360$$

but

$\theta_3 = 90^0$

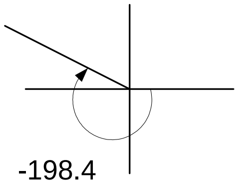
$\theta_2 = 90^0 + \tan^{-1}(1/1) = 135^0$

$\theta_1 = 90^0 + \tan^{-1}(2/1) = 90^0 + 63.4^0 = 153.4^0$

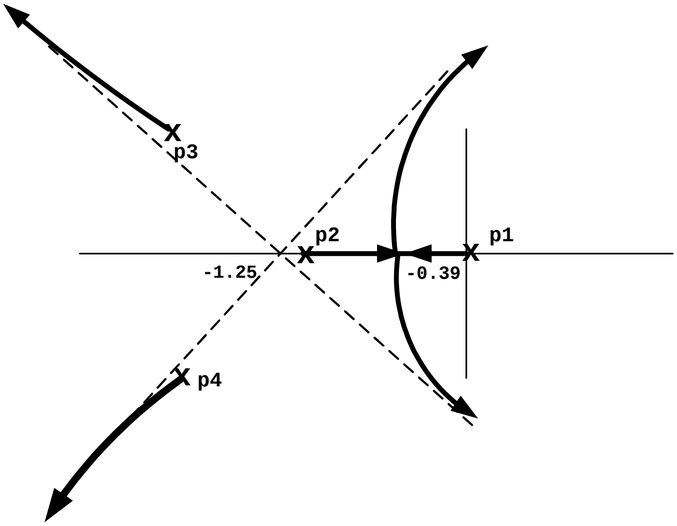
hence

$$\begin{aligned} \alpha + 90 + 135 + 153.4 &= 180^0 \pm n360 \\ \alpha + 378.4 &= 180^0 \pm n360 \\ \alpha &= 180^0 \pm n360 - 378.4 \\ &= -198.4 \pm n360 \end{aligned}$$

Since angles are positive anticlockwise, then this angle of -198.4 means this:



Hence, this is the angle of departures of p3. By symmetry we know the angle of departure of p4. Hence the plot now looks like this:



Note: at the break away point, -0.39, the loci is at 90^0 from the real axis.

step 6

Find where loci cross the j axis.

Looking at the charaterstic equation for the close loop $s(s+1)(s^2+4s+5)+k$, set $s=j\omega$ and solve

$$s(s+1)\left(s^2+4s+5\right)+k=0$$

Hence

$$\begin{aligned}\left(s^2+s\right)\left(s^2+4s+5\right)+k &= 0\\ 5s+9s^2+5s^3+s^4+k &= 0\end{aligned}$$

Let $s=j\omega$

$$5j\omega-9\omega^2-5j\omega^3+\omega^4+k=0$$

Hence equating real parts and imaginary parts, we get

$$\begin{aligned}5\omega-5\omega^3 &= 0\\ -9\omega^2+\omega^4+k &= 0\end{aligned}$$

From first equation, we get $1-\omega^2=0$ or $\omega=\ \pm1$

Hence the loci crosses the imaginary axis at $\pm i$

To find the gain k at these point, using the second equation, we get

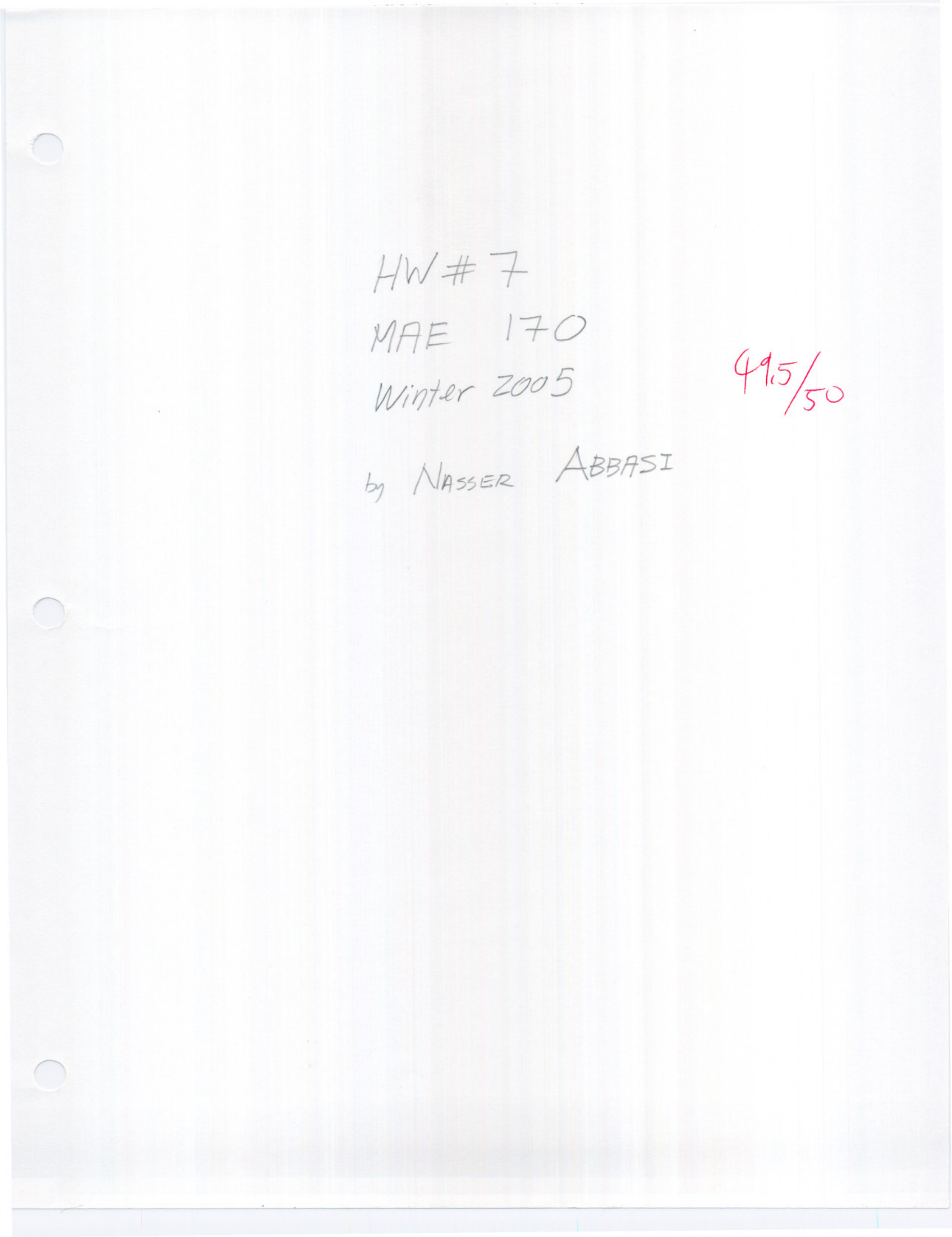
$$\begin{aligned} -9i^2+i^4+k &= 0 \\ +9+1 &= -k \end{aligned}$$

hence gain is 10 (negative gain) where it cross the point $\boxed{(0,i)}$ by symmetry, the gain will be 10 (positive gain) where it cross at point $\boxed{(0,-i)}$

3.7 HW 7

Local contents

3.7.1 my solution



HW # 7

MAE 170

Winter 2005

by Nasser Abbasi

49.5/50

HW#7

Problem B-8-2

Consider system whose closed loop TF is $\frac{C(s)}{R(s)} = \frac{K(T_2 s+1)}{T_1 s+1}$

Obtain steady state output of the system when it is subjected to input $r(t) = R \sin \omega t$.

solution

let $G(s) = \frac{C(s)}{R(s)}$

since input is sinosidal, and system is LTI, then output is $R \left| G(s) \right|_{s=j\omega} \sin(\omega t + \phi)$

where $\phi = \angle G(s)|_{s=j\omega}$

$$G(s)|_{s=j\omega} = \frac{K(T_2 j\omega + 1)}{T_1 j\omega + 1} = \frac{j(KT_2\omega) + K}{j(T_1\omega) + 1}$$

$$\text{so } |G(s)|_{s=j\omega} = \frac{\sqrt{(KT_2\omega)^2 + K^2}}{\sqrt{T_1^2\omega^2 + 1}} = \frac{K\sqrt{T_2^2\omega^2+1}}{\sqrt{T_1^2\omega^2+1}}$$

$$\angle G(j\omega) = \tan^{-1}\frac{KT_2\omega}{K} - \tan^{-1} T_1\omega$$

$$= \tan^{-1} T_2\omega - \tan^{-1} T_1\omega .$$

$$\text{so } \boxed{c(\infty) = R \frac{K\sqrt{T_2^2\omega^2+1}}{\sqrt{T_1^2\omega^2+1}} \sin\left(\omega t + \left(\tan^{-1} T_2\omega - \tan^{-1} T_1\omega\right)\right)}$$

10

The above is the answer.

side question why can't I seem to be able to solve above using Final value theorm? what Am I doing wrong below?

$C(s) = X(s)\,G(s)$. but $X(s) = R \frac{\omega}{s^2+\omega^2}$.

hence $C(s) = \frac{R\omega}{(s^2+\omega^2)} \cdot \frac{K(T_2 s+1)}{(T_1 s+1)} \Rightarrow C(\infty) = \lim_{s\to 0} S\, C(s)$

i.e $C(\infty) = \lim_{s\to 0} S . \frac{R\omega\, K(T_2 s+1)}{(s^2+\omega^2)(T_1 s+1)} = \lim_{s\to 0} \frac{s^2 R\omega K T_2 + S R\omega K}{T_1 s^3 + s^2 + T_1\omega^2 s + \omega^2}$

$= \lim_{s\to 0} \frac{R\omega K T_2 + \frac{R\omega K}{s}}{T_1 s + 1 + T_1\omega^2/s + \omega^2/s} \longrightarrow \frac{R\omega K T_2}{\infty} = 0$?? how is this equivalent to (no oscillation) ?

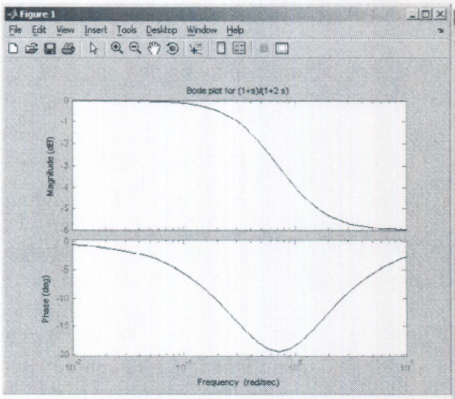
see pg. 25
FVT only applies when f(t) settles down to a definite value for $t \to \infty$

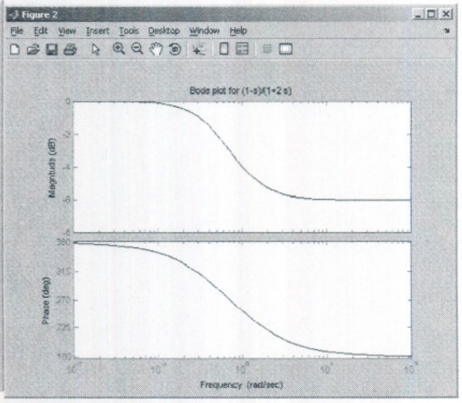
```
close all;
clear all;
%problem B-8-3, by Nasser Abbasi

s=tf('s');
sys=(1+s)/(1+2*s);
bode(sys);
title('Bode plot for (1+s)/(1+2 s)');

sys=(1-s)/(1+2*s);
figure;
bode(sys);
title('Bode plot for (1-s)/(1+2 s)');
```

10





$$\frac{1+s}{1+2s} \qquad \frac{1-s}{1+2s}$$

HW#7

Problem B-8-4

Sketch Bode diagram of

a) $G(s) = \frac{T_1 s+1}{T_2 s+1}$ $T_1 > T_2 > 0$

b) $G(s) = \frac{T_1 s-1}{T_2 s+1}$ $T_1 > T_2 > 0$

c) $G(s) = \frac{-T_1 s+1}{T_2 s+1}$ $T_1 > T_2 > 0$

note:
after doing this I see that $|G(s)|$ for parts a,b,c is the same only phase diagrams changed. I should have taken advantage of this to reduce work needed.

yup

9.5

Answer

a) let $G(s) = G_1(s)\,G_2(s)$ where $G_1(s) = T_1 s+1$, $G_2(s) = \frac{1}{T_2 s+1}$

now plot bode plot for G_1 and G_2 separatly and Combine plots.

G_1 $T_1 S+1$, then $G_1(j\omega) = T_1 j\omega + 1$, $|G_1(j\omega)| = \sqrt{(T_1\omega)^2+1}$

so $\log|G_1(j\omega)| = \frac{1}{2}\log\left(1+(T_1\omega)^2\right)$

For small ω, $\log|G_1| \to \frac{1}{2}\log(1) = 0$

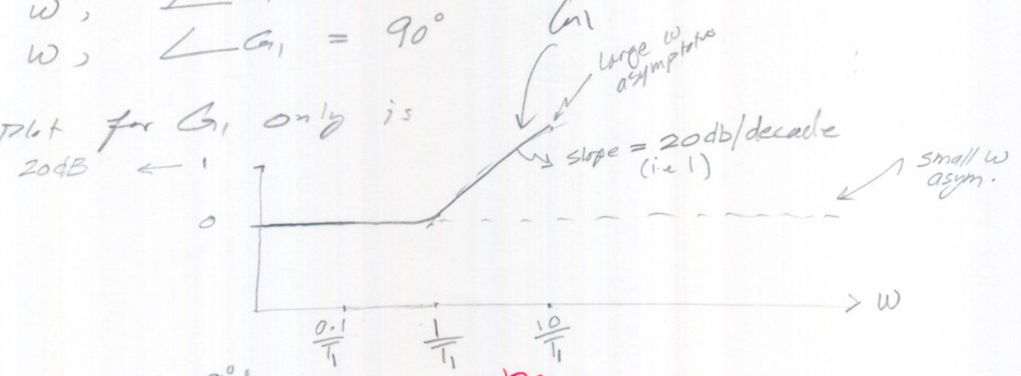
for Larg ω, $\log|G_1| \to \frac{1}{2}\log(T\omega)^2 = \log T\omega$

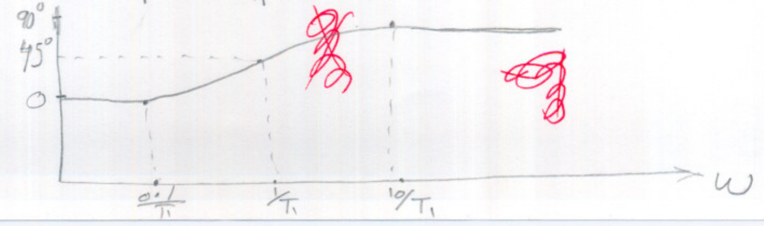
$\angle G_1 = \tan^{-1} T_1\omega$.

For small ω, $\angle G_1 = 0°$

for Large ω, $\angle G_1 = 90°$

So Bode plot for G_1 only is





next I do bode for G_2

G_2 $\quad \frac{1}{T_2 s+1}$ so $G_2(j\omega) = \frac{1}{T_2 j\omega+1}$

$|G_2| = \frac{1}{\sqrt{1+(T_2\omega)^2}} \rightarrow \log|G_2| = \log(1) - \log\sqrt{1+(T_2\omega)^2}$

$= 0 - \frac{1}{2}\log\left(1+(T_2\omega)^2\right)$

for small ω, $\log|G_2| \rightarrow -\frac{1}{2}\log(1) = 0$

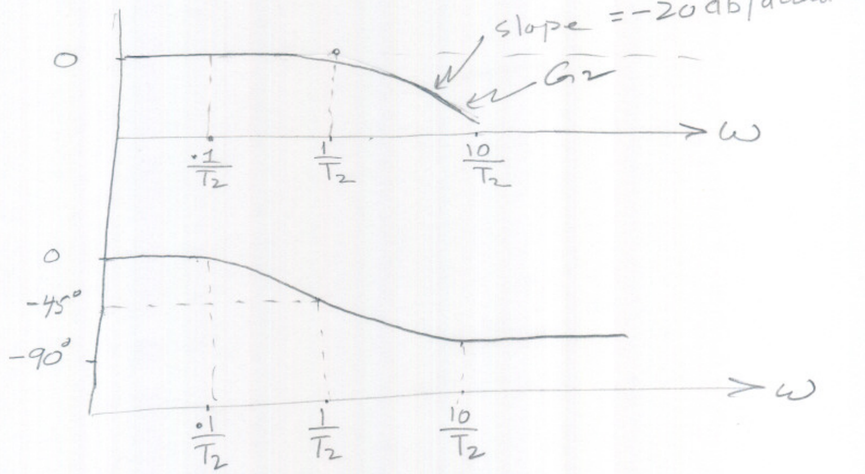
for Large ω, $\log|G_2| \rightarrow -\log T_2\omega$

$\angle G_2(j\omega) = 0 - \tan^{-1} T_2\omega$

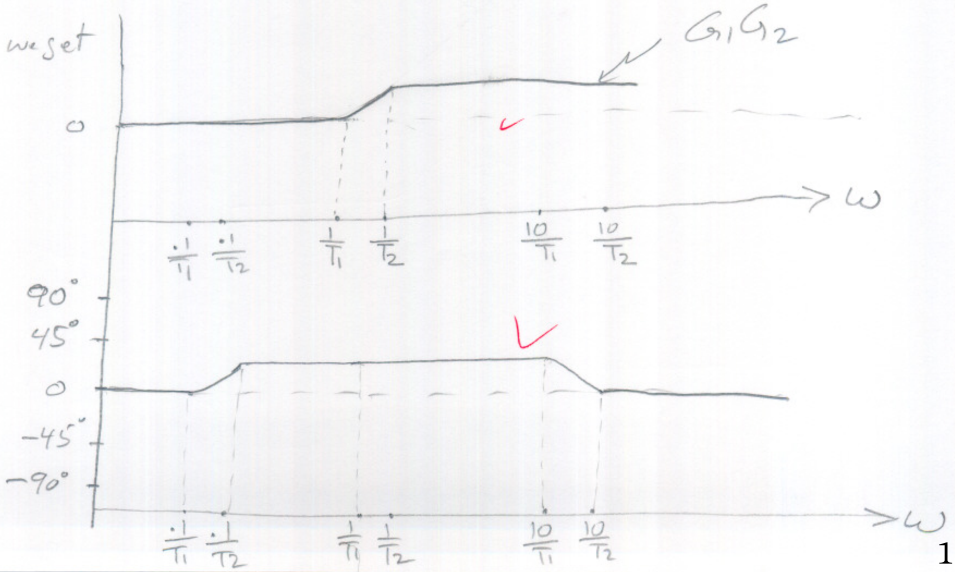
for small ω, $\angle G_2 = 0 - \tan^{-1} 0 = 0$

for Large ω, $\angle G_2 = 0 - \tan^{-1}\infty = -90^\circ$

G_2



so now add G_1, G_2 we get



Part b

$G(s) = \frac{T_1 s - 1}{T_2 s + 1}$ $T_1 > T_2 > 0$

let $G(s) = G_1 G_2$ where $G_1 = T_1 s - 1$, $G_2 = \frac{1}{T_2 s + 1}$

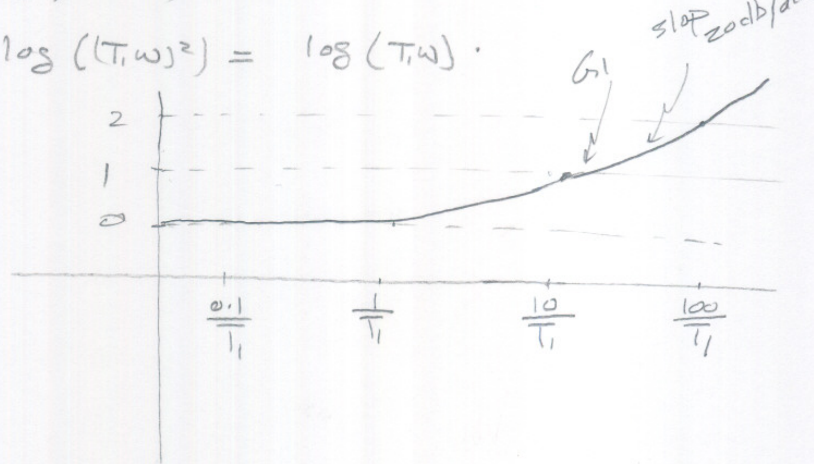
G_1

$G_1(j\omega) = T_1 j\omega - 1$

$|G_1(j\omega)| = \sqrt{(T_1\omega)^2 + 1}$, $\log|G(j\omega)| = \frac{1}{2}\log((T_1\omega)^2 + 1)$.

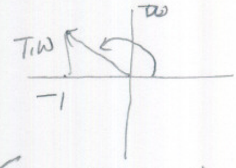
small ω, $\log|G(j\omega)| \to \frac{1}{2}\log((\cancel{T_1}\omega)^2 + 1) = 0$

large ω, $\log|G_1(j\omega)| \to \frac{1}{2}\log((T_1\omega)^2) = \log(T_1\omega)$.



To Find Phase:

$\angle G_1 = \tan^{-1}\frac{T_1\omega}{-1}$

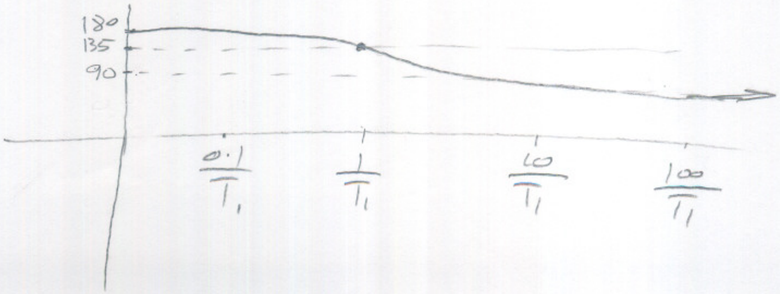


so for small ω, we see that $\angle G_1$ goes to 180°

and for Large ω, we see that $\angle G_1$ goes to 90°

when $\omega = \frac{1}{T_1}$, $\angle G_1 = \tan^{-1} -1$ i.e 135°

when $\omega = \frac{10}{T_1}$, $\angle G_1 = \tan^{-1} -10$, etc.



next I do G_2 →

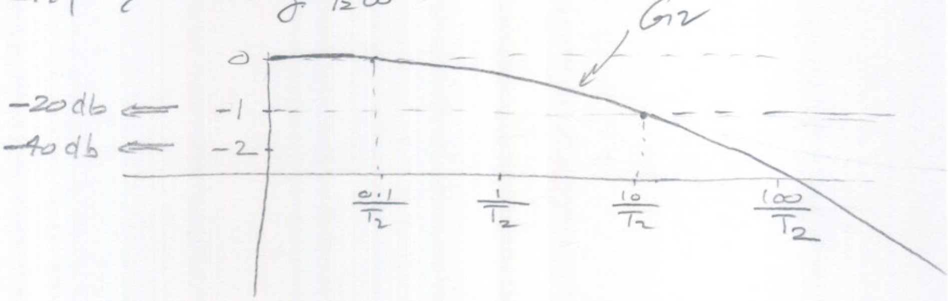
$$G_2 = \frac{1}{T_2 s+1}$$

$$G_2(j\omega) = \frac{1}{T_2 j\omega+1} \qquad |G_2(j\omega)| = \frac{1}{\sqrt{(T_2\omega)^2+1}}$$

$$\log|G_2| = 0 - \log\sqrt{(T_2\omega)^2+1} = -\frac{1}{2}\log\left((T_2\omega)^2+1\right)$$

For small ω, $\log|G_2| \to 0$

For Large ω, $\log|G_2| \to -\log T_2\omega$.

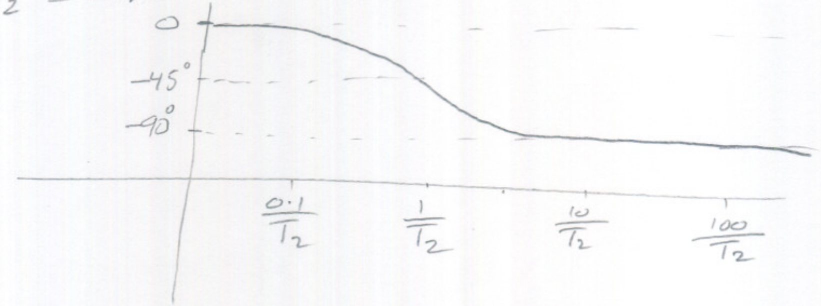


For phase

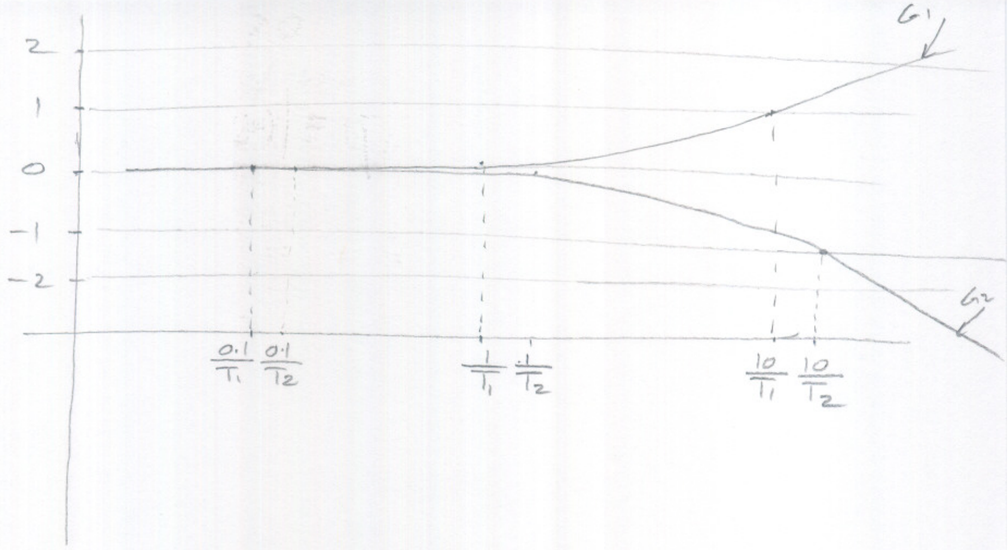
$$\angle G_2 = 0 - \tan^{-1}(T_2\omega) = -\tan^{-1}T_2\omega \, .$$

For small ω, $\angle G_2 = 0$

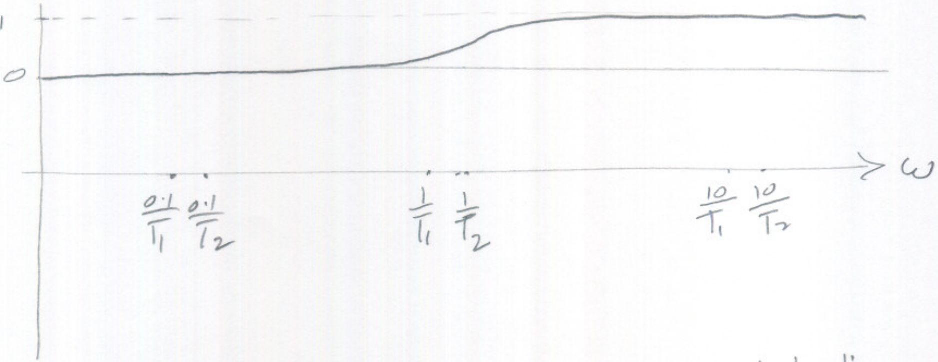
for Large ω, $\angle G_2 = -90°$



so now I put G_1, G_2 together ⟹



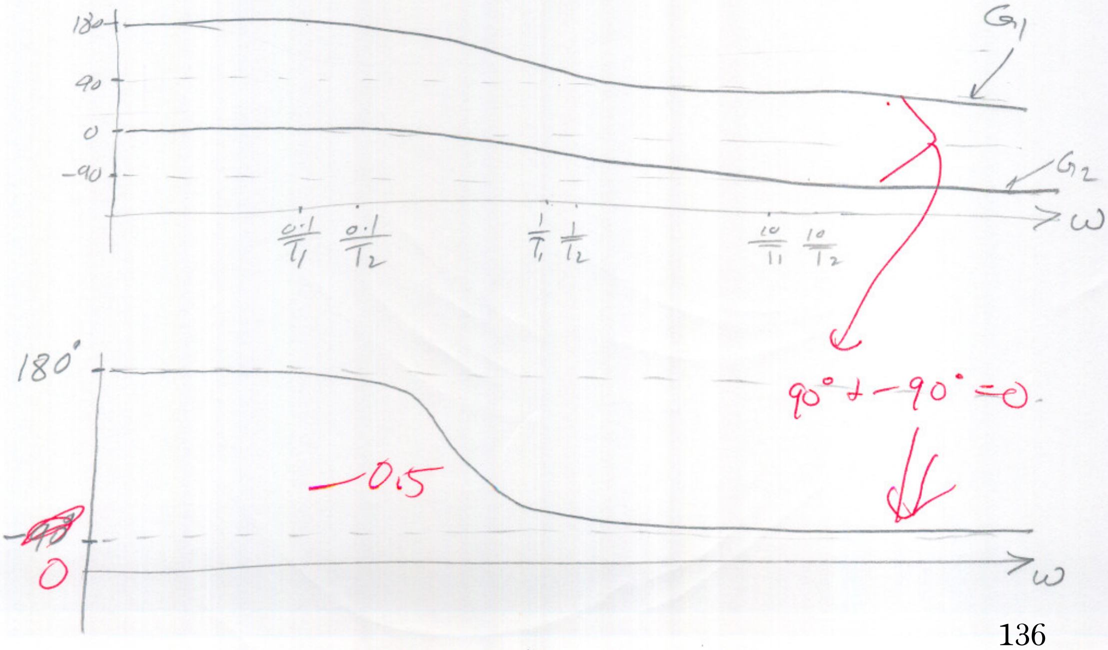
Combine ⇒



at large ω, G_1 slope cancell G_2, we get straight line.

at low ω, G_1 rises befor G_2 since $T_1 > T_2$.

for phase:



Combine

part c

$$G(s) = \frac{-T_1 s+1}{T_2 s+1} \qquad T_1 > T_2 > 0$$

let $G_1(s) = -T_1 s+1$, $\quad G_2(s) = \frac{1}{T_2 s+1}$

now do Bode plot for each G separatly and combine results.

G1

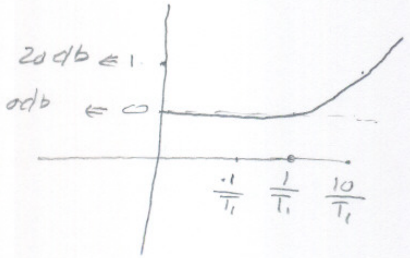
$G_1(j\omega) = -T_1 j\omega + 1$

$|G_1| = \sqrt{(T_1\omega)^2+1}$

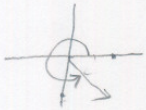
$\log|G_1| = \frac{1}{2}\log[(T_1\omega)^2+1]$

for Large ω, $\log|G_1| \rightarrow \log(T_1\omega)$

for small ω, $\log|G_1| \rightarrow 0$

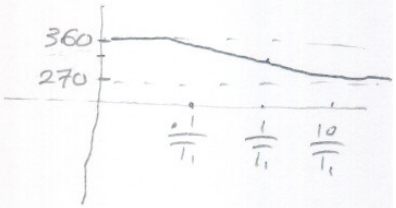


$\angle G_1 = \tan^{-1} -T_1\omega$



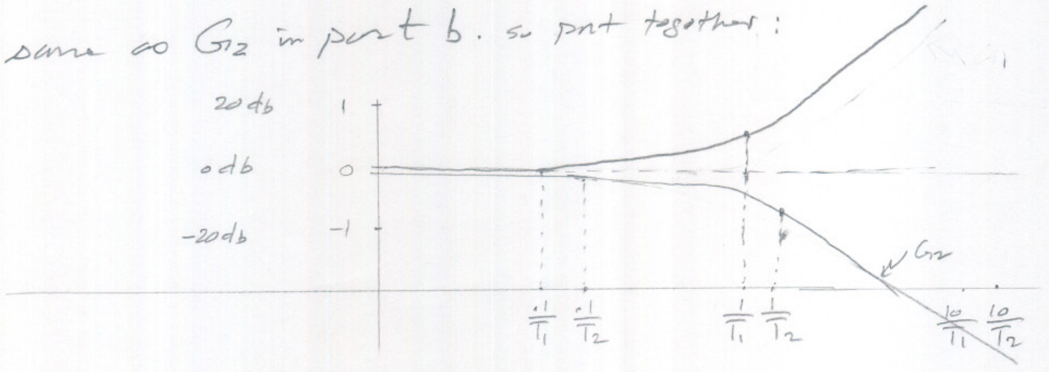
so for small ω, $\angle G_1 = 360°$ (could also say 0)

for Large ω, $\angle G_1 = 270°$ (could also say -90)



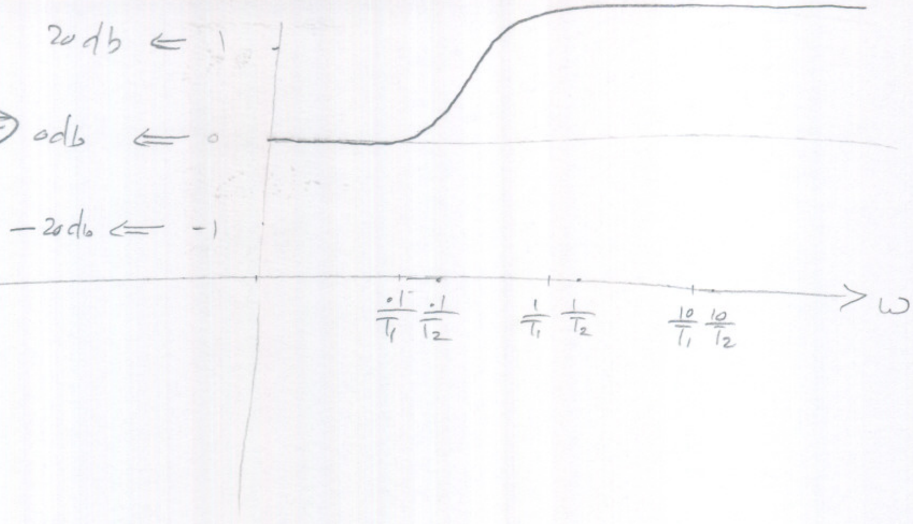
Now do G2

This is the same as G2 in part b. so put together:



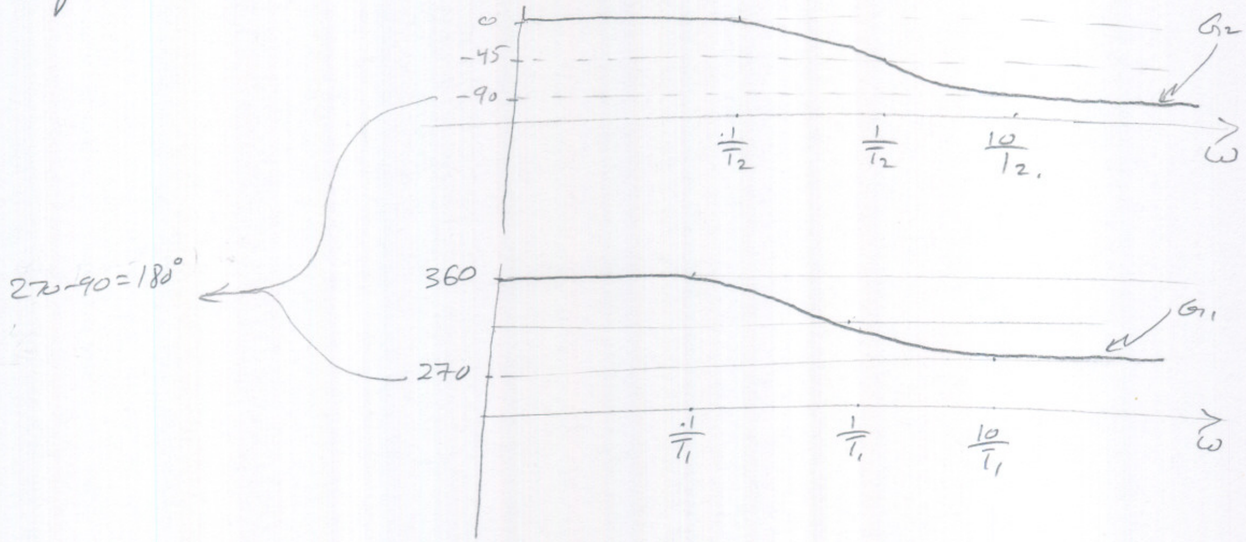
Combine, we get →

$G_1(s)=\dfrac{-T_1s+1}{T_2s+1} \Rightarrow$

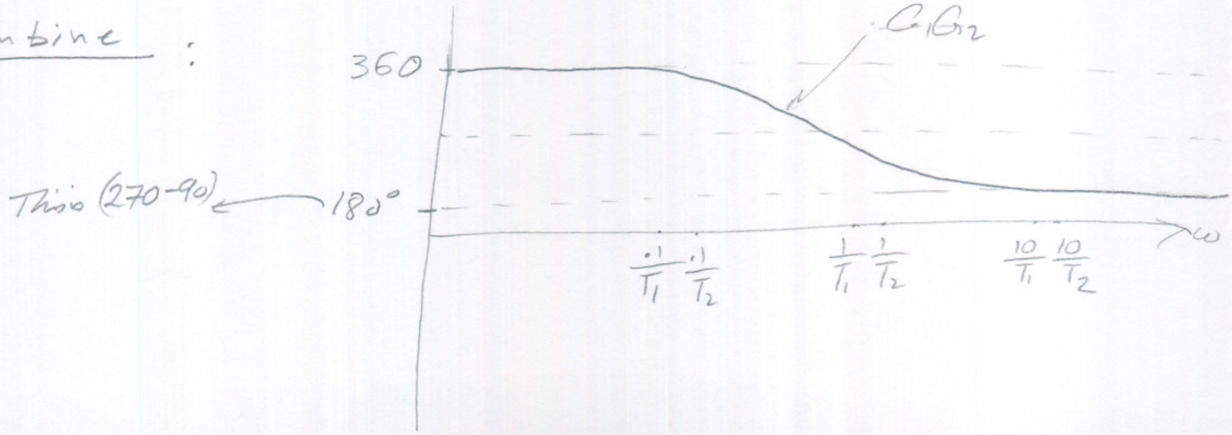


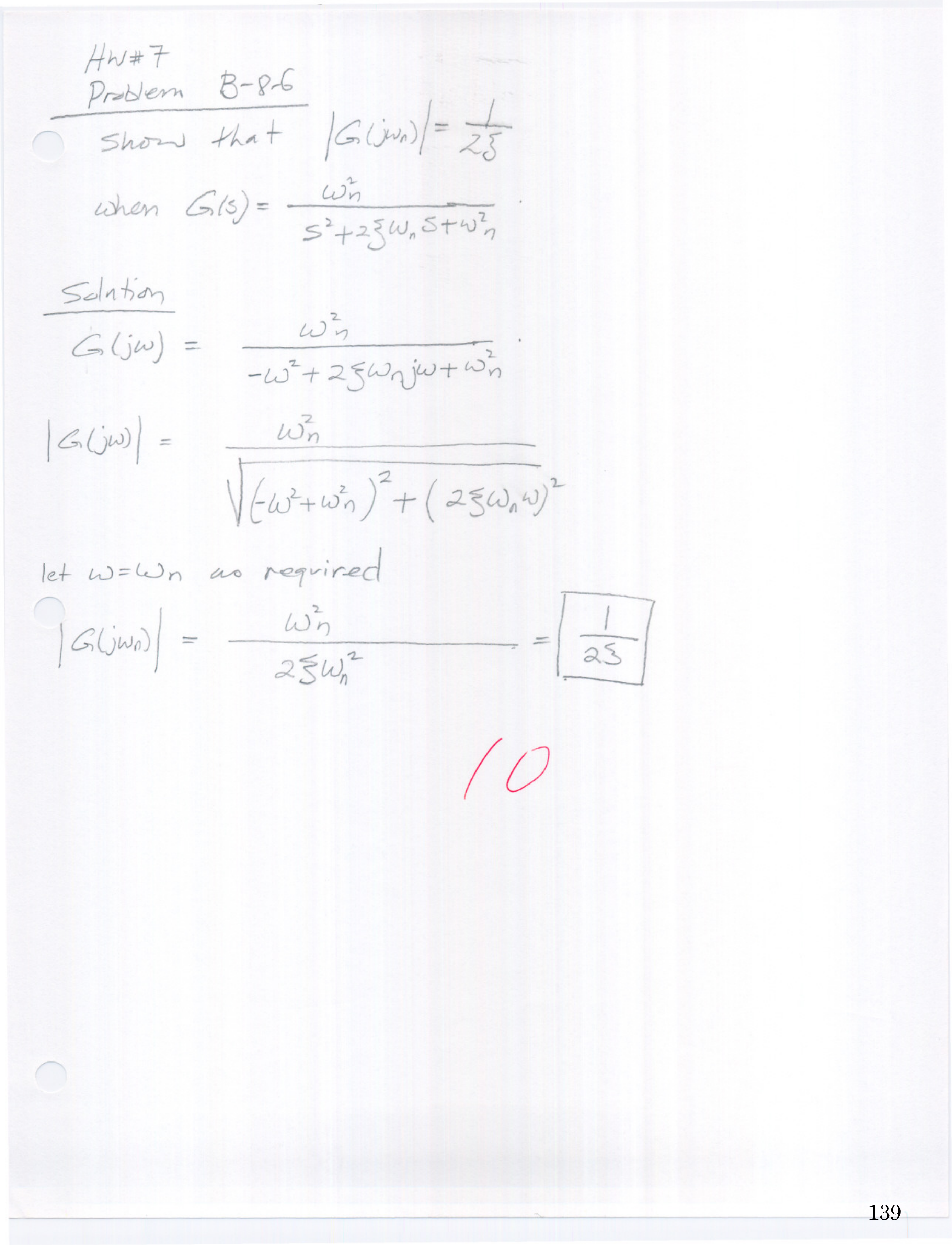
Now do phase

∠ for G_2 is same as part(b) G_2 which is



combine :





HW#7

Problem B-8-6

Show that $|G(j\omega_n)| = \frac{1}{2\zeta}$

when $G(s) = \dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$.

Solution

$$G(j\omega) = \frac{\omega_n^2}{-\omega^2+2\zeta\omega_n j\omega+\omega_n^2}.$$

$$|G(j\omega)| = \frac{\omega_n^2}{\sqrt{(-\omega^2+\omega_n^2)^2+(2\zeta\omega_n\omega)^2}}$$

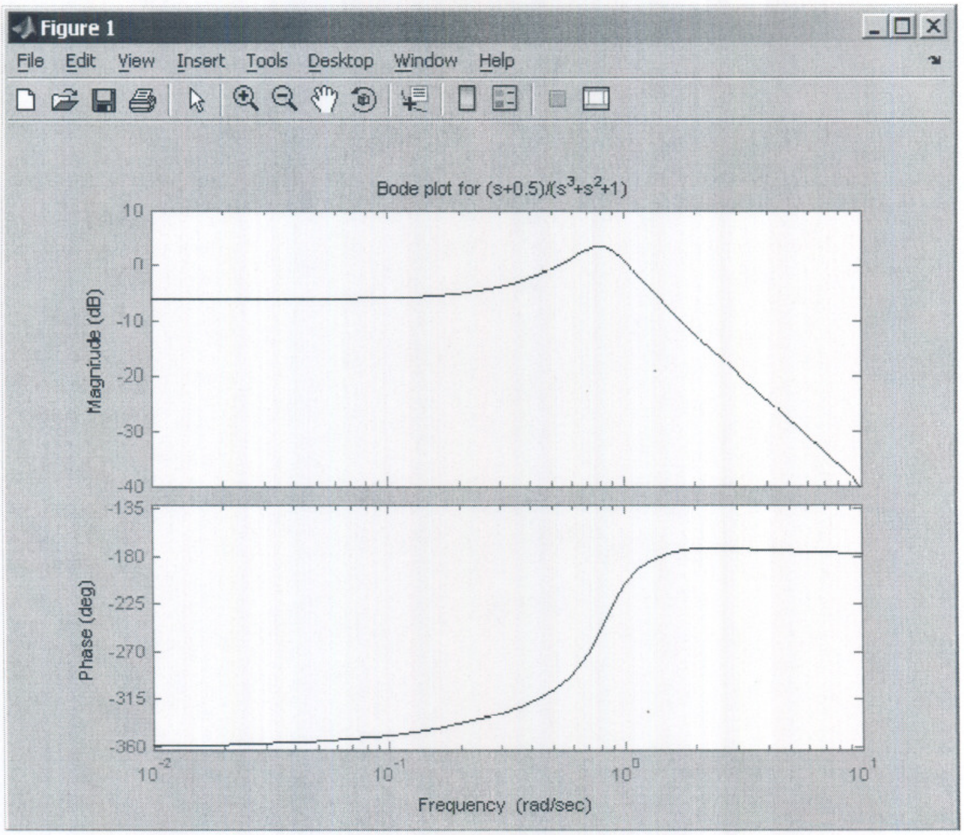
let $\omega=\omega_n$ as required

$$|G(j\omega_n)| = \frac{\omega_n^2}{2\zeta\omega_n^2} = \boxed{\frac{1}{2\zeta}}$$

10

```
close all;
clear all;
%problem B-8-7, by Nasser Abbasi

s=tf('s');
sys=(s+0.5)/(s^3+s^2+1);
bode(sys);
title('Bode plot for (s+0.5)/(s^3+s^2+1)');
```



(1)

→

now need to explain phase diagram.

Phase as shown by matlab starts from -360° and ends at -180°. This is the same as starting from 0° and ending at 180° as per problem statement.

$$G(j\omega) = \frac{j\omega+1}{-j\omega^3-\omega^2+1}$$

$$\text{so } \angle G(j\omega) = \angle j\omega+1 - \angle -j\omega^3-\omega^2+1$$

$$= \tan^{-1}\omega - \tan^{-1}\left(\frac{-\omega^3}{-\omega^2+1}\right).$$

For small ω, $\frac{-\omega^3}{-\omega^2+1} \longrightarrow \frac{-\omega}{1} \longrightarrow 0$

So we get in the limit

$$\angle G(j\omega)_{\omega\to 0} = \tan^{-1}\omega - \tan -\omega = 0-360^\circ$$
$$= -360^\circ$$

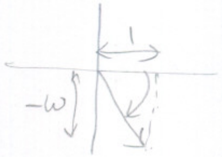
which the result given by Matlab. but $-360^\circ = 0^\circ$ also.

now I consider what happens when $\omega \rightarrow \infty$.

$$\lim_{\omega\to\infty} \angle G(j\omega) = \tan^{-1}\omega_{\omega\to\infty} - \tan^{-1}\left(\frac{-\omega^3}{-\omega^2+1}\right)_{\omega\to\infty}$$

$$= 90^\circ - \tan^{-1}(-\omega)_{\omega\to\infty}$$

$$= 90^\circ - (-90^\circ)$$



$$= 180^\circ$$

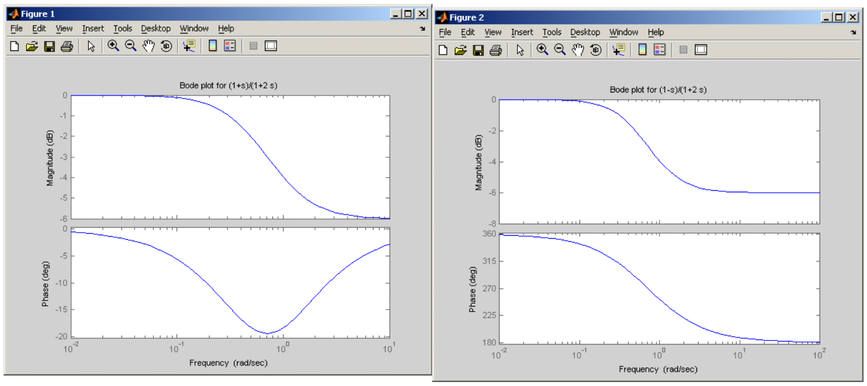
This explains the asymptotic behavior given.

3.7.1.1 problem B 8.3

```
close all;
clear all;
%problem B-8-3, by Nasser Abbasi

s=tf('s');
sys=(1+s)/(1+2*s);
bode(sys);
title('Bode plot for (1+s)/(1+2 s)');

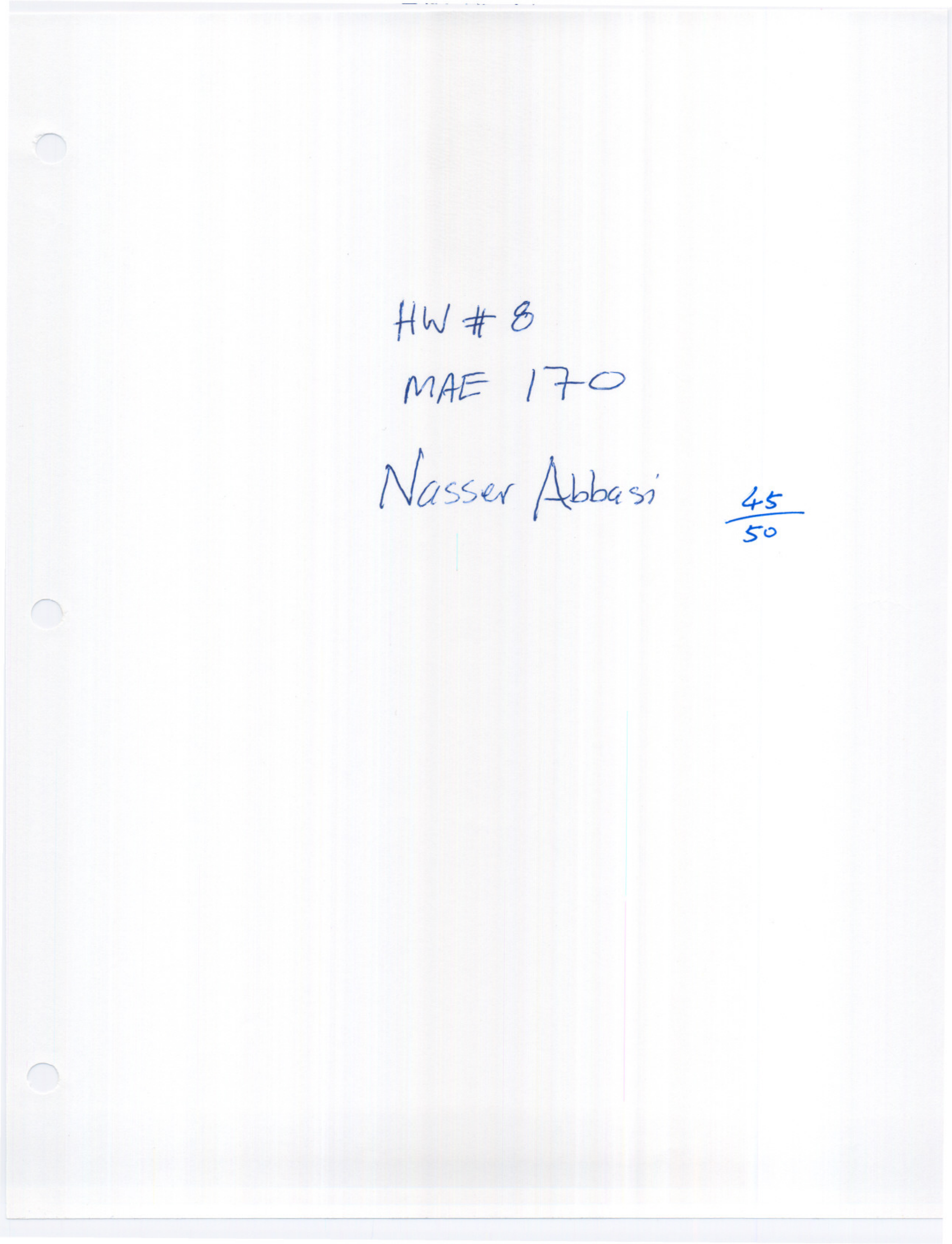
sys=(1-s)/(1+2*s);
figure;
bode(sys);
title('Bode plot for (1-s)/(1+2 s)');
```



3.8 HW 8

Local contents

3.8.1 my solution



HW # 8

MAE 170

Nasser Abbasi

$\frac{45}{50}$

problem B-8-14

$$G(s) = \frac{1}{s(s^2+0.8s+1)}$$

ntion . let $s=j\omega$,

$$G(j\omega) = \frac{1}{j\omega(-\omega^2+0.8j\omega+1)} \quad \circ \quad |G(j\omega)| = \frac{1}{\omega\sqrt{(1-\omega^2)^2+(0.8\omega)^2}}$$

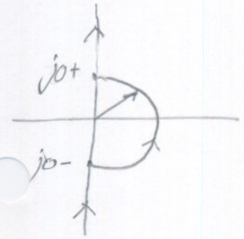
For $j0+$, $\angle G(j\omega) = 0 - \left[90^\circ + \tan^{-1}\frac{0.8\omega}{1-\omega^2}\right] = -90^\circ$

for $j0-$, $\angle G(j\omega) = 0 - \left[-90^\circ + \tan^{-1}\frac{0.8\omega}{1-\omega^2}\right] = +90^\circ$.

For both cases, $|G(j\omega)| = \infty$.

$\boxed{\text{so the point } j0+ \text{ maps to } -j\infty \text{ and point } j0- \text{ maps to } +j\infty}$

now consider the small semicircle around the origion.



let point on this semicircle be $s = \varepsilon e^{j\theta}$ and change θ from -90° to $+90^\circ$.

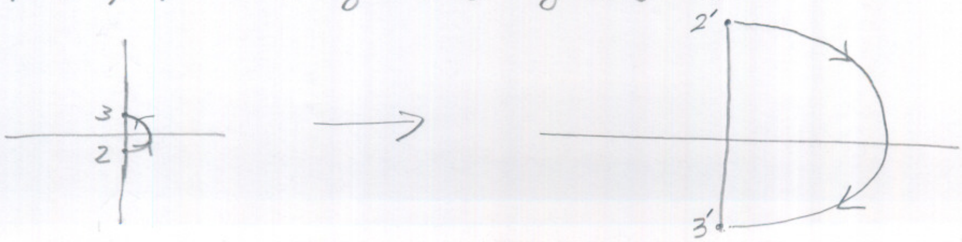
write $$G(s) = \frac{1}{\varepsilon e^{j\theta}((\varepsilon e^{j\theta})^2+0.8\varepsilon e^{j\theta}+1)} = \frac{1}{\varepsilon e^{j\theta}(\varepsilon^2 e^{2j\theta}+0.8\varepsilon e^{j\theta}+1)}$$

$$= \frac{1}{\varepsilon^3 e^{j3\theta}+0.8\varepsilon^2 e^{j2\theta}+\varepsilon e^{j\theta}}$$

since $\varepsilon \ll 1$, then $\varepsilon^3 \to 0$, $\varepsilon^2 \to 0$ and above can be approximated as

$$G(s) \simeq \frac{1}{\varepsilon e^{j\theta}} = \frac{1}{\varepsilon}e^{-j\theta} = \boxed{\frac{1}{\varepsilon}\angle -\theta}$$

now change θ from -90° to $+90^\circ$. we see that this maps to infinite Large circle from $+90^\circ$ to -90°. so now can draw the mapings of path along imaginary axis as



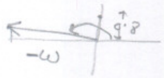
For points $+j\infty$, and $-j\infty$:

From $|G(j\omega)|$ we see that $|G(j\omega)| \to 0$ in both Cases.

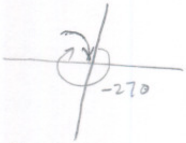
For phase, at point $+j\infty$, $\angle G(j\omega) = 0 - \left[90^\circ + \tan^{-1}\frac{0.8\omega}{1-\omega^2}\right]$

$$= -\left[90^\circ + \tan^{-1}\frac{0.8}{\underbrace{\frac{1}{\omega}}_{=0}-\omega}\right] = -\left[90^\circ + \tan^{-1}\frac{0.8}{-\omega}\right]$$

so $\angle G(j\omega)\Big|_{\omega\to+\infty} = -\left[90^\circ + 180\right] = -270^\circ$



so point $+j\infty$ maps to orision, at phase -270°



For point $-j\infty$, $\angle G(j\omega) = 0 - \left[-90^\circ + \tan^{-1}\frac{0.8}{-\omega}\right] = -\left[-90^\circ + 0\right] = +90^\circ$

Now For large semi circle.

let $s = Re^{j\theta}$.

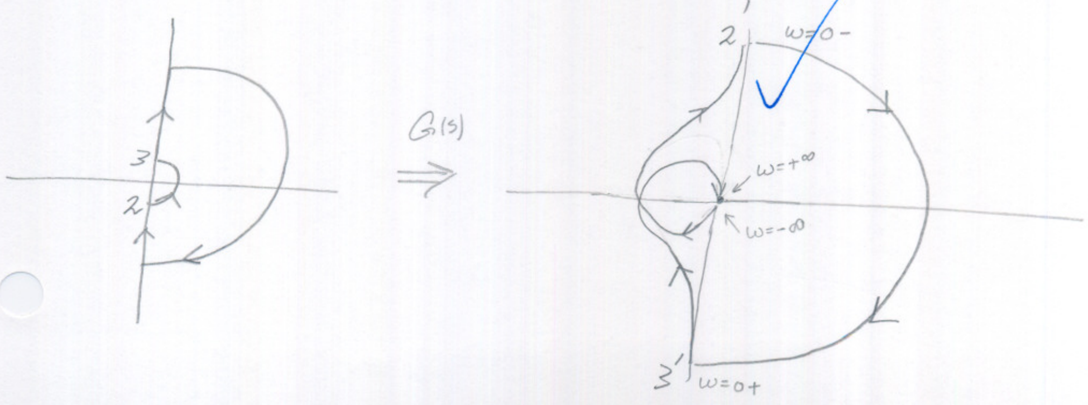
so $G(s) = \dfrac{1}{R^3 e^{j3\theta} + 0.8R^2 e^{j2\theta} + Re^{j\theta}}$

so $|G(s)| = \dfrac{1}{R^3+R^2+R} \to 0$ as $R \to \infty$.

$\angle G(s) = 0 - [3\theta + 2\theta + \theta] = -6\theta$.

so the whole Large Circle maps to orision.

hence Nyquist plot is



HW # 8
Problem B-8-26

open loop $G(s) = \frac{as+1}{s^2}$. Find value of a so that phase margin = 45°

Solution

$$G(j\omega) = \frac{aj\omega+1}{(j\omega)^2}$$

$$\angle G(j\omega) = \angle aj\omega+1 - 2\angle j\omega$$

$$\angle G(j\omega) = \tan^{-1} a\omega - 2(90^\circ) = \tan^{-1} a\omega - 180^\circ$$

we want $45^\circ = 180 + \angle G(j\omega_c)$ where ω_c is cross over freq.

so

$$\tan^{-1} a\omega_c - 180^\circ = -135^\circ$$

$$\tan^{-1} a\omega_c = 45^\circ$$

$$\boxed{a\omega_c = 1} \checkmark \text{———} ①$$

Now need another equation to solve for a.

Since at $\omega = \omega_c$, $|G(j\omega)| = 1$

Then $1 = \frac{\sqrt{(a\omega_c)^2+1}}{\omega_c} \Rightarrow \omega_c = \sqrt{(a\omega_c)^2+1}$

so $\omega_c^2 - a^2\omega_c^2 - 1 = 0$

$$\omega_c^2(1-a^2) = 1$$

$$\omega_c^2 = \frac{1}{1-a^2} \Rightarrow \boxed{\omega_c = \frac{1}{\sqrt{1-a^2}}} \text{———} ②$$

Plug ② into ① we get

$$a\frac{1}{\sqrt{1-a^2}} = 1 \Rightarrow a^2 = 1-a^2 \Rightarrow 2a^2 = 1 \Rightarrow \boxed{a = \sqrt{\tfrac{1}{2}}}$$

-5

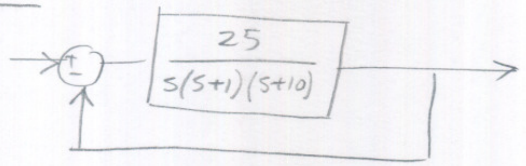
X $\boxed{a = 0.707}$

This is the value of a such as phase margin = 45°

HW#8

Problem B-8-27

Consider



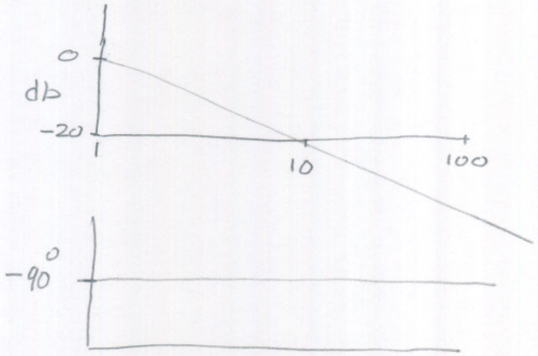
Draw Bode plot of open loop. Determine phase and gain margin.

Solution

For $\frac{1}{s}$ bode plot is

$\phi = -90^{\circ}$

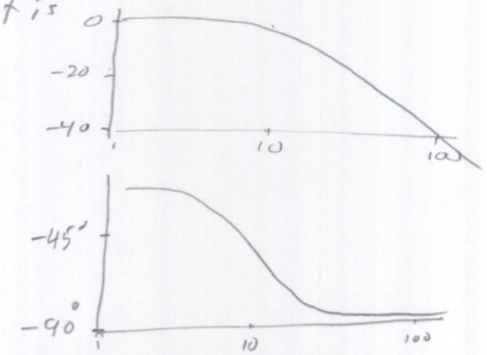
$|G(j\omega)| = \frac{1}{\omega}$



for $\frac{1}{s+1}$, bode plot is

$\phi = -\tan^{-1} \omega$

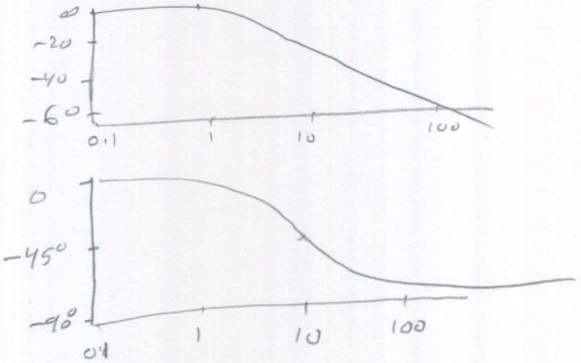
$|G(j\omega)| = \frac{1}{\sqrt{w^2+1}}$



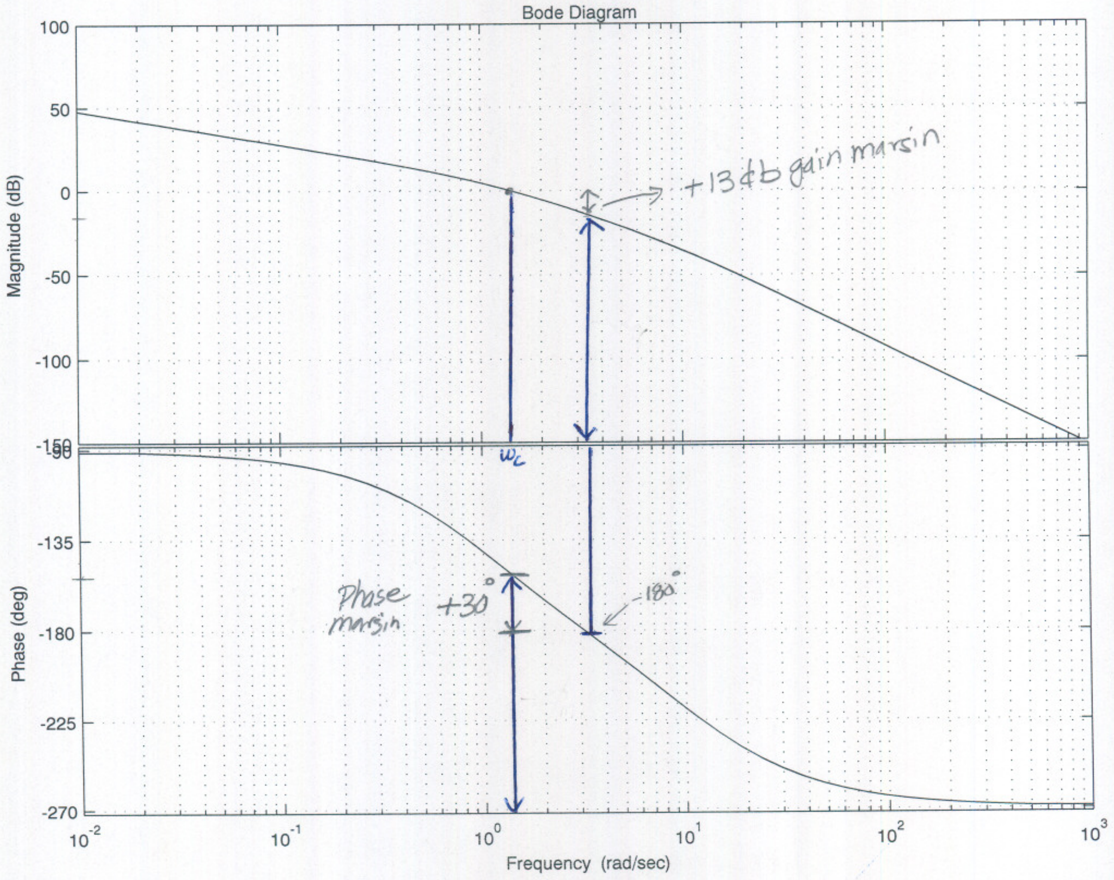
for $\frac{1}{s+10}$

$\phi = -\tan^{-1} \frac{w}{10}$

$|G(j\omega)| = \frac{1}{\sqrt{w^2+100}}$



adding all these together gives bode plot. to get accurate phase and gain margin I did the above on top of matlab output →

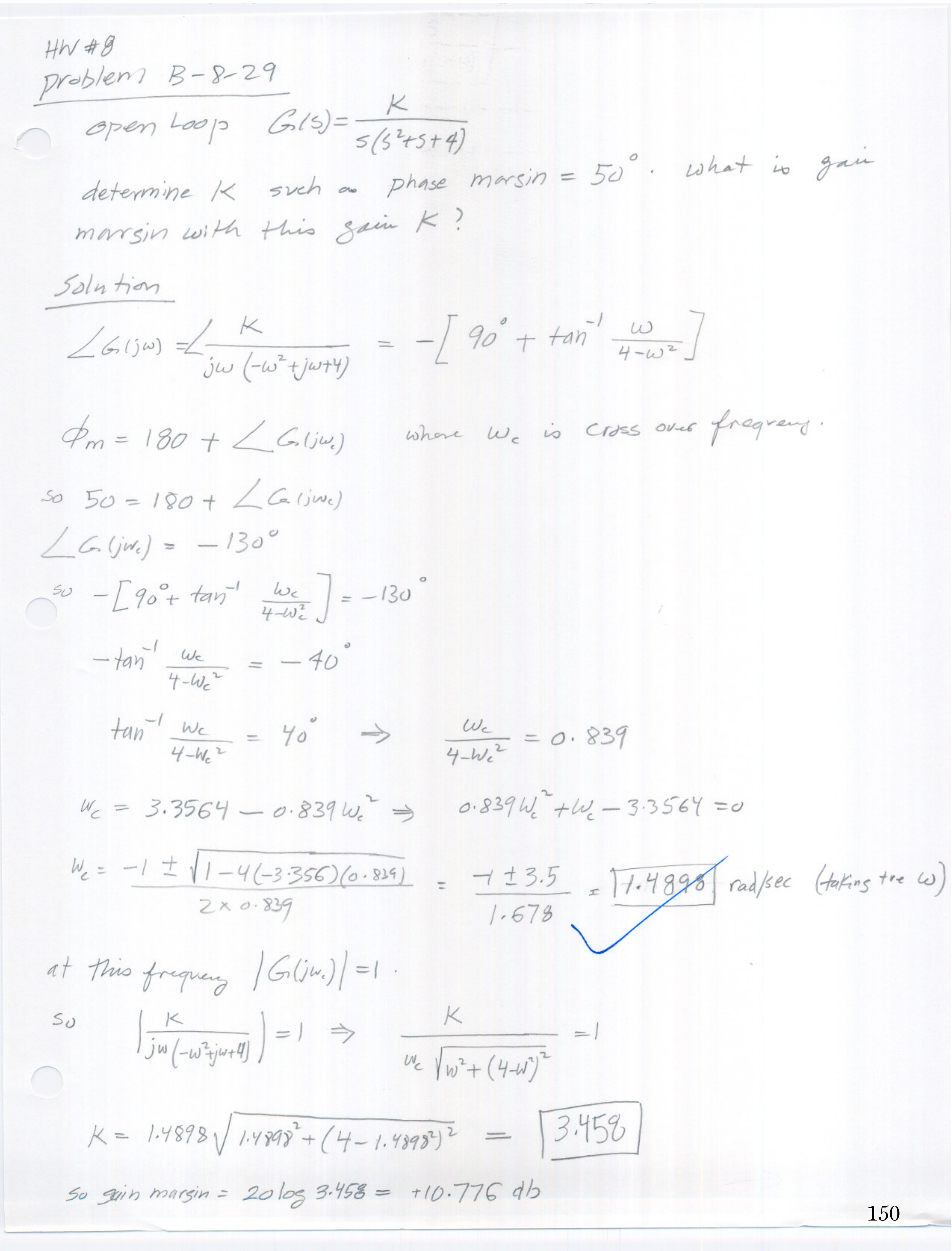


From diagram, when $\omega = \omega_c \cong 1.03$ rad/sec , is Cross over freq.

at this frequency, $\angle G(j\omega_c) \cong -150^\circ$

so $\phi_m = 180 - 150 = \boxed{30^\circ}$ a positive phase margin.

when $\angle G(j\omega) = -180^\circ$, $|G(j\omega)| \cong \boxed{+13 \text{ db}}$. (since below the 0db line)



HW #8

problem B-8-29

open Loop $G(s)=\frac{K}{s(s^2+s+4)}$

determine K such as phase margin = 50°. what is gain margin with this gain K?

Solution

$$\angle G(j\omega)=\angle\frac{K}{j\omega(-\omega^2+j\omega+4)}=-\left[90^\circ+\tan^{-1}\frac{\omega}{4-\omega^2}\right]$$

$\phi_m=180+\angle G(j\omega_c)$ where ω_c is cross over frequency.

so $50=180+\angle G(j\omega_c)$

$\angle G(j\omega_c)=-130^\circ$

so $-\left[90^\circ+\tan^{-1}\frac{\omega_c}{4-\omega_c^2}\right]=-130^\circ$

$$-\tan^{-1}\frac{\omega_c}{4-\omega_c^2}=-40^\circ$$

$$\tan^{-1}\frac{\omega_c}{4-\omega_c^2}=40^\circ\Rightarrow\frac{\omega_c}{4-\omega_c^2}=0.839$$

$$\omega_c=3.3564-0.839\,\omega_c^2\Rightarrow 0.839\,\omega_c^2+\omega_c-3.3564=0$$

$$\omega_c=\frac{-1\pm\sqrt{1-4(-3.356)(0.839)}}{2\times0.839}=\frac{-1\pm3.5}{1.678}=\boxed{1.4898}\text{ rad/sec (taking +ve }\omega)$$

at this frequency $|G(j\omega_c)|=1$.

so $\left|\frac{K}{j\omega(-\omega^2+j\omega+4)}\right|=1\Rightarrow\frac{K}{\omega_c\sqrt{\omega^2+(4-\omega^2)^2}}=1$

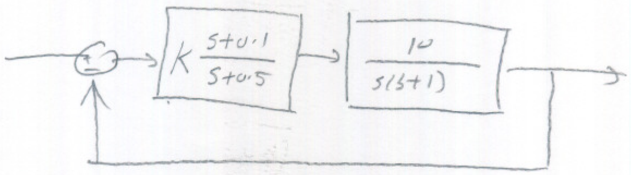
$$K=1.4898\sqrt{1.4898^2+(4-1.4898^2)^2}=\boxed{3.458}$$

so gain margin = 20 log 3.458 = +10.776 db

HW#8

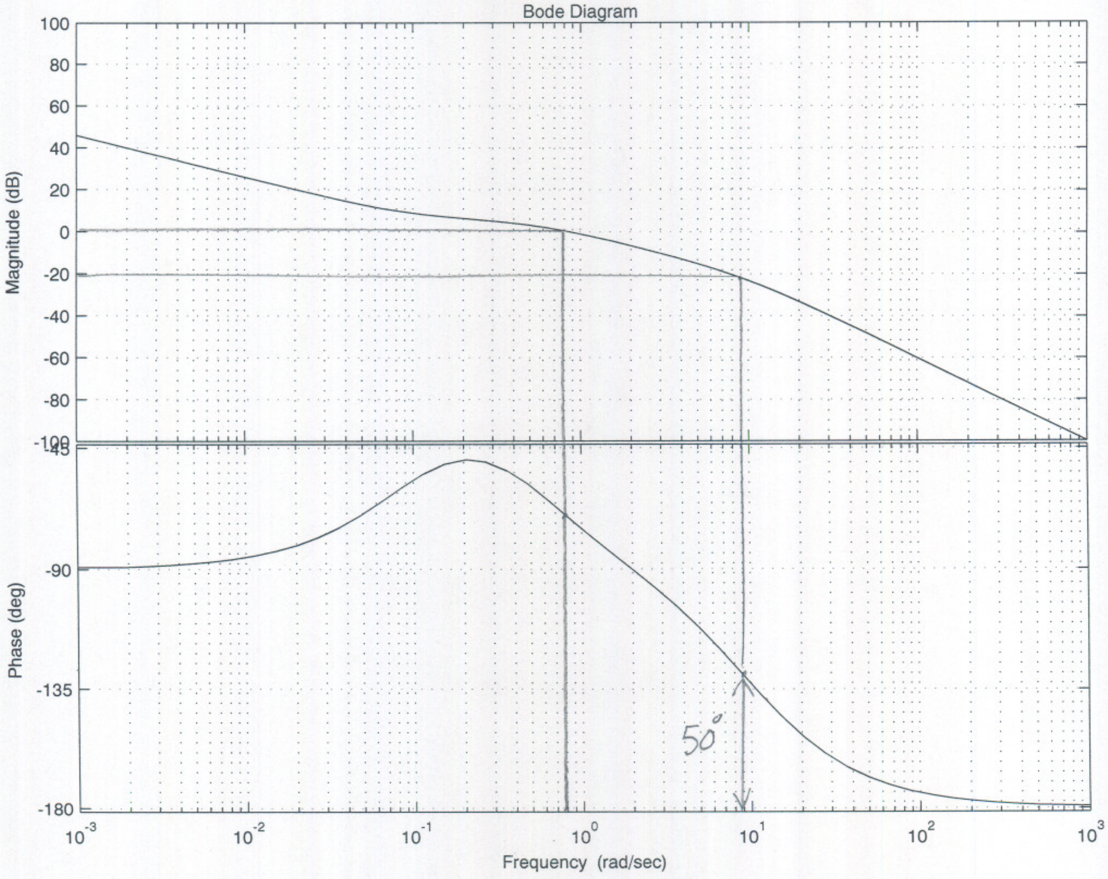
B-8-30

Consider



Draw Bode plot for open loop, determine K such as phase margin = 50°. what is the gain margin of this system with this gain K?

→



we see that at 50° phase margin $|G(j\omega)| = 0.1$ (i.e -20 db).

so $\frac{10K\,|s+0.1|}{|s+0.5|\,|s|\,|s+1|} = 0.1$. from plot we see $\omega \cong 9$ rad/sec.

so $\frac{10K\sqrt{\omega^2+0.1^2}}{\sqrt{\omega^2+0.5^2}\;\omega\;\sqrt{\omega^2+1}} = 0.1 \Rightarrow K = \frac{(0.1)\sqrt{81+0.5^2}\;(9)\sqrt{81+1}}{10\sqrt{81+0.1^2}} = \frac{73.46}{90} = \boxed{0.81}$

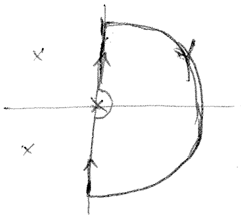
so gain margin with this gain is $20\log 0.81 = \boxed{-1.76}$ db

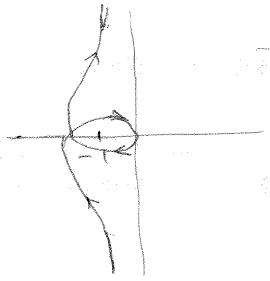
3.8.2 key solution

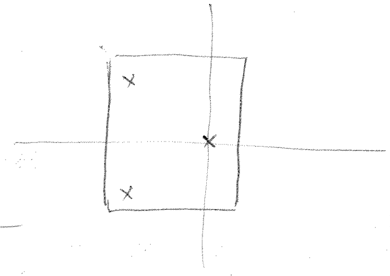
B - 8 - 14

HW 8 solutions.

$$G(s) = \frac{1}{s(s^2+0.8s+1)}$$

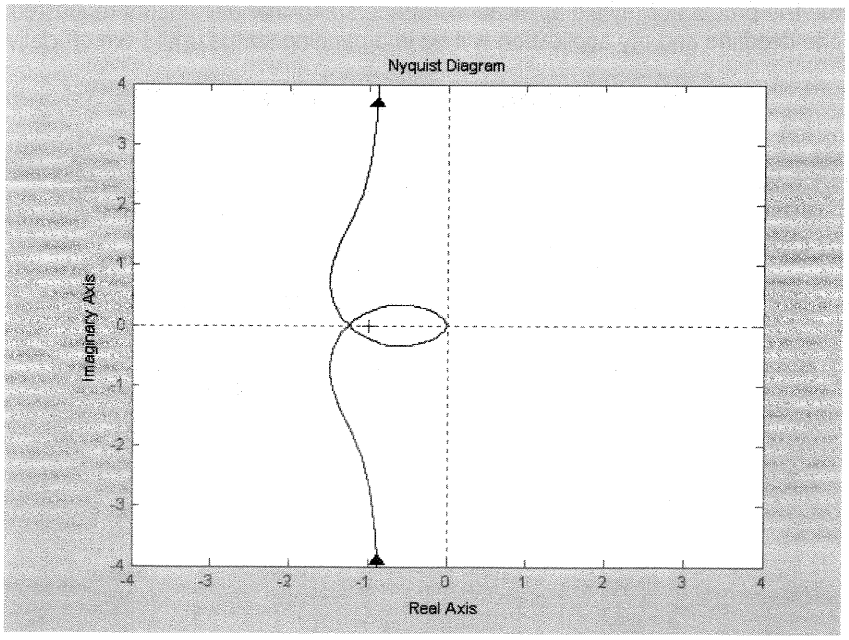






P = 0 N = 1 => Z = 1 unstable.

```
nyquist([0 0 0 1], [1 0.8 1 0])
axis([-4 4 -4 4]
```



8 - 26

$$G_{OL}(s) = \frac{as+1}{s^2}$$

$$\left| G_{OL}(j\omega) \right| = \sqrt{\frac{a^2\omega^2+1}{\omega^4}} = \frac{\sqrt{a^2\omega^2+1}}{\omega^2}$$

$$\angle G_{OL}(j\omega) = \tan^{-1} a\omega - \tan^{-1}\left(\frac{0}{-\omega^2}\right) = \tan^{-1} a\omega - 180°$$

$\phi_m = 45°$ given

$$\phi_m = 180° + \angle G_{OL}(j\omega_1)$$

$$\Rightarrow \tan^{-1} a\omega_1 = 45° \Rightarrow \underline{a\omega_1 = 1}$$

$$\left| G(j\omega_1) \right| = 1 = \sqrt{\frac{(a\omega_1)^2+1}{\omega_1^2}}$$

$$\frac{\sqrt{1^2+1}}{\omega_1^2} = \frac{\sqrt{2}}{\omega_1^2} = 1 \Rightarrow \omega_1 = 2^{\frac{1}{4}}$$

$$a = \frac{1}{\omega_1} = \frac{1}{2^{\frac{1}{4}}} = \boxed{0.841}$$

8-27

$$G_{OL}(S) = \frac{25}{S(S+1)(S+10)} = \frac{25}{S^3+11S^2+10S}$$

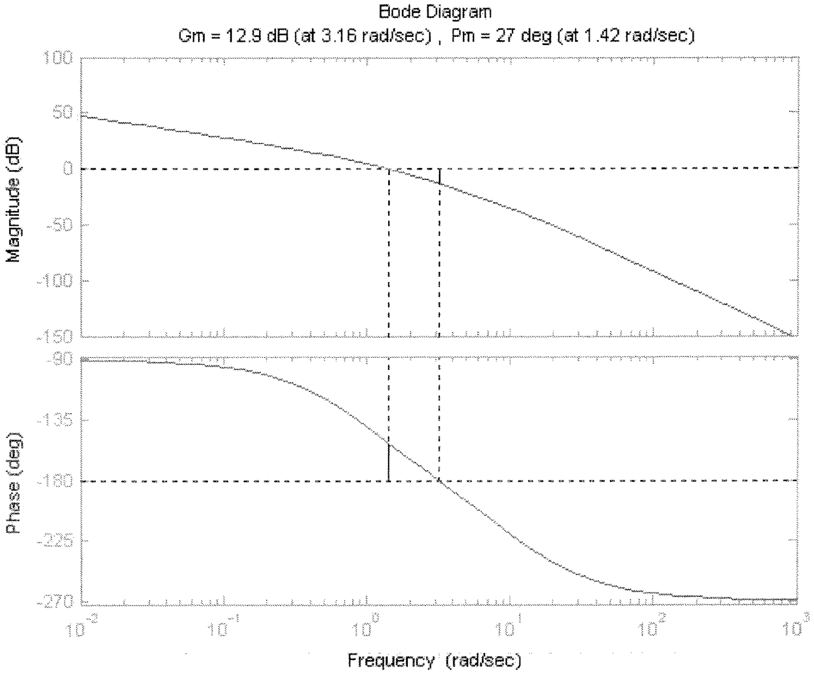
```
num = [0   0   0   25];
den = [1   11  10  0 ];
[gm, Pm, wcp, wcg] = margin(num, den)
```

$gm = 4.4 \longrightarrow gm\_dB = 20\log(gm) = 12.9\,dB$

$pm = 26.4973^\circ$

$wcp = 3.1623$ rad/sec

$wcg = 1.4230$ rad/sec.



8-29

$$G_{OL}(s) = \frac{K}{s(s^2+s+4)}$$

$$\phi_m = 180° + \underline{\angle G(j\omega_1)}$$

$$50° = 180° + 0° - 90° - \underline{\angle -\omega_1^2 + j\omega_1 + 4}$$

$$-40° = -\tan^{-1}\frac{\omega_1}{4-\omega_1^2}$$

$$\omega_1 = (4-\omega_1^2)(0.8391)$$

$$+0.8391\,\omega_1^2 + \omega_1 - 4(0.8391) = 0$$

$$\Rightarrow \underline{\omega_1 = 1.491 \text{ rad/sec}}$$

$$|G(j\omega_1)| = 1 = \left|\frac{K}{-j\omega_1^3 - \omega_1^2 + j4\omega_1}\right| = \frac{K}{3.46}$$

phase crossover frequency

$$\Rightarrow \boxed{K = 3.46}$$

$$\underline{\angle G(j\omega_{cp})} = -\overset{90°}{\cancel{\angle j\omega_{cp}}} - \underline{\angle -\omega_{cp}^2 + j\omega_{cp} + 4} = -180°$$

$$\tan^{-1}\frac{\omega_{cp}}{4-\omega_{cp}^2} = 90°$$

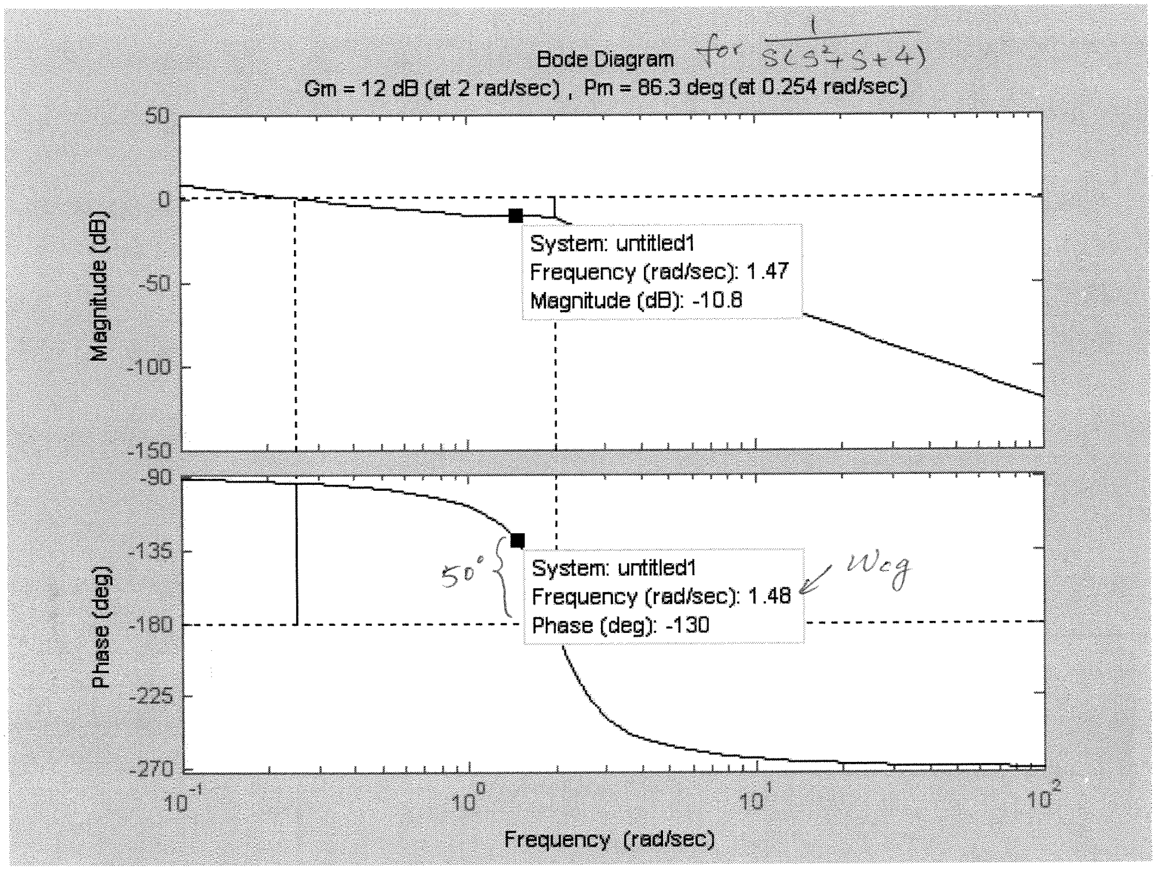
$$\Rightarrow \frac{\omega_{cp}}{4-\omega_{cp}^2} = \infty \Rightarrow \underline{\omega_{cp} = 2 \text{ rad/sec}}$$

$$gm = |G(j2)| = \left|\frac{3.46}{(j2)(-4+j2+4)}\right| = 0.865$$

$$|gm\_dB| = |20\log(0.865)| = |-1.26\,dB| = \boxed{1.26\,dB}$$

gain margin

Method 2 (B-8-29)



W_{cg} for $50^\circ = \phi_m$ is 1.48 rad/sec

$$|G(jW_{cg})| = |G(j1.48)| = 10.8\,dB$$

$$K = 10^{\frac{10.8}{20}} = \boxed{3.467}$$ same as method 1

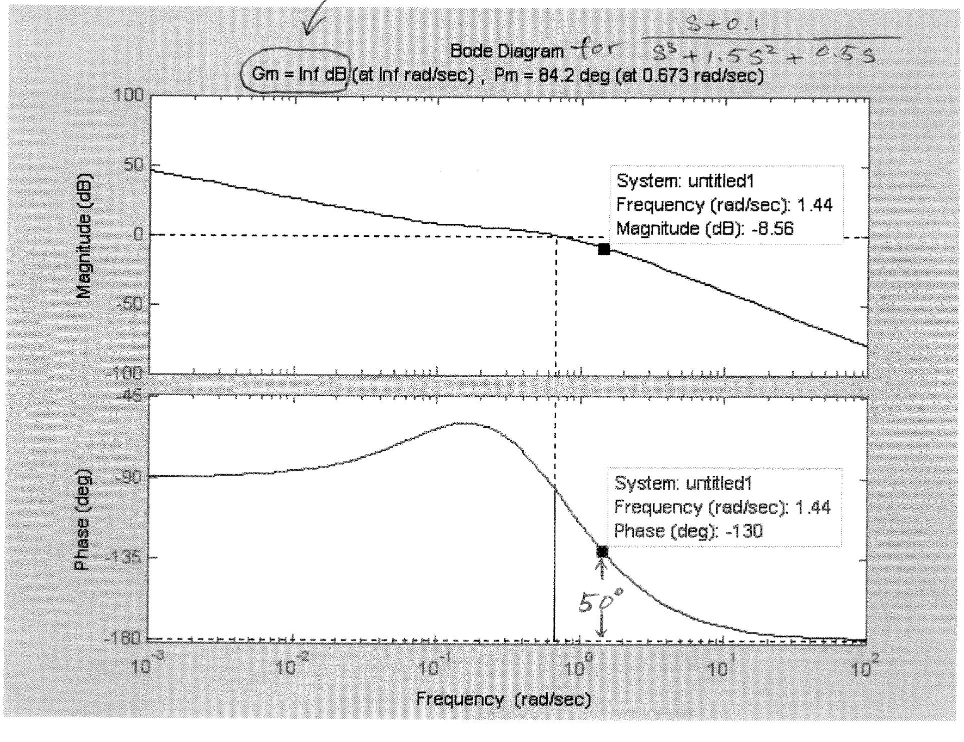
8-30

$$G_{OL}(s) = \frac{K(s+0.1)10}{s(s+0.5)(s+1)} = \frac{10K(s+0.1)}{s^3+1.5s^2+0.5s}$$

We will solve this problem graphically
first we let $10K = 1$, and plot ~~the~~ bode diagram
for $sys = \frac{s+0.1}{s^3+1.5s^2+0.5s}$. Since we require the phase margine to be 50°, ω is found to be at 1.44 rad/sec according to our bode plot. Since the diagram indicates that $|G(j1.44)| = 8.56dB$, we need to choose $10K = 8.56dB$

$$8.56dB = 20\log(10K) \Rightarrow \boxed{K = 0.268}$$

Since the phase curve lies above the -180° line for all ω, the gain margin is ∞ dB.



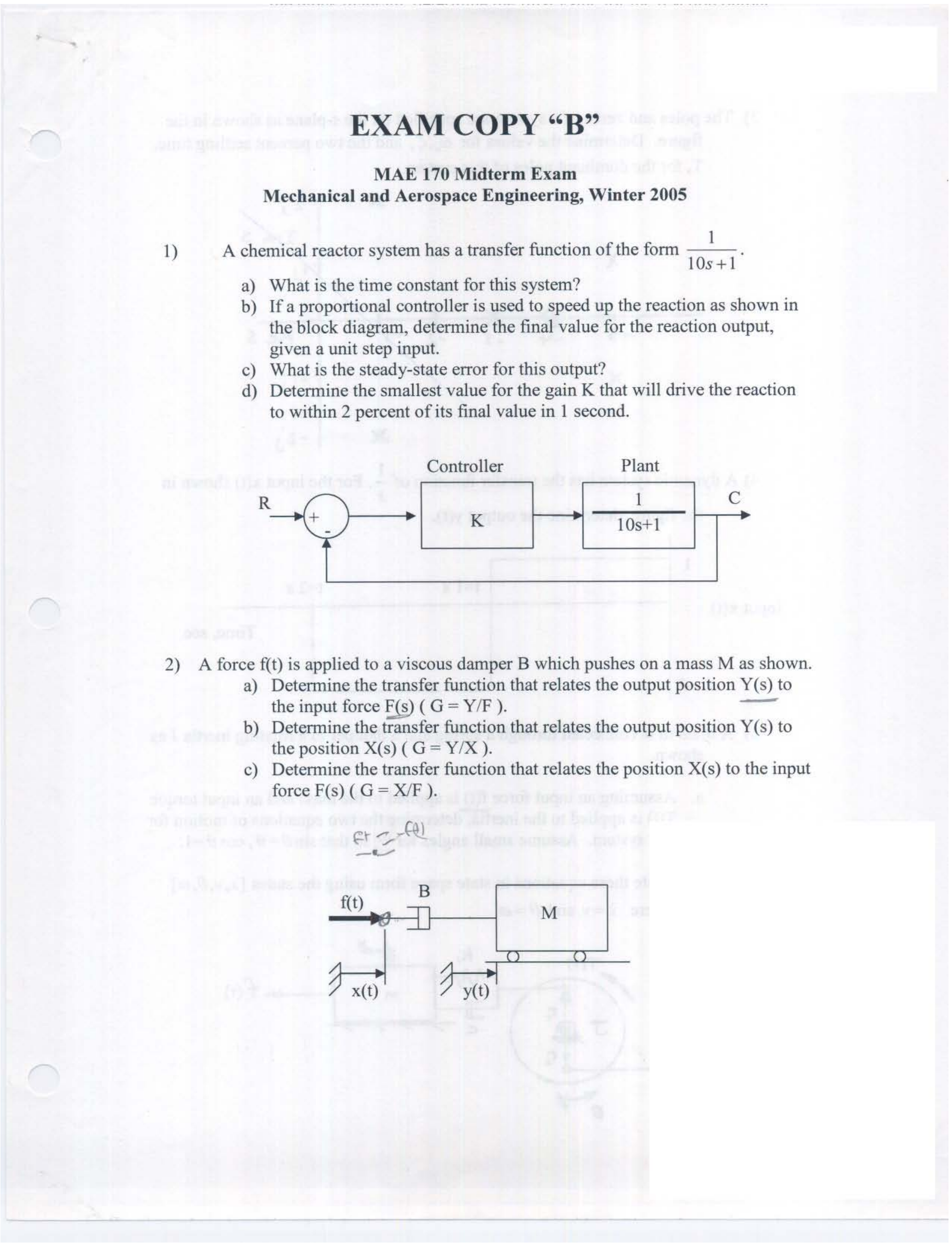
Chapter 4

Exams

Local contents

4.1 midterm

4.1.1 midterm questions



EXAM COPY "B"

MAE 170 Midterm Exam
Mechanical and Aerospace Engineering, Winter 2005

1) A chemical reactor system has a transfer function of the form $\frac{1}{10s+1}$.

a) What is the time constant for this system?
b) If a proportional controller is used to speed up the reaction as shown in the block diagram, determine the final value for the reaction output, given a unit step input.
c) What is the steady-state error for this output?
d) Determine the smallest value for the gain K that will drive the reaction to within 2 percent of its final value in 1 second.

2) A force f(t) is applied to a viscous damper B which pushes on a mass M as shown.

a) Determine the transfer function that relates the output position Y(s) to the input force F(s) (G = Y/F).
b) Determine the transfer function that relates the output position Y(s) to the position X(s) (G = Y/X).
c) Determine the transfer function that relates the position X(s) to the input force F(s) (G = X/F).

Figure 4.1: questions