Problem 30.4 HW#7

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Use the implicit method to solve for the temp distribution of a long thin rod with length 10 cm and $k^{'} = 0.49$ cal/(s.cm.C^0), $\Delta x = 2$ cm and $\Delta t = 0.1s$. At t=0, the temp. of the rod is 0^0C and the boundary conditions is that $\partial T/\partial x = 0$ at $x = 10$ cm and $T(0) = 100^0$C for all time. note that the rode is aluminum with C=0.2147 cal/(g.C^0) and $\rho = 2.7$ g/cm^3. therefor k=0.49/(2.7×0.2174)=0.835 cm^2/s and $\lambda = 0.835(0.1)/2^2 = 0.020875$

Solution

The only difference between this problem and the example shown is in the boundary conditions. Here, $\partial T/\partial x = 0$ at $x = 10$ implies that $T_6^l = T_4^l$. The left point boundary condition is not changed, and T will remain at 100 degrees at x=0 for all time.
We now need to solve a parabolic PDE using the implicit method.
Recall that the solution using the parabolic PDE for the implicit method is written as

$$(1 + 2\lambda)\, T_i^{l+1} - \lambda T_{i+1}^{l+1} = T_i^l + \lambda T_{i-1}^{l+1}$$

We just need to pay attention to the edge nodes (i=0 and i=5 in our case). At i=0, $f_0\left(t^{l+1}\right) = 100$ and $f_5\left(t^{l+1}\right) = 0$
Now apply the PDE solution with the above boundary conditions in place. note that $(1 + 2\lambda) = 1.04175$
Apply one time step. now at time=0.1 seconds:

at i=1
$1.04175\ T_1^1 - 0.02875\ T_2^1 = T_1^0 + 0.02875 T_0^1$
or
$1.04175\ T_1^1 - 0.02875\ T_2^1 = 0 + 0.02875\,(100)$
$1.04175\ T_1^1 - 0.02875\ T_2^1 = 2.0875$

Continue as above for the rest of the points on the rod
i=2
$1.04175\ T_2^1 - 0.02875\ T_3^1 = T_2^0 + 0.02875 T_1^1$
$1.04175\ T_2^1 - 0.02875\ T_3^1 = 0 + 0.02875 T_1^1$
$1.04175\ T_2^1 - 0.02875\ T_3^1 - 0.02875 T_1^1 = 0$

i=3
$1.04175\ T_3^1 - 0.02875\ T_4^1 = T_3^0 + 0.02875 T_2^1$
$1.04175\ T_3^1 - 0.02875\ T_4^1 = 0 + 0.02875 T_2^1$
$1.04175\ T_3^1 - 0.02875\ T_4^1 - 0.02875 T_2^1 = 0$

i=4
$1.04175\ T_4^1 - 0.02875\ T_5^1 = T_4^0 + 0.02875 T_3^1$
$1.04175\ T_4^1 - 0.02875\ \ (0) = 0 + 0.02875 T_3^1$
$1.04175\ T_4^1 - 0.02875 T_3^1 = 0$

i=5

$1.04175\ T_5^1 - 0.02875\ T_6^1 = T_5^0 + 0.02875 T_4^1$

replace T_6^1 with T_4^1

$1.04175\ T_5^1 - 0.02875\ T_4^1 = T_5^0 + 0.02875 T_4^1$

$1.04175\ T_5^1 - 0.02875\ T_4^1 - 0.02875 T_4^1 = 0$

$1.04175\ T_5^1 \ = 0$

The above leads to the following set of simulatanous equations

$$\begin{bmatrix} 1.04175 & -0.02875 & 0 & 0 & 0 \\ -0.02875 & 1.04175 & -0.2875 & 0 & 0 \\ 0 & -0.02875 & 1.04175 & -0.02875 & 0 \\ 0 & 0 & -0.020875 & 1.04175 & 0 \\ 0 & 0 & 0 & 0 & 1.04175 \end{bmatrix} \begin{bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \\ T_5^1 \end{bmatrix} = \begin{bmatrix} 2.0875 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve using matlab, we get

```
>> A

    1.0418   -0.0288         0         0         0
   -0.0288    1.0418   -0.2875         0         0
         0   -0.0288    1.0418   -0.0288         0
         0         0   -0.0209    1.0418         0
         0         0         0         0    1.0418

>>  b=[2.0875;0;0;0;0]

b =

    2.0875
         0
         0
         0
         0
>>
>> x=inv(A)*b

x =

   2.00538141884130
   0.05586410705841
   0.00154518911589
   0.00003096311283
                  0
```

So solution is at t=0.1 seconds:

$T_1^1 = 2.00538141884130$
$T_2^1 = 0.05586410705841$
$T_3^1 = 0.00154518911589$
$T_4^1 = 0.00003096311283$
$T_5^1 = 0$
To solve for t=0.2 seconds
at i=1
$1.04175\ T_1^2 - 0.02875\ T_2^2 = T_1^1 + 0.02875 T_0^2$
or
$1.04175\ T_1^2 - 0.02875\ T_2^2 = 2.00538141884130 + 0.02875\,(100)$
$1.04175\ T_1^2 - 0.02875\ T_2^2 = 4.88038141884130$

Continue as above for the rest of the points on the rod
i=2
$1.04175\ T_2^2 - 0.02875\ T_3^2 = T_2^1 + 0.02875 T_1^2$
$1.04175\ T_2^2 - 0.02875\ T_3^2 = 0.05586410705841 + 0.02875 T_1^2$
$1.04175\ T_2^2 - 0.02875\ T_3^2 - 0.02875 T_1^2 = 0.05586410705841$

i=3
$1.04175\ T_3^2 - 0.02875\ T_4^2 = T_3^1 + 0.02875 T_2^2$
$1.04175\ T_3^2 - 0.02875\ T_4^2 = 0.00154518911589 + 0.02875 T_2^2$
$1.04175\ T_3^2 - 0.02875\ T_4^2 - 0.02875 T_2^2 = 0.00154518911589$

i=4
$1.04175\ T_4^2 - 0.02875\ T_5^2 = T_4^1 + 0.02875 T_3^2$
$1.04175\ T_4^2 - 0.02875\ \ (0) = 0.00003096311283 + 0.02875 T_3^2$
$1.04175\ T_4^2 - 0.02875 T_3^2 = 0.00003096311283$

i=5
$1.04175\ T_5^2 - 0.02875\ T_6^2 = T_5^1 + 0.02875 T_4^2$
replace T_6^2 with T_4^2
$1.04175\ T_5^2 - 0.02875\ T_4^2 = T_5^1 + 0.02875 T_4^2$
$1.04175\ T_5^2 - 0.02875\ T_4^2 - 0.02875 T_4^2 = 0$
$1.04175\ T_5^2\ \ = 0$

The above leads to the following set of simulatanous equations

$$\begin{bmatrix} 1.04175 & -0.02875 & 0 & 0 & 0 \\ -0.02875 & 1.04175 & -0.2875 & 0 & 0 \\ 0 & -0.02875 & 1.04175 & -0.02875 & 0 \\ 0 & 0 & -0.020875 & 1.04175 & 0 \\ 0 & 0 & 0 & 0 & 1.04175 \end{bmatrix} \begin{bmatrix} T_1^2 \\ T_2^2 \\ T_3^2 \\ T_4^2 \\ T_5^2 \end{bmatrix} = \begin{bmatrix} 4.88038141884130 \\ 0.05586410705841 \\ 0.00154518911589 \\ 0.00003096311283 \\ 0 \end{bmatrix}$$

Solve using matlab, we get

```
b =

   4.88038141884130
   0.05586410705841
   0.00154518911589
   0.00003096311283
                  0

x =

   4.68989959331789
   0.18509504339506
   0.00660452474090
   0.00016206629882
                  0
```

So solution is at t=0.2 seconds:
$T_1^2 = 4.68989959331789$
$T_2^2 = 0.18509504339506$
$T_3^2 = 0.00660452474090$
$T_4^2 = 0.00016206629882$
$T_5^2 = 0$

So we see that the heat is moving from the left edge to the right edge with time.
Compare the above with the example (where T was fixed at 50^0 on the right edge of the rod compared with being fixed at zero in this case)

$T_1^2 = 3.9305$
$T_2^2 = 0.119$
$T_3^2 = 0.0618$
$T_4^2 = 1.9653$