Problem 30.2 HW#7

Nasser Abbasi. UCI. MAE 185. May 24, 2003

Use explicit method to solve for the temp distribution of a long thin rod with length 10 cm and $k^{'} = 0.49$ cal/(s.cm.C^0), $\Delta x = 2$ cm and $\Delta t = 0.1s$. At t=0, the temp. of the rod is 50^{0}C and the boundary conditions are $\partial T/\partial x = 0$ at $x = 10$ cm and $\partial T/\partial x = 1$ at $x = 0\ cm$. note that the rode is aluminum with C=0.2147 cal/(g.C^0) and $\rho = 2.7$ g/cm^3. therefor k=0.49/(2.7×0.2174)=0.835 cm^2/s and $\lambda = 0.835(0.1)/2^2 = 0.020875$.

Solution

We need to solve a parabolic PDE using the explicit method.
Recall that the solution using the parabolic PDE is written as

$$\begin{aligned} T_i^{l+1} &= T_i^l + k\frac{\Delta t}{(\Delta x)^2}\left(T_{i+1}^l - 2T_i^l + T_{i-1}^l\right) \\ T_i^{l+1} &= T_i^l + \lambda\left(T_{i+1}^l - 2T_i^l + T_{i-1}^l\right) \end{aligned}$$

The only difference between this problem an the example shown is in the boundary conditions at x=0 and x=10 cm. Here, boundary conditions are $\partial T/\partial x = 0$ at $x = 10$ cm and $\partial T/\partial x = 1$ at $x = 0\ cm$.

Looking at x=0 cm, $\partial T/\partial x = 1$ means that $\frac{T_{i+1}^l - T_{i-1}^l}{2\Delta x} = 1$, or $T_{i-1}^l = T_{i+1}^l - 2\Delta x$, this applies for $i = 0$, i.e. on the left edge, at x=0. The point $i-1$ when $i = 0$ is ofcourse outside the bar, so it is an imaginary point, but its T value is found by the above relation from knowing the T at the inside point at $i = 1$. So, for $\Delta x = 2$ the above relation become $T_{-1}^l = T_1^l - 2\,(2)$.
Looking at x=10 cm, $\partial T/\partial x = 0$ means that $\frac{T_{i+1}^l - T_{i-1}^l}{2\Delta x} = 0$, or $T_{i+1}^l = T_{i-1}^l$, this applied for the righ t side, where i is at the last edge point. So, for $\Delta x = 2$ the above relation become $T_6^l = T_4^l$.

Now apply the PDE solution with the above boundary conditions in place.
Apply one time step. now at time=0.1 seconds:

Everywhere we see T_{-1} we will replace that with $T_1 - 4$ and everywhere we see T_6 will be replaced with T_4. notice that at initial condition (time=0), $T_0^0 = 50^0$ and $T_5^0 = 50^0$

$T_0^1 = T_0^0 + \lambda\left(T_1^0 - 2T_0^0 + T_{-1}^0\right) = T_0^0 + \lambda\left(T_1^0 - 2T_0^0 + T_1^0 - 4\right) = 50 + 0.020875\left(50 - 2(50) + 50 - 4\right) = 49.9165$

$T_1^1 = T_1^0 + \lambda\left(T_2^0 - 2T_1^0 + T_0^0\right) = 50 + 0.020875\left(50 - 2(50) + 50\right) = 50$

$T_2^1 = T_2^0 + \lambda\left(T_3^0 - 2T_2^0 + T_1^0\right) = 50 + 0.020875\left(50 - 2(50) + 50\right) = 50$

$T_3^1 = T_3^0 + \lambda\left(T_4^0 - 2T_3^0 + T_2^0\right) = 50 + 0.020875\left(50 - 2(50) + 50\right) = 50$

$T_4^1 = T_4^0 + \lambda\left(T_5^0 - 2T_4^0 + T_3^0\right) = 50 + 0.020875\left(50 - 2(50) + 50\right) = 50$

$T_5^1 = T_5^0 + \lambda\left(T_6^0 - 2T_5^0 + T_4^0\right) = T_5^0 + \lambda\left(T_4^0 - 2T_5^0 + T_4^0\right) = 50 + 0.020875\left(50 - 2(50) + 50\right) = 50$

Ok, now continue. Let t=0.2 seconds, i.e. $l = 2$. Apply the same process as above:

$T_0^2 = T_0^1 + \lambda\left(T_1^1 - 2T_0^1 + T_{-1}^1\right) = T_0^1 + \lambda\left(T_1^1 - 2T_0^1 + T_1^1 - 4\right) = 49.9165 + 0.020875\left(50 - 2\left(49.9165\right) + 50 - 4\right) = 49.8365$

$T_1^2 = T_1^1 + \lambda\left(T_2^1 - 2T_1^1 + T_0^1\right) = 50 + 0.020875\left(50 - 2(50) + 49.9165\right) = 49.9983$

$T_2^2 = T_2^1 + \lambda\left(T_3^1 - 2T_2^1 + T_1^1\right) = 50 + 0.020875\left(50 - 2(50) + 50\right) = 50$

$T_3^2 = T_3^1 + \lambda\left(T_4^1 - 2T_3^1 + T_2^1\right) = 50 + 0.020875\left(50 - 2(50) + 50\right) = 50$

$T_4^2 = T_4^1 + \lambda\left(T_5^1 - 2T_4^1 + T_3^1\right) = 50 + 0.020875\left(50 - 2(50) + 50\right) = 50$

$T_5^2 = T_5^1 + \lambda\left(T_6^1 - 2T_5^1 + T_4^1\right) = T_5^1 + \lambda\left(T_4^1 - 2T_5^1 + T_4^1\right) = 50 + 0.020875\left(50 - 2\left(50\right) + 50\right) = 50$

So we see the effect of the boundary conditions. On the left edge, the temperature is slowly going down (heat escapes from the left edge), while on the right edge, the temperature is not changing, hence insulated edge.

To Verify hand solution, I have written a matlab functions that solves parabolic PDE using explicit method for fixed boundary conditions (as in example 30.1) and also a function that solves the same problem but using boundary condition as in this problem so I can see the difference more clearly for larger runs. The functions are flexible in that they accept any value for the rate at either edge and for any value of Δt and Δx. The functions will issue a warning if condition of instability is detected.

First, this is the result for the fixed boundary condition as in example 30.1, where we have the left edge at 100^0 all the time, and the right edge at 50^0 al the time, and the rod is at 0^0 initially. This display shows the output for running the solution for number of steps of 30 steps, where $\Delta t = 0.1$ seconds. Next to it is shown the solution using the bundary conditions as specifid in this problem, that is, rod initially at 50^0 and derivative at x=0 is 1, and derivative at x=10 is 0.