Problem 30.1 HW#7

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Use explicit method to solve for the temp distribution of a long thin rod with length 10 cm and $k^{'} = 0.49$ cal/(s.cm.C^0), $\Delta x = 2$ cm and $\Delta t = 0.1s$. At t=0, the tem. of the rod is zero and the boundary conditions are fixed for all time at $T(0) = 100^0C$ and $T(0) = 50^0C$. note that the rode is aluminum with C=0.2147 cal/(g.C^0) and $\rho = 2.7$ g/cm^3. therefor k=0.49/(2.7×0.2174)=0.835 cm^2/s and $\lambda = 0.835(0.1)/2^2 = 0.020875$. Use the predictor-corrector method to solve this (called Heun method) but without iterator the corrector).

Solution

We need to solve a parabolic PDE using the explicit method. But we need to apply the predictor-corrector method as well.

Recall that the solution using the parabolic PDE is written as

$$_pT_i^{l+1} = T_i^l + k\frac{\Delta t}{(\Delta x)^2}\left(T_{i+1}^l - 2T_i^l + T_{i-1}^l\right)$$

Looking at the above, we see that the future value of T is found from the current value of T plus a constant times a quantity evaluated all at the current time. This is in the same form as taking an ordinary Euler step:

$$y_{i+1} = y_i + \Delta x\ y_i^{'}$$

Hence, the above gives the idea of how to apply the corrector-predictor for the PDE case, by following the method one would use on the Euler step. In the Euler step case, to apply the predictor-correct once, we write

$y_{i+1}^p = y_i + \Delta x\ y_i^{'}$

Now find $y^{'}$ at the predicted value. Then take the average of the $y^{'}$ at the predcited value with the $y^{'}$ at the orginal y position, and use this average to find $y^{'}$ again. Write this down, we have

$y_{i+1}^c = y_i + \Delta x(\frac{y_{i+1}^{'}+y_i^{'}}{2})$

Apply the above approach to the PDE case, we have

$$_pT_i^{l+1} = T_i^l + k\frac{\Delta t}{(\Delta x)^2}\left(T_{i+1}^l - 2T_i^l + T_{i-1}^l\right) \quad (1)$$

$$_cT_i^{l+1} = T_i^l + k\frac{\Delta t}{(\Delta x)^2}\left(\frac{\left(T_{i+1}^l - 2T_i^l + T_{i-1}^l\right) + \left({}_pT_{i+1}^{l+1} - 2\ {}_pT_i^{l+1} +\ {}_pT_{i-1}^{l+1}\right)}{2}\right) \quad (2)$$

Where I write $_pT_i^{l+1}$ to mean the predicted T value, and write $_cT_i^{l+1}$ to mean the corrected T value.

Looking at the above equation, we see that to find the corrected T value at one future time unit for the position i, we need to find the predicted T value at one future time unit for the positions $i+1, i, i-1$. Hence this means we must first find the predicted T values at one unit time ahead for all the positions on the rod, before we start the correction process.

The above sets up the approach of how to solve this problem. To summariz:

1. Find the T values at time t+1 for all positions on the rod.
2. Using the above values, find the corrected T values at time t+1 for all positions on the rod
3. Now advance the time t=t+1
4. go to step 1

Now I will show the above computation for $t = 0.1$ sec and $t = 0.2$ sec
We have nodes on the rod at $x = 2, x = 4, x = 6$ and $x = 8$ since the rod is 10 cm long and we are asked to use 2cm as the position step size.
At t=0.1 , apply equation (1) above, and noting that $k\frac{\Delta t}{(\Delta x)^2} = 0.020875$ we get:
${}_pT_1^1 = 0 + 0.020875\,(0 - 2(0) + 100) = 2.0875$
${}_pT_2^1 = 0 + 0.020875\,(0 - 2(0) + 0) = 0$
${}_pT_3^1 = 0 + 0.020875\,(0 - 2(0) + 100) = 0$
${}_pT_4^1 = 0 + 0.020875\,(50 - 2(0) + 0) = 1.0438$
Now apply the correction per equation (2), which is written here again using actuall values of time for the superscript to see things more clearly:

$$ {}_cT_i^1 = T_i^0 + 0.020875\left(\frac{\left(T_{i+1}^0 - 2T_i^0 + T_{i-1}^0\right) + \left({}_pT_{i+1}^1 - 2\ \ {}_pT_i^1 +\ {}_pT_{i-1}^1\right)}{2}\right) $$

So, now start the substitusion in the above equation for i=2,4,6 and 8
${}_cT_{i=1}^1 = T_i^0 + 0.020875\left(\frac{\left(T_{i+1}^0-2T_i^0+T_{i-1}^0\right)+\left({}_pT_{i+1}^1-2\ \ {}_pT_i^1+\ {}_pT_{i-1}^1\right)}{2}\right)$
${}_cT_{i=2}^1 = T_i^0 + 0.020875\left(\frac{\left(T_{i+1}^0-2T_i^0+T_{i-1}^0\right)+\left({}_pT_{i+1}^1-2\ \ {}_pT_i^1+\ {}_pT_{i-1}^1\right)}{2}\right)$
${}_cT_{i=3}^1 = T_i^0 + 0.020875\left(\frac{\left(T_{i+1}^0-2T_i^0+T_{i-1}^0\right)+\left({}_pT_{i+1}^1-2\ \ {}_pT_i^1+\ {}_pT_{i-1}^1\right)}{2}\right)$
${}_cT_{i=4}^1 = T_i^0 + 0.020875\left(\frac{\left(T_{i+1}^0-2T_i^0+T_{i-1}^0\right)+\left({}_pT_{i+1}^1-2\ \ {}_pT_i^1+\ {}_pT_{i-1}^1\right)}{2}\right)$
Now subtitiute for correct i value in the above, they become
${}_cT_{i=1}^1 = T_1^0 + 0.020875\left(\frac{\left(T_2^0-2T_1^0+T_0^0\right)+\left({}_pT_2^1-2\ \ {}_pT_1^1+\ {}_pT_0^1\right)}{2}\right)$
${}_cT_{i=2}^1 = T_2^0 + 0.020875\left(\frac{\left(T_3^0-2T_2^0+T_1^0\right)+\left({}_pT_3^1-2\ \ {}_pT_2^1+\ {}_pT_1^1\right)}{2}\right)$
${}_cT_{i=3}^1 = T_3^0 + 0.020875\left(\frac{\left(T_4^0-2T_3^0+T_2^0\right)+\left({}_pT_4^1-2\ \ {}_pT_3^1+\ {}_pT_2^1\right)}{2}\right)$
${}_cT_{i=4}^1 = T_4^0 + 0.020875\left(\frac{\left(T_5^0-2T_4^0+T_3^0\right)+\left({}_pT_5^1-2\ \ {}_pT_4^1+\ {}_pT_3^1\right)}{2}\right)$
Hence we get
${}_cT_{i=1}^1\ = 0{+}0.020875\left(\frac{(0-2(0)+100)+(0-2\ \ (2.0875)+\ 100)}{2}\right) = 0.020875\left(\frac{(100)+(104.1750)}{2}\right) = 2.1311$
${}_cT_{i=2}^1 =\ 0 + 0.020875\left(\frac{(0-2(0)+0)+(0-2\ \ (0)+\ 2.0875)}{2}\right) = 0.020875\left(\frac{(0)+(\ 2.0875)}{2}\right) = 0.0218$
${}_cT_{i=3}^1 = 0 + 0.020875\left(\frac{(0-2(0)+0)+(1.0438-2\ \ (0)+\ 0)}{2}\right) = 0.020875\left(\frac{(0)+(1.0438\ )}{2}\right) = 0.0109$
${}_cT_{i=4}^1 =\ 0 + 0.020875\left(\frac{(50-2(0)+0)+(50-2\ \ (1.0438)+\ 0)}{2}\right) = 0.020875\left(\frac{(50\ )+(47.9124\ )}{2}\right) = 1.0220$

This completes one time step. So now the above corrected T values become the T values to use for evalutating T for the next time step. I.e. we have now

$T_{i=1}^{1} = 2.1311$
$T_{i=2}^{1} = 0.0218$
$T_{i=3}^{1} = 0.0109$
$T_{i=4}^{1} = 1.0220$

Ok, now continue. Let t=0.2 seconds, i.e. $l = 2$. Apply the same process as above:

$$
\begin{aligned}
{}_pT_{i=1}^{2} &= T_i^{1} + 0.020875\left(T_{i+1}^{1} - 2T_i^{1} + T_{i-1}^{1}\right) = T_1^{1} + 0.020875\left(T_2^{1} - 2T_1^{1} + T_0^{1}\right)\\
{}_pT_{i=2}^{2} &= T_i^{l-1} + 0.020875\left(T_{i+1}^{1} - 2T_i^{1} + T_{i-1}^{1}\right) = T_2^{1} + 0.020875\left(T_3^{1} - 2T_2^{1} + T_1^{1}\right)\\
{}_pT_{i=3}^{2} &= T_i^{l-1} + 0.020875\left(T_{i+1}^{1} - 2T_i^{1} + T_{i-1}^{1}\right) = T_3^{1} + 0.020875\left(T_4^{1} - 2T_3^{1} + T_2^{1}\right)\\
{}_pT_{i=4}^{2} &= T_i^{l-1} + 0.020875\left(T_{i+1}^{1} - 2T_i^{1} + T_{i-1}^{1}\right) = T_4^{1} + 0.020875\left(T_5^{1} - 2T_4^{1} + T_3^{1}\right)
\end{aligned}
$$

These equation now become

$$
\begin{aligned}
{}_pT_{i=1}^{2} &= 2.1311 + 0.020875\,(0.0218 - 2\,(2.1311) + 100) = 4.1301\\
{}_pT_{i=2}^{2} &= 0.0218 + 0.020875\,(0.0109 - 2\,(0.0218) + 2.1311) = 0.0656\\
{}_pT_{i=3}^{2} &= 0.0109 + 0.020875\,(1.022 - 2\,(0.0109) + 0.0218) = 0.0322\\
{}_pT_{i=4}^{2} &= 1.022 + 0.020875\,(50 - 2\,(1.022) + 0.0109) = 2.0233
\end{aligned}
$$

so

${}_pT_1^{2} = 4.1301$
${}_pT_2^{2} = 0.0656$
${}_pT_3^{2} = 0.0322$
${}_pT_4^{2} = 2.0233$

Now apply the correction

$$
\begin{aligned}
{}_cT_{i=1}^{2} &= T_1^{1} + 0.020875\left(\frac{\left(T_{i+1}^{1} - 2T_i^{1} + T_{i-1}^{1}\right) + \left({}_pT_{i+1}^{2} - 2\ {}_pT_i^{2} + {}_pT_{i-1}^{2}\right)}{2}\right)\\
{}_cT_{i=2}^{2} &= T_2^{1} + 0.020875\left(\frac{\left(T_{i+1}^{1} - 2T_i^{1} + T_{i-1}^{1}\right) + \left({}_pT_{i+1}^{2} - 2\ {}_pT_i^{2} + {}_pT_{i-1}^{2}\right)}{2}\right)\\
{}_cT_{i=3}^{2} &= T_3^{1} + 0.020875\left(\frac{\left(T_{i+1}^{1} - 2T_i^{1} + T_{i-1}^{1}\right) + \left({}_pT_{i+1}^{2} - 2\ {}_pT_i^{2} + {}_pT_{i-1}^{2}\right)}{2}\right)\\
{}_cT_{i=4}^{2} &= T_4^{1} + 0.020875\left(\frac{\left(T_{i+1}^{1} - 2T_i^{1} + T_{i-1}^{1}\right) + \left({}_pT_{i+1}^{2} - 2\ {}_pT_i^{2} + {}_pT_{i-1}^{2}\right)}{2}\right)
\end{aligned}
$$

Now subtitiute for correct i value in the above RHS, they become

$$
\begin{aligned}
{}_cT_1^{2} &= T_1^{1} + 0.020875\left(\frac{\left(T_2^{1} - 2T_1^{1} + T_0^{1}\right) + \left({}_pT_2^{2} - 2\ {}_pT_1^{2} + {}_pT_0^{2}\right)}{2}\right)\\
{}_cT_2^{2} &= T_2^{1} + 0.020875\left(\frac{\left(T_3^{1} - 2T_2^{1} + T_1^{1}\right) + \left({}_pT_3^{2} - 2\ {}_pT_2^{2} + {}_pT_1^{2}\right)}{2}\right)\\
{}_cT_3^{2} &= T_3^{1} + 0.020875\left(\frac{\left(T_4^{1} - 2T_3^{1} + T_2^{1}\right) + \left({}_pT_4^{2} - 2\ {}_pT_3^{2} + {}_pT_2^{2}\right)}{2}\right)\\
{}_cT_4^{2} &= T_4^{1} + 0.020875\left(\frac{\left(T_5^{1} - 2T_4^{1} + T_3^{1}\right) + \left({}_pT_5^{2} - 2\ {}_pT_4^{2} + {}_pT_3^{2}\right)}{2}\right)
\end{aligned}
$$

Hence we get

$$
{}_cT_1^{2} = 2.1311 + 0.020875\left(\frac{(0.0218 - 2(2.1311) + 100) + (0.0656 - 2\ (4.1301) + 100)}{2}\right) = 4.0888
$$

$$ {}_cT_2^2 = 0.0218 + 0.020875\left(\frac{(0.0109-2(0.0218)+2.1311)+(0.0322-2\ \ (0.0656)+\ 4.1301)}{2}\right) = 0.0858 $$
$$ {}_cT_3^2 = 0.0109 + 0.020875\left(\frac{(\ 1.0220-2(0.0109)+\ 0.0218)+(2.0233-2\ \ (0.0322)+\ 0.0656)}{2}\right) = 0.0427 $$
$$ {}_cT_4^2 = \ \ 1.0220 + 0.020875\left(\frac{(50-2(1.0220)+0.0109)+(50-2\ \ (2.0233)+\ 0.0322)}{2}\right) = 2.0026 $$

i.e.
${}_cT_1^2 = 4.0888$
${}_cT_2^2 = 0.0858$
${}_cT_3^2 = 0.0427$
${}_cT_4^2 = 2.0026$

Compare the above to T found at second time step without the use of the predictor-corrector method as shown in the example:
$T_1^2 = 4.0878$
$T_2^2 = 0.043577$
$T_3^2 = 0.021788$
$T_4^2 = 2.0439$