HW5, Problem 25.3.

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Problem: Use heun method with h=0.5 to solve $\frac{dy}{dx} = yx^2 - 1.2y$ over the interval $x = 0$ to 2 with $y(0) = 1$. Iterate the corrector to $\varepsilon_s = 1\%$

Solution:

We are asked to stop the corrector iteration when $\varepsilon_s = 1\%$ this means $|\varepsilon_a| < \varepsilon_s$, where ε_a is the relative error normalized to the approximation result. i.e. $\varepsilon_a = \left|\frac{approx.\ step\ i\ -\ approx\ step\ i-1}{approx\ step\ i}\right| \asymp 100\%$

I illustrate the method, then write a matlab function to implement it.

Recall that in Euler method, we obtain an estimate to future value of y (at next step) by using the value for the rate of change of y at the current step. i.e. $y_{i+1} = y_i + h\ y_i^{'}$. i.e. we assume that the rate of change of y has remained the same over the duration of the step size.

In Heun method, y', is taken as the average rate of change of y between the start and end of the step, i.e. between i and $i+1$, instead of just at start of the step, i.e. at i.

Using this averaged value for y', we now obtain the future value of y (called the corrected value for y).

In Order to find the average rate of change over the step duration, we therefor need to first find the rate of change at the end of the step. For this, we use predicted value for y_{i+1} in the expression for y'. Once we found the rate of change at the end of the step, we take the average, and find the corrected value for y. If this corrected value for y has a relative error $\varepsilon_a > \varepsilon_s$, then we repeat the process. This is continued, until $|\varepsilon_a| < \varepsilon_s$. Then the final corrected value of y is used for the calculation of the next step.

Algorithm

```
make_a_step:
   Find derivative of y at start of current step. Use initial conditions for y at start of f
   call this the start_step_derivative.
   Use this derivative to make a Euler step to find y at end of current step. This is predic

   try_again:
      Find derivative of y at end of current step using the found predicted y.
      call this the end_step_derivative.
      Take the average of the above two derivatives: the start_step_derivative and end_step_
      Use this average to make a Euler step from start of current step to find y again at en
      This y is the corrected y.
      Compare the corrected y with the predicted y.
      IF difference is less than tolerance THEN
         y=corrected_y ------- i.e. use the corrected y as the final y for current step
         advance x to start of next step. next step become currect step.
         GOTO make_a_step
      ELSE
         predicted Y = corrected Y.
         GOTO try_again
```

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END
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Things to notice: The start_step_derivative is fixed and do not change during the predictor-corrector loop.

What changes is the end_step_derivative only. Then the average is taken again between the above 2 derivatives.

As the y solution at the end of the steps improves, the end_step_derivatives keeps changes (it becomes better and better) but the start_step_derivative is fixed. But since we are taking the average of those two, we are getting a better final y.

Intuitively, the predictor-corrector makes sense. Imagine the derivative to be the velocity of a car, and y to be the position of the car.

When we use the velocity at the start of a time step to find the new position of the car after some delta time interval, we assume that the velocity did not change during this delta time.

If the velocity changes with time, i.e. velocity is not constant, then we expect the final position of the car to be different from the true position. What huen method does is take the average velocity between the end of the time step and the start of the time step to find where the car will be if it had this average velocity at the start of the step. In Each step, it checks if the position of the car has not changed much by using this average velocity. If the position at the end of the delta time is still changing, then the velocity of the car is calculated again at the new position, and the average of the velocities is found again, and the new position is found. This process is repeated until we found that the new position of the car does not change by too much.

$h = 0.5$
step $i = 0$
$y_0' = y_0\left(x_0^2 - 1.2\right) = 1\left(0 - 1.2\right) = -1.2$
$y_1 = y_0 + h\ y_0' = 1 + 0.5\left(-1.2\right) = 0.4$
Use y_1 to find the rate of change at end of the step:
$y_1' = y_1\left(x_1^2 - 1.2\right) = 0.4\left(0.5^2 - 1.2\right) = -0.38$
Now take the average $y_{average}' = \frac{y_0' + y_1'}{2} = \frac{-1.2 - 0.38}{2} = -0.79$
Now, use this averaged value to find y (predicted value) at the end of the step again
$y_1 = y_0 + h\ y_{average}' = 1 + 0.5\left(-0.79\right) = 0.605$
Now, find the relative error between the two solution for y obtained above
$\varepsilon_a = \left|\frac{y_{corrected} - y}{y_{corrected}}\right| 100 = \left|\frac{0.605 - 0.4}{0.605}\right| 100 = 33.8\%$
This value is larger than $\varepsilon_s = 1\%$ so must repeat the process again.

Use the newly predicted value of y to find the rate of change at $i + 1$ and take the average again
$y_1' = y_1\left(x_1^2 - 1.2\right) = 0.605\left(0.5^2 - 1.2\right) = -0.57475$

$y'_{average} = \frac{y'_0 + y'_1}{2} = \frac{-1.2 - 0.57475}{2} = -0.887375$

Use this new averaged value of rate of change to find a new predicted value for y

$y_1 = y_0 + h\ y'_{average} = 1 + 0.5\,(-0.887375) = 0.5563125$

Now, find the relative error between the two solution for y obtained above

$\varepsilon_a = \left| \frac{y_{corrected} - y}{y_{corrected}} \right| 100 = \left| \frac{0.5563125 - 0.605}{0.5563125} \right| 100 = 8.751\%$

This value is larger than $\varepsilon_s = 1\%$ so must repeat the process again.

Use the newly predicted value of y to find the rate of change at $i+1$ and take the average again

$y'_1 = y_1 \left(x_1^2 - 1.2\right) = 0.5563125 \left(0.5^2 - 1.2\right) = -0.528496875$

$y'_{average} = \frac{y'_0 + y'_1}{2} = \frac{-1.2 - 0.528496875}{2} = -0.8642484375$

Use this new averaged value of rate of change to find a new predicted value for y

$y_1 = y_0 + h\ y'_{average} = 1 + 0.5\,(-0.8642484375) = 0.56787578125$

Now, find the relative error between the two solution for y obtained above

$\varepsilon_a = \left| \frac{y_{corrected} - y}{y_{corrected}} \right| 100 = \left| \frac{0.56787578125 - 0.5563125}{0.56787578125} \right| 100 = 2.036\ \%$

This value is larger than $\varepsilon_s = 1\%$ so must repeat the process again.

Use the newly predicted value of y to find the rate of change at $i+1$ and take the average again

$y'_1 = y_1 \left(x_1^2 - 1.2\right) = 0.56787578125 \left(0.5^2 - 1.2\right) = -0.539482$

$y'_{average} = \frac{y'_0 + y'_1}{2} = \frac{-1.2 - 0.539482}{2} = -0.869741$

Use this new averaged value of rate of change to find a new predicted value for y

$y_1 = y_0 + h\ y'_{average} = 1 + 0.5\,(-0.869741) = 0.5651295$

Now, find the relative error between the two solution for y obtained above

$\varepsilon_a = \left| \frac{y_{corrected} - y}{y_{corrected}} \right| 100 = \left| \frac{0.5651295\ - 0.56787578125}{0.5651295} \right| 100 = 0.485\ \%$

This value is smaller than $\varepsilon_s = 1\%$ so we take this value for y as the final corrected value to use for the next step.

step $i = 1$

$y'_1 = y_1 \left(x_1^2 - 1.2\right) = 0.5651295\ \left(0.5^2 - 1.2\right) = -0.536872$

$y_2 = y_1 + h\ y'_1 = 0.5651295 + 0.5\,(-0.536872) = 0.2966935$

Use y_2 to find the rate of change at end of the step:

$y'_2 = y_2 \left(x_2^2 - 1.2\right) = 0.2966935 \left(1^2 - 1.2\right) = -0.0593387$

$y'_{average} = \frac{y'_1 + y'_2}{2} = \frac{-0.536872 - 0.0593387}{2} =$ -0.29810535

Now use the above averaged value to find a corrected value for y_2

$y_2 = y_1 + h\ y'_{average} = 0.5651295 + 0.5\,(-0.29810535) = 0.416076825$

Now, find the relative error between the two solution for y obtained above

$\varepsilon_a = \left| \frac{y_{corrected} - y}{y_{corrected}} \right| 100 = \left| \frac{0.416076825\ - 0.2966935}{0.416076825} \right| 100 = 28.69\ \%$

This value is larger than $\varepsilon_s = 1\%$ so must repeat the process again.

The above process is repeated until all of values for y over the domain x are found.

A Matlab function called nma_euler_heun2.m is written to implement the above. It accepts as input the function that represents the rate of change, i.e. $y^{'} = f(x, y)$, and accepts h the step size, and the domain range for the independent variable (starting and ending values), and ε_s the tolerance to use for the Heun algorithm (in percentage). As output it will generate a table showing the progress of the iteration at each step to refine the corrected y from the predicted y, and then it will plot the solution of the ODE using this method and compares it against the true and the Euler.