HW 5. CEE 247. Structural Dynamics. UCI. Fall 2006.

Nasser Abbasi

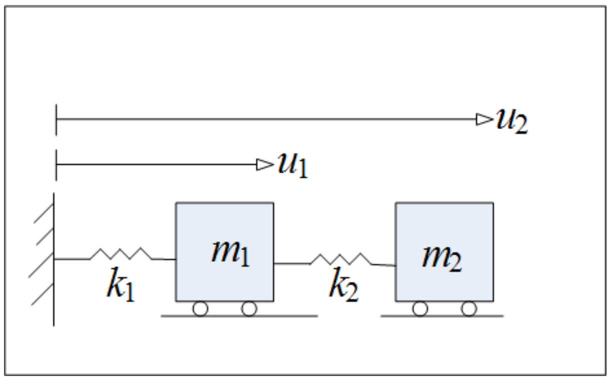
April 16, 2008

Contents

1 Problem 7.1

Solution

The idealized physical system is the following



The Lagrangian of the system is

$$
\begin{aligned}
L &= KE - PE \\
&= \left(\frac{1}{2}m_1\dot{u}_1^2 + \frac{1}{2}m_1\dot{u}_1^2\right) - \left(\frac{1}{2}k_1u_1^2 + \frac{1}{2}k_2\left(u_2 - u_1\right)^2\right) \\
&= \frac{1}{2}m_1\dot{u}_1^2 + \frac{1}{2}m_1\dot{u}_1^2 - \frac{1}{2}k_1u_1^2 - \frac{1}{2}k_2\left(u_2 - u_1\right)^2
\end{aligned}
$$

Now apply Euler equation on the Lagrangian to obtain the equation of motion for each degree of freedom. Given L the equation of motion for u_i is given by $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_i}\right) - \frac{\partial L}{\partial u_i} = 0$

Hence the equation of motion associated with u_1 is given by

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_1}\right) - \frac{\partial L}{\partial u_1} &= 0 \\
\frac{d}{dt}\left(m_1\dot{u}_1\right) - \left(-k_1u_1 - k_2\left(u_2 - u_1\right) \times -1\right) &= 0 \\
m_1\ddot{u}_1 + k_1u_1 - k_2u_2 + k_2u_1 &= 0 \\
m_1\ddot{u}_1 + u_1\left(k_1 + k_2\right) - u_2k_2 &= 0 \qquad (1)
\end{aligned}
$$

And the equation of motion associated with u_2 is given by

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_2}\right) - \frac{\partial L}{\partial u_2} &= 0 \\
\frac{d}{dt}\left(m_2 \dot{u}_2\right) + k_2\left(u_2 - u_1\right) &= 0 \\
m_2 \ddot{u}_2 - u_1 k_2 + u_2 k_2 &= 0 \qquad (1)
\end{aligned}$$

Hence the equation of motions are

$$\begin{aligned}
m_1 \ddot{u}_1 + u_1\left(k_1 + k_2\right) - u_2 k_2 &= 0 \\
m_2 \ddot{u}_2 - u_1 k_2 + u_2 k_2 &= 0
\end{aligned}$$

Hence the overall system EQM can be put in a matrix form as follows

$$\boxed{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0} \tag{3}$$

Notice that the mass matrix M and the stiffness matrix K are symmetric. This will always be the case for conservative systems.

Equation (3) can be written as

$$[M]\{\ddot{u}\} + [K]\{u\} = 0 \tag{4}$$

Now assume the solution is given by

$$\{u\} = \{a\} e^{i\omega t} \tag{5}$$

Substitute (5) into (4) we obtain

$$-\omega^2 [M]\{a\} e^{i\omega t} + [K]\{a\} e^{i\omega t} = 0$$

Since $e^{i\omega t} \neq 0$ we divide by it and obtain

$$-\omega^2 [M]\{a\} + [K]\{a\} = 0$$

Factor out $\{a\}$

$$\left([K] - \omega^2 [M]\right)\{a\} = 0$$

To have a non-trivial solution for the motion the above implies that the determinant of $([K] - \omega^2 [M])$ must be zero. Hence we need to solve

$$\det \left([K] - \omega^2 [M]\right) = 0$$

Let $\lambda = \omega^2$, and expand the matrices and rewrite we obtain

$$\begin{aligned}
\det\left(\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} - \lambda \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right) &= 0 \\
\det\left(\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} - \begin{bmatrix} \lambda m_1 & 0 \\ 0 & \lambda m_2 \end{bmatrix}\right) &= 0 \\
\det\begin{bmatrix} k_1+k_2-\lambda m_1 & -k_2 \\ -k_2 & k_2-\lambda m_2 \end{bmatrix} &= 0 \\
(k_1+k_2-\lambda m_1)(k_2-\lambda m_2) - k_2^2 &= 0 \\
k_1k_2 - k_1\lambda m_2 + k_2^2 - k_2\lambda m_2 - k_2\lambda m_1 + \lambda^2 m_1 m_2 - k_2^2 &= 0 \\
k_1k_2 - k_1\lambda m_2 - k_2\lambda m_2 - k_2\lambda m_1 + \lambda^2 m_1 m_2 &= 0 \\
\lambda^2 m_1 m_2 - k_1\lambda m_2 - k_2\lambda m_2 - k_2\lambda m_1 + k_1k_2 &= 0 \qquad (1) \\
\lambda^2 m_1 m_2 - \lambda\left((k_1+k_2)m_2 + k_2 m_1\right) + k_1k_2 &= 0 \\
\lambda^2 - \lambda\frac{((k_1+k_2)m_2 + k_2m_1)}{m_1m_2} + \frac{k_1k_2}{m_1m_2} &= 0 \qquad (6)
\end{aligned}$$

Now find the numerical values for k_1, k_2, m_1, m_2 and plug into the above equation to find $\lambda_{1,2}$

$$k_1 = \frac{12\ EI}{L^3} = \frac{12\ (5\times 10^8)}{(15\times 12)^3} = 1028.8 \text{ lb/in}$$

$$k_2 = \frac{12\ EI}{L^3} = \frac{12\ (2.5\times 10^8)}{(12\times 12)^3} = 1004.7 \text{ lb/in}$$

and

$$m_1 = \frac{W_1}{g} = \frac{3860}{386} = 10\ lb$$

$$m_2 = \frac{W_2}{g} = \frac{1930}{386} = 5\ lb$$

Hence eq (6) above becomes

$$\lambda^2 - \lambda\frac{((1028.8+1004.7)\,5 + 1004.7\times 10)}{50} + \frac{1028.8\times 1004.7}{50} = 0$$

Hence

$$\boxed{\lambda^2 - 404.29\ \lambda + 20673 = 0}$$

Hence this is now in standard quadratic format, solve for λ

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{404.29 \pm \sqrt{(-404.29)^2 - 4 \times 20673}}{2}$$

$$\lambda = \frac{404.29 \pm 284.18}{2}$$

Hence

$$\lambda_1 = \frac{404.29 - 284.18}{2} = 60.055$$

and

$$\lambda_2 = \frac{404.29 + 284.18}{2} = 344.24$$

Since $\lambda_1 = \omega_1^2$ then

$$\omega_1 = \sqrt{60.055} = \boxed{7.7495 \text{ rad/sec}}$$

and similarly

$$\omega_2 = \sqrt{344.24} = \boxed{18.554 \text{ rad/sec}}$$

Now to find the eigenvectors, since

$$\left([K] - \omega^2 [M]\right) \{a\} = 0$$

Then

$$\begin{bmatrix} k_1 + k_2 - \omega_i^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega_i^2 m_2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_i = 0$$

$$\begin{bmatrix} 1028.8 + 1004.7 - 10\omega_i^2 & -1004.7 \\ -1004.7 & 1004.7 - 5\omega_i^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_i = 0$$

For the first eigenvalue $\omega_1 = 7.7495$ the above becomes

$$\begin{bmatrix} 1028.8 + 1004.7 - 10 \times 7.7495^2 & -1004.7 \\ -1004.7 & 1004.7 - 5 \times 7.7495^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_1 = 0$$

$$\begin{bmatrix} 1433.0 & -1004.7 \\ -1004.7 & 704.43 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0$$

From first equation we obtain

$$1433.0a_1 - 1004.7a_2 = 0$$

Hence

$$\frac{a_1}{a_2} = \frac{1004.7}{1433.0} = 0.70112$$

Hence we choose the first eigenvector to be

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_1 = \boxed{\begin{Bmatrix} 0.70112 \\ 1 \end{Bmatrix}}$$

For the second eigenvalue $\omega_2 = 18.554$

$$\begin{bmatrix} 1028.8 + 1004.7 - 10 \times 18.554^2 & -1004.7 \\ -1004.7 & 1004.7 - 5 \times 18.554^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_2 = 0$$

$$\begin{bmatrix} -1409.0 & -1004.7 \\ -1004.7 & -716.55 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_2 = 0$$

From first equation we obtain

$$-1409\ a_1 - 1004.7 a_2 = 0$$

Hence

$$\frac{a_1}{a_2} = \frac{1004.7}{-1409} = -0.71306$$

Hence we choose the second eigenvector to be

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_2 = \boxed{\begin{Bmatrix} -0.71306 \\ 1 \end{Bmatrix}}$$

Conclusion

$$\omega_1 = \boxed{7.7495 \text{ rad/sec}}$$
$$\omega_2 = \boxed{18.554 \text{ rad/sec}}$$

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_1 = \boxed{\begin{Bmatrix} 0.70112 \\ 1 \end{Bmatrix}}$$

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_2 = \boxed{\begin{Bmatrix} -0.71306 \\ 1 \end{Bmatrix}}$$

2 Problem 7.6

Answer

I wrote a Mathematica program to solve this. This is the result, and below that I attach step by step run of the program