HW 3. CEE 247. Structural Dynamics. UCI. Fall 2006

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Contents

1 Formulas used

Integration by parts is used in many problems below to solve $\int_a^b \tau \sin\omega_n (t-\tau)\, d\tau$. I derive it once.

Let $u=\tau$ and $dv = \sin\omega_n (t-\tau)$ hence $du = 1$ and $v = \int \sin\omega_n (t-\tau) = \frac{\cos\omega_n (t-\tau)}{\omega_n}$

Hence

$$\begin{aligned}
\int_a^b \tau \sin(t-\tau)\, d\tau &\equiv \int u dv \\
&= uv - \int v du \\
&= \left[\tau \frac{\cos\omega_n (t-\tau)}{\omega_n}\right]_a^b - \frac{1}{\omega_n}\int_a^b \cos\omega_n (t-\tau)\, d\tau \\
&= \frac{1}{\omega_n}\left[\tau \cos\omega_n (t-\tau)\right]_a^b - \frac{1}{\omega_n}\left[\frac{\sin\omega_n (t-\tau)}{-\omega_n}\right]_a^b \\
&= \frac{1}{\omega_n}\left[\tau \cos\omega_n (t-\tau)\right]_a^b + \frac{1}{\omega_n^2}\left[\sin\omega_n (t-\tau)\right]_a^b
\end{aligned}$$

Hence the integral I becomes

$$\boxed{\int_a^b \tau \sin(t-\tau)\, d\tau = \frac{1}{\omega_n^2}\left[\tau\omega \cos\omega_n (t-\tau) + \sin\omega_n (t-\tau)\right]_a^b}$$

The above is the form to remember.

or

$$I = \frac{1}{\omega_n}\left[b\cos\omega_n (t-b) - a\cos\omega_n (t-a)\right] + \frac{1}{\omega_n^2}\left[\sin\omega_n (t-b) - \sin\omega_n (t-a)\right]$$

For example, when $a=0$, $b=t$, we obtain

$$\int_0^t \tau \sin\omega_n (t-\tau)\, d\tau = \frac{t}{\omega_n} - \frac{\sin\omega_n t}{\omega_n^2}$$

2 Problem 4.3

Problem 4.3

The steel frame shown in Fig. P4.3 is subjected to a horizontal force F(t) applied at the girder level. The force decreases linearly from 5 kip at time $t = 0$ to zero at $t = 0.6$ sec. Determine: (a) the horizontal deflection at $t = 0.5$ sec and (b) the maximum horizontal deflection. Assume the columns massless and the girder rigid. Neglect damping.

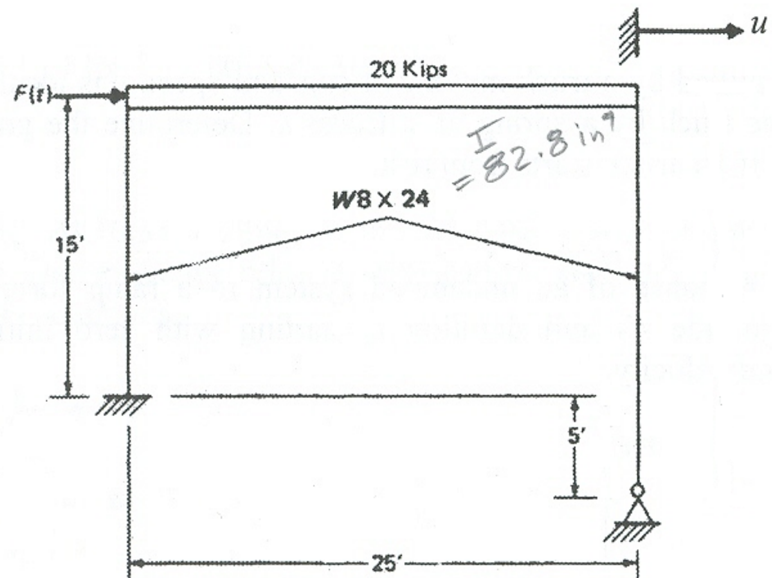
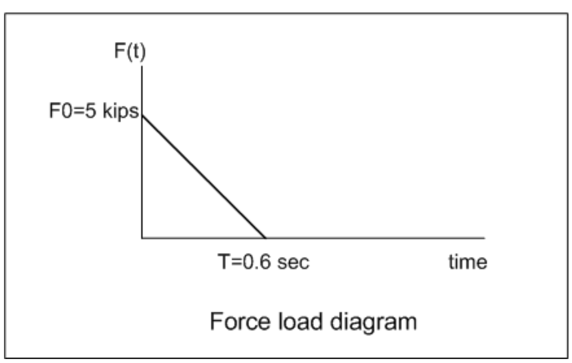


Fig. P4.3

Solution

We first assume that the initial absolute state of the girder is $u(0) = 0$, and $\dot{u}(0) = 0$

This is the force load diagram

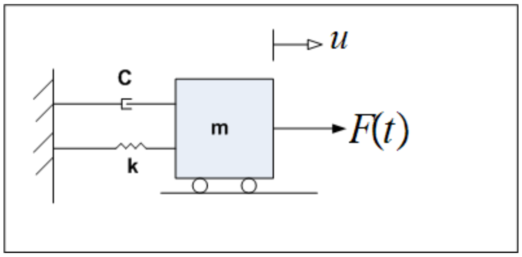


Force load diagram

The intercept is F_0 and the slope is $-\frac{F_0}{t_d}$ hence since the general line equation for $y(x)$ is $y = intercept + slope * x$, we see that the equation for force loading is

$$\boxed{F(t) = F_0 - \frac{F_0}{t_d} t = F_0 \left(1 - \frac{t}{t_d}\right)} \tag{1}$$

First we draw the physical model diagram



Using Duhamel integral, the displacement $u(t)$ is (using the assumption of no damping)

$$u(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin\omega_n (t - \tau)\, d\tau$$

Substitute (1) into the above and carry the integration.

$$\begin{aligned}
u(t) &= \frac{F_0}{m\omega_n} \int_0^t \left(1 - \frac{\tau}{t_d}\right) \sin\omega_n (t-\tau)\, d\tau \\
&= \frac{F_0}{m\omega_n} \int_0^t \sin\omega_n (t-\tau)\, d\tau - \frac{F_0}{m\omega_n t_d} \overbrace{\int_0^t \tau \sin\omega_n (t-\tau)\, d\tau}^{\text{by parts}} \\
&= \frac{F_0}{m\omega_n^2} \left(\cos\omega_n (t-\tau)\right)_0^t - \frac{F_0}{m\omega_n t_d} \left(\frac{1}{\omega_n^2} \left[\tau\omega \cos\omega_n (t-\tau) + \sin\omega_n (t-\tau)\right]_0^t\right) \\
&= \frac{F_0}{m\omega_n^2} (1 - \cos\omega_n t) - \frac{F_0}{m\omega_n^3 t_d} (t\omega - \sin\omega_n t) \\
&= \frac{F_0}{m\omega_n^2} (1 - \cos\omega_n t) + \frac{F_0}{m\omega_n^2 t_d} \left(\frac{\sin\omega_n t}{\omega_n} - t\right)
\end{aligned}$$

But $\omega_n^2 = \frac{k}{m}$ hence the above becomes

$$u(t) = \frac{F_0}{k} (1 - \cos\omega_n t) + \frac{F_0}{k\ t_d} \left(\frac{\sin\omega_n t}{\omega_n} - t\right) \tag{2}$$

Now to find the stiffness k :

$$
\begin{aligned}
k &= \frac{12EI}{L_{left}^3} + \frac{3EI}{L_{right}^3} \\
&= 3EI\left(\frac{4}{(15\times 12)^3} + \frac{1}{(20\times 12)^3}\right) \\
&= 3\times 30\times 10^6 \times 82.8\left(\frac{4}{(15\times 12)^3} + \frac{1}{(20\times 12)^3}\right) \\
&= \boxed{5650.2 \ \ \text{lb/in}}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\omega_n &= \sqrt{\frac{k}{m}} \\
&= \sqrt{\frac{5650.2\times 386}{20\times 10^3}} \\
&= \boxed{10.443 \text{ rad/sec}}
\end{aligned}
$$

Now substitute the above results for k and ω_n in equation (2), and evaluate at $t = 0.5$ we obtain

$$
\begin{aligned}
u(t) &= \frac{F_0}{k}(1-\cos\omega_n t) + \frac{F_0}{k\ t_d}\left(\frac{\sin\omega_n t}{\omega_n} - t\right) \\
&= \frac{5\times 10^3}{5650.2}(1-\cos(10.443\times 0.5)) + \frac{5\times 10^3}{5650.2\times\ 0.6}\left(\frac{\sin(10.443\times 0.5)}{10.443} - 0.5\right) \\
&= \boxed{-0.40715 \text{ inch}}
\end{aligned}
$$

part(b)

To find maximum displacement $u_{\max}(t)$ we use the response spectrum shown on page 107 of the 5th edition of the text book. First we find the natural period T.

$$
\begin{aligned}
T &= \frac{2\pi}{\omega_n} = \frac{2\pi}{10.443} \\
&= \boxed{0.60166 \text{ sec}}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{t_d}{T} &= \frac{0.6}{0.60166} \\
&= \boxed{0.99724}
\end{aligned}
$$

Hence from the spectrum on page 107, we see that

$$
DLF_{\max} = 1.55
$$

approximately

But

$$DLF_{\max} = \frac{u_{\max}}{u_{st}} = \frac{u_{\max}}{\frac{F_0}{k}}$$

Hence

$$\begin{aligned} u_{\max} &= DLF_{\max}\left(\frac{F_0}{k}\right) \\ &= 1.55\left(\frac{5\times 10^3}{5650.2}\right) \\ &= \boxed{1.3716 \quad \text{inch}} \end{aligned}$$

This is a small program to plot u(t) itself. We see that $u(t)$ became maximum before t_d. $u(t)$ maximum as at about $t = 0.25\,\text{sec}$

```
Remove["Global`*"];
(*verification code for problem 4.3, CEE 477. By Nasser Abbasi*)
m= 20*10^3/386 ; k=5650.2; td=0.6; f0=5*10^3;
f[t_] := Piecewise[{ {0, t<0}, {f0 (1- t/td), t≥0&&t≤td}, {0, t>td} }]
Plot[f[t], {t, 0, 2}, PlotLabel-> "F(t) for problem 4.3", AxesLabel→{"t", "F(t)"}]
sol= Flatten[DSolve[{m u''[t] + k u[t] == f[t], u[0] == 0, u'[0] == 0}, u[t], t]];
Plot[u[t] /. sol, {t, 0, 4}, PlotLabel-> "u(t) for problem 4.3", AxesLabel→{"t", "u(t)"}]
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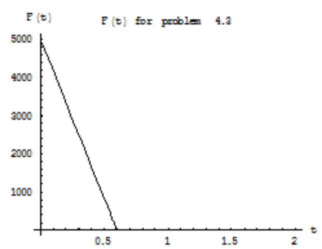
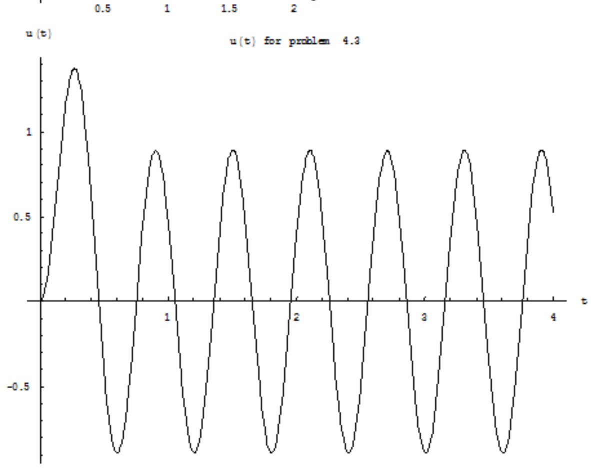




Out[115]= - Graphics -

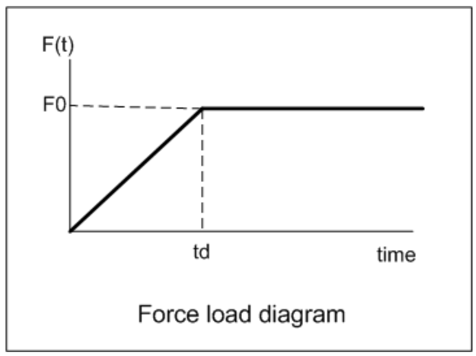
3 Problem 4.5

Problem 4.5

For the load-time function in Fig. P4.5, derive the expression for the dynamic load factor for the undamped simple oscillator as a function of t, ω, and t_d.

solution

fig P4.5 is



Force load diagram

$$DLF = \frac{u(t)}{u_{st}}$$

Hence we need to find $u(t)$

For $t \leq t_d$ and for an undamped simple oscillator, using Duhamel integral, the displacement $u(t)$ is

$$\begin{aligned}
u(t) &= \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau \\
&= \frac{1}{m\omega_n}\int_0^t \frac{F_0}{t_d}\tau\sin\omega_n(t-\tau)\,d\tau \\
&= \frac{F_0}{t_d m\omega_n}\overbrace{\int_0^t \tau\sin\omega_n(t-\tau)\,d\tau}^{\text{integrate by parts}} \\
&= \frac{F_0}{t_d m\omega_n}\left(\frac{1}{\omega_n^2}\left[\tau\omega\cos\omega_n(t-\tau)+\sin\omega_n(t-\tau)\right]_0^t\right) \\
&= \frac{F_0}{t_d m\omega_n}\frac{1}{\omega_n^2}\left(t\omega_n-\sin\omega_n t\right)
\end{aligned}$$

$\omega_n^2 = \frac{k}{m}$ hence the above becomes

$$\boxed{u(t) = \frac{F_0}{t_d\omega_n}\frac{1}{k}\left(t\omega_n-\sin\omega_n t\right)}$$

Now we find DLF

$$
\begin{aligned}
DLF &= \frac{u(t)}{u_{st}} = \frac{u(t)\,k}{F_0} \\
&= \frac{\frac{F_0}{t_d\omega_n}\frac{1}{k}\left(t\omega_n - \sin\omega_n t\right)k}{F_0} \\
&= \frac{t\omega_n - \sin\omega_n t}{t_d\omega_n}
\end{aligned}
$$

Hence

$$
\boxed{DLF = \tfrac{t}{t_d} - \tfrac{\sin\omega_n t}{t_d\omega_n}}
$$

Now we do the case for $t \geq t_d$

$$
\begin{aligned}
u(t) &= \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau \\
&= \frac{1}{m\omega_n}\left(\int_0^{t_d}\frac{F_0}{t_d}\tau\sin\omega_n(t-\tau)\,d\tau + \int_{t_d}^t F_0\sin\omega_n(t-\tau)\,d\tau\right) \\
&= \frac{F_0}{m\omega_n}\left(\frac{1}{t_d}\overbrace{\int_0^{t_d}\tau\sin\omega_n(t-\tau)\,d\tau}^{\text{integrate by parts}} + \int_{t_d}^t \sin\omega_n(t-\tau)\,d\tau\right) \\
&= \frac{F_0}{m\omega_n}\left(\frac{1}{t_d}\left\{\frac{\tau\omega_n\cos\omega_n(t-\tau)+\sin\omega_n(t-\tau)}{\omega_n^2}\right\}_0^{t_d} + \left\{\frac{\cos\omega_n(t-\tau)}{\omega_n}\right\}_{t_d}^t\right) \\
&= \frac{F_0}{m\omega_n^2}\left(\frac{1}{\omega_n t_d}\left\{t_d\omega_n\cos\omega_n(t-t_d)+\sin\omega_n(t-t_d)-\sin\omega_n t\right\}\right) + \\
&\quad \frac{F_0}{m\omega_n^2}\left\{\cos\omega_n(t-t)-\cos\omega_n(t-t_d)\right\} \\
&= \frac{F_0}{m\omega_n^2}\left(\cos\omega_n(t-t_d)+\frac{1}{\omega_n t_d}\sin\omega_n(t-t_d)-\frac{1}{\omega_n t_d}\sin\omega_n t\right) + \\
&\quad \frac{F_0}{m\omega_n^2}\left(1-\cos\omega_n(t-t_d)\right) \\
&= \frac{F_0}{m\omega_n^2}\left(1+\frac{1}{\omega_n t_d}\sin\omega_n(t-t_d)-\frac{1}{\omega_n t_d}\sin\omega_n t\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
DLF &= \frac{u(t)}{u_{st}} = \frac{u(t)\,k}{F_0} \\
&= \frac{\frac{F_0}{m\omega_n^2}\left(1+\frac{1}{\omega_n t_d}\sin\omega_n(t-t_d)-\frac{1}{\omega_n t_d}\sin\omega_n t\right)k}{F_0}
\end{aligned}
$$

But $\omega_n^2 = \frac{k}{m}$ hence the above becomes

$$
\begin{aligned}
DLF &= 1+\frac{1}{\omega_n t_d}\sin\omega_n(t-t_d)-\frac{1}{\omega_n t_d}\sin\omega_n t \\
&= \boxed{1+\frac{1}{\omega_n t_d}\left(\sin\omega_n(t-t_d)-\sin\omega_n t\right)}
\end{aligned}
$$

Notice there is a sign difference with the answer on the back of the book. The back of the book gives

$$
DLF = 1+\frac{1}{\omega_n t_d}\left(\sin\omega_n t-\sin\omega_n(t+t_d)\right)
$$

I think the answer in the back of the book is wrong. One way to obtain the book answer from my answer is to replace t by $-t$.

4 Problem 4.6

Frame shown in problem 4.3 above is subjected to sudden acceleration of 0.5 g applied to the foundation. Determine the maximum shear force in the columns. Neglect damping.

solution

The equation for motion when the system is subjected to ground acceleration can be written as

$$m\ddot{u}_r + ku_r = -m\ddot{u}_g$$

Where u_r is the relative motion of the girder to the ground, and $\ddot{u}_g$ is the ground acceleration (absolute). Hence $-m\ddot{u}_g$ is the effective force F_e

Hence this is the same problem as

$$m\ddot{u}_r + ku_r = F_e$$

which has the solution

$$\begin{aligned} u_r &= \frac{1}{m\omega_n}\int_0^t F_e \sin\omega_n (t-\tau)\, d\tau \\ &= \frac{F_e}{m\omega_n^2}\left(\cos\omega_n (t-\tau)\right)_0^t \\ &= \frac{F_e}{k}\left(1-\cos\omega_n t\right) \end{aligned}$$

Hence

$$u_{r\max} = \frac{2F_e}{k}$$

But from problem 4.3, we calculated k to be 5650.2 lb/in, hence

$$\begin{aligned} u_{r\max} &= \frac{2F_e}{k} \\ &= \frac{2\left(-m\ddot{u}_g\right)}{k} \\ &= \frac{-2\times\frac{20\times10^3}{g}\times 0.5\times g}{5650.2} \\ &= \frac{-20\times10^3}{5650.2} \\ &= \boxed{-3.5397} \text{ inch} \end{aligned}$$

Now maximum shear is given by $ku_{\max}$, hence for the left column we have (I will take absolute value of displacement, since we are only interested in maximum value, the sense of shear is not relevant).

$$\begin{aligned}\text{maximum shear} &= \frac{12EI}{L_{left}^3} u_{\max} \\ &= \frac{12 \times 30 \times 10^6 \times 82.8}{(15 \times 12)^3} \times 3.5397 \\ &= \boxed{18,092 \quad \text{lb}}\end{aligned}$$

and for the right column

$$\begin{aligned}\text{maximum shear} &= \frac{3EI}{L_{right}^3} u_{\max} \\ &= \frac{3 \times 30 \times 10^6 \times 82.8}{(20 \times 12)^3} \times 3.5397 \\ &= \boxed{1,908.1 \quad \text{lb}}\end{aligned}$$

5 Problem 4.14

Problem 4.14.

For the dynamic system shown in Fig. P4.14, determine and plot the displacement as a function of time for the interval $0 \le t \le 0.5$ sec. Neglect damping.

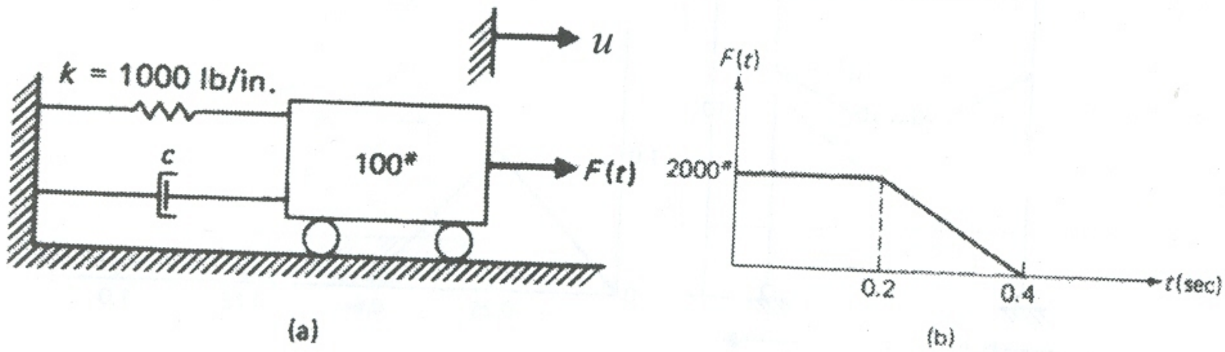


Fig. P4.14

solution

$m = 100$ lb

$F_0 = 2000$ lb

$k = 1000$ lb/in

Let $t_1 = 0.2\,\text{sec}$

$t_2 = 0.4\,\text{sec}$

Hence for $t \le t_1$, $\boxed{F(t) = F_0}$

For $t_1 \le t \le t_2$, $\boxed{F(t) = F_0(1 - \frac{(t-t_1)}{0.2})}$

For an undamped simple oscillator, using Duhamel integral, the displacement $u(t)$ is

Hence for $0 \le t \le t_1$

$$\begin{aligned} u_1(t) &= \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau \\ &= \frac{F_0}{m\omega_n}\left(\frac{\cos\omega_n(t-\tau)}{\omega_n}\right)_0^t \\ &= \boxed{\frac{F_0}{m\omega_n^2}(1-\cos\omega_n t)} \end{aligned}$$

Hence

$$\dot{u}_1(t) = \boxed{\frac{F_0}{m\omega_n}\sin\omega_n t}$$

Note that

$$u_1(t_1) = \frac{F_0}{m\omega_n^2}(1-\cos 0.2\omega_n)$$

and

$$\dot{u}_1(t_1) = \frac{F_0}{m\omega_n}\sin 0.2\omega_n$$

Now for $t_1 \le t \le t_2$

$$u_2(t) = \overbrace{u_1(t_1)\cos\omega_n(t-t_1) + \frac{\dot{u}_1(t_1)}{\omega_n}\sin\omega_n(t-t_1)}^{\text{response due to initial conditions at t\_1 free vibration}} + \overbrace{\frac{1}{m\omega_n}\int_{t_1}^{t} F(\tau)\sin\omega_n(t-\tau)\,d\tau}^{\text{second integral}} \tag{1}$$

But the free vibration response is , using $t_1 = 0.2$

$$\begin{aligned}
&= \overbrace{\frac{F_0}{m\omega_n^2}(1-\cos 0.2\omega_n)}^{u_1(t_1)}\cos\omega_n(t-0.2) + \frac{\overbrace{\frac{F_0}{m\omega_n}\sin 0.2\omega_n}^{\dot{u}_1(t_1)}}{\omega_n}\sin\omega_n(t-0.2)\\
&= \frac{F_0}{m\omega_n^2}\left\{\cos\omega_n(t-0.2) - \cos 0.2\omega_n\cos\omega_n(t-0.2)\right\} + \frac{F_0}{m\omega_n^2}\sin 0.2\omega_n\sin\omega_n(t-0.2)\\
&= \frac{F_0}{m\omega_n^2}\left\{\cos\omega_n(t-0.2) - \cos 0.2\omega_n\cos\omega_n(t-0.2) + \sin 0.2\omega_n\sin\omega_n(t-0.2)\right\}
\end{aligned}$$

and the second integral is

$$\begin{aligned}
I_2 &= \int_{t_1}^{t} F(\tau)\sin\omega_n(t-\tau)\,d\tau\\
&= \int_{t_1}^{t} F_0\left(1-\frac{(\tau-t_1)}{0.2}\right)\sin\omega_n(t-\tau)\,d\tau\\
&= F_0\int_{t_1}^{t}\sin\omega_n(t-\tau)\,d\tau - \frac{F_0}{0.2}\int_{t_1}^{t}(\tau-t_1)\sin\omega_n(t-\tau)\,d\tau\\
&= F_0\int_{t_1}^{t}\sin\omega_n(t-\tau)\,d\tau - \frac{F_0}{0.2}\overbrace{\int_{t_1}^{t}\tau\sin\omega_n(t-\tau)\,d\tau}^{\text{use integration by parts}} + \frac{t_1F_0}{0.2}\int_{t_1}^{t}\sin\omega_n(t-\tau)\,d\tau\\
&= \left(F_0+\frac{t_1F_0}{0.2}\right)\left(\frac{\cos\omega_n(t-\tau)}{\omega_n}\right)_{t_1}^{t} - \frac{F_0}{0.2}\left\{\frac{\tau\omega_n\cos\omega_n(t-\tau)+\sin\omega_n(t-\tau)}{\omega_n^2}\right\}_{t_1}^{t}\\
&= \left(F_0+\frac{t_1F_0}{0.2}\right)\frac{1}{\omega_n}\left\{\cos\omega_n(t-t) - \cos\omega_n(t-t_1)\right\}\\
&\quad - \frac{F_0}{0.2\omega_n^2}\left\{t\omega_n\cos\omega_n(t-t) + \sin\omega_n(t-t) - [t_1\omega_n\cos\omega_n(t-t_1) + \sin\omega_n(t-t_1)]\right\}
\end{aligned}$$

Simplify to

$$I_2 = \frac{F_0}{\omega_n}\left(1+\frac{t_1}{0.2}\right)\left(1-\cos\omega_n\left(t-t_1\right)\right) - \frac{F_0}{0.2\omega_n^2}\left\{t\omega_n - t_1\omega_n\cos\omega_n\left(t-t_1\right) - \sin\omega_n\left(t-t_1\right)\right\}$$

Hence $u_2\left(t\right)$ for $t_1 \leq t \leq t_2$ is by putting the above result back into (1) we obtain

$$\begin{aligned} u_2\left(t\right) &= \frac{F_0}{m\omega_n^2}\left(\cos\omega_n\left(t-0.2\right) - \cos 0.2\omega_n\cos\omega_n\left(t-0.2\right) + \sin 0.2\omega_n\sin\omega_n\left(t-0.2\right)\right) \\ &+ \frac{1}{m\omega_n}\left(\frac{F_0}{\omega_n}\left(1+\frac{t_1}{0.2}\right)\left(1-\cos\omega_n\left(t-t_1\right)\right) - \frac{F_0}{0.2\omega_n^2}\left\{t\omega_n - t_1\omega_n\cos\omega_n\left(t-t_1\right) - \sin\omega_n\left(t-t_1\right)\right\}\right) \end{aligned}$$

But $t_1 = 0.2$, hence

$$\begin{aligned} u_2\left(t\right) &= \frac{F_0}{m\omega_n^2}\left(\cos\omega_n\left(t-0.2\right) - \cos\left(0.2\omega_n\right)\cos\omega_n\left(t-0.2\right) + \sin\left(0.2\omega_n\right)\sin\omega_n\left(t-0.2\right)\right) \\ &+ \frac{1}{m\omega_n}\left(\frac{F_0}{\omega_n}\left(1+\frac{0.2}{0.2}\right)\left(1-\cos\omega_n\left(t-0.2\right)\right) - \frac{F_0}{0.2\omega_n^2}\left\{t\omega_n - 0.2\omega_n\cos\omega_n\left(t-0.2\right) - \sin\omega_n\left(t-0.2\right)\right\}\right) \\ &= \frac{F_0}{m\omega_n^2}\left(\cos\omega_n\left(t-0.2\right) - \cos\left(0.2\omega_n\right)\cos\omega_n\left(t-0.2\right) + \sin\left(0.2\omega_n\right)\sin\omega_n\left(t-0.2\right)\right) \\ &+ \frac{1}{m\omega_n}\left(\frac{2F_0}{\omega_n} - \frac{2F_0}{\omega_n}\cos\omega_n\left(t-0.2\right) - t\frac{F_0}{0.2\omega_n} + \frac{F_0}{\omega_n}\cos\omega_n\left(t-0.2\right) + \frac{F_0}{0.2\omega_n^2}\sin\omega_n\left(t-0.2\right)\right) \end{aligned}$$

simplify

$$\begin{aligned} u_2\left(t\right) = &\overbrace{\frac{F_0}{m\omega_n^2}\cos\omega_n\left(t-0.2\right)} - \frac{F_0}{m\omega_n^2}\cos\left(0.2\omega_n\right)\cos\omega_n\left(t-0.2\right) + \frac{F_0}{m\omega_n^2}\sin\left(0.2\omega_n\right)\sin\omega_n\left(t-0.2\right) \\ &+ \frac{2F_0}{m\omega_n^2} - \overbrace{\frac{2F_0}{m\omega_n^2}\cos\omega_n\left(t-0.2\right)} - \frac{t\ F_0}{0.2m\omega_n^2} + \overbrace{\frac{F_0}{m\omega_n^2}\cos\omega_n\left(t-0.2\right)} + \frac{F_0}{0.2m\omega_n^3}\sin\omega_n\left(t-0.2\right) \end{aligned}$$

Hence

$$\begin{aligned} u_2\left(t\right) &= -\frac{F_0}{m\omega_n^2}\cos\left(0.2\omega_n\right)\cos\omega_n\left(t-0.2\right) + \frac{F_0}{m\omega_n^2}\sin\left(0.2\omega_n\right)\sin\omega_n\left(t-0.2\right) + \frac{2F_0}{m\omega_n^2} \\ &- \frac{t\ F_0}{0.2m\omega_n^2} + \frac{F_0}{0.2m\omega_n^3}\sin\omega_n\left(t-0.2\right) \\ &= \boxed{\frac{F_0}{m\omega_n^2}\left(2 - \cos\left(0.2\omega_n\right)\cos\omega_n\left(t-0.2\right) + \sin\left(0.2\omega_n\right)\sin\omega_n\left(t-0.2\right) - 5t + \frac{5}{\omega_n}\sin\omega_n\left(t-0.2\right)\right)} \end{aligned}$$

Now

$$\dot{u}_2(t) = \boxed{\frac{F_0}{m\omega_n^2}\left(\omega_n \cos(0.2\omega_n)\sin\omega_n(t-0.2) + \omega_n \sin(0.2\omega_n)\cos\omega_n(t-0.2) - \frac{1}{0.2} + \frac{1}{0.2}\cos\omega_n(t-0.2)\right)}$$

Note that at t_2 we have

$$\begin{aligned}
u_2(t_2) &= \frac{F_0}{m\omega_n^2}\left(2 - \cos(0.2\omega_n)\cos\omega_n(0.4-0.2) + \sin(0.2\omega_n)\sin\omega_n(0.4-0.2) - \frac{0.4}{0.2} + \frac{1}{0.2\omega_n}\sin\omega_n(0.4-0.2)\right) \\
&= \frac{F_0}{m\omega_n^2}\left(2 - \cos(0.2\omega_n)\cos(0.2\omega_n) + \sin(0.2\omega_n)\sin(0.2\omega_n) - 2 + \frac{1}{0.2\omega_n}\sin 0.2\omega_n\right) \\
&= \boxed{\frac{F_0}{m\omega_n^2}\left(-\cos^2(0.2\omega_n) + \sin^2(0.2\omega_n) + \frac{5}{\omega_n}\sin(0.2\omega_n)\right)}
\end{aligned}$$

and at $t_2 = 0.4$ we have

$$\dot{u}_2(t_2) = \frac{F_0}{m\omega_n^2}\left(\omega_n\cos(0.2\omega_n)\sin(0.2\omega_n) + \omega_n\sin(0.2\omega_n)\cos(0.2\omega_n) - \frac{1}{0.2} + \frac{1}{0.2}\cos 0.2\omega_n\right)$$

Now for $t \geq t_2$ since no force is applied, we use the free vibration solution using the above $u_2(t_2)$ and $\dot{u}_2(t_2)$ as initial conditions

$$u_3(t) = u_2(t_2)\cos\omega_n(t-t_2) + \frac{\dot{u}_2(t_2)}{\omega_n}\sin\omega_n(t-t_2)$$

$$\begin{aligned}
&= \frac{F_0}{m\omega_n^2}\left(-\cos^2(0.2\omega_n) + \sin^2(0.2\omega_n) + \frac{5}{\omega_n}\sin(0.2\omega_n)\right)\cos\omega_n(t-t_2) \\
&+ \frac{F_0}{m\omega_n^3}\left(\omega_n\cos(0.2\omega_n)\sin 0.2\omega_n + \omega_n\sin(0.2\omega_n)\cos 0.2\omega_n - 5 + 5\cos 0.2\omega_n\right)\sin\omega_n(t-t_2)
\end{aligned}$$

Now that we have $u(t)$ for each time segment, we can plot the solution. Here it is for up to $t = 0.5$ sec

```
In[168]:= Remove["Global`*"];
(*Plot for problem 4.14, HW3. Nasser Abbasi*)

m = 100/386; k = 1000; t1 = 0.2; t2 = 0.4; f0 = 2000; w = Sqrt[k/m];

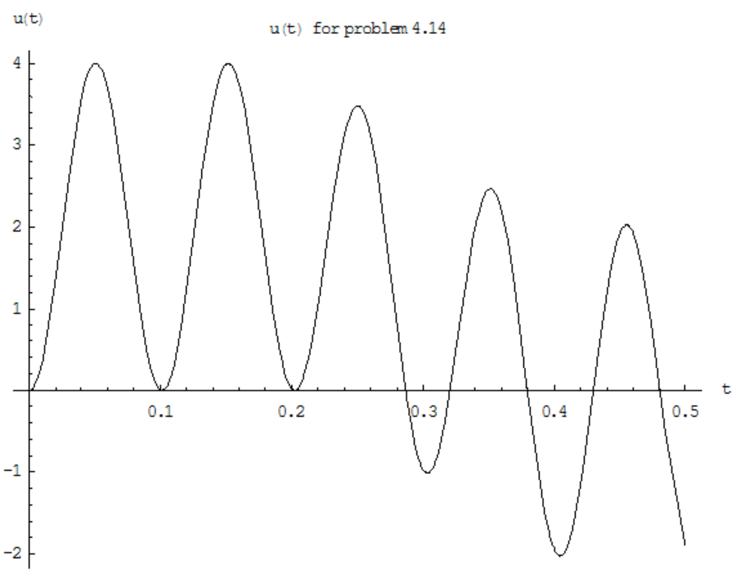
u1[t_] := f0/(m (w)^2) (1 - Cos[w t]);

u2[t_] :=
  f0/(m w^2) (2 - Cos[0.2 w] Cos[w (t - 0.2)] + Sin[0.2 w] Sin[w (t - 0.2)] - 5 t + 5/w Sin[w (t - 0.2)])

u3[t_] := f0/(m w^2) (- Cos[0.2 w]^2 + Sin[0.2 w]^2 + 5/w Sin[0.2 w]) Cos[w (t - t2)] +
   f0/(m w^3) (w Cos[0.2 w] Sin[0.2 w] + w Sin[0.2 w] Cos[0.2 w] - 5 + 5 Cos[0.2 w]) Sin[w (t - t2)]

u[t_] := Piecewise[{ {u1[t], t < t1}, {u2[t], t1 ≤ t ≤ t2}, {u3[t], t > t2} }];

Plot[Evaluate[u[t]], {t, 0, .5}, PlotLabel -> "u(t) for problem 4.14",
  AxesLabel → {"t", "u(t)"}, PlotRange → All];
```



Here is the solution for up to $t = 1.5\,\text{sec}$