Quizz 2

Math 2520
Differential Equations and Linear Algebra

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Contents

1 Problem 1

Let $\vec{v} = \begin{bmatrix} 5 \\ 3 \\ -6 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$ be in $\mathbb{R}^3$. Let $W = span\,(v_1, v_2)$. Determine if $\vec{v}$ is in W

Solution

$\vec{v}$ is in W if $\vec{v}$ can be expressed as a linear combination of basis $\vec{v}_1, \vec{v}_2$. To find if this is the case, we solve

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{v}$$

If a solution exists, then $\vec{v}$ is in W. Writing the above in matrix form gives

$$c_1 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 \\ 1 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -6 \end{bmatrix} \tag{1}$$

Hence the augmented matrix is

$$\begin{bmatrix} -1 & 3 & 5 \\ 1 & 1 & 3 \\ 2 & -4 & -6 \end{bmatrix}$$

$R_2 = R_2 + R_1$ gives

$$\begin{bmatrix} -1 & 3 & 5 \\ 0 & 4 & 8 \\ 2 & -4 & -6 \end{bmatrix}$$

$R_3 = R_3 + 2R_1$ gives

$$\begin{bmatrix} -1 & 3 & 5 \\ 0 & 4 & 8 \\ 0 & 2 & 4 \end{bmatrix}$$

$R_3 = R_3 - \frac{R_2}{2}$ gives

$$\begin{bmatrix} -1 & 3 & 5 \\ 0 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

The pivots columns are the first two columns. The system (1) now becomes

$$\begin{bmatrix} -1 & 3 \\ 0 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 0 \end{bmatrix}$$

Last row provides no information as It just says $0 = 0$. Second row gives $4c_2 = 8$ or $c_2 = 2$. First row gives $-c_1 + 3c_2 = 5$ or $-c_1 = 5 - 3(2)$ or $c_1 = 1$. Hence the solution is

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Since a solution is found, this means $\vec{v}$ can be expressed as a linear combination of $\vec{v}_1, \vec{v}_2$ given by

$$\vec{v}_1 + 2\vec{v}_2 = \vec{v}$$

Hence $\vec{v}$ is in W.

2 Problem 2

Determine the component vector of $p(x) = -4x^2+2x+6$ in the given vector space $V = P_2(\mathbb{R})$ relative to the given ordered basis $B = \left\{x^2 + x, 2 + 2x, 1\right\}$

Solution

The problem is asking us to find c_1, c_2, c_3 such that

$$\begin{aligned} c_1\left(x^2+x\right) + c_2(2+2x) + c_3(1) &= p(x) \\ &= -4x^2 + 2x + 6 \end{aligned}$$

Expanding gives

$$\begin{aligned} c_1x^2 + c_1x + 2c_2 + 2c_2x + c_3 &= -4x^2 + 2x + 6 \\ x^2(c_1) + x(c_1 + 2c_2) + (2c_2 + c_3) &= -4x^2 + 2x + 6 \end{aligned}$$

For these to be equal, the corresponding coefficients of the polynomials must be the same. Therefore equating coefficients of each power of x gives

$$\begin{aligned} c_1 &= -4 \\ c_1 + 2c_2 &= 2 \\ 2c_2 + c_3 &= 6 \end{aligned}$$

Or

$$\begin{aligned} c_1 &= -4 \\ -4 + 2c_2 &= 2 \\ 2c_2 + c_3 &= 6 \end{aligned}$$

Or

$$\begin{aligned} c_1 &= -4 \\ c_2 &= 3 \\ 2c_2 + c_3 &= 6 \end{aligned}$$

Or

$$\begin{aligned} c_1 &= -4 \\ c_2 &= 3 \\ 2(3) + c_3 &= 6 \end{aligned}$$

Or

$$\begin{aligned} c_1 &= -4 \\ c_2 &= 3 \\ c_3 &= 0 \end{aligned}$$

Hence the component vector of $p(x)$ is $\{-4, 3, 0\}$ relative to the basis B.

3 Problem 3

Determine if the given linear transformation is a) one-to-one and b) onto. Justify your answer.

$$T(x,y,z) = (x, x+y+z)$$

Solution

In Matrix forum the above is

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ x+y+z \end{bmatrix}$$

Therefore A matrix must have dimensions 2×3. The above becomes

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ x+y+z \end{bmatrix}$$

First row gives $a_{11}x + a_{12}y + a_{13}z = x$. Equating coefficients of the polynomials on each side gives $a_{11} = 1, a_{12} = 0, a_{13} = 0$. Second row gives $a_{21}x + a_{22}y + a_{23}z = x + y + z$ which implies that $a_{21} = 1, a_{22} = 1, a_{23} = 1$. Hence the matrix A is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Now that we have found the matrix representation of the linear transformation T, we can answer parts a and b.

Using Theorem 6.4.8 which says for the linear transformation $T : V \to W$

1. one-to-one iff $\ker(T) = \{\vec{0}\}$
2. onto iff $Rng(T) = W$

In this problem V is $\mathbb{R}^3$ and W is $\mathbb{R}^2$. To show one-to-one, we need to find $\ker(T)$ by solving $A\vec{x} = \vec{0}$ and check if it is the zero vector or not.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$R_2 = R_2 - R_1$ gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Therefore the base variables are x, y and the free variable is $z = t$. Hence (2) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

First row gives $x = 0$ and second row gives $y + z = 0$ or $y = -t$ Therefore the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix}$$

$$= t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Therefore null-space is $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$. Since null-space is not the zero vector, then T is not one-to-one.

To check if it is onto, the $Rng(T)$ is the column space of A. From the above RREF, we found that the first two columns are the pivot columns. These correspond to the first two columns of A. Therefore

$$Rng(T) = \left\{ \vec{v} \in \mathbb{R}^3 : \vec{v} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}, c_1, c_2 \in \mathbb{R} \right\}$$

The question now is, does the above span all of W which is $\mathbb{R}^2$? it is clear it does, since $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly independent of each others and any two linearly independent vectors in $\mathbb{R}^2$ span all of $\mathbb{R}^2$. Hence it is onto.

If we need to show this, then this can be done by solving

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

First row gives $c_1 = 0$ and second row gives $c_2 = 0$. Hence only solution to $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is $c_1 = c_2 = 0$. Therefore these two vectors are linearly independent vectors in $\mathbb{R}^2$. Hence $Rng(T)$ is all W. Therefore T is onto.

4 Problem 4

Given the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by the equation, find the standard matrix for the inverse transformation T^{-1}

$$\begin{aligned} w_1 &= x_1 + 4x_2 - x_3 \\ w_2 &= 2x_1 + 7x_2 + x_3 \\ w_3 &= x_1 + 3x_2 \end{aligned}$$

<u>Solution</u>

In matrix form the above is

$$\begin{bmatrix} 1 & 4 & -1 \\ 2 & 7 & 1 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_2 \end{bmatrix}$$

Where $A = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 7 & 1 \\ 1 & 3 & 0 \end{bmatrix}$ represents the linear transformation T. Hence T^{-1} is represented by A^{-1}. Therefore we need to find A^{-1}. Setting up the augmented matrix for finding the inverse is setup by adding the identity matrix to the right half as follows

$$\begin{bmatrix} 1 & 4 & -1 & 1 & 0 & 0 \\ 2 & 7 & 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$

$$\begin{bmatrix} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 - R_1$

$$\begin{bmatrix} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$\begin{bmatrix} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 & -1 & 1 \end{bmatrix}$$

$R_3 = \frac{-R_3}{2}, R_2 = -R_2$

$$\begin{bmatrix} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$$

Now we start the reduced Echelon phase. $R_2 = R_2 + 3R_3$

$$\begin{bmatrix} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2+3\left(\frac{-1}{2}\right) & -1+3\left(\frac{1}{2}\right) & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$R_1 = R_1 + R_3$

$$\begin{bmatrix} 1 & 4 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$R_1 = R_1 - 4R_2$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}-4\left(\frac{1}{2}\right) & \frac{1}{2}-4\left(\frac{1}{2}\right) & -\frac{1}{2}-4\left(-\frac{3}{2}\right) \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Now that the left half above is the identity matrix, then the right half is A^{-1}. Therefore

$$A^{-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Which is the standard matrix for the representation of T^{-1}

5 Problem 5

Answer the following questions by writing TRUE or FALSE.

Solution

a The rank of a matrix equals the dimension of its column space. **TRUE**

b The number of variables in the equations $Ax = 0$ equals the dimension of the nullsapce of A. **FALSE**

c If A is 3×5 matrix and T is a transformation defined by $T(x) = Ax$, then the domain of T is $\mathbb{R}^3$. **FALSE**

d If a 4×7 matrix A has four pivot columns then the $nullity(A) = 3$. **TRUE**

e A linearly independent set in a subspace H is a basis for H. **FALSE**