University Course

Math 2520
Differential Equations and Linear Algebra

Normandale college, Bloomington, Minnesota
Summer 2021

My Class Notes

Nasser M. Abbasi

Summer 2021

Contents

Chapter 1

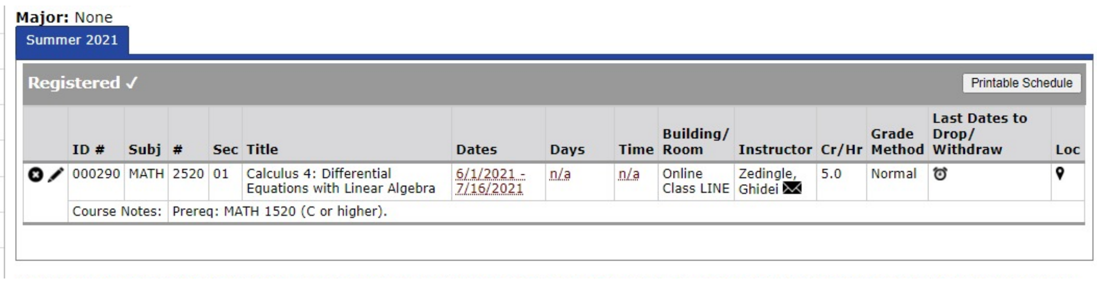
Introduction

Local contents

1.1 Links

1. D2l web page `https://normandale.learn.minnstate.edu/`

1.2 Schedule



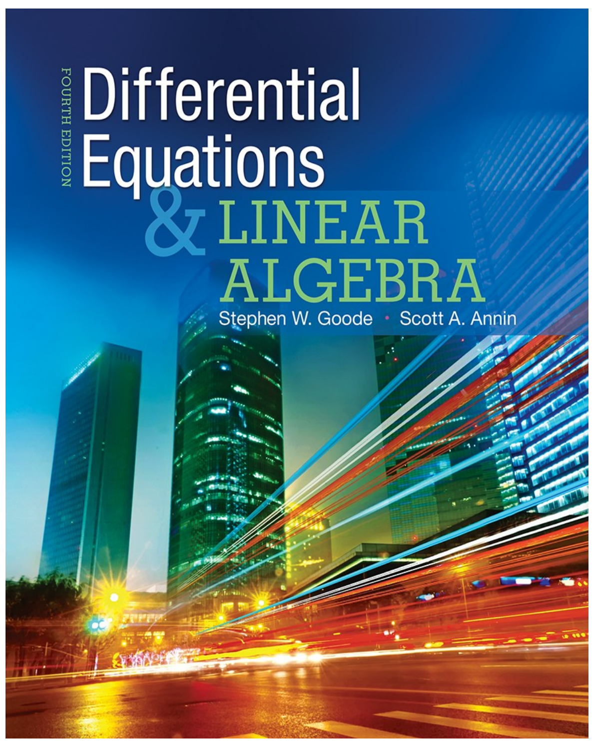
Major: None

Summer 2021

Registered ✓ Printable Schedule

	ID #	Subj	#	Sec	Title	Dates	Days	Time	Building/ Room	Instructor	Cr/Hr	Grade Method	Last Dates to Drop/ Withdraw	Loc
	000290	MATH	2520	01	Calculus 4: Differential Equations with Linear Algebra	6/1/2021 - 7/16/2021	n/a	n/a	Online Class LINE	Zedingle, Ghidei	5.0	Normal		
	Course Notes:	Prereq: MATH 1520 (C or higher).												

1.3 Text book



1.4 syllabus

NORMANDALE COMMUNITY COLLEGE
SUMMER 2021, COURSE SYLLABUS
MATH 2520-01 DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA (*Online*)

I. **IDENTIFYING INFORMATION**

A. Mathematics 2520 - 01: Differential Equations and Linear Algebra (Online)
B. Instructor: Ghidei Zedingle
C. Summer Semester, 2021
D. 5 credits
E. Prerequisite: Math 1520 with a grade of C or higher, or approved equivalent preparation.
F. Office:
G. Office hours: MWF 9:00 - 10:00am, TTh 5:00 – 6:00; other times by appointment. (**Using zoom meeting**)
H. Office phone: (952) 358- 8362
I. Office e-mail: ghidei.zedingle@normandale.edu
J. Normandale fax number: (952) 358 – 8101

II. **COURSE DESCRIPTION**

Matrices and systems, vector spaces, subspaces, linear independence, basis, dimension, linear transformation, eigenvectors; first and second order differential equations, Euler's Method, phase plane analysis of linear and nonlinear systems, extensive modeling. Laplace transforms and power series solution.

III. **LEARNING OUTCOMES**

Upon successful completion of Math2520, students will be able to**:**

- Classify and solve first order differential equations of various types: separable, exact and linear including initial-value problems.
- Apply existence and uniqueness theorems.
- Use direct field to illustrate solutions of differential equations.
- Approximate solutions to first order equation using Euler's method.
- Compute algebraically with matrices, products, inverses and determinants.
- Apply matrix reduction method to solve and describe solution sets of linear systems.
- Describe the structure and characteristics of vector spaces, subspaces and linear transformation between vector spaces.
- Compute eigenvalues and eigenvectors.
- Solve nth order linear differential equations with constant coefficients using undetermined coefficients and variation of parameter, including initial and boundary value problems.
- Analyze linear and nonlinear systems of differential equations using eigenvalue and phase plane methods.
- Model a variety of applied situations with differential equations and dynamical systems (e.g. harmonic oscillator and predatory-prey).
- Solve problems using Laplace transforms and power series solutions.

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IV MAJOR TOPICS: (Based on Lecture notes)

Chapter 1:	FIRST ORDER DIFFERENTIAL EQUATIONS
Chapter 2:	MATRICES AND SYSTEMS OF LINEAR EQUATIONS
Chapter 3:	DETERMINANTS
Chapter 4:	VECTOR SPACES
Chapter 5:	LINEAR TRANSFORMATION
Chapter 6:	EIGENVALUES AND EIGENVECTORS
Chapter 7:	LINEAR DIFFERENTIAL EQUATIONS OF ORDER *n*
Chapter 8:	SYSTEMS OF DIFFERENTIAL EQUATIONS
Chapter 9:	THE LAPLACE TRANSFORM AND SOME ELEMENTARY APPLICATIONS
Chapter 10:	SERIES SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS

V BREAKDOWN OF SECTIONS BY WEEK

Breakdown of topics by week will be posted as reading material on D2L content area. Lecture notes will also be posted on D2L in the content area. Because some chapters, some sections from chapters and even topics from sections are omitted, I recommend that you start reading from the lecture notes and go to the text book for more examples and reading on the subject matter.

VI. MATERIALS NEEDED

A. Text book: Differential Equations and Linear Algebra by Goode and Annin, 4th edition, published by Pearson.
B. Graphic Calculator(recommended): TI-83/TI-84/TI-89.

VII. EVALUATION

A. Homework (25%)

There will be weekly assignment questions that will be posted on D2L content area. All home-works will be done in paper and pencil (pen) and will be submitted on D2L under Assessment -> Assignment. Due dates will be indicated with the questions. Check for the due date ahead of time. I didn't know (or I forgot) is not an excuse for not completing the assignment on time. You can review your work and any comments after it is corrected. Lowest one assignment will be dropped.

B. Quizzes (20%)

There will be three quizzes as indicated in the table below. Quizzes will be posted on D2L content area. Same as with the homework, quizzes will be done in paper and pencil and will be submitted on D2L. Due dates will be indicated with the questions. Check for the due date ahead of time. I didn't know (or I forgot) is not an excuse for not completing the quizzes on time. You can review your work and any comments after it is corrected.

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Week 2, 06/11	**Q1**
Week 4 , 06/25	**Q2**
Week 6, 07/9	**Q3**

C. Exams. (55%)

Two exams will be given. One mid semester and one final exam. Mid-semester exam is comprehensive and will be on June 18, 2021. Final exam will be comprehensive and will take place on July 16, 2021.

D. Discussion Board

Discussion board will be set up on D2L. You can post questions or respond to questions posted on the discussion board. The questions can be on the course material and use of technology (TI-83/84, Maple etc.). If you have personal or grade related questions the best way will be to use e-mail and make sure to use your Normandale e-mail account.

VIII. Technology

The course being mostly on line, you will depend heavily on technology and in particular in computer. You are expected to have your own computer or have continuous access to computer that works effectively. Assignments posted for 4 – 5 days so that you will have time to check your equipment and to finish them on time. If for some reason it didn't work, you will have time to fix it or to look for an alternative, such as public library to complete it on time. Have a backup plan. Normandale's Computer Center is also open.

VIII. COURSE POLICIES

A. *Grading*. The final course grade will be based on the total accrued in the course as follows:

Homework	25%
Quiz	20%
Mid Exams	25%
Final exam	30%
Total	100%

The final course grade will be based on these percentages of the total points:

90% - 100% = A
80% - 89.9% = B
70% - 79.9% = C
60% - 69.9% = D
Below 60% = F .

Incomplete (I) grades generally will not be given. They are reserved for students who have successfully completed a great majority of course work, but due to extreme circumstances cannot complete some essential component, e.g. final exam and the student has a passing grade. Withdrawing from class (W) is a student-initiated action Last date of withdrawal is **07/07/2021**. Please talk to your counselor about withdrawing before you take any action to do so. Any student who stops

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working on class activity without officially withdrawing will receive an F grade.

B. *Late assignment policy*. Each homework has a due date. Check ahead of time. There will be 24 hrs automatic extension of time with a 10% penalty. Any make-up quiz or exam will alo be subject to 10% penalty.

VIII. ADDITIONAL HELPS

A. Communication with instructor through e-mail or zoom meeting during the office hrs or by appointment.
B. Online tutoring from Normandale.

IX. Disability Services and Accessibility

If you qualify to receive classroom accommodations based on a disability, please contact the Office for students with Disabilities Director, Debbie Tillman, at 952-358-8623 or osd@normandale.edu to discuss how accommodations may be implemented in all of your Normandale classes.

Chapter 2

HWs

Local contents

2.1 HW 1

Local contents

2.1.1 Problems listing

Math2520-01 **Assignment I**

<u>INSTRUCTION</u>: Show all the necessary work. Writ your answer on a separate sheet preferably hand written clear and legible. Post your answer sheet on D2L by Monday June 6. Late June 7.

1. Determine the order of the differential equation.

   a) $\left(\frac{dy}{dx}\right)^3 + y^2 = \sin x$ b) $t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = \sin(t)$

2. Verify that for $t > 0$, $y(t) = \ln t$ is a solution to the differential equation

$$2\left(\frac{dy}{dt}\right)^3 = \frac{d^3y}{dt^3}.$$

3. Determine whether the differential equation is linear or nonlinear.

   a) $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} = xy^2 + \tan x.$ b) $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = \sin(t)$

4. Prove (show) that the initial-value problem

$$y' = x\sin(x+y), \quad y(0) = 1$$

   has a unique solution using the existence and uniqueness theorem.

5. Let

$$y' = (y-2)(y+1).$$

   a) Determine all equilibrium solutions.
   b) Determine the region in the xy-plane where the solutions are increasing, and where the solutions are decreasing.
   c)

6. Solve the following differential equations.

   a) $\frac{dy}{dx} = \frac{y}{x\ln x}$ b) $(x^2+1)y' + y^2 = -1,\ y(0) = 1$

7. Solve the following differential equations.

   a) $\frac{dy}{dx} + \frac{2}{x}y = 5x^2,\ x > 0.$ b) $t\frac{dx}{dt} + 2x = 4e^t,\ t > 0$

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8. A container initially containing10 L of water in which there is 20 g of salt dissolved. A solution containing 4 g/L of salt is pumped into the container at a rate of 2 L/min, and the well-stilled mixture runs out at a rate of 1 L/min. How much salt is in the tank after 40 min?

9. Consider the RC circuit (See page 65 in the text) which has R = 5 Ω, C = $\frac{1}{50}$ F and $E(t) = 100$ V. If the capacitor is uncharged initially, determine the current in the circuit for $t \geq 0$.

10. Solve the initial-value problem.

$$\frac{dy}{dx} = \frac{2x - y}{x + 4y}, \quad y(1) = 1$$

11. Solve the given differential equation.

$$y' + 2x^{-1}y = 6y^2x^4$$

12. Determine whether the given differential equation is exact. Show the work.

$$2xe^y dx + (3y^2 + x^2e^y)dy = 0$$

13. Solve the given differential equation.

$$(y^2 + \cos x)dx + (2xy + \sin y)dy = 0$$

14. Determine an integrating factor for the given differential equation and hence find the general solution.

$$(xy - 1)dx + x^2dy = 0$$

2.1.2 Problem 1

Determine the order of the differential equation a) $\left(\frac{dy}{dx}\right)^3 + y^2 = \sin x$, b) $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = \sin(t)$

Solution

For (a), the order is one. Since highest derivative $\frac{dy}{dx}$ is of order one. For (b) the order is second. Since highest derivative $\frac{d^2y}{dt^2}$ is of order two.

2.1.3 Problem 2

Verify for $t > 0, y(t) = \ln t$ is a solution to $2\left(\frac{dy}{dt}\right)^3 = \frac{d^3y}{dt^3}$

Solution

The verification is done by substituting the solution into the ODE, if the result is an identity (both sides of the equation are the same), then it is verified, otherwise it is not. Since solution is $y(t) = \ln t$ then $\frac{dy}{dt} = \frac{1}{t}$ and $\frac{d^2y}{dt^2} = \frac{-1}{t^2}$ and $\frac{d^3y}{dt^3} = \frac{2}{t^3}$. Substituting these into the ODE gives

$$\begin{aligned} 2\left(\frac{1}{t}\right)^3 &= \frac{2}{t^3} \\ \frac{2}{t^3} &= \frac{2}{t^3} \end{aligned}$$

Which is an identity. Hence $y(t) = \ln t$ is a solution to the ODE.

2.1.4 Problem 3

Determine whether the differential equation is linear or nonlinear a) $y''' + 4y'' + \sin xy' = xy^2 + \tan x$, b) $t^2y'' + ty' + 2y = \sin t$

Solution

ODE (a) is not linear, due to presence of the term y^2 while ODE (b) is linear, since all derivative terms of the dependent variable and the dependent variable are linear.

2.1.5 Problem 4

Prove (show) that the initial-value problem $y' = x\sin\left(x+y\right)$, $y(0) = 1$ has a unique solution using the existence and uniqueness theorem

Solution

Writing the ODE as

$$\begin{aligned} y' &= x\sin\left(x+y\right) \\ &= f\left(x,y\right) \end{aligned}$$

Shows that $f\left(x,y\right)$ is continuous everywhere, since x and sin function are continuous everywhere. And

$$\frac{\partial f}{\partial y} = x\cos\left(x+y\right)$$

Which is also continuous everywhere. This shows there exists an interval I which must contain $x_0 = 0$ where the initial value ODE given above has a solution and the solution is unique for all x in I.

2.1.6 Problem 5

Let $y' = (y-2)(y+1)$. a) Determine all equilibrium solutions. b) Determine the region in the xy-plane where the solutions are increasing, and where the solutions are decreasing.

Solution

2.1.6.1 Part a

The equilibrium solutions are given by solution to $y' = 0$ which gives $y = 2, y = -1$.

2.1.6.2 Part b

The equilibrium solutions divide the solution domain into three regions. One is $y > 2$ and one is where $-1 < y < 2$ and one where $y < -1$.

When $y > 2$, we see that $(y-2)(y+1)$ is always positive. Hence y' is positive, which means the solution is increasing.

When $y < -1$, then $(y-2) < 0$ and also $(y+1) < 0$. Hence the product is positive, This means for $y < -1$, the slope is positive and the solution is increasing.

For $-1 < y < 2$, the term $(y-2)$ is negative and the term $(y+1)$ is positive. Hence the product is negative. This means the slope is negative and the solution is decreasing. Therefore

$$\begin{cases} y > 2 & \text{increasing} \\ -1 < y < 2 & \text{decreasing} \\ y < -1 & \text{increasing} \end{cases}$$

To verify this, the following is a plot of the solution curves. It shows the 3 regions which agrees with the above result.

```
restart;
ode:=diff(y(x),x)=(y(x)-2)*(y(x)+1):
p1:=DEtools:-DEplot(ode,y(x),x=-4..4, y=-4..4):
p2:=plot([-1,2],x=-4..4,color=blue):
plots:-display([p1,p2],axes=boxed, scaling=constrained,title="Regions of
   solution")
```

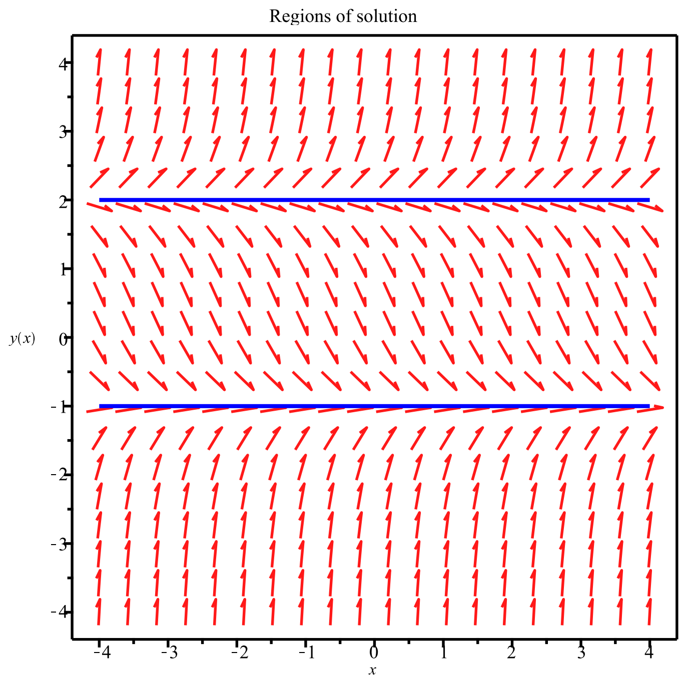


Figure 2.1: Solution curves

2.1.7 Problem 6

Solve the following differential equations a) $\frac{dy}{dx} = \frac{y}{x \ln x}$. b) $\left(x^2+1\right) y' + y^2 = -1, y(0) = 1$

Solution

2.1.7.1 Part a

This is separable ODE, it can be written as

$$\frac{dy}{y} = \frac{dx}{x \ln x}$$

Integrating gives

$$\ln |y| = \int \frac{dx}{x \ln x} + C_1$$

To find $\int \frac{dx}{x \ln x}$, we notice that, by definition $\frac{d}{dx} \ln(\ln x) = \frac{\frac{d}{dx} \ln x}{\ln x} = \frac{1}{x \ln x}$. This shows that $\ln(\ln x)$ is the antiderivative. Hence the above becomes

$$\ln |y| = \ln(\ln |x|) + C_1$$

Taking the exponential of both sides gives

$$y = Ce^{\ln(\ln|x|)}$$

Where the sign is absorbed by the constant C. Hence

$$y = C \ln x$$

2.1.7.2 Part b

The ODE is

$$\begin{aligned} y' &= \frac{-1}{\left(x^2+1\right)} - \frac{y^2}{\left(x^2+1\right)} \\ &= \frac{\left(-1-y^2\right)}{\left(x^2+1\right)} \\ &= \frac{-1}{\left(x^2+1\right)}\left(1+y^2\right) \end{aligned}$$

This is now separable.

$$\frac{dy}{\left(1+y^2\right)} = \frac{-dx}{\left(x^2+1\right)}$$

Integrating gives

$$\arctan(y) = -\arctan(x) + C$$

or

$$y = -\tan\left(\arctan(x) + C\right) \tag{1}$$

Applying initial conditions $y(0) = 1$ to the above gives

$$\begin{aligned} 1 &= -\tan\left(\arctan(0) + C\right) \\ &= -\tan(C) \end{aligned}$$

Hence $C = -\frac{\pi}{4}$. Therefore the general solution (1) becomes

$$\begin{aligned} y &= -\tan\left(\arctan(x) - \frac{\pi}{4}\right) \\ &= \tan\left(\frac{\pi}{4} - \arctan(x)\right) \end{aligned}$$

2.1.8 Problem 7

Solve the following differential equations a) $\frac{dy}{dx} + \frac{2}{x}y = 5x^2, x > 0$, b) $t\frac{dx}{dt} + 2x = 4e^t, t > 0$

Solution

2.1.8.1 Part a

This is a linear ODE in y. It is of the form $y' + p(x)y = q(x)$, where $p(x) = \frac{2}{x}, q(x) = 5x^2$. Hence the integrating factor is $I = e^{\int p(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x}$ or $I = x^2$. Multiplying both sides by this integrating factor make the LHS complete differential giving

$$\begin{aligned} \frac{d}{dx}\left(Iy\right) &= I\left(5x^2\right) \\ \frac{d}{dx}\left(yx^2\right) &= 5x^4 \\ yx^2 &= \int 5x^4 dx \\ yx^2 &= 5\frac{x^5}{5} + C \\ yx^2 &= x^5 + C \\ y &= x^3 + \frac{C}{x^2} \qquad x \neq 0 \end{aligned}$$

The above is the general solution.

2.1.8.2 Part b

Writing the ODE as

$$\frac{dx}{dt} + \frac{2}{t}x = 4\frac{e^t}{t} \qquad t \neq 0$$

Show this is a linear ODE in x. It is of the form $x' + p(t)x = q(t)$, where $p(t) = \frac{2}{t}, q(t) = 4\frac{e^t}{t}$. Hence the integrating factor is $I = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln t}$ or $I = t^2$. Multiplying both sides by this integrating factor make the LHS complete differential giving

$$\frac{d}{dt}\left(xt^2\right) = 4te^t$$

Integrating gives

$$xt^2 = 4\int te^t dt \tag{1}$$

Integration by parts. $\int u dv = uv - \int v du$. Let $u = t, dv = e^t, du = dt, v = e^t$, therefore

$$\begin{aligned}\int te^t dt &= te^t - \int e^t dt \\ &= te^t - e^t\end{aligned}$$

Hence (1) becomes

$$xt^2 = 4\left(te^t - e^t\right) + C$$

Where C is constant of integration. Therefore

$$\begin{aligned}x(t) &= \frac{4\left(te^t - e^t\right)}{t^2} + \frac{C}{t^2} \\ &= \frac{4e^t(t-1)}{t^2} + \frac{C}{t^2} \qquad t \neq 0\end{aligned}$$

2.1.9 Problem 8

A container initially containing 10 L of water in which there is 20 g of salt dissolved. A solution containing 4 g/L of salt is pumped into the container at a rate of 2 L/min, and the well-stilled mixture runs out at a rate of 1 L/min. How much salt is in the tank after 40 min?

Solution

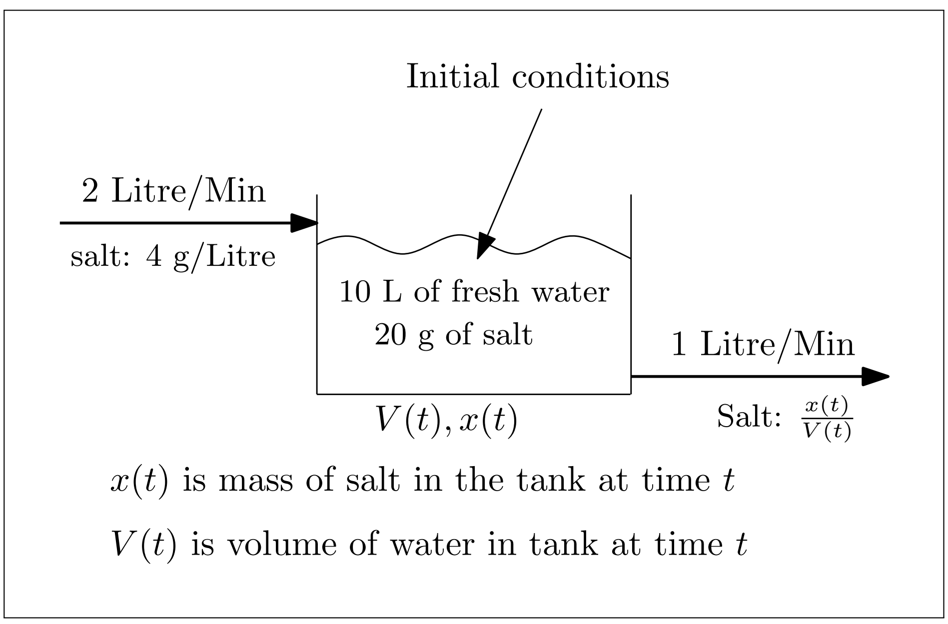


Figure 2.2: Showing tank flow

Let $x(t)$ be mass of salt (in grams) in tank at time t. Let $V(t)$ be the volume of water (in litre) in the tank at time t. Using the equilibrium equation for change of mass of salt

$$\frac{dx}{dt} = \text{rate of salt mass in } - \text{ rate of salt mass out}$$

Which becomes

$$\begin{aligned}\frac{dx}{dt} &= \left(2\ \frac{\text{L}}{\text{min}}\right)\left(4\ \frac{\text{g}}{\text{L}}\right) - \left(1\ \frac{\text{L}}{\text{min}}\right)\left(\frac{x(t)}{V(t)}\ \frac{\text{g}}{\text{L}}\right) \\ &= 8 - \frac{x(t)}{V(t)}\end{aligned} \tag{1}$$

But

$$\begin{aligned} V(t) &= V(0) + (\text{rate of mixture volume in } - \text{ rate of mixture volume out})\, t \\ &= V(0) + (2-1)\, t \\ &= V(0) + t \end{aligned}$$

But we are given that $V(0) = 10$ L. Hence

$$V(t) = 10 + t$$

Substituting the above in (1) gives

$$\frac{dx}{dt} = 8 - \frac{x}{10+t}$$

The solution to above ODE gives the mass x of salt in tank at time t.

$$\frac{dx}{dt} + \frac{x}{10+t} = 8$$

This is linear ODE. The integrating factor is $I = e^{\int \frac{1}{10+t} dt} = e^{\ln(10+t)} = 10 + t$. Multiplying both sides of the above ODE by this integrating factor gives

$$\frac{d}{dt}\left((10+t)\, x\right) = 8\,(10+t)$$

Integrating gives

$$\begin{aligned} (10+t)\, x &= 8 \int (10+t)\, dt \\ &= 8\left(10t + \frac{t^2}{2}\right) + C \end{aligned}$$

Hence

$$\begin{aligned} x &= 8\frac{t\left(10+\frac{t}{2}\right)}{(10+t)} + \frac{C}{(10+t)} \\ &= 4\frac{t\,(20+t)}{(10+t)} + \frac{C}{(10+t)} \end{aligned} \tag{1}$$

At $t = 0$, we are given that $x(0) = 20$ (g). Hence the above becomes

$$\begin{aligned} 20 &= \frac{C}{10} \\ C &= 200 \end{aligned}$$

Therefore (1) becomes

$$x = 4\frac{t\,(20+t)}{(10+t)} + \frac{200}{(10+t)} \tag{2}$$

At $t = 40$, the above gives

$$\begin{aligned} x\,(40) &= 4\frac{40\,(20+40)}{(10+40)} + \frac{200}{(10+40)} \\ &= 196 \text{ grams} \end{aligned}$$

2.1.10 Problem 9

Consider the RC circuit (See page 65 in the text) which has $R = 5\Omega, C = \frac{1}{50}F$ and $E(t) = 100V$. If the capacitor is uncharged initially, determine the current in the circuit for $t \geq 0$.

Solution

The equation for RC circuit is given by equation 1.7.16 in the text book as

$$\frac{dq}{dt} + \frac{1}{RC}q = \frac{E(t)}{R}$$

Where $q(t)$ is the charge on the plates of the capacitor We are told that at $t = 0, q = 0$. Using the numerical values given, the above ODE becomes

$$\begin{aligned}\frac{dq}{dt} + \frac{1}{5\left(\frac{1}{50}\right)}q &= \frac{100}{5}\\ \frac{dq}{dt} + 10q &= 20\end{aligned}$$

This is linear ODE in q. The integrating factor is $I = e^{\int 10dt} = e^{10t}$. Multiplying both sides by this integrating factor gives

$$\frac{d}{dt}\left(qe^{10t}\right) = 20e^{10t}$$

Integrating

$$\begin{aligned}qe^{10t} &= 20\int e^{10t}dt\\ &= 20\frac{e^{10t}}{10} + C\end{aligned}$$

Hence

$$q(t) = 2 + Ce^{-10t}$$

Using initial conditions $q(0) = 0$ shows that $0 = 2 + C$ or $C = -2$. Hence

$$\begin{aligned}q(t) &= 2 - 2e^{-10t}\\ &= 2\left(1 - e^{-10t}\right)\end{aligned}$$

Hence the current in the circuit is

$$\begin{aligned}i(t) &= \frac{dq}{dt}\\ &= 2\frac{d}{dt}\left(1 - e^{-10t}\right)\\ &= 2\left(10e^{-10t}\right)\\ &= 20e^{-10t}\end{aligned}$$

2.1.11 Problem 10

Solve the initial-value problem

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x-y}{x+4y}\\ y(1) &= 1\end{aligned}$$

Solution

Let us first check if a solution exists, and unique. $f\left(x,y\right) = \frac{2x-y}{x+4y}$. This is continuos for all x,y except when $y = -\frac{1}{4}x$. And $\frac{\partial f}{\partial y} = \frac{-9x}{\left(x+4y\right)^2}$. This is also continuos for all x,y except when $y = -\frac{1}{4}x$. Since initial conditions satisfies $y \neq -\frac{1}{4}x$, then there is an interval I that includes $x_0 = 0$ where a solution exists and is unique for all x in this interval. Now we can solve the ODE.

Let $v=\frac{y}{x}$. Hence $y=xv$. Therefore $\frac{dy}{dx}=v+x\frac{dv}{dx}$. The given ODE can be written as

$$\frac{dy}{dx}=\frac{2-\frac{y}{x}}{1+4\frac{y}{x}}\qquad x\neq 0$$

In terms of the new dependent variable $v(x)$, the above becomes

$$\begin{aligned}
v+x\frac{dv}{dx}&=\frac{2-v}{1+4v}\\
x\frac{dv}{dx}&=\frac{2-v}{1+4v}-v\\
&=\frac{(2-v)-v(1+4v)}{1+4v}\\
&=\frac{2-v-v-4v^2}{1+4v}\\
&=\frac{2-2v-4v^2}{1+4v}
\end{aligned}$$

The above ODE is separable. Therefore

$$\frac{1+4v}{2-2v-4v^2}dv=\frac{1}{x}dx$$

Integrating gives

$$\int\frac{1+4v}{2-2v-4v^2}dv=\int\frac{1}{x}dx$$

We notice that $\frac{d}{dx}\ln\left(2-2v-4v^2\right)=\frac{-2-8v}{2-2v-4v^2}$. Therefore $-\frac{1}{2}\frac{d}{dx}\ln\left(2-2v-4v^2\right)=\frac{1+4v}{2-2v-4v^2}$ which is the integrand. This shows that $-\frac{1}{2}\ln\left(2-2v-4v^2\right)$ is the anti derivative of the integral of the LHS above. Therefore the above becomes

$$\begin{aligned}
-\frac{1}{2}\ln\left(2-2v-4v^2\right)&=\ln x+C_1\\
\ln\left(2-2v-4v^2\right)&=-2\ln x-2C_1\\
2-2v-4v^2&=e^{-2C_1}\frac{1}{x^2}
\end{aligned}$$

Let $c=e^{-2C_1}$ be new constant. The above becomes

$$\begin{aligned}
2-2v-4v^2&=\frac{c}{x^2}\\
4v^2+2v-2+\frac{c}{x^2}&=0\\
v^2+\frac{1}{2}v-\frac{1}{2}+\frac{c}{4x^2}&=0
\end{aligned}$$

Solving for v gives

$$\begin{aligned}
v&=-\frac{b}{2a}\pm\frac{1}{2a}\sqrt{b^2-4ac}\\
&=-\frac{1}{4}\pm\frac{1}{2}\sqrt{\frac{1}{4}-4\left(-\frac{1}{2}+\frac{c}{4x^2}\right)}\\
&=-\frac{1}{4}\pm\frac{1}{2}\sqrt{\frac{1}{4}+2-\frac{c}{x^2}}\\
&=-\frac{1}{4}\pm\frac{1}{2}\sqrt{\frac{9}{4}-\frac{c}{x^2}}\\
&=-\frac{1}{4}\pm\frac{1}{2}\sqrt{\frac{9x^2-4c}{4x^2}}\\
&=-\frac{1}{4}\pm\frac{1}{4x}\sqrt{9x^2-4c}\qquad x>0
\end{aligned}$$

Since $v = \frac{y}{x}$, then there are two general solutions

$$
\begin{aligned}
y_1(x) &= -\frac{1}{4} + \frac{1}{4x}\sqrt{9x^2 - 4c} \\
y_2(x) &= -\frac{1}{4} - \frac{1}{4x}\sqrt{9x^2 - 4c}
\end{aligned}
$$

Initial conditions are now used to find a particular soluton. For $y_1(x)$

$$
\begin{aligned}
1 &= -\frac{1}{4} + \frac{1}{4}\sqrt{9 - 4c} \\
\frac{5}{4} &= \frac{1}{4}\sqrt{9 - 4c} \\
5 &= \sqrt{9 - 4c} \\
25 &= 9 - 4c \\
c &= -4
\end{aligned}
$$

Hence one solution is

$$
y_1(x) = -\frac{x}{4} + \frac{1}{4}\sqrt{9x^2 + 16}
$$

And for $y_2(x)$

$$
\begin{aligned}
1 &= -\frac{1}{4} - \frac{1}{4}\sqrt{9 - 4c} \\
\frac{5}{4} &= -\frac{1}{4}\sqrt{9 - 4c} \\
-5 &= \sqrt{9 - 4c}
\end{aligned}
$$

There is no solution for c in this equation since sqrt of a real number must be positive (principal root). Hence the only particular solution is the first one which is

$$
\boxed{y_1(x) = -\frac{x}{4} + \frac{1}{4}\sqrt{9x^2 + 16}}
$$

The above verifies the existence and uniqueness theorem, as only one solution is found which includes $x_0 = 1$.

2.1.12 Problem 11

Solve the given differential equation

$$
y' + 2\frac{y}{x} = 6y^2x^4
$$

Solution

In canonical form the ODE is

$$
y' = -\frac{2}{x}y + 6x^4y^2
$$

We see that this is Bernoulli ODE of the form $y' = p(x)\,y + q(x)y^n$ where $n = 2$. Dividing both sides by y^2 gives

$$
\frac{y'}{y^2} + \frac{2}{x}\frac{1}{y} = 6x^4
$$

Let $v = \frac{1}{y}$. Then $\frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$. Or $\frac{dy}{dx} = -y^2\frac{dv}{dx}$. Substituting this in the above ODE gives

$$
\begin{aligned}
-y^2\frac{dv}{dx}\frac{1}{y^2} + \frac{2}{x}v &= 6x^4 \\
\frac{dv}{dx} - \frac{2}{x}v &= -6x^4
\end{aligned}
$$

This is now linear in v. The integrating factor is $I = e^{\int -\frac{2}{x} dx} = e^{-2\ln x} = \frac{1}{x^2}$. Multiplying both sides by this integrating factor gives

$$\frac{d}{dx}\left(\frac{v}{x^2}\right) = -6x^2$$

Integrating

$$\begin{aligned}\frac{v}{x^2} &= -6\int x^2 dx + C\\ &= -2x^3 + C\\ v &= -2x^5 + Cx^2\end{aligned}$$

But $y = \frac{1}{v}$. Therefore the final solution is

$$y(x) = \frac{1}{-2x^5 + Cx^2}$$

2.1.13 Problem 12

Determine whether the given differential equation is exact

$$2xe^y dx + \left(3y^2 + x^2e^y\right) dy = 0$$

Solution

Writing the ODE as

$$M\left(x,y\right) dx + N\left(x,y\right) dy = 0$$

Where

$$\begin{aligned}M &= 2xe^y\\ N &= 3y^2 + x^2e^y\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial M}{\partial y} &= 2xe^y\\ \frac{\partial N}{\partial x} &= 2xe^y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then the ode is exact.

2.1.14 Problem 13

Solve the given differential equation

$$\left(y^2 + \cos x\right) dx + \left(2xy + \sin y\right) dy = 0 \tag{1}$$

Solution

The first step is to determine if the ODE is exact or not. Writing the ODE as

$$M\left(x,y\right) dx + N\left(x,y\right) dy = 0$$

Therefore

$$\begin{aligned}\frac{\partial M}{\partial y} &= 2y\\ \frac{\partial N}{\partial x} &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. This implies there exists potential function $\phi(x,y)$ such that its differential is

$$\begin{aligned} d\phi &= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy \\ &= 0 \end{aligned} \tag{2}$$

This implies $\phi(x,y) = C_1$. Comparing (1,2) shows that

$$\frac{\partial \phi}{\partial x} = M \tag{3}$$

$$\frac{\partial \phi}{\partial y} = N \tag{4}$$

Integrating (3) w.r.t. x gives

$$\phi = \int M dx + f(y)$$

Where $f(y)$ acts as the integration constant, but since ϕ depends on both x, y, it becomes an arbitrary function of y instead. The above becomes

$$\begin{aligned} \phi &= \int \left(y^2 + \cos x\right) dx + f(y) \\ &= xy^2 + \sin x + f(y) \end{aligned} \tag{5}$$

Taking derivative w.r.t. y of the above gives

$$\frac{\partial \phi}{\partial y} = 2xy + f'(y) \tag{6}$$

Comparing (6) and (4) shows that

$$\begin{aligned} N &= 2xy + f'(y) \\ 2xy + \sin y &= 2xy + f'(y) \\ \sin y &= f'(y) \end{aligned}$$

Therefore $f(y) = -\cos x + C_2$ where C_2 is arbitrary constant. Substituting $f(y)$ back in (5) gives

$$\phi(x,y) = xy^2 + \sin x - \cos x + C_2$$

But since $\phi(x,y)$ is a constant function, say C_1 then the above becomes

$$\boxed{xy^2 + \sin x - \cos x = C}$$

Where the constants C_1, C_2 are combined to one constant C. The above is the solution to the ODE. It can be left in implicit form as the above, or we can solve for y explicitly. Solving for y gives

$$y^2 = \frac{C + \cos x - \sin x}{x}$$

Hence

$$y(x) = \pm\sqrt{\frac{C + \cos x - \sin x}{x}} \qquad x \neq 0$$

2.1.15 Problem 14

Determine an integrating factor for the given differential equation and hence find the general solution

$$\left(xy - 1\right) dx + x^2 dy = 0 \tag{1}$$

Solution

Writing the ODE as

$$M\left(x,y\right)dx+N\left(x,y\right)dy=0$$

Where

$$\begin{aligned}M&=xy-1\\N&=x^2\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial M}{\partial y}&=x\\\frac{\partial N}{\partial x}&=2x\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ then the ode is <u>not exact</u>. Applying theorem 1.9.11 part(1).

$$\begin{aligned}\frac{M_y-N_x}{N}&=\frac{x-2x}{x^2}\\&=-\frac{1}{x}\\&=f(x)\end{aligned}$$

Since this depends on x only, then there exists an integrating factor that depends on x only which makes the ODE exact. The integrating factor is therefore

$$\begin{aligned}I&=e^{\int f(x)dx}\\&=e^{\int-\frac{1}{x}dx}\\&=e^{-\ln x}\\&=\frac{1}{x}\end{aligned}$$

Multiplying the given ODE (1) by this integrating factor gives

$$\begin{aligned}\frac{1}{x}\left(xy-1\right)dx+\frac{1}{x}x^2dy&=0\\\left(y-\frac{1}{x}\right)dx+xdy&=0\end{aligned}$$

Where now

$$\begin{aligned}M&=y-\frac{1}{x}\\N&=x\end{aligned}$$

Let us first verify the above is indeed exact.

$$\begin{aligned}\frac{\partial M}{\partial y}&=1\\\frac{\partial N}{\partial x}&=1\end{aligned}$$

This shows it is exact as expected. Hence now we need to find $\phi\left(x,y\right)$ by solving the following two equations

$$\frac{\partial\phi}{\partial x}=M=y-\frac{1}{x}\tag{3}$$

$$\frac{\partial\phi}{\partial y}=N=x\tag{4}$$

Integrating (3) w.r.t. x gives

$$\phi = \int M dx + f(y)$$

Where $f(y)$ acts as the integration constant, but since ϕ depends on both x, y, it becomes an arbitrary function of y instead. The above becomes

$$\begin{aligned}\phi &= \int \left(y - \frac{1}{x}\right) dx + f(y) \\ &= xy - \ln x + f(y) \qquad (5)\end{aligned}$$

Taking derivative w.r.t. y of the above gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \qquad (6)$$

Comparing (6) and (4) shows that

$$\begin{aligned}N &= x + f'(y) \\ x &= x + f'(y) \\ 0 &= f'(y)\end{aligned}$$

Therefore $f(y) = C_2$ where C_2 is arbitrary constant. Substituting $f(y)$ back in (5) gives

$$\phi(x, y) = xy - \ln x + C_2$$

But since $\phi(x, y)$ is a constant function, say C_1 then the above becomes

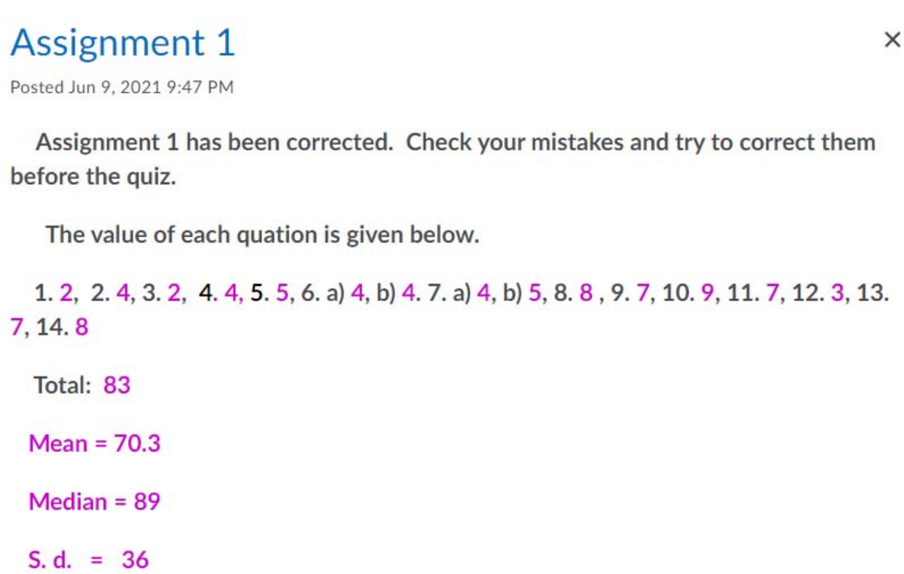
$$xy - \ln x = C$$

Where the constants C_1, C_2 are combined to one constant C. Solving for y gives

$$\boxed{y = \frac{C + \ln x}{x}}$$

Where $x \neq 0$

2.1.16 Marks per problem


Assignment 1

Posted Jun 9, 2021 9:47 PM

Assignment 1 has been corrected. Check your mistakes and try to correct them before the quiz.

The value of each quation is given below.

1. 2, 2. 4, 3. 2, 4. 4, 5. 5, 6. a) 4, b) 4. 7. a) 4, b) 5, 8. 8 , 9. 7, 10. 9, 11. 7, 12. 3, 13. 7, 14. 8

Total: 83

Mean = 70.3

Median = 89

S. d. = 36

Figure 2.3: marks

2.2 HW 2

Local contents

2.2.1 Problems listing

Math2520-01 **Assignment 2**

INSTRUCTION: Show all the necessary work. Write your answer on a separate sheet preferably hand written clear and legible. Post your answer sheet as a PDF on D2L by Sunday June 13. Late June 14.

1. Let

$$A=\begin{bmatrix}2 & 0 & -1\\ 4 & -5 & 2\end{bmatrix},\quad B=\begin{bmatrix}7 & -5 & 1\\ 1 & -4 & -3\end{bmatrix},\quad C=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}$$

Compute: a) $A + 2B$ b) A^T c) $A+C$ d) CB

If the expression is undefined explain why.

2. Determine the solution set to the given system. If using calculator, show three steps:
   1. Write the augmented matrix and determine the size of the matrix.
   2. Find the rref(augmented matrix)
   3. Write the solution.

a) $$\begin{aligned} x_1+2x_2+x_3&=1\\ 3x_1+5x_2+x_3&=3\\ 2x_1+6x_2+7x_3&=1\end{aligned}$$

b) $$\begin{aligned} x_1-2x_2-x_3+3x_4&=0\\ -2x_1+4x_2+5x_3-5x_4&=3\\ 3x_1-6x_2-6x_3+8x_4&=2\end{aligned}$$

If the system is inconsistent state why.

3. Given the following matrix function,

$$A(t)=\begin{bmatrix}-7 & t^2\\ 1+t & \cos(\frac{\pi t}{2})\end{bmatrix},$$

a) determine the derivative $\frac{dA}{dt}$,

b) determine $\int_0^1 A(t)dt$.

1

4. Write the system of equations with the given coefficient matrix and right-hand side vector.

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & 1 & -2 & 6 \\ 3 & 1 & 4 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

5. Reduce to row-echelon form and determine the rank of the matrix.

a) $A = \begin{bmatrix} 2 & 1 & 4 \\ 2 & -3 & 4 \\ 3 & -2 & 6 \end{bmatrix}$ b) $B = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

6. Reduce the matrix to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix and list the pivot columns.

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

7. Determine A^{-1} , if possible, using the Gauss-Jordan method.

a) $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$ b) $B = \begin{bmatrix} 1 & 3 & -4 \\ -1 & -2 & 5 \\ 2 & 6 & 7 \end{bmatrix}$

8. Use A^{-1} to find the solution to the given system.

$$\begin{aligned} 6x_1 + 20x_2 &= -8 \\ 2x_1 + 7x_2 &= 4 \end{aligned}$$

9. Use the cofactor expansion theorem to evaluate the given determinant along the specified row or column.

a) $A = \begin{vmatrix} 2 & 1 & 4 \\ 2 & -3 & 4 \\ 3 & -2 & 6 \end{vmatrix}$, column 3 b) $B = \begin{bmatrix} 6 & -1 & 2 \\ -4 & 7 & 1 \\ 0 & 3 & 1 \end{bmatrix}$, second row

10. Find A^{-1} .

$$A=\begin{bmatrix} 3e^{t} & e^{2t} \\ 2e^{t} & 2e^{2t} \end{bmatrix}$$

11. Let A and B be 3×3 matrices with $\det(A)=3$ and $\det(B)=-4$. Compute $\det(B^{-1}AB)^{2}$.

12. Use Cramer's rule to solve the given linear system.

$$\begin{aligned} 3x_1-2x_2 &= 6 \\ -5x_1+4x_2 &= 8 \end{aligned}$$

2.2.2 Problem 1

Let $A=\begin{bmatrix}2 & 0 & -1\\ 4 & -5 & 2\end{bmatrix}, B=\begin{bmatrix}7 & -5 & 1\\ 1 & -4 & -3\end{bmatrix}, C=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}$. Compute a) $A+2B$. b) A^T. c)$A+C$, d)CB

Solution

2.2.2.1 part a

$$
\begin{aligned}
A+2B &= \begin{bmatrix}2 & 0 & -1\\ 4 & -5 & 2\end{bmatrix} + 2\begin{bmatrix}7 & -5 & 1\\ 1 & -4 & -3\end{bmatrix}\\
&= \begin{bmatrix}2 & 0 & -1\\ 4 & -5 & 2\end{bmatrix} + \begin{bmatrix}14 & -10 & 2\\ 2 & -8 & -6\end{bmatrix}\\
&= \begin{bmatrix}2+14 & 0-10 & -1+2\\ 4+2 & -5-8 & 2-6\end{bmatrix}\\
&= \begin{bmatrix}16 & -10 & 1\\ 6 & -13 & -4\end{bmatrix}
\end{aligned}
$$

2.2.2.2 part b

The transpose operation exchanges rows with columns, therefore

$$
\begin{aligned}
A^T &= \begin{bmatrix}2 & 0 & -1\\ 4 & -5 & 2\end{bmatrix}^T\\
&= \begin{bmatrix}2 & 4\\ 0 & -5\\ -1 & 2\end{bmatrix}
\end{aligned}
$$

2.2.2.3 part c

$$
A+C = \begin{bmatrix}2 & 0 & -1\\ 4 & -5 & 2\end{bmatrix} + \begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}
$$

This is undefined because the size of the matrices must be the same in order to add them to together, because addition is done element by element.

2.2.2.4 part d

$$
CB = \begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}\begin{bmatrix}7 & -5 & 1\\ 1 & -4 & -3\end{bmatrix}
$$

C has size 2×2 and B has size 2×3. Since inner dimensions agree, then the matrix product is defined. The result will be 2×3. Doing the matrix multiplication using the standard rows times columns method, gives

$$
CB = \begin{bmatrix}9 & -13 & -5\\ -13 & 6 & -5\end{bmatrix}
$$

2.2.3 Problem 2

Determine the solution set to the given system. If using calculator, show three steps: 1. Write the augmented matrix and determine the size of the matrix. 2. Find the rref(augmented matrix) 3. Write the solution.

a)

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 + 5x_2 + x_3 &= 3 \\ 2x_1 + 6x_2 + 7x_3 &= 1 \end{aligned}$$

b)

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2 \end{aligned}$$

If the system is inconsistent state why.

Solution

2.2.3.1 part a

In matrix form $Ax = b$ the system becomes

$$Ax = b$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 5 & 1 \\ 2 & 6 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

The size of the A matrix is 3×3. Now the augmented matrix is setup in order to solve the system.

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 5 & 1 & 3 \\ 2 & 6 & 7 & 1 \end{bmatrix}$$

In the above, the vector b was appended to the right side of the A matrix. The augmented matrix has size 3×4. The augmented matrix is now converted to Echelon form using the allowed row operations. (Multiplication by constant, or adding multiples of another row to the row). In all the following, the notation $R_i = R_i + R_j$ means to replace row i with row i added to row j.

$R_2 = 3R_1 - R_2$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 2 & 6 & 7 & 1 \end{bmatrix}$$

$R_3 = 2R_1 - R_3$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & -5 & 1 \end{bmatrix}$$

$R_3 = 2R_2 + R_3$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Now it is in row echelon form. Next step is to convert to row reduced echelon form. Multiplying last row by −1 gives

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_2 = R_2 - 2R_3$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_1 = R_1 - R_3$ gives

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_1 = R_1 - 2R_2$ gives

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

This is now in row reduced Echelon form. The basic variables are x_1, x_2, x_3. There are no free variables. Hence the original system now becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

The last row gives $x_3 = -1$. Second row gives $x_2 = 2$ and first row gives $x_1 = -2$. The solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

2.2.3.2 part b

In matrix form $Ax = b$ the system becomes

$$Ax = b$$

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ -2 & 4 & 5 & -5 \\ 3 & -6 & -6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

The A matrix size is 3×4. The augmented matrix is

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 1 \end{bmatrix}$$

$R_2 = R_2 + 2R_1$ gives

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & 1 & 5 \\ 3 & -6 & -6 & 8 & 1 \end{bmatrix}$$

$R_3 = R_3 - 3R_1$ gives

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & 1 & 5 \\ 0 & 0 & -3 & -1 & -2 \end{bmatrix}$$

$R_3 = R_3 + R_2$ gives

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

The system is inconsistent. Because the last row says $0 = 3$. Therefore there are no solutions.

2.2.4 Problem 3

Given the following matrix function $A(t) = \begin{bmatrix} -7 & t^2 \\ 1+t & \cos\left(\frac{\pi t}{2}\right) \end{bmatrix}$. (a) determine the derivative $\frac{dA}{dt}$. (b) Determine $\int_0^t A(\tau)\, d\tau$

Solution

2.2.4.1 Part a

$$\begin{aligned} \frac{dA}{dt} &= \frac{d}{dt}\begin{bmatrix} -7 & t^2 \\ 1+t & \cos\left(\frac{\pi t}{2}\right) \end{bmatrix} \\ &= \begin{bmatrix} \frac{d}{dt}(-7) & \frac{d}{dt}\left(t^2\right) \\ \frac{d}{dt}(1+t) & \frac{d}{dt}\cos\left(\frac{\pi t}{2}\right) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2t \\ 1 & -\frac{\pi}{2}\sin\left(\frac{\pi t}{2}\right) \end{bmatrix} \end{aligned}$$

2.2.4.2 Part b

$$\int_0^t A(\tau)\,d\tau = \int_0^t \begin{bmatrix} -7 & \tau^2 \\ 1+\tau & \cos\left(\frac{\pi\tau}{2}\right) \end{bmatrix} dt$$

$$= \begin{bmatrix} -\int_0^t 7dt & \int_0^t \tau^2 dt \\ \int_0^t (1+\tau)\,dt & \int_0^t \cos\left(\frac{\pi\tau}{2}\right) dt \end{bmatrix}$$

$$= \begin{bmatrix} -[7\tau]_0^t & \frac{1}{3}\left[\tau^3\right]_0^t \\ \left[\tau + \frac{\tau^2}{2}\right]_0^t & \frac{2}{\pi}\left[\sin\left(\frac{\pi\tau}{2}\right)\right]_0^t \end{bmatrix}$$

$$= \begin{bmatrix} -7t & \frac{1}{3}t^3 \\ t + \frac{t^2}{2} & \frac{2}{\pi}\sin\left(\frac{\pi t}{2}\right) \end{bmatrix}$$

2.2.5 Problem 4

Write the system of equations with the given coefficient matrix and right-hand side vector.

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & 1 & -2 & 6 \\ 3 & 1 & 4 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Solution

The problem is asking to write $Ax = b$. Let the variables be x_1, x_2, x_3, x_4. Hence the above becomes

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & 1 & -2 & 6 \\ 3 & 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Carrying the matrix vector multiplication using standard method of a row times a column gives the system of equations

$$\begin{aligned} x_1 - x_2 + 2x_3 + 3x_4 &= 1 \\ x_1 + x_2 - 2x_3 + 6x_4 &= -1 \\ 3x_1 + x_2 + 4x_3 + 2x_4 &= 2 \end{aligned}$$

2.2.6 Problem 5

Reduce to row-echelon form and determine the rank of the matrix

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 2 & -3 & 4 \\ 3 & -2 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Solution

2.2.6.1 Part a

$$A=\begin{bmatrix}2 & 1 & 4\\ 2 & -3 & 4\\ 3 & -2 & 6\end{bmatrix}$$

$R_2 = R_2 - R_1$ gives

$$\begin{bmatrix}2 & 1 & 4\\ 0 & -4 & 0\\ 3 & -2 & 6\end{bmatrix}$$

$R_3 = 2R_3, R_1 = 3R_1$ gives

$$\begin{bmatrix}6 & 3 & 12\\ 0 & -4 & 0\\ 6 & -4 & 12\end{bmatrix}$$

$R_3 = R_3 - R_1$ gives

$$\begin{bmatrix}6 & 3 & 12\\ 0 & -4 & 0\\ 0 & -7 & 0\end{bmatrix}$$

$R_3 = R_3 + \frac{7}{4}R_2$ gives

$$\begin{bmatrix}6 & 3 & 12\\ 0 & -4 & 0\\ 0 & 0 & 0\end{bmatrix}$$

The above is row echelon form. Pivots are $A(1,1)$ and $A(2,2)$. Number of pivots is 2. Hence Rank is 2

2.2.6.2 Part b

$$B=\begin{bmatrix}2 & -1 & 3\\ 3 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$$

$R_1 = 3R_1, R_2 = 2R_2$ gives

$$\begin{bmatrix}6 & -3 & 9\\ 6 & 2 & -4\\ 2 & -2 & 1\end{bmatrix}$$

$R_2 = R_2 - R_1$ gives

$$\begin{bmatrix}6 & -3 & 9\\ 0 & 5 & -13\\ 2 & -2 & 1\end{bmatrix}$$

$R_3 = 3R_3$ gives

$$\begin{bmatrix}6 & -3 & 9\\ 0 & 5 & -13\\ 6 & -6 & 3\end{bmatrix}$$

$R_3 = R_3 - R_1$ gives

$$\begin{bmatrix}6 & -3 & 9\\ 0 & 5 & -13\\ 0 & -3 & -6\end{bmatrix}$$

$R_3 = 5R_3, R_2 = 3R_2$ gives

$$\begin{bmatrix} 6 & -3 & 9 \\ 0 & 15 & -39 \\ 0 & -15 & -30 \end{bmatrix}$$

$R_3 = R_3 + R_2$ gives

$$\begin{bmatrix} 6 & -3 & 9 \\ 0 & 15 & -39 \\ 0 & 0 & -69 \end{bmatrix}$$

Pivots are $A(1,1), A(2,2), A(3,3)$. Three pivots. Hence rank is 3.

2.2.7 Problem 6

Reduce the matrix to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix and list the pivot columns

$$A = \begin{bmatrix} 1 & 3 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

Solution

$R_2 = R_2 - 2R_1$ gives

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & -2 & -2 & -8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

$R_3 = R_3 - 3R_1$ gives

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & -2 & -2 & -8 \\ 0 & -3 & -3 & -12 \end{bmatrix}$$

$R_2 = 3R_2, R_3 = 2R_3$ gives

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & -6 & -6 & -24 \\ 0 & -6 & -6 & -24 \end{bmatrix}$$

$R_3 = R_3 - R_2$ gives

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & -6 & -6 & -24 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 = \frac{R_2}{6}$ gives (to simplify)

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & -1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basic variables are x_1, x_2. Free variables are x_3, x_4. The above is row echelon form. Now we apply the reduced Echelon phase by zeroing all entries in the pivot columns above the pivots. In the above, the pivots are $A(1,1)$ and $A(2,2)$. First, all pivot entries are changed to 1

$R_2 = -R_2$ gives

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 = R_1 - 3R_2$ gives

$$\begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is the reduced echelon form. I do not know how to circle the pivots using Latex. But they are $A(1,1) = 1$ and $A(2,2) = 1$ in the above final matrix. I the original matrix they are $A(1,1) = 1$ and $A(2,2) = 4$.

The pivot columns are the first and second columns. In the final matrix, these are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

2.2.8 Problem 7

Determine A^{-1} if possible using Gauss-Jordan method

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & -4 \\ -1 & -2 & 5 \\ 2 & 6 & 7 \end{bmatrix}$$

Solution

2.2.8.1 Part a

The augmented matrix becomes, after adding the identity 3×3 matrix to the right side of the original matrix

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix}$$

Elimination is now applied to transform the left half of the above matrix to becomes the identity matrix. What then results on the right half will be A^{-1}.

$R_2 = R_2 + R_1$ gives

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & -1 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 - 5R_1$ gives

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & -1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 - 2R_2$ gives

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & -1 & 1 & 0 \\ 0 & 0 & 0 & -3 & -2 & 1 \end{bmatrix}$$

Since the last row in the left half is all zeros, it means it is not possible to obtain the identity matrix on the left side. Therefore this matrix has no inverse since it is singular. The rank is only 2. We stop here.

2.2.8.2 Part b

The augmented matrix becomes, after adding the identity matrix to the right side

$$\begin{bmatrix} 1 & 3 & -4 & 1 & 0 & 0 \\ -1 & -2 & 5 & 0 & 1 & 0 \\ 2 & 6 & 7 & 0 & 0 & 1 \end{bmatrix}$$

Now elimination is applied to transform the left half to the identity matrix. What results on the right side will be A^{-1}.

$R_2 = R_2 + R_1$ gives

$$\begin{bmatrix} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 2 & 6 & 7 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 - 2R_1$ gives

$$\begin{bmatrix} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 15 & -2 & 0 & 1 \end{bmatrix}$$

Now reduced echelon phase starts.

$R_3 = \frac{R_3}{15}$ gives

$$\begin{bmatrix} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{2}{15} & 0 & \frac{1}{15} \end{bmatrix}$$

$R_2 = R_2 - R_3$ gives

$$\begin{bmatrix} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{17}{15} & 1 & -\frac{1}{15} \\ 0 & 0 & 1 & -\frac{2}{15} & 0 & \frac{1}{15} \end{bmatrix}$$

$R_1 = R_1 + 4R_3$ gives

$$\begin{bmatrix} 1 & 3 & 0 & 1+4\left(-\frac{2}{15}\right) & 0 & 4\left(\frac{1}{15}\right) \\ 0 & 1 & 0 & \frac{17}{15} & 1 & -\frac{1}{15} \\ 0 & 0 & 1 & -\frac{2}{15} & 0 & \frac{1}{15} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & \frac{7}{15} & 0 & \frac{4}{15} \\ 0 & 1 & 0 & \frac{17}{15} & 1 & -\frac{1}{15} \\ 0 & 0 & 1 & -\frac{2}{15} & 0 & \frac{1}{15} \end{bmatrix}$$

$R_1 = R_1 - 3R_2$ gives

$$\begin{bmatrix} 1 & 0 & 0 & \frac{7}{15}-3\left(\frac{17}{15}\right) & -3 & \frac{4}{15}-3\left(-\frac{1}{15}\right) \\ 0 & 1 & 0 & \frac{17}{15} & 1 & -\frac{1}{15} \\ 0 & 0 & 1 & -\frac{2}{15} & 0 & \frac{1}{15} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -\frac{44}{15} & -3 & \frac{7}{15} \\ 0 & 1 & 0 & \frac{17}{15} & 1 & -\frac{1}{15} \\ 0 & 0 & 1 & -\frac{2}{15} & 0 & \frac{1}{15} \end{bmatrix}$$

Since the left half is the identity matrix, then the right half is the matrix inverse of the original matrix. Therefore

$$B^{-1} = \begin{bmatrix} -\frac{44}{15} & -3 & \frac{7}{15} \\ \frac{17}{15} & 1 & -\frac{1}{15} \\ -\frac{2}{15} & 0 & \frac{1}{15} \end{bmatrix}$$

2.2.9 Problem 8

Use A^{-1} to find the solution to the given system

$$\begin{aligned} 6x_1 + 20x_2 &= -8 \\ 2x_1 + 7x_2 &= 4 \end{aligned}$$

<u>Solution</u>

The first step is to find A^{-1}. In matrix form, the above system is

$$Ax = b$$

$$\begin{bmatrix} 6 & 20 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

Hence the augmented matrix to find A^{-1} is

$$\begin{bmatrix} 6 & 20 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix}$$

Now elimination is applied to transform the left half to the identity matrix. What results on the right side will be A^{-1}.

$R_2 = 3R_2$ gives

$$\begin{bmatrix} 6 & 20 & 1 & 0 \\ 6 & 21 & 0 & 3 \end{bmatrix}$$

$R_2 = R_2 - R_1$ gives

$$\begin{bmatrix} 6 & 20 & 1 & 0 \\ 0 & 1 & -1 & 3 \end{bmatrix}$$

$R_1 = \frac{R_1}{6}$ gives

$$\begin{bmatrix} 1 & \frac{20}{6} & \frac{1}{6} & 0 \\ 0 & 1 & -1 & 3 \end{bmatrix}$$

$R_1 = R_1 - \frac{20}{6}R_2$ gives

$$\begin{bmatrix} 1 & 0 & \frac{1}{6} + \left(\frac{20}{6}\right) & -10 \\ 0 & 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{7}{2} & -10 \\ 0 & 1 & -1 & 3 \end{bmatrix}$$

Since the left half is now the identity matrix, then the right half is the inverse. Therefore

$$A^{-1} = \begin{bmatrix} \frac{7}{2} & -10 \\ -1 & 3 \end{bmatrix}$$

Now that A^{-1} is found, then the solution is found as follows. By premultiplying both sides of the equation by A^{-1}

$$A^{-1}Ax = A^{-1}b$$

But $A^{-1}A = I$, the identity matrix. Hence the above simplifies to

$$x = A^{-1}b$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & -10 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

Applying the standard matrix vector multiplications on the right side gives the solution as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -68 \\ 20 \end{bmatrix}$$

2.2.10 Problem 9

Use the cofactor expansion theorem to evaluate the given determinant along the specified row or column.

$$A = \begin{vmatrix} 2 & 1 & 4 \\ 2 & -3 & 4 \\ 3 & -2 & 6 \end{vmatrix} \qquad \text{column 3}$$

$$B = \begin{vmatrix} 6 & -1 & 2 \\ -4 & 7 & 1 \\ 0 & 3 & 1 \end{vmatrix} \qquad \text{row 2}$$

Solution

2.2.10.1 Part a

The expansion along column 3 is given by

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & 1 & 4 \\ 2 & -3 & 4 \\ 3 & -2 & 6 \end{vmatrix} \\ &= (-1)^{1+3}(4)\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix} + (-1)^{2+3}(4)\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} + (-1)^{3+3}(6)\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} \\ &= 4\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix} - 4\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} + 6\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} \\ &= (4)(5) - (4)(-7) + (6)(-8) \\ &= 20 + 28 - 48 \\ &= 0 \end{aligned}$$

Since the determinant is zero, the matrix is singular.

2.2.10.2 Part b

The expansion along row 2 is given by (the sign of each cofactor is found using $(-1)^{i+j}$ where i is the row number and j is the column number).

$$
\begin{aligned}
\det(B) &= \begin{vmatrix} 6 & -1 & 2 \\ -4 & 7 & 1 \\ 0 & 3 & 1 \end{vmatrix} \\
&= (-1)^{2+1}(-4)\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} + (-1)^{2+2}(7)\begin{vmatrix} 6 & 2 \\ 0 & 1 \end{vmatrix} + (-1)^{2+3}(1)\begin{vmatrix} 6 & -1 \\ 0 & 3 \end{vmatrix} \\
&= 4\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} + 7\begin{vmatrix} 6 & 2 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 6 & -1 \\ 0 & 3 \end{vmatrix} \\
&= (4)(-7) + (7)(6) - (18) \\
&= -28 + 42 - 18 \\
&= -4
\end{aligned}
$$

2.2.11 Problem 10

Find A^{-1}

$$
A = \begin{bmatrix} 3e^t & e^{2t} \\ 2e^t & 2e^{2t} \end{bmatrix}
$$

Solution

The augment matrix is

$$
\begin{bmatrix} 3e^t & e^{2t} & 1 & 0 \\ 2e^t & 2e^{2t} & 0 & 1 \end{bmatrix}
$$

$R_1 = 2R_1, R_2 = 3R_2$ gives

$$
\begin{bmatrix} 6e^t & 2e^{2t} & 2 & 0 \\ 6e^t & 6e^{2t} & 0 & 3 \end{bmatrix}
$$

$R_2 = R_2 - R_1$ gives

$$
\begin{bmatrix} 6e^t & 2e^{2t} & 2 & 0 \\ 0 & 4e^{2t} & -2 & 3 \end{bmatrix}
$$

$R_1 = R_1 - \frac{1}{2}R_2$ gives

$$
\begin{bmatrix} 6e^t & 0 & 3 & -\frac{3}{2} \\ 0 & 4e^{2t} & -2 & 3 \end{bmatrix}
$$

$R_1 = \frac{R_1}{6}$ gives

$$
\begin{bmatrix} e^t & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 4e^{2t} & -2 & 3 \end{bmatrix}
$$

$R_2 = \frac{R_2}{4}$ gives

$$
\begin{bmatrix} e^t & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & e^{2t} & -\frac{1}{2} & \frac{3}{4} \end{bmatrix}
$$

$R_1 = \frac{R_1}{e^t}$ (since $e^t \neq 0$) gives

$$\begin{bmatrix} 1 & 0 & \frac{1}{2}e^{-t} & -\frac{1}{4}e^{-t} \\ 0 & e^{2t} & -\frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

$R_2 = \frac{R_1}{e^{2t}}$ (since $e^{2t} \neq 0$) gives

$$\begin{bmatrix} 1 & 0 & \frac{1}{2}e^{-t} & -\frac{1}{4}e^{-t} \\ 0 & 1 & -\frac{1}{2}e^{-2t} & \frac{3}{4}e^{-2t} \end{bmatrix}$$

Since the left half is now the identity matrix, then the right half is the inverse. Therefore

$$A^{-1} = \begin{bmatrix} \frac{1}{2}e^{-t} & -\frac{1}{4}e^{-t} \\ -\frac{1}{2}e^{-2t} & \frac{3}{4}e^{-2t} \end{bmatrix}$$

2.2.12 Problem 11

Let A and B be 3×3 matrices with $\det(A) = 3$ and $\det(B) = -4$. Compute $\det\left(B^{-1}AB\right)^2$

Solution

The determinant of a product of matrices is the product of their determinants. Hence

$$\det\left(B^{-1}AB\right)^2 = \det\left(B^{-1}AB\right)\det\left(B^{-1}AB\right) \tag{1}$$

But

$$\det\left(B^{-1}AB\right) = \det\left(B^{-1}\right)\det(A)\det(B) \tag{2}$$

Also, the determinant of the inverse of an invertible matrix is the inverse of the determinant. Hence $\det\left(B^{-1}\right) = \frac{1}{\det(B)}$ and the above reduces to

$$\begin{aligned} \det\left(B^{-1}AB\right) &= \frac{1}{\det(B)}\det(A)\det(B) \\ &= \det(A) \\ &= 3 \end{aligned}$$

Substituting the above into (1) gives

$$\begin{aligned} \det\left(B^{-1}AB\right)^2 &= (3)(3) \\ &= 9 \end{aligned}$$

Note that knowing $\det(B) = -4$ was not really needed, as it cancels out. We just needed to know that $\det(B) \neq 0$. (in other words, that B is not singular).

2.2.13 Problem 12

Use Cramer's rule to solve the given linear system

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$$

Solution

The system is matrix form is

$$Ax = b$$

$$\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

Hence, using Cramer's rule gives

$$x_1 = \frac{\begin{vmatrix} 6 & -2 \\ 8 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -5 & 4 \end{vmatrix}}$$

$$x_2 = \frac{\begin{vmatrix} 3 & 6 \\ -5 & 8 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -5 & 4 \end{vmatrix}}$$

But $\begin{vmatrix} 3 & -2 \\ -5 & 4 \end{vmatrix} = 12 - 10 = 2$. The above becomes

$$x_1 = \frac{\begin{vmatrix} 6 & -2 \\ 8 & 4 \end{vmatrix}}{2} = \frac{24+16}{2} = \frac{40}{2} = 20$$

$$x_2 = \frac{\begin{vmatrix} 3 & 6 \\ -5 & 8 \end{vmatrix}}{2} = \frac{24+30}{2} = \frac{54}{2} = 27$$

Therefore the solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 27 \end{bmatrix}$$

2.3 HW 3

Local contents

2.3.1 Problems listing

Math2520-01 **Assignment 3**

INSTRUCTION: Show all the necessary work. Write your answer on a separate sheet preferably hand written clear and legible. Post your answer sheet on D2L by Sunday June 20.

1. If $x=(-3,9,9)$ and $y=(3,0,-5)$, find a vector z in R^3 such that $4x-y+2z=0$ and its additive inverse.

2. Determine whether the given set S of vectors is closed under addition and is closed under scalar multiplication. The set of scalars is the set of all real numbers. Justify your answer.

   a) The set S = Q , the set of all rational numbers.

   b) The set S of all solutions to the differential equation

      $y'+3y=0$ (do not solve the differential equation)

3. Let $S=\{(x,y)\in R^2 : x\geq 0, y\geq 0\}$. Is S a subspace of R^2. Justify your answer.

4. Let $V=C^2(I)$ and S is a subset of V consisting of those functions satisfying the differential equation

$$y''+2y'-y=0,$$

   On I. Determine if S is a subspace of V.

5. a) Determine the null space of the given matrix A, nullsapce(A) .

$$A=\begin{bmatrix} 2 & 6 & 4 \\ -3 & 2 & 5 \\ -5 & -4 & 1 \end{bmatrix}$$

   b) Determine if $w=\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is in the nullsapce(A).

6. Let $v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ vectors in R^2 . Express the vector $v = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$ as a linear combination of v_1, v_2.

7. Let $v = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ be in R^3. Let $W = span(v_1, v_2)$. Determine if *v* is in *W*.

8. Determine whether the given set $\{(1,-1,0),(0,1,-1),(1,1,1)\}$ in R^3 is linearly independent or linearly dependent.

9. Use the Wronskian to show that the given functions are linearly independent on the given interval *I*.

$$f_1(x) = 1,\ \ f_2(x) = 3x,\ \ f_3(x) = x^2 - 1,\ \ I = (-\infty, \infty)$$

10. Determine whether the set of vectors,

$$S = \{(1,1,0,2),(2,1,3,-1),(-1,1,1,-2),(2,-1,1,2)\}$$

is a basis for R^4.

11. Determine whether the set $S = \{1-3x^2, 2x+5x^2, 1-x+3x^2\}$ is basis for $P_2(R)$.

12. Find the dimension of the null space of the given matrix A.

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 2 & 3 & -2 \\ 1 & 2 & -2 \end{bmatrix}$$

13. Determine the component vector of the given vector space V relative to the given ordered basis B.

$V = R^2$; $B = \{(2,-2),(1,4)\}$; $v = (5,-10)$.

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14. a) find n such that rowspace(A) is a subspace of R^n and determine the basis for rowspace(A).
   a) find m such that colspace(A) is a subspace of R^m, and determine a basis for colspace(A).

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & 1 & -2 & 6 \\ 3 & 1 & 4 & 2 \end{bmatrix}$$

Note: 1. You can use a theorem whenever applicable.

2. Check the video clips posted on D2L related to this chapter.

3

2.3.2 Problem 1

If $\vec{x} = (-3, 9, 9)$ and $\vec{y} = (3, 0, -5)$ find a vector $\vec{z}$ in $\mathbb{R}^3$ such that $4\vec{x} - \vec{y} + 2\vec{z} = \vec{0}$ and its additive inverse.

Solution

$$4\begin{bmatrix}-3\\9\\9\end{bmatrix} - \begin{bmatrix}3\\0\\-5\end{bmatrix} + 2\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-15\\36\\41\end{bmatrix} + \begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix} = \begin{bmatrix}15\\-36\\-41\end{bmatrix}$$

Now it is in $Az = b$ form. The A matrix is already in rref form. Last row gives $2z_3 = -41$ or $z_3 = \frac{-41}{2}$. Second row gives $2z_2 = -36$ or $z_2 = -18$ and first row gives $2z_1 = 15$ or $z_1 = \frac{15}{2}$. Hence the vector z is

$$\vec{z} = \begin{bmatrix}\frac{15}{2}\\-18\\-\frac{41}{2}\end{bmatrix}$$

Therefore its additive inverse is

$$\begin{bmatrix}-\frac{15}{2}\\18\\\frac{41}{2}\end{bmatrix}$$

2.3.3 Problem 2

Determine whether the given set S of vectors is closed under addition and is closed under scalar multiplication. The set of scalars is the set of all real numbers. Justify your answer

a) The set $S = \mathbb{Q}$, the set of all rational numbers b) The set S of all solutions to the differential equation $y' + 3y = 0$

Solution

2.3.3.1 Part a

Let x_1, x_2 be any two rational numbers in S. Then $x_1 + x_2$ is also a rational number, since the sum of two rational numbers is a rational number. Hence $x_1 + x_2 \in \mathbb{Q}$ which means S is closed under addition.

Let a be any real scalar and x a rational number in S. The type of the product ax_i will depend on if the real number a can be represented as rational number or not. Since not all real numbers are rational, then it is possible to find scalar a which is not a rational number which make ax not rational (for an example if $a = \pi$ or $a = \sqrt{2}$). Therefore the set S is not closed under scalar multiplication by real numbers.

2.3.3.2 Part b

The general solution to above first order ODE are given by $y(x) = Cf(x)$. Where C is an arbitrary constant. Let $y_1(x)$ be one general solution given by $y_1(x) = c_1 f(x)$ where c_1 is arbitrary constant of integration. And let Let $y_2(x)$ be another general solution of the ODE given by $y_2(x) = c_2 f(x)$ where c_2 is arbitrary constant of integration. Hence

$$\begin{aligned} y_1(x) + y_2(x) &= c_1 f(x) + c_2 f(x) \\ &= (c_1 + c_2) f(x) \end{aligned}$$

Let $c_1 + c_2 = C_0$ be new constant. Hence the above can be written as

$$y_1(x) + y_2(x) = C_0 f(x)$$

This shows it is closed under the sum since it has the same form. Similarly, let a be any scalar from the reals . Then

$$ay_1(x) = a\left(c_1 f(x)\right)$$

Let ac_1 be new constant C_0. The above becomes

$$ay_1(x) = C_0 f(x)$$

This shows it is closed under scalar multiplication since it has the same form.

2.3.4 Problem 3

Let $S = \left\{\left(x, y\right) \in \mathbb{R}^2 : x \geq 0, y \geq 0\right\}$. Is S a subspace of $\mathbb{R}^2$. Justify your answer

Solution

The set S contains all vectors in the first quadrant in $\mathbb{R}^2$. First, we see that the zero vector is in S which is when $x = 0, y = 0$. This is requirements for all subspaces. Now we check to see if S is closed closed under addition and scalar multiplication.

Let v_1, v_2 be two arbitrary vectors selected from first quadrant. Hence

$$\begin{aligned} \vec{v}_1 + \vec{v}_2 &= \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \end{aligned}$$

But since $x_1 \geq 0$ and $x_2 \geq 0$ then $x_1 + x_2 \geq 0$. Similarly since $y_1 \geq 0$ and $y_2 \geq 0$ then $y_1 + y_2 \geq 0$. Hence $\vec{v}_1 + \vec{v}_2 \in S$ which means closed under addition. Now, let a be real scalar. Hence

$$\begin{aligned} a\vec{v} &= a \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} ax \\ ay \end{bmatrix} \end{aligned}$$

But this is not closed for all a. For example if $a = -1$ then $ax \leq 0$ and $ay \leq 0$. Hence not closed under scalar multiplication.

This shows S is not a subspace, since it is not closed under scalar multiplication.

2.3.5 Problem 4

Let $V = C^2(I)$ and S is a subset of V consisting of those functions satisfying the differential equation $y'' + 2y' - y = 0$ on I. Determine if S is a subspace of V

Solution

The first step is to check for the zero solution. Since $y = 0$ is a solution to the ode (since it satisfies it), then the zero solution is in S. Now we need to check if S is closed under additions. Since the general solution to second order ode of constant coefficients can be written as (assuming the independent variable is t)

$$y(x) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Where C_1, C_2 are arbitrary constants, and r_1, r_2 are the roots of the auxiliary equation $r^2 + 2r - 1 = 0$. We do not have to solve the ODE, but the roots are distinct in this case, hence the above is a valid general solution form.

Let $y_1(x) = A_1 e^{r_1 t} + A_2 e^{r_2 t}$ be one solution which satisfies the ODE on I and let $y_2(x) = B_1 e^{r_1 t} + B_2 e^{r_2 t}$ be another solution which satisfies the ODE on I. Both are twice differentiable. Therefore

$$\begin{aligned} y_1(t) + y_2(t) &= \left(A_1 e^{r_1 t} + A_2 e^{r_2 t}\right) + \left(B_1 e^{r_1 t} + B_2 e^{r_2 t}\right) \\ &= (A_1 + B_1) e^{r_1 t} + (A_2 + B_2) e^{r_2 t} \\ &= C_1 e^{r_1 t} + C_2 e^{r_2 t} \end{aligned}$$

Where $C_1 = (A_1 + B_1)$ is new constant, and $C_2 = (A_2 + B_2)$. This shows it is <u>closed under addition</u> since it has the same form and this is twice differentiable as well because the exponential functions are.

Now we show if it is closed under scalar multiplication. Let a be a scalar. Then, let $y(x) = Ae^{r_1 t} + Be^{r_2 t}$ be a solution which satisfies the ODE on I (it is also twice differentiable). Hence

$$\begin{aligned} ay(t) &= a\left(Ae^{r_1 t} + Be^{r_2 t}\right) \\ &= aAe^{r_1 t} + aBe^{r_2 t} \end{aligned}$$

Let $aA = C_1$ be new constant and let $aB = C_2$ be new constant. The above becomes

$$ay = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

This shows it is <u>closed under scalar</u> multiplication since it has the same form and this is twice differentiable.

2.3.6 Problem 5

a) Determine the null space of the given matrix A, null-space(A)

$$A = \begin{bmatrix} 2 & 6 & 4 \\ -3 & 2 & 5 \\ -5 & -4 & 1 \end{bmatrix}$$

b) Determine if $w = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is in the null-space(A)

<u>Solution</u>

2.3.6.1 Part a

A is 3×3. The null-space of A is the set of all 3×1 vectors $\vec{x}$ which satisfies $A\vec{x} = \vec{0}$. To find this set, we need to solve

$$\begin{bmatrix} 2 & 6 & 4 \\ -3 & 2 & 5 \\ -5 & -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 2 & 6 & 4 & 0 \\ -3 & 2 & 5 & 0 \\ -5 & -4 & 1 & 0 \end{bmatrix}$$

$R_1 = \frac{R_1}{2}$ gives

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ -3 & 2 & 5 & 0 \\ -5 & -4 & 1 & 0 \end{bmatrix}$$

$R_2 = R_2 + 3R_1$ gives

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 11 & 11 & 0 \\ -5 & -4 & 1 & 0 \end{bmatrix}$$

$R_3 = R_3 + 5R_1$ gives

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 11 & 11 & 0 \\ 0 & 11 & 11 & 0 \end{bmatrix}$$

$R_3 = R_3 - R_2$ gives

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 11 & 11 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This shows that x_1, x_2 basic variables and x_3 is free variable. There is no need to go all the way to rref to find the solution. But we can also do that and same solution will results. Let the free variable be $x_3 = s$.

Second row gives $11x_2 + 11x_3 = 0$ or $x_2 = -s$. First row gives $x_1 + 3x_2 + 2x_3 = 0$ or $x_1 = -3x_2 - 2x_3$ or $x_1 = 3s - 2s = s$. Hence the solution is

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} s \\ -s \\ s \end{bmatrix} \\ &= s \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

There are infinite number of solutions, one for different s value. Therefore the null-space(A) is the set of all vectors which are scalar multiples of $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

2.3.6.2 Part b

Yes. Since when $s = 1$ then $\vec{w} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is in the null-space(A). Hence $\vec{w}$ is scalar multiple of $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

2.3.7 Problem 6

Let $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$. Express the vector $\vec{v} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$ as linear combinations of v_1, v_2

Solution

We want to find scalars c_1, c_2 such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{v}$$

Therefore

$$\begin{aligned} c_1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 2 \end{bmatrix} &= \begin{bmatrix} 5 \\ -7 \end{bmatrix} \\ \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 5 \\ -7 \end{bmatrix} \end{aligned} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 2 & 3 & 5 \\ -1 & 2 & -7 \end{bmatrix}$$

$R_2 = 2R_2 + R_1$ gives

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 7 & -9 \end{bmatrix}$$

$R_1 = \frac{R_1}{2}, R_2 = \frac{R_2}{7}$ gives

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & -\frac{9}{7} \end{bmatrix}$$

$R_1 = R_1 - \frac{3}{2}R_2$ gives

$$\begin{bmatrix} 1 & 0 & \frac{5}{2} - \frac{3}{2}\left(-\frac{9}{7}\right) \\ 0 & 1 & -\frac{9}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{31}{7} \\ 0 & 1 & -\frac{9}{7} \end{bmatrix}$$

This is rref form. Hence the original system (1) now becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{31}{7} \\ -\frac{9}{7} \end{bmatrix}$$

Last row gives $c_2 = -\frac{9}{7}$ and first row gives $c_1 = \frac{31}{7}$. Therefore the combination is

$$\begin{aligned} c_1\vec{v}_1 + c_2\vec{v}_2 &= \vec{v} \\ \frac{31}{7}\vec{v}_1 - \frac{9}{7}\vec{v}_2 &= \vec{v} \end{aligned}$$

To verify

$$
\begin{aligned}
\frac{31}{7}\begin{bmatrix}2\\-1\end{bmatrix}-\frac{9}{7}\begin{bmatrix}3\\2\end{bmatrix} &= \begin{bmatrix}2\left(\frac{31}{7}\right)\\-1\left(\frac{31}{7}\right)\end{bmatrix}-\begin{bmatrix}3\left(\frac{9}{7}\right)\\2\left(\frac{9}{7}\right)\end{bmatrix}\\
&= \begin{bmatrix}\frac{62}{7}\\-\frac{31}{7}\end{bmatrix}-\begin{bmatrix}\frac{27}{7}\\\frac{18}{7}\end{bmatrix}\\
&= \begin{bmatrix}5\\-7\end{bmatrix}
\end{aligned}
$$

Which is $\vec{v}$

2.3.8 Problem 7

Let $\vec{v}=\begin{bmatrix}3\\3\\4\end{bmatrix}, \vec{v}_1=\begin{bmatrix}1\\-1\\2\end{bmatrix}, \vec{v}_2=\begin{bmatrix}2\\1\\3\end{bmatrix}$ be vectors in $\mathbb{R}^3$. Let $W=span\left(\vec{v}_1,\vec{v}_2\right)$. Determine if $\vec{v}$ is in W.

Solution

To find if $\vec{v}$ is in W means if $\vec{v}$ can be reached using the vectors $\vec{v}_1,\vec{v}_2$. This implies we can find solution c_1,c_2 to

$$c_1\vec{v}_1+c_2\vec{v}_2=\vec{v}$$

In this context, c_1,c_2 are called the coordinates of $\vec{v}$ using the basis $\vec{v}_1,\vec{v}_2$. Setting the above gives

$$c_1\begin{bmatrix}1\\-1\\2\end{bmatrix}+c_2\begin{bmatrix}2\\1\\3\end{bmatrix}=\begin{bmatrix}3\\3\\4\end{bmatrix}$$
$$\begin{bmatrix}1&2\\-1&1\\2&3\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}3\\3\\4\end{bmatrix}$$

The augmented matrix becomes

$$\begin{bmatrix}1&2&3\\-1&1&3\\2&3&4\end{bmatrix}$$

$R_2=R_2+R_1$ gives

$$\begin{bmatrix}1&2&3\\0&3&6\\2&3&4\end{bmatrix}$$

$R_3=R_3-2R_1$ gives

$$\begin{bmatrix}1&2&3\\0&3&6\\0&-1&-2\end{bmatrix}$$

$R_3 = 3R_3 + R_2$ gives

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 = \frac{R_2}{3}$ gives

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 = R_1 - 2R_2$ gives

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The above is the rref form. Hence the system becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

The last row provides no information. The second row gives $c_2 = 2$. First row gives $c_1 = -1$. Since solution is found, then $\vec{v}$ is in W. The vector $\vec{v}$ can be expressed as a linear combination of the basis vectors given.

$$-\vec{v}_1 + 2\vec{v}_2 = \vec{v}$$

2.3.9 Problem 8

Determine whether the given set $\{(1,-1,0), (0,1,-1), (1,1,1)\}$ in $\mathbb{R}^3$ is linearly independent or linearly dependent

Solution

We need to find c_1, c_2, c_3 such that

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If we can find c_1, c_2, c_3 not all zero that solves the above, then the set is linearly dependent. If the only solution is $c_1 = c_2 = c_3 = 0$ then the set is linearly independent. Writing the above in matrix form $Ax = b$ gives

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

Therefore, the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$R_2 = R_2 + R_1$ gives

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$R_3 = R_3 + R_2$ gives

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$R_3 = \frac{R_3}{3}$ gives

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 = R_2 - R_3$ gives

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_1 = R_1 - R_3$ gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The above is rref form. Hence the system (1) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This shows that $c_1 = 0, c_2 = 0, c_3 = 0$. Since the only solution is $c_i = 0$, then the set is linearly independent. Another way we could have solved this is by finding the determinant of $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$. If the determinant is not zero, then $\vec{x} = \vec{0}$ is the only solution and hence the columns are linearly independent. In this example $\det(A)$ can be found to be 3, which confirms the above result.

2.3.10 Problem 9

Use the Wronskian to show that the given functions are linearly independent on the given interval $I = (-\infty, \infty)$

$$f_1(x) = 1 \qquad f_2(x) = 3x \qquad f_3(x) = x^2 - 1$$

Solution

The Wronskian is

$$\begin{aligned} W &= \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} \\ &= \begin{vmatrix} 1 & 2x & x^2-1 \\ 0 & 3 & 2x \\ 0 & 0 & 2 \end{vmatrix} \end{aligned}$$

To find the determinant, it is easiest to expand along the last row as that has the most zeros (Also the first column will do). Therefore the determinant is

$$\begin{aligned} W &= (-1)^{3+3}(2)\begin{vmatrix} 1 & 2x \\ 0 & 3 \end{vmatrix} \\ &= 2(3) \\ &= 6 \end{aligned}$$

Since $W \neq 0$ then the functions are linearly independent.

2.3.11 Problem 10

Determine whether the set of vectors

$$S = \{(1,1,0,2),(2,2,3,-1),(-1,1,1,-2),(2,-1,1,2)\}$$

is a basis for $\mathbb{R}^4$.

Solution

Since there are four vectors given, they can be used as basis for $\mathbb{R}^4$ if they are linearly independent of each others. To find this, we need to find c_1, c_2, c_3, c_4 which solves

$$c_1\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 2 \\ 3 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix} + c_4\begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If we can find c_1, c_2, c_3, c_4 not all zero that solves the above, then the set is linearly dependent and they can not be used as basis for $\mathbb{R}^4$. If the only solution is $c_1 = c_2 = c_3 = c_4 = 0$ then they are basis for $\mathbb{R}^4$. Writing the above in matix form $Ax = b$ gives

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 1 & 2 & 1 & -1 \\ 0 & 3 & 1 & 1 \\ 2 & -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 1 & 2 & 1 & -1 & 0 \\ 0 & 3 & 1 & 1 & 0 \\ 2 & -1 & -2 & 2 & 0 \end{bmatrix}$$

$R_2 = R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 3 & 1 & 1 & 0 \\ 2 & -1 & -2 & 2 & 0 \end{bmatrix}$$

$R_4 = R_4 - 2R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 3 & 1 & 1 & 0 \\ 0 & -3 & 0 & -2 & 0 \end{bmatrix}$$

Swapping R_3, R_2 so the pivot is non-zero

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 0 & 3 & 1 & 1 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & -3 & 0 & -2 & 0 \end{bmatrix}$$

$R_4 = R_4 + R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 0 & 3 & 1 & 1 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

$R_4 = 2R_4$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 0 & 3 & 1 & 1 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 2 & -2 & 0 \end{bmatrix}$$

$R_4 = R_4 - R_3$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 0 & 3 & 1 & 1 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_3 = \frac{R_3}{2}$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 0 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 = \frac{R_2}{3}$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_3 = R_3 + \frac{3}{2}R_4$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 = R_2 - \frac{1}{3}R_4$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 0 & 1 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_1 = R_1 - 2R_4$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 = R_2 - \frac{1}{3}R_3$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_1 = R_1 + R_3$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_1 = R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This is now rref. Hence the original system (1) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Which implies $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0$. Therefore the set S is a basis for $\mathbb{R}^4$. Another way to solve this is to find the determinant of A. If it is not zero, then the set S is basis.

2.3.12 Problem 11

Determine whether the set

$$S = \left\{1 - 3x^2, 2x + 5x^2, 1 - x + 3x^2\right\}$$

is a basis for $p_2(R)$

Solution

Let $p_1(x) = 1 - 3x^2, p_2(x) = 2x + 5x^2, p_3(x) = 1 - x + 3x^2$, hence the Wronskian is

$$\begin{aligned} W &= \begin{vmatrix} p_1 & p_2 & p_3 \\ p_1' & p_2' & p_3' \\ p_1'' & p_2'' & p_3'' \end{vmatrix} \\ &= \begin{vmatrix} 1 - 3x^2 & 2x + 5x^2 & 1 - x + 3x^2 \\ -6x & 2 + 10x & -1 + 6x \\ -6 & 10 & 6 \end{vmatrix} \end{aligned}$$

Expanding along last row gives

$$\begin{aligned} W &= (-1)^{3+1}(-6)\begin{vmatrix} 2x + 5x^2 & 1 - x + 3x^2 \\ 2 + 10x & -1 + 6x \end{vmatrix} + (-1)^{3+2}(10)\begin{vmatrix} 1 - 3x^2 & 1 - x + 3x^2 \\ -6x & -1 + 6x \end{vmatrix} + (-1)^{3+3}(6)\begin{vmatrix} 1 - 3x^2 & 2x + 5x^2 \\ -6x & 2 + 10x \end{vmatrix} \\ &= -6\begin{vmatrix} 2x + 5x^2 & 1 - x + 3x^2 \\ 2 + 10x & -1 + 6x \end{vmatrix} - 10\begin{vmatrix} 1 - 3x^2 & 1 - x + 3x^2 \\ -6x & -1 + 6x \end{vmatrix} + 6\begin{vmatrix} 1 - 3x^2 & 2x + 5x^2 \\ -6x & 2 + 10x \end{vmatrix} \\ &= -6\left(\left(2x + 5x^2\right)(-1 + 6x) - \left(1 - x + 3x^2\right)(2 + 10x)\right) \\ &\quad - 10\left(\left(1 - 3x^2\right)(-1 + 6x) - \left(1 - x + 3x^2\right)(-6x)\right) \\ &\quad + 6\left(\left(1 - 3x^2\right)(2 + 10x) - \left(2x + 5x^2\right)(-6x)\right) \end{aligned}$$

or

$$\begin{aligned} W &= -6\left(\left(30x^3 + 7x^2 - 2x\right) - \left(30x^3 - 4x^2 + 8x + 2\right)\right) \\ &\quad - 10\left(\left(-18x^3 + 3x^2 + 6x - 1\right) - \left(-18x^3 + 6x^2 - 6x\right)\right) \\ &\quad + 6\left(\left(-30x^3 - 6x^2 + 10x + 2\right) - \left(-30x^3 - 12x^2\right)\right) \end{aligned}$$

or

$$\begin{aligned} W &= -6\left(11x^2 - 10x - 2\right) - 10\left(-3x^2 + 12x - 1\right) + 6\left(6x^2 + 10x + 2\right) \\ &= -66x^2 + 60x + 12 + 30x^2 - 120x + 10 + 36x^2 + 60x + 12 \\ &= 34 \end{aligned}$$

Since the Wronskian is not zero, then set S is basis for $p_2(R)$

2.3.13 Problem 12

Find the dimension of the null space of the given matrix A

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 2 & 3 & -2 \\ 1 & 2 & -2 \end{bmatrix}$$

Solution

$R_2 = R_2 - 2R_1$ gives

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 5 & -11 \\ 1 & 2 & -2 \end{bmatrix}$$

$R_3 = R_3 - R_1$ gives

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 5 & -11 \\ 0 & 4 & -62 \end{bmatrix}$$

$R_2 = 4R_2, R_3 = 5R_3$ gives

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 20 & -44 \\ 0 & 20 & -310 \end{bmatrix}$$

$R_3 = R_3 - R_2$ gives

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 20 & -44 \\ 0 & 0 & -266 \end{bmatrix}$$

The above shows there are 3 pivot columns, which means the rank is 3 which is the same as the dimension of the column space. The dimension of A is 3. Using the Rank–nullity theorem (4.9.1, in textbook at page 325) which says, for matrix A of dimensions $m \times n$

$$Rank(A) + nullity(A) = n$$

Therefore, since $n = 3$ in this case (it is the number of columns)

$$3 + nullity(A) = 3$$

Hence

$$\begin{aligned} nullity(A) &= 3 - 3 \\ &= 0 \end{aligned}$$

This means the dimension of the null space of A is zero. The *nullity* (A) is the dimension of null-space(A).

2.3.14 Problem 13

Determine the component vector of the given vector space V relative to the given ordered basis B.

$$V = \mathbb{R}^2 \qquad B = \{(2,-2),(1,4)\} \qquad v = (5,-10)$$

<u>Solution</u>

Let

$$c_1 \begin{bmatrix} 2 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

In $Ax = b$ form the above becomes

$$\begin{bmatrix} 2 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 2 & 1 & 5 \\ -2 & 4 & -10 \end{bmatrix}$$

$R_2 = R_2 + R_1$

$$\begin{bmatrix} 2 & 1 & 5 \\ 0 & 5 & -5 \end{bmatrix}$$

$R_2 = \frac{R_2}{5}, R_1 = \frac{R_1}{2}$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 1 & -1 \end{bmatrix}$$

$R_1 = R_1 - \frac{1}{2}R_2$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

This is rref form. Hence the original system (1) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Which means $c_2 = -1, c_1 = 3$, Therefore the component vector is $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

2.3.15 Problem 14

a) find n such that rowspace(A) is a subspace of $\mathbb{R}^n$ and determine the basis for rowspace(A).

b) find m such that colspace(A) is a subspace of $\mathbb{R}^m$ and determine a basis for colspace(A)

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & 1 & -2 & 6 \\ 3 & 1 & 4 & 2 \end{bmatrix}$$

Solution

2.3.15.1 Part a

$R_2 = R_2 - R_1$ gives

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -4 & 3 \\ 3 & 1 & 4 & 2 \end{bmatrix}$$

$R_3 = R_3 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -4 & 3 \\ 0 & 4 & -2 & -7 \end{bmatrix}$$

$R_3 = R_3 - 2R_2$

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -4 & 3 \\ 0 & 0 & 6 & -13 \end{bmatrix}$$

Pivots are $A(1,1), A(2,2), A(3,3)$.

$R_3 = \frac{R_3}{6}$ gives

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -4 & 3 \\ 0 & 0 & 1 & -\frac{13}{6} \end{bmatrix}$$

$R_2 = \frac{R_2}{2}$

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -2 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{13}{6} \end{bmatrix}$$

$R_2 = R_2 + 2R_3$

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 0 & \frac{3}{2} + 2\left(-\frac{13}{6}\right) \\ 0 & 0 & 1 & -\frac{13}{6} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 0 & -\frac{17}{6} \\ 0 & 0 & 1 & -\frac{13}{6} \end{bmatrix}$$

$R_1 = R_1 - 2R_3$

$$\begin{bmatrix} 1 & -1 & 0 & 3 - 2\left(-\frac{13}{6}\right) \\ 0 & 1 & 0 & -\frac{17}{6} \\ 0 & 0 & 1 & -\frac{13}{6} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & \frac{22}{3} \\ 0 & 1 & 0 & -\frac{17}{6} \\ 0 & 0 & 1 & -\frac{13}{6} \end{bmatrix}$$

$R_1 = R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{22}{3} - \frac{17}{6} \\ 0 & 1 & 0 & -\frac{17}{6} \\ 0 & 0 & 1 & -\frac{13}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \frac{9}{2} \\ 0 & 1 & 0 & -\frac{17}{6} \\ 0 & 0 & 1 & -\frac{13}{6} \end{bmatrix}$$

The above is rref form. Pivot columns are $1,2,3$. The set of nonzero row vectors in the above rref form are basis for rowspace(A). Hence rowspace is

$$\left\{\left(1,0,0,\frac{9}{2}\right),\left(0,1,0,-\frac{17}{6}\right),\left(0,0,1,-\frac{13}{6}\right)\right\}$$

The rowspace is 3 dimensional in $\mathbb{R}^4$.

2.3.15.2 Part b

From part(a) we found that the pivot columns are $1,2,3$. Therefore the column space is given by the corresponding columns in the original vector A. Hence the column space is

$$\left\{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}\right\}$$

It is 3 dimensional in $\mathbb{R}^3$

2.4 HW 4

Local contents

2.4.1 Problems listing

Math2520-01 **Assignment 4**

INSTRUCTION: ***Show all the necessary work***. Write your answer on a separate sheet preferably hand written clear and legible. Post your answer sheet on D2L by Sunday June 27. In this section you may need to remember the theorems and how to apply them to answer the questions.

1. Determine the null space of A and verify the Rank-Nullity Theorem.

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{bmatrix}$$

2. Using the definition of linear transformation, verify that the given transformation is linear.

$$T: R^2 \to R^2 \text{ defined by } T(x, y) = (x + 2y, 2x - y).$$

3. Determine the matrix of the given linear transformation.

$$T: R^3 \to R^2 \text{ defined by } T(x, y, z) = (x - y + z, z - x)$$

4. Let $T: R^2 \to R^2$ be a linear transformation that maps $u = (5, 2)$ into $(2,1)$ and $v = (1, 3)$ into $(-1, 3)$. Use the fact that T is linear to find the image under T of $3u + 2v$.

5. Assume that T defines a linear transformation and use the given information to find the matrix of T.

   $T: R^2 \to R^4$ such that $T(0,1) = (1, 0, -2, 2)$ and $T(1, 2) = (-3,1,1,1)$.

6. Find the Ker(T) and Rng(T) and their dimensions.

   $T: R^3 \to R^2$ defined by $T(x) = Ax$, where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 3 & -6 \end{bmatrix}.$$

7. Let $T: R^3 \to R^3$ be a linear transformation defined by $Tx = Ax$ where

1

$$A = \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix}.$$

Show that T is both one-to-one and onto.

8. Determine all eigenvalues and corresponding eigenvectors of the given matrix.

i) $A = \begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix}$.

ii) $A = \begin{bmatrix} 7 & 4 \\ -1 & 3 \end{bmatrix}$.

iii) $A = \begin{bmatrix} 7 & 3 \\ -6 & 1 \end{bmatrix}$

9. If $v_1 = (1,-1)$ and $v_2 = (2,1)$ be eigenvectors of the matrix A corresponding to the eigenvalues $\lambda_1 = 2, \lambda_2 = -3$, respectively find $A(3v_1 - v_2)$.

10. Determine the multiplicity of each eigenvalue and a basis for each eigenspace of the given matrix A. Determine the dimension of each eigenspce and state whether the matrix is defective or nondefective.

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}.$$

11. Determine whether the given matrix A is diagonalizable.

$$A = \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix}$$

12. Determine the general solution to the given differential equation.

a) $y'' - y' - 2y = 0$

b) $y'' + 10y' + 25y = 0$

c) $y'' + 6y' + 11y = 0$

2.4.2 Problem 1

Determine the null space of A and verify the Rank-Nullity Theorem

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{bmatrix}$$

Solution

The null space of A is the solution $A\vec{x} = \vec{0}$. Therefore

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 4 & 0 \\ 3 & 8 & 7 & 20 & 0 \\ 2 & 7 & 9 & 23 & 0 \end{bmatrix}$$

$R_2 = R_2 - 3R_1$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 4 & 0 \\ 0 & 2 & 4 & 8 & 0 \\ 2 & 7 & 9 & 23 & 0 \end{bmatrix}$$

$R_3 = R_3 - 2R_1$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 4 & 0 \\ 0 & 2 & 4 & 8 & 0 \\ 0 & 3 & 7 & 15 & 0 \end{bmatrix}$$

$R_2 = \frac{R_2}{2}$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 4 & 0 \\ 0 & 1 & 2 & 4 & 0 \\ 0 & 3 & 7 & 15 & 0 \end{bmatrix}$$

$R_3 = R_3 - 3R_2$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 4 & 0 \\ 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix}$$

Now the reduced echelon phase starts.

$R_2 = R_2 - 2R_3$

$$\begin{bmatrix} 1 & 2 & 1 & 4 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix}$$

$R_1 = R_1 - R_3$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix}$$

$R_1 = R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix}$$

The above in RREF form. There are 3 pivots. They are $A(1,1), A(2,2), A(3,3)$. Hence original system (1) becomes

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The base variables are x_1, x_2, x_3 and the free variable is $x_4 = s$. Last row gives $x_3 + 3x_4 = 0$ or $x_3 = -3s$. Second row gives $x_2 - 2x_4 = 0$ or $x_2 = 2s$. First row gives $x_1 + 5x_4 = 0$ or $x_1 = -5s$. Hence the solution is

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} -5s \\ 2s \\ -3s \\ s \end{bmatrix} \\ &= s \begin{bmatrix} -5 \\ 2 \\ -3 \\ 1 \end{bmatrix} \end{aligned}$$

It is one parameter solution. Hence the <u>dimension of the null space</u> is 1. (it is subspace of $\mathbb{R}^n$ or $\mathbb{R}^4$ in this case). Any scalar multiple of $\begin{bmatrix} -5 \\ 2 \\ -3 \\ 1 \end{bmatrix}$ is basis for the null space. For verification, using the Rank–nullity theorem (4.9.1, in textbook at page 325) which says, for matrix A of dimensions $m \times n$

$$Rank(A) + nullity(A) = n$$

Therefore, since $n = 4$ in this case (it is the number of columns), and rank is 3 (since there are 3 pivots) then

$$3 + nullity(A) = 4$$

Hence

$$\begin{aligned} nullity(A) &= 4 - 3 \\ &= 1 \end{aligned}$$

This means the dimension of the null space of A is 1. The $nullity(A)$ is the dimension of null-space(A), which is also the <u>number of free</u> variables at the end of the RREF phase. This verifies the result found above.

2.4.3 Problem 2

Using the definition of linear transformation, verify that the given transformation is linear. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + 2y, 2x - y)$

Solution

The mapping is linear if it satisfies the following two properties

$$T\left(\vec{u}+\vec{v}\right) = T\left(\vec{u}\right) + T\left(\vec{v}\right) \qquad \text{for all } \vec{u},\vec{v} \in V$$
$$T\left(c\vec{u}\right) = cT\left(\vec{u}\right) \qquad \text{for all } \vec{u} \in V \text{ and all scalars } c$$

T above is the linear mapping that assigns each vector $\vec{v} \in V$ one vector $w \in W$, where V, W are vector spaces. V is called the domain of T and W is called the codomain of T. The range of T is the subset of vectors in W which can be reached by the mapping T applied to all vectors in V. i.e. $Rng(T) = \left\{T\left(\vec{v}\right) : \vec{v} \in V\right\}$. To find if T is linear, we need to check both properties above. Let $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \vec{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Then (Please note that = below is used as a place holder since we do not know yet if LHS is equal to RHS. It should really be $\stackrel{?}{=}$ but this gives a Latex issue when used)

$$\begin{aligned}
T\left(\vec{u}+\vec{v}\right) &= T\left(\vec{u}\right) + T\left(\vec{v}\right) \\
T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) &= T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) \\
T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) &= \begin{bmatrix} x_1 + 2y_1 \\ 2x_1 - y_1 \end{bmatrix} + \begin{bmatrix} x_2 + 2y_2 \\ 2x_2 - y_2 \end{bmatrix} \\
\begin{bmatrix} (x_1 + x_2) + 2\left(y_1 + y_2\right) \\ 2\left(x_1 + x_2\right) - \left(y_1 + y_2\right) \end{bmatrix} &= \begin{bmatrix} x_1 + 2y_1 + x_2 + 2y_2 \\ 2x_1 - y_1 + 2x_2 - y_2 \end{bmatrix} \\
\begin{bmatrix} x_1 + x_2 + 2y_1 + 2y_2 \\ 2x_1 + 2x_2 - y_1 - y_2 \end{bmatrix} &= \begin{bmatrix} x_1 + x_2 + 2y_1 + 2y_2 \\ 2x_1 + 2x_2 - y_1 - y_2 \end{bmatrix}
\end{aligned}$$

<u>Comparing both sides</u> shows they are indeed the same. Hence the first property is satisfied. Now the second property is checked. Let c be scalar and let $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ then

$$\begin{aligned}
T\left(c\vec{u}\right) &= cT\left(\vec{u}\right) \\
T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) &= cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \\
T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) &= c\begin{bmatrix} x + 2y \\ 2x - y \end{bmatrix} \\
\begin{bmatrix} cx + 2cy \\ 2cx - cy \end{bmatrix} &= \begin{bmatrix} cx + 2cy \\ 2cx - cy \end{bmatrix}
\end{aligned}$$

<u>Comparing both sides</u> shows they are the same. Hence the second property is satisfied. This verifies that the given transformation T is linear

2.4.4 Problem 3

Determine the matrix of the given linear transformation

$$T : \mathbb{R}^3 \to \mathbb{R}^2 \qquad \text{defined by } T\left(x,y,z\right) = \left(x - y + z, z - x\right)$$

Solution

Let the matrix of the transformation be $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be some vector in the domain of T, then we need to solve

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - y + z \\ z - x \end{bmatrix}$$

For the unknowns $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$. The first row equation is

$$a_{11}x + a_{12}y + a_{13} = x - y + z \tag{1}$$

Comparing coefficients for each of the variables x, y, x gives $a_{11} = 1, a_{12} = -1, a_{13} = 1$. The second row equation is

$$a_{21}x + a_{22}y + a_{23}z = z - x \tag{2}$$

Comparing coefficients again gives $a_{21} = -1, a_{22} = 0, a_{23} = 1$. Hence the matrix A is

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

2.4.5 Problem 4

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\vec{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Use the fact that T is linear to find the image under T of $3\vec{u} + 2\vec{v}$

Solution

The mapping is linear if it satisfies the following two properties

$$T\left(\vec{u} + \vec{v}\right) = T\left(\vec{u}\right) + T\left(\vec{v}\right) \qquad \text{for all } \vec{u}, \vec{v} \in V$$
$$T\left(c\vec{u}\right) = cT\left(\vec{u}\right) \qquad \text{for all } \vec{u} \in V \text{ and all scalars } c$$

By using first property above we can then write

$$T\left(3\vec{u} + 2\vec{v}\right) = T\left(3\vec{u}\right) + T\left(2\vec{v}\right)$$

And by using the second property the RHS above can be written as

$$T\left(3\vec{u} + 2\vec{v}\right) = 3T\left(\vec{u}\right) + 2T\left(\vec{v}\right)$$

But we are given that $T\left(\vec{u}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, T\left(\vec{v}\right) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Substituting these in the above gives

$$\begin{aligned} T\left(3\vec{u} + 2\vec{v}\right) &= 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 6-2 \\ 6+6 \end{bmatrix} \end{aligned}$$

Hence the image under T of $3\vec{u} + 2\vec{v}$ is

$$T\left(3\vec{u} + 2\vec{v}\right) = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$$

2.4.6 Problem 5

Assume that T defines a linear transformation and use the given information to find the matrix of T.

$$T : \mathbb{R}^2 \to \mathbb{R}^4$$

Such that $T(0,1) = (1,0,-2,2)$ and $T(1,2) = (-3,1,1,1)$

Solution

Let A be the representation of the linear transformation and let $\vec{x}$ vector in the domain of T. Hence

$$A\vec{x} = \vec{b}$$

Where $b \in \mathbb{R}^4$, hence it has dimensions 4×1 and since $\vec{x} \in \mathbb{R}^2$ then it has dimensions 2×1. Therefore

$$(m \times n)(2 \times 1) = (4 \times 1)$$

Since inner dimensions between A and $\vec{x}$ must be the same for the multiplication to be valid, then $n = 2$. Therefore $m = 4$. Hence A must have dimensions 4×2. Let A be

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix}$$

Using $T(0,1) = (1,0,-2,2)$, then we can write

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 2 \end{bmatrix}$$

or by carrying out the multiplication

$$\begin{bmatrix} a_{11}(0) + a_{12}(1) \\ a_{21}(0) + a_{22}(1) \\ a_{31}(0) + a_{32}(1) \\ a_{41}(0) + a_{42}(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 2 \end{bmatrix} \tag{1}$$

And using the second relation $T(1,2) = (-3,1,1,1)$ gives

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11}(1) + a_{12}(2) \\ a_{21}(1) + a_{22}(2) \\ a_{31}(1) + a_{32}(2) \\ a_{41}(1) + a_{42}(2) \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} + 2a_{12} \\ a_{21} + 2a_{22} \\ a_{31} + 2a_{32} \\ a_{41} + 2a_{42} \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Substituting values found in (1) into the above gives

$$\begin{bmatrix} a_{11} + 2(1) \\ a_{21} + 2(0) \\ a_{31} + 2(-2) \\ a_{41} + 2(2) \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} + 2 \\ a_{21} \\ a_{31} - 4 \\ a_{41} + 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} = \begin{bmatrix} -3 - 2 \\ 1 \\ 1 + 4 \\ 1 - 4 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 1 \\ 5 \\ -3 \end{bmatrix}$$

All entries of A are now found. Therefore the matrix representation of T is

$$A = \begin{bmatrix} -5 & 1 \\ 1 & 0 \\ 5 & -2 \\ -3 & 2 \end{bmatrix}$$

2.4.7 Problem 6

Find the ker(T) and $Rng(T)$ and their dimensions. $T:\mathbb{R}^3\rightarrow\mathbb{R}^2$ defined by $T(x)=Ax$ where

$$A=\begin{bmatrix}1 & -1 & 2\\ -3 & 3 & -6\end{bmatrix}$$

Solution

$Rng(T)$ are all vectors in $\mathbb{R}^2$ (subspace of $\mathbb{R}^m$) which can be reached by T for every vector in domain of T which is $\mathbb{R}^3$. It is the same as the column space of A.

$Ker(T)$ are all vectors in $\mathbb{R}^3$ which map to the zero vector in $\mathbb{R}^2$. They are the solution of $A\vec{x}=\vec{0}$. $Ker(T)$ is the same as null-space of A where A is the matrix representation of the linear mapping T. To find $Ker(T)$, we then need to solve the system $A\vec{x}=\vec{0}$.

$$\begin{bmatrix}1 & -1 & 2\\ -3 & 3 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\tag{1}$$

The augmented matrix is

$$\begin{bmatrix}1 & -1 & 2 & 0\\ -3 & 3 & -6 & 0\end{bmatrix}$$

$R_2=R_2+3R_1$ gives

$$\begin{bmatrix}1 & -1 & 2 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Base variable is x_1. Free variables are $x_2=s,x_3=t$. Pivot column is the first column. Hence (1) becomes

$$\begin{bmatrix}1 & -1 & 2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

First row gives $x_1-s+2t=0$ or $x_1=s-2t$. Hence the solution is

$$\begin{aligned}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}&=\begin{bmatrix}s-2t\\ s\\ t\end{bmatrix}\\ &=\begin{bmatrix}s\\ s\\ 0\end{bmatrix}+\begin{bmatrix}-2t\\ 0\\ t\end{bmatrix}\\ &=s\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}+t\begin{bmatrix}-2\\ 0\\ 1\end{bmatrix}\end{aligned}$$

It is two parameters system. The dimension of the null-space is therefore 2. (it is also the number of the free variables). The null-space is subspace of $\mathbb{R}^3$. Hence

$$\ker(T)=\left\{\vec{v}\in\mathbb{R}^3:\vec{v}=s\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}+t\begin{bmatrix}-2\\ 0\\ 1\end{bmatrix},s,t\in\mathbb{R}\right\}$$

Now $Rng(T)$ is the column space. From above we found that the first column was the pivot column. This corresponds to the first column in A given by $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Therefore

$$Rng(T) = \left\{ \vec{v} \in \mathbb{R}^2 : \vec{v} = s \begin{bmatrix} 1 \\ -3 \end{bmatrix}, s \in \mathbb{R} \right\}$$

It is one dimension subspace of $\mathbb{R}^2$.

2.4.8 Problem 7

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be linear transformation defined by $Tx = Ax$ where

$$A = \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix}$$

Show that T is both one-to-one and onto.

Solution

Using Theorem 6.4.8 which says, the linear transformation $T : V \to W$ is

1. one-to-one iff $\ker(T) = \left\{\vec{0}\right\}$
2. onto iff $Rng(T) = W$

To show one-to-one, we need to find $\ker(T)$ by solving the system $A\vec{x} = \vec{0}$.

$$\begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

Augmented matrix is

$$\begin{bmatrix} 3 & 5 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 6 & 7 & 0 \end{bmatrix}$$

Swapping R_2, R_1 gives (it is simpler to have the pivot be 1 to avoid fractions)

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 5 & 1 & 0 \\ 2 & 6 & 7 & 0 \end{bmatrix}$$

$R_2 = R_2 - 3R_1$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 2 & 6 & 7 & 0 \end{bmatrix}$$

$R_3 = R_3 - 2R_1$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & 5 & 0 \end{bmatrix}$$

$R_3 = R_3 + 2R_2$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 = -R_2$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 = R_2 - 2R_3$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_1 = R_1 - R_3$ gives

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_1 = R_1 - 2R_2$ gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

There are no free variables. Number of pivots is 3. The system (1) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which shows that the solution is $x_1 = 0, x_2 = x_3 = 0$. Hence $\ker(T) = \left\{\vec{0}\right\}$. Since number of free variables is zero, then we see that the dimension of the null space is zero. Therefore T is <u>one-to-one</u>.

Now we need to show if it is onto. The matrix A is 3×3. Therefore the mapping is $\mathbb{R}^3 \to \mathbb{R}^3$. Hence W is $\mathbb{R}^3$. But $Rng(T)$ is the column space of A. From above, we find that there are 3 pivots. So the 3 columns of A are pivots columns. Hence

$$Rng(T) = \left\{ \vec{v} \in \mathbb{R}^3 : \vec{v} = c_1 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}, c_1, c_2, c_3 \in \mathbb{R} \right\}$$

Which is all of W, since there are 3 independent basis vectors which span all of $\mathbb{R}^3$ and W is $\mathbb{R}^3$. Hence <u>onto</u>.

2.4.9 Problem 8

Determine all eigenvalues and corresponding eigenvectors of the given matrix 1) $\begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix}$, 2) $\begin{bmatrix} 7 & 4 \\ -1 & 3 \end{bmatrix}$, 3) $\begin{bmatrix} 7 & 3 \\ -6 & 1 \end{bmatrix}$

<u>Solution</u>

2.4.9.1 Part 1

$$A=\begin{bmatrix}5 & -4\\ 8 & -7\end{bmatrix}$$

The eigenvalues are found by solving

$$\begin{aligned}
|A-\lambda I| &= 0\\
\det\left(\begin{bmatrix}5 & -4\\ 8 & -7\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right) &= 0\\
\begin{vmatrix}5-\lambda & -4\\ 8 & -7-\lambda\end{vmatrix} &= 0\\
(5-\lambda)(-7-\lambda)-(-4)(8) &= 0\\
(5-\lambda)(-7-\lambda)+32 &= 0\\
\lambda^2+2\lambda-35+32 &= 0\\
\lambda^2+2\lambda-3 &= 0\\
(\lambda-1)(\lambda+3) &= 0
\end{aligned}$$

Hence the eigenvalues are $\lambda_1 = 1, \lambda_2 = -3$. For each eigenvalues, we now find the corresponding eigenvector.

$\underline{\lambda_1 = 1}$

We need to solve $A\vec{v} = \lambda_1\vec{v}$ for vector $\vec{v}$. This gives

$$\begin{aligned}
\begin{bmatrix}5 & -4\\ 8 & -7\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \lambda_1\begin{bmatrix}v_1\\ v_2\end{bmatrix}\\
\begin{bmatrix}5 & -4\\ 8 & -7\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}-\lambda_1\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\\
\begin{bmatrix}5-\lambda_1 & -4\\ 8 & -7-\lambda_1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}
\end{aligned}$$

But $\lambda_1 = 1$. The above becomes

$$\begin{bmatrix}4 & -4\\ 8 & -8\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$R_2 = R_2 - 2R_1$ gives $\begin{bmatrix}4 & -4\\ 0 & 0\end{bmatrix}$. Hence v_1 is the base variable and $v_2 = t$ is the free variable. Therefore the system becomes

$$\begin{bmatrix}4 & -4\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Using first row gives

$$\begin{aligned}
4v_1 - 4v_2 &= 0\\
v_1 &= v_2\\
&= t
\end{aligned}$$

Then the eigenvector is

$$\vec{v}_{\lambda_1=1} = \begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}t\\ t\end{bmatrix} = t\begin{bmatrix}1\\ 1\end{bmatrix}$$

Choosing $t = 1$. (any arbitrary value will work), then the eigenvector is

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda_2 = -3}$

We need to solve $A\vec{v} = \lambda_2 \vec{v}$ for vector $\vec{v}$. This gives (as was done above)

$$\begin{bmatrix} 5-\lambda_2 & -4 \\ 8 & -7-\lambda_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 8 & -4 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - R_1$ gives $\begin{bmatrix} 8 & -4 \\ 0 & 0 \end{bmatrix}$. Hence v_1 is the base variable and $v_2 = t$ is the free variable. Therefore the system becomes

$$\begin{bmatrix} 8 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using first row gives

$$8v_1 - 4v_2 = 0$$
$$v_1 = \frac{1}{2}v_2 = \frac{1}{2}t$$

Therefore the eigenvector is

$$\vec{v}_{\lambda_2=3} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Choosing $t = 1$. (any arbitrary value will work), then the eigenvector is

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Summary table

eigenvalue	Algebraic multiplicity	Geometric multiplicity	defective?	eigenvector
$\lambda_1 = 1$	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\lambda_2 = -3$	1	1	No	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

2.4.9.2 Part 2

$$A = \begin{bmatrix} 7 & 4 \\ -1 & 3 \end{bmatrix}$$

The eigenvalues are found by solving

$$
\begin{aligned}
|A-\lambda I| &= 0\\
\det\left(\begin{bmatrix}7 & 4\\ -1 & 3\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right) &= 0\\
\begin{vmatrix}7-\lambda & 4\\ -1 & 3-\lambda\end{vmatrix} &= 0\\
(7-\lambda)(3-\lambda)+4 &= 0\\
\lambda^2-10\lambda+21+4 &= 0\\
\lambda^2-10\lambda+25 &= 0\\
(\lambda-5)(\lambda-5) &= 0
\end{aligned}
$$

Hence the roots is $\lambda = 5$ which is a repeated root. (its algebraic multiplicity is 2)

$\lambda = 5$

We need to solve $A\vec{v} = \lambda_1\vec{v}$ for vector $\vec{v}$. This gives

$$
\begin{aligned}
\begin{bmatrix}7-\lambda & 4\\ -1 & 3-\lambda\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\\
\begin{bmatrix}7-5 & 4\\ -1 & 3-5\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\\
\begin{bmatrix}2 & 4\\ -1 & -2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}
\end{aligned}
$$

$R_2 = R_2 + \frac{1}{2}R_1$ gives $\begin{bmatrix}2 & 4\\ 0 & 0\end{bmatrix}$. Hence v_1 is base variable and $v_2 = t$ is free variable. Therefore the system becomes

$$
\begin{bmatrix}2 & 4\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}
$$

The first row gives

$$
\begin{aligned}
2v_1 + 4v_2 &= 0\\
2v_1 &= -4v_2\\
v_1 &= -2v_2\\
&= -2t
\end{aligned}
$$

Therefore the first eigenvector is

$$
\vec{v} = \begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}-2t\\ t\end{bmatrix} = t\begin{bmatrix}-2\\ 1\end{bmatrix}
$$

Choosing $t = 1$. (any arbitrary value will work), then the eigenvector is

$$
\vec{v} = \begin{bmatrix}-2\\ 1\end{bmatrix}
$$

Since we are able to obtain only one eigenvector from $\lambda = 5$, then this is a defective eigenvalue. It has an algebraic multiplicity of 2 but its geometric multiplicity is only 1. When the geometric multiplicity is less than the algebraic multiplicity then the eigenvalue is defective.
Summary table

eigenvalue	Algebraic multiplicity	Geometric multiplicity	defective?	eigenvector
$\lambda = 5$	2	1	yes	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

The matrix is defective and hence not diagonalizable.

2.4.9.3 Part 3

$$A = \begin{bmatrix} 7 & 3 \\ -6 & 1 \end{bmatrix}$$

The eigenvalues are found by solving

$$\begin{aligned}
|A - \lambda I| &= 0 \\
\det\left(\begin{bmatrix} 7 & 3 \\ -6 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) &= 0 \\
\begin{vmatrix} 7-\lambda & 3 \\ -6 & 1-\lambda \end{vmatrix} &= 0 \\
(7-\lambda)(1-\lambda) + 18 &= 0 \\
\lambda^2 - 8\lambda + 7 + 18 &= 0 \\
\lambda^2 - 8\lambda + 25 &= 0
\end{aligned}$$

Using quadratic formula $\lambda = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$ gives

$$\begin{aligned}
\lambda &= \frac{8}{2} \pm \frac{1}{2}\sqrt{64 - 4(25)} \\
&= 4 \pm \frac{1}{2}\sqrt{64 - 100} \\
&= 4 \pm \frac{1}{2}\sqrt{-36} \\
&= 4 \pm \frac{6}{2}i \\
&= 4 \pm 3i
\end{aligned}$$

Hence the eigenvalues are complex conjugates of each other. They are $\lambda_1 = 4 + 3i, \lambda_2 = 4 - 3i$ .For each eigenvalues, we now find the corresponding eigenvector.

$\lambda_1 = 4 + 3i$

We need to solve $A\vec{v} = \lambda_1\vec{v}$ for vector $\vec{v}$. This gives

$$\begin{bmatrix} 7-\lambda_1 & 3 \\ -6 & 1-\lambda_1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

But $\lambda_1 = 4 + 3i$. The above becomes

$$\begin{aligned}
\begin{bmatrix} 7-(4+3i) & 3 \\ -6 & 1-(4+3i) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 3-3i & 3 \\ -6 & -3-3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

$R_1 = R_1\left(\frac{1}{6}+\frac{1}{6}i\right)$ gives $\begin{bmatrix}(3-3i)\left(\frac{1}{6}+\frac{1}{6}i\right) & 3\left(\frac{1}{6}+\frac{1}{6}i\right)\\ -6 & -3-3i\end{bmatrix} = \begin{bmatrix}1 & \frac{1}{2}+\frac{1}{2}i\\ -6 & -3-3i\end{bmatrix}$ and now $R_2 = R_2 + 6R_1$ gives

$$\begin{bmatrix}1 & \frac{1}{2}+\frac{1}{2}i\\ 0 & (-3-3i)+6\left(\frac{1}{2}+\frac{1}{2}i\right)\end{bmatrix} = \begin{bmatrix}1 & \frac{1}{2}+\frac{1}{2}i\\ 0 & 0\end{bmatrix}$$

Hence the system using RREF becomes

$$\begin{bmatrix}1 & \frac{1}{2}+\frac{1}{2}i\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

v_1 is the base variable and $v_2 = t$ is the free variable. First row gives

$$v_1 + \left(\frac{1}{2}+\frac{1}{2}i\right)v_2 = 0$$

$$\begin{aligned} v_1 &= \left(-\frac{1}{2}-\frac{1}{2}i\right)v_2\\ &= \left(-\frac{1}{2}-\frac{1}{2}i\right)t \end{aligned}$$

Therefore the eigenvector is

$$\vec{v}_{\lambda_1} = \begin{bmatrix}v_1\\ v_2\end{bmatrix} = t\begin{bmatrix}\frac{-1}{2}-\frac{1}{2}i\\ 1\end{bmatrix} = t\begin{bmatrix}-1-i\\ 2\end{bmatrix}$$

Choosing $t = 1$. (any arbitrary value will work), then the eigenvector is

$$\vec{v}_{\lambda_1} = \begin{bmatrix}-1-i\\ 2\end{bmatrix}$$

$\lambda_2 = 4 - 3i$

We need to solve $A\vec{v} = \lambda_2\vec{v}$ for vector $\vec{v}$. We could follow the same steps above to find the second eigenvector, but since the eigenvectors are complex, then they must come as complex conjugate pairs. Hence $\vec{v}_{\lambda_2}$ can directly be found using

$$\begin{aligned} \vec{v}_{\lambda_2} &= \left(\vec{v}_{\lambda_2}\right)^*\\ &= \begin{bmatrix}-1+i\\ 2\end{bmatrix} \end{aligned}$$

Summary table

eigenvalue	Algebraic multiplicity	Geometric multiplicity	defective?	eigenvector
$\lambda_1 = 4+3i$	1	1	No	$\begin{bmatrix}-1-i\\ 2\end{bmatrix}$
$\lambda_2 = 4-3i$	1	1	No	$\begin{bmatrix}-1+i\\ 2\end{bmatrix}$

2.4.10 Problem 9

If $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ eigenvectors of the matrix A corresponding to the eigenvalues $\lambda_1 = 2, \lambda_2 = -3$ respectively. Find $A(3v_1 - v_2)$

Solution

By definition

$$Av = \lambda v$$

Where λ is the eigenvalue and v is the corresponding eigenvector. Therefore by linearity of operator A

$$\begin{aligned} A(3v_1 - v_2) &= A(3v_1) - Av_2 \\ &= 3Av_1 - Av_2 \\ &= 3(\lambda_1 v_1) - (\lambda_2 v_2) \\ &= 3\left(2\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) - \left(-3\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) \\ &= 3\left(\begin{bmatrix} 2 \\ -2 \end{bmatrix}\right) + \begin{bmatrix} 6 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ -6 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 6+6 \\ -6+3 \end{bmatrix} \\ &= \begin{bmatrix} 12 \\ -3 \end{bmatrix} \end{aligned}$$

2.4.11 Problem 10

Determine the multiplicity of each eigenvalue and a basis for each eigenspace of the given matrix A. Determine the dimension of each eigenspace and state whether the matrix is defective or nondefective.

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Solution

The eigenvalues are found by solving

$$\begin{aligned} |A - \lambda I| &= 0 \\ \det\left(\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) &= 0 \\ \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} &= 0 \\ (1-\lambda)(3-\lambda) - 8 &= 0 \\ \lambda^2 - 4\lambda + 3 - 8 &= 0 \\ \lambda^2 - 4\lambda - 5 &= 0 \\ (\lambda - 5)(\lambda + 1) &= 0 \end{aligned}$$

Hence the eigenvalues are $\lambda_1 = 5$ with multiplicity 1, and $\lambda_2 = -1$ with multiplicity 1. For each eigenvalues, we now find the corresponding eigenvector.

$\underline{\lambda_1 = 5}$

We need to solve $A\vec{v} = \lambda_1\vec{v}$ for vector $\vec{v}$. This gives

$$\begin{bmatrix} 1-\lambda_1 & 4 \\ 2 & 3-\lambda_1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

But $\lambda_1 = 5$. The above becomes

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 + \frac{1}{2}R_1$ gives $\begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix}$. Hence v_1 is base variable and $v_2 = t$ is free variable. Therefore the system becomes

$$\begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using first row gives

$$\begin{aligned} -4v_1 + 4v_2 &= 0 \\ v_1 &= v_2 \\ &= t \end{aligned}$$

$$\vec{v}_{\lambda_1} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

By choosing $t = 1$

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda_2 = -1}$

We need to solve $A\vec{v} = \lambda_2\vec{v}$ for vector $\vec{v}$. This gives (as was done above)

$$\begin{bmatrix} 1-\lambda_1 & 4 \\ 2 & 3-\lambda_1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

But $\lambda_2 = -1$. The above becomes

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - R_1$ gives $\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}$. Hence v_1 is base variable and $v_2 = t$ is free variable. Therefore the system becomes

$$\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

First row gives

$$\begin{aligned} 2v_1 + 4v_2 &= 0 \\ v_1 &= -2v_2 \\ &= -2t \end{aligned}$$

Choosing $t = 1$ the eigenvector is

$$\vec{v}_{\lambda_2=3} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Summary table

eigenvalue	eigenvector
$\lambda_1 = 5$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\lambda_2 = -1$	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

The matrix is not defective because we are able to find two unique eigenvalues for a 2×2 matrix. The dimension of eigenspace corresponding to each eigenvalue is given by the dimension of the null space of $A - \lambda I$ where λ is the eigenvalue and I is the identity matrix. For $\lambda_1 = 5$, since there was one free variable, then the dimension of this eigenspace is one.

Similarly for $\lambda_2 = -1$ since there was one free variable, then the dimension of this eigenspace is one.

2.4.12 Problem 11

Determine whether the given matrix A is diagonalizable

$$A = \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix}$$

Solution

A matrix is diagonalizable if it is not defective. The eigenvalues are found by solving

$$|A - \lambda I| = 0$$

$$\det\left(\begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} -1-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(2-\lambda) - 4 = 0$$

$$\lambda^2 - \lambda - 2 - 4 = 0$$

$$\lambda^2 - \lambda - 6 = 0$$

$$(\lambda - 3)(\lambda + 2) = 0$$

Hence the eigenvalues are $\lambda_1 = 3, \lambda_2 = -2$. For each eigenvalues, we now find the corresponding eigenvector.

$\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda_1 \vec{v}$ for vector $\vec{v}$. This gives

$$\begin{bmatrix} -1-\lambda_1 & -2 \\ -2 & 2-\lambda_1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

But $\lambda_1 = 3$. The above becomes

$$\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - \frac{1}{2}R_1$ gives $\begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix}$. Hence v_1 is base variable and $v_2 = t$ is free variable. Therefore the system becomes

$$\begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

First row gives

$$\begin{aligned} -4v_1 - 2v_2 &= 0 \\ v_2 &= -\frac{1}{2}v_2 \\ &= -\frac{1}{2}t \end{aligned}$$

Therefore the eigenvector is

$$\vec{v}_{\lambda_1} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Choosing $t = 1$ then

$$\vec{v}_{\lambda_1} == \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$\underline{\lambda_2 = -2}$

We need to solve $A\vec{v} = \lambda_2 \vec{v}$ for vector $\vec{v}$. This gives

$$\begin{bmatrix} -1 - \lambda_2 & -2 \\ -2 & 2 - \lambda_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

But $\lambda_2 = -2$. The above becomes

$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 + 2R_1$ gives $\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$. Hence v_1 is base variable and $v_2 = t$ is free variable. Therefore the system becomes

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

First row gives

$$\begin{aligned} v_1 - 2v_2 &= 0 \\ v_1 &= 2v_2 \\ &= 2t \end{aligned}$$

Therefore the eigenvector is

$$\vec{v}_{\lambda_2} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Choosing $t = 1$ gives

$$\vec{v}_{\lambda_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Summary table

eigenvalue	eigenvector
$\lambda_1 = 3$	$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$
$\lambda_2 = -2$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Since the matrix is not defective (because it has two unique eigenvalues), then it is diagonalizable. To show this, let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where $D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ and $P = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$. Hence

$$\begin{aligned} A &= \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (-1)(3) & 2(-2) \\ 2(3) & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} -3 & -4 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} \end{aligned}$$

But $\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{(-1)-(4)} \begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix} = \frac{1}{-5} \begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix}$. Hence the above becomes

$$\begin{aligned} A &= \frac{1}{-5} \begin{bmatrix} -3 & -4 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix} \\ &= \frac{1}{-5} \begin{bmatrix} (-3)(1) + (-4)(-2) & (-3)(-2) + (-4)(-1) \\ (6)(1) + (-2)(-2) & (6)(-2) + (-2)(-1) \end{bmatrix} \\ &= \frac{1}{-5} \begin{bmatrix} 5 & 10 \\ 10 & -10 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} \end{aligned}$$

Verified.

2.4.13 Problem 12

Determine the general solution to the given differential equations a) $y'' - y' - 2y = 0$. b) $y'' + 10y' + 25y = 0$. c) $y'' + 6y' + 11y = 0$

Solution

2.4.13.1 Part a

This is a constant coefficients second order linear ODE. Hence it is solved using the characteristic polynomial method. Assuming solution is $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 2e^{\lambda x} = 0$$

Since $e^{\lambda x} \neq 0$, the above simplifies to

$$\begin{aligned} \lambda^2 - \lambda - 2 &= 0 \\ (\lambda + 1)(\lambda - 2) &= 0 \end{aligned}$$

The roots are $\lambda_1 = -1, \lambda_2 = 2$. Therefore there are two basis solutions, they are $y_1 = e^{\lambda_1 x} = e^{-x}$ and $y_2 = e^{\lambda_2 x} = e^{2x}$. The general solution is a linear combination of these basis solutions. The general solution is

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 e^{-x} + c_2 e^{2x} \end{aligned}$$

Where c_1, c_2 are the constants of integration.

2.4.13.2 Part b

This is a constant coefficients second order linear ODE. Hence it is solved using the characteristic polynomial method. Assuming solution is $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 10\lambda e^{\lambda x} + 25 e^{\lambda x} = 0$$

Since $e^{\lambda x} \neq 0$, then the above simplifies to

$$\begin{aligned} \lambda^2 + 10\lambda + 25 &= 0 \\ (\lambda + 5)(\lambda + 5) &= 0 \end{aligned}$$

Hence the roots are $\lambda = -5$, which is double root. Since the root is double, then the first basis solution is $y_1 = e^{-5x}$ and the second is x times the first, which gives $y_2 = xe^{-5x}$.

The general solution is a linear combination of these basis solutions

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 e^{-5x} + c_2 x e^{-5x} \end{aligned}$$

2.4.13.3 Part c

This is a constant coefficients second order linear ODE. Hence it is solved using the characteristic polynomial method. Assuming solution is $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda e^{\lambda x} + 11 e^{\lambda x} = 0$$

Since $e^{\lambda x} \neq 0$, then the above simplifies to

$$\lambda^2 + 6\lambda + 11 = 0$$

Using quadratic formula $\lambda = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$ gives

$$\begin{aligned} \lambda &= \frac{-6}{2} \pm \frac{1}{2}\sqrt{36 - 4(11)} \\ &= -3 \pm \frac{1}{2}\sqrt{36 - 44} \\ &= -3 \pm \frac{1}{2}\sqrt{-8} \\ &= -3 \pm \sqrt{-2} \\ &= -3 \pm i\sqrt{2} \end{aligned}$$

Hence roots are $\lambda_1 = -3 + i\sqrt{2}, \lambda_2 = -3 - i\sqrt{2}$. Hence there are two basis solutions, they are

$$\begin{aligned} y_1 &= e^{\lambda_1 x} \\ &= e^{\left(-3 + i\sqrt{2}\right)x} \\ &= e^{-3x} e^{i\sqrt{2}x} \end{aligned}$$

And

$$\begin{aligned} y_2 &= e^{\lambda_2 x} \\ &= e^{\left(-3-i\sqrt{2}\right)x} \\ &= e^{-3x}e^{-i\sqrt{2}x} \end{aligned}$$

The general solution is a linear combination of these basis solutions. Therefore

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 e^{-3x} e^{i\sqrt{2}x} + c_2 e^{-3x} e^{-i\sqrt{2}x} \\ &= e^{-3x}\left(c_1 e^{i\sqrt{2}x} + c_2 e^{-i\sqrt{2}x}\right) \end{aligned}$$

Using Euler formula $e^{i\sqrt{2}x} = \cos\left(\sqrt{2}x\right) + i\sin\left(\sqrt{2}x\right)$ and $e^{-i\sqrt{2}x} = \cos\left(\sqrt{2}x\right) - i\sin\left(\sqrt{2}x\right)$. The above becomes

$$\begin{aligned} y(x) &= e^{-3x}\left(c_1\left(\cos\left(\sqrt{2}x\right) + i\sin\left(\sqrt{2}x\right)\right) + c_2\left(\cos\left(\sqrt{2}x\right) - i\sin\left(\sqrt{2}x\right)\right)\right) \\ &= e^{-3x}\left(\cos\left(\sqrt{2}x\right)(c_1 + c_2) + \sin\left(\sqrt{2}x\right)(ic_1 + ic_2)\right) \end{aligned}$$

Let $(c_1 + c_2) = C_1$ and $(ic_1 + ic_2) = C_2$ be new constants. Hence the above becomes

$$y(x) = e^{-3x}\left(C_1\cos\left(\sqrt{2}x\right) + C_2\sin\left(\sqrt{2}x\right)\right)$$

2.5 HW 5

Local contents

2.5.1 Problems listing

Math2520-01 **Assignment 5**

<u>INSTRUCTION</u>: ***Show all the necessary work***. Write your answer on a separate sheet preferably hand written clear and legible. Post your answer sheet on D2L by Monday July 5.

1. Solve the following Differential Equations.

   a) $y'' - y' - 2y = 5e^{2x}$

   b) $y'' + 16y = 4\cos x$

   c) $y'' - 4y' + 3y = 9x^2 + 4,\ y(0) = 6,\ y'(0) = 8$

2. Use the variation of parameters method to find the general solution to the given differential equation.

   $y'' + y = \tan^2(x)$

3. Show that the given vector functions are linearly independent on $(-\infty, \infty)$.

$$x_1(t) = \begin{bmatrix} t \\ t \end{bmatrix}, x_2(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$$

4. Show that the given vector functions are linearly dependent on $(-\infty, \infty)$.

$$x_1(t) = \begin{bmatrix} e^t \\ 2e^{2t} \end{bmatrix}, x_2(t) = \begin{bmatrix} 4e^t \\ 8e^{2t} \end{bmatrix}$$

5. Show that the given functions are solutions of the system $x'(t) = A(x)x(t)$ for the given matrix A and hence find the general solution to the system (remember to check linear independence). Then find the particular solution for the given auxiliary conditions.

$$x_1(t) = \begin{bmatrix} e^{4t} \\ 2e^{4t} \end{bmatrix}, x_2(t) = \begin{bmatrix} 3e^{-t} \\ e^{-t} \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix}, x(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

1

6. Solve the initial-value problem $x' = Ax,\ x(0) = x_0$.

$$A = \begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

7. Use the variation of parameters technique to find a particular solution x_p to $x' = Ax + b$ for the given A and b. Also obtain the general solution to the system of differential equations.

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 4e^t \end{bmatrix}$$

2.5.2 Problem 1

Solve the following Differential Equations

a $y'' - y' - 2y = 5e^{2x}$

b $y'' + 16y = 4\cos x$

c $y'' - 4y' + 3y = 9x^2 + 4, y(0) = 6, y'(0) = 8$

Solution

2.5.2.1 part a

$$y'' - y' - 2y = 5e^{2x}$$

This is a nonhomogeneous linear second order ODE with constant coefficients. The general solution is given by

$$y = y_h + y_p \tag{1}$$

Where y_h is the solution to the homogeneous part and y_p is a particular solution. The first step is to determine y_h which is solution to $y'' - y' - 2y = 0$. The characteristic equation becomes (by assuming the solution to be $y = e^{\lambda x}$ and substituting this into the ODE and simplifying)

$$\begin{aligned} \lambda^2 - \lambda - 2 &= 0 \\ (\lambda - 2)(\lambda + 1) &= 0 \end{aligned}$$

The roots are $\lambda_1 = 2, \lambda_2 = -1$. Therefore the basis for y_h are $\left\{e^{2x}, e^{-x}\right\}$ and y_h is linear combination of these basis which is

$$y_h = c_1 e^{2x} + c_2 e^{-x} \tag{2}$$

Looking at RHS of the ODE which is $5e^{2x}$ shows that the basis function for this is $\left\{e^{2x}\right\}$. But e^{2x} is also also a basis function for y_h. Therefore this is adjusted by multiplying it by x and it becomes $\left\{xe^{2x}\right\}$ which no longer a basis for y_h.Therefore the trial solution is

$$y_p = Axe^{2x}$$

Hence

$$\begin{aligned} y_p' &= Ae^{2x} + 2Axe^{2x} \\ y_p'' &= 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x} \end{aligned}$$

Substituting the above in the given ode gives

$$\begin{aligned} y_p'' - y_p' - 2y_p &= 5e^{2x} \\ \left(2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}\right) - \left(Ae^{2x} + 2Axe^{2x}\right) - 2\left(Axe^{2x}\right) &= 5e^{2x} \end{aligned}$$

Since $e^{2x} \neq 0$, the above simplifies to

$$\begin{aligned} 2A + 2A + 4Ax - A - 2Ax - 2Ax &= 5 \\ A(2 + 2 - 1) + x(4A - 2A - 2A) &= 5 \\ 3A &= 5 \\ A &= \frac{5}{3} \end{aligned}$$

Therefore the particular solution is

$$y_p = \frac{5}{3}xe^{2x} \tag{3}$$

Substituting (2,3) into (1) gives the general solution as

$$\begin{aligned} y(x) &= y_h + y_p \\ &= c_1 e^{2x} + c_2 e^{-x} + \frac{5}{3}xe^{2x} \end{aligned}$$

2.5.2.2 part b

$$y'' + 16y = 4\cos x$$

This is a nonhomogeneous linear second order ODE with constant coefficients. The general solution is given by

$$y = y_h + y_p \tag{1}$$

Where y_h is the solution to the homogeneous part and y_p is a particular solution. The first step is to determine y_h which is solution to $y'' + 16y = 0$. The characteristic equation is

$$\begin{aligned} \lambda^2 + 16 &= 0 \\ \lambda^2 &= -16 \\ \lambda &= \pm 4i \end{aligned}$$

The roots are $\lambda_1 = 4i, \lambda_2 = -4i$. Therefore the basis for y_h are $\left\{e^{4ix}, e^{-4ix}\right\}$. These are converted to trigonometric functions using the Euler relation $e^{ix} = \cos(x) + i\sin(x)$ as was done in the last HW and the basis become $\{\cos(4x), \sin(4x)\}$. y_h is a linear combination of these basis.

$$y_h = c_1 \cos(4x) + c_2 \sin(4x) \tag{2}$$

Looking at RHS of the ode which is $4\cos x$ shows that the basis function for y_p is $\{\cos x\}$. Taking all possible derivatives (and ignoring any sign change and constants that appear), results in the basis for y_p as the set $\{\cos x, \sin x\}$. There are no duplications with the basis for y_h found above. Hence the trial solution is

$$y_p = A\cos x + B\sin x$$

Therefore

$$\begin{aligned} y_p' &= -A\sin x + B\cos x \\ y_p'' &= -A\cos x - B\sin x \end{aligned}$$

Substituting the above in the given ode gives

$$\begin{aligned} y_p'' + 16y_p &= 4\cos x \\ (-A\cos x - B\sin x) + 16(A\cos x + B\sin x) &= 4\cos x \\ \cos(x)(-A + 16A) + \sin(x)(-B + 16B) &= 4\cos x \end{aligned}$$

Comparing coefficients gives

$$\begin{aligned} -A + 16A &= 4 \\ -B + 16B &= 0 \end{aligned}$$

Or

$$\begin{aligned} A &= \frac{4}{15} \\ B &= 0 \end{aligned}$$

Therefore the particular solution is

$$y_p = \frac{4}{15}\cos x \tag{3}$$

Substituting (2,3) into (1) gives the general solution

$$\begin{aligned} y(x) &= y_h + y_p \\ &= c_1 \cos(4x) + c_2 \sin(4x) + \frac{4}{15}\cos x \end{aligned}$$

2.5.2.3 part c

$$\begin{aligned} y'' - 4y' + 3y &= 9x^2 + 4 \\ y(0) &= 6 \\ y'(0) &= 8 \end{aligned}$$

This is a nonhomogeneous linear second order ODE with constant coefficients. The general solution is given by

$$y = y_h + y_p \tag{1}$$

Where y_h is the solution to the homogeneous part and y_p is a particular solution. The first step is to determine y_h which is solution to $y'' - 4y' + 3y = 0$. The characteristic equation is

$$\begin{aligned} \lambda^2 - 4\lambda + 3 &= 0 \\ (\lambda - 3)(\lambda - 1) &= 0 \end{aligned}$$

The roots are $\lambda_1 = 3, \lambda_2 = 1$. Therefore the basis for y_h are $\left\{e^{3x}, e^x\right\}$. y_h is a linear combination of these basis.

$$y_h = c_1 e^{3x} + c_2 e^x \tag{2}$$

Looking at RHS of the ode $9x^2+4$ shows that the basis functions for this are the set $\left\{1, x^2\right\}$. Taking all possible derivatives (and ignoring any sign change and constant multipliers that appear) results in the set $\left\{1, x, x^2\right\}$.There are no duplications with the basis for y_h. Hence the trial solution is linear combination of these basis which is

$$y_p = A + Bx + Cx^2$$

Hence

$$\begin{aligned} y_p' &= B + 2Cx \\ y_p'' &= 2C \end{aligned}$$

Substituting the above in the given ode gives

$$\begin{aligned} y_p'' - 4y_p' + 3y_p &= 9x^2 + 4 \\ (2C) - 4(B + 2Cx) + 3\left(A + Bx + Cx^2\right) &= 9x^2 + 4 \\ x^2(3C) + x(-8C + 3B) + (2C - 4B + 3A) &= 9x^2 + 4 \end{aligned}$$

Comparing coefficients gives

$$\begin{aligned} 3C &= 9 \\ -8C + 3B &= 0 \\ 2C - 4B + 3A &= 4 \end{aligned}$$

First equation gives $C = 3$. Substituting in second equation gives $-24 + 3B = 0$ or $B = 8$. Third equation now becomes

$$\begin{aligned} 2(3) - 4(8) + 3A &= 4 \\ A &= 10 \end{aligned}$$

Therefore the particular solution is

$$y_p = 10 + 8x + 3x^2 \tag{3}$$

Substituting (2,3) into (1) gives the general solution

$$\begin{aligned} y(x) &= y_h + y_p \\ &= c_1 e^{3x} + c_2 e^x + 10 + 8x + 3x^2 \end{aligned} \tag{4}$$

Initial conditions are now used to determine c_1, c_2. $y(0) = 6$ gives

$$\begin{aligned} 6 &= c_1 + c_2 + 10 \\ c_1 + c_2 &= -4 \end{aligned} \tag{5}$$

Taking derivative of (4)

$$y' = 3c_1 e^{3x} + c_2 e^{x} + 8 + 6x$$

Using $y'(0) = 8$ the above becomes

$$\begin{aligned} 8 &= 3c_1 + c_2 + 8 \\ 3c_1 + c_2 &= 0 \end{aligned} \tag{6}$$

Eq (5,6) are solved for c_1, c_2. From (5) $c_1 = -4 - c_2$. Substituting in (6) gives $3(-4 - c_2) + c_2 = 0$, $c_2 = -6$. Hence $c_1 = -4 + 6 = 2$. Therefore the solution (4) now becomes

$$y(x) = 2e^{3x} - 6e^{x} + 10 + 8x + 3x^2$$

2.5.3 Problem 2

Use the variation of parameters method to find the general solution to the given differential equation.

$$y'' + y = \tan^2(x)$$

Solution

This is a nonhomogeneous linear second order ODE with constant coefficients. The general solution is given by

$$y = y_h + y_p \tag{1}$$

where y_h is the solution to the homogeneous part and y_p is a particular solution. The first step is to determine y_h which is solution to $y'' + y = 0$. The characteristic equation is

$$\begin{aligned} \lambda^2 + 1 &= 0 \\ \lambda &= \pm i \end{aligned}$$

The roots are $\lambda_1 = i, \lambda_2 = -i$. Therefore the basis for y_h are $\left\{e^{ix}, e^{-ix}\right\}$. Using Euler relation these become $\{\cos x, \sin x\}$. Hence y_h is a linear combination of these basis

$$y_h = c_1 \cos x + c_2 \sin x \tag{2}$$

Using variation of parameters, let $y_p = y_1 u_1 + y_2 u_2$, where

$$\begin{aligned} y_1 &= \cos x \\ y_2 &= \sin x \end{aligned}$$

Are the basis of y_h found above, and u_1, u_2 are functions yet to be determined. Applying reduction of order as shown in the textbook (not repeated here) gives

$$u_1 = -\int \frac{y_2 g(x)}{W(x)} dx \tag{3}$$

$$u_2 = \int \frac{y_1 g(x)}{W(x)} dx \tag{4}$$

Where in the above $W(x)$ is the Wronskian and $g(x)$ is the forcing function which is $g(x) = \tan^2(x)$ in this case. The first step is to calculate $W(x)$

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \\ &= \cos^2 x + \sin^2 x \\ &= 1 \end{aligned}$$

Therefore (3) becomes

$$u_1 = -\int \sin x \tan^2 x \, dx$$

But $\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1$. Hence the above becomes

$$\begin{aligned} u_1 &= -\int \sin x \left(\frac{1}{\cos^2 x} - 1 \right) dx \\ &= -\int \left(\frac{\sin x}{\cos^2 x} - \sin x \right) dx \\ &= -\int \frac{\sin x}{\cos^2 x} dx + \int \sin x \, dx \\ &= -\int \tan x \frac{1}{\cos x} dx + \int \sin x \, dx \end{aligned} \tag{5}$$

To find the first integral in (5), let $u = \frac{1}{\cos x}$. Then $du = -(\cos x)^{-2} (-\sin x) \, dx = \frac{\sin x}{\cos^2 x} dx$. Hence $dx = \frac{\cos^2 x}{\sin x} du = \frac{\cos x}{\tan x} du$. Therefore the first integral in (5) becomes

$$\begin{aligned} -\int \tan x \frac{1}{\cos x} dx &= -\int (\tan x) \, u \left(\frac{\cos x}{\tan x} du \right) \\ &= -\int u \cos x \, du \end{aligned}$$

But $\cos x = \frac{1}{u}$. The above becomes

$$\begin{aligned} -\int \tan x \frac{1}{\cos x} dx &= -\int du \\ &= -u \\ &= -\frac{1}{\cos x} \end{aligned}$$

The second integral in (5) is just $\int \sin x \, dx = -\cos x$. Therefore (5) becomes

$$\begin{aligned} u_1 &= -\frac{1}{\cos x} - \cos x \\ &= \frac{-1 - \cos^2 x}{\cos x} \\ &= -\frac{1 + \cos^2 x}{\cos x} \end{aligned}$$

Now u_2 in (4) is found.

$$\begin{aligned} u_2 &= \int \cos x \tan^2 x \, dx \\ &= \int \cos x \frac{\sin^2 x}{\cos^2 x} dx \\ &= \int \frac{\sin^2 x}{\cos x} dx \\ &= \int \frac{1 - \cos^2 x}{\cos x} dx \\ &= \int \left(\frac{1}{\cos x} - \cos x \right) dx \\ &= \int \frac{1}{\cos x} dx - \int \cos x \, dx \\ &= \int \sec x dx - \int \cos x \, dx \end{aligned} \tag{6}$$

To find $\int \sec x \, dx$, we start by multiplying the integrand by $\frac{\sec x + \tan x}{\sec x + \tan x} = 1$. Hence

$$\begin{aligned} \int \sec x \, dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \end{aligned} \tag{7}$$

Using the substitution

$$u = \sec x + \tan x$$

Then

$$\frac{du}{dx} = \frac{d}{dx}\sec x + \frac{d}{dx}\tan x \tag{7A}$$

But $\frac{d}{dx}\sec x = \frac{d}{dx}(\cos x) = -(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x} = \sin x \sec^2 x = \sec x \tan x$. And

$$\begin{aligned}\frac{d}{dx}\tan x &= 1 + \tan^2 x\\ &= 1 + \frac{\sin^2 x}{\cos^2 x}\\ &= \frac{\cos^2 + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\\ &= \sec^2 x\end{aligned}$$

Hence (7A) becomes

$$\frac{du}{dx} = \sec x \tan x + \sec^2 x$$

Therefore (7) now becomes

$$\begin{aligned}\int \sec x\, dx &= \int \frac{\sec^2 x + \sec x \tan x}{u} \frac{du}{\sec x \tan x + \sec^2 x}\\ &= \int \frac{du}{u}\\ &= \ln u\\ &= \ln(\sec x + \tan x)\end{aligned}$$

Eq (6) now becomes

$$\begin{aligned}u_2 &= \int \sec x dx - \int \cos x\, dx\\ &= \ln(\sec x + \tan x) - \sin x\end{aligned}$$

Now that u_1, u_2 are found, then $y_p = y_1 u_1 + y_2 u_2$ gives

$$\begin{aligned}y_p &= -\cos x\left(\frac{1+\cos^2 x}{\cos x}\right) + \sin x(\ln(\sec x + \tan x) - \sin x)\\ &= -\left(1+\cos^2 x\right) + \sin x(\ln(\sec x + \tan x) - \sin x)\\ &= -1 - \cos^2 x + \sin x \ln(\sec x + \tan x) - \sin^2 x\\ &= -1 - \left(\cos^2 x + \sin^2 x\right) + \sin x \ln(\sec x + \tan x)\\ &= -2 + \sin x \ln(\sec x + \tan x)\\ &= -2 + \sin x \ln\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)\\ &= -2 + \sin x \ln\left(\frac{1+\sin x}{\cos x}\right)\end{aligned}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\ &= c_1 \cos x + c_2 \sin x + \sin x \ln\left(\frac{1+\sin x}{\cos x}\right) - 2\end{aligned}$$

2.5.4 Problem 3

Show that the given vector functions are linearly independent on $(-\infty, \infty)$

$$\vec{x}_1(t) = \begin{bmatrix} t \\ t \end{bmatrix} \qquad \vec{x}_2(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$$

Solution

The Wronskian of these vectors is

$$W(t) = \begin{vmatrix} t & t \\ t & t^2 \end{vmatrix}$$

If the above is nonzero at some point in the interval $(-\infty, \infty)$ then $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent.

$$\begin{aligned} W(t) &= t^3 - t^2 \\ &= t^2 (t-1) \end{aligned}$$

Any point other than $t = 0, t = 1$, then $W(t) \neq 0$. For example at $t = 2$, $W(2) = 4 \neq 0$. Hence $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent .

2.5.5 Problem 4

Show that the given vector functions are linearly independent on $(-\infty, \infty)$

$$\vec{x}_1(t) = \begin{bmatrix} e^t \\ 2e^t \end{bmatrix} \qquad \vec{x}_2(t) = \begin{bmatrix} 4e^t \\ 8e^{2t} \end{bmatrix}$$

Solution

The Wronskian of these vectors is

$$W(t) = \begin{vmatrix} e^t & 4e^t \\ 2e^t & 8e^{2t} \end{vmatrix}$$

If the above is nonzero at some point in the interval $(-\infty, \infty)$ then $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent.

$$\begin{aligned} W(t) &= 8e^{3t} - 6e^{2t} \\ &= e^{2t} \left(8e^t - 6\right) \end{aligned}$$

Choosing say $t = 0$ then the above becomes $W(0) = 2 \neq 0$. Therefore $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent.

2.5.6 Problem 5

Show that the given functions are solutions of the system $x'(t) = A(x)x(t)$ for the given matrix A and hence find the general solution to the system (remember to check linear independence). Then find the particular solution for the given auxiliary conditions.

$$\vec{x}_1(t) = \begin{bmatrix} e^{4t} \\ 2e^{4t} \end{bmatrix} \qquad \vec{x}_2(t) = \begin{bmatrix} 3e^{-t} \\ e^{-t} \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix} \qquad x(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Solution

The system to solve is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

We need to first find the eigenvalues and eigenvectors of A. The eigenvalues are solution to $|A - \lambda I| = 0$ or

$$\begin{aligned} \begin{vmatrix} -2-\lambda & 3 \\ -2 & 5-\lambda \end{vmatrix} &= 0 \\ (-2-\lambda)(5-\lambda)+6 &= 0 \\ \lambda^2 - 3\lambda - 10 + 6 &= 0 \\ \lambda^2 - 3\lambda - 4 &= 0 \\ (\lambda-4)(\lambda+1) &= 0 \end{aligned}$$

Hence the eigenvalues are $\lambda_1 = 4, \lambda_2 = -1$.

$\underline{\lambda_1 = 4}$

$$\begin{aligned} \begin{bmatrix} -2-\lambda & 3 \\ -2 & 5-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -2-4 & 3 \\ -2 & 5-4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -6 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

$R_2 = R_2 + \frac{1}{3}R_1$

$$\begin{bmatrix} -6 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence v_1 is base variable and $v_2 = t$ is free variable. First row gives $-6v_1 = -3t$ or $v_1 = \frac{1}{2}t$. The eigenvector is then

$$\vec{v}_{\lambda_1} = \begin{bmatrix} v_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Choosing $t = 1$, then

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore the first basis vector solution is given by

$$\begin{aligned} \vec{x}_1(t) &= e^{\lambda_1 t} \vec{v}_{\lambda_1} \\ &= e^{4t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} e^{4t} \\ 2e^{4t} \end{bmatrix} \end{aligned}$$

$\underline{\lambda_1 = -1}$

$$\begin{bmatrix} -2-\lambda & 3 \\ -2 & 5-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2+1 & 3 \\ -2 & 5+1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 3 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$

$$\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence v_1 is base variable and $v_2 = t$ is free variable. First row gives $-v_1 = -3t$ or $v_1 = 3t$. The eigenvector is then

$$\vec{v}_{\lambda_2} = \begin{bmatrix} v_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Choosing $t = 1$, then

$$\vec{v}_{\lambda_2} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Therefore the second basis vector solution is given by

$$\begin{aligned} \vec{x}_2(t) &= e^{\lambda_2 t}\, \vec{v}_{\lambda_2} \\ &= e^{-t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3e^{-t} \\ e^{-t} \end{bmatrix} \end{aligned}$$

The above result shows that the solution to $x'(t) = A(x)x(t)$ is

$$\begin{aligned} \vec{x}(t) &= c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) \qquad (1) \\ &= c_1 \begin{bmatrix} e^{4t} \\ 2e^{4t} \end{bmatrix} + c_2 \begin{bmatrix} 3e^{-t} \\ e^{-t} \end{bmatrix} \end{aligned}$$

Now we check that $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent (they have to be, since they are eigenvectors of A, but the problem is asking to verify this result). The Wronskian of these vectors is

$$W(t) = \begin{vmatrix} e^{4t} & 3e^{-t} \\ 2e^{4t} & e^{-t} \end{vmatrix}$$

If the above is nonzero at some point in the interval $(-\infty, \infty)$ then $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent.

$$\begin{aligned} W(t) &= e^{3t} - 6e^{3t} \\ &= -5e^{3t} \end{aligned}$$

Choosing say $t = 0$ then the above becomes $W(0) = -5$. Since we found at least one point where $W(t) \neq 0$ then $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent and (1) is the general solution to given system of differential equations. This answers the first part of the question by showing that the given functions are solutions of the system $x'(t) = A(x)\, x(t)$.

The final step is to find the particular solution to the given initial conditions $x(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. At $t = 0$ the solution in (1) becomes

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Which can be written as

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \tag{2}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 1 & 1 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -5 & 5 \end{bmatrix}$$

Hence (2) becomes

$$\begin{bmatrix} 1 & 3 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} \tag{3}$$

Second row gives $-5c_2 = 5$ or $c_2 = -1$. First row gives $c_1 + 3c_2 = -2$ or $c_1 = -2 - 3(-1) = -2 + 3 = 1$. Hence

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore the solution (1) becomes

$$\vec{x}(t) = \begin{bmatrix} e^{4t} \\ 2e^{4t} \end{bmatrix} - \begin{bmatrix} 3e^{-t} \\ e^{-t} \end{bmatrix}$$

Or

$$x_1(t) = e^{4t} - 3e^{-t}$$
$$x_2(t) = 2e^{4t} - e^{-t}$$

2.5.7 Problem 6

Solve the initial-value problem $x' = Ax, x(0) = x_0$

$$A = \begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix} \qquad x(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Solution

The system is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

We need to first find the eigenvalues and eigenvectors of A. The eigenvalues are solutions to $|A - \lambda I| = 0$ or

$$\begin{vmatrix} -1-\lambda & 4 \\ 2 & -3-\lambda \end{vmatrix} = 0$$
$$(-1-\lambda)(-3-\lambda) - 8 = 0$$
$$\lambda^2 + 4\lambda + 3 - 8 = 0$$
$$\lambda^2 + 4\lambda - 5 = 0$$
$$(\lambda - 1)(\lambda + 5) = 0$$

Hence the eigenvalues are $\lambda_1 = 1, \lambda_2 = -5$.

$\underline{\lambda_1 = 1}$

$$\begin{bmatrix} -1-\lambda & 4 \\ 2 & -3-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 + R_1$

$$\begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

v_1 is base variables and $v_2 = t$ is free variable. First row gives $-2v_1 = -4t$ or $v_1 = 2t$. Hence the first eigenvector is

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Or for $t = 1$

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The first basis vector solution is therefore

$$\begin{aligned} \vec{x}_1 &= e^{\lambda_1 t}\, \vec{v}_{\lambda_1} \\ &= e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2e^t \\ e^t \end{bmatrix} \end{aligned} \tag{1}$$

$\underline{\lambda_1 = -5}$

$$\begin{bmatrix} -1-\lambda & 4 \\ 2 & -3-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - \frac{1}{2}R_1$

$$\begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

v_1 is base variables and $v_2 = t$ is free variable. First row gives $4v_1 = -4t$ or $v_1 = -t$. Hence the second eigenvector is

$$\vec{v}_{\lambda_2} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Or for $t = 1$

$$\vec{v}_{\lambda_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The second basis vector solution is therefore

$$\begin{aligned} \vec{x}_2 &= e^{\lambda_2 t}\, \vec{v}_{\lambda_2} \\ &= e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -e^{-5t} \\ e^{-5t} \end{bmatrix} \end{aligned} \tag{2}$$

From (1,2), the general solution is linear combination of (1,2) which is

$$\begin{aligned} \vec{x}(t) &= c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) \\ &= c_1 \begin{bmatrix} 2e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} -e^{-5t} \\ e^{-5t} \end{bmatrix} \end{aligned} \tag{3}$$

Now $\vec{x}_1(t), \vec{x}_2(t)$ are verified to be linearly independent using the Wronskian.

$$\begin{aligned} W(t) &= \begin{vmatrix} 2e^t & -e^{-5t} \\ e^t & e^{-5t} \end{vmatrix} \\ &= 2e^{-4t} + e^{-4t} \\ &= 3e^{-4t} \end{aligned}$$

At $t = 0$, $W(0) = 3 \neq 0$. Hence $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent. c_1, c_2 are now found from initial conditions. At $t = 0$, (3) becomes

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Which can be written as

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \tag{4}$$

The augmented matrix is

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

$R_2 = 2R_2 - R_1$

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

Hence (2) becomes

$$\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

Second row gives $c_2 = -1$. First row gives $2c_1 - c_1 = 3$ or $2c_1 = 3 - 1 = 2$. Hence $c_1 = 1$.

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore the solution (3) becomes

$$\vec{x}(t) = \begin{bmatrix} 2e^t \\ e^t \end{bmatrix} - \begin{bmatrix} -e^{-5t} \\ e^{-5t} \end{bmatrix}$$

Or

$$x_1(t) = 2e^t + e^{-5t}$$
$$x_2(t) = e^t - e^{-5t}$$

2.5.8 Problem 7

Use the variation of parameters technique to find a particular solution x_p to $x' = Ax + b$ for the given A, b. Also obtain the general solution to the system of differential equations

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 4e^t \end{bmatrix}$$

Solution

The system to solve is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 4e^t \end{bmatrix}$$

The solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the solution to the homogeneous system $x' = Ax$ and $\vec{x}_p(t)$ is a particular solution. First $\vec{x}_h(t)$ is solved for. The eigenvalues and eigenvectors of A are now found. The eigenvalues are solutions to $|A - \lambda I| = 0$ or

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$
$$(2-\lambda)(2-\lambda) - 1 = 0$$
$$\lambda^2 - 4\lambda + 4 - 1 = 0$$
$$\lambda^2 - 4\lambda + 3 = 0$$
$$(\lambda - 3)(\lambda - 1) = 0$$

Hence the eigenvalues are $\lambda_1 = 3, \lambda_2 = 1$.

$\lambda_1 = 1$

$$\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 + R_1$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

v_1 is base variables and $v_2 = t$ is free variable. First row gives $v_1 = t$. Hence the first eigenvector is

$$\vec{v}_1 = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Or for $t = 1$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The first basis vector solution is therefore

$$\begin{aligned} \vec{x}_1(t) &= e^{\lambda_1 t}\, \vec{v}_1 \\ &= e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^t \\ e^t \end{bmatrix} \end{aligned} \tag{1}$$

$\lambda_1 = 3$

$$\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - R_1$

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

v_1 is base variables and $v_2 = t$ is free variable. First row gives $v_1 = -t$. Hence the second eigenvector is

$$\vec{v}_2 = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Or for $t = 1$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The second basis vector solution is therefore

$$\begin{aligned} \vec{x}_2(t) &= e^{\lambda_2 t}\, \vec{v}_2 \\ &= e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -e^{3t} \\ e^{3t} \end{bmatrix} \end{aligned} \tag{2}$$

From $(1,2)$, the homogeneous is

$$\begin{aligned} \vec{x}_h(t) &= c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) \\ &= c_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} -e^{3t} \\ e^{3t} \end{bmatrix} \end{aligned} \tag{2A}$$

The Wronskian can be used to show that $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent

$$W(t) = \begin{vmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{vmatrix} = e^{4t} + e^{4t} = 2e^{4t}$$

Which is not zero at a point, say at $t = 0$. Variation of parameters is now used to find the particular solution $\vec{x}_p(t)$. The fundamental matrix is the matrix whose columns are $\vec{x}_1(t), \vec{x}_2(t)$

$$\begin{aligned} \Phi &= \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) \end{bmatrix} \\ &= \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix} \end{aligned}$$

Therefore

$$\begin{aligned} \vec{x}_p(t) &= \Phi \int \Phi^{-1} \vec{b}(t) dt \\ &= \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix} \int \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 4e^t \end{bmatrix} dt \end{aligned} \tag{3}$$

But

$$\begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix}^{-1} = \frac{\begin{bmatrix} e^{3t} & e^{3t} \\ -e^t & e^t \end{bmatrix}}{|\Phi|}$$

And

$$|\Phi| = \begin{vmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{vmatrix} = e^{4t} + e^{4t} = 2e^{4t}$$

Therefore

$$\begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} e^{-t} & e^{-t} \\ -e^{-3t} & e^{-3t} \end{bmatrix}$$

Substituting the above in (3) gives

$$\vec{x}_p(t) = \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix} \frac{1}{2} \int \begin{bmatrix} e^{-t} & e^{-t} \\ -e^{-3t} & e^{-3t} \end{bmatrix} \begin{bmatrix} 0 \\ 4e^t \end{bmatrix} dt$$

But $\begin{bmatrix} e^{-t} & e^{-t} \\ -e^{-3t} & e^{-3t} \end{bmatrix} \begin{bmatrix} 0 \\ 4e^t \end{bmatrix} = \begin{bmatrix} 4 \\ 4e^{-2t} \end{bmatrix}$. Hence the above becomes

$$\vec{x}_p(t) = \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix} \int \begin{bmatrix} 2 \\ 2e^{-2t} \end{bmatrix} dt$$

Carrying the integration element by element gives

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix} \begin{bmatrix} \int 2dt \\ \int 2e^{-2t} dt \end{bmatrix} \\ &= \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix} \begin{bmatrix} 2t \\ -e^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} 2te^t + e^t \\ 2te^t - e^t \end{bmatrix} \end{aligned} \tag{4}$$

Substituting (2A) and (4) into $\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$ gives the final solution as

$$
\begin{aligned}
\vec{x}(t) &= c_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} -e^{3t} \\ e^{3t} \end{bmatrix} + \begin{bmatrix} 2te^t + e^t \\ 2te^t - e^t \end{bmatrix} \\
&= \begin{bmatrix} c_1 e^t - c_2 e^{3t} + 2te^t + e^t \\ c_1 e^t + c_2 e^{3t} + 2te^t - e^t \end{bmatrix}
\end{aligned}
$$

Or

$$
\begin{aligned}
x_1(t) &= c_1 e^t - c_2 e^{3t} + 2te^t + e^t \\
x_2(t) &= c_1 e^t + c_2 e^{3t} + 2te^t - e^t
\end{aligned}
$$

2.6 HW 6

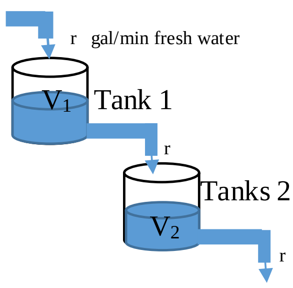
Local contents

2.6.1 Problems listing

Math2520-01 **Assignment 6**

<u>INSTRUCTION</u>: ***Show all the necessary work***. Write your answer on a separate sheet preferably hand written clear and legible. Post your answer sheet on D2L by Sunday July 11.

1. Consider the following figure below that shows two brine tanks



containing V_1 and V_2 gallons of brine respectively. Fresh water flows into tank 1, while mixed brine flows from tank 1 into tank 2. Let $x_i(t)$ denote the amount (in pounds) of salt in tank i at time t for $i=1,2$. If each flow rate is r gallons per minute, then a simple account of salt concentration yields the first-order system

$$x_1' = -k_1 x_1$$
$$x_2' = k_1 x_1 - k_2 x_2$$

where

$$k_i = \frac{r}{V_i}, \quad i = 1, 2.$$

If $V_1 = 25$, $V_2 = 50$, $r = 10(gal/\min)$, and the initial amounts of salt in the two brine tanks, in pounds, are

$$x_1(0) = 15, \ x_2(0) = 0,$$

a) Find the amount of salt in each tank at time $t \geq 0$.
b) Find the maximum amount of salt ever in tank 2.

2. Determine all the equilibrium points of the given system.

$$x' = x - x^2 - xy$$
$$y' = 3y - xy - 2y^2$$

3. Using the definition of Laplace transform, determine $L\{f\}$.

$f(t) = te^t$

4. Find the inverse Laplace transform of the given functions.

a) $F(s) = \dfrac{2}{s(s-2)}$

b) $F(s) = \dfrac{2s+2}{s^2+2s+5}$

5. Use the Laplace transform to solve the following given initial-value problems.

a) $y' + y = 8e^{3t}, \; y(0) = 2$

b) $y'' + y' - 2y = 10e^{-t}, \; y(0) = 0, \; y'(0) = 1$

2.6.2 Problem 1

$$
\begin{aligned}
x_1' &= -k_1 x_1\\
x_2' &= k_1 x_1 - k_2 x_2
\end{aligned}
$$

Where

$$
\begin{aligned}
k_1 &= \frac{10}{V_1(t)} = \frac{10}{25} = \frac{2}{5}\\
k_2 &= \frac{10}{V_2(t)} = \frac{10}{50} = \frac{1}{5}\\
x_1(0) &= 15\\
x_2(0) &= 0
\end{aligned}
$$

a) Find the amount of salt in each tank at time $t \geq 0$. b) Find the maximum amount of salt ever in tank 2.

Solution

2.6.2.1 Part a

The system in matrix form is

$$
\begin{aligned}
x' &= Ax\\
\begin{bmatrix} x_1'(t)\\ x_2'(t)\end{bmatrix} &= \begin{bmatrix} -k_1 & 0\\ k_1 & -k_2\end{bmatrix}\begin{bmatrix} x_1\\ x_2\end{bmatrix}\\
&= \begin{bmatrix} -\frac{2}{5} & 0\\ \frac{2}{5} & -\frac{1}{5}\end{bmatrix}\begin{bmatrix} x_1\\ x_2\end{bmatrix}
\end{aligned}
$$

The eigenvalues of A are found from solving $\det(A - \lambda I) = 0$ or

$$
\begin{aligned}
\begin{vmatrix} -\frac{2}{5} - \lambda & 0\\ \frac{2}{5} & -\frac{1}{5} - \lambda\end{vmatrix} &= 0\\
\left(-\frac{2}{5} - \lambda\right)\left(-\frac{1}{5} - \lambda\right) &= 0
\end{aligned}
$$

Hence $\lambda_1 = -\frac{2}{5}, \lambda_2 = -\frac{1}{5}$. Now the eigenvector for each eigenvalue is found.

$\lambda_1 = -\frac{2}{5}$

$$
\begin{aligned}
\begin{bmatrix} -\frac{2}{5} - \lambda_1 & 0\\ \frac{2}{5} & -\frac{1}{5} - \lambda_1\end{bmatrix}\begin{bmatrix} v_1\\ v_2\end{bmatrix} &= \begin{bmatrix} 0\\ 0\end{bmatrix}\\
\begin{bmatrix} -\frac{2}{5} - \left(-\frac{2}{5}\right) & 0\\ \frac{2}{5} & -\frac{1}{5} - \left(-\frac{2}{5}\right)\end{bmatrix}\begin{bmatrix} v_1\\ v_2\end{bmatrix} &= \begin{bmatrix} 0\\ 0\end{bmatrix}\\
\begin{bmatrix} 0 & 0\\ \frac{2}{5} & \frac{1}{5}\end{bmatrix}\begin{bmatrix} v_1\\ v_2\end{bmatrix} &= \begin{bmatrix} 0\\ 0\end{bmatrix}
\end{aligned}
$$

Hence $v_1 = t$ is free variable and v_2 is base variable. From second row $\frac{2}{5}t + \frac{1}{5}v_2 = 0$ or $2t + v_2 = 0$ or $v_2 = -2t$. The solution is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} t \\ -2t \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Choosing $t = 1$ this gives the first eigenvector

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\underline{\lambda_2 = -\frac{1}{5}}$

$$\begin{bmatrix} -\frac{2}{5} - \lambda_2 & 0 \\ \frac{2}{5} & -\frac{1}{5} - \lambda_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{2}{5} - \left(-\frac{1}{5}\right) & 0 \\ \frac{2}{5} & -\frac{1}{5} - \left(-\frac{1}{5}\right) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{5} & 0 \\ \frac{2}{5} & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - 2R_2$

$$\begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence v_1 is base variable and $v_2 = t$ is free variable. Therefore $\frac{1}{5}v_1 = 0$ or $v_1 = 0$. The eigenvector is

$$\vec{v}_2 = \begin{bmatrix} 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Choosing $t = 1$ this gives the second eigenvector is

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore the solution basis are

$$\vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1 = e^{-\frac{2}{5}t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\vec{x}_2(t) = e^{\lambda_2 t}\vec{v}_2 = e^{\frac{-1}{5}t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

And the solution is linear combination of the above basis, which gives

$$\vec{x}(t) = c_1\begin{bmatrix} e^{-\frac{2}{5}t} \\ -2e^{-\frac{2}{5}t} \end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{\frac{-1}{5}t} \end{bmatrix} \tag{1}$$

The scalar solutions are therefore

$$x_1(t) = c_1 e^{-\frac{2}{5}t}$$
$$x_2(t) = -2c_1 e^{-\frac{2}{5}t} + c_2 e^{\frac{-1}{5}t}$$

Now c_1, c_2 are found from initial conditions. At $t = 0, x_1(0) = 15, x_2(0) = 0$. Hence (1) becomes

$$\begin{bmatrix} 15 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Or

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \end{bmatrix} \tag{2}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 0 & 15 \\ -2 & 1 & 0 \end{bmatrix}$$

$R_2 = R_2 + 2R_1$ gives

$$\begin{bmatrix} 1 & 0 & 15 \\ 0 & 1 & 30 \end{bmatrix}$$

Hence (2) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 30 \end{bmatrix}$$

Second row gives $c_2 = 30$ and first row gives $c_1 = 15$. Hence

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 30 \end{bmatrix}$$

And the solution (1) becomes

$$x_1(t) = 15e^{-\frac{2}{5}t}$$
$$x_2(t) = -2(15)e^{-\frac{2}{5}t} + 30e^{\frac{1}{5}t}$$

Or

$$x_1(t) = 15e^{-\frac{2t}{5}} \tag{3}$$
$$x_2(t) = -30e^{-\frac{2t}{5}} + 30e^{\frac{-t}{5}}$$

The above is the amount of salt in each tank for $t \geq 0$.

2.6.2.2 Part b

The solution in (3) above shows that at $t \to \infty$ then $x_2(t) \to 0$ because both exponential are raised to negative power of t. This is as expected, as with time, and with more fresh water coming in and mixture discharges, we expect the initial salt in the tank to eventually vanish leaving only pure water in the tank. The following plot shows how the amount of salt changes in each tank as function of time

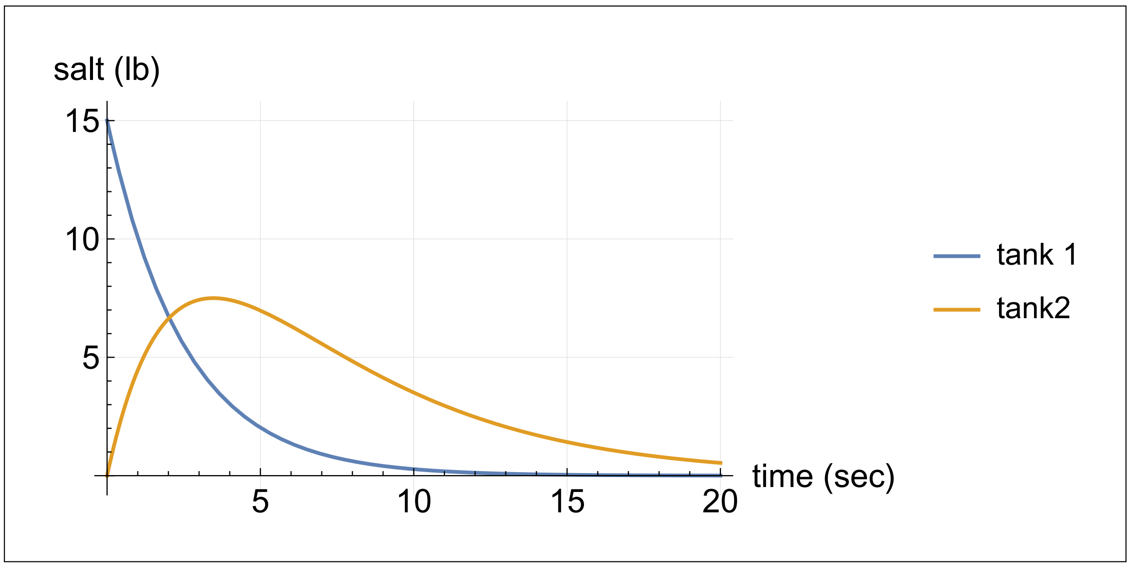


Figure 2.4: salt amount vs. time for each tank

The above shows that salt starts in tank 1 at amount $x_1(0) = 15$ which is the initial value, and continues to decreases exponentially where is becomes close to zero after about 15 seconds. While for tank 2, which initially has no salt, its salt content initially increases to a maximum value after about 3 seconds and then starts to decrease where it will eventually becomes zero. The initial increase in tank 2 is because of the salt coming from tank 1 in the mixture. But as salt decrease in tank 1 with time, so will the salt in tank 2 as well.

Code used for the plot above is

```
ClearAll[x1,x2,t];
x1[t_] := 15*Exp[-2/5*t]
x2[t_] := -30*Exp[-2/5*t]+30*Exp[-t/5]

Plot[{x1[t],x2[t]},{t,0,20},
   AxesLabel->{"time (sec)","salt (lb)"},
   BaseStyle->14,
   GridLines->Automatic,GridLinesStyle->LightGray,
   PlotLegends->{"tank 1","tank2"}
]
```

2.6.3 Problem 2

Determine all the equilibrium points of the given system.

$$\begin{aligned} x' &= x - x^2 - xy \\ y' &= 3y - xy - 2y^2 \end{aligned}$$

Solution

The equilibrium points are the solutions x, y to

$$x - x^2 - xy = 0 \tag{1}$$
$$3y - xy - 2y^2 = 0 \tag{2}$$

We can start with either equation, find one unknown from it, and use that to solve for the second unknown using the second equation. Starting with (1) and solving for x. Writing (1) as

$$\begin{aligned} x^2 + x\left(y - 1\right) &= 0 \\ x\left(x + \left(y - 1\right)\right) &= 0 \end{aligned}$$

Then the solutions are

$$x = 0$$
$$x + (y - 1) = 0$$

Or

$$x = 0 \tag{3}$$
$$x = 1 - y \tag{4}$$

For each one of the above solutions, we go back to (2) and solve for y now. When $x = 0$, then (2) gives

$$3y - 2y^2 = 0$$
$$y(3 - 2y) = 0$$

Hence $y = 0, y = \frac{3}{2}$ are the solutions. So now we have the following solutions found for the case when $x = 0$

$$\left\{(0,0), \left(0, \frac{3}{2}\right)\right\} \tag{5}$$

And when $x = 1 - y$ then (2) gives

$$3y - (1 - y)y - 2y^2 = 0$$
$$2y - y^2 = 0$$
$$y(2 - y) = 0$$

Hence $y = 0, y = 2$ are the solutions. When $y = 0$, the corresponding x from $x = 1 - y$ is 1. And when $y = 2$, the corresponding x is -1, So now we have the following solutions found

$$\{(1,0), (-1,2)\} \tag{6}$$

Adding (6,5) gives the list of equilibrium points as

$$\left\{(0,0), \left(0, \frac{3}{2}\right), (1,0), (-1,2)\right\}$$

2.6.4 Problem 3

Using the definition of Laplace transform, determine Laplace transform of

$$f(t) = te^t$$

Solution

By definition

$$\mathscr{L}(f(t)) = \lim_{N\to\infty} \int_0^N f(t)e^{-st}dt$$

Therefore

$$\begin{aligned}\mathscr{L}(f(t)) &= \lim_{N\to\infty} \int_0^N te^t e^{-st}dt \\ &= \lim_{N\to\infty} \int_0^N te^{-st+t}dt \\ &= \lim_{N\to\infty} \int_0^N te^{t(1-s)}dt\end{aligned}$$

Integration by parts. $\int udv = uv - \int vdu$. Let $u = t, dv = e^{t(1-s)}$, therefore $du = dt, v = \frac{e^{t(1-s)}}{1-s}$. Hence the above becomes

$$\begin{aligned}\mathscr{L}\left(f(t)\right) &= \frac{1}{1-s}\lim_{N\to\infty}\left[te^{t(1-s)}\right]_0^N - \lim_{N\to\infty}\int_0^N \frac{e^{t(1-s)}}{1-s}dt \qquad (1)\\ &= \frac{1}{1-s}\lim_{N\to\infty}\left[te^{t(1-s)}\right]_0^N - \frac{1}{1-s}\lim_{N\to\infty}\int_0^N e^{t(1-s)}dt\end{aligned}$$

But

$$\begin{aligned}\lim_{N\to\infty}\left[te^{t(1-s)}\right]_0^N &= \lim_{N\to\infty}\left(Ne^{N(1-s)}\right) - 0\\ &= \lim_{N\to\infty}\left(Ne^{N(1-s)}\right)\end{aligned}$$

But

$$\begin{aligned}\lim_{N\to\infty}\left(Ne^{N(1-s)}\right) &= \lim_{N\to\infty} N \lim_{N\to\infty} e^{N(1-s)}\\ &= (\infty)\left(\lim_{N\to\infty} e^{N(1-s)}\right)\end{aligned}$$

For $s > 1$, $\lim_{N\to\infty} e^{N(1-s)} = e^{-\infty} = 0$ since $1-s < 0$, and therefore the exponential is raised to negative infinity. Hence the above becomes

$$\begin{aligned}\lim_{N\to\infty}\left(Ne^{N(1-s)}\right) &= (\infty)(0)\\ &= 0\end{aligned}$$

Therefore (1) simplifies to

$$\begin{aligned}\mathscr{L}\left(f(t)\right) &= \frac{1}{1-s}\lim_{N\to\infty}\int_0^N e^{t(1-s)}dt\\ &= \frac{1}{s-1}\frac{1}{1-s}\lim_{N\to\infty}\left[e^{t(1-s)}\right]_0^\infty\\ &= \frac{1}{(s-1)(1-s)}\lim_{N\to\infty}\left[e^{t(1-s)}\right]_0^\infty\end{aligned}$$

But for $s > 1$ then $\lim_{N\to\infty}\left[e^{t(1-s)}\right]_0^\infty = \lim_{N\to\infty} e^{N(1-s)} - 1 = 0 - 1 = -1$. The above then becomes

$$\begin{aligned}\mathscr{L}\left(f(t)\right) &= \frac{-1}{(s-1)(1-s)}\\ &= \frac{1}{(s-1)(s-1)}\\ &= \frac{1}{(s-1)^2}\end{aligned}$$

For $s > 1$.

2.6.5 Problem 4

Find the inverse Laplace transform of the given functions

a $F(s) = \frac{2}{s(s-2)}$

b $F(s) = \frac{2s+2}{s^2+2s+5}$

<u>Solution</u>

2.6.5.1 Part a

Using partial fractions, let

$$\frac{2}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

Therefore

$$A = \left.\frac{2}{(s-2)}\right|_{s=0} = -1$$
$$B = \left.\frac{2}{s}\right|_{s=2} = 1$$

Hence

$$\frac{2}{s(s-2)} = -\frac{1}{s} + \frac{1}{s-2}$$

Using Table 10.2.1 in textbook,

$$\mathscr{L}^{-1}\left(\frac{1}{s}\right) = 1 \qquad s > 0$$
$$\mathscr{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t} \qquad s > 2$$

Therefore, by linearity of $\mathscr{L}^{-1}$

$$\begin{aligned}\mathscr{L}^{-1}\left(\frac{2}{s(s-2)}\right) &= -\mathscr{L}^{-1}\left(\frac{1}{s}\right) + \mathscr{L}^{-1}\left(\frac{1}{s-2}\right)\\ &= -1 + e^{2t} \qquad s > 2\end{aligned}$$

2.6.5.2 Part b

$$F(s) = \frac{2s+2}{s^2+2s+5}$$

Completing the squares in the denominator

$$\begin{aligned}s^2 + 2s + 5 &= (s+A)^2 + B\\ &= s^2 + 2sA + A^2 + B\end{aligned}$$

Comparing coefficients of s shows that

$$\begin{aligned}2A &= 2\\ A^2 + B &= 5\end{aligned}$$

Hence $A = 1$ and $B = 4$. Therefore

$$\begin{aligned}\frac{2s+2}{s^2+2s+5} &= \frac{2s+2}{(s+1)^2+4}\\ &= \frac{2(s+1)}{(s+1)^2+4}\end{aligned}$$

Using the first shifting property (theorem 10.5.1 in book, which says)

$$\mathscr{L}\left(e^{at} f(t)\right) = F(s-a)$$

Then for $a = -1$, we see that

$$\mathscr{L}\left(e^{-t} f(t)\right) = F(s+1) \tag{1}$$

Where $f(t) = \mathscr{L}^{-1}(F(s))$. Therefore we just now need to find $f(t)$ using

$$f(t) = \mathscr{L}^{-1}\left(\frac{2s}{s^2+4}\right)$$

To complete the solution. But $\mathscr{L}^{-1}\left(\frac{s}{s^2+4}\right) = \cos 2t$ for $s > 0$ from Table 10.2.1. Hence

$$\mathscr{L}^{-1}\left(\frac{2s}{s^2+4}\right) = 2\cos(2t) \qquad s > 0$$

Therefore using (1) the final result is given by

$$\begin{aligned}\mathscr{L}^{-1}\left(\frac{2s+2}{s^2+2s+5}\right) &= e^{-t}\mathscr{L}^{-1}\left(\frac{2s}{s^2+4}\right)\\ &= e^{-t}\left(2\cos(2t)\right)\\ &= 2e^{-t}\cos s\,(2t) \qquad s > 0\end{aligned}$$

2.6.6 Problem 5

Use the Laplace transform to solve the following given initial-value problems

a $y' + y = 8e^{3t}, y(0) = 2$

b $y'' + y' - 2y = 10e^{-t}, y(0) = 0, y'(0) = 1$

Solution

2.6.6.1 Part a

Taking the Laplace transform of both sides of $y' + y = 8e^{3t}$ gives (using linearity)

$$\mathscr{L}(y') + \mathscr{L}(y) = 8\mathscr{L}\left(e^{3t}\right)$$

Assuming $\mathscr{L}(y) = Y(s)$, and using the property that $\mathscr{L}(y') = s\mathscr{L}(y) - y(0)$ and from table 10.2.1 $\mathscr{L}\left(e^{3t}\right) = \frac{1}{s-3}, s > 3$, then the above becomes

$$sY(s) - y(0) + Y(s) = \frac{8}{s-3}$$

but $y(0) = 2$, hence the above simplifies to

$$\begin{aligned}sY(s) - 2 + Y(s) &= \frac{8}{s-3}\\ Y(s)(s+1) - 2 &= \frac{8}{s-3}\\ Y(s)(s+1) &= \frac{8}{s-3} + 2\\ Y(s) &= \frac{8}{(s-3)(s+1)} + \frac{2}{(s+1)}\end{aligned} \tag{1}$$

Looking at first term above, and using partial fractions

$$\frac{8}{(s-3)(s+1)} = \frac{A}{(s-3)} + \frac{B}{(s+1)}$$

Therefore

$$A = \left.\frac{8}{(s+1)}\right|_{s=3} = \frac{8}{4} = 2$$

And

$$B = \left.\frac{8}{(s-3)}\right|_{s=-1} = \frac{8}{-4} = -2$$

Therefore (1) becomes

$$\begin{aligned}Y(s) &= \frac{2}{(s-3)} - \frac{2}{(s+1)} + \frac{2}{(s+1)}\\ &= \frac{2}{(s-3)}\end{aligned}$$

From table 10.2.1 $\mathscr{L}\left(e^{3t}\right) = \frac{1}{s-3}, s > 3$. Hence

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left(\frac{2}{(s-3)}\right) \\ &= 2\mathscr{L}^{-1}\left(\frac{1}{(s-3)}\right) \\ &= 2e^{3t} \end{aligned}$$

2.6.6.2 Part b

Taking the Laplace transform of both sides of $y'' + y' - 2y = 10e^{-t}$ gives (using linearity)

$$\mathscr{L}\left(y''\right) + \mathscr{L}\left(y'\right) - 2\mathscr{L}(y) = 10\mathscr{L}\left(e^{-t}\right) \tag{1}$$

Assuming $\mathscr{L}(y) = Y(s)$, and using the property that

$$\mathscr{L}\left(y'\right) = s\mathscr{L}(y) - y(0)$$

And

$$\mathscr{L}\left(y''\right) = s^2\mathscr{L}(y) - sy(0) - y'(0)$$

And from table 10.2.1 $\mathscr{L}\left(e^{-t}\right) = \frac{1}{s+1}, s > -1$, then (1) becomes

$$\begin{aligned} \left(s^2\mathscr{L}(y) - sy(0) - y'(0)\right) + \left(s\mathscr{L}(y) - y(0)\right) - 2\mathscr{L}(y) &= 10\left(\frac{1}{s+1}\right) \\ \left(s^2 Y - s(0) - 1\right) + (sY - 0) - 2Y &= 10\left(\frac{1}{s+1}\right) \\ s^2 Y - 1 + sY - 2Y &= 10\left(\frac{1}{s+1}\right) \\ Y\left(s^2 + s - 2\right) &= 10\left(\frac{1}{s+1}\right) + 1 \\ Y &= \frac{10}{(s+1)\left(s^2+s-2\right)} + \frac{1}{\left(s^2+s-2\right)} \end{aligned}$$

But $\left(s^2 + s - 2\right) = (s+2)(s-1)$. The above becomes

$$Y = \frac{10}{(s+1)(s+2)(s-1)} + \frac{1}{(s+2)(s-1)} \tag{2}$$

Using partial fractions to simplify the above, the first term becomes

$$\frac{10}{(s+1)(s+2)(s-1)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-1}$$

Hence

$$\begin{aligned} A &= \left.\frac{10}{(s+2)(s-1)}\right|_{s=-1} = \frac{10}{(-1+2)(-1-1)} = -5 \\ B &= \left.\frac{10}{(s+1)(s-1)}\right|_{s=-2} = \frac{10}{(-2+1)(-2-1)} = \frac{10}{3} \\ C &= \left.\frac{10}{(s+1)(s+2)}\right|_{s=1} = \frac{10}{(1+1)(1+2)} = \frac{5}{3} \end{aligned}$$

And for the second term in (2)

$$\frac{1}{(s+2)(s-1)} = \frac{D}{(s+2)} + \frac{E}{(s-1)}$$

Hence

$$
\begin{aligned}
D &= \left.\frac{1}{(s-1)}\right|_{s=-2} = \frac{1}{(-2-1)} = -\frac{1}{3} \\
E &= \left.\frac{1}{(s+2)}\right|_{s=1} = \frac{1}{(1+2)} = \frac{1}{3}
\end{aligned}
$$

Using all the above in (2) gives

$$
\begin{aligned}
Y &= \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-1} + \frac{D}{(s+2)} + \frac{E}{(s-1)} \\
&= -5\frac{1}{s+1} + \frac{10}{3}\frac{1}{s+2} + \frac{5}{3}\frac{1}{s-1} - \frac{1}{3}\frac{1}{(s+2)} + \frac{1}{3}\frac{1}{(s-1)} \\
&= -5\frac{1}{s+1} + 3\frac{1}{s+2} + 2\frac{1}{s-1} \qquad (3)
\end{aligned}
$$

But from table 10.2.1

$$
\begin{aligned}
\mathscr{L}^{-1}\left(\frac{1}{s+1}\right) &= e^{-t} \qquad s > -1 \\
\mathscr{L}^{-1}\left(\frac{1}{s+2}\right) &= e^{-2t} \qquad s > -2 \\
\mathscr{L}^{-1}\left(\frac{1}{s-1}\right) &= e^{t} \qquad s > 1
\end{aligned}
$$

Using these results in (3) gives the final solution as

$$
\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left(Y(s)\right) \\
&= -5\mathscr{L}^{-1}\left(\frac{1}{s+1}\right) + 3\mathscr{L}^{-1}\left(\frac{1}{s+2}\right) + 2\mathscr{L}^{-1}\left(\frac{1}{s-1}\right) \\
&= -5e^{-t} + 3e^{-2t} + 2e^{t}
\end{aligned}
$$

Chapter 3

Quizzes

Local contents

3.1 Quizz 1

Local contents

3.1.1 Problem 1

Find the general solution of the following differential equation

$$\frac{dy}{dx} = y + 2xe^{2x}$$

Solution

Writing the ODE as

$$\frac{dy}{dx} - y = 2xe^{2x} \tag{1}$$

Shows it is a linear ode as it has for form $y' + p(x)y = q(x)$ where in this case $p(x) = -1$ and $q(x) = 2xe^{2x}$. The first step is to determine the integrating factor, which is given by

$$\begin{aligned} I &= e^{\int p(x)dx} \\ &= e^{\int -1dx} \\ &= e^{-x} \end{aligned}$$

Multiplying both sides of (1) by this integrating factor gives

$$\begin{aligned} e^{-x}\left(\frac{dy}{dx} - y\right) &= 2xe^{2x}e^{-x} \\ \left(\frac{dy}{dx}e^{-x} - ye^{-x}\right) &= 2xe^{x} \end{aligned}$$

But $\left(\frac{dy}{dx}e^{-x} - ye^{-x}\right) = \frac{d}{dx}\left(ye^{-x}\right)$ by the product rule. Hence the above becomes

$$\begin{aligned} \frac{d}{dx}\left(ye^{-x}\right) &= 2xe^{x} \\ d\left(ye^{-x}\right) &= 2xe^{x}dx \end{aligned}$$

Integrating both sides gives

$$\begin{aligned} \int d\left(ye^{-x}\right) &= \int 2xe^{x}dx \\ ye^{-x} &= 2\int xe^{x}dx + C \end{aligned} \tag{2}$$

Where C is the constant of integration. What is left is to solve the integral $\int xe^{x}dx$. Using integration by parts

$$\int udv = uv - \int vdu$$

Let $u = x, dv = e^{x}dx$, therefore $du = dx$ and $v = e^{x}$. Therefore

$$\begin{aligned} \int xe^{x}dx &= uv - \int vdu \\ &= xe^{x} - \int e^{x}dx \end{aligned}$$

But $\int e^{x}dx = e^{x}$. Hence the above becomes

$$\int xe^{x}dx = xe^{x} - e^{x} \tag{3}$$

Note that a constant of integration is not needed in (3), since constant of integration was already added in (2) earlier. Substituting (3) in (2) gives

$$ye^{-x} = 2\left(xe^{x} - e^{x}\right) + C$$

Solving for y from the above (by multiplying both sides of the above equation by e^{x}), gives the general solution

$$\begin{aligned} y &= 2\left(xe^{x} - e^{x}\right)e^{x} + Ce^{x} \\ &= 2\left(xe^{2x} - e^{2x}\right) + Ce^{x} \end{aligned}$$

Therefore

$$y(x) = 2e^{2x}(x-1) + Ce^{x}$$

The solution contains one constant of integration as expected since the order of the ODE is one.

3.1.2 Problem 2

Solve

$$\frac{dy}{dx} = 2xy\left(4-y\right)$$

Solution

This ode is separable because it can be written as

$$y' = p(y)q(x)$$

Where the function $p(y)$ depends explicitly on y only and the function $q(x)$ depends on x only. In the given ODE, $p(y) = y\left(4-y\right)$ and $q(x) = 2x$. Hence the ODE can be be written as

$$\begin{aligned}\frac{dy}{dx} &= p(y)q(x)\\ \frac{1}{p(y)}dy &= q(x)dx \qquad y \neq 4\end{aligned}$$

Integrating both sides gives

$$\int \frac{1}{p(y)}dy = \int q(x)dx$$

Replacing $p(y) = y\left(4-y\right)$ and $q(x) = 2x$ into the above gives

$$\int \frac{1}{y\left(4-y\right)}dy = \int 2xdx \tag{1}$$

The integral on the right side is

$$\begin{aligned}\int 2xdx &= \frac{2x^2}{2} + C_1\\ &= x^2 + C_1 \end{aligned} \tag{2}$$

Where C_1 is the constant of integration. The integral on the left side is solved using partial fractions. Let

$$\begin{aligned}\frac{1}{y\left(4-y\right)} &= \frac{A}{y} + \frac{B}{4-y}\\ &= \frac{A\left(4-y\right)+By}{y\left(4-y\right)}\\ &= \frac{4A-yA+By}{y\left(4-y\right)}\\ &= \frac{4A-y\left(A-B\right)}{y\left(4-y\right)}\end{aligned}$$

Comparing the numerators shows that

$$1 = 4A - y\left(A-B\right)$$

Which implies, by comparing coefficients of y on each side that $4A = 1$ and $(A-B) = 0$. This means $A = \frac{1}{4}$ and $B = \frac{1}{4}$. Therefore

$$\begin{aligned}\frac{1}{y\left(4-y\right)} &= \frac{1}{4}\frac{1}{y} + \frac{1}{4}\frac{1}{4-y}\\ &= \frac{1}{4}\frac{1}{y} - \frac{1}{4}\frac{1}{y-4}\end{aligned}$$

Hence the left integral in (1) now can be written as

$$\int \frac{1}{y\left(4-y\right)}dy = \frac{1}{4}\int \frac{1}{y}dy - \frac{1}{4}\int \frac{1}{y-4}dy \tag{3}$$

But $\int \frac{1}{y}dy = \ln\left|y\right|$. To find $\int \frac{1}{y-4}dy$, let $y-4=u$. Hence $dy = du$. Therefore

$$\begin{aligned}\int \frac{1}{y-4}dy &= \int \frac{1}{u}du\\ &= \ln\left|u\right|\\ &= \ln\left(\left|y-4\right|\right)\end{aligned}$$

Substituting all of this back in (3) gives

$$\begin{aligned}\int \frac{1}{y\left(4-y\right)}dy &= \frac{1}{4}\ln\left|y\right| - \frac{1}{4}\ln\left(\left|y-4\right|\right)\\ &= \frac{1}{4}\left(\ln\left|y\right| - \ln\left(\left|4-y\right|\right)\right)\\ &= \frac{1}{4}\ln\left|\frac{y}{4-y}\right| \qquad (4)\end{aligned}$$

Now that both integrals are found, substituting (4) and (2) back in (1) gives

$$\begin{aligned}\frac{1}{4}\ln\left|\frac{y}{4-y}\right| &= x^2 + C_1\\ \ln\left|\frac{y}{4-y}\right| &= 4x^2 + 4C_1\end{aligned}$$

Let $4C_1 = C_2$, a new constant.

$$\ln\left|\frac{y}{4-y}\right| = 4x^2 + C_2$$

Raising both sides to exponential gives

$$\begin{aligned}\left|\frac{y}{4-y}\right| &= e^{4x^2+C_2}\\ &= e^{C_2}e^{4x^2}\end{aligned}$$

Let $e^{C_2} = C$ a new constant. The above becomes

$$\frac{y}{4-y} = Ce^{4x^2}$$

The absolute on the left side can be removed by letting the new constant C absorbs the sign for either positive or negative..

Solving for y from the above in order to obtain an explicit solution gives

$$\begin{aligned}y &= \left(4-y\right)Ce^{4x^2}\\ &= 4Ce^{4x^2} - yCe^{4x^2}\\ y + yCe^{4x^2} &= 4Ce^{4x^2}\\ y\left(1+Ce^{4x^2}\right) &= 4Ce^{4x^2}\end{aligned}$$

Hence the solution is

$$y = \frac{4Ce^{4x^2}}{1+Ce^{4x^2}}$$

The above can be simplified further by dividing numerator and denominator by $Ce^{4x^2} \neq 0$ which gives

$$y = \frac{4}{\frac{1}{C}e^{-4x^2}+1}$$

Let $\frac{1}{C} = C_0$ a new constant. Hence the final general solution becomes

$$y(x) = \frac{4}{1 + C_0 e^{-4x^2}}$$

Where C_0 is the constant of integration which can be found from initial conditions when given.

3.1.3 Problem 3

Solve

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$$

Solution

This is homogeneous ODE. Dividing both the numerator and denominator of the right side by $xy \neq$ gives

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{x^2}{xy} + 3\frac{y^2}{xy}}{2\frac{xy}{xy}} \\
&= \frac{\frac{x}{y} + 3\frac{y}{x}}{2} \qquad (1)
\end{aligned}$$

Let $v(x) = \frac{y(x)}{x}$ be a new dependent variable. Hence $y = vx$. By the product rule

$$\frac{dy}{dx} = v + \frac{dv}{dx}x$$

Substituting $v = \frac{y}{x}$ and replacing $\frac{dy}{dx}$ by the above into (1) gives a new ODE in $v(x)$ which is now separable

$$\begin{aligned}
v + \frac{dv}{dx}x &= \frac{\frac{1}{v} + 3v}{2} \\
\frac{dv}{dx}x &= \frac{\frac{1}{v} + 3v}{2} - v \\
&= \frac{\frac{1+3v^2}{v}}{2} - v \\
&= \frac{1 + 3v^2}{2v} - v \\
&= \frac{1 + 3v^2 - 2v^2}{2v} \\
&= \frac{1 + v^2}{2v}
\end{aligned}$$

The above ODE is separable. It can be written as

$$\begin{aligned}
\frac{2v}{1 + v^2}\frac{dv}{dx} &= \frac{1}{x} \qquad x \neq 0 \\
\frac{2v}{1 + v^2}dv &= \frac{dx}{x}
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\int \frac{2v}{1 + v^2}dv &= \int \frac{dx}{x} \\
&= \ln|x| + C_1 \qquad (2)
\end{aligned}$$

Where C_1 is constant of integration. To evaluate the left integral $\int \frac{2v}{1+v^2}dv$, let $u = 1 + v^2$. Hence $du = 2vdv$. Therefore

$$\begin{aligned}\int \frac{2v}{1+v^2}dv &= \int \frac{2v}{u}\left(\frac{du}{2v}\right)\\ &= \int \frac{du}{u}\\ &= \ln|u|\\ &= \ln\left|1+v^2\right|\end{aligned}$$

But $\left|1+v^2\right|$ is always positive so the absolute sign is not needed. Therefore

$$\int \frac{2v}{1+v^2}dv = \ln\left(1+v^2\right)$$

Substituting the above in (2) gives

$$\ln\left(1+v^2\right) = \ln|x| + C_1$$

Raising both sides to exponential gives

$$\begin{aligned}1+v^2 &= e^{\ln|x|+C_1}\\ &= xe^{C_1}\end{aligned}$$

Since exponential function is never negative. Let $e^{C_1} = C$ be a new constant. The above becomes

$$\begin{aligned}1+v^2 &= Cx\\ v^2 &= Cx-1\\ v &= \pm\sqrt{Cx-1}\end{aligned}$$

But $v = \frac{y}{x}$. Hence the above becomes

$$\frac{y}{x} = \pm\sqrt{Cx-1}$$

Which implies

$$y = \pm x\sqrt{Cx-1} \qquad x \neq 0$$

There are two solutions. They are

$$\begin{aligned}y_1(x) &= x\sqrt{Cx-1}\\ y_2(x) &= -x\sqrt{Cx-1}\end{aligned}$$

3.1.4 Problem 4

A tank initially contains 120 L of pure water. A mixture containing of α g/L of salt enters the tank at a rate of 2 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of α for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$

Solution

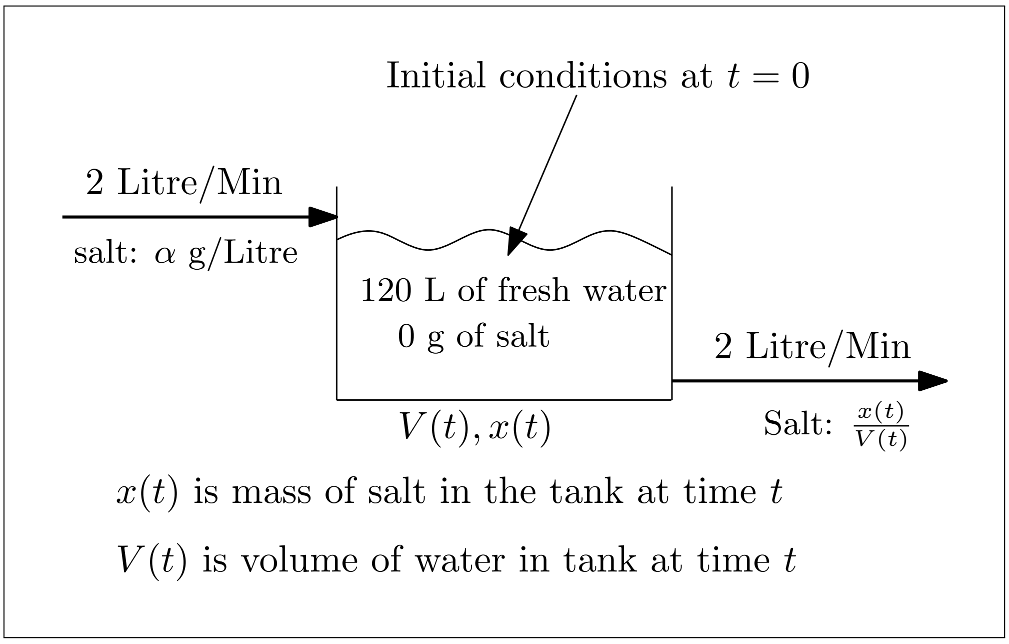


Figure 3.1: Showing tank flow and initial conditions

Let $x(t)$ be mass (amount) of salt (in grams) in the tank at time t. Let $V(t)$ be the volume of water (mixture) (in litre) in the tank at time t. Using the equilibrium equation for change of mass of salt

$$\frac{dx}{dt} = \text{rate of salt mass in} - \text{rate of salt mass out}$$

Which becomes

$$\begin{aligned}\frac{dx}{dt} &= \left(2\,\frac{\text{L}}{\text{min}}\right)\left(\alpha\,\frac{\text{g}}{\text{L}}\right) - \left(2\,\frac{\text{L}}{\text{min}}\right)\left(\frac{x(t)}{V(t)}\,\frac{\text{g}}{\text{L}}\right)\\ &= 2\alpha - 2\frac{x(t)}{V(t)} \qquad (1)\end{aligned}$$

But

$$\begin{aligned}V(t) &= V(0) + (\text{rate of mixture volume in} - \text{rate of mixture volume out})\,t\\ &= V(0) + (2-2)\,t\\ &= V(0)\end{aligned}$$

But we are given that $V(0) = 120$ L. Hence

$$V(t) = 120$$

Which means the volume of mixture remains constant in the tank (this is as expected, since rate of flow in is the same as rate of flow out). Substituting the above in (1) gives the ODE to solve

$$\begin{aligned}\frac{dx}{dt} &= 2\alpha - 2\frac{x}{120}\\ &= 2\alpha - \frac{x}{60}\end{aligned}$$

The solution to above ODE gives the mass $x(t)$ of salt in tank at time t.

$$\frac{dx}{dt} + \frac{x}{60} = 2\alpha \qquad (2)$$

This is linear ODE as it has the form $x' + p(t)x = q(t)$. The integrating factor is $I = e^{\int p(t)dt} = e^{\int \frac{1}{60}dt} = e^{\frac{t}{60}}$. Multiplying both sides of the above ODE (2) by this integrating factor gives

$$\begin{aligned}e^{\frac{t}{60}}\left(\frac{dx}{dt} + \frac{x}{60}\right) &= 2\alpha e^{\frac{t}{60}}\\ \left(\frac{dx}{dt}e^{\frac{t}{60}} + \frac{x}{60}e^{\frac{t}{60}}\right) &= 2\alpha e^{\frac{t}{60}}\end{aligned}$$

But $\left(\frac{dx}{dt}e^{\frac{t}{60}} + \frac{x}{60}e^{\frac{t}{60}}\right) = \frac{d}{dt}\left(xe^{\frac{t}{60}}\right)$ by the product rule. Hence the above becomes

$$\frac{d}{dt}\left(xe^{\frac{t}{60}}\right) = 2\alpha e^{\frac{t}{60}}$$

Integrating both sides gives

$$\begin{aligned}\int d\left(xe^{\frac{t}{60}}\right) &= \int 2\alpha e^{\frac{t}{60}}dt\\ xe^{\frac{t}{60}} &= 2\alpha\int e^{\frac{t}{60}}dt + C\end{aligned}$$

Where C is the constant of integration. To evaluate $\int e^{\frac{t}{60}}dt$, let $\frac{t}{60} = u$, then $\frac{1}{60}dt = du$. Hence $\int e^{\frac{t}{60}}dt = \int e^{u}(60du) = 60\int e^{u}du = 60e^{u}$ or $60e^{\frac{t}{60}}$. Therefore the above becomes

$$\begin{aligned}xe^{\frac{t}{60}} &= 2\alpha\left(60e^{\frac{t}{60}}\right) + C\\ &= 120\alpha e^{\frac{t}{60}} + C\end{aligned}$$

Multiplying both sides by $e^{-\frac{t}{60}}$ gives the general solution as

$$x(t) = 120\alpha + Ce^{\frac{-t}{60}} \tag{3}$$

Initial conditions are now used to find C. At $t = 0$, we are given that $x(0) = 0$ since there was no salt in the tank initially. Hence the above becomes at $t = 0$

$$\begin{aligned}0 &= 120\alpha + C\\ C &= -120\alpha\end{aligned}$$

Therefore (3) becomes the particular solution given by

$$x(t) = 120\alpha - 120\alpha e^{\frac{-t}{60}} \tag{4}$$

To answer the final part, as $t \to \infty$ then $e^{\frac{-t}{60}} \to e^{-\infty} \to 0$ and the above gives

$$\lim_{t\to\infty} x(t) = 120\alpha$$

In grams. The above is the limiting mass (amount) of salt in the tank. For example if $\alpha = 1$ grams per liter, then the maximum possible mass of salt in the tank will be 120 grams. The amount of salt is initially zero in the tank, and increases exponentially before leveling off at the limit given by 120α grams. The following is a plot that illustrates this for $\alpha = 1$.

```
restart;
x:=(alpha,t)->120*alpha-120*alpha*exp(-t/60):
plot(x(1,t),t=0..500,gridlines=true,axes=boxed,color=blue,labels=["time (sec)
    ","salt (g)"],view=[default, 0..125], labelfont = ["TimesNewRoman", 16])
```

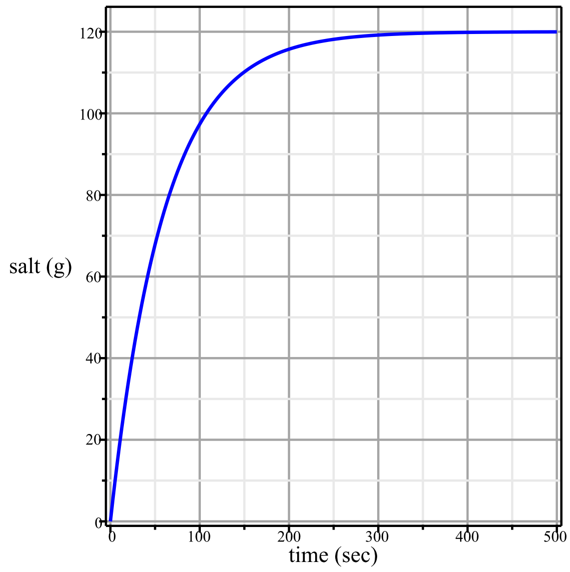


Figure 3.2: Showing limiting value of amount of salt for $\alpha = 1$

3.2 Quizz 2

Local contents

3.2.1 Problem 1

Let $\vec{v} = \begin{bmatrix} 5 \\ 3 \\ -6 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$ be in $\mathbb{R}^3$. Let $W = span\,(v_1, v_2)$. Determine if $\vec{v}$ is in W

Solution

$\vec{v}$ is in W if $\vec{v}$ can be expressed as a linear combination of basis $\vec{v}_1, \vec{v}_2$. To find if this is the case, we solve

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{v}$$

If a solution exists, then $\vec{v}$ is in W. Writing the above in matrix form gives

$$c_1 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 \\ 1 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -6 \end{bmatrix} \tag{1}$$

Hence the augmented matrix is

$$\begin{bmatrix} -1 & 3 & 5 \\ 1 & 1 & 3 \\ 2 & -4 & -6 \end{bmatrix}$$

$R_2 = R_2 + R_1$ gives

$$\begin{bmatrix} -1 & 3 & 5 \\ 0 & 4 & 8 \\ 2 & -4 & -6 \end{bmatrix}$$

$R_3 = R_3 + 2R_1$ gives

$$\begin{bmatrix} -1 & 3 & 5 \\ 0 & 4 & 8 \\ 0 & 2 & 4 \end{bmatrix}$$

$R_3 = R_3 - \frac{R_2}{2}$ gives

$$\begin{bmatrix} -1 & 3 & 5 \\ 0 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

The pivots columns are the first two columns. The system (1) now becomes

$$\begin{bmatrix} -1 & 3 \\ 0 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 0 \end{bmatrix}$$

Last row provides no information as It just says $0 = 0$. Second row gives $4c_2 = 8$ or $c_2 = 2$. First row gives $-c_1 + 3c_2 = 5$ or $-c_1 = 5 - 3(2)$ or $c_1 = 1$. Hence the solution is

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Since a solution is found, this means $\vec{v}$ can be expressed as a linear combination of $\vec{v}_1, \vec{v}_2$ given by

$$\vec{v}_1 + 2\vec{v}_2 = \vec{v}$$

Hence $\vec{v}$ is in W.

3.2.2 Problem 2

Determine the component vector of $p(x) = -4x^2 + 2x + 6$ in the given vector space $V = P_2(\mathbb{R})$ relative to the given ordered basis $B = \left\{x^2 + x, 2 + 2x, 1\right\}$

Solution

The problem is asking us to find c_1, c_2, c_3 such that

$$\begin{aligned} c_1\left(x^2 + x\right) + c_2(2 + 2x) + c_3(1) &= p(x) \\ &= -4x^2 + 2x + 6 \end{aligned}$$

Expanding gives

$$\begin{aligned} c_1x^2 + c_1x + 2c_2 + 2c_2x + c_3 &= -4x^2 + 2x + 6 \\ x^2(c_1) + x(c_1 + 2c_2) + (2c_2 + c_3) &= -4x^2 + 2x + 6 \end{aligned}$$

For these to be equal, the corresponding coefficients of the polynomials must be the same. Therefore equating coefficients of each power of x gives

$$\begin{aligned} c_1 &= -4 \\ c_1 + 2c_2 &= 2 \\ 2c_2 + c_3 &= 6 \end{aligned}$$

Or

$$\begin{aligned} c_1 &= -4 \\ -4 + 2c_2 &= 2 \\ 2c_2 + c_3 &= 6 \end{aligned}$$

Or

$$\begin{aligned} c_1 &= -4 \\ c_2 &= 3 \\ 2c_2 + c_3 &= 6 \end{aligned}$$

Or

$$\begin{aligned} c_1 &= -4 \\ c_2 &= 3 \\ 2(3) + c_3 &= 6 \end{aligned}$$

Or

$$\begin{aligned} c_1 &= -4 \\ c_2 &= 3 \\ c_3 &= 0 \end{aligned}$$

Hence the component vector of $p(x)$ is $\{-4, 3, 0\}$ relative to the basis B.

3.2.3 Problem 3

Determine if the given linear transformation is a) one-to-one and b) onto. Justify your answer.

$$T\left(x, y, z\right) = \left(x, x + y + z\right)$$

Solution

In Matrix forum the above is

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ x+y+z \end{bmatrix}$$

Therefore A matrix must have dimensions 2×3. The above becomes

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ x+y+z \end{bmatrix}$$

First row gives $a_{11}x + a_{12}y + a_{13}z = x$. Equating coefficients of the polynomials on each side gives $a_{11} = 1, a_{12} = 0, a_{13} = 0$. Second row gives $a_{21}x + a_{22}y + a_{23}z = x + y + z$ which implies that $a_{21} = 1, a_{22} = 1, a_{23} = 1$. Hence the matrix A is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Now that we have found the matrix representation of the linear transformation T, we can answer parts a and b.

Using Theorem 6.4.8 which says for the linear transformation $T : V \to W$

1. one-to-one iff $\ker(T) = \{\vec{0}\}$
2. onto iff $Rng(T) = W$

In this problem V is $\mathbb{R}^3$ and W is $\mathbb{R}^2$. To show one-to-one, we need to find $\ker(T)$ by solving $A\vec{x} = \vec{0}$ and check if it is the zero vector or not.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$R_2 = R_2 - R_1$ gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Therefore the base variables are x, y and the free variable is $z = t$. Hence (2) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

First row gives $x = 0$ and second row gives $y + z = 0$ or $y = -t$ Therefore the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix}$$

$$= t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Therefore null-space is $\left\{\begin{bmatrix}0\\-1\\1\end{bmatrix}\right\}$. Since null-space is not the zero vector, then T is not one-to-one.

To check if it is onto, the $Rng(T)$ is the column space of A. From the above RREF, we found that the first two columns are the pivot columns. These correspond to the first two columns of A. Therefore

$$Rng(T) = \left\{\vec{v} \in \mathbb{R}^3 : \vec{v} = c_1 \begin{bmatrix}1\\1\end{bmatrix} + c_2 \begin{bmatrix}0\\1\end{bmatrix}, c_1, c_2 \in \mathbb{R}\right\}$$

The question now is, does the above span all of W which is $\mathbb{R}^2$? it is clear it does, since $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ are linearly independent of each others and any two linearly independent vectors in $\mathbb{R}^2$ span all of $\mathbb{R}^2$. Hence it is onto.

If we need to show this, then this can be done by solving

$$c_1 \begin{bmatrix}1\\1\end{bmatrix} + c_2 \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

First row gives $c_1 = 0$ and second row gives $c_2 = 0$. Hence only solution to $c_1 \begin{bmatrix}1\\1\end{bmatrix} + c_2 \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ is $c_1 = c_2 = 0$. Therefore these two vectors are linearly independent vectors in $\mathbb{R}^2$. Hence $Rng(T)$ is all W. Therefore T is onto.

3.2.4 Problem 4

Given the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by the equation, find the standard matrix for the inverse transformation T^{-1}

$$\begin{aligned} w_1 &= x_1 + 4x_2 - x_3 \\ w_2 &= 2x_1 + 7x_2 + x_3 \\ w_3 &= x_1 + 3x_2 \end{aligned}$$

Solution

In matrix form the above is

$$\begin{bmatrix}1 & 4 & -1\\2 & 7 & 1\\1 & 3 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}w_1\\w_2\\w_2\end{bmatrix}$$

Where $A = \begin{bmatrix}1 & 4 & -1\\2 & 7 & 1\\1 & 3 & 0\end{bmatrix}$ represents the linear transformation T. Hence T^{-1} is represented by A^{-1}. Therefore we need to find A^{-1}. Setting up the augmented matrix for finding the inverse is setup by adding the identity matrix to the right half as follows

$$\begin{bmatrix}1 & 4 & -1 & 1 & 0 & 0\\2 & 7 & 1 & 0 & 1 & 0\\1 & 3 & 0 & 0 & 0 & 1\end{bmatrix}$$

$R_2 = R_2 - 2R_1$

$$\begin{bmatrix} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 - R_1$

$$\begin{bmatrix} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$\begin{bmatrix} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 & -1 & 1 \end{bmatrix}$$

$R_3 = \frac{-R_3}{2}, R_2 = -R_2$

$$\begin{bmatrix} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$$

Now we start the reduced Echelon phase. $R_2 = R_2 + 3R_3$

$$\begin{bmatrix} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2+3\left(\frac{-1}{2}\right) & -1+3\left(\frac{1}{2}\right) & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$R_1 = R_1 + R_3$

$$\begin{bmatrix} 1 & 4 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$R_1 = R_1 - 4R_2$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}-4\left(\frac{1}{2}\right) & \frac{1}{2}-4\left(\frac{1}{2}\right) & -\frac{1}{2}-4\left(-\frac{3}{2}\right) \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Now that the left half above is the identity matrix, then the right half is A^{-1}. Therefore

$$A^{-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Which is the standard matrix for the representation of T^{-1}

3.2.5 Problem 5

Answer the following questions by writing TRUE or FALSE.

Solution

a The rank of a matrix equals the dimension of its column space. **TRUE**

b The number of variables in the equations $Ax = 0$ equals the dimension of the null-sapce of A. **FALSE**

c If A is 3×5 matrix and T is a transformation defined by $T(x) = Ax$, then the domain of T is $\mathbb{R}^3$. **FALSE**

d If a 4×7 matrix A has four pivot columns then the $\mathit{nullity}(A) = 3$. **TRUE**

e A linearly independent set in a subspace H is a basis for H. **FALSE**

3.3 Quizz 3

Local contents

3.3.1 Problem 1

Determine whether or not the given matrix A is diagonalizable. If it is find a diagonalizing matrix P and a diagonal matrix D such that $P^{-1}AP = D$

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

Solution

The first step is to find the eigenvalues. This is found by solving

$$\det(A - \lambda I) = 0$$
$$\begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix} = 0$$
$$(3-\lambda)(-1-\lambda) = 0$$

Hence the eigenvalues are $\lambda_1 = 3, \lambda_2 = -1$. Since the eigenvalues are unique, then the matrix is diagonalizable. We need to determine the corresponding eigenvector in order to find P

$\lambda_1 = 3$

Solving

$$\begin{bmatrix} 3-\lambda_1 & 0 \\ 8 & -1-\lambda_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3-3 & 0 \\ 8 & -1-3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence $v_1 = t$ is free variable and v_2 is base variable. Second row gives $8t - 4v_2 = 0$ or $v_2 = 2t$. Therefore the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Let $t = 1$, then

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\lambda_2 = -1$

Solving

$$\begin{bmatrix} 3-\lambda_2 & 0 \\ 8 & -1-\lambda_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3+1 & 0 \\ 8 & -1+1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 0 \\ 8 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$ gives

$$\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence $v_2 = t$ is free variable and v_1 is base variable. First row gives $4v_1 = 0$ or $v_1 = 0$. Therefore the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$, then

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Now that both eigenvectors are found, then

$$\begin{aligned} P &= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \end{aligned}$$

And D is the diagonal matrix of the eigenvalues arranged in same order as the corresponding eigenvectors. (Will verify below). Hence

$$D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

Therefore

$$P^{-1}AP = D$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \tag{1}$$

To verify the above, the LHS of (1) is evaluated directly, to confirm that D is indeed the result and it is diagonal of the eigenvalues. The first step is to find $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1}$. Since this is 2×2 then

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{\det(P)} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

But $\det(P) = 1$. The above becomes

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

Therefore the LHS of (1) becomes

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 3 & 0 \\ -2 \times 3 + 1 \times 8 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ -6 + 8 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} \end{aligned}$$

And now LHS of (1) becomes

$$\begin{bmatrix}3 & 0\\2 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix} = \begin{bmatrix}3 & 0\\2\times 1-1\times 2 & -1\end{bmatrix}$$
$$= \begin{bmatrix}3 & 0\\0 & -1\end{bmatrix}$$

Hence

$$D = \begin{bmatrix}3 & 0\\0 & -1\end{bmatrix}$$

Which confirms D is the matrix whose diagonal elements are the eigenvalues of A.

3.3.2 Problem 2

Find the general solution of the homogeneous differential equation $y''' + y' - 10y = 0$

Solution

This is a linear 3rd order constant coefficient ODE. Hence the method of characteristic equation will be used. Let the solution be $y = Ae^{\lambda x}$. Substituting this into the ODE gives

$$A\lambda^3 e^{\lambda x} + A\lambda e^{\lambda x} - 10Ae^{\lambda x} = 0$$
$$Ae^{\lambda x}\left(\lambda^3 + \lambda - 10\right) = 0$$

Which simplifies (for non-trivial y) to the characteristic equation which is a polynomial in λ

$$\lambda^3 + \lambda - 10 = 0$$

By inspection, we see that $\lambda = 2$ is a root. Therefore a factor of the equation is $(\lambda - 2)$. Now doing long division $\frac{\lambda^3+\lambda-10}{(\lambda-2)}$ gives $\lambda^2 + 2\lambda + 5$.

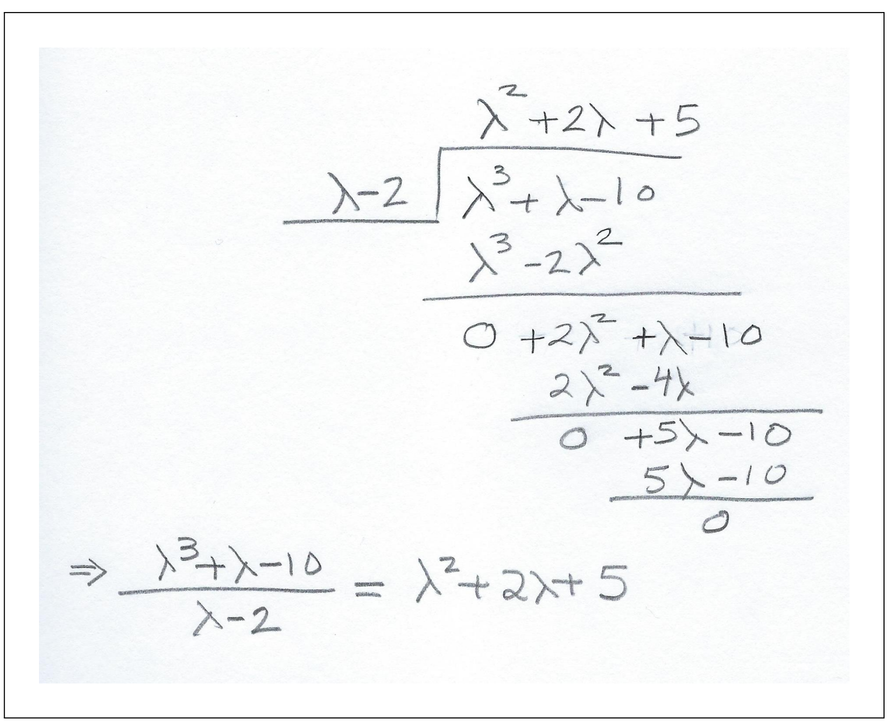


Figure 3.3: Polynmial long division to find remainder

Hence the above polynomial can be written as

$$(\lambda - 2)\left(\lambda^2 + 2\lambda + 5\right) = 0 \tag{1}$$

Now the roots for $\left(\lambda^2+2\lambda+5\right)=0$ are found using the quadratic formula.

$$\begin{aligned}\lambda &= -\frac{b}{2a}\pm\frac{1}{2a}\sqrt{b^2-4ac}\\ &= -\frac{2}{2}\pm\frac{1}{2}\sqrt{4-4\times 5}\\ &= -1\pm\frac{1}{2}\sqrt{-16}\\ &= -1\pm\frac{1}{2}(4i)\\ &= -1\pm 2i\end{aligned}$$

Hence the roots of the characteristic equation are

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -1+2i\\ \lambda_3 &= -1-2i\end{aligned}$$

Therefore the basis solution are $\left\{e^{2x}, e^{(-1+2i)x}, e^{(-1-2i)x}\right\}$ and the general solution is a linear combination of these basis solutions which gives

$$y = Ae^{2x} + Be^{(-1+2i)x} + Ce^{(-1-2i)x}$$

Which can be simplified to

$$y = Ae^{2x} + e^{-x}\left(Be^{2ix} + Ce^{-2ix}\right) \tag{2}$$

By using Euler formula, the above can be simplified further as follows

$$\begin{aligned}Be^{2ix}+Ce^{-2ix} &= B\left(\cos(2x)+i\sin(2x)\right)+C\left(\cos(2x)-i\sin(2x)\right)\\ &= (B+C)\cos(2x)+\sin(2x)\left(i(B-C)\right)\end{aligned}$$

Let $(B+C)=B_0$ a new constant and let $i(B-C)=C_0$ a new constant, the above becomes

$$Be^{2ix}+Ce^{-2ix} = B_0\cos(2x)+C_0\sin(2x)$$

Substituting the above back in (2) gives the general solution as

$$y = Ae^{2x}+e^{-x}\left(B_0\cos(2x)+C_0\sin(2x)\right)$$

The constants A, B_0, C_0 can be found from initial conditions if given.

3.3.3 Problem 3

Using the method of undermined coefficients, compute the general solution of the given equation $y''+3y'+2y=2\sin(x)$

Solution

The solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $y''+3y'+2y=0$ and y_p is a particular solution to the given ODE. The ode $y''+3y'+2y=0$ is linear second order constant coefficient ODE. Hence the method of characteristic equation will be used. Let the solution be $y_h=Ae^{\lambda x}$. Substituting this into $y''+3y'+2y=0$

$$\begin{aligned}A\lambda^2e^{\lambda x}+A\lambda 3e^{\lambda x}+2Ae^{\lambda x} &= 0\\ Ae^{\lambda x}\left(\lambda^2+3\lambda+2\right) &= 0\end{aligned}$$

And for non trivial solution the above simplifies to

$$\begin{aligned}\lambda^2+3\lambda+2 &= 0\\ (\lambda+1)(\lambda+2) &= 0\end{aligned}$$

Hence the roots are $\lambda = -1, \lambda = -2$. Therefore the basis solutions for y_h are $\left\{e^{-x}, e^{-2x}\right\}$ and y_h is linear combination of these basis. Therefore

$$y_h = c_1 e^{-x} + c_2 e^{-2x} \tag{1}$$

Now y_p is found. Since the RHS is $\sin(x)$ then the trial solution is

$$y_p = A\cos(x) + B\sin(x) \tag{2}$$

This shows that the basis for y_p are $\{\sin x, \cos x\}$. There are no duplication between these basis and the basis for y_h, so no need to multiply by an extra x. Using (2) gives

$$y_p' = -A\sin(x) + B\cos(x) \tag{3}$$
$$y_p'' = -A\cos(x) - B\sin(x) \tag{4}$$

Substituting (2,3,4) back into the given ODE gives

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= 2\sin(x) \\ (-A\cos(x) - B\sin(x)) + 3(-A\sin(x) + B\cos(x)) + 2(A\cos(x) + B\sin(x)) &= 2\sin(x) \\ \cos(x)(-A + 3B + 2A) + \sin(x)(-B - 3A + 2B) &= 2\sin(x) \\ \cos(x)(3B + A) + \sin(x)(-3A + B) &= 2\sin(x) \end{aligned}$$

Comparing coefficients on both sides gives two equations to solve for A, B

$$\begin{aligned} 3B + A &= 0 \\ -3A + B &= 2 \end{aligned}$$

Multiplying the second equation by -3 gives

$$\begin{aligned} 3B + A &= 0 \\ 9A - 3B &= -6 \end{aligned}$$

Adding the above two equations gives $10A = -6$. Hence $A = -\frac{3}{5}$ and therefore $3B = \frac{3}{5}$ or $B = \frac{1}{5}$. Substituting these values of A, B into (2) gives

$$y_p = -\frac{3}{5}\cos(x) + \frac{1}{5}\sin(x)$$

Hence the solution becomes

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x}\right) + \left(-\frac{3}{5}\cos(x) + \frac{1}{5}\sin(x)\right) \\ &= c_1 e^{-x} + c_2 e^{-2x} - \frac{3}{5}\cos(x) + \frac{1}{5}\sin(x) \end{aligned}$$

3.3.4 Problem 4

Show that the given vector functions are linearly independent

$$\vec{x}_1(t) = \begin{bmatrix} e^t \\ 2e^t \end{bmatrix} \qquad \vec{x}_2(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

Solution

These functions are defined for all t. Hence domain is $t \in (-\infty, \infty)$. The Wronskian of these vectors is

$$\begin{aligned} W(t) &= \begin{vmatrix} e^t & \sin t \\ 2e^t & \cos t \end{vmatrix} \\ &= e^t\cos t - 2e^t\sin t \\ &= e^t(\cos t - 2\sin t) \end{aligned}$$

We just need find one value t_0 where $W(t_0) \neq 0$ to show linearly independence. At $t = 0$ the above becomes

$$W(t = 0) = 1$$

Therefore the given vector functions are linearly independent. An <u>alternative method</u> is to write

$$c_1\vec{x}_1(t) + c_2\vec{x}_2(t) = \vec{0}$$

$$c_1\begin{bmatrix} e^t \\ 2e^t \end{bmatrix} + c_2\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If the above is true only for $c_1 = 0, c_2 = 0$ then $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent. The above can be written as

$$\begin{bmatrix} e^t & \sin t \\ 2e^t & \cos t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$

$$\begin{bmatrix} e^t & \sin t \\ 0 & \cos t - 2\sin t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Row two gives

$$c_2(\cos t - 2\sin t) = 0$$

For this to be true for <u>any</u> t in the interval $t \in (-\infty, \infty)$, then only solution is $c_2 = 0$. First row now gives

$$c_1 e^t = 0$$

But e^t is never zero which means $c_1 = 0$.

Since the only solution to $c_1\vec{x}_1 + c_2\vec{x}_2 = \vec{0}$ is $c_1 = c_2 = 0$, then this shows that $\vec{x}_1, \vec{x}_2$ are linearly independent.

Chapter 4

Exams

Local contents

4.1 Mid term exam

Local contents

4.1.1 Problems listing

Math2520 **Calculus IV** **Mid-Semester Exam.**

Name: ______________________________________

INSTRUCTION:

1. Write you name on the answer sheet.
2. Write clearly and legibly (bigger and darker) so that it easy to read when printed.
3. You can only post once, so make sure that all the pages/questions are posted.
4. You can use your own paper if you cannot print it.

1. Solve the initial-value problem. (**4 pts**)

$$x\frac{dy}{dx} - y = 2x^2 y, \quad y(1) = 1$$

2. Solve: (**5 pts**)

   $$x^2 \frac{dy}{dx} + 2xy - y^3 = 0, x > 0$$

2

3. Verify that the given differential equation is exact; then solve it. (**6 pts**)

$$(x^3 + \frac{y}{x})dx + (y^2 + \ln x)dy = 0$$

4. a) Solve the initial value problem (**4 pts**)

$$\frac{dy}{dx} = 3 + x - y, \quad y(0) = 1$$

b) Apply Euler's methods to the initial value problem with step size h = 0.1 and complete the following table. You can use calculator or excel. (**4 pts**)

x	Euler method y	Exact y	Absolute Error
0.1			
0.2			
0.3			
0.4			

5. Solve the following system of equations and write the solution in parametric vector form. (**4 pts**)

$$x_1 + 2x_2 + x_3 = 1$$
$$2x_1 - x_2 + 2x_3 = 2$$
$$3x_1 + x_2 + 3x_3 = -8$$

6. Given the matrix $A = \begin{bmatrix} 3 & 4 \\ 4 & -2 \end{bmatrix}$, **(5 pts)**

   a) Find A^{-1} , the inverse matrix of *A*.

   b) Use A^{-1} to solve the system of equations

$$\begin{aligned} 3x + 4y &= 7 \\ 4x - 2y &= 5 \end{aligned}$$

7. Use the cofactor expansion to evaluate the given determinant along the 2nd row.

$$\begin{vmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{vmatrix}$$

8. Let H be the set of points in the $xy-plane$ given by,

$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$. Show that H is not a subspace of R^2. (**3 pts**)

9. Determine if the set of vectors span R^3. Justify our answer. **(3 pts)**

$$\{(1,-2,1),(2,3,1),(4,-1,2)\}$$

10. Mark each statement **TRUE** or **FALSE**. (**5 pts**)

a) An integrating factor for the differential equation $\frac{dy}{dx} = x^2 y$ is $e^{\int x^2 dx}$. ______________

b) The equation $Ax = 0$ has the nontrivial solution if and only if there are free variables.____________

c) If A is $n \times n$ matrix, then $\det(cA) = c \det A$, c constant. ___________________

d) The solution set of a homogeneous linear system $Ax = 0$ of *m* equation and *n* unknowns is a subspace of R^n. ______________________

e) If **x** is a vector in the first quadrant of R^2, then any scalar multiple *k***x** of **x** is still a vector in the first quadrant of R^2. ___________________

4.1.2 Problem 1

Solve the initial value problem

$$x\frac{dy}{dx} - y = 2x^2 y$$
$$y(1) = 1$$

Solution

It is a good idea to start by first applying the uniqueness theorem in order to find if we expect the solution to exist and if it unique and the interval I it is valid on. Writing the ODE as

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x^2 y + y}{x}\\ &= \frac{y\left(2x^2+1\right)}{x}\\ &= f\left(x,y\right)\end{aligned}$$

The above shows that $f\left(x,y\right)$ is continuous for all y and x except at $x = 0$. Taking derivative w.r.t. y gives

$$\frac{\partial f\left(x,y\right)}{\partial y} = \frac{2x^2+1}{x}$$

Which is continuous for all x except at $x = 0$. This shows that interval I can not contain $x = 0$. And since the initial condition are at $x = 1$ (to the right side of $x = 0$), then the interval must contain $x = 1$ and since interval can not cross $x = 0$, then this means x must be positive and there exists an interval I that contains $x = 1$ and for $x > 0$ where the solution exists and is unique. Now we know that x must be positive, we can solve the ODE.

Dividing the given ODE through by $x \neq 0$ gives

$$\frac{dy}{dx} - \frac{y}{x} = 2xy$$

Collecting on y gives

$$\begin{aligned}\frac{dy}{dx} - \frac{y}{x} - 2xy &= 0\\ \frac{dy}{dx} + y\left(\frac{-1}{x} - 2x\right) &= 0 \qquad (1)\end{aligned}$$

This has the form $y' + p(x)y = 0$. It is therefore a linear ODE in y. The integrating factor is

$$I = e^{\int p(x)dx} \qquad (2)$$

But $p(x) = \frac{-1}{x} - 2x$ in this case. Hence

$$\begin{aligned}\int p(x)dx &= -\int \frac{1}{x}dx - 2\int x dx\\ &= -\ln|x| - x^2\end{aligned}$$

But since $x > 0$, then the above simplifies to

$$\int p(x)dx = -\ln x - x^2$$

Substituting this in (2) gives

$$\begin{aligned}I &= e^{-\ln x - x^2}\\ &= e^{-\ln x}e^{-x^2}\\ &= \frac{1}{x}e^{-x^2}\end{aligned}$$

Multiplying both sides of (1) by this integrating factor gives

$$\begin{aligned}\frac{1}{x}e^{-x^2}\left(\frac{dy}{dx}+y\left(\frac{-1}{x}-2x\right)\right)&=0\\ \left(\frac{dy}{dx}\frac{e^{-x^2}}{x}+y\left(-\frac{1}{x}-2x\right)e^{-x^2}\right)&=0\end{aligned}$$

But $\left(\frac{dy}{dx}\frac{e^{-x^2}}{x}+y\left(\frac{-1}{x}-2x\right)e^{-x^2}\right)=\frac{d}{dx}\left(y\frac{e^{-x^2}}{x}\right)$ by the product rule. Hence the above becomes

$$\frac{d}{dx}\left(y\frac{e^{-x^2}}{x}\right)=0$$

Integrating gives

$$\begin{aligned}y\frac{e^{-x^2}}{x}&=C\\ y&=Cxe^{x^2}\end{aligned}\tag{3}$$

The constant of integration C in the above general soltion is found from the given initial conditions $y(1)=1$. Substituting initial conditions in (3) gives

$$\begin{aligned}1&=Ce\\ C&=e^{-1}\end{aligned}$$

Hence (3) becomes

$$y=xe^{-1}e^{x^2}$$

Therefore the particular solution is

$$y(x)=xe^{x^2-1}\qquad x>0$$

4.1.3 Problem 2

Solve the initial value problem

$$\begin{aligned}x^2\frac{dy}{dx}+2xy-y^3&=0\\ x&>0\end{aligned}$$

<u>Solution</u>

Since $x\neq0$, then dividing the given ODE throughout by x^2 gives

$$\begin{aligned}\frac{dy}{dx}+\frac{2y}{x}-\frac{1}{x^2}y^3&=0\\ \frac{dy}{dx}&=-\frac{2}{x}y+\frac{1}{x^2}y^3\end{aligned}$$

This ODE has the form $y'=p(x)y+q(x)y^n$ where $n>1$. Therefore it is a <u>Bernoulli</u> ODE. In this case $p(x)=\frac{-2}{x},q(x)=\frac{1}{x^2}$ and $n=3$. Dividing the above by y^3 for $y\neq0$ gives

$$\frac{1}{y^3}\frac{dy}{dx}=-\frac{2}{x}y^{-2}+\frac{1}{x^2}\tag{1}$$

Let

$$u(x)=y^{-2}(x)\tag{2}$$

be a new dependent variable. Taking derivative w.r.t x and applying the chain rule to the above gives

$$\frac{du}{dx}=-2y^{-3}\frac{dy}{dx}$$

Which means

$$\frac{dy}{dx} = -\frac{1}{2}y^3\frac{du}{dx} \tag{3}$$

Substituting equations (2,3) back into (1) gives a new ODE in $u(x)$

$$\begin{aligned}
\frac{1}{y^3}\left(-\frac{1}{2}y^3\frac{du}{dx}\right) &= -\frac{2}{x}u + \frac{1}{x^2}\\
-\frac{1}{2}\frac{du}{dx} &= -\frac{2}{x}u + \frac{1}{x^2}\\
\frac{du}{dx} &= \frac{4}{x}u - \frac{2}{x^2}\\
\frac{du}{dx} - \frac{4}{x}u &= -\frac{2}{x^2} \qquad (4)
\end{aligned}$$

The above has the form $u' + p(x)u = q(x)$. Therefore it is linear in u. The integrating factor is

$$I = e^{\int p(x)dx}$$

But $p(x) = -\frac{4}{x}$. Hence

$$\begin{aligned}
I &= e^{\int -\frac{4}{x}dx}\\
&= e^{-4\ln|x|}
\end{aligned}$$

But $x > 0$, therefore the above simplifies to

$$\begin{aligned}
I &= e^{-4\ln x}\\
&= \frac{1}{x^4}
\end{aligned}$$

Multiplying both sides of (4) by the above integrating factor gives

$$\begin{aligned}
\frac{1}{x^4}\left(\frac{du}{dx} - \frac{4}{x}u\right) &= \frac{1}{x^4}\left(-\frac{2}{x^2}\right)\\
\left(\frac{du}{dx}\frac{1}{x^4} - \frac{1}{x^4}\frac{4}{x}u\right) &= -\frac{2}{x^6}
\end{aligned}$$

But $\left(\frac{du}{dx}\frac{1}{x^4} - \frac{1}{x^4}\frac{4}{x}u\right) = \frac{d}{dx}\left(u\frac{1}{x^4}\right)$ by the product rule. The above simplifies to

$$\begin{aligned}
\frac{d}{dx}\left(u\frac{1}{x^4}\right) &= -\frac{2}{x^6}\\
d\left(u\frac{1}{x^4}\right) &= -\frac{2}{x^6}dx
\end{aligned}$$

Integrating both sides gives

$$\begin{aligned}
\frac{u}{x^4} &= -2\int \frac{1}{x^6}dx + C\\
&= -2\int x^{-6}dx + C\\
&= -2\left(\frac{x^{-5}}{-5}\right) + C\\
&= \frac{2}{5}x^{-5} + C
\end{aligned}$$

Hence the solution in u is

$$u = \frac{2}{5x} + Cx^4$$

But from (2), $u = y^{-2}$. Therefore the above becomes

$$\begin{aligned} y^{-2} &= \frac{2}{5x} + Cx^4 \\ &= \frac{2 + 5Cx^5}{5x} \end{aligned}$$

Or

$$y^2 = \frac{5x}{2 + 5Cx^5}$$

We can simplify this more by letting $5C = C_0$ be a new constant. The above becomes

$$y^2 = \frac{5x}{2 + C_0 x^5}$$

There are two solutions given by

$$\begin{aligned} y_1(x) &= \sqrt{\frac{5x}{2 + C_0 x^5}} \qquad x > 0 \\ y_2(x) &= -\sqrt{\frac{5x}{2 + C_0 x^5}} \qquad x > 0 \end{aligned}$$

4.1.4 Problem 3

Verify that the given differential equation is exact, then solve it.

$$\left(x^3 + \frac{y}{x}\right) dx + \left(y^2 + \ln x\right) dy = 0$$

Solution

The ODE has the form

$$M\left(x,y\right) dx + N\left(x,y\right) dy = 0 \tag{1}$$

This is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Comparing (1) with the given ODE shows that

$$\begin{aligned} M\left(x,y\right) &= x^3 + \frac{y}{x} \\ N\left(x,y\right) &= y^2 + \ln x \end{aligned}$$

Hence

$$\frac{\partial M}{\partial y} = \frac{1}{x}$$

And

$$\frac{\partial N}{\partial x} = \frac{1}{x}$$

Therefore it is exact since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Let $\phi\left(x,y\right)$ be some constant function, which means $d\left(\phi\left(x,y\right)\right) = 0$ or by the chain rule

$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\frac{\partial \phi}{\partial x} = M \tag{3}$$

$$\frac{\partial \phi}{\partial y} = N \tag{4}$$

Therefore, if we can find such a function $\phi\left(x,y\right)$, then the solution to the ODE becomes $\phi\left(x,y\right) = C_1$, where C_1 is some constant. $\phi\left(x,y\right) = C_1$ is a solution since it satisfies the given ODE (1). To find $\phi\left(x,y\right)$ we start with Eq. (3). (we could also start with Eq. (4) and same result will show up). Substituting $M = x^3 + \frac{y}{x}$ in (3) gives

$$\frac{\partial \phi}{\partial x} = x^3 + \frac{y}{x}$$

Integrating both sides w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int x^3 + \frac{y}{x} dx$$

$$\phi = \frac{x^4}{4} + y \ln x + f(y) \tag{5}$$

Where in the above $f(y)$ acts as the constant of integration but now it is a function of y only since ϕ is function of both x, y and the integration was done w.r.t. x. Taking derivative of the above w.r.t. y gives

$$\frac{\partial \phi}{\partial y} = \ln x + f'(y) \tag{6}$$

Comparing (4,6) shows that

$$\ln x + f'(y) = N$$

But $N = y^2 + \ln x$, hence the above becomes

$$\ln x + f'(y) = y^2 + \ln x$$
$$f'(y) = y^2$$

Integrating both sides w.r.t y gives

$$\int \frac{df(y)}{dy} dy = \int y^2 dy$$
$$\int df(y) = \frac{y^3}{3} + C$$
$$f(y) = \frac{y^3}{3} + C$$

Now that $f(y)$ is found, substituting it back into Eq. (5) gives

$$\phi = \frac{x^4}{4} + y \ln x + \left(\frac{y^3}{3} + C\right)$$

But since ϕ is constant, say C_1. Then the above gives

$$C_1 = \frac{x^4}{4} + y \ln x + \left(\frac{y^3}{3} + C\right)$$

Combining the two constants into one and calling the new constant C_0 then the above becomes

$$C_0 = \frac{x^4}{4} + y \ln x + \frac{y^3}{3}$$

The above is the final solution. It is kept in implicit form. C_0 is the constant of integration.

4.1.5 Problem 4

a) Solve the initial value problem

$$\frac{dy}{dx} = 3 + x - y$$
$$y(0) = 1$$

b) Apply Euler's methods to the initial value problem with step size $h = 0.1$ and complete the following table

x	Euler method y	Exact y	Absolute error
0.1			
0.2			
0.3			
0.4			

Solution

4.1.5.1 Part (a)

Writing the ODE as

$$\frac{dy}{dx} + y = 3 + x \tag{1}$$

Shows it is linear ODE since it has the form $y' + p(x)y = q(x)$ where $p(x) = 1, q(x) = 3 + x$. The integrating factor is $I = e^{\int p(x)dx} = e^{\int dx} = e^x$. Multiplying both sides of (1) by this integration factor gives

$$e^x\left(\frac{dy}{dx} + y\right) = e^x(3 + x)$$
$$\left(\frac{dy}{dx}e^x + ye^x\right) = 3e^x + xe^x$$

But $\left(\frac{dy}{dx}e^x + ye^x\right) = \frac{d}{dx}\left(ye^x\right)$ by the product rule. Hence the above becomes

$$\frac{d}{dx}\left(ye^x\right) = 3e^x + xe^x$$
$$d\left(ye^x\right) = (3e^x + xe^x)\,dx$$

Integrating both sides gives

$$ye^x = 3\int e^x dx + \int xe^x dx + C \tag{2}$$

The integral $\int e^x dx = e^x$. For the second $\int xe^x dx$ we apply integration by parts. $\int u dv = uv - \int v du$. Let $u = x, dv = e^x$, then $du = dx$ and $v = e^x$. Hence the second integral becomes

$$\int xe^x dx = xe^x - \int e^x dx$$
$$= xe^x - e^x$$
$$= e^x(x - 1)$$

Putting these results back in (2) gives

$$ye^x = 3e^x + e^x(x - 1) + C$$

Multiplying both sides by e^{-x} gives

$$y = 3 + x - 1 + Ce^{-x}$$
$$= x + 2 + Ce^{-x} \tag{3}$$

Initial conditions are now used to find C. Since $y(0) = 1$, then the above becomes

$$1 = 2 + C$$
$$C = -1$$

Substituting the above back in (3) gives the particular solution as

$$y(x) = x + 2 - e^{-x}$$

4.1.5.2 Part (b)

Euler method is given by

$$\begin{aligned} y_1 &= y_0 + hf\left(x_0, y_0\right) \\ y_2 &= y_1 + hf\left(x_1, y_1\right) \\ &\vdots \\ y_{n+1} &= y_n + hf\left(x_n, y_n\right) \end{aligned}$$

In this problem $f\left(x, y\right) = 3 + x - y$ and $x_0 = 0$ and $y_0 = 1$ because initial conditions are $y(0) = 1$. And $h = 0.1$. We found the exact solution in part (a) as $y_{exact}(x) = x + 2 - e^{-x}$. Therefore,

$\underline{x = 0.1}$

$$\begin{aligned} y_1 &= y_0 + hf\left(x_0, y_0\right) \\ &= (1) + (0.1)\left(3 + x_0 - y_0\right) \\ &= (1) + (0.1)(3 + 0 - 1) \\ &= 1.2 \end{aligned}$$

And exact is

$$\begin{aligned} y_{exact}(0.1) &= x + 2 - e^{-x} \\ &= 0.1 + 2 - e^{-0.1} \\ &= 1.1952 \end{aligned}$$

$\underline{x = 0.2}$

Now, using $x_1 = 0.1$ gives

$$\begin{aligned} y_2 &= y_1 + hf\left(x_1, y_1\right) \\ &= 1.2 + (0.1)\left(3 + x_1 - y_1\right) \\ &= 1.2 + (0.1)(3 + 0.1 - 1.2) \\ &= 1.39 \end{aligned}$$

And exact is

$$\begin{aligned} y_{exact}(0.2) &= 0.2 + 2 - e^{-0.2} \\ &= 1.3813 \end{aligned}$$

$\underline{x = 0.3}$

Using using $x_2 = x_1 + h = 0.2$ gives

$$\begin{aligned} y_3 &= y_2 + hf\left(x_2, y_2\right) \\ &= 1.39 + (0.1)\left(3 + x_2 - y_2\right) \\ &= 1.39 + (0.1)(3 + 0.2 - 1.39) \\ &= 1.571 \end{aligned}$$

And exact is

$$\begin{aligned} y_{exact}(0.3) &= 0.3 + 2 - e^{-0.3} \\ &= 1.5592 \end{aligned}$$

$\underline{x = 0.4}$

Using $x_3 = x_2 + h = 0.3$ gives

$$\begin{aligned} y_4 &= y_3 + hf\left(x_3, y_3\right) \\ &= 1.571 + (0.1)\left(3 + x_3 - y_3\right) \\ &= 1.571 + (0.1)(3 + 0.3 - 1.571) \\ &= 1.7439 \end{aligned}$$

And exact is

$$\begin{aligned} y_{exact}(0.4) &= 0.4 + 2 - e^{-0.4} \\ &= 1.7297 \end{aligned}$$

The table becomes

x	Euler method y	Exact y	Absolute error
0.1	1.2	1.1952	0.0048
0.2	1.39	1.3813	0.0087
0.3	1.571	1.5592	0.0118
0.4	1.7439	1.7297	0.0142

The above shows that the absolute error increases as more steps are taken. Reducing h will reduce the magnitude of the error.

4.1.6 Problem 5

Solve the following system of equations and write the solution in parametric vector form

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 + 2x_3 &= 2 \\ 3x_1 + x_2 + 3x_3 &= -8 \end{aligned}$$

Solution

In Matrix form $Ax = b$ the above becomes

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix} \tag{1}$$

Therefore the augmented matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 3 & -8 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -5 & 0 & 0 \\ 3 & 1 & 3 & -8 \end{bmatrix}$$

$R_3 = R_3 - 3R_1$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & -5 & 0 & -11 \end{bmatrix}$$

$R_3 = R_3 - R_2$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & -11 \end{bmatrix}$$

$R_2 = -\frac{R_2}{5}$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -11 \end{bmatrix}$$

$R_1 = R_1 - 2R_2$ gives

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -11 \end{bmatrix}$$

But from the last row, it says $0 = -11$. Hence there is no solution. Inconsistent system. Unable to find solution in parametric vector form.

4.1.7 Problem 6

Given the matrix $A = \begin{bmatrix} 3 & 4 \\ 4 & -2 \end{bmatrix}$ a) Find A^{-1}. b) Use A^{-1} to solve the system of equations

$$\begin{aligned} 3x + 4y &= 7 \\ 4x - 2y &= 5 \end{aligned}$$

Solution

4.1.7.1 Part a

Since this is a 2×2 system, then if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its inverse is given by $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. For the matrix A, its determinant is $(-6) - (16) = -22$. Therefore

$$\begin{aligned} A^{-1} &= \frac{1}{-22} \begin{bmatrix} -2 & -4 \\ -4 & 3 \end{bmatrix} \\ &= \frac{1}{22} \begin{bmatrix} 2 & 4 \\ 4 & -3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{11} & \frac{2}{11} \\ \frac{2}{11} & -\frac{3}{22} \end{bmatrix} \end{aligned}$$

4.1.7.2 Part b

The system of equations given can be written in matrix form $Ax = b$ as

$$\begin{bmatrix} 3 & 4 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

And since A is non singular as we found in part (a), then premultiplying both sides by A^{-1} gives

$$\begin{bmatrix} 3 & 4 \\ 4 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 4 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

But $A^{-1}A$ is the identity matrix. The above simplifies to

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

Using result of part (a) the above becomes

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{11} & \frac{2}{11} \\ \frac{2}{11} & -\frac{3}{22} \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

But

$$\begin{bmatrix} \frac{1}{11} & \frac{2}{11} \\ \frac{2}{11} & -\frac{3}{22} \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{11}(7) + \frac{2}{11}(5) \\ \frac{2}{11}(7) - \frac{3}{22}(5) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{7}{11} + \frac{10}{11} \\ \frac{14}{11} - \frac{15}{22} \end{bmatrix}$$

Hence the solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{17}{11} \\ \frac{13}{22} \end{bmatrix}$$

4.1.8 Problem 7

Use the cofactor expansion to evaluate the given determinant along the second row

$$\begin{vmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{vmatrix}$$

Solution

Using second row then (where below, $(-1)^{i+j}$ means row i and column j. This is used to obtain the sign of each cofactor).

$$\begin{aligned} \det(A) &= (-1)^{2+1}(-2)\begin{vmatrix} 2 & -3 \\ -5 & 0 \end{vmatrix} + (-1)^{2+2}(0)\begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} + (-1)^{2+3}(5)\begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix} \\ &= (-1)(-2)\begin{vmatrix} 2 & -3 \\ -5 & 0 \end{vmatrix} + (-1)(5)\begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix} \\ &= 2\begin{vmatrix} 2 & -3 \\ -5 & 0 \end{vmatrix} - 5\begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix} \\ &= 2((2\times 0) - (-3\times -5)) - 5((0\times -5) - (2\times 3)) \\ &= 2(-15) - 5(-6) \\ &= -30 + 30 \end{aligned}$$

Hence

$$\det(A) = 0$$

4.1.9 Problem 8

Let H be the set of points in the xy plane given by $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$. Show that H is not a subspace of $\mathbb{R}^2$

Solution

The first thing to check if the zero vector is in H. It is, since x, y are allowed to be zero and that will satisfy $xy = 0$ part. Hence $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in H.

Now we need to check if H is closed under addition. Let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ such that $x_1 y_1 \geq 0$ and $\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ such that $x_2 y_2 \geq 0$, which means v_1, v_2 are in H. Then

$$\begin{aligned} \vec{v}_1 + \vec{v}_2 &= \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\ \vec{v}_3 &= \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \end{aligned}$$

And therefore

$$(x_1 + x_2)\left(y_1 + y_2\right) = x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2 \tag{1}$$

We know that $x_1 y_1 \geq 0$ and that $x_2 y_2 \geq 0$ because $\vec{v}_1, \vec{v}_2$ are in H. But it is possible that $x_1 y_2$ or $x_2 y_1$ can be negative leading to an overall result which is negative. All what we need is one example that shows this. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, which satisfies $xy \geq 0$ and let $\vec{v}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ which satisfies $xy \geq 0$. Eq. (1) now becomes

$$\begin{aligned} (x_1 + x_2)\left(y_1 + y_2\right) &= (1-3)(2-1) \\ &= (-2)(1) \\ &= -2 \end{aligned} \tag{2}$$

This shows that $xy < 0$ in this case. Therefore not closed under addition. We do not need to check if closed under scalar multiplication since the first test above failed. The above shows that H is not a subspace of $\mathbb{R}^2$.

4.1.10 Problem 9

Determine if the set of vectors span $\mathbb{R}^3$. Justify our answer

$$\{(1,-2,1),(2,3,1),(4,-1,2)\}$$

Solution

The set spans $\mathbb{R}^3$ if the vectors are linearly independent. One way to find this is to solve

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

For c_1, c_2, c_3 and see if the only solution is $c_1 = 0, c_2 = 0, c_3 = 0$ or not. If it is, then the vectors are linearly independent and therefore span $\mathbb{R}^3$. The system to solve is

$$c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In Matrix $Ax = b$ form it becomes

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ -2 & 3 & -1 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

$R_2 = R_2 + 2R_1$ gives

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 7 & 7 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

$R_3 = R_3 - R_1$ gives

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix}$$

$R_2 = \frac{R_2}{7}$ gives

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix}$$

$R_3 = R_3 + R_2$ gives

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$R_3 = -R_3$ gives

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 = R_2 - R_3$ gives

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_1 = R_1 - 4R_3$ gives

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_1 = R_1 - 2R_2$ gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The above is in RREF. The original system (1) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

The above shows that $c_3 = 0, c_2 = 0, c_1 = 0$. Since this is the only solution, therefore the set of vectors given <u>span</u> $\mathbb{R}^3$ because they are linearly independent.

4.1.11 Problem 10

Mark each statement TRUE or FALSE

Solution

a An integrating factor for the differential equation $\frac{dy}{dx} = x^2 y$ is $e^{\int x^2 dx}$. FALSE.

b The equation $Ax = 0$ has the nontrivial solution if and only if there are free variables. TRUE.

c If A is $n \times n$ matrix, then $\det(cA) = c\det(A)$, c is constant. FALSE.

d The solution set of a homogeneous linear system $Ax = 0$ of m equation and n unknowns is a subspace of $\mathbb{R}^n$. FALSE

e If $\vec{x}$ is a vector in the first quadrant of $\mathbb{R}^2$, then any scalar multiple $k\vec{x}$ of $\vec{x}$ is still a vector in the first quadrant of $\mathbb{R}^2$. FALSE

4.2 Final exam

Local contents

4.2.1 Problems listing

Math2520 Calculus IV FINAL EXAM

<u>Instruction</u>:

1. **Print this question paper and show the necessary steps on the space provided under each question.**
2. **If you don't have access to printer, use white paper.**
3. **Use black ink and write bigger fonts so that it is visible when posted and printed again.**
4. **Follow the instruction given for each question.**
5. **You can only post once. So make sure all the pages/questions are posted properly.**
6. **Post on D2L as one PDF file by July 17, 2021.**
7. **Write your name on the answer sheet.**

0

MATH2520 CALCULUS 4 FINAL EXAM

Name: ____________________________

INSTRUCTION: Show all the necessary steps on the space provided under each question or on a separate sheet.

1. Solve the following system of equations and write the solution as a parametric vector form. (**4 pts**)

$$\begin{aligned} x+2y-3z &= 5 \\ 2x+y-3z &= 13 \\ -x+y \quad\quad &= -8 \end{aligned}$$

2. Compute the determinant using a cofactor expansion. (**3 pts**)

$$\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

3. Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $\boldsymbol{u} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$.

a) Is ***u*** in Nullspace(A)? Justify your answer. (**3 pts**)

b) Is ***u*** in Columnspace(A) ? Justify your answer. (**3 pts**)

c) Determine the rank A and Nullity of A. Show your work. (**2 pts**)

4. a) Using the definition, verify that the given transformation is linear transformation.
$T : C^2(I) \rightarrow C^0(I)$ defined by $T(y) = y'' + y$ (**4 pts**)

b) Find the kernel of *T*. (**4 pts**)

5. Solve: **(5 pts)**

$(y+3x^2)dx+xdy=0$

6. Using the **method of undermined coefficients**, find the general solution of the given differential equation. (**10 pts**)
   $y'' - y' - 2y = e^{-x} + 2\cos x$

7. Use the **Laplace transform** to solve the given initial-value problems. (**10 pts**)
   You can use the table of transformation.

$y'' + y = e^{2t}, \quad y(0) = 0, \quad y'(0) = 1$

8. Find a **series solution** in powers of x of the differential equation. (**10 pts**)
   $y'' + x^2 y' + y = 0$

9. a) Determine all the equilibrium points of the given system.
   b) Select two equilibrium points and classify them as saddle, node, spiral or center and whether they are stable or unstable.

(**10 pts**)

$$x' = 2x - x^2 - xy$$
$$y' = 3y - 3xy - 2y^2$$

4.2.2 Problem 1

Solve the following system of equations and write the solution as a parametric vector form

$$\begin{aligned} x + 2y - 3z &= 5 \\ 2x + y - 3z &= 13 \\ -x + y &= -8 \end{aligned}$$

<u>Solution</u>

In matrix form $Ax = b$, the above system is

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ -8 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 2 & 1 & -3 & 13 \\ -1 & 1 & 0 & -8 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ -1 & 1 & 0 & -8 \end{bmatrix}$$

$R_3 = R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ 0 & 3 & -3 & -3 \end{bmatrix}$$

$R_3 = R_3 + R_2$

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence original system (1) becomes

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix} \tag{2}$$

The above shows that $z = t$ is a <u>free variable</u> and x, y are basic variables. Second row gives $-3y + 3t = 3$ or $-y + t = 1$ or $y = t - 1$. First row gives $x + 2y - 3t = 5$ or $x = 5 + 3t - 2(t - 1)$ or $x = 7 + t$. Hence the solution is

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 7 + t \\ t - 1 \\ t \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ t \\ t \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

The above is the solution in parametric vector form. For any value of the parameter t, a solution exist.

4.2.3 Problem 2

Compute the determinant using a cofactor expansion

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

solution

Expanding along the last row since it has most number of zeros gives (only the element $A(3,2) \neq 0$)

$$\begin{aligned} \det(A) &= (-1)^{3+2}(-2)\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \\ &= 2\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \\ &= 2(-1) \\ &= -2 \end{aligned}$$

4.2.4 Problem 3

Let

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$$

$$u = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$$

a is u in NullSpace of A ? Justify your answer.

b Is u in columnspace of A ? Justify your answer.

c Determine the rank A and the Nullity of A. Show your work

solution

4.2.4.1 Part a

For an $m \times n$ matrix, the solution set corresponding to $A\vec{x} = \vec{0}$ is called the NullSpace(A). Therefore we need to first find this solution set by solving

$$\begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -3 & -4 & 0 \\ -4 & 6 & -2 & 0 \\ -3 & 7 & 6 & 0 \end{bmatrix}$$

$R_2 = R_2 + 4R_1$

$$\begin{bmatrix} 1 & -3 & -4 & 0 \\ 0 & -6 & -18 & 0 \\ -3 & 7 & 6 & 0 \end{bmatrix}$$

$R_3 = R_3 + 3R_1$

$$\begin{bmatrix} 1 & -3 & -4 & 0 \\ 0 & -6 & -18 & 0 \\ 0 & -2 & -6 & 0 \end{bmatrix}$$

$R_3 = R_3 - \frac{1}{3}R_2$

$$\begin{bmatrix} 1 & -3 & -4 & 0 \\ 0 & -6 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence (1) becomes

$$\begin{bmatrix} 1 & -3 & -4 \\ 0 & -6 & -18 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

The above shows that $z = t$ is free variable and x, y are basic variables. Second row gives $-6y - 18t = 0$ or $y = -3t$. First row gives $x - 3y - 4t = 0$ or $x = 3(-3t) + 4t$ or $x = -5t$. Hence the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5t \\ -3t \\ t \end{bmatrix} = t \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$$

Now we are ready to answer the question if $u = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$ is in the NullSpace(A). In other words, does there exist t which make $t \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$. It is clear there is no such t. To show this, looking at the second row, it says $-3t = 3$ or $t = -1$. But third row says $t = -4$. Therefore there is no t which makes u in NullSpace(A). Hence u is not in NullSpace(A)

4.2.4.2 Part b

The columnspace of A is the set of all linear combinations of the columns of A. The basis for the columnspace are columns of A that correspond to the pivot columns are doing the above REF. From part A we found that column 1, 2 are the pivot columns. Hence the basis of columnspace of A are

$$\left\{ \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ 7 \end{bmatrix} \right\}$$

Hence the columnspace of A is two dimensional subspace of $\mathbb{R}^3$. To find if u is in columnspace of A, we need to find if there exists a linear combination of these basis which gives u. Therefore we need to solve

$$c_1 \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$$

For c_1, c_2 to see if a solution exist. In matrix form the above becomes

$$\begin{bmatrix} 1 & -3 \\ -4 & 6 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -3 & 3 \\ -4 & 6 & 3 \\ -3 & 7 & -4 \end{bmatrix}$$

$R_2 = R_2 + 4R_1$

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & -6 & 15 \\ -3 & 7 & -4 \end{bmatrix}$$

$R_3 = R_3 + 3R_1$

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & -6 & 15 \\ 0 & -2 & 5 \end{bmatrix}$$

$R_3 = R_3 - \frac{1}{3}R_2$

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & -6 & 15 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 = -\frac{R_2}{6}$

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 = R_1 + 3R_2$

$$\begin{bmatrix} 1 & 0 & -\frac{9}{2} \\ 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Hence (1) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} \\ -\frac{5}{2} \\ 0 \end{bmatrix}$$

Therefore the solution is

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} \\ -\frac{5}{2} \end{bmatrix}$$

Therefore a solution exists. This means

$$-\frac{9}{2}\begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} -3 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$$

This means u is in the columnspace of A.

4.2.4.3 Part c

The rank of A is the dimension of the columnspace of A. Which is the same as the number of pivot columns found. In this case, it is 2 as found in part b above. Hence $rank(A) = 2$.

Nullity of A is the dimension of the nullspace of A. From part (a) we found that the nullspace of A is given by the one parameter vector $t\begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$. Hence the dimension is 1. It is the number of the free variables. Therefore Nullity of $A = 1$. To verify this, we can use the rank-nullity theorem, which says for a matrix $m \times n$,

$$rank(A) + nullity(A) = n$$

Since $n = 3$ and since $rank(A) = 2$ then $nullity\,(A) = 1$.

4.2.5 Problem 4

a Using the definition, verify that the given transformation is linear transformation $T : C^2(I) \to C^0(I)$ defined by $T(y) = y'' + y$

b Find the kernel of T

solution

4.2.5.1 Part a

The transformation T is linear if

1. $T(u + v) = T(u) + T(v)$ for all u, v in $C^2(I)$
2. $T(cu) = cT(u)$ for all scalars c and u in $C^2(I)$

To show property 1:

$$T(u + v) = (u + v)'' + (u + v)$$

By linearity of second derivatives (and since u, v are in $C^2(I)$) the above becomes

$$\begin{aligned} T(u + v) &= (u'' + v'') + (u + v) \\ &= u'' + u + v'' + v \\ &= (u'' + u) + (v'' + v) \end{aligned}$$

But $(u'' + u) = T(u)$ and $(v'' + v) = T(v)$, Hence the above becomes

$$T(u + v) = T(u) + T(v)$$

To show property 2:

$$T(cu) = (cu)'' + (cu)$$

But since c is a scalar, we can move it outside the derivative and the above becomes

$$\begin{aligned} T(cu) &= c(u)'' + cu \\ &= c(u'' + u) \end{aligned}$$

But $u'' + u = T(u)$. Hence the above becomes

$$T(cu) = cT(u)$$

Both properties are satisfied. Hence T is linear transformation.

4.2.5.2 Part b

The kernel of $T : V \to W$ is $\ker(T) = \{u \in V : Tu = 0\}$. In this case $V = C^2(I)$ and $W = C^0(I)$. Hence we need to find all u, such that $T(u) = 0$. Which is the same as saying all u which satisfies $u'' + u = 0$. Hence $\ker(T)$ is the solution of this ode.

This is linear constant coefficient ode. The characteristic equation is $\lambda^2 + 1 = 0$. The solutions are $\lambda = \pm i$. Hence the basis functions are $\left\{e^{ix}, e^{-ix}\right\}$ (assuming the independent variable is x), or using Euler relation $\{\cos x, \sin x\}$. Therefore the solution is linear combination of these basis given by $u = c_1 \cos x + c_2 \sin x$ where c_1, c_2 are arbitrary constants. Hence

$$\ker(T) = \{u : u = c_1 \cos x + c_2 \sin x\}$$

4.2.6 Problem 5

Solve $\left(y + 3x^2\right) dx + xdy = 0$

solution

Writing the ODE as

$$Mdx + Ndy = 0$$

Where $M = y + 3x^2, N = x$. Checking if the ODE is exact

$$\begin{aligned} \frac{\partial M}{\partial y} &= 1 \\ \frac{\partial N}{\partial x} &= 1 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then it is exact in some region R. Let there exists constant function $\Phi\left(x, y\right) = c$ which satisfies

$$\frac{\partial \Phi}{\partial x} = M = y + 3x^2 \tag{1}$$

$$\frac{\partial \Phi}{\partial y} = N = x \tag{2}$$

For all $\left(x, y\right)$ in R. Integrating (1) w.r.t. x gives

$$\int \frac{\partial \Phi}{\partial x} dx = \int y + 3x^2 dx$$

$$\Phi = yx + x^3 + g(y) \tag{3}$$

Taking derivative of the above w.r.t. y gives

$$\frac{\partial \Phi}{\partial y} = x + g'(y) \tag{4}$$

Comparing (4) and (2) gives

$$\begin{aligned} x + g'(y) &= x \\ g'(y) &= 0 \end{aligned}$$

Hence $g(y) = c_1$ a constant. Substituting this in (3) gives

$$\Phi = yx + x^3 + c_1$$

But $\Phi = c$. Combining the constants c, c_1 into one constant, say C, the above becomes

$$C = yx + x^3$$

Solving for y gives

$$yx = C - x^3$$

For $x \neq 0$

$$y = \frac{C - x^3}{x}$$

4.2.7 Problem 6

Using the method of undermined coefficients, find the general solution of the given differential equation

$$y'' - y' - 2y = e^{-x} + 2\cos x \tag{1}$$

solution

The solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $y'' - y' - 2y = 0$ and y_p is a particular solution to the ode. We start by finding y_h. Since this is linear second order with constant coefficient, then the characteristic equation method is used. The characteristic equation for $y'' - y' - 2y = 0$ is

$$\lambda^2 - \lambda - 2 = 0$$
$$(\lambda - 2)(\lambda + 1) = 0$$

Hence the roots are $\lambda_1 = 2, \lambda_2 = -1$. Therefore the basis set of solutions for y_h is the set

$$\left\{e^{2x}, e^{-x}\right\} \tag{2}$$

And y_h is linear combination of these basis. Therefore

$$y_h = c_1 e^{2x} + c_2 e^{-x} \tag{3}$$

Looking at RHS of (1) shows it is linear combination of basis $[e^{-x}, \cos x]$. For each basis in this list, we generate all possible derivatives. Which gives (ignoring sign changes and any leading constants as they will be parts of the unknowns to be found later on). This results in the following list

$$[\{e^{-x}\}, \{\cos x, \sin x\}] \tag{4}$$

Now we compare each basis in (2) with each basis in (4) to see if there is any duplication. We see that e^{-x} is in (4) as well in (2). We now multiply e^{-x} in (4) by an extra x and obtain new list

$$[\{xe^{-x}\}, \{\cos x, \sin x\}] \tag{4A}$$

We repeat this process again, checking if (2) still has any duplication in (4A). There are no duplication now. Hence the trial solution is linear combination of the basis in (4A). Which gives

$$y_p = Axe^{-x} + B\cos x + C\sin x \tag{5}$$

To determine A, B, C, we substitute y_p back in the ODE (1) and solve for these unknowns by comparing terms.

$$y_p' = Ae^{-x} - Axe^{-x} - B\sin x + C\cos x \tag{6}$$

$$\begin{aligned} y_p'' &= -Ae^{-x} - Ae^{-x} + Axe^{-x} - B\cos x - C\sin x \\ &= -2Ae^{-x} + Axe^{-x} - B\cos x - C\sin x \end{aligned} \tag{7}$$

Substituting (5,6,7) into the ODE (1) gives

$$
\begin{aligned}
(-2Ae^{-x} + Axe^{-x} - B\cos x - C\sin x) - (Ae^{-x} - Axe^{-x} - B\sin x + C\cos x) - 2(Axe^{-x} + B\cos x + C\sin x) &= e^{-x} + 2\cos x \\
-2Ae^{-x} + Axe^{-x} - B\cos x - C\sin x - Ae^{-x} + Axe^{-x} + B\sin x - C\cos x - 2Axe^{-x} - 2B\cos x - 2C\sin x &= e^{-x} + 2\cos x
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
-3Ae^{-x} - 3B\cos x - 3C\sin x + B\sin x - C\cos x &= e^{-x} + 2\cos x \\
-3Ae^{-x} + \cos x(-3B - C) + \sin x(-3C + B) &= e^{-x} + 2\cos x
\end{aligned}
$$

Comparing terms on each side gives 3 equations to solve for A, B, C

$$
\begin{aligned}
-3A &= 1 \\
-3B - C &= 2 \\
-3C + B &= 0
\end{aligned}
$$

First equation gives $A = -\frac{1}{3}$. Multiplying second equation by -3 and adding the result to third equation gives

$$
\begin{aligned}
9B + 3C &= -6 \\
-3C + B &= 0
\end{aligned}
$$

Adding gives

$$
\begin{aligned}
9B + B &= -6 \\
10B &= -6 \\
B &= -\frac{6}{10} \\
&= -\frac{3}{5}
\end{aligned}
$$

From $-3B-C = 2$ we now find $-3\left(-\frac{6}{10}\right)-C = 2$, or $C = -\frac{1}{5}$. Hence the particular solution (5) becomes

$$
y_p = -\frac{1}{3}xe^{-x} - \frac{3}{5}\cos x - \frac{1}{5}\sin x \tag{8}
$$

Substituting (8) and (3) in $y = y_h + y_p$ gives the final solution as

$$
y = c_1e^{2x} + c_2e^{-x} - \frac{1}{3}xe^{-x} - \frac{3}{5}\cos x - \frac{1}{5}\sin x
$$

4.2.8 Problem 7

Use the Laplace transform to solve the given initial-value problems. You can use the table of transformation

$$
\begin{aligned}
y'' + y &= e^{2t} \\
y(0) &= 0 \\
y'(0) &= 1
\end{aligned}
$$

solution

Taking the Laplace transform of both sides of $y'' + y = e^{2t}$ gives (using linearity)

$$
\mathscr{L}\left(y''\right) + \mathscr{L}(y) = \mathscr{L}\left(e^{2t}\right) \tag{1}
$$

Assuming $\mathscr{L}(y) = Y(s)$, and using the property that

$$
\mathscr{L}\left(y''\right) = s^2\mathscr{L}(y) - sy(0) - y'(0)
$$

And from table 10.2.1 $\mathscr{L}\left(e^{2t}\right) = \frac{1}{s-2}, s > 2$, then the ode becomes

$$
\begin{aligned}
\left(s^2\mathscr{L}(y) - sy(0) - y'(0)\right) + \mathscr{L}(y) &= \frac{1}{s-2}\\
\left(s^2Y - s(0) - 1\right) + Y &= \frac{1}{s-2}\\
s^2Y - 1 + Y &= \frac{1}{s-2}\\
Y\left(s^2+1\right) &= \frac{1}{s-2} + 1\\
Y &= \frac{1}{(s-2)\left(s^2+1\right)} + \frac{1}{s^2+1} \qquad (1A)
\end{aligned}
$$

Using partial fractions on the first term in the RHS above gives

$$
\begin{aligned}
\frac{1}{(s-2)\left(s^2+1\right)} &= \frac{A}{s-2} + \frac{Bs+C}{s^2+1}\\
&= \frac{A\left(s^2+1\right) + (Bs+C)(s-2)}{(s-2)\left(s^2+1\right)}\\
&= \frac{As^2 + A + \left(Bs^2 - 2Bs + Cs - 2C\right)}{(s-2)\left(s^2+1\right)}\\
&= \frac{As^2 + A + Bs^2 - 2Bs + Cs - 2C}{(s-2)\left(s^2+1\right)}\\
&= \frac{s^2(A+B) + s(C-2B) + (A-2C)}{(s-2)\left(s^2+1\right)}
\end{aligned}
$$

Therefore

$$1 = s^2(A+B) + s(C-2B) + (A-2C)$$

Comparing terms gives

$$
\begin{aligned}
A + B &= 0\\
C - 2B &= 0\\
A - 2C &= 1
\end{aligned}
$$

In matrix form the above is

$$\begin{bmatrix} 1 & 1 & 0\\ 0 & -2 & 1\\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} A\\ B\\ C \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 1 \end{bmatrix} \qquad (2)$$

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 0\\ 0 & -2 & 1 & 0\\ 1 & 0 & -2 & 1 \end{bmatrix}$$

$R_3 = R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 0\\ 0 & -2 & 1 & 0\\ 0 & -1 & -2 & 1 \end{bmatrix}$$

$R_3 = 2R_3$

$$\begin{bmatrix} 1 & 1 & 0 & 0\\ 0 & -2 & 1 & 0\\ 0 & -2 & -4 & 2 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -5 & 2 \end{bmatrix}$$

$R_3 = \frac{-1}{5}R_3, R_2 = -\frac{1}{2}R_2$ gives

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 1 & \frac{-2}{5} \end{bmatrix}$$

$R_2 = R_2 + \frac{1}{2}R_3$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{-1}{5} \\ 0 & 0 & 1 & \frac{-2}{5} \end{bmatrix}$$

$R_1 = R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{5} \\ 0 & 1 & 0 & \frac{-1}{5} \\ 0 & 0 & 1 & \frac{-2}{5} \end{bmatrix}$$

Hence (2) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{-1}{5} \\ \frac{-2}{5} \end{bmatrix} \tag{3}$$

Therefore, since now in RREF form, the solution is

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{-1}{5} \\ \frac{-2}{5} \end{bmatrix}$$

Hence

$$\begin{aligned} \frac{1}{(s-2)\left(s^2+1\right)} &= \frac{A}{s-2} + \frac{Bs+C}{s^2+1} \\ &= \frac{1}{5}\frac{1}{s-2} + \frac{-\frac{1}{5}s - \frac{2}{5}}{s^2+1} \\ &= \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{s+2}{s^2+1} \\ &= \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} \end{aligned} \tag{4}$$

Substituting (4) back in (1A) gives

$$\begin{aligned} Y &= \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} + \frac{1}{s^2+1} \\ &= \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{s}{s^2+1} + \frac{3}{5}\frac{1}{s^2+1} \end{aligned} \tag{5}$$

Now we will use tables to do the inverse Laplace transform. From table 10.2.1

$$\mathscr{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t} \qquad s > 2$$
$$\mathscr{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos t \qquad s > 0$$
$$\mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t \qquad s > 0$$

Using this result in (5) and since $\mathscr{L}^{-1}(Y(s)) = y(t)$ then (5) becomes

$$y(t) = \frac{1}{5}e^{2t} - \frac{1}{5}\cos t + \frac{3}{5}\sin t$$

4.2.9 Problem 8

Find a series solution in powers of x of the differential equation $y'' + x^2 y' + y = 0$

solution

Let the solution be

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

Then

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} n a_n x^{n-1} \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} \end{aligned} \tag{2}$$

And

$$\begin{aligned} y''(x) &= \sum_{n=1}^{\infty} n(n-1) a_n x^{n-2} \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned} \tag{3}$$

Substituting (1,2,3) into the given ODE gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$
$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0 \tag{3A}$$

Now we make all powers of x the same by rewriting $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$ and $\sum_{n=1}^{\infty} n a_n x^{n+1} = \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n$. The way the above is done is by using the rule: When adding a value to the summation index n inside the sum, then we must at same time subtract the same value from the starting index n.

Hence (3A) now becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

To be able to compare coefficients of x, we expand up to $n = 1$ the sums in order to make all sums start from $n = 2$. This gives

$$(2)(1)a_2 + (1+2)(1+1) a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n + (a_0 + a_1 x) + \sum_{n=2}^{\infty} a_n x^n = 0$$
$$(2a_2 + a_0) + x(6a_3 + a_1) + \left(\sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} + \sum_{n=2}^{\infty} (n-1) a_{n-1} + \sum_{n=2}^{\infty} a_n\right) x^n = 0$$

Now we are to compare coefficients on each power of x. The above gives the three equations

$$\begin{aligned} 2a_2 + a_0 &= 0 \\ 6a_3 + a_1 &= 0 \\ (n+2)(n+1)a_{n+2} + (n-1)a_{n-1} + a_n &= 0 \qquad n \geq 2 \end{aligned} \tag{4}$$

First equation above gives

$$a_2 = -\frac{1}{2}a_0$$

Second equation in (4) gives

$$a_3 = -\frac{a_1}{6}$$

And the third equation in (4) gives the recursive equation which allows us to find all a_n after these

$$(n+2)(n+1)a_{n+2} + (n-1)a_{n-1} + a_n = 0 \qquad n \geq 2$$

Or

$$a_{n+2} = \frac{-a_n - (n-1)a_{n-1}}{(n+2)(n+1)} \qquad n \geq 2 \tag{5}$$

Therefore, for $n = 2$ the above gives

$$a_4 = \frac{-a_2 - a_1}{(2+2)(2+1)} = \frac{-a_2 - a_1}{12}$$

But $a_2 = -\frac{1}{2}a_0$, therefore the above becomes

$$a_4 = \frac{-\left(-\frac{1}{2}a_0\right) - a_1}{(4)(3)} = \frac{\frac{1}{2}a_0 - a_1}{12} = \frac{a_0 - 2a_1}{24} \tag{6}$$

And for $n = 3$ (5) gives

$$a_5 = \frac{-a_3 - (3-1)a_2}{(3+2)(3+1)} = \frac{-a_3 - 2a_2}{20}$$

But $a_2 = -\frac{1}{2}a_0, a_3 = -\frac{a_1}{(2)(3)}$, the above becomes

$$a_5 = \frac{-\left(-\frac{a_1}{(2)(3)}\right) - 2\left(-\frac{1}{2}a_0\right)}{20} = \frac{\frac{a_1}{(2)(3)} + a_0}{20} = \frac{a_1 + 6a_0}{120} \tag{7}$$

And for $n = 4$ (5) gives

$$a_6 = \frac{-a_4 - (4-1)a_3}{(4+2)(4+1)} = \frac{-a_4 - 3a_3}{30}$$

But $a_4 = \frac{a_0 - 2a_1}{24}$ and $a_3 = -\frac{a_1}{6}$. The above becomes

$$a_6 = \frac{-\left(\frac{a_0-2a_1}{24}\right) - 3\left(-\frac{a_1}{6}\right)}{30} = \frac{\frac{-a_0+2a_1}{24} + \frac{a_1}{2}}{30} = \frac{-a_0 + 2a_1 + 12a_1}{(30)(24)} = \frac{-a_0 + 14a_1}{(30)(24)} = \frac{-a_0 + 14a_1}{720}$$

And so on. Therefore, from (1)

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots \\ &= a_0 + a_1 x - \frac{1}{2}a_0 x^2 - \frac{a_1}{6}x^3 + \left(\frac{a_0 - 2a_1}{24}\right)x^4 + \left(\frac{a_1 + 6a_0}{120}\right)x^5 + \left(\frac{-a_0 + 14a_1}{720}\right)x^6 + \cdots \\ &= a_0 + a_1 x - \frac{1}{2}a_0 x^2 - \frac{a_1}{6}x^3 + \left(\frac{a_0}{24} - \frac{a_1}{12}\right)x^4 + \left(\frac{a_1}{120} + \frac{a_0}{20}\right)x^5 + \left(\frac{-a_0}{720} + \frac{7a_1}{360}\right)x^6 + \cdots \end{aligned}$$

Therefore

$$y(x)=a_0\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4+\frac{1}{20}x^5-\frac{1}{720}x^6+\cdots\right)+a_1\left(x-\frac{1}{6}x^3-\frac{1}{12}x^4+\frac{1}{120}x^5+\frac{7}{360}x^6+\cdots\right) \tag{8}$$

The series solution above contains two unknowns a_0, a_1. There are the same as the constant of integrations. Since this is a second order ODE, then there will be two unknowns in the general solutions. These can be found from initial conditions. For example, assuming $y(0)=y_0, y'(0)=y'_0$. Then from (8) at $x=0$, it gives $y(0)=a$. Taking one derivative of (8) gives

$$y'(x)=a_0\left(-x+\frac{4}{24}x^3+\cdots\right)+a_1\left(1-\frac{3}{6}x^2+\cdots\right) \tag{9}$$

At $x=0$ the above becomes $y'_0=a_1$. Therefore (8) can be written as

$$y(x)=y(0)\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4+\frac{1}{20}x^5-\frac{1}{720}x^6+\cdots\right)+y'(0)\left(x-\frac{1}{6}x^3-\frac{1}{12}x^4+\frac{1}{120}x^5+\frac{7}{360}x^6+\cdots\right) \tag{10}$$

4.2.10 Problem 9

a) Determine all the equilibrium points of the given system. b) Select two equilibrium points and classify them as saddle, node, spiral or center and whether they are stable or unstable.

$$\begin{aligned} x' &= 2x-x^2-xy \\ y' &= 3y-3xy-2y^2 \end{aligned}$$

solution

4.2.10.1 Part a

equilibrium points are the solutions in x,y of

$$2x-x^2-xy=0 \tag{1}$$
$$3y-3xy-2y^2=0 \tag{2}$$

Which can be written as

$$x\left(2-x-y\right)=0 \tag{1}$$
$$y\left(3-3x-2y\right)=0 \tag{2}$$

From (1), we see that

$$x=0 \tag{3}$$

is a solution and $2-x-y=0$ or

$$x=2-y \tag{4}$$

Is another solution. For each x value in (3,4), now we solve for y from (2). When $x=0$ then (2) becomes

$$y\left(3-2y\right)=0$$

Which has solutions $y=0, y=\frac{3}{2}$. Therefore $\{0,0\}$ and $\left\{0,\frac{3}{2}\right\}$ are two solutions found so far. And when $x=2-y$ then (2) becomes

$$\begin{aligned} y\left(3-3\left(2-y\right)-2y\right)&=0 \\ y\left(3-6+3y-2y\right)&=0 \\ y\left(3-6+y\right)&=0 \end{aligned}$$

Which has solutions $y=0$ and $y=3$. When $y=0$ then $x=2-y$ gives $x=2$. Therefore $\{2,0\}$ is a solution, and when $y=3$ then $x=2-y$ gives $x=2-3=-1$. Hence $\{-1,3\}$ is another solution. Putting all these together gives the solutions as

$$\{0,0\},\left\{0,\frac{3}{2}\right\},\{2,0\},\{-1,3\}$$

4.2.10.2 Part b

The given system is matrix form is

$$
\begin{aligned}
\begin{bmatrix} x' \\ y' \end{bmatrix} &= \mathbf{F} \\
&= \begin{bmatrix} 2x - x^2 - xy \\ 3y - 3xy - 2y^2 \end{bmatrix} \\
&= \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}
\end{aligned}
$$

The Jacobian matrix for the system is given by the gradient of $\mathbf{F}$

$$
\begin{aligned}
J &= \nabla \mathbf{F} \\
J &= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial(2x-x^2-xy)}{\partial x} & \frac{\partial(2x-x^2-xy)}{\partial y} \\ \frac{\partial(3y-3xy-2y^2)}{\partial x} & \frac{\partial(3y-3xy-2y^2)}{\partial y} \end{bmatrix} \\
&= \begin{bmatrix} 2-2x-y & -x \\ -3y & 3-3x-4y \end{bmatrix}
\end{aligned}
$$

Selecting points $\{0,0\}$ and $\left\{0,\frac{3}{2}\right\}$ for analysis.

At Point $\{0,0\}$ the linearized system matrix A is the Jacobian matrix evaluated at this equilibrium point. Hence

$$
\begin{aligned}
A &= \begin{bmatrix} 2-2x-y & -x \\ -3y & 3-3x-4y \end{bmatrix}_{\substack{x=0 \\ y=0}} \\
&= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}
\end{aligned}
$$

The eigenvalues are found by solving $\det(A - \lambda I) = 0$ or

$$
\begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0
$$

$$
(2-\lambda)(3-\lambda) = 0
$$

Hence $\lambda_1 = 2$ and $\lambda_2 = 3$. Since both eigenvalues are positive, then this is unstable critical point. It is a negative attractor also called a node.

At Point $\left\{0,\frac{3}{2}\right\}$ the linearized system matrix A is the Jacobian matrix evaluated at this equilibrium point. Hence

$$A = \begin{bmatrix} 2-2x-y & -x \\ -3y & 3-3x-4y \end{bmatrix}_{\substack{x=0 \\ y=\frac{3}{2}}}$$
$$= \begin{bmatrix} 2-\frac{3}{2} & 0 \\ -3\left(\frac{3}{2}\right) & 3-4\left(\frac{3}{2}\right) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{9}{2} & -3 \end{bmatrix}$$

The eigenvalues are found by solving $\det(A-\lambda I)=0$ or

$$\begin{vmatrix} \frac{1}{2}-\lambda & 0 \\ -\frac{9}{2} & -3-\lambda \end{vmatrix} = 0$$
$$\left(\frac{1}{2}-\lambda\right)(-3-\lambda) = 0$$

Hence $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -3$. Since one eigenvalue is positive, and one eigenvalue is negative, then this is unstable critical point. It is a saddle point.