HW 5

Math 2520
Differential Equations and Linear Algebra

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Normandale college, Bloomington, Minnesota.

Nasser M. Abbasi

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Contents

1 Problem 1

Solve the following Differential Equations

a $y'' - y' - 2y = 5e^{2x}$

b $y'' + 16y = 4\cos x$

c $y'' - 4y' + 3y = 9x^2 + 4, y(0) = 6, y'(0) = 8$

Solution

1.1 part a

$$y'' - y' - 2y = 5e^{2x}$$

This is a nonhomogeneous linear second order ODE with constant coefficients. The general solution is given by

$$y = y_h + y_p \tag{1}$$

Where y_h is the solution to the homogeneous part and y_p is a particular solution. The first step is to determine y_h which is solution to $y'' - y' - 2y = 0$. The characteristic equation becomes (by assuming the solution to be $y = e^{\lambda x}$ and substituting this into the ODE and simplifying)

$$\begin{aligned} \lambda^2 - \lambda - 2 &= 0 \\ (\lambda - 2)(\lambda + 1) &= 0 \end{aligned}$$

The roots are $\lambda_1 = 2, \lambda_2 = -1$. Therefore the basis for y_h are $\left\{e^{2x}, e^{-x}\right\}$ and y_h is linear combination of these basis which is

$$y_h = c_1 e^{2x} + c_2 e^{-x} \tag{2}$$

Looking at RHS of the ODE which is $5e^{2x}$ shows that the basis function for this is $\left\{e^{2x}\right\}$. But e^{2x} is also also a basis function for y_h. Therefore this is adjusted by multiplying it by x and it becomes $\left\{xe^{2x}\right\}$ which no longer a basis for y_h.Therefore the trial solution is

$$y_p = Axe^{2x}$$

Hence

$$\begin{aligned} y_p' &= Ae^{2x} + 2Axe^{2x} \\ y_p'' &= 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x} \end{aligned}$$

Substituting the above in the given ode gives

$$\begin{aligned} y_p'' - y_p' - 2y_p &= 5e^{2x} \\ \left(2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}\right) - \left(Ae^{2x} + 2Axe^{2x}\right) - 2\left(Axe^{2x}\right) &= 5e^{2x} \end{aligned}$$

Since $e^{2x} \neq 0$, the above simplifies to

$$\begin{aligned} 2A + 2A + 4Ax - A - 2Ax - 2Ax &= 5 \\ A(2+2-1) + x(4A - 2A - 2A) &= 5 \\ 3A &= 5 \\ A &= \frac{5}{3} \end{aligned}$$

Therefore the particular solution is

$$y_p = \frac{5}{3} x e^{2x} \tag{3}$$

Substituting (2,3) into (1) gives the general solution as

$$\begin{aligned} y(x) &= y_h + y_p \\ &= c_1 e^{2x} + c_2 e^{-x} + \frac{5}{3} x e^{2x} \end{aligned}$$

1.2 part b

$$y'' + 16y = 4\cos x$$

This is a nonhomogeneous linear second order ODE with constant coefficients. The general solution is given by

$$y = y_h + y_p \tag{1}$$

Where y_h is the solution to the homogeneous part and y_p is a particular solution. The first step is to determine y_h which is solution to $y'' + 16y = 0$. The characteristic equation is

$$\begin{aligned} \lambda^2 + 16 &= 0 \\ \lambda^2 &= -16 \\ \lambda &= \pm 4i \end{aligned}$$

The roots are $\lambda_1 = 4i, \lambda_2 = -4i$. Therefore the basis for y_h are $\left\{e^{4ix}, e^{-4ix}\right\}$. These are converted to trigonometric functions using the Euler relation $e^{ix} = \cos(x) + i\sin(x)$ as was done in the last HW and the basis become $\{\cos(4x), \sin(4x)\}$. y_h is a linear combination of these basis.

$$y_h = c_1 \cos(4x) + c_2 \sin(4x) \tag{2}$$

Looking at RHS of the ode which is $4\cos x$ shows that the basis function for y_p is $\{\cos x\}$. Taking all possible derivatives (and ignoring any sign change and constants that appear), results in the basis for y_p as the set $\{\cos x, \sin x\}$. There are no duplications with the basis for y_h found above. Hence the trial solution is

$$y_p = A\cos x + B\sin x$$

Therefore

$$\begin{aligned} y_p' &= -A\sin x + B\cos x \\ y_p'' &= -A\cos x - B\sin x \end{aligned}$$

Substituting the above in the given ode gives

$$\begin{aligned} y_p'' + 16y_p &= 4\cos x \\ (-A\cos x - B\sin x) + 16\,(A\cos x + B\sin x) &= 4\cos x \\ \cos(x)\,(-A + 16A) + \sin(x)\,(-B + 16B) &= 4\cos x \end{aligned}$$

Comparing coefficients gives

$$\begin{aligned} -A + 16A &= 4 \\ -B + 16B &= 0 \end{aligned}$$

Or

$$\begin{aligned} A &= \frac{4}{15} \\ B &= 0 \end{aligned}$$

Therefore the particular solution is

$$y_p = \frac{4}{15}\cos x \tag{3}$$

Substituting (2,3) into (1) gives the general solution

$$\begin{aligned} y(x) &= y_h + y_p \\ &= c_1\cos(4x) + c_2\sin(4x) + \frac{4}{15}\cos x \end{aligned}$$

1.3 part c

$$\begin{aligned} y'' - 4y' + 3y &= 9x^2 + 4 \\ y(0) &= 6 \\ y'(0) &= 8 \end{aligned}$$

This is a nonhomogeneous linear second order ODE with constant coefficients. The general solution is given by

$$y = y_h + y_p \tag{1}$$

Where y_h is the solution to the homogeneous part and y_p is a particular solution. The first step is to determine y_h which is solution to $y'' - 4y' + 3y = 0$. The characteristic equation is

$$
\begin{aligned}
\lambda^2 - 4\lambda + 3 &= 0 \\
(\lambda - 3)(\lambda - 1) &= 0
\end{aligned}
$$

The roots are $\lambda_1 = 3, \lambda_2 = 1$. Therefore the basis for y_h are $\left\{e^{3x}, e^x\right\}$. y_h is a linear combination of these basis.

$$
y_h = c_1 e^{3x} + c_2 e^x \tag{2}
$$

Looking at RHS of the ode $9x^2 + 4$ shows that the basis functions for this are the set $\left\{1, x^2\right\}$. Taking all possible derivatives (and ignoring any sign change and constant multipliers that appear) results in the set $\left\{1, x, x^2\right\}$.There are no duplications with the basis for y_h. Hence the trial solution is linear combination of these basis which is

$$
y_p = A + Bx + Cx^2
$$

Hence

$$
\begin{aligned}
y_p' &= B + 2Cx \\
y_p'' &= 2C
\end{aligned}
$$

Substituting the above in the given ode gives

$$
\begin{aligned}
y_p'' - 4y_p' + 3y_p &= 9x^2 + 4 \\
(2C) - 4(B + 2Cx) + 3\left(A + Bx + Cx^2\right) &= 9x^2 + 4 \\
x^2(3C) + x(-8C + 3B) + (2C - 4B + 3A) &= 9x^2 + 4
\end{aligned}
$$

Comparing coefficients gives

$$
\begin{aligned}
3C &= 9 \\
-8C + 3B &= 0 \\
2C - 4B + 3A &= 4
\end{aligned}
$$

First equation gives $C = 3$. Substituting in second equation gives $-24 + 3B = 0$ or $B = 8$. Third equation now becomes

$$
\begin{aligned}
2(3) - 4(8) + 3A &= 4 \\
A &= 10
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = 10 + 8x + 3x^2 \tag{3}
$$

Substituting (2,3) into (1) gives the general solution

$$\begin{aligned} y(x) &= y_h + y_p \\ &= c_1 e^{3x} + c_2 e^x + 10 + 8x + 3x^2 \end{aligned} \tag{4}$$

Initial conditions are now used to determine c_1, c_2. $y(0) = 6$ gives

$$\begin{aligned} 6 &= c_1 + c_2 + 10 \\ c_1 + c_2 &= -4 \end{aligned} \tag{5}$$

Taking derivative of (4)

$$y' = 3c_1 e^{3x} + c_2 e^x + 8 + 6x$$

Using $y'(0) = 8$ the above becomes

$$\begin{aligned} 8 &= 3c_1 + c_2 + 8 \\ 3c_1 + c_2 &= 0 \end{aligned} \tag{6}$$

Eq (5,6) are solved for c_1, c_2. From (5) $c_1 = -4 - c_2$. Substituting in (6) gives $3(-4 - c_2) + c_2 = 0$, $c_2 = -6$. Hence $c_1 = -4 + 6 = 2$. Therefore the solution (4) now becomes

$$y(x) = 2e^{3x} - 6e^x + 10 + 8x + 3x^2$$

2 Problem 2

Use the variation of parameters method to find the general solution to the given differential equation.

$$y'' + y = \tan^2(x)$$

Solution

This is a nonhomogeneous linear second order ODE with constant coefficients. The general solution is given by

$$y = y_h + y_p \tag{1}$$

where y_h is the solution to the homogeneous part and y_p is a particular solution. The first step is to determine y_h which is solution to $y'' + y = 0$. The characteristic equation is

$$\begin{aligned} \lambda^2 + 1 &= 0 \\ \lambda &= \pm i \end{aligned}$$

The roots are $\lambda_1 = i, \lambda_2 = -i$. Therefore the basis for y_h are $\left\{e^{ix}, e^{-ix}\right\}$. Using Euler relation these become $\{\cos x, \sin x\}$. Hence y_h is a linear combination of these basis

$$y_h = c_1 \cos x + c_2 \sin x \tag{2}$$

Using variation of parameters, let $y_p = y_1 u_1 + y_2 u_2$, where

$$\begin{aligned} y_1 &= \cos x \\ y_2 &= \sin x \end{aligned}$$

Are the basis of y_h found above, and u_1, u_2 are functions yet to be determined. Applying reduction of order as shown in the textbook (not repeated here) gives

$$u_1 = -\int \frac{y_2 g(x)}{W(x)} dx \tag{3}$$

$$u_2 = \int \frac{y_1 g(x)}{W(x)} dx \tag{4}$$

Where in the above $W(x)$ is the Wronskian and $g(x)$ is the forcing function which is $g(x) = \tan^2(x)$ in this case. The first step is to calculate $W(x)$

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \\ &= \cos^2 x + \sin^2 x \\ &= 1 \end{aligned}$$

Therefore (3) becomes

$$u_1 = -\int \sin x \tan^2 x \, dx$$

But $\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1$. Hence the above becomes

$$\begin{aligned} u_1 &= -\int \sin x \left(\frac{1}{\cos^2 x} - 1 \right) dx \\ &= -\int \left(\frac{\sin x}{\cos^2 x} - \sin x \right) dx \\ &= -\int \frac{\sin x}{\cos^2 x} dx + \int \sin x \, dx \\ &= -\int \tan x \frac{1}{\cos x} dx + \int \sin x \, dx \end{aligned} \tag{5}$$

To find the first integral in (5), let $u = \frac{1}{\cos x}$. Then $du = -(\cos x)^{-2} (-\sin x) \, dx = \frac{\sin x}{\cos^2 x} dx$. Hence $dx = \frac{\cos^2 x}{\sin x} du = \frac{\cos x}{\tan x} du$. Therefore the first integral in (5) becomes

$$\begin{aligned} -\int \tan x \frac{1}{\cos x} dx &= -\int (\tan x) \, u \left(\frac{\cos x}{\tan x} du \right) \\ &= -\int u \cos x \, du \end{aligned}$$

But $\cos x = \frac{1}{u}$. The above becomes

$$\begin{aligned} -\int \tan x \frac{1}{\cos x} dx &= -\int du \\ &= -u \\ &= -\frac{1}{\cos x} \end{aligned}$$

The second integral in (5) is just $\int \sin x \, dx = -\cos x$. Therefore (5) becomes

$$\begin{aligned} u_1 &= -\frac{1}{\cos x} - \cos x \\ &= \frac{-1 - \cos^2 x}{\cos x} \\ &= -\frac{1 + \cos^2 x}{\cos x} \end{aligned}$$

Now u_2 in (4) is found.

$$\begin{aligned}
u_2 &= \int \cos x \tan^2 x \, dx \\
&= \int \cos x \frac{\sin^2 x}{\cos^2 x} \, dx \\
&= \int \frac{\sin^2 x}{\cos x} \, dx \\
&= \int \frac{1 - \cos^2 x}{\cos x} \, dx \\
&= \int \left(\frac{1}{\cos x} - \cos x \right) \, dx \\
&= \int \frac{1}{\cos x} dx - \int \cos x \, dx \\
&= \int \sec x dx - \int \cos x \, dx && (6)
\end{aligned}$$

To find $\int \sec x \, dx$, we start by multiplying the integrand by $\frac{\sec x + \tan x}{\sec x + \tan x} = 1$. Hence

$$\begin{aligned}
\int \sec x \, dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx && (7)
\end{aligned}$$

Using the substitution

$$u = \sec x + \tan x$$

Then

$$\frac{du}{dx} = \frac{d}{dx} \sec x + \frac{d}{dx} \tan x \tag{7A}$$

But $\frac{d}{dx} \sec x = \frac{d}{dx} (\cos x) = -(\cos x)^{-2} (-\sin x) = \frac{\sin x}{\cos^2 x} = \sin x \sec^2 x = \sec x \tan x$. And

$$\begin{aligned}
\frac{d}{dx} \tan x &= 1 + \tan^2 x \\
&= 1 + \frac{\sin^2 x}{\cos^2 x} \\
&= \frac{\cos^2 + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}$$

Hence (7A) becomes

$$\frac{du}{dx} = \sec x \tan x + \sec^2 x$$

Therefore (7) now becomes

$$\begin{aligned}\int \sec x \, dx &= \int \frac{\sec^2 x + \sec x \tan x}{u} \frac{du}{\sec x \tan x + \sec^2 x} \\ &= \int \frac{du}{u} \\ &= \ln u \\ &= \ln(\sec x + \tan x)\end{aligned}$$

Eq (6) now becomes

$$\begin{aligned}u_2 &= \int \sec x dx - \int \cos x \, dx \\ &= \ln(\sec x + \tan x) - \sin x\end{aligned}$$

Now that u_1, u_2 are found, then $y_p = y_1 u_1 + y_2 u_2$ gives

$$\begin{aligned}y_p &= -\cos x \left(\frac{1+\cos^2 x}{\cos x}\right) + \sin x (\ln(\sec x + \tan x) - \sin x) \\ &= -\left(1+\cos^2 x\right) + \sin x (\ln(\sec x + \tan x) - \sin x) \\ &= -1 - \cos^2 x + \sin x \ln(\sec x + \tan x) - \sin^2 x \\ &= -1 - \left(\cos^2 x + \sin^2 x\right) + \sin x \ln(\sec x + \tan x) \\ &= -2 + \sin x \ln(\sec x + \tan x) \\ &= -2 + \sin x \ln\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) \\ &= -2 + \sin x \ln\left(\frac{1+\sin x}{\cos x}\right)\end{aligned}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= c_1 \cos x + c_2 \sin x + \sin x \ln\left(\frac{1+\sin x}{\cos x}\right) - 2\end{aligned}$$

3 Problem 3

Show that the given vector functions are linearly independent on $(-\infty, \infty)$

$$\vec{x}_1(t) = \begin{bmatrix} t \\ t \end{bmatrix} \qquad \vec{x}_2(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$$

Solution

The Wronskian of these vectors is

$$W(t) = \begin{vmatrix} t & t \\ t & t^2 \end{vmatrix}$$

If the above is nonzero at some point in the interval $(-\infty, \infty)$ then $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent.

$$\begin{aligned} W(t) &= t^3 - t^2 \\ &= t^2 (t-1) \end{aligned}$$

Any point other than $t = 0, t = 1$, then $W(t) \neq 0$. For example at $t = 2$, $W(2) = 4 \neq 0$. Hence $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent .

4 Problem 4

Show that the given vector functions are linearly independent on $(-\infty, \infty)$

$$\vec{x}_1(t) = \begin{bmatrix} e^t \\ 2e^t \end{bmatrix} \qquad \vec{x}_2(t) = \begin{bmatrix} 4e^t \\ 8e^{2t} \end{bmatrix}$$

Solution

The Wronskian of these vectors is

$$W(t) = \begin{vmatrix} e^t & 4e^t \\ 2e^t & 8e^{2t} \end{vmatrix}$$

If the above is nonzero at some point in the interval $(-\infty, \infty)$ then $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent.

$$\begin{aligned} W(t) &= 8e^{3t} - 6e^{2t} \\ &= e^{2t}\left(8e^t - 6\right) \end{aligned}$$

Choosing say $t = 0$ then the above becomes $W(0) = 2 \neq 0$. Therefore $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent.

5 Problem 5

Show that the given functions are solutions of the system $x'(t) = A(x)x(t)$ for the given matrix A and hence find the general solution to the system (remember to check linear independence). Then find the particular solution for the given auxiliary conditions.

$$\vec{x}_1(t) = \begin{bmatrix} e^{4t} \\ 2e^{4t} \end{bmatrix} \qquad \vec{x}_2(t) = \begin{bmatrix} 3e^{-t} \\ e^{-t} \end{bmatrix}$$
$$A = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix} \qquad x(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

<u>Solution</u>

The system to solve is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

We need to first find the eigenvalues and eigenvectors of A. The eigenvalues are solution to $|A - \lambda I| = 0$ or

$$\begin{vmatrix} -2-\lambda & 3 \\ -2 & 5-\lambda \end{vmatrix} = 0$$
$$(-2-\lambda)(5-\lambda) + 6 = 0$$
$$\lambda^2 - 3\lambda - 10 + 6 = 0$$
$$\lambda^2 - 3\lambda - 4 = 0$$
$$(\lambda - 4)(\lambda + 1) = 0$$

Hence the eigenvalues are $\lambda_1 = 4, \lambda_2 = -1$.

<u>$\lambda_1 = 4$</u>

$$\begin{bmatrix} -2-\lambda & 3 \\ -2 & 5-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2-4 & 3 \\ -2 & 5-4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -6 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 + \frac{1}{3}R_1$

$$\begin{bmatrix} -6 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence v_1 is base variable and $v_2 = t$ is free variable. First row gives $-6v_1 = -3t$ or $v_1 = \frac{1}{2}t$. The eigenvector is then

$$\vec{v}_{\lambda_1} = \begin{bmatrix} v_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Choosing $t = 1$, then

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore the first basis vector solution is given by

$$\begin{aligned} \vec{x}_1(t) &= e^{\lambda_1 t}\vec{v}_{\lambda_1} \\ &= e^{4t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} e^{4t} \\ 2e^{4t} \end{bmatrix} \end{aligned}$$

$\underline{\lambda_1 = -1}$

$$\begin{bmatrix} -2-\lambda & 3 \\ -2 & 5-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2+1 & 3 \\ -2 & 5+1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$

$$\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence v_1 is base variable and $v_2 = t$ is free variable. First row gives $-v_1 = -3t$ or $v_1 = 3t$. The eigenvector is then

$$\vec{v}_{\lambda_2} = \begin{bmatrix} v_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Choosing $t = 1$, then

$$\vec{v}_{\lambda_2} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Therefore the second basis vector solution is given by

$$\begin{aligned}\vec{x}_2(t) &= e^{\lambda_2 t}\,\vec{v}_{\lambda_2} \\ &= e^{-t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3e^{-t} \\ e^{-t} \end{bmatrix}\end{aligned}$$

The above result shows that the solution to $x'(t) = A(x)x(t)$ is

$$\begin{aligned}\vec{x}(t) &= c_1\vec{x}_1(t) + c_2\vec{x}_2(t) \qquad (1) \\ &= c_1\begin{bmatrix} e^{4t} \\ 2e^{4t} \end{bmatrix} + c_2\begin{bmatrix} 3e^{-t} \\ e^{-t} \end{bmatrix}\end{aligned}$$

Now we check that $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent (they have to be, since they are eigenvectors of A, but the problem is asking to verify this result). The Wronskian of these vectors is

$$W(t) = \begin{vmatrix} e^{4t} & 3e^{-t} \\ 2e^{4t} & e^{-t} \end{vmatrix}$$

If the above is nonzero at some point in the interval $(-\infty, \infty)$ then $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent.

$$\begin{aligned}W(t) &= e^{3t} - 6e^{3t} \\ &= -5e^{3t}\end{aligned}$$

Choosing say $t = 0$ then the above becomes $W(0) = -5$. Since we found at least one point where $W(t) \neq 0$ then $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent and (1) is the general solution to given system of differential equations. This answers the first part of the question by showing that the given functions are solutions of the system $x'(t) = A(x)\,x(t)$.

The final step is to find the particular solution to the given initial conditions $x(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

At $t = 0$ the solution in (1) becomes

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Which can be written as

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \tag{2}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 1 & 1 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -5 & 5 \end{bmatrix}$$

Hence (2) becomes

$$\begin{bmatrix} 1 & 3 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} \tag{3}$$

Second row gives $-5c_2 = 5$ or $c_2 = -1$. First row gives $c_1 + 3c_2 = -2$ or $c_1 = -2 - 3(-1) = -2 + 3 = 1$. Hence

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore the solution (1) becomes

$$\vec{x}(t) = \begin{bmatrix} e^{4t} \\ 2e^{4t} \end{bmatrix} - \begin{bmatrix} 3e^{-t} \\ e^{-t} \end{bmatrix}$$

Or

$$\begin{aligned} x_1(t) &= e^{4t} - 3e^{-t} \\ x_2(t) &= 2e^{4t} - e^{-t} \end{aligned}$$

6 Problem 6

Solve the initial-value problem $x' = Ax, x(0) = x_0$

$$A = \begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix} \qquad x(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

<u>Solution</u>

The system is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

We need to first find the eigenvalues and eigenvectors of A. The eigenvalues are solutions to $|A - \lambda I| = 0$ or

$$\begin{vmatrix} -1-\lambda & 4 \\ 2 & -3-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(-3-\lambda) - 8 = 0$$

$$\lambda^2 + 4\lambda + 3 - 8 = 0$$

$$\lambda^2 + 4\lambda - 5 = 0$$

$$(\lambda - 1)(\lambda + 5) = 0$$

Hence the eigenvalues are $\lambda_1 = 1, \lambda_2 = -5$.

<u>$\lambda_1 = 1$</u>

$$\begin{bmatrix} -1-\lambda & 4 \\ 2 & -3-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 + R_1$

$$\begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

v_1 is base variables and $v_2 = t$ is free variable. First row gives $-2v_1 = -4t$ or $v_1 = 2t$. Hence the first eigenvector is

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Or for $t = 1$

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The first basis vector solution is therefore

$$\begin{aligned} \vec{x}_1 &= e^{\lambda_1 t} \vec{v}_{\lambda_1} \\ &= e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2e^t \\ e^t \end{bmatrix} \end{aligned} \tag{1}$$

$\underline{\lambda_1 = -5}$

$$\begin{bmatrix} -1-\lambda & 4 \\ 2 & -3-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - \frac{1}{2} R_1$

$$\begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

v_1 is base variables and $v_2 = t$ is free variable. First row gives $4v_1 = -4t$ or $v_1 = -t$. Hence the second eigenvector is

$$\vec{v}_{\lambda_2} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Or for $t = 1$

$$\vec{v}_{\lambda_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The second basis vector solution is therefore

$$\begin{aligned} \vec{x}_2 &= e^{\lambda_2 t} \vec{v}_{\lambda_2} \\ &= e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -e^{-5t} \\ e^{-5t} \end{bmatrix} \end{aligned} \tag{2}$$

From (1,2), the general solution is linear combination of (1,2) which is

$$\begin{aligned}\vec{x}(t) &= c_1\vec{x}_1(t) + c_2\vec{x}_2(t)\\ &= c_1\begin{bmatrix}2e^t\\ e^t\end{bmatrix} + c_2\begin{bmatrix}-e^{-5t}\\ e^{-5t}\end{bmatrix}\end{aligned}\tag{3}$$

Now $\vec{x}_1(t), \vec{x}_2(t)$ are verified to be linearly independent using the Wronskian.

$$\begin{aligned}W(t) &= \begin{vmatrix}2e^t & -e^{-5t}\\ e^t & e^{-5t}\end{vmatrix}\\ &= 2e^{-4t} + e^{-4t}\\ &= 3e^{-4t}\end{aligned}$$

At $t = 0$, $W(0) = 3 \neq 0$. Hence $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent. c_1, c_2 are now found from initial conditions. At $t = 0$, (3) becomes

$$\begin{bmatrix}3\\ 0\end{bmatrix} = c_1\begin{bmatrix}2\\ 1\end{bmatrix} + c_2\begin{bmatrix}-1\\ 1\end{bmatrix}$$

Which can be written as

$$\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}3\\ 0\end{bmatrix}\tag{4}$$

The augmented matrix is

$$\begin{bmatrix}2 & -1 & 3\\ 1 & 1 & 0\end{bmatrix}$$

$R_2 = 2R_2 - R_1$

$$\begin{bmatrix}2 & -1 & 3\\ 0 & 3 & -3\end{bmatrix}$$

Hence (2) becomes

$$\begin{bmatrix}2 & -1\\ 0 & 3\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}3\\ -3\end{bmatrix}$$

Second row gives $c_2 = -1$. First row gives $2c_1 - c_1 = 3$ or $2c_1 = 3 - 1 = 2$. Hence $c_1 = 1$.

$$\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}1\\ -1\end{bmatrix}$$

Therefore the solution (3) becomes

$$\vec{x}(t) = \begin{bmatrix} 2e^{t} \\ e^{t} \end{bmatrix} - \begin{bmatrix} -e^{-5t} \\ e^{-5t} \end{bmatrix}$$

Or

$$x_1(t) = 2e^{t} + e^{-5t}$$
$$x_2(t) = e^{t} - e^{-5t}$$

7 Problem 7

Use the variation of parameters technique to find a particular solution x_p to $x' = Ax + b$ for the given A, b. Also obtain the general solution to the system of differential equations

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 4e^t \end{bmatrix}$$

<u>Solution</u>

The system to solve is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 4e^t \end{bmatrix}$$

The solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the solution to the homogeneous system $x' = Ax$ and $\vec{x}_p(t)$ is a particular solution. First $\vec{x}_h(t)$ is solved for. The eigenvalues and eigenvectors of A are now found. The eigenvalues are solutions to $|A - \lambda I| = 0$ or

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$\begin{aligned} (2-\lambda)(2-\lambda) - 1 &= 0 \\ \lambda^2 - 4\lambda + 4 - 1 &= 0 \\ \lambda^2 - 4\lambda + 3 &= 0 \\ (\lambda - 3)(\lambda - 1) &= 0 \end{aligned}$$

Hence the eigenvalues are $\lambda_1 = 3, \lambda_2 = 1$.

<u>$\lambda_1 = 1$</u>

$$\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 + R_1$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

v_1 is base variables and $v_2 = t$ is free variable. First row gives $v_1 = t$. Hence the first eigenvector is

$$\vec{v}_1 = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Or for $t = 1$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The first basis vector solution is therefore

$$\begin{aligned} \vec{x}_1(t) &= e^{\lambda_1 t}\, \vec{v}_1 \\ &= e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix} \end{aligned} \tag{1}$$

<u>$\lambda_1 = 3$</u>

$$\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - R_1$

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

v_1 is base variables and $v_2 = t$ is free variable. First row gives $v_1 = -t$. Hence the second eigenvector is

$$\vec{v}_2 = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Or for $t = 1$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The second basis vector solution is therefore

$$\begin{aligned}\vec{x}_2(t) &= e^{\lambda_2 t}\,\vec{v}_2\\ &= e^{3t}\begin{bmatrix}-1\\1\end{bmatrix}\\ &= \begin{bmatrix}-e^{3t}\\e^{3t}\end{bmatrix}\end{aligned} \tag{2}$$

From (1,2), the homogeneous is

$$\begin{aligned}\vec{x}_h(t) &= c_1\vec{x}_1(t) + c_2\vec{x}_2(t)\\ &= c_1\begin{bmatrix}e^t\\e^t\end{bmatrix} + c_2\begin{bmatrix}-e^{3t}\\e^{3t}\end{bmatrix}\end{aligned} \tag{2A}$$

The Wronskian can be used to show that $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent

$$W(t) = \begin{vmatrix}e^t & -e^{3t}\\e^t & e^{3t}\end{vmatrix} = e^{4t} + e^{4t} = 2e^{4t}$$

Which is not zero at a point, say at $t = 0$. Variation of parameters is now used to find the particular solution $\vec{x}_p(t)$. The fundamental matrix is the matrix whose columns are $\vec{x}_1(t), \vec{x}_2(t)$

$$\begin{aligned}\Phi &= \begin{bmatrix}\vec{x}_1(t) & \vec{x}_2(t)\end{bmatrix}\\ &= \begin{bmatrix}e^t & -e^{3t}\\e^t & e^{3t}\end{bmatrix}\end{aligned}$$

Therefore

$$\begin{aligned}\vec{x}_p(t) &= \Phi\int\Phi^{-1}\vec{b}(t)dt\\ &= \begin{bmatrix}e^t & -e^{3t}\\e^t & e^{3t}\end{bmatrix}\int\begin{bmatrix}e^t & -e^{3t}\\e^t & e^{3t}\end{bmatrix}^{-1}\begin{bmatrix}0\\4e^t\end{bmatrix}dt\end{aligned} \tag{3}$$

But

$$\begin{bmatrix}e^t & -e^{3t}\\e^t & e^{3t}\end{bmatrix}^{-1} = \frac{\begin{bmatrix}e^{3t} & e^{3t}\\-e^t & e^t\end{bmatrix}}{|\Phi|}$$

And

$$|\Phi| = \begin{vmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{vmatrix} = e^{4t} + e^{4t} = 2e^{4t}$$

Therefore

$$\begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} e^{-t} & e^{-t} \\ -e^{-3t} & e^{-3t} \end{bmatrix}$$

Substituting the above in (3) gives

$$\vec{x}_p(t) = \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix} \frac{1}{2} \int \begin{bmatrix} e^{-t} & e^{-t} \\ -e^{-3t} & e^{-3t} \end{bmatrix} \begin{bmatrix} 0 \\ 4e^t \end{bmatrix} dt$$

But $\begin{bmatrix} e^{-t} & e^{-t} \\ -e^{-3t} & e^{-3t} \end{bmatrix} \begin{bmatrix} 0 \\ 4e^t \end{bmatrix} = \begin{bmatrix} 4 \\ 4e^{-2t} \end{bmatrix}$. Hence the above becomes

$$\vec{x}_p(t) = \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix} \int \begin{bmatrix} 2 \\ 2e^{-2t} \end{bmatrix} dt$$

Carrying the integration element by element gives

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix} \begin{bmatrix} \int 2dt \\ \int 2e^{-2t} dt \end{bmatrix} \\ &= \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix} \begin{bmatrix} 2t \\ -e^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} 2te^t + e^t \\ 2te^t - e^t \end{bmatrix} \end{aligned} \tag{4}$$

Substituting (2A) and (4) into $\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$ gives the final solution as

$$\begin{aligned} \vec{x}(t) &= c_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} -e^{3t} \\ e^{3t} \end{bmatrix} + \begin{bmatrix} 2te^t + e^t \\ 2te^t - e^t \end{bmatrix} \\ &= \begin{bmatrix} c_1 e^t - c_2 e^{3t} + 2te^t + e^t \\ c_1 e^t + c_2 e^{3t} + 2te^t - e^t \end{bmatrix} \end{aligned}$$

Or

$$\begin{aligned} x_1(t) &= c_1 e^t - c_2 e^{3t} + 2te^t + e^t \\ x_2(t) &= c_1 e^t + c_2 e^{3t} + 2te^t - e^t \end{aligned}$$