Using MLE and EM for Image Reconstruction

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1 Maximum Likelihood Estimation(MLE)

Suppose the unknown object is represented by a set of N volume elements or voxels. The object can then be described by an $N \times 1$ column vector f , the nth element of which is f_n. Now we map f to projection space with a $M \times N$ matrix H, then the data consist of a set of M measurements $\mathbf{g}_m, m = 1, \ldots, M$, which can be regarded as components of an $M \times 1$ column vector g. the index m specifies both a particular detector element and the projection angle. $\mathbf{g}_m$, is the number of photons detected in the mth data bin. We can see H_{mn} as probability of detecting a photon from voxel n and detected by mth measurement. For a fixed f each $\mathbf{g}_m$ is a Poisson random variable. We can thus write the conditional probability $\mathbf{g}_m$ conditioned on a particular f, as

$$\mathbf{P}\left(\mathbf{g}_m|f\right) = \left[\frac{(\bar{\mathbf{g}}_m)^{\mathbf{g}_m}}{\mathbf{g}_m}\right]\mathbf{e}^{-\bar{\mathbf{g}}_m}, \tag{1}$$

Where $\mathbf{g}_m$ is the conditional expectation value of $\mathbf{g}_m$ given f. or $\mathbf{E}(\mathbf{g}_m|f)$. For given f the components of g are independent, so the joint probability of $\mathbf{g}_m$'s is

$$\mathbf{P}\left(\mathbf{g}|\mathbf{f}\right) = \prod_{\mathbf{m=1}}^{\mathbf{M}} \mathbf{P}\left(\mathbf{g}_m|f\right) = \prod_{\mathbf{m=1}}^{\mathbf{M}} \left[\frac{(\bar{\mathbf{g}}_m)^{\mathbf{g}_m}}{\mathbf{g}_m}\right]\mathbf{e}^{-\bar{\mathbf{g}}_m}, \tag{2}$$

and this is nothing but likelihood. Where $\bar{\mathbf{g}}_m = \sum_{m=1}^{M} H_{mn} f_n$. If we maximize log of equation (2) we will get,

$$f_n = f_n \frac{1}{s_n} \sum_{m=1}^{M} \frac{\mathbf{g}_m}{(Hf)_m} H_{mn}, \tag{3}$$

Where $s_n = \sum_{m=1}^{M} H_{mn}$ Shepp and Verdi(1982) claimed that with no loss of generality we can assume that $\sum_{m=1}^{M} H_{mn} = 1$

2 EM Algorithm

We start with initial estimate of f, call it f^{old}, and $f^{old} > 0$ for $n = 1, \ldots, N$. also $\mathbf{g}_m$ which is number of of photon detected in measurement m is our incomplete data. We define complete data $\mathbf{g}_{mn}$, as number of photon from voxel nth and detected in measurement mth, Where $n = 1, \ldots, N$ and $m = 1, \ldots, M$. For a given f^{old} then our E step of EM algorithm is

$$\hat{\mathbf{g}_n} = E\left(\mathbf{g}_n | f^{old}, \mathbf{g}_m, m = 1, \ldots, M\right) \tag{4}$$

Where $\mathbf{g}_n$ is is number of photon has been detected any where and $\hat{\mathbf{g}_n}$ is our estimate for number of detected voxel n. If $\hat{\mathbf{g}_n}$ is our estimate of number of voxel n it is also the maximum likelihood estimate for density of f_n; this is our M step. So we have

$$f_n^{new} = \hat{\mathbf{g}_n} = E\left(\mathbf{g}_n | f^{old}, \mathbf{g}_m, m = 1, \ldots, M\right)$$

replacing $\mathbf{g}_n$ by $\sum_{m=1}^{M} \mathbf{g}_{mn}$ we get

$$f_n^{new} = E\left(\sum_{m=1}^{M} \mathbf{g}_{mn} | f^{old}, \mathbf{g}_m, m = 1, \ldots, M\right)$$

or

$$f_n^{new} = \sum_{m=1}^{M} E\left(\mathbf{g}_{mn} | f^{old}, \mathbf{g}_m, m = 1, \ldots, M\right)$$

because $\mathbf{g}_m$'s are independent we have

$$f_n^{new} = \sum_{m=1}^{M} E\left(\mathbf{g}_{mn} | f^{old}, \mathbf{g}_m\right) \tag{5}$$

Notice that $\mathbf{g}_{mn}$'s have independent Poisson distributions with parameters $lambda_{mn}$ which is equal to $f_n H_{mn}$. Also we use this property that if X_i for $i = 1, \ldots, n$ are independent Poisson variables with parameters λ_i then conditional distribution of $X_j | \sum_{i=1}^{n} X_i = x^*$ is Binomial$(x^*, \lambda_j / \sum_{i=1}^{n} \lambda_i)$ Using this property we can write

$$E\left(\mathbf{g}_{mn} | f^{old}, \mathbf{g}_m\right) = \frac{\mathbf{g}_m f_n^{old} H_{mn}}{\sum_{n=1}^{N} f_n^{old} H_{mn}}$$

combining this and (5) we will get

$$f_n^{new} = f_n^{old} \sum_{m=1}^{M} \frac{\mathbf{g}_m H_{mn}}{\sum_{n=1}^{N} f_n^{old} H_{mn}} \tag{6}$$