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Mathematical Relation of HYPR to MLEM

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Reference HYPR: Rafael O'Halloran

EM Theory: Harrison H Barrett

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What We Need (HYPR)

$$s_t = R_{\phi_t}\left[I_t\right]$$

$$J_n = \frac{1}{N_p} C_n \cdot R_\phi^u \left(\frac{s}{R_\phi(C)} \right)_n$$

R_ϕ – *Radon Transform*

R_ϕ^u – *Unfiltered backprojection*

N_p – *Projections per time frame*

s – *Original projection*

C – *Composite image*

Deriving MLEM based on [1]

- ***Eq 1 shows likelihood function to be maximized over*** I_t
- ***Eq 2 shows log likelihood, easier to manipulate***

$$s_t = R_{\Phi_t}[I_t]$$

(1) $$\Pr(s_t|I_t) = \prod_{m=1}^{M} \exp[(-R_\Phi I_t)_m] * ([(R_\Phi I_t)_m]^{s_{t_m}} / (s_{t_m}!))$$ ¿

(2) $$Ln[\Pr(s_t|I_t)] = \sum_{m=1}^{M}$$ ¿¿¿

¿

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Deriving MLEM based on [1]

- ***To maximize, differentiate Eq. 2 and set = 0***

$$\partial / \partial I_{t_j}\left(Ln\left[\Pr\left(S_t \middle| I_t\right)\right]\right) = \sum_{m=0}^{M} ¿¿¿¿_¿$$

- ***Re-writing as fixed-point iteration***

¿

(3)

- ***Eq. 3 defines MLEM formula***

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MLEM – 1 Algorithm

$$\hat{I}_{tn}^{(1)} = \hat{I}_{tn}^{(0)} * (1/z_n) * \{R_\Phi^{\mathrm{T}}[\mathrm{s}/\mathrm{R}_\Phi I^0]\}_n$$

R_Φ : Radon Transform

R_Φ^T : Unfiltered backprojection

z_n : For an nth pixel of unit intensity, this is sum of its contributions to all detector cells

s : Original projection

I^0 : Composite image

Comparison of MLEM & HYPR

MLEM-1 Algorithm		**HYPR**
$s_t = R_{\phi_t}\left[I_t\right]$	------------	$s_t = R_{\phi_t}\left[I_t\right]$
$\hat{I}_{tn}^{(1)} = \hat{I}_{tn}^{(0)} * (1/z_n) * \{R_\Phi{}^T\,[\mathrm{s}/\mathrm{R}_\Phi I^0]\}_n$	------------	$J_n = \frac{1}{N_p} C_n \cdot R_\phi^u \left(\frac{s}{R_\phi(C)} \right)_n$
R_Φ	------------	R_ϕ – *Radon Transform*
$R_\Phi{}^T$	------------	R_ϕ^u – *Unfiltered backprojection*
z_n	------------	N_p – *Projections per time frame*
s	------------	s – *Original projection*
I^0	------------	C – *Composite image*

Original HYPR equation modified using equation below to get J_n Eq. above

$$\frac{R_\phi^u(s)}{R_\phi^u(s_c)} = R_\phi^u\left(\frac{s}{s_c}\right)$$

The ratio of unfiltered backprojections is the unfiltered backprojection of the ratio. **So MLEM-1=HYPR**

Computational Comparison

- Disk of varying intensity used to explore similarity of MLEM-1 and HYPR
  - Starts out black, gets steadily brighter (white at end)
  - 16 projections used, 4 projections per time frame
  - Bit-reversed ordering
- Results shown in following slides
  - Mean of true image for time frame 4 (last frame)
  - Composite Image
  - MLEM-1 reconstruction for last time frame
  - HYPR reconstruction not shown—indistinguishable from MLEM-1
  - Spatial, temporal profiles: Wright HYPR [3], Original HYPR and MLEM-1
  - Component-by-component visual comparison of Original HYPR and MLEM-1
- Results for MLEM-1, Original HYPR very close

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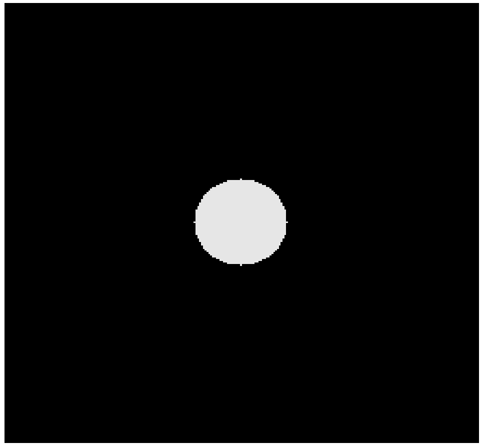
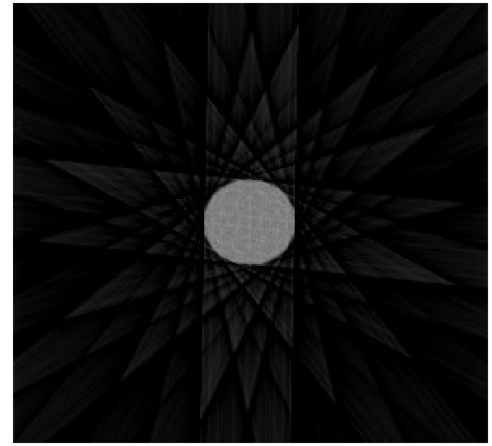
Computational Comparison

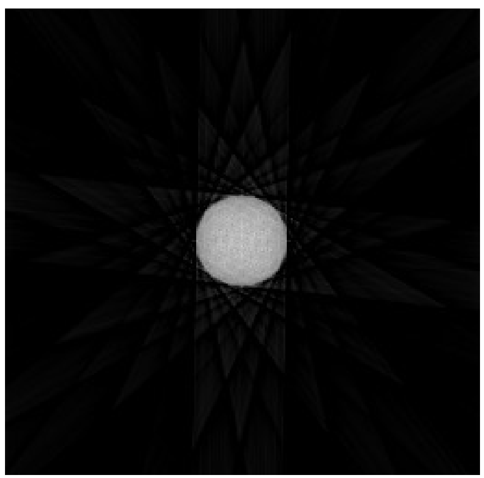
Mean of Last 4 True Images (13-16)



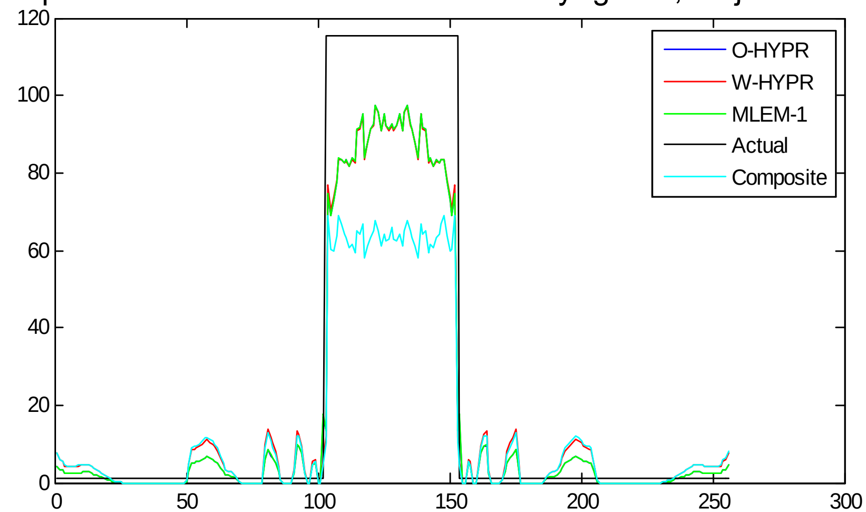
Composite



MLEM-1 Reconstruction for Projections 13-16



Comparison of Reconstructions for Time-Varying Disk, Projections 13-16



For time-varying disk, with no noise, MLEM-1, O-HYPR indistinguishable.

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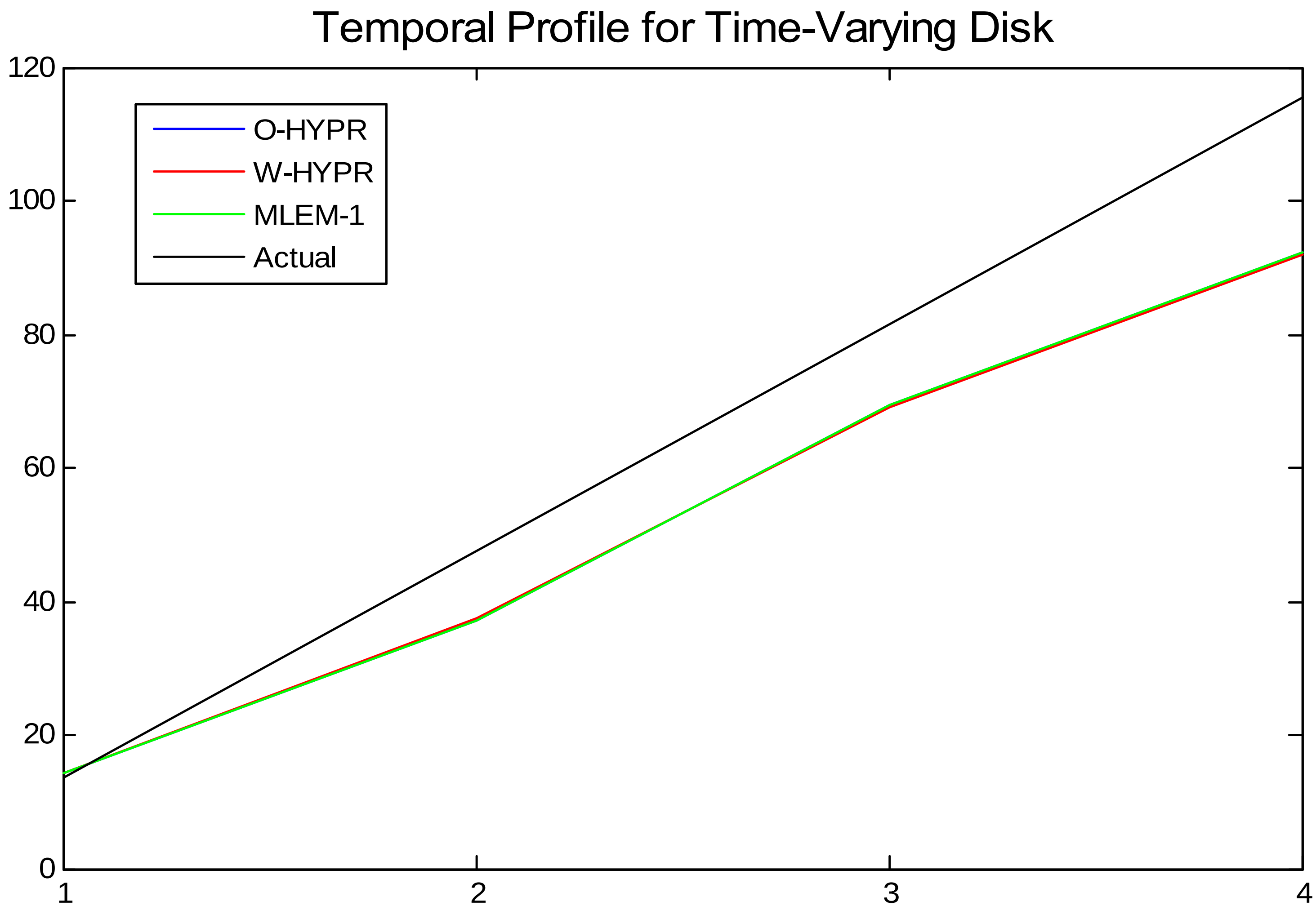
Computational Comparison



MLEM-1, O-HYPR again indistinguishable

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Computational Comparison

- Next slide illustrates how MLEM-1, HYPR arrive at same result via different paths
- Time frame 4 is used (Projections 13-16)
- Each row of images illustrates respective algorithms, excluding multiplication of common composite
- Top row is MLEM-1, bottom is Original HYPR
- For all images, 0=black, 2=white
- Final column demonstrates equality of MLEM-1 and Original HYPR

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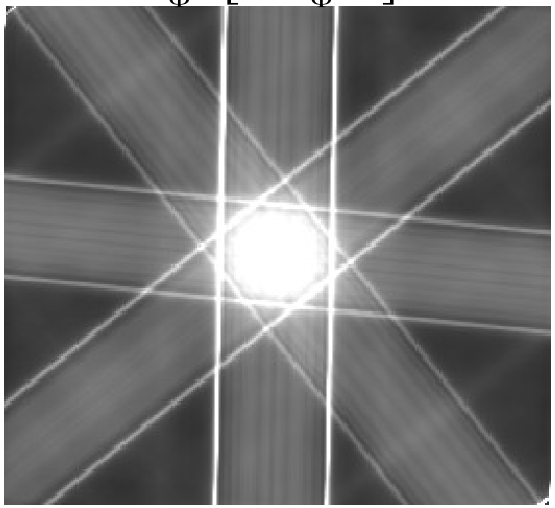
Computational Comparison

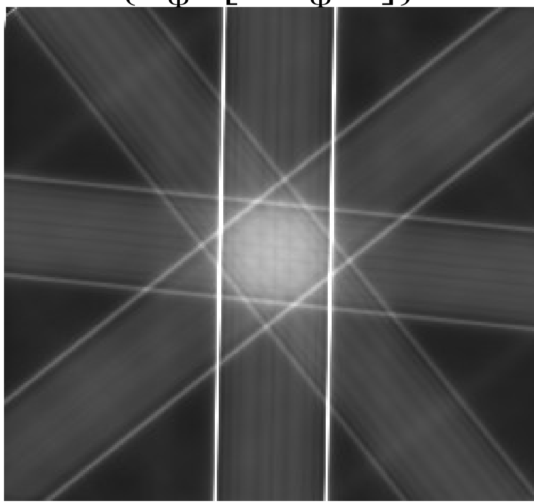
$R_{\Phi}^{T}[s/R_{\Phi}I^{0}]$



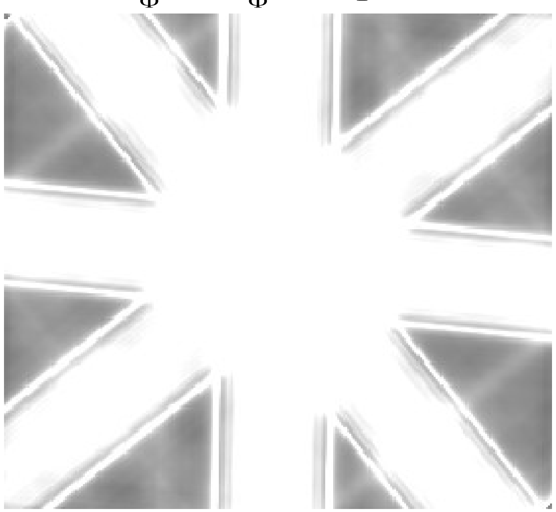
MLEM-1

z: BP of Unit Projections (=2/π)

$\{R_{\Phi}^{T}[s/R_{\Phi}I^{0}]\}/z$



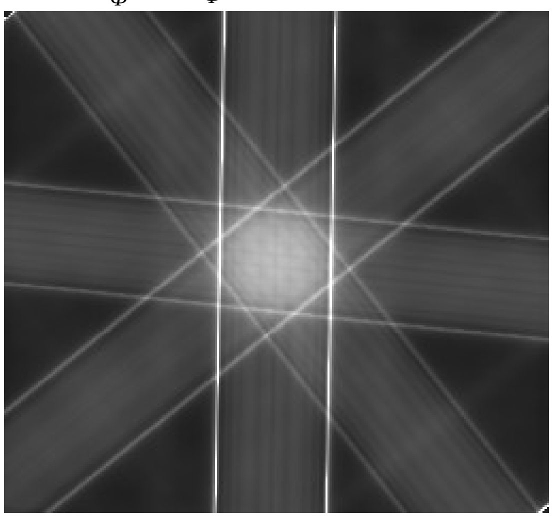
$R_{\Phi}^{u}s/R_{\Phi}^{u}(R_{\Phi}(C))$



O-HYPR

$1/N_p$: One over Num Proj. (=1/4)

$(R_{\Phi}^{u}s/R_{\Phi}^{u}(R_{\Phi}(C)))/N_p$



Conclusion

- MLEM-1 and Original HYPR same method, but with different implementation
- Equality dependent on $\frac{R_\phi^u(s)}{R_\phi^u(s_c)} = R_\phi^u\left(\frac{s}{s_c}\right)$ (not quite true)
  - Numerically, BP of ratio not equal to ratio of BP
  - In discrete implementation, $N_{p,}$ z_n terms not equal for MLEM-1, O-HYPR
  - N_p makes up for difference in ratio of BP vs. BP of ratio

References

1. Barrett, Myers. *Foundations of Image Science*. Wiley-Interscience 2003.

2. (HYPR Reference) Rafael L. O'Halloran, Zhifei Wen, James H. Holmes, and Sean B. Fain. "**Iterative Projection Reconstruction of Time-Resolved Images Using Highly-Constrained Back-Projection (HYPR)**" Magnetic Resonance in Medicine 59:132-139, 2008

3. Yuexi Huang GAW. "Time-resolved MR angiography with limited projections". Magn Reson Med 2007;58:316-325.

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