Hand out 5/27/08
(HYPER) & (LEVEL SET)

the $\hat{\mathbf{f}}$ found by minimizing Q is a linear (or at least affine) function of $\mathbf{g}$ [*cf.* (1.195) or (1.200)], but more generally $\hat{\mathbf{f}}$ is a nonlinear function of $\mathbf{g}$.

Implicit formulations often lead to iterative algorithms for finding the $\hat{\mathbf{f}}$ that minimizes the functional. In these algorithms, successive estimates $\hat{\mathbf{f}}^{(k)}$ are generated according to a recursion rule with the general form,

$$\hat{\mathbf{f}}^{(k+1)} = \mathcal{O}^{(k)} \left\{ \hat{\mathbf{f}}^{(k)}, \mathbf{g} \right\}, \tag{15.5}$$

where $\mathcal{O}^{(k)}\{\hat{\mathbf{f}}^{(k)}, \mathbf{g}\}$ is some operator with two operands, so that its output at each step depends (often nonlinearly) on both the previous estimate $\hat{\mathbf{f}}^{(k)}$ and the original data $\mathbf{g}$. The superscript on $\mathcal{O}$ indicates that the operator itself can change as the iteration proceeds. Often the recursion rule (15.5) will be chosen so that the argmin solution of (15.4) will coincide with $\hat{\mathbf{f}}^{(\infty)}$.

15.1.2 Discretization dilemma

← look at this for Monday 6/1/08

In Chap. 7 we discussed in detail various approximate object representations. The general form of a linear approximation to an object function was given in (7.27) as

$$f_a(\mathbf{r}) = \sum_{n=1}^{N} \theta_n \phi_n(\mathbf{r}), \tag{15.6}$$

where the subscript a denotes *approximate*, and $\{\phi_n(\mathbf{r}), n = 1, ..., N\}$ is any convenient set of expansion functions. In a more compact operator notation, (15.6) becomes

$$\mathbf{f}_a = \mathcal{D}_\phi^\dagger \boldsymbol{\theta}, \tag{15.7}$$

where $\mathcal{D}_\phi$ is a CD discretization operator and $\mathcal{D}_\phi^\dagger$ is its adjoint (hence a DC operator).

If the coefficients $\{\theta_n\}$ in (15.6) are derived linearly from the object, we know from (7.33) that they can be written as

$$\theta_n = \int_\infty d^q r \; \chi_n^*(\mathbf{r}) \, f(\mathbf{r}), \tag{15.8}$$

or in operator form as

$$\boldsymbol{\theta} = \mathcal{D}_\chi \mathbf{f}. \tag{15.9}$$

Mapping a discrete object representation through a CD system In Sec. 7.3 we saw how object functions map through a linear CD system to form discrete data. If the CD system acquires M noisy measurements, the discrete data vector $\mathbf{g}$ is an $M \times 1$ random vector given by

$$\mathbf{g} = \mathcal{H}\mathbf{f} + \mathbf{n}, \tag{15.10}$$

where $\mathcal{H}$ is the linear CD operator defined in Sec. 7.3.1 and $\mathbf{n}$ is an $M \times 1$ noise vector. A simple mathematical tautology allows us to write

$$\mathbf{g} = \mathcal{H}\mathbf{f}_a + \mathcal{H}\mathbf{f} - \mathcal{H}\mathbf{f}_a + \mathbf{n} \equiv \mathbf{H}\boldsymbol{\theta} + \boldsymbol{\epsilon}, \tag{15.11}$$

where the overall error $\boldsymbol{\epsilon}$ (modeling error plus noise) is given by

$$\boldsymbol{\epsilon} = \mathcal{H}\mathbf{f} - \mathcal{H}\mathbf{f}_a + \mathbf{n}, \tag{15.12}$$

and the system matrix $\mathbf{H}$ is given in operator form by (7.307):

$$\mathbf{H} \equiv \mathcal{H}\mathcal{D}_{\phi}^{\dagger}. \tag{15.13}$$

More specifically, the elements of $\mathbf{H}$ are given by (7.304):

$$H_{mn} = \int_{\mathbf{S}_f} d^q r \; h_m(\mathbf{r})\, \phi_n(\mathbf{r}). \tag{15.14}$$

Thus the $(mn)^{th}$ element is the n^{th} expansion function as imaged onto the m^{th} detector element.

Considerations on choosing a discretization scheme The linear discretization problem boils down to selecting two sets of functions, $\{\phi_n(\mathbf{r})\}$ and $\{\chi_n(\mathbf{r})\}$, or equivalently, two sets of Hilbert-space vectors $\{\boldsymbol{\phi}_n\}$ and $\{\boldsymbol{\chi}_n\}$. These vectors affect the accuracy with which $f_a(\mathbf{r})$ approximates the actual object $f(\mathbf{r})$, and they also affect the form and dimensions of the system matrix $\mathbf{H}$ as well as the size and nature of the error vector $\boldsymbol{\epsilon}$. Finally, through (15.8), the set $\{\boldsymbol{\chi}_n\}$ affects the meaning of the parameters $\{\theta_n\}$ that we want to determine.

There are several ways we can approach the problem of choosing $\{\boldsymbol{\phi}_n\}$ and $\{\boldsymbol{\chi}_n\}$. We might say at the outset that we are interested in certain functionals of the object, such as pixel values, and select the functions $\{\boldsymbol{\chi}_n\}$ accordingly. That would leave us free to choose $\{\boldsymbol{\phi}_n\}$ by some other criterion, such as minimizing the data-space modeling error $\mathcal{H}\mathbf{f} - \mathcal{H}\mathbf{f}_a$. Unfortunately, as we shall discover in Sec. 15.1.3, most functionals that we might choose to estimate do not admit of an unambiguous estimate. That is, we cannot determine them from the data even in the absence of noise.

Alternatively, we might want to construct as accurate a representation of the object as possible for a specified number of terms in the expansion (15.6). In Secs. 7.1.4 and 7.1.5, we learned how to choose $\{\boldsymbol{\chi}_n\}$ for minimum object error once $\{\boldsymbol{\phi}_n\}$ was specified. In particular, if $\{\boldsymbol{\phi}_n\}$ is an orthonormal set, then $\{\boldsymbol{\chi}_n\}$ should be chosen to be the same orthonormal set if representational accuracy is our concern.

The choice of $\{\boldsymbol{\phi}_n\}$ itself might be dictated by statistical considerations. If we consider a statistical ensemble of objects, then the ensemble-average representational error is minimized by using the N eigenfunctions of the object covariance matrix corresponding to the N largest eigenvalues as $\{\boldsymbol{\phi}_n\}$. As noted in Sec. 7.1.4, this representation is called the Karhunen-Loève or KL expansion. One problem is that we do not usually have enough information about the ensemble to be able to compute the eigenvectors needed in a KL expansion.

No matter how we choose $\{\boldsymbol{\phi}_n\}$, representational accuracy can be improved by increasing the number of terms N in the representation. The rank of the matrix $\mathbf{H}$, however, cannot exceed the rank R of the CD operator $\mathcal{H}$, which in turn cannot exceed the number of measurements M. Whenever N exceeds R, therefore, the problem of finding $\boldsymbol{\theta}$ when given $\mathbf{g}$ is underdetermined. Moreover, the nature of the effective noise term is unknown, except that $\boldsymbol{\epsilon} \to \mathbf{n}$ as $N \to \infty$ with any sensible choice of expansion functions.

We are thus faced with a conundrum: If we use an accurate object model (large N), we cannot possibly find all the coefficients, and if we use a less accurate model ($N \leq R$), we make unknown modeling errors before even starting to estimate coefficients, and the approximate object $\mathbf{f}_a$ may not resemble the actual object $\mathbf{f}$, even if the coefficients can be determined exactly. It is the view of the authors that the only satisfactory way to resolve this problem is via task-based assessment of image quality, as introduced in Chap. 14 and discussed further in Sec. 15.1.5.

15.1.3 Estimability

If $N < R$ in a linear expansion like (15.6), the number of unknowns is less than the rank of the system operator, but it is still not evident that we can estimate the coefficients $\{\theta_n\}$ from the data $\mathbf{g}$, even in the absence of noise. The coefficients may not be *estimable parameters.* The concept of estimability was briefly introduced in Sec. 13.3; we shall revisit the subject here from the viewpoint of image reconstruction.

Estimability of a single linear parameter Consider a scalar parameter θ defined by the linear functional,

$$\theta = \int_\infty d^q r \; \chi^*(\mathbf{r}) \, f(\mathbf{r}) \,. \tag{15.15}$$

For a mental image, think of θ as the integral of the object over a region defined by a 0-1 function $\chi(\mathbf{r})$, but the mathematics will be more general. If we are given a noise-free data vector $\mathbf{g} = \boldsymbol{\mathcal{H}}\mathbf{f}$, where $\boldsymbol{\mathcal{H}}$ is a linear CD operator, we would like to know whether we can determine θ uniquely from $\mathbf{g}$. An equivalent question is whether we can find an unbiased estimate of θ when zero-mean noise is present.

To answer these questions, note that we can write θ as a scalar product in object space:

$$\theta = (\boldsymbol{\chi}, \mathbf{f}) \,. \tag{15.16}$$

If the system operator $\boldsymbol{\mathcal{H}}$ is linear, we can define two orthogonal subspaces of object space, called measurement space and null space, with the latter defined as the space of all vectors $\mathbf{a}_{null}$ such that $\boldsymbol{\mathcal{H}}\mathbf{a}_{null} = 0$. The vectors $\mathbf{f}$ and $\boldsymbol{\chi}$ can be uniquely decomposed as

$$\mathbf{f} = \mathbf{f}_{meas} + \mathbf{f}_{null} \,, \tag{15.17}$$

$$\boldsymbol{\chi} = \boldsymbol{\chi}_{meas} + \boldsymbol{\chi}_{null} \,. \tag{15.18}$$

Since these two subspaces are orthogonal, we can write

$$\theta = (\boldsymbol{\chi}_{meas}, \mathbf{f}_{meas}) + (\boldsymbol{\chi}_{null}, \mathbf{f}_{null}) \,. \tag{15.19}$$

Since the data vector is insensitive to null components, the first term represents what one can learn about θ from noise-free data, and the second term is the component of θ that cannot cannot be measured with the system in question. This term is zero if *either* $\mathbf{f}_{null} = 0$ *or* $\boldsymbol{\chi}_{null} = 0$. The first condition is often satisfied in simulation studies but seldom in reality; we have no control over the object our imaging system is pointed at, so we cannot assert that $\mathbf{f}_{null} = 0$. We can, however, choose the function $\chi(\mathbf{r})$ defining θ, so we can make the error zero by choosing it so that $\boldsymbol{\chi}_{null} = 0$.

If $\boldsymbol{\chi}_{null} = 0$, the associated parameter θ is said to be *estimable* or *identifiable.* If θ is not estimable, there is an inevitable error of unknown magnitude arising from the second term in (15.19). Objects differing by null functions will give the same data, and hence the same value for any estimate derived from the data, even though they might have vastly different true values for θ.