Derivation of Equation (7) in the paper by Sungwon Yoon, A Pineda, and R. Fahrig

The first term in equation (3) measures the error between the data y and the projection of the object function u. This term has the form $E(u)$, where

$$E(u) = \frac{1}{2} \parallel Pu - y \parallel^2 .$$

The object function u depends on the coefficients c_j, and the level set functions ϕ_1 and ϕ_2 In the following, we derive only the partial derivative with respect either of these level set functions. To simplify notation, then, denote either ϕ_1 or ϕ_2 by ϕ. Now, writing $J(\phi) = E(u)$, and using the chain rule, the partial derivative of J with respect to ϕ is

$$\frac{\partial J}{\partial \phi} = E'(u) \frac{\partial u}{\partial \phi} .$$

The partial derivatives of u respect to ϕ_1 and to ϕ_2 are given in equations (8) and (9). It remains then to find the derivative of E.

For a nonzero vector v, set $q(t) = E(u + tv)$. Then $E'(u)$ is the linear functional such that $q'(0) = E'(u) \cdot v$. To find the derivative of q, note first that

$$q(t) = \frac{1}{2} (Pu - y + tPv, Pu - y + tPv)$$

where that parentheses denote the inner product associated with the given norm. We assume here that all the terms are real. Then, writing out this expression we get

$$q(t) = \frac{1}{2} \parallel Pu - y \parallel^2 + t (Pu - y, Pv) + \frac{1}{2} t^2 \parallel Pv \parallel^2 .$$

Differentiating this expression with respect to t, and then setting $t = 0$, we get

$$q'(0) = (Pu - y, Pv) = (P^*(Pu - y), v) .$$

Thus,

$$E'(u) \cdot v = (P^*(Pu - y), v) .$$

Returning to the expression above for $\partial J/\partial\phi$, we can think of the partial dervitive of J with respect to ϕ as

$$\frac{\partial J}{\partial \phi} = P^*(Pu - y)\frac{\partial u}{\partial \phi} \, .$$

The second term in equation (3), except for the scalar β, is the sum of two terms of the form $S(\phi) = G(u)$, where $u = H(\phi)$, and again ϕ stands for either ϕ_1 or ϕ_2, and where

L2 Norm

$$G(u) = \int_\Omega |\nabla u| \, d^2r = \int_\Omega \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right]^{1/2} d^2r \, .$$

Here we use r to denote the point (x, y) in 2D space.

Using the chain rule, the partial derivative of S with respect to ϕ is

$$\frac{\partial S}{\partial \phi} = G'(u)H'(\phi) = G'(u)\delta(\phi) \, .$$

where we recall that the derivative of the Heaviside function H is the dirac-delta function. It suffices to find the derivative of G.

For a nonzero vector v, set $q(t) = G(u + tv)$. Then $G'(u)$ is the linear functional such that $q'(0) = G'(u) \cdot v$. To find the derivative of q, note first that

$$\begin{aligned} q(t) &= \int_\Omega |\nabla u + t\nabla v| \, d^2r \\ &= \int_\Omega \left[\left(\frac{\partial u}{\partial x} + t\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + t\frac{\partial v}{\partial y}\right)^2 \right]^{1/2} d^2r \, . \end{aligned}$$

Differentiating this expression with respect to t, and then setting $t = 0$, we get

$$q'(0) = \int_\Omega \frac{1}{|\nabla u|} \nabla u \cdot \nabla v \, d^2r \, .$$

Next apply integration by parts to this formula to obtain

$$q'(0) = -\int_\Omega \nabla \cdot \left(\frac{1}{|\nabla u|} \nabla u \right) v \, d^2 r \, .$$

To see how this expression comes about, set

$$w = \frac{1}{|\nabla u|} \nabla u \, ,$$

and note that

$$0 = \int_\Omega \nabla (w \, v) d^2 r = \int_\Omega \nabla w \, v \, d^2 r + \int_\Omega w \, \nabla v \, d^2 r \, .$$

The first integral is zero, since it can written as an integral over the boundary of Ω, and function v is zero on this boundary.

Finally, since $u = H(\phi)$, we have $\nabla u = H'(\phi) \nabla \phi = \delta(\phi) \nabla \phi$. Substituting this expression into the formula for $q'(0)$ gives us

$$q'(0) = -\int_\Omega \nabla \cdot \left(\frac{1}{|\nabla \phi|} \nabla \phi \right) v \, d^2 r \, .$$

Thus, returning to the expression for $\partial S / \partial \phi$ above, we can think of the partial dervitive of S with respect to ϕ as

$$\frac{\partial S}{\partial \phi} = \nabla \cdot \left(\frac{1}{|\nabla \phi|} \nabla \phi \right) \delta(\phi) \, .$$

Combining the two partial derivatives above yields formula (7) of the paper, which is

$$\frac{\partial F}{\partial \phi} = P^*(Pu - y) \frac{\partial u}{\partial \phi} - \beta \, \nabla \cdot \left(\frac{1}{|\nabla \phi|} \nabla \phi \right) \delta(\phi) \, .$$