HW 6 Mathematics 503, Mathematical Modeling, CSUF , June 24, 2007

Nasser M. Abbasi

June 15, 2014

Contents

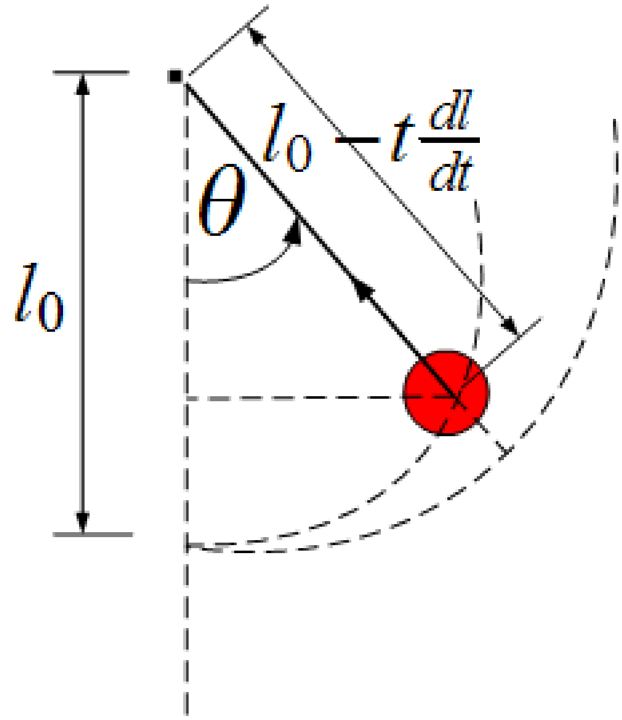
1 Problem 1 (section 3.5,#9, page 197)

problem:

Consider a simple plane pendulum with a bob of mass m attached to a string of length l. After the pendulum is set in motion the string is shortened by a constant rate $\frac{dl}{dt} = -\alpha$. Formulate Hamilton's principle and determine the equation of motion. Compare the Hamiltonian to the total energy. Is energy conserved?

Solution:

$$\frac{dl}{dt} = -\alpha$$



Assume initial string length is l, and assume $t(0) = 0$, then at time t we have

$$r(t) = l - \alpha t$$

K.E. First note that

$$\begin{aligned} \dot{x} &= \frac{d}{dt}\left(r(t)\sin\theta(t)\right) \\ &= \dot{r}\sin\theta + r\cos\theta\dot{\theta} \end{aligned}$$

and

$$\begin{aligned} \dot{y} &= \frac{d}{dt}\left(r(t)\cos\theta(t)\right) \\ &= \dot{r}\cos\theta - r\sin\theta\dot{\theta} \end{aligned}$$

Now

$$\begin{aligned} T &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \\ &= \frac{1}{2}m\left(\left(\dot{r}\sin\theta + r\cos\theta\dot{\theta}\right)^2 + \left(\dot{r}\cos\theta - r\sin\theta\dot{\theta}\right)^2\right) \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \\ &= \frac{1}{2}m\left(\alpha^2 + r^2\dot{\theta}^2\right) \end{aligned}$$

P.E.

$$\begin{aligned} V &= mgl - mg(r\cos\theta) \\ &= mg(l - r\cos\theta) \end{aligned}$$

Hence

$$\begin{aligned} J(\theta) &= \int_0^T (T - V)\,dt \\ &= \int_0^T \frac{1}{2}m\left(\alpha^2 + r^2\dot{\theta}^2\right) - mg(l - r\cos\theta)\; dt \end{aligned}$$

Hence

$$L\left(t, \theta(t), \dot{\theta}(t)\right) = \frac{1}{2}m\left(\alpha^2 + r^2\dot{\theta}^2\right) - mg(l - r\cos\theta) \tag{1}$$

Hence the Euler-Lagrange equations are

$$L_\theta - \frac{d}{dt}\left(L_{\dot{\theta}}\right) = 0 \tag{2}$$

But

$$\boxed{L_\theta = -mgr\sin\theta}$$

and

$$\boxed{L_{\dot{\theta}} = mr^2\dot{\theta}}$$

and

$$\frac{d}{dt}\left(L_{\dot{\theta}}\right) = m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right)$$

But $\dot{r} = -\alpha$, the above becomes

$$\boxed{\tfrac{d}{dt}\left(L_{\dot{\theta}}\right) = m\left(r^2\ddot{\theta} - 2r\alpha\dot{\theta}\right)}$$

Hence $L_\theta - \frac{d}{dt}\left(L_{\dot{\theta}}\right) = 0$ becomes

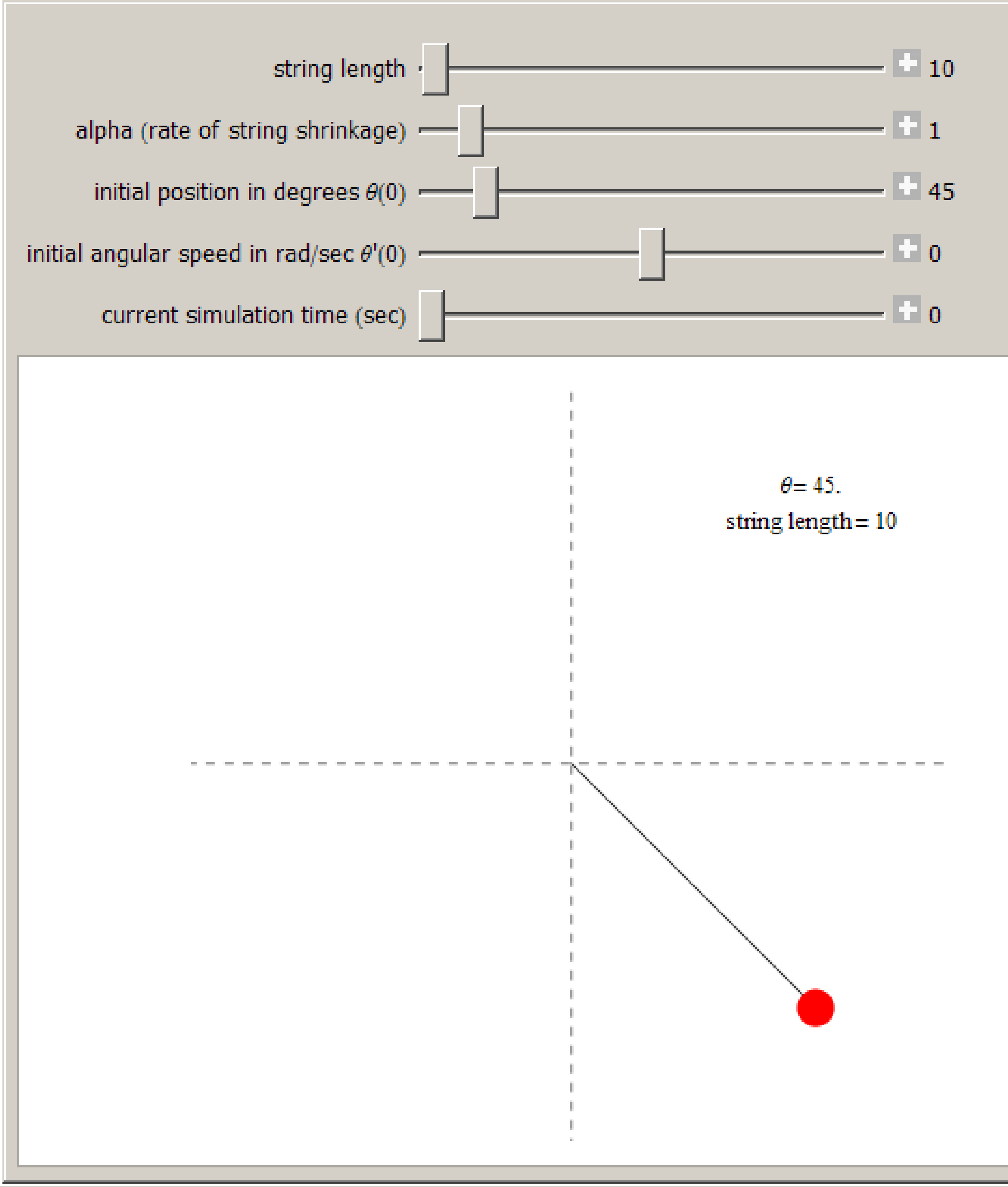
$$\begin{aligned} -mgr\sin\theta - m\left(r^2\ddot{\theta} - 2\alpha\dot{\theta}r\right) &= 0 \\ r^2\ddot{\theta} - 2\alpha\dot{\theta}r + gr\sin\theta &= 0 \end{aligned}$$

Hence the ODE becomes, after dividing by common factor r

$$\boxed{r\ddot{\theta} - 2\alpha\dot{\theta} + g\sin\theta = 0}$$

This is a second order nonlinear differential equation. Notice that when $l = \alpha t$ it will mean that the string has been pulled all the way back to the pivot and $r(t) = 0$. So when running the solution it needs to run from $t = 0$ up to $t = \frac{l}{\alpha}$.

A small simulation was done for the above solution which can be run for different parameters to see the effect more easily. Here is a screen shot.


string length
10
alpha (rate of string shrinkage)
1
initial position in degrees θ(0)
45
initial angular speed in rad/sec θ'(0)
0
current simulation time (sec)
0
θ= 45.
string length= 10

Now we need to determine the Hamiltonian of the system.

$$H = -L(t,\theta,\phi(t,\theta,p)) + \phi(t,\theta,p)\,p \tag{3}$$

Where we define a new variable p called the canonical momentum by

$$\begin{aligned} p &\equiv L_{\dot{\theta}}\left(t,\theta,\dot{\theta}\right) \\ &= mr^2\dot{\theta} \end{aligned}$$

Hence

$$\dot{\theta} = \frac{p}{mr^2}$$

This implies that

$$\phi(t,\theta,p) = \frac{p}{mr^2}$$

Then from (1) and (3), we now calculate H

$$\begin{aligned} H &= -L(t,\theta,\phi(t,\theta,p)) + \phi(t,\theta,p)\,p \\ &= -\left\{\frac{1}{2}m\left(\alpha^2 + r^2\left(\frac{p}{mr^2}\right)^2\right) - mg(l - r\cos\theta)\right\} + \left(\frac{p}{mr^2}\right)p \\ &= -\frac{1}{2}m\left(\alpha^2 + \frac{p^2}{m^2r^2}\right) + mg(l - r\cos\theta) + \frac{p^2}{mr^2} \\ &= -\frac{1}{2}m\alpha^2 - \frac{1}{2}\frac{p^2}{mr^2} + mg(l - r\cos\theta) + \frac{p^2}{mr^2} \end{aligned}$$

Hence the Hamiltonian is

$$H = \frac{1}{2}\frac{p^2}{mr^2} + mg(l - r\cos\theta) - \frac{1}{2}m\alpha^2 \tag{5}$$

Now we are asked to compare H to the total energy. The total instantaneous energy of the system is $(T+V)$, hence we need to determine if $H = T+V$ or not.

$$T + V = \frac{1}{2}m\left(\alpha^2 + r^2\dot{\theta}^2\right) + mg(l - r\cos\theta) \tag{6}$$

To make comparing (5) and (6) easier, I need to either replace p by $mr^2\dot{\theta}$ in (5) or replace $\dot{\theta}$ by $\frac{p}{mr^2}$. Lets do the later, hence (6) becomes

$$\begin{aligned} T + V &= \frac{1}{2}m\left(\alpha^2 + r^2\left(\frac{p}{mr^2}\right)^2\right) + mg(l - r\cos\theta) \\ &= \frac{1}{2}m\left(\alpha^2 + \frac{p^2}{m^2r^2}\right) + mg(l - r\cos\theta) \\ &= \frac{1}{2}m\alpha^2 + \frac{1}{2}\frac{p^2}{mr^2} + mg(l - r\cos\theta) \end{aligned} \tag{7}$$

If H is the total energy, then (7)-(6) should come out to be zero, lets find out

$$H-(T+V)=\left(\frac{1}{2}\frac{p^2}{mr^2}+mg\left(l-r\cos\theta\right)-\frac{1}{2}m\alpha^2\right)-\left(\frac{1}{2}m\alpha^2+\frac{1}{2}\frac{p^2}{mr^2}+mg\left(l-r\cos\theta\right)\right)$$
$$=-m\alpha^2$$

Hence we see that

$$\boxed{H-(T+V)\neq 0}$$

Hence H does not represents the total energy, and the $\boxed{\text{energy of the system is not conserved}}$.[1]

[1]Reading a reference on Noether's theorem, total energy is written as $-(T+V)$ not $(T+V)$, this would not make a difference in showing that $H\neq$ total energy, just different calculations results as shown below, but the same conclusion

$$-(T+V)=-\left(\frac{1}{2}m\left(\alpha^2+r^2\dot{\theta}^2\right)+mg\left(l-r\cos\theta\right)\right) \tag{6}$$

To make comparing (5) and (6) easier, I need to either replace p by $mr^2\dot{\theta}$ in (5) or replace $\dot{\theta}$ by $\frac{p}{mr^2}$, let do the later, hence (6) becomes

$$\begin{aligned}-(T+V)&=-\left(\frac{1}{2}m\left(\alpha^2+r^2\left(\frac{p}{mr^2}\right)^2\right)+mg\left(l-r\cos\theta\right)\right)\\&=-\left(\frac{1}{2}m\left(\alpha^2+\frac{p^2}{m^2r^2}\right)+mg\left(l-r\cos\theta\right)\right)\\&=-\left(\frac{1}{2}m\alpha^2+\frac{1}{2}\frac{p^2}{mr^2}+mg\left(l-r\cos\theta\right)\right) \qquad (7)\end{aligned}$$

If H is the total energy, then (7)-(6) should come out to be zero, lets find out

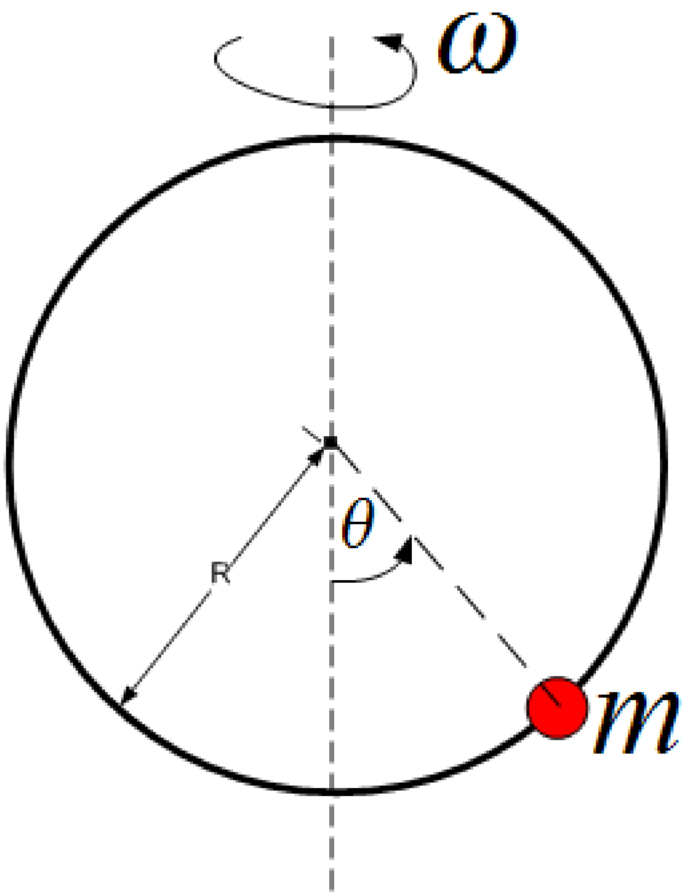
$$H-(-(T+V))=\left(\frac{1}{2}\frac{p^2}{mr^2}+mg\left(l-r\cos\theta\right)-\frac{1}{2}m\alpha^2\right)+\left(\frac{1}{2}m\alpha^2+\frac{1}{2}\frac{p^2}{mr^2}+mg\left(l-r\cos\theta\right)\right)$$
$$=\frac{p^2}{mr^2}+2mg\left(l-r\cos\theta\right)$$

Now we ask, can the above be zero? Since $\frac{p^2}{mr^2}$ is always ≥ 0, and since $(l-r\cos\theta)$ represents the remaining length of the string, hence it is a positive quantity (until such time the string has been pulled all the way in), Therefore RHS above >0. Hence H does not represent the total energy of the system. Hence the energy is not conserved.

2 Problem 1 (section 3.5,#9, page 197)

problem: A bead of mass m slides down the rim of a circular hoop of radius R. The hoop stands vertically and rotates about its diameter with angular velocity ω. Determine the equation of motion of the bead.

Answer:



Kinetic energy T is made up of 2 components, one due to motion of the bead along the hoop itself with speed $R\dot{\theta}$, and another due to motion with angular speed ω which has speed given by $R\sin\theta\omega$

Hence

$$
\begin{aligned}
T &= \frac{1}{2}m\left(\left(R\dot{\theta}\right)^2 + \left(R\sin\theta\omega\right)^2\right) \\
&= \frac{1}{2}mR^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right)
\end{aligned}
$$

P.E. V is due to the bead movement up and down the hoop, which is the standard V for a pendulum given by

$$
V = mgR\left(1-\cos\theta\right)
$$

Hence

$$
\begin{aligned}
L &= T - V \\
&= \frac{1}{2} m R^2 \left(\dot{\theta}^2 + \omega^2 \sin^2 \theta\right) - mgR\left(1 - \cos\theta\right)
\end{aligned}
$$

Hence

$$
L_\theta = mR^2\left(\omega^2 \sin\theta \cos\theta\right) - mgR\sin\theta
$$

and

$$
L_{\dot{\theta}} = mR^2 \dot{\theta}
$$

Hence

$$
\frac{d}{dt} L_{\dot{\theta}} = mR^2 \ddot{\theta}
$$

Hence

$$
\begin{aligned}
L_\theta - \frac{d}{dt} L_{\dot{\theta}} &= 0 \\
mR^2\left(\omega^2 \sin\theta\cos\theta\right) - mgR\sin\theta - mR^2\ddot{\theta} &= 0 \\
\omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta - \ddot{\theta} &= 0
\end{aligned}
$$

Hence the ODE is

$$
\boxed{\ddot{\theta} + \sin\theta\left(\tfrac{g}{R} - \omega^2\cos\theta\right) = 0}
$$

With initial conditions $\theta(0) = \theta_0, \dot{\theta}(0) = \dot{\theta}_0$