HW 5 Mathematics 503, Mathematical Modeling, CSUF , June 18, 2007

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Contents

1 Problem 1 (section 3.5, 5(b), page 185)

problem: Find extrermals for the following functional:

(b) $J(y)=\int_0^3 e^{2x}\left(y'^2-y^2\right)dx$

$y(0)=1, y(3)=$free

Solution:

$$L\left(x,y,y'\right)=e^{2x}\left(y'^2-y^2\right)$$

Starting from first principles. First the preliminary standard setup:

Let $J:(A\subset V)\to\mathbb{R}$, where A is the set of admissible functions, and $V: C^2[a,b]$, hence $A=\{y\in V:\ y(a)=0, y(b)=\text{free}\}$

Let $v(x)$ be the set $A_d(y)$ of the permissible directions defined as $A_d(y)=\{v\in V: y+tv\in A\}$ for some real scalar $-\xi<t<\xi$

And $L_y(x,y,y')\equiv\frac{\partial L}{\partial y}L(x,y,y')$ and $L_{y'}(x,y,y')\equiv\frac{\partial L}{\partial y'}L(x,y,y')$

Now we write

$$\begin{aligned}\delta J(y,v)&=\frac{d}{dt}J(y+tv)|_{t=0}\\&=\int_a^b L_y\left(x,y,y'\right)v+L_{y'}\left(x,y,y'\right)v'\ dx\end{aligned}\qquad\text{(see 3.14 in book)}$$

Therefor a necessary condition for $y(x)\in A$ to be a local minimum for the functional $J(y)$ is that $\delta J(y,v)=0$ for all $v\in A_d$, which means

$$\int_a^b L_y\left(x,y,y'\right)v + L_{y'}\left(x,y,y'\right)v'\,dx = 0$$

Integrating by parts the second term above results in the general expression for the necessary condition for $y(x)$ to be a local minimum for $J(y)$, which is

$$\int_a^b \left\{L_y\left(x,y,y'\right) - \frac{d}{dx}L_{y'}\left(x,y,y'\right)\right\} v\,dx + \left[L_{y'}\left(x,y,y'\right)v\right]_a^b = 0 \qquad \text{(see 3.15 in text)}$$

Since $v(a) = 0$, the second term above simplifies, and the above equation becomes

$$\int_a^b \left\{L_y\left(x,y,y'\right) - \frac{d}{dx}L_{y'}\left(x,y,y'\right)\right\} v\,dx + L_{y'}\left(b,y(b),y'(b)\right)v(b) = 0 \tag{1}$$

Now we apply the following argument: Out of all functions $v \in A_d$, we can find a set which has the property such that $v(b) = 0$. For these $v's$ only (1) becomes

$$\int_a^b \left\{L_y\left(x,y,y'\right) - \frac{d}{dx}L_{y'}\left(x,y,y'\right)\right\} v\,dx = 0$$

Where now we apply the other standard argument: Since the above is true for every arbitrary v (but remember now v is such that $v(b) = 0$, but since there are so many such $v's$ still, then the argument still holds) , then it must mean that

$$L_y\left(x,y,y'\right) - \frac{d}{dx}L_{y'}\left(x,y,y'\right) = 0 \tag{2}$$

This will generate a second order ODE, which we will solve, with the boundary conditions $y(0) = 1$

But we need another boundary condition. Then we hold off solving this for one moment. Let us now consider those functions $v \in A_d$ which have the property that $v(b) \neq 0$. For these v's, and for the second term in (1) to become zero, we now must have

$$L_{y'}\left(b,y(b),y'(b)\right) = 0 \tag{3}$$

Now from (3) we have $\frac{\partial L}{\partial y'} = \frac{\partial}{\partial y'}e^{2x}\left(y'^2 - y^2\right) = 2e^{2x}y'$, which means

$$2e^{2x}y'|_{x=b} = 0$$
$$2e^{2b}y'(b) = 0$$

Hence

$$\boxed{y'(b) = 0}$$

This gives us the second boundary condition we needed to solve (2).

Hence to summarize the problem becomes that of solving for y given

$$L_y\left(x,y,y'\right) - \frac{d}{dx}L_{y'}\left(x,y,y'\right) = 0$$

with the boundary conditions $y(0) = 1$ and $y'(3) = 0$

Now (2) can be written as

$$\begin{aligned}\frac{\partial}{\partial y}e^{2x}\left(y'^2-y^2\right)-\frac{d}{dx}\left(2e^{2x}y'\right)&=0\\ -2e^{2x}y-2\left(2e^{2x}y'+e^{2x}y''\right)&=0\\ -2y-4y'-2y''&=0\end{aligned}$$

Hence

$$\boxed{y''+2y'+y=0 \qquad\qquad y(0)=1,\, y'(3)=0}$$

Assume $y=Ae^{mx}$, hence the characteristic equation is $m^2+2m+1=0\rightarrow m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{4-4}}{2}=\boxed{-1}$

Since we have *repeated root*, then the solution is

$$\boxed{y(x)=c_1e^{-x}+c_2xe^{-x}}$$

When $x=0,\ y=1$, hence $\boxed{c_1=1}$

$$y'(x)=-e^{-x}+c_2\left(e^{-x}-xe^{-x}\right)$$

when $x=3, y'=0$, hence $0=-e^{-3}+c_2\left(e^{-3}-3e^{-3}\right)\rightarrow c_2=-\frac{e^{-3}}{e^{-3}(2)}\rightarrow \boxed{c_2=-\frac{1}{2}}$

Hence the solution is

$$y(x)=e^{-x}-\frac{1}{2}xe^{-x}$$

or

$$\boxed{y(x)=e^{-x}\left(1-\tfrac{1}{2}x\right)}$$

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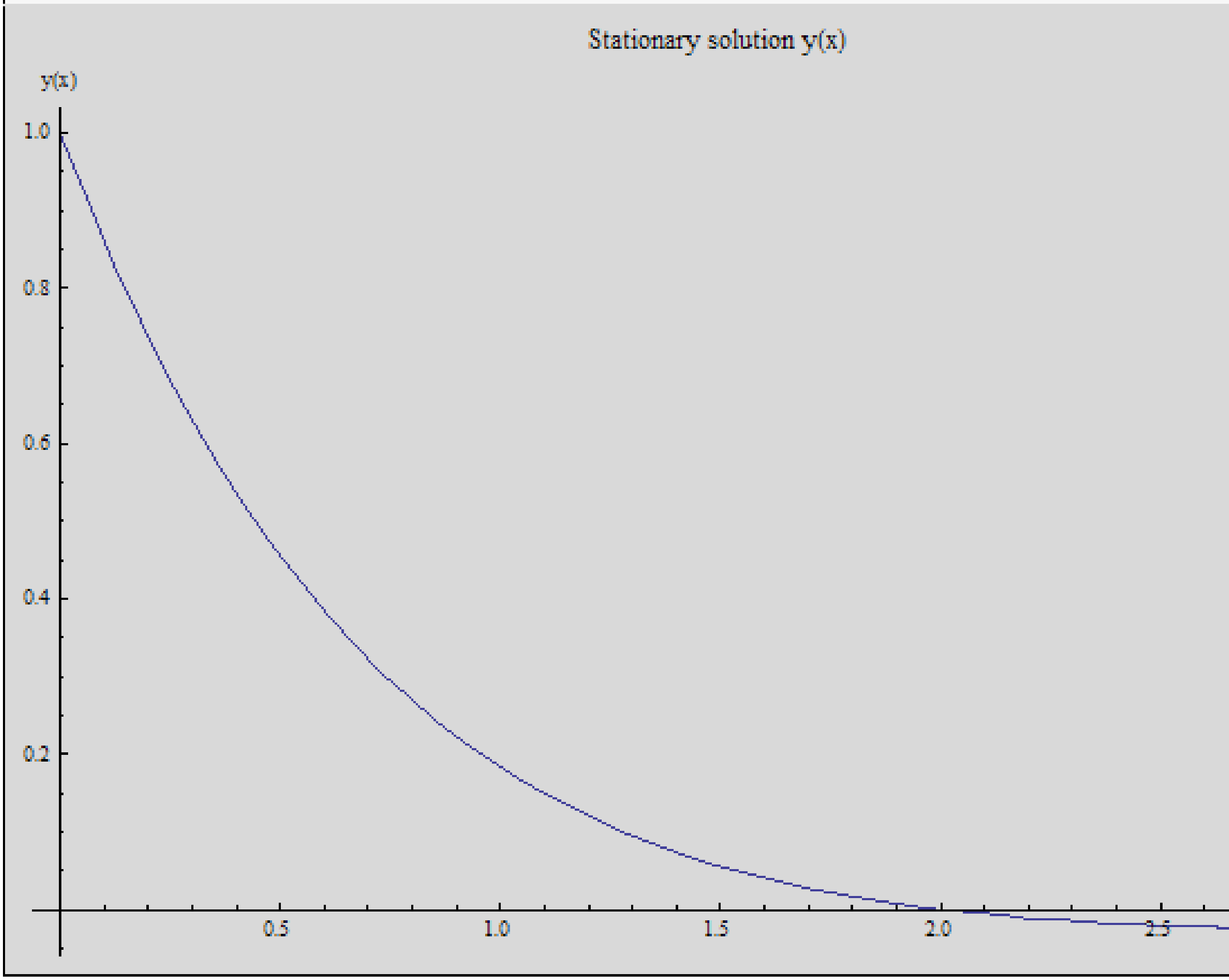

2 Problem 2 (section 3.5,#6, page 185)

problem: determine the natural boundary condition at $x=b$ for the variational problem defined by $J(y)=\int_a^b L(x,y,y')\,dx+G(y(b))$ where $y\in C^2[a,b], y(a)=y_0$ and G is a given differentiable function on $\mathbb{R}$

Solution:

Starting from first principles, first the preliminary standard setup.

Let $J:(A\subset V)\to\mathbb{R}$, where A is the set of admissible functions, and $V: C^2[a,b]$ Hence $A=\{y\in V:\ y(a)=0, y(b)=free\}$ .Let $v(x)$ be a set $A_d(y)$ of permissible directions defined as $A_d(y)=\{v\in V: y+tv\in A\}$ for some real scalar $-\xi<t<\xi$, and Let $L_y(x,y,y')\equiv\frac{\partial L}{\partial y}L(x,y,y')$, and $L_{y'}(x,y,y')\equiv\frac{\partial L}{\partial y'}L(x,y,y')$

Now we write

$$\begin{aligned}\delta J(y,v)&=\frac{d}{dt}J(y+tv)\,|_{t=0}\\&=\int_a^b L_y\left(x,y,y'\right)v+L_{y'}\left(x,y,y'\right)v'\;dx+\frac{d}{dt}G(y(b)+tv(b))\,|_{t=0}\\&=\int_a^b L_y\left(x,y,y'\right)v+L_{y'}\left(x,y,y'\right)v'\;dx+v(b)\,G'(y(b))\end{aligned}$$

Therefor a necessary condition for $y(x)\in A$ to be a local minimum for $J(y)$ is that $\delta J(y,v)=0$ for all $v\in A_d$, which means

$$\left(\int_a^b L_y\left(x,y,y'\right)v+L_{y'}\left(x,y,y'\right)v'\;dx\right)+v(b)\,G'(y(b))=0$$

Integrating the second term in the integral above by parts results in the general expression for the necessary condition for $y(x)$ to be a local minimum for $J(y)$, which is

$$\int_a^b\left\{L_y\left(x,y,y'\right)-\frac{d}{dx}L_{y'}\left(x,y,y'\right)\right\}v\;dx+\left[L_{y'}\left(x,y,y'\right)v\right]_a^b+v(b)\,G'(y(b))=0$$

Hence

$$\begin{aligned}&\int_a^b\left\{L_y\left(x,y,y'\right)-\frac{d}{dx}L_{y'}\left(x,y,y'\right)\right\}v\;dx+\\&\qquad L_{y'}\left(b,y(b),y'(b)\right)v(b)-L_{y'}\left(a,y(a),y'(a)\right)v(a)+v(b)\,G'(y(b))=0\end{aligned}$$

Since $y(a)=y_0$, we must have $v(a)=0$, then the above simplifies to

$$\int_a^b\left\{L_y\left(x,y,y'\right)-\frac{d}{dx}L_{y'}\left(x,y,y'\right)\right\}v\;dx+\left\{L_{y'}\left(b,y(b),y'(b)\right)+G'(y(b))\right\}v(b)=0 \tag{1}$$

Let us now consider those functions $v\in A_d$ which have the property that $v(b)\neq0$. For these v's, for the second term in (1) to become zero, we now must have

$$L_{y'}\left(b,y(b),y'(b)\right)+G'(y(b))=0$$

Hence

$$\boxed{\left.\frac{\partial L}{\partial y'}\right|_{x=b} = -\left. G'\left(y\left(x\right)\right)\right|_{x=b}}$$

Hence the natural boundary condition on $y(x)$ at $x=b$ must satisfy the above. (I do not see how can one go further without being given what L and G are.)