HW 4 Mathematics 503, Mathematical Modeling, CSUF , June 9, 2007

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Contents

1 Problem 1 (section 3.3, 2(b), page 175)

problem: Find extrermals for the following functional:

(b) $\int_a^b y^2 + (y')^2 + 2ye^x \ dx$

Solution:

Assume first that $y(x)$ has normal conditions on the boundaries. I.e. $y(a) = y_a, y(b) = y_b$

$$L(y, y', x) = y^2 + (y')^2 + 2ye^x$$

We have the functional

$$J(y) = \int_a^b L(y, y', x) \ dx$$

and we seek to find a function $y(x)$ which minimizes this functional.

Let the vector space from which we can pick $y(x)$ from be

$$V = C^2[a, b]$$

And let the set of admissible functions (within V) (Is this set a subspace?) be defined as

$$A = \{y(x) \text{ s.t. } y(x) \in V \text{ and } y(a) = y_0, y(b) = y_1\}$$

And let the set of admissible directions $v(x)$ be

$$A_d = \{v(x) \in V \text{ s.t. } v(a) = 0, v(b) = 0\}$$

Use the variational method since the Lagrangian contains a quadratic terms.

$$
\begin{aligned}
J(y+v) &= \int_a^b L\left((y+v),(y+v)',x\right)\ dx \\
&= \int_a^b (y+v)^2 + \left((y+v)'\right)^2 + 2(y+v)e^x\ dx \\
&= \int_a^b \left(y^2+v^2+2yv\right) + \left(y'+v'\right)^2 + 2\left(ye^x+ve^x\right)\ dx \\
&= \int_a^b y^2+v^2+2yv+\left(y'\right)^2+\left(v'\right)^2+2y'v'+2ye^x+2ve^x\ dx
\end{aligned}
$$

rearrange terms

$$
J(y+v) = \overbrace{\int_a^b y^2+\left(y'\right)^2+2ye^x}^{J(y)} + \overbrace{v^2\ +\left(v'\right)^2}^{+ve} + 2yv+2y'v'+2ve^x\ dx
$$

$$
J(y+v) = \ J(y) + \overbrace{\int_a^b v^2\ +\left(v'\right)^2\ dx}^{+ve} + \overbrace{2\int_a^b yv+y'v'\ +ve^x\,dx}^{\text{make this zero}}
$$

Hence if we can find $\tilde{y}(x)$ which will make the last term above zero, then $J(y)$ will have been minimized by this $\tilde{y}(x)$

Therefor the problem now becomes of solving for $y(x)$ the following integral equation

$$
\int_a^b yv+y'v'+ve^x\ dx = 0 \tag{1}
$$

We need to try to convert the above into something like $\int_a^b f\left(y,y',e^x\right)v(x)\ \ dx = 0$ so that we can say that $f\left(y,y',e^x\right)=0$, so this means in (1) we need to do integration by parts on the term $y'v'$. Hence (1) can be written as

$$
\int_a^b yv dx + \int_a^b y'v' dx + \int_a^b ve^x\ dx = 0
$$

Now since $\int u\,dz = [uz]_a^b - \int_a^b z\,du$, now let $u=y' \rightarrow du = y''$, and let $dz = v'dx \rightarrow z = v$, hence we have

$$
\int_a^b y'v'dx = \overbrace{\left[y'v\right]_a^b}^{0} - \int_a^b vy''\,dx
$$

Hence (1) can be written as

$$
\begin{aligned}
0 &= \int_a^b yv+y'v'+ve^x\ dx \\
&= \int_a^b yv - vy'' + ve^x\ dx \\
&= \int_a^b \left(y-y''+e^x\right)v\ \ dx
\end{aligned}
$$

Now we apply the standard argument and say that since $v(x)$ is arbitrary function, and the integral above is always zero, then it must be that

$$\left(y - y'' + e^x\right) = 0$$

or

$$\boxed{y'' - y = e^x}$$

This is a linear second order ODE with constant coefficients with a forcing function. The homogeneous ODE will have 2 independent solutions, say $y_1(t)$ and $y_2(t)$, so the total solution is

$$\begin{aligned} y &= y_h(x) + y_p(x) \\ &= c_1 y_1(x) + c_1 y_2(x) + y_p(x) \end{aligned}$$

To solve the homogeneous ODE

$$y'' - y = 0$$

Assume the solution is $y = Ae^{mx}$, hence the characteristic equation is $m^2 - 1 = 0 \rightarrow m = \pm 1$, hence the solution is $y_1(x) = e^x, y_2(x) = e^{-x}$, so

$$\begin{aligned} y_h(t) &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^x + c_2 e^{-x} \end{aligned}$$

Or the solution can be written in hyperbolic sin and cosine

$$y_h(t) = c_1 \cosh(x) + c_2 \sinh(x)$$

Now to find particular solution, use variation of parameters. Assume

$$y_p = -y_1 u_1 + y_2 u_2$$

where

$$\begin{aligned} u_1 &= \int \frac{y_2 e^x}{W} dx \\ u_2 &= \int \frac{y_1 e^x}{W} dx \end{aligned}$$

Where

$$\begin{aligned} W &= y_1 y_2' - y_2 y_1' = -e^x e^{-x} - e^{-x} e^x \\ &= -1 - 1 \\ &= -2 \end{aligned}$$

Hence

$$
\begin{aligned}
u_1 &= -\frac{1}{2}\int e^{-x}e^{x}dx \\
&= -\frac{1}{2}\int dx \\
&= -\frac{1}{2}x
\end{aligned}
$$

and

$$
\begin{aligned}
u_2 &= -\frac{1}{2}\int e^{x}e^{x}dx \\
&= -\frac{1}{2}\int e^{2x}dx \\
&= -\frac{1}{4}e^{2x}
\end{aligned}
$$

Hence the solution is

$$
\begin{aligned}
y &= c_1e^{x}+c_2e^{-x}+y_p \\
&= c_1e^{x}+c_2e^{-x}+(-u_1y_1+u_2y_2) \\
&= c_1e^{x}+c_2e^{-x}+\left(\frac{1}{2}xe^{x}+-\frac{1}{4}e^{2x}e^{-x}\right)
\end{aligned}
$$

Hence

$$
\boxed{y(x)=c_1e^{x}+c_2e^{-x}+\tfrac{1}{2}xe^{x}-\tfrac{1}{4}e^{x}}
$$

2 Problem 1 (section 3.3,#10, page 176)

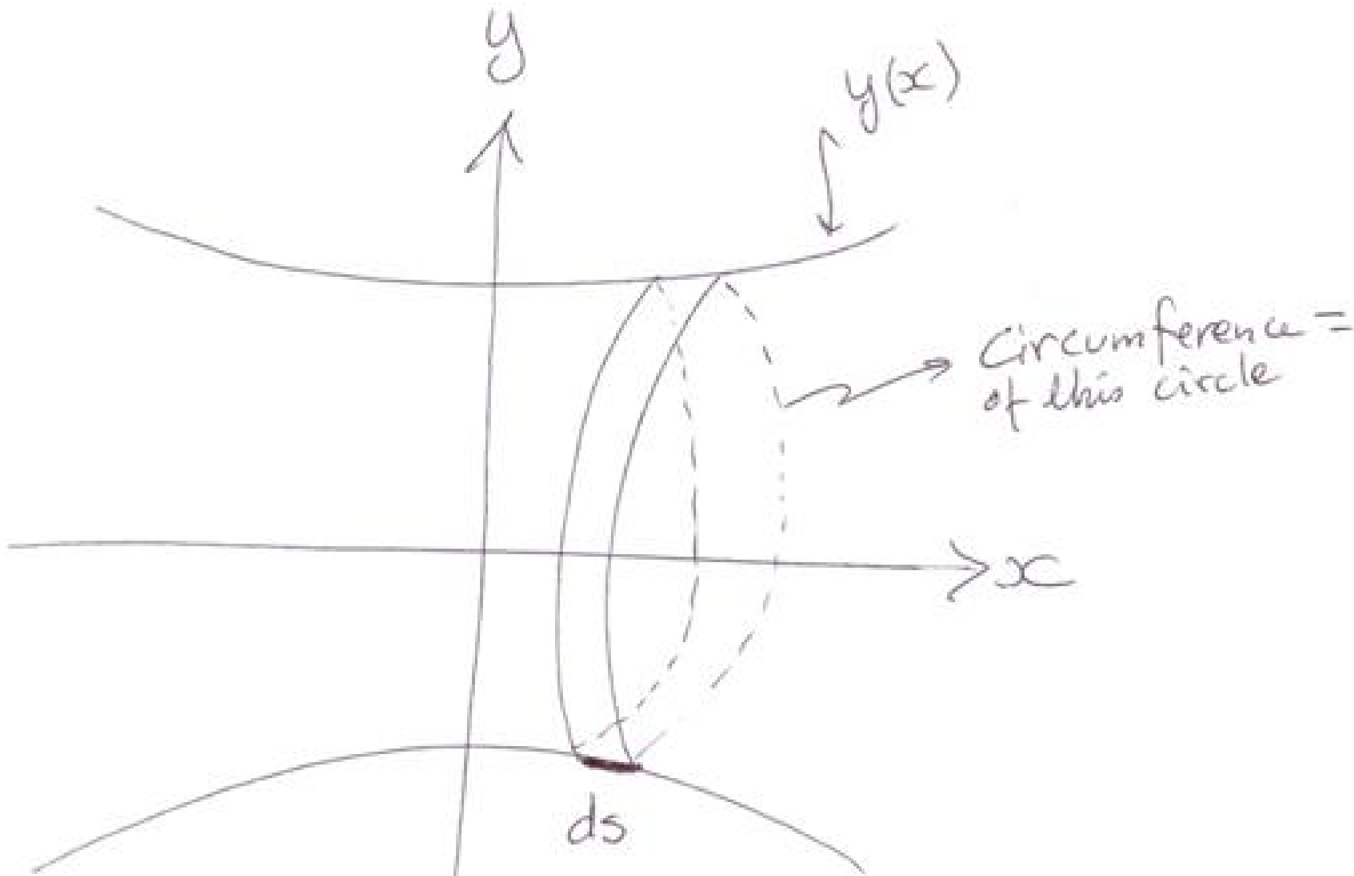
problem: Show that the minimal area of a surface of revolution in a catenoid, that is, the surface found by revolving a catenary

$$y = c_1 \cosh\left(\frac{x + c_2}{c_1}\right)$$

about the x axis

solution:

First we assume that $y'(x) > 0$ over the integration range. And that the lower end of the integration $x = a$ is smaller than the upper limit $x = b$



dA = area of strip $= ds\,(2\pi\, y(x))$

so $A = \int 2\pi\, y(x)\, ds$

but $ds = \sqrt{dx^2 + dy^2}$

so $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx$

so $A = \int 2\pi\, y(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; d$

If we view a small ds at $y(x)$ we see that it has a length of

$$ds^2 = dy^2 + dx^2$$

Hence

$$ds = \sqrt{dy^2 + dx^2}$$
$$\boxed{ds = dx\sqrt{\left(\tfrac{dy}{dx}\right)^2 + 1}} \tag{1}$$

or we can also write

$$ds = \sqrt{dy^2 + dx^2}$$
$$\boxed{ds = dy\sqrt{\left(\tfrac{dx}{dy}\right)^2 + 1}} \tag{2}$$

So there are 2 ways to solve this depending if we use (1) or (2). Let us leave the choice open for a little longer.

Now a size of a differential area dA of a strip of width ds and a length given by the circumference of the circle generated by rotation is

$$dA = 2\pi y(x)\,ds$$

Hence the total surface area is the integral of the above over the range which $y(x)$ is defined at. Let this be from $x = a$,to $x = b$ is given by

$$\begin{aligned} A &= \int_{x=a}^{x=b} dA \\ &= 2\pi \int_{x=a}^{x=b} y(x)\,ds \end{aligned}$$

Since we have $y(x)$ already in the Lagrangian, let then pick expression (2) from the above.

$$A = 2\pi \int_{x=a}^{x=b} y(x)\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}dy$$

Now I need to change the limits. Let $y(a) = y_1$, and let $y(b) = y_2$ hence

$$A = 2\pi \int_{y=y(a)}^{y=y(b)} y(x)\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}dy$$

If we write the above in the more standard format, we have

$$
\begin{aligned}
A &= 2\pi \int_{y=y(a)}^{y=y(b)} y(x)\sqrt{1+(x')^2}dy \\
&= 2\pi \int_{y=y(a)}^{y=y(b)} L\left(y, x(y), x'(y)\right)
\end{aligned}
$$

Remember now that y is the independent variable, and x is the dependent variable. This is different from the normal way.

Hence the Lagrangian L is

$$L\left(y,x,x'\right) = y\sqrt{1+(x')^2} \tag{3}$$

If I had picked expression (1) instead, I would have obtained the Lagrangian as

$$L\left(x,y,y'\right) = y\sqrt{1+(y')^2} \tag{4}$$

Both will give the same answer but with (3) we have $\frac{\partial L}{\partial x} - \frac{d}{dy}\left(\frac{\partial L}{\partial x'}\right) = 0$ and the first term $\frac{\partial L}{\partial x} = 0$ since L does not depend on x, and now we can just say that $\frac{\partial L}{\partial x'}$ =constant,. While with (4) we have $\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$ now $\frac{\partial L}{\partial y}$ is not zero.

Now we continue, and we will use (3) as our lagrangian.

We start the solution of the problem. We seek a function $\tilde{x}(y)$ which minimizes $J(x) = \int_{y=y(a)}^{y=y(b)} L(y,x,x')\,dy$.

Where $\tilde{x}(y) \in V$ s.t. $C^2[a,b]$, and let the set of admissible functions

$A(x(y)) = \{x(y)\,|x \in V \text{ and } x(a) = x_1, x(b) = x_2\}$,

and let the set of admissible directions $v(y) = \{v(y)\,|v \in V \text{ and } v(a) = 0 \text{ and } v(b) = 0\}$

Now that we have written down all the formal definitions, we can just solve this by applying Euler-Lagrange equation since the Lagrangian above meets the conditions of using Euler-Lagrange equations (L is a function of x, x', y and x is defined at the boundary conditions with a dirichlet type boundary conditions).

The Euler Lagrangian equation is

$$\frac{\partial L}{\partial x} - \frac{d}{dy}\left(\frac{\partial L}{\partial x'}\right) = 0$$

Since L does NOT depend on x then $\frac{\partial L}{\partial x} = 0$, and the above reduces to

$$\frac{d}{dy}\left(\frac{\partial L}{\partial x'}\right) = 0$$

Since the derivative is zero, then we can write that

$$\frac{\partial L}{\partial x'} = c_1$$

Where c_1 is some constant. So the above becomes

$$y\frac{2x'}{2\sqrt{1+(x')^2}} = c_1$$
$$\frac{yx'}{\sqrt{1+(x')^2}} = c_1$$
$$\frac{(yx')^2}{1+(x')^2} = c_1^2$$

Hence we have

$$\left(yx'\right)^2 = c_1^2\left(1+\left(x'\right)^2\right)$$
$$= c_1^2 + c_1^2\left(x'\right)^2$$
$$\left(x'\right)^2\left(y^2-c_1^2\right) = c_1^2$$

Hence the final ODE is

$$\boxed{x'(y) = \frac{c_1}{\sqrt{\left(y^2-c_1^2\right)}}}$$

This is a linear ODE Its solution is found by integration both side as follows

$$\int x'(y)\,dy = \int \frac{c_1}{\sqrt{\left(y^2-c_1^2\right)}}dy$$
$$x(y) = c_1\int \frac{dy}{c_1\sqrt{\left(\frac{y}{c_1}\right)^2-1}}$$
$$x(y) = \int \frac{dy}{\sqrt{\left(\frac{y}{c_1}\right)^2-1}}$$

Let $\frac{y}{c_1} = u$ hence $dy = c_1 du$ and the above becomes (do not need to worry about limits of integrations, as I will flip back the earlier variable in a minute)

$$x(y) = c_1\int \frac{du}{\sqrt{u^2-1}}$$

Which from table is given by

$$x(y) = c_1 \ln\left(u+\sqrt{u^2-1}\right)+c_2$$

Where c_2 is constant of integration. Hence going back to our variables, we have

$$x(y) = c_1 \ln\left(\frac{y}{k}+\sqrt{\left(\frac{y}{k}\right)^2-1}\right)+c_2 \tag{3}$$

From tables I found that $\cosh^{-1}(z) = \ln\left(z + \sqrt{z^2 - 1}\right)$
Hence (3) can now be written as

$$x(y) = c_1 \cosh^{-1}\left(\frac{y}{k}\right) + c_2$$
$$\frac{x(y) - c_2}{c_1} = \cosh^{-1}\left(\frac{y}{c_1}\right)$$

Or

$$\frac{y}{c_1} = \cosh\left(\frac{x - c_2}{c_1}\right)$$

Then

$$\boxed{y(x) = c_1 \cosh\left(\frac{x-c_2}{c_2}\right)}$$

So the above curve will minimize the surface area.

Problem 1 (section 3.3,#14, page 177)

14. (Economics) Let $y = y(t)$ be a indivi
let $r = r(t)$ be the rate that capital is
enjoyment, then his total enjoyment ov

$$E = \int_0^T e^{-\beta t} U$$

where the exponential factor reflects th
counted over time. Initially, his capita
Because his capital gains interest at rat

$$y' = \alpha y -$$

Assume $\alpha < 2\beta < 2\alpha$. Determine $r(t)$
total enjoyment is maximized if his enj

answer:
$y(0) = Y, y(T) = 0$

$$E = \int_0^T e^{-\beta t} U(r(t))\, dt$$

but

$$r(t) = \alpha y - y'$$

Hence

$$E=\int_0^T e^{-\beta t}U\left(\alpha y(t)-y'(t)\right)dt$$

Hence the Lagrangian is

$$\boxed{L(t,y,y')=e^{-\beta t}U(\alpha y(t)-y'(t))}$$

since $y(t)$ is defined at boundaries of the interval, we can use Euler-Lagrange equations

$$\frac{d}{dy}L\left(t,y,y'\right)-\frac{d}{dt}\left(\frac{d}{dy'}L\left(t,y,y'\right)\right)=0$$

Now the first term above is

$$\begin{aligned}\frac{d}{dy}L\left(t,y,y'\right)&=e^{-\beta t}U'\frac{d}{dy}\left(\alpha y(t)-y'(t)\right)\\&=\alpha e^{-\beta t}U'\end{aligned}$$

and the second term is

$$\begin{aligned}\frac{d}{dt}\left(\frac{d}{dy'}L\left(t,y,y'\right)\right)&=\frac{d}{dt}\left(e^{-\beta t}U'\frac{d}{dy'}\left(\alpha y(t)-y'(t)\right)\right)\\&=\frac{d}{dt}\left(-e^{-\beta t}U'\right)\end{aligned}$$

Hence our E-L equations now looks like

$$\alpha e^{-\beta t}U'+\frac{d}{dt}\left(e^{-\beta t}U'\right)=0 \tag{1}$$

Since U is a function of $r(t)$, then

$$\frac{d}{dt}\left(e^{-\beta t}U'\right)=-\beta e^{-\beta t}U'+e^{-\beta t}U''r'(t)$$

And (1) now becomes

$$\begin{aligned}\alpha e^{-\beta t}U'-\beta e^{-\beta t}U'+e^{-\beta t}U''r'(t)&=0\\(\alpha-\beta)U'+U''r'(t)&=0\end{aligned}$$

This is separable ODE, hence

$$\begin{aligned}\frac{U''}{U'}\frac{dr}{dt}&=-(\alpha-\beta)\\\frac{U''}{U'}dr&=-(\alpha-\beta)dt\end{aligned}$$

Integrate both sides

$$\ln\left(U'(r)\right) = -(\alpha-\beta)\int dt$$
$$\ln\left(U'(r)\right) = -(\alpha-\beta)t + k$$

where k is constant on integration

$$U'(r) = e^{-(\alpha-\beta)t+k}$$
$$= ce^{-(\alpha-\beta)t}$$

where $c = e^k$ is another constant
But $U(r) = 2\sqrt{r}$ hence $\frac{dU}{dr} = \frac{1}{\sqrt{r}}$ and the above becomes

$$\frac{1}{\sqrt{r}} = ce^{-(\alpha-\beta)t}$$
$$\frac{1}{r} = c^2 e^{-2(\alpha-\beta)t}$$
$$\boxed{r(t) = c_2 e^{2(\alpha-\beta)t}}$$

Where since c^{-2} is constant, I call it c_2
Now, Since

$$y'(t) = \alpha y(t) - r(t)$$

Then

$$\boxed{y'(t) - \alpha y(t) = c_2 e^{2(\alpha-\beta)t}}$$

The solution is

$$y = y_h + y_p$$

Assume $y_h = Ae^{mt}$, hence $Ame^{mt} - \alpha Ae^{mt} = 0 \rightarrow m = \alpha$
So the solution is

$$y_h = c_1 e^{\alpha t}$$

For the particular solution, guess a solution. Since the forcing function is of the form ce^t, guess

$$y_p = Ae^{kt}$$

so $y_p' = Ate^{kt}$ and we substitute this solution in the ODE above, we obtain

$$Ake^{kt} - \alpha Ae^{kt} = c_2 e^{2(\alpha-\beta)t}$$
$$A(k-\alpha)e^{kt} = c_2 e^{2(\alpha-\beta)t}$$

so by comparing exponents, we see that $k = 2(\alpha - \beta)$ and $A(k-\alpha) = c_2$ hence $A = \frac{c_2}{k-\alpha} = \frac{c_2}{2(\alpha-\beta)-\alpha} = \frac{c_2}{\alpha-2\beta}$

Therefore

$$y_p = \frac{c_2}{\alpha - 2\beta} e^{2(\alpha-\beta)t}$$

Hence, since

$$y = y_h + y_p$$

Then

$$\boxed{y(t) = c_1 e^{\alpha t} + \frac{c_2}{\alpha-2\beta} e^{2(\alpha-\beta)t}}$$

We now find c_1 and c_2 from I.C. At $t = 0, y = Y$, hence

$$\begin{aligned} Y &= c_1 + \frac{c_2}{\alpha - 2\beta} \\ c_1 &= Y - \frac{c_2}{\alpha - 2\beta} \end{aligned} \tag{2}$$

at $t = T, y = 0$, hence

$$\begin{aligned} 0 &= \left(Y - \frac{c_2}{\alpha - 2\beta}\right) e^{\alpha T} + \frac{c_2}{\alpha - 2\beta} e^{2(\alpha-\beta)T} \\ c_2 &= \frac{(\alpha - 2\beta) Y e^{\alpha T}}{e^{\alpha T} - e^{2(\alpha-\beta)T}} \end{aligned}$$

so from (2)

$$c_1 = Y - \frac{(\alpha - 2\beta) Y e^{\alpha T}}{(\alpha - 2\beta)\left(e^{\alpha T} - e^{2(\alpha-\beta)T}\right)}$$

Hence

$$\boxed{y(t) = \left(Y - \frac{(\alpha-2\beta) Y e^{\alpha T}}{(\alpha-2\beta)\left(e^{\alpha T} - e^{2(\alpha-\beta)T}\right)}\right) e^{\alpha t} + \frac{(1-\alpha+2\beta Y) e^{(-\alpha+2\beta)T}}{\alpha-2\beta} e^{2(\alpha-\beta)t}}$$

and

$$\boxed{r(t) = (1 - \alpha + 2\beta Y) e^{(-\alpha+2\beta)T} \; e^{2(\alpha-\beta)t}}$$

Analysis on results:
These are 3 plots showing $y(t)$ and $r(t)$. The first is for $\alpha = 0.03$

```
α = .03; β = (α / 2) + 0.001; T = 80; Y = 1000;
c2 = ((α - 2 β) ⅇ^(α T) Y) / (ⅇ^(α T) - ⅇ^(2 (α - β) T));
c1 = Y - c2 / (α - 2 β);
r = c2 Exp[2 (α - β) t];
y = c1 Exp[α t] + c2 / (α - 2 β) Exp[2 (α - β) t];

py = Plot[y, {t, 0, T}, PlotLabel → "y(t) over lifetime",
   AxesLabel → {"T in years", "y(t) capital at time t"}, PlotRange → All,
   AxesOrigin → {0, 0}];

pr = Plot[r, {t, 0, T}, PlotLabel → "r(t) over lifetime",
   AxesLabel → {"T in years", "r(t) rate capital is spend "}];

Show[GraphicsArray[{py, pr}]]
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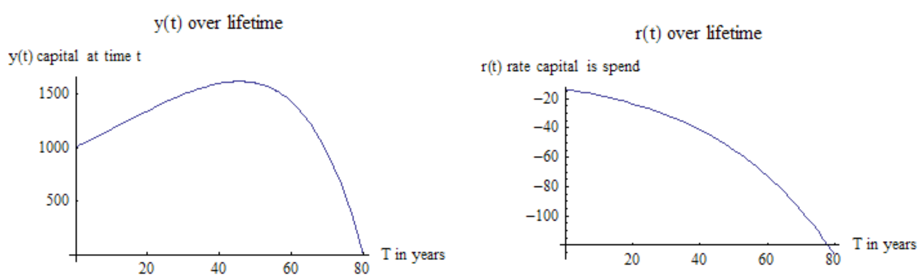
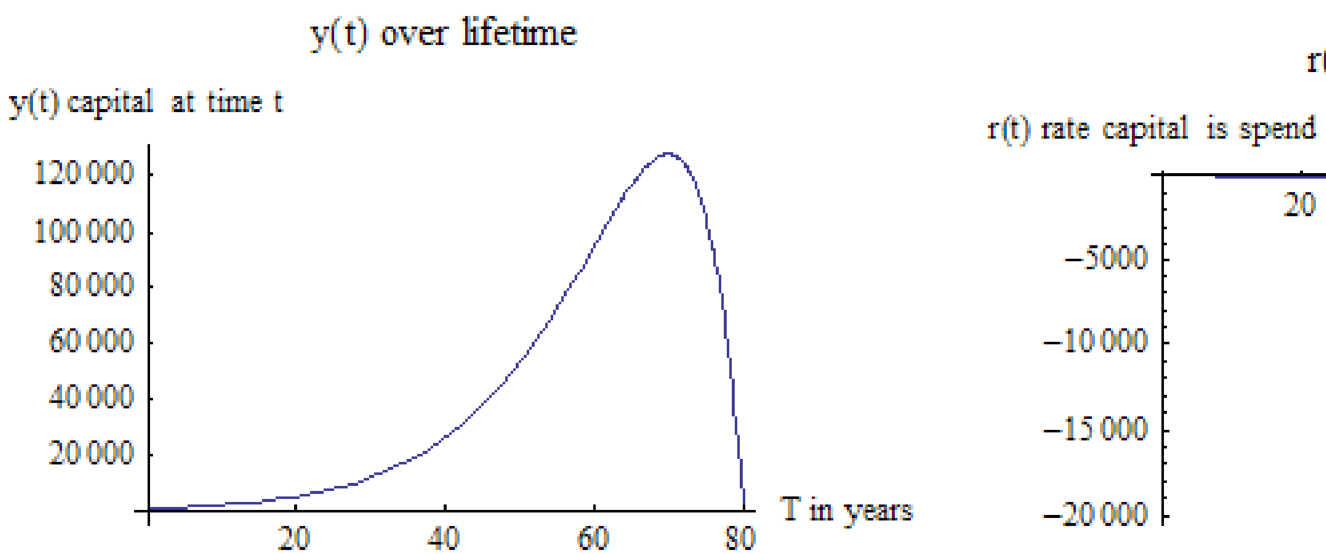
Out[232]=



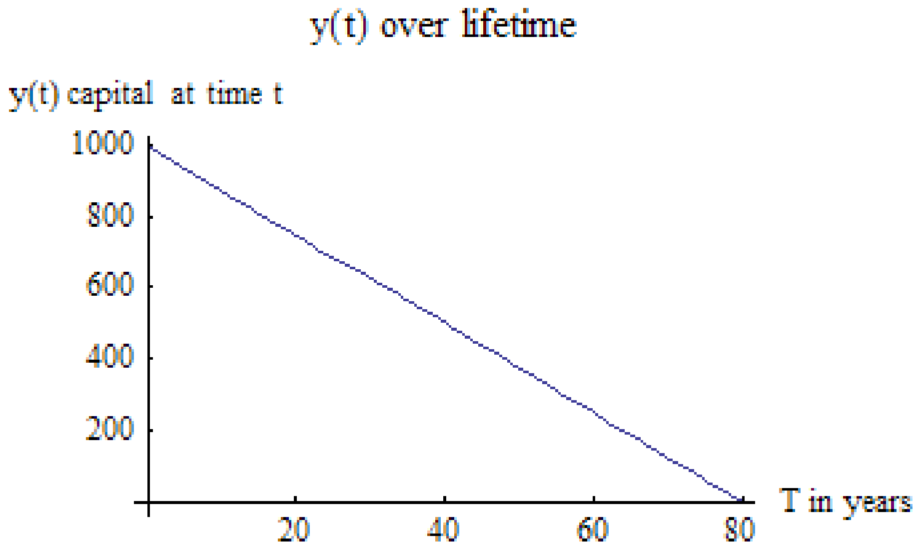
This one is for $\alpha = 0.1$

Out[248]=

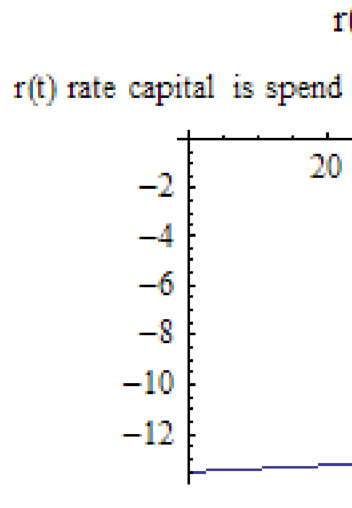


We notice that the higher the interest rate α is the more capital will accumulate, which means to achieve the goal of zero capital at death the rate $r(t)$ is more steep near the end. If the money hardly accumulate during life time, i.e. when the interest rate is very low, then we should expect a straight line for $y(t)$, which is verified by this plot below when I set $\alpha = 0.001$

Out[272]=



y(t) over lifetime
y(t) capital at time t
1000
800
600
400
200
20
40
60
80
T in years



r(t) rate capital is spend
20
−2
−4
−6
−8
−10
−12