HW 3 Mathematics 503, Mathematical Modeling, CSUF June 2 2007

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Contents

1 Problem 1 (section 1.3,#1, page 40)

problem: Find the general solution of the following differential equations

(m) $u'' + \omega^2 u = \sin\beta t$, $\omega \neq \beta$

solution:

We start by assuming ω is real, hence ω^2 must be positive.

Now, the general solution is

$$\begin{aligned} u(t) &= u_h(t) + u_p(t) \\ &= c_1 u_1(t) + c_2 u_2(t) + u_p(t) \end{aligned}$$

Where $u_1(t), u_2(t)$ are the 2 independent solutions of the homogeneous differential equation

$$u'' + \omega^2 u = 0$$

and $u_p(t)$ is the particular solution.

To find $u_1(t)$ and $u_2(t)$, we assume the homogeneous solution is $u_h(t) = Ae^{mt}$ for some constants A, m and substitute this assumed solution in the ODE. We obtain the characteristic equation $Am^2e^{mt} + \omega^2 Ae^{mt} = 0 \rightarrow m^2 + \omega^2 = 0 \rightarrow m^2 = -\omega^2$ or $m = \pm i\omega$, hence $u_1(t) = A_1 e^{i\omega t}$ and $u_2(t) = A_2 e^{-i\omega t}$.

Since the homogeneous solution is a linear combination of all the independent solutions, we take the sum and the difference of these solutions, and using Euler relation which converts the complex exponential to the trigonometric sin and cos functions we obtain

$$\begin{aligned} u_1(t) &= \cos\omega t \\ u_2(t) &= \sin\omega t \end{aligned}$$

and we now write

$$u_h(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

Now to obtain $u_p(t)$, we use the method of undetermined coefficients. Assume

$$u_p(t) = a \cos \beta t + b \sin \beta t$$

and plug into the original ODE, we obtain

$$\begin{aligned}
(a\cos\beta t + b\sin\beta t)'' + \omega^2 (a\cos\beta t + b\sin\beta t) &= \sin\beta t \\
(-\beta a\sin\beta t + \beta b\cos\beta t)' + \omega^2 (a\cos\beta t + b\sin\beta t) &= \sin\beta t \\
\left(-\beta^2 a\cos\beta t - \beta^2 b\sin\beta t\right) + \omega^2 (a\cos\beta t + b\sin\beta t) &= \sin\beta t \\
\left(-\beta^2 a + \omega^2 a\right)\cos\beta t + \left(-\beta^2 b + \omega^2 b\right)\sin\beta t &= \sin\beta t
\end{aligned}$$

Hence by comparing coefficients we obtain

$$\begin{aligned}
a\left(\omega^2 - \beta^2\right) &= 0 \\
b\left(\omega^2 - \beta^2\right) &= 1
\end{aligned}$$

Since $\omega \neq \beta$ (given), then $\omega^2 - \beta^2 \neq 0$, hence this must mean the following

$$\begin{aligned}
a &= 0 \\
b &= \frac{1}{\omega^2 - \beta^2}
\end{aligned}$$

Therefor, the particular solution now can be written as

$$\begin{aligned}
u_p &= b\sin\beta t \\
&= \frac{\sin\beta t}{\omega^2 - \beta^2}
\end{aligned}$$

Hence the general solution, which is $y(t) = y_h(t) + y_p(t)$ is given by

$$\boxed{y(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{\sin\beta t}{\omega^2 - \beta^2}}$$

Where c_1 and c_2 are constants that can be found from the initial conditions.

Verify my solution to HW3, problem 1.m, page 40, Math 503. by

In[93]:=
```
Remove["Global`*"]
ode = u''[t] + ω^2 u[t] == Sin[β t];
s = u[t] /. DSolve[ode, u[t], t];
s = Simplify[s, Element[ω, Reals] && β ≠ ω];
s = s /. {C[1] → c1, C[2] → c2}
```

Out[97]=

$$\left\{\frac{\text{Sin}[t\,\beta] - (\beta^2 - \omega^2)\,\text{Cos}[t\,\omega]\,c_1 + (-\beta^2 + \omega^2)\,\text{Sin}[t\,\omega]\,c_2}{-\beta^2 + \omega^2}\right\}$$

In[98]:=
```
u = Collect[s, Sin[t β], Simplify]
```

Out[98]=

$$\left\{\frac{\text{Sin}[t\,\beta]}{-\beta^2 + \omega^2} + \text{Cos}[t\,\omega]\,c_1 + \text{Sin}[t\,\omega]\,c_2\right\}$$

Extract the particular and the homogenous solutions and plot
arbitratry constants since these are not given in the problem

In[99]:=
```
up = u[[1, 1]]
```

Out[99]=

$$\frac{\text{Sin}[t\,\beta]}{-\beta^2 + \omega^2}$$

In[100]:=
```
uh = u[[1, 2 ;;]]
```

Out[100]=

$$\text{Cos}[t\,\omega]\,c_1 + \text{Sin}[t\,\omega]\,c_2$$

Give some constants so that we can plot the solution

In[138]:=
```
β = .5; ω = 2; c1 = 1; c2 = .5;
```


Now plot the solutions. Plot the homogenous, particular and t

2 Problem 2 (section 1.3,#1, page 40)

problem: Find the general solution of the following differential equations

(n) $u'' + \omega^2 u = \cos \omega t$

solution:

First, let the forcing function be called $f(t)$, hence $f(t) = \cos \omega t$ in this example.

From (m) we found the homogeneous solution to be $u_h(t) = c_1 u_1(t) + c_2 u_2(t)$ where

$$u_1(t) = \cos \omega t$$

and

$$u_2(t) = \sin \omega t$$

Now to find the particular solution we can not use the method of undetermined coefficients since the forcing frequency is the same as the undamped natural frequency of the system ω and this will lead to the denominator going to zero. Hence use the method of variation of parameters which is a general method to find particular solution which will work with this case.

$$u_p(t) = -u_1(t)\int \frac{u_2(t) f(t)}{W} dt + u_2(t)\int \frac{u_1(t) f(t)}{W} dt$$

Where W is the Wronskian of u_1, u_2 given by $W = u_1 u_2' - u_2 u_1'$

Hence $W(t) = \cos \omega t (\omega \cos \omega t) - \sin \omega t (-\omega \sin \omega t) = \omega \left(\cos^2 \omega t + \sin^2 \omega t\right) = \omega$

Hence

$$\begin{aligned} u_p(t) &= -\frac{\cos \omega t}{\omega}\int \sin(\omega t) f(t) dt + \frac{\sin \omega t}{\omega}\int \cos(\omega t) f(t) dt \\ &= -\frac{\cos \omega t}{\omega}\int \sin(\omega t)\cos(\omega t) dt + \frac{\sin \omega t}{\omega}\int \cos(\omega t)\cos(\omega t) dt \\ &= I_1 + I_2 \end{aligned}$$

We now evaluate I_1 and I_2. Start with the easy one, I_2

$$\begin{aligned} I_2 &= \frac{\sin \omega t}{\omega}\int \cos(\omega t)\cos(\omega t) dt \\ &= \frac{\sin \omega t}{\omega}\int \cos^2(\omega t) dt \\ &= \frac{\sin \omega t}{\omega}\left(\frac{1}{4\omega}(\sin 2t\omega + 2t\omega)\right) \\ &= \frac{\sin \omega t}{4\omega^2}(\sin 2t\omega + 2t\omega) \end{aligned}$$

and now I_1

$$I_1 = -\frac{\cos \omega t}{\omega}\int \sin(\omega t)\cos(\omega t) dt$$

We can use integration by parts (do it twice) or use an trigonometric identity. From tables, Using the formula of

$$2\sin\left(\frac{\lambda+\zeta}{2}\right)\cos\left(\frac{\lambda-\zeta}{2}\right) = \sin(\lambda) + \sin(\zeta)$$

so if we let $\lambda = 2\omega t$ and $\zeta = 0$ we obtain the integrand above, hence

$$2\sin(\omega t)\cos(\omega t) = \sin(2\omega t)$$

Substitute into I_1

$$\begin{aligned} I_1 &= -\frac{\cos \omega t}{\omega} \int \frac{1}{2} \sin(2\omega t)\, dt \\ &= -\frac{\cos \omega t}{2\omega} \left(\frac{-1}{2\omega} \cos(2\omega t) \right) \\ &= \frac{\cos \omega t}{4\omega^2} \cos(2\omega t) \end{aligned}$$

Therefor

$$\begin{aligned} u_p(t) &= I_1 + I_2 \\ &= \frac{\cos \omega t}{4\omega^2} \cos(2\omega t) + \frac{\sin \omega t}{4\omega^2} (\sin 2\omega t + 2\omega t) \\ &= \frac{\cos(\omega t)\cos(2\omega t) + \sin \omega t\,(\sin 2\omega t + 2\omega t)}{4\omega^2} \end{aligned}$$

Hence the general solution is

$$\begin{aligned} u(t) &= c_1 u_1(t) + c_2 u_2(t) + u_p(t) \\ &= c_1 \cos \omega t + c_2 \sin \omega t + \frac{\cos(\omega t)\cos(2\omega t) + \sin \omega t\,(\sin 2\omega t + 2\omega t)}{4\omega^2} \end{aligned}$$

Verify using Mathematica

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Verify my solution to HW3, problem 1.n, page
Abbasi

In[36]:=
Remove["Global`*"]
ode = u''[t] + ω^2 u[t] == Cos[ω t];
s = u[t] /. DSolve[ode, u[t], t];
s = s /. {C[1] → c1, C[2] → c2}

Out[39]=
{(Cos[t ω] Cos[2 t ω] + 2 t ω Sin[t ω] + Sin[t ω] S
 ------------------------------------------------
                      4 ω^2
  Cos[t ω] c1 + Sin[t ω] c2}
```