HW 2 Mathematics 503, Mathematical Modeling, CSUF
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Contents

1   Problem 1 (section 1.2,#9, page 33)

problem:

9. The temperature $T = T(t)$ of a chemical sa governed by the initial value problem

$$\frac{dT}{dt} = qe^{-A/T} - k(T - T_f)$$

where T_0 is the initial temperature of the sa the furnace, and q, k, and A are positive co right side is the heat generation term, and t given by Newton's law of cooling.

a) What are the dimensions of the constar

b) Reduce the problem to dimensionless for scale and choosing a time scale to be on the heat loss term is large compared reaction.

solution:
Let $[\Theta]$ be the dimension of temperature.
A small note on how to scale the dependent variable and the independent variable in the problem.
For the dependent variable, say $y(t)$, if the problem does not say which problem parameters to scale against, then look for a parameter or a combination of these parameters with the appropriate dimension of that of the dependent variable such the parameter is the largest value. This way we are measuring the depndent variable relative to the largest parameter in the problem. Hence when we write $\tilde{y} = \frac{y}{s}$ where s is the *largest* parameter in the problem with dimension of y.

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For the indepenent variable (typically this will be time), we look for the larget rate of change in the problem involving this variable, and scale the time relative to this rate. Hence we write $\tilde{t} = \frac{t}{s}$ where now s is the *smallest* parameter in the problem with dimension of t.

Now we start with the solution of the problem.

(a) $\frac{dT}{dt} = q \times$number, hence

$$[q] = \left[\frac{\Theta}{T}\right]$$

i.e. the dimension of q is temperature over time.

Now to find the dimension of k. since $\frac{dT}{dt} = k\left(T - T_f\right)$, then $\left[\frac{\Theta}{T}\right] = [k]\,[\Theta]$ hence

$$[k] = \left[\frac{1}{T}\right]$$

Now to find the dimension of A. Since $e^{-\frac{A}{T}}$ must be a numerical dimensionless value, then A must have the same dimensions as T (which has dimension $[\Theta]$), hence

$$[A] = [\Theta]$$

(b) The constants in this problem are T_f, T_0, q, k, A, and the dependent variable is T which is the temperature of the sample, and the independent variable is t which is time.

$$\frac{dT}{dt} = qe^{\frac{-A}{T}} - k\left(T - T_f\right)$$

Hence to reduce the above ODE to dimensionless form, we need to transform the dependent and the independent variables to dimensionless variables. Start with the dependent variable T.

We write $\tilde{T} = \frac{T}{s}$, so we need to find s as a combination using the constants T_f, T_0, q, k, A in the ODE which would have the dimension of temperature to make $\tilde{T}$ dimensionless.

This is easy, since we are told to use T_f, hence $s = T_f$ and so

$$\tilde{T} = \frac{T}{T_f} \tag{1}$$

Now we need to transform the independent variable, which is time t, hence we write $\tilde{t} = \frac{t}{s}$, and we need to find s as a combination of the constants T_f, T_0, q, k, A which has a dimension of time.

Looking above at the dimensions of these constants, we see that the following combinations are possible: $\left\{\frac{1}{k}, \frac{A}{q}, \frac{T_f}{q}, \frac{T_0}{q}\right\}$. Now since we want the new time scale to be large when the heat loss is large, and since the heat loss is proportional to k, hence we pick $s = \frac{1}{k}$, this will make s small and $\frac{t}{s}$ will then be large. Therefor

$$\tilde{t} = kt \tag{2}$$

Now we convert the ODE to non dimensional using (1) and (2)

$$\frac{dT}{dt} = \frac{dT}{d\tilde{t}}\frac{d\tilde{t}}{dt}$$

but from (2) we have $\frac{d\tilde{t}}{dt} = k$, hence

$$\frac{dT}{dt} = k\frac{dT}{d\tilde{t}}$$

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and since $T = T_f \tilde{T}$, then the above becomes

$$\frac{dT}{dt} = T_f k \frac{d\tilde{T}}{d\tilde{t}}$$

Hence the ODE $\frac{dT}{dt} = qe^{\frac{-A}{T}} - k\left(T - T_f\right)$ becomes (after replacing T by $T_f\tilde{T}$ in the RHS)

$$T_f k \frac{d\tilde{T}}{d\tilde{t}} = qe^{\frac{-A}{T_f\tilde{T}}} - k\left(T_f\tilde{T} - T_f\right)$$

$$\frac{d\tilde{T}}{d\tilde{t}} = \frac{q}{T_f k} e^{\frac{-A}{T_f\tilde{T}}} - \left(\tilde{T} - 1\right)$$

Now $\frac{q}{T_f k}$ must be dimensionless. Verify $\frac{\left[\frac{\Theta}{T}\right]}{[\Theta]\left[\frac{1}{T}\right]} = 1$. OK, so the above ODE can be written as

$$\boxed{\frac{d\tilde{T}}{d\tilde{t}} = \beta e^{\frac{-A}{T_f\tilde{T}}} - \left(\tilde{T} - 1\right)}$$

where $\beta = \frac{q}{T_f k}$

Note $e^{\frac{-A}{T_f\tilde{T}}}$ must be dimensionless also, since $\frac{-A}{T_f\tilde{T}}$ is dimensionless.

For the initial conditions, now it becomes $\tilde{T}_o = \frac{T}{T_f}$

2   Problem 2 (section 1.1, #11, page 18)

problem:

10. A ball of mass m is tossed upward with in
force caused by air resistance is proportion
of the ball and the gravitational field is cons
problem for the height of the ball at any
length and time scales and recast the probl

solution:

First draw a free body diagram. Assume the ball is moving upwards, and assume the drag force is given by $F_d = k(y'(t))^2$ where k is constant of proportionality, and weight of ball is given by $W = mg$
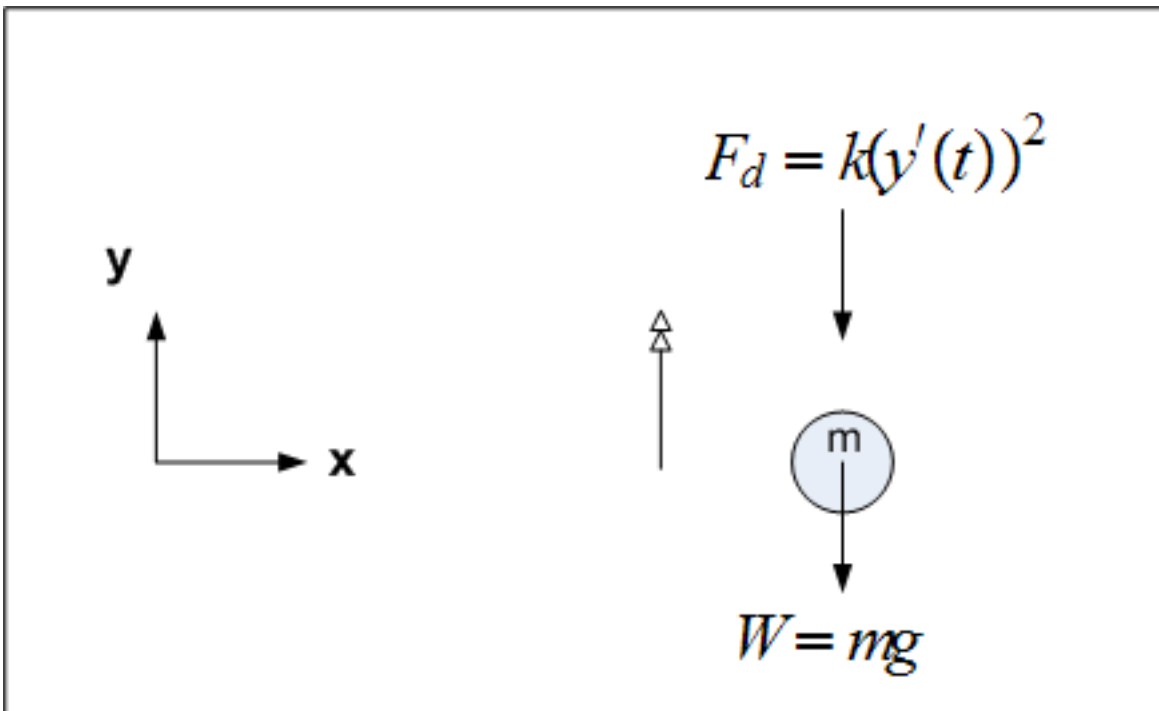
Now to obtain the equation of motion, apply Newtons second law $F = ma$, hence we obtain

$$-k\left(\frac{dy(t)}{dt}\right)^2 - mg = m\frac{d^2y(t)}{dt^2}$$

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Notice that we assumed the positive y direction is upwards. Hence we obtain the equation of motion

$$m\frac{d^2y(t)}{dt^2} + k\left(\frac{dy(t)}{dt}\right)^2 + mg = 0$$

with the initial conditions

$$y(0) = y_0, \frac{dy(0)}{dt} = V$$

This problem has the following constants: $\{V, m, g, k, y_0\}$. To find the dimensions of k, since $k\left(\frac{dy(t)}{dt}\right)^2$ must have the dimension of Newton $[N]$ then we write

$$[N] = \left[k\left(\frac{dy(t)}{dt}\right)^2\right] \rightarrow \left[M\frac{L}{T^2}\right] = [k]\left[\frac{L}{T}\right]^2$$

$$[k] = \left[\frac{M}{L}\right]$$

and $[g] = \left[\frac{L}{T^2}\right]$, and $[m] = [M]$, and $[V] = \left[\frac{L}{T}\right]$, and $y_0 = [L]$, hence we write

$$\text{constants} \rightarrow \left\{V = \left[\frac{L}{T}\right], m = [M], g = \left[\frac{L}{T^2}\right], k = \left[\frac{M}{L}\right], y_0 = [L]\right\}$$

Now we can start scaling the ODE.

The dependent variable is $y(t)$, and the independent variable is t. Hence we write $\tilde{y} = \frac{y}{s}$, where $[s] = [L]$, so using the above constants, we need to find a combination which has the dimension of length $[L]$. Clearly y_0 is one possibility. There are other combinations which give the dimension of length. We find the following: $\left\{y_0, \frac{m}{k}, \frac{V^2}{g}\right\}$.

Now we get to the hard part of these scaling problems. Which combination to choose? The problem did not give us a hint on this. I choose $s = \frac{V^2}{g}$, therefor

$$\tilde{y} = \frac{y}{V^2}g$$

Now for the time scale. We write $\tilde{t} = \frac{t}{s}$ where $[s] = [T]$, looking at the above constants, we need a combination with dimension $[T]$, we see that $s = \frac{V}{g}$ has the dimension of time, so we have

$$\tilde{t} = \frac{t}{V}g$$

Now that we have the scaling completed, we apply them to scale the ODE. In otherwords, we need to rewrite the ODE so that instead of using $\{y, t\}$ it will use $\{\tilde{y}, \tilde{t}\}$

$\frac{dy}{dt} = \frac{dy}{d\tilde{t}}\frac{d\tilde{t}}{dt}$, but $\frac{d\tilde{t}}{dt} = \frac{g}{V}$, hence

$$\frac{dy}{dt} = \frac{g}{V}\frac{dy}{d\tilde{t}}$$

and $\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{d\tilde{t}}\left(\frac{dy}{dt}\right)\frac{d\tilde{t}}{dt} = \frac{d}{d\tilde{t}}\left(\frac{g}{V}\frac{dy}{d\tilde{t}}\right)\frac{g}{V}$ hence

$$\frac{d^2y}{dt^2} = \left(\frac{g}{V}\right)^2\frac{d^2y}{d\tilde{t}^2}$$

Hence the scaled ODE is

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$$m\frac{d^2y(t)}{dt^2} + k\left(\frac{dy(t)}{dt}\right)^2 + mg = 0$$

$$m\left(\frac{g}{V}\right)^2\frac{d^2y}{d\tilde{t}^2} + k\left(\frac{g}{V}\frac{dy}{d\tilde{t}}\right)^2 + mg = 0$$

$$m\frac{g^2}{V^2}\frac{d^2y}{d\tilde{t}^2} + k\frac{g^2}{V^2}\left(\frac{dy}{d\tilde{t}}\right)^2 + mg = 0$$

$$\frac{d^2y}{d\tilde{t}^2} + \frac{k}{m}\left(\frac{dy}{d\tilde{t}}\right)^2 + \frac{V^2}{g} = 0$$

Now replace $y \to \frac{V^2}{g}\tilde{y}$

$$\frac{V^2}{g}\frac{d^2\tilde{y}}{d\tilde{t}^2} + \frac{k}{m}\left(\frac{V^2}{g}\frac{d\tilde{y}}{d\tilde{t}}\right)^2 + \frac{V^2}{g} = 0$$

$$\frac{V^2}{g}\frac{d^2\tilde{y}}{d\tilde{t}^2} + \frac{k}{m}\frac{V^4}{g^2}\left(\frac{d\tilde{y}}{d\tilde{t}}\right)^2 + \frac{V^2}{g} = 0$$

$$\frac{d^2\tilde{y}}{d\tilde{t}^2} + \frac{k}{m}\frac{V^2}{g}\left(\frac{d\tilde{y}}{d\tilde{t}}\right)^2 + 1 = 0$$

Lets us now verify the above is dimensionless. Looking at $\left[\frac{k}{m}\frac{V^2}{g}\right] \to \left[\frac{M}{L}\right]\left[\frac{1}{M}\right]\left[\frac{L}{T}\right]^2\left[\frac{1}{\frac{L}{T^2}}\right] \to$
$\left[\frac{M}{L}\right]\left[\frac{1}{M}\right]\left[\frac{L^2}{T^2}\right]\left[\frac{T^2}{L}\right]$

Hence $\left[\frac{k}{m}\frac{V^2}{g}\right] \to 1$ (like magic)

So the dimensionless ODE is

$$\frac{d^2\tilde{y}}{d\tilde{t}^2} + \beta\left(\frac{d\tilde{y}}{d\tilde{t}}\right)^2 + 1 = 0$$

where $\beta = \frac{k}{m}\frac{V^2}{g}$

Now for the initial conditions in terms of the new scaled variables. Since

$$y(0) = y_0, \frac{dy(0)}{dt} = V$$

hence

$$\tilde{y}(0) = \frac{y_0}{V^2}g$$

and $V = \frac{dy(0)}{dt} = \frac{dy(0)}{d\tilde{t}}\frac{d\tilde{t}}{dt} = \frac{V^2}{g}\frac{d\tilde{y}(0)}{d\tilde{t}}\frac{d\tilde{t}}{dt}$, but $\frac{d\tilde{t}}{dt} = \frac{g}{V}$ hence
$\frac{dy(0)}{dt} = \frac{V^2}{g}\frac{d\tilde{y}(0)}{d\tilde{t}}\frac{g}{V} \to V\frac{d\tilde{y}(0)}{d\tilde{t}}$

So $\frac{d\tilde{y}(0)}{d\tilde{t}} = \frac{1}{V}\frac{dy(0)}{dt}$, but $\frac{dy(0)}{dt} = V$, therefor

$$\frac{d\tilde{y}(0)}{d\tilde{t}} = 1$$

So the equation of motion in scaled dimensionless form is

$$\frac{d^2\tilde{y}}{d\tilde{t}^2} + \beta \left(\frac{d\tilde{y}}{d\tilde{t}}\right)^2 + 1 = 0$$

with initial conditions

$$\tilde{y}(0) = \frac{y_0}{V^2}g$$

and

$$\frac{d\tilde{y}(0)}{d\tilde{t}} = 1$$