HW 15 Mathematics 503, Mathematical Modeling, CSUF , August 6, 2007

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Contents

1 Problem 6 page 346 section 6.2 (PDE's)

problem:

A fluid having density ρ, specific heat C, and heat conductivity K, flows at a constant velocity V in a cylindrical tube of length L, and radius R. The temperature at position x is $T = T(x,t)$ and diffusion of heat is ignored. As it flows, heat is lost though the lateral side at a rate jointly proportional to the area and to the difference between the temperature T_e of the external environment and the temperature $T(x,t)$ of the fluid (Newton's law of cooling). Derive a PDE model for the temperature $T(x,t)$

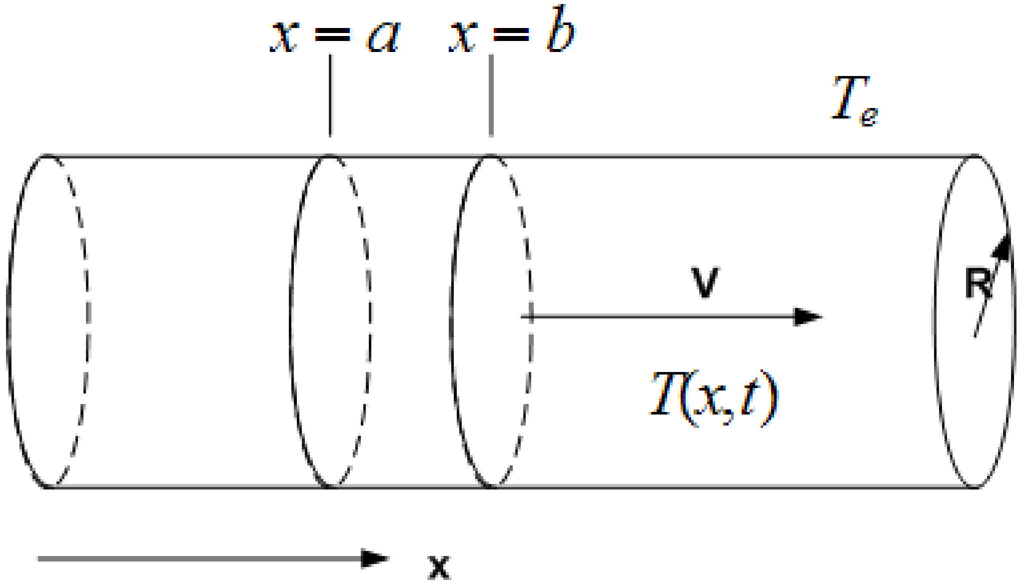
Find the general solution of the equation by transforming to a moving coordinates system $z = x - Vt, \tau = t$

Answer:

Term	Definition	Units (SI)
heat conductivity K	The quantity of heat that passes in unit time through unit area of a substance whose thickness is unity, when its opposite faces differ in temperature by one degree	$\frac{ML}{T^3\theta}$
Specific heat C	the amount of heat needed to raise a unit mass by one degree.	$\frac{L^2}{T^2\theta}$
Density ρ	density	$\frac{M}{L^3}$

First we need to decide on the quantity over which we are applying the balance equation over. We solve this problem 2 times. The first by using the amount of heat as the quantity. Next we solve the problem by using the density of heat as the quantity.

Using amount of heat as the quantity to apply balance equation on



Apply the energy balance equation. Consider 2 cross sectional area at $x=a, x=b$.

Total amount of heat inside this volume = mass of fluid inside this volume $\times$ specific heat $\times$ its temperature $T(x,t)$

Hence amount of heat between $x=a, x=b$ is

$$\int_a^b \left(\underbrace{\overbrace{\pi r^2 dx}^{\text{volume}} \rho}_{\text{mass}} c T(x,t) \right)$$

Now consider the rate that heat flows in/out the volume between $x=a$ and $x=b$, this is given by

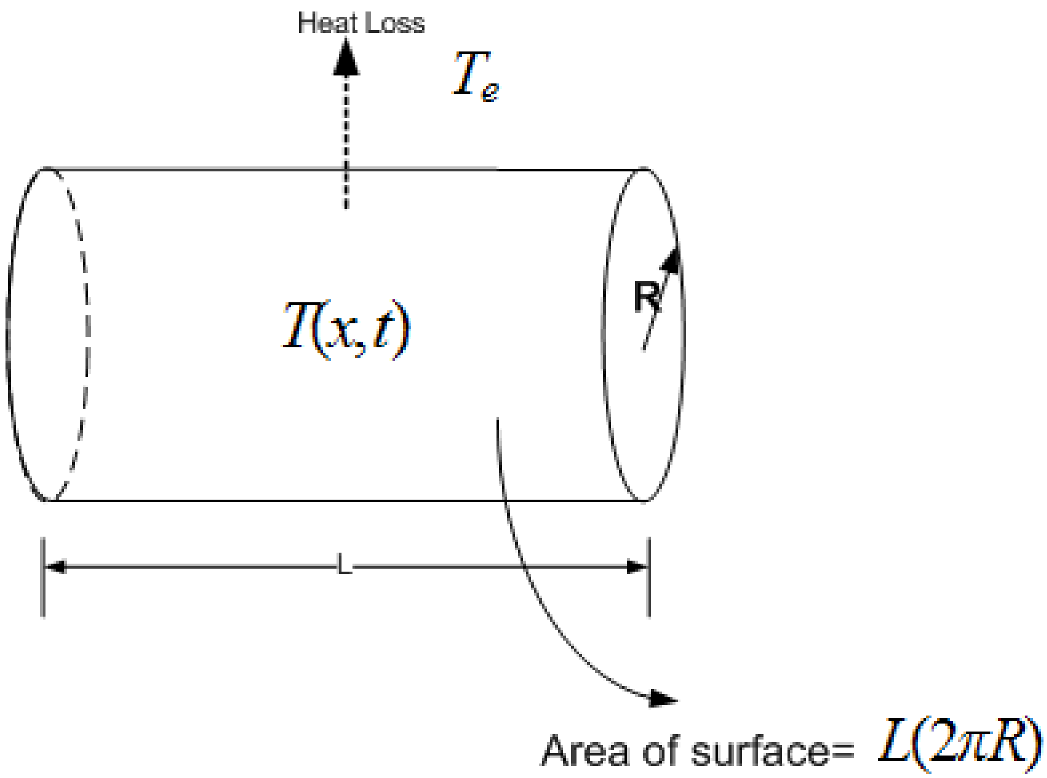
$$A_{x=a} J(a,t) - A_{x=b} J(b,t)$$

and since the area does not change, this is the same as

$$A\left(J(a,t) - J(b,t)\right)$$

where $J(x,t)$ is the amount of heat that flows though a unit area per unit time. (the Flux)

Now there is some heat loss due to cooling via the lateral side. Use Newton law of cooling which says that the amount of heat a body loses per second is proportional to temperature difference between it and its surrounding temperature, and proportional to the area of the body though which heat is lost. The constant of proportionality is k. The area of the body in this case is the surface of the cylinder of strip of width dx which is given by $dx(2\pi r)$ when considering a differential length.



Hence lateral heat loss rate between $x=a$ and $x=b$ is $\left[-\int_a^b k\,dx\,(2\pi r)\,(T(x,t)-T_e)\right]$. The minus sign is to say that this is considered to be a heat loss, not gain, into the volume under consideration. Now need to find what the units of this k is (this is NOT the heat conductivity coefficient). Using dimensional analysis, we write

$$[rate\ of\ heat] = [k]\left[L^2\right][\theta] \rightarrow [k] = \frac{[rate\ of\ heat]}{\left[L^2\right][\theta]} = \frac{[heat]}{\left[L^2\right][\theta][T]} = \frac{\left[M\frac{L^2}{T^2}\right]}{\left[L^2\right][\theta][T]} = \boxed{\frac{[M]}{[\theta]\left[T^3\right]}}$$

So to obtain the energy balance we write

$$\frac{d}{dt}\int_a^b \pi r^2 c\rho T(x,t)\,dx = A\,(J(a,t)-J(b,t)) - \int_a^b k\,dx\,(2\pi r)\,(T(x,t)-T_e)$$

Move the differentiation inside and assuming constant density throughout

$$\pi r^2 c\rho \int_a^b \frac{\partial}{\partial t}T(x,t)\,dx = \pi r^2\,(J(a,t)-J(b,t)) - 2\pi rk\int_a^b (T(x,t)-T_e)\,dx$$
$$rc\rho \int_a^b \frac{\partial}{\partial t}T(x,t)\,dx = r\,(J(a,t)-J(b,t)) - 2k\int_a^b (T(x,t)-T_e)\,dx$$

Now apply fundamental law of calculus to the flux term in the RHS we have

$$rc\rho \int_a^b \frac{\partial}{\partial t}T(x,t)\,dx = r\left(\int_b^a \frac{\partial}{\partial x}J(x,t)\,dx\right) - 2k\int_a^b (T(x,t)-T_e)\,dx$$

Changing the limits of integration on the RHS to make it match the LHS and canceling A we obtain

$$rc\rho \int_a^b \frac{\partial}{\partial t}T(x,t)\,dx = -r\int_a^b \frac{\partial}{\partial x}J(x,t)\,dx - 2k\int_a^b (T(x,t)-T_e)\,dx$$

Because the above holds for all intervals of integration and the functions involved are continuous, then we can remove the integrals and just write

$$rc\rho \frac{\partial}{\partial t} T(x,t) = -r \frac{\partial}{\partial x} J(x,t) - 2k \ (T(x,t) - T_e)$$

For flux, consider the advection model, hence $J(x,t) = Vu(x,t)$ where $u(x,t)$ is amount of heat per unit volume at x, but this is the same as $\rho c T(x,t)$, hence $\frac{\partial}{\partial x} J(x,t) = \frac{\partial}{\partial x} V \rho c T(x,t) = V \rho c T_x(x,t)$ since V is constant, then the above becomes

$$\begin{aligned} rc\rho \frac{\partial}{\partial t} T(x,t) &= -rV\rho c T_x(x,t) - 2k \ (T(x,t) - T_e) \\ rc\rho T_t &= -rV\rho c T_x - 2k \ (T - T_e) \end{aligned}$$

Hence the PDE is

$$\boxed{T_t(x,t) + VT_x(x,t) + \tfrac{2k}{rc\rho} \ T(x,t) = \tfrac{2k}{rc\rho} T_e}$$

Let me check dimensionality. Each term above should have the same units.

T_t has units $\left[\frac{\theta}{T}\right]$

VT_x has units $\left[\frac{L}{T}\frac{\theta}{L}\right] = \left[\frac{\theta}{T}\right]$

$\frac{2k}{rc\rho} T$ has units $\left[\frac{\frac{M}{T^3 \theta}\theta}{L \frac{L^2}{T^2 \theta} \frac{M}{L^3}}\right] = \left[\frac{\theta}{T}\right]$

Hence the PDE is dimensionality correct. Now I have more confidence it is correct. Now we solve the same problem by using the density of heat.

Using density of amount of heat as the quantity to apply balance equation on

density of heat is $c\rho T(x,t)$, hence rate of change of density of heat is $\frac{d}{dt} c\rho T(x,t)$

For the flux term, using the advection model, we obtain that

$$\begin{aligned} J(x,t) &= V \,(\text{density of heat flow though unit area}) \\ &= Vc\rho T(x,t) \end{aligned}$$

Hence

$$J_x(x,t) = -Vc\rho T_x(x,t)$$

Now for heat loss,

$$\begin{aligned} f &= \frac{(2\pi r \ dx) k (T - T_e)}{\pi r^2 \ dx} \\ &= \frac{2k (T - T_e)}{r} \end{aligned}$$

Hence the balance equation now becomes

$$\frac{d}{dt}c\rho T(x,t) = -Vc\rho T_x(x,t) - \frac{2k(T-T_e)}{r}$$

Hence

$$\boxed{T_t(x,t) + VT_x(x,t) + \tfrac{2k}{c\rho r}T(x,t) = \tfrac{2k}{c\rho r}T_e}$$

Which matches the results obtained earlier.

Now complete the problem. Find a general solution.

Let $z = x - Vt, \tau = t$,Hence

$$\begin{aligned}\frac{\partial x}{\partial \tau} &= \frac{\partial (z + V\tau)}{\partial \tau} \\ &= V\end{aligned}$$

and

$$\frac{\partial t}{\partial \tau} = 1$$

Let $W(z,\tau) \equiv T(x,t) = T(z + V\tau, \tau)$, therefore

$$\begin{aligned}W_\tau(z,\tau) &= \frac{\partial T(x,t)}{\partial x}\frac{\partial x}{\partial \tau} + \frac{\partial T(x,t)}{\partial t}\frac{\partial t}{\partial \tau} \\ W_\tau(z,\tau) &= T_x(x,t)V + T_t(x,t)\end{aligned}$$

But from the PDE itself which we derived above, we see that $T_x(x,t)V + T_t(x,t) = -\frac{2k(T-T_e)}{r}$,Hence

$$\boxed{W_\tau(z,\tau) = -\tfrac{2k}{r}\left[T(z+V\tau,\tau) - T_e\right]}$$

Integrating over time we obtain

$$W(z,\tau) = \int_0^t -\frac{2k}{r}\left[T(z+V\tau,\tau) - T_e\right]\, d\tau + g(z)$$

For an arbitrary function $g(z)$, Substituting back for $z = x - Vt$ and $\tau = t$ we obtain

$$\begin{aligned}T(x,t) &= \int_0^t -\frac{2k}{r}\left(T\left((x-Vt) + V\tau, t\right) - T_e\right)d\tau + g(x - Vt) \\ &= \int_0^t -\frac{2k}{r}\left(T\left(x + V(\tau - t), t\right) - T_e\right)d\tau + g(x - Vt)\end{aligned}$$

Hence general solution is

$$\boxed{T(x,t) = \tfrac{-2k}{r}\int_0^t \left[T\left(x + V(\tau - t), t\right) - T_e\right]d\tau \; + g(x - Vt)}$$

The function $g(\cdot)$ can be found from initial conditions that specific $T(x,0)$

2 Problem 6 page 372 section 6.3

problem:

Prove that if the Dirchlet problem

$$-\nabla^2 u = \lambda u \qquad \mathbf{x} \in \Omega$$

$$u = 0 \qquad \mathbf{x} \in \partial\Omega$$

has nontrivial solution then the constant λ must be positive.

solution:

Using Green first identity,

$$\int_\Omega u\nabla^2 w \, dx + \int_\Omega \nabla u \cdot \nabla w \, dx = \int_{\partial\Omega} u \frac{du}{d\mathbf{n}} dA$$

Let $w = u$ in the above

$$\int_\Omega u\nabla^2 u \, dx + \int_\Omega \|\nabla u\|^2 \, dx = \int_{\partial\Omega} u \frac{du}{d\mathbf{n}} dA$$

But $\boxed{\int_{\partial\Omega} u \frac{du}{d\mathbf{n}} dA = 0}$ since $u = 0$ on $\partial\Omega$, hence

$$\int_\Omega u\nabla^2 u \, dx = -\int_\Omega \|\nabla u\|^2 \, dx$$

Now $\nabla^2 u = -\lambda$ from the PDE itself, hence the above becomes

$$-\lambda \int_\Omega u^2 \, dx = -\int_\Omega \|\nabla u\|^2 \, dx$$

$$\lambda \int_\Omega u^2 \, dx = \int_\Omega \|\nabla u\|^2 \, dx$$

The integral in the RHS is ≥ 0

Consider the case if the integral in the RHS is zero. This implies that $\|\nabla u\| = 0$. But $\|\nabla u\|$ is a norm of a vector. (this vector being the gradient of the scalar field function $u(x_i)$). The only way for a norm of a vector to be zero is for the vector itself to be zero. For example, look at $n = 2$, $\nabla u(x_1, x_2) = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} \rightarrow \|\nabla u(x_1, x_2)\|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$.Since this sum is zero, and we are adding nonnegative numbers, then this means $\frac{\partial u}{\partial x_i} = 0$, $i = 1..n$. This implies that u itself is either zero everywhere, or u is not zero, but $\frac{\partial u}{\partial x_i} = 0$, i.e. u is a constant function w.r.t to each of its independent variables. But we can't use the case when $u = 0$ everywhere since we are looking for nontrivial solution, hence the other possibility is for u is a constant function w.r.t to each of its independent variables. But this case also fails, because we are told $u = 0$ on $\partial\Omega$, and so if it is constant, this means it is zero also everywhere inside Ω which means the trivial solution again, which we do not like. Hence we conclude that $\|\nabla u\|$ can NOT be zero. Hence only other option is for it to be strictly positive, i.e. $\int_\Omega \|\nabla u\|^2 \, dx$ must be > 0

Hence $\lambda \int_\Omega u^2 \, dx = k$, where $k > 0$. But since u is not zero, then $\int_\Omega u^2 \, dx > 0$, hence $\boxed{\lambda \text{ is strictly positive}}$

3 Problem 10 page 373 section 6.3

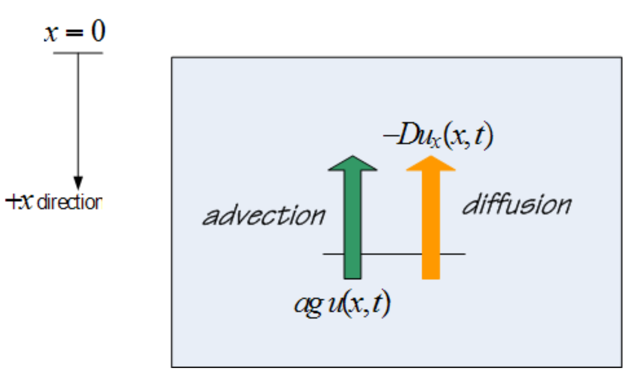
question:

10. The biomass density u (mass per unit volume) of zooplankton deep lake varies as a function of depth x and time t. Zooplankt vertically with diffusion constant D, and buoyancy effects caus migrate toward the surface at a constant speed of ag, where g celeration due to gravity and a is a positive constant. At any ti vertical tube of unit cross-sectional area, the total biomass of zo is a constant U.

   a) From first principles derive a partial differential equation mo biomass density of zooplankton.

   b) Find the *steady-state* biomass density as a function of depth late any reasonable boundary conditions, or other auxiliary c that are needed to solve this problem.)

solution:



Part(a)

Apply balance equation. Using the volume density of the material as the quantity to conserve. There is no loss in this problem, hence $f=0$

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = J_a(x,t) - J_b(x,t)$$
$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \int_b^a J_x(x,t)\,dx$$
$$\int_a^b u_t(x,t)\,dx = -\int_a^b J_x(x,t)\,dx$$

Hence

$$\boxed{u_t(x,t) = -J_x(x,t)}$$

Now the flux is made up of 2 parts in this problem, due to advection and due to diffusion. Hence

$$J(x,t) = -Du_x(x,t) - agu(x,t)$$

Where the minus sign next to the advection term is due to speed being in opposite direction to the x-axis. Hence the PDE is

$$u_t(x,t) = \frac{\partial}{\partial x}\left[Du_x(x,t) + agu(x,t)\right]$$

$$\boxed{u_t(x,t) = Du_{x,x}(x,t) + agu_x(x,t)}$$

Part(b)

At steady state, $u_t(x,t) = 0$, hence we obtain

$$\frac{d^2u(x)}{dx^2} + \frac{ag}{D}\frac{du(x)}{dx} = 0$$

We can solve this by separation of variables. Let $y = u'$

$$y' + ky = 0$$
$$\frac{1}{y}dy = -kdx$$
$$\ln y = -\frac{1}{2}kx + C$$
$$y = Ae^{\frac{-ag}{2D}x}$$

Let $k = \frac{ag}{2D}$Since $y = u'$

$$u(x) = \int Ae^{-kx}dx + B$$

Hence

$$\boxed{u(x) = -\tfrac{A}{k}e^{-kx} + B}$$

We consider that at $x=\infty, u=0$, (i.e. since very deep lake, we take x very large)

$$\boxed{B=0}$$

Then

$$u(x) = -\frac{A}{k}e^{-kx}$$

We also know that $\int_0^\infty u(x)\,dx = U$, hence

$$\begin{aligned}\frac{-A}{k}\int_0^\infty e^{-kx}dx &= U\\ \frac{A}{k^2}\left[e^{-kx}\right]_0^\infty &= U\\ \frac{A}{k}[0-1] &= U\\ \frac{-A}{k} &= U\end{aligned}$$

Hence $\boxed{A=-kU}$

Hence the solution is

$$u(x) = -\frac{A}{k}e^{-kx}$$

Then

$$\boxed{u(x) = Ue^{-\frac{ag}{2D}x}}$$

To verify, as $x \to \infty$ we obtain $u(x)=0$ and at $x=0$, $u(x)=U$

Note, it does not seem to make too much sense to me that in *any tube length*, total amount is always U, for this implies, for some length h that $\int_0^h u(x)\,dx = U$ but $\int_0^h u(x)\,dx = \int_0^h Ue^{-kx}dx = \frac{U}{-k}\left[e^{-kx}\right]_0^h = \frac{U}{-k}\left[e^{-kh}-1\right]$

Hence $\frac{U}{-k}\left[e^{-kh}-1\right] = U$ which means $\frac{1-e^{-kh}}{k} = 1$ or $1-e^{-kh} = k$, which means $h = \frac{-\ln(1-k)}{k}$which at $k=1$ blows up. So this means $k<1$, i.e. $\frac{ag}{2D}$ must be <1.