HW 14 Mathematics 503, Mathematical Modeling, CSUF , August 6, 2007

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1 Problem 9 page 346 section 6.2 (PDE's)

problem:

Find all solutions to the heat equation $u_t = \kappa u_{xx}$ of the form $u(x,t) = U(z)$ where $z = \frac{x}{\sqrt{\kappa t}}$

answer:

We have that $z(x,t) = \frac{x}{\sqrt{\kappa t}}$, hence $\frac{\partial z}{\partial x} = \frac{1}{\sqrt{\kappa t}}$ and $\frac{\partial^2 z}{\partial x^2} = 0$ and $\frac{\partial z}{\partial t} = -\frac{x}{2}(\kappa t)^{-\frac{3}{2}}\kappa = \frac{-x}{2}\frac{t^{-\frac{3}{2}}}{\sqrt{k}}$

Now

$$\begin{aligned} u_x(x,t) &= U'(z)\frac{\partial z}{\partial x} \\ &= U'(z)\frac{1}{\sqrt{\kappa t}} \end{aligned}$$

and

$$\begin{aligned} u_{xx}(x,t) &= U''(z)\frac{1}{\sqrt{\kappa t}}\frac{\partial z}{\partial x} \\ &= U''(z)\frac{1}{\kappa t} \end{aligned}$$

and

$$
\begin{aligned}
u_t(x,t) &= U'(z)\frac{\partial z}{\partial t} \\
&= \frac{-x}{2}U'(z)\frac{t^{-\frac{3}{2}}}{\sqrt{k}}
\end{aligned}
$$

Plug in the above expressions into the PDE we obtain

$$
\begin{aligned}
u_t &= \kappa u_{xx} \\
\frac{-x}{2}U'(z)\frac{t^{-\frac{3}{2}}}{\sqrt{k}} &= \kappa U''(z)\frac{1}{\kappa t} \\
\frac{-x}{2\sqrt{kt}}U'(z) &= U''(z)
\end{aligned}
$$

But $z=\frac{x}{\sqrt{\kappa t}}$, hence the above becomes

$$
-\frac{1}{2}z\,U'(z) = U''(z)
$$

or

$$
\boxed{U''(z)+\tfrac{1}{2}z\,U'(z)=0}
$$

Let $U'(z)=y(z)$, hence the above becomes

$$
\begin{aligned}
y'+\frac{1}{2}z\,y &= 0 \\
\frac{y'}{y} &= -\frac{1}{2}z \\
\frac{1}{y}\frac{dy}{dz} &= -\frac{1}{2}z \\
\frac{1}{y}dy &= -\frac{1}{2}zdz
\end{aligned}
$$

Integrate both sides

$$
\ln y = -\frac{1}{4}z^2 + C
$$

Hence

$$
y(z) = Ae^{\frac{-1}{4}z^2}
$$

But since $U'(z)=y(z)$, then

$$
\begin{aligned}
U(z) &= \int y(z)\,dz + B \\
&= A\int e^{\frac{-1}{4}z^2}dz + B
\end{aligned}
$$

I think now I need to write the above in terms of x,t again. Fix time, and change x and so we have $dz=\frac{\partial z}{\partial x}dx=\frac{1}{\sqrt{\kappa t}}dx$ and the above integral becomes

$$u(x,t;\xi)=\int A(\xi)\,e^{\frac{-(x-\xi)^2}{4\kappa t}}\frac{1}{\sqrt{\kappa t}}d\xi+B(\xi)$$

for any ξ location along the space dimension x, where $A(\xi),B(\xi)$ are functions that depend on the value ξ

2 Problem 3 page 365 section 6.2 (PDE's)

problem:

Use the energy method to prove the uniqueness for the problem

$$\begin{aligned} u_t &= \nabla^2 u & \mathbf{x} \in \Omega, t > 0 \\ u(\mathbf{x},0) &= f(\mathbf{x}) & \mathbf{x} \in \Omega \\ u(\mathbf{x},t) &= g(\mathbf{x}) & \mathbf{x} \in \partial\Omega, t > 0 \end{aligned}$$

Solution

First note that $\nabla^2 u \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}$ i.e. the Laplacian.

Proof by contradiction. Assume there is no unique solution. Let $u_1(x,t)$ and $u_2(x,t)$ be 2 different solutions to the above PDE. Let $w(x,t)$ be the difference between these 2 solutions. i.e. $w(x,t) = u_1(x,t) - u_2(x,t)$, hence $w(x,t)$ must satisfy the following conditions: it must be zero at the boundaries $\mathbf{x} \in \partial\Omega$ for all time, and also it must be zero inside Ω initially. Hence

$$\begin{aligned} w(\mathbf{x},0) &= 0 & \mathbf{x} \in \Omega \\ w(\mathbf{x},t) &= 0 & \mathbf{x} \in \partial\Omega, t > 0 \end{aligned}$$

Now if we can show that $w(\mathbf{x},t) = 0$ for $t > 0$ inside Ω, then this would imply that $u_1(x,t) = u_2(x,t)$, showing a contradiction, hence completing the proof.

i.e. we need to show that $w_t(\mathbf{x},t) = \nabla^2 w(\mathbf{x},t)$ yields a solution $w(\mathbf{x},t) = 0$ for $\mathbf{x} \in \Omega, t > 0$

Using the energy argument, we write

$$E(t) = \int_\Omega w^2(\mathbf{x},t)\, d\mathbf{x}$$

First we note that $E(0) = 0$ since $w(\mathbf{x},0) = 0$ from the initial conditions above.

$$\begin{aligned} E'(t) &= \frac{\partial}{\partial t}\int_\Omega w^2(\mathbf{x},t)\, d\mathbf{x} \\ &= \int_\Omega \frac{\partial}{\partial t} w^2(\mathbf{x},t)\, d\mathbf{x} \\ &= \int_\Omega 2w(\mathbf{x},t)\ \frac{\partial}{\partial t} w(\mathbf{x},t)\ d\mathbf{x} \end{aligned}$$

But $\frac{\partial}{\partial t} w(\mathbf{x},t) = \nabla^2 w(\mathbf{x},t)$ from the PDE itself, hence the above becomes

$$E'(t) = 2\int_\Omega w(\mathbf{x},t)\ \nabla^2 w(\mathbf{x},t)\ d\mathbf{x} \tag{1}$$

But from Green first identity which states the following

$$\int_\Omega \left(u\nabla^2 w + \nabla u \cdot \nabla w\right) d\mathbf{x} = \int_{\partial\Omega} u\frac{dw}{dn} dA$$

Replace u by w in the above, we obtain

$$\int_{\Omega} \left(w\nabla^2 w + \nabla w \cdot \nabla w \right) d\mathbf{x} = \int_{\partial\Omega} w \frac{dw}{dn} dA$$
$$\int_{\Omega} w\nabla^2 w \, d\mathbf{x} + \int_{\Omega} \nabla w \cdot \nabla w \, d\mathbf{x} = \int_{\partial\Omega} w \frac{dw}{dn} dA$$
$$\int_{\Omega} w\nabla^2 w \, d\mathbf{x} = \int_{\partial\Omega} w \frac{dw}{dn} dA - \int_{\Omega} \nabla w \cdot \nabla w \, d\mathbf{x} \tag{2}$$

Comparing (1) and (2) we see that LHS of (2) is $\frac{1}{2}E'(t)$ Hence the above become

$$\frac{1}{2}E'(t) = \int_{\partial\Omega} w \frac{dw}{dn} dA - \int_{\Omega} \nabla w \cdot \nabla w \, d\mathbf{x}$$

But $\nabla w \cdot \nabla w = \|\nabla w\|^2$, so the above becomes

$$\frac{1}{2}E'(t) = \int_{\partial\Omega} w \frac{dw}{dn} dA - \int_{\Omega} \|\nabla w\|^2 \, d\mathbf{x}$$

But $w(\mathbf{x},t) = 0$ on $\partial\Omega$ for $t > 0$, since this is the boundary conditions. Hence the above becomes

$$E'(t) = -2\int_{\Omega} \|\nabla w\|^2 \, d\mathbf{x}$$

Therefore we showed that $E'(t)$ is ≤ 0 since $\int_{\Omega} \|\nabla w\|^2 \, d\mathbf{x} \geq \mathbf{0}$

So energy inside Ω is nonincreasing with time. But since $E(0) = 0$ then $E(t) = 0$ (since energy can not be negative, this is the only choice left).

Therefore, from $E(t) = \int_{\Omega} w^2(\mathbf{x},t)\, d\mathbf{x}$, we conclude that $w(\mathbf{x},t) = 0$ everywhere in Ω for $t > 0$ since $w(\mathbf{x},t)$ is continuous in both its arguments.

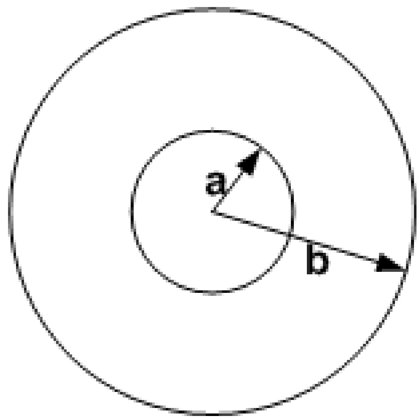
Hence we conclude since $w(\mathbf{x},t) = u_1(\mathbf{x},t) - u_2(\mathbf{x},t) = 0$ then $u_1(\mathbf{x},t) = u_2(\mathbf{x},t)$, then the PDE solution is unique.

3 Problem 5 page 365 section 6.2 Conservation laws

problem:

In absence of sources derive the diffusion equation for radial motion in the plane $u_t = \frac{D}{r}(ru_r)_r$ from first principles. That is, take an arbitrary domain between circles $r=a, r=b$ and apply conservation law for the density $u=u(r,t)$ assuming the flux is $J(r,t) = -Du_r$. Assume no sources.

Answer:



First note that the density $u(r,t)$ is measured in quantity per unit volume.

Consider a cross sectional area through circle $r_a = a$. This area is $2\pi h r_a$ where h is the width of the strip.

Let $J(r,t)$ be the flux at r at time t , measured in quantity per unit area per unit time.

Hence amount u that passes though cross sectional area at r_a , per unit time, is $A(r_a)J(r_a,t)$ where $A(r_a) = 2\pi h r_a$

Similarly, amount u that passes though cross sectional area at r_b , per unit time, is $A(r_b)J(r_a,t)$ where $A(r_b) = 2\pi h r_b$

Hence the net amount that flows, per unit time, between r_b and r_a is $A(r_a)J(r_a,t) - A(r_b)J(r_b,t)$

Since there is no source nor sink inside this region, then the above equal the rate at which the amount u itself changes between r_b and r_a, which is $\frac{d}{dt}(u(r,t) \times \text{volume between } r_a \text{ and } r_b)$.

Hence we have

$$\frac{d}{dt}\int_a^b u(r,t)A(r)\,dr = A(r_a)J(r_a,t) - A(r_b)J(r_b,t)$$
$$\int_a^b u_t(r,t)A(r)\,dr = A(r_a)J(r_a,t) - A(r_b)J(r_b,t)$$

Apply fundamental theorem of calculus on the RHS above where $J(a,t) - J(b,t) = \int_b^a J_r dr$ hence the above becomes

$$\int_a^b u_t(r,t)A(r)\,dr = \int_b^a \frac{\partial}{\partial r}[A(r)J(r,t)]\,dr$$

But $A(r) = 2\pi rh$ so the above becomes

$$\int_a^b u_t(r,t)\,r dr = \int_b^a \frac{\partial}{\partial r}[rJ(r,t)]\,dr$$

Changing the limits on the integral in the RHS above to make it match the LHS, we obtain

$$\int_a^b u_t(r,t)\, r dr = -\int_a^b \frac{\partial}{\partial r}\left[rJ(r,t)\right] dr$$

Because the above holds for all intervals of integration and the functions involved are continuous, then we can remove the integrals and just write

$$u_t(r,t)\, r = -\frac{\partial}{\partial r}\left[rJ(r,t)\right]$$

Now assuming diffusion model for the flux, i.e. $J(r,t) = -Du_r(r,t)$, then the above becomes

$$u_t(r,t)\, r = D\frac{\partial}{\partial r}\left[ru_r(r,t)\right]$$

Hence

$$\boxed{u_t = \tfrac{D}{r}\left[ru_r\right]_r}$$