HW 13 Mathematics 503, Mathematical Modeling, CSUF , August 2, 2007

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June 15, 2014

Contents

1 Problem 10 page 268 section 4.5 (Distributions)

problem:

Find fundamental solution associated with operator L defined by $Lu = -x^2u'' - xu' + u, 0 < x < 1$, such that $u(x,\xi) = x$ for $0 < x < \xi$.

answer:

The fundamental solution can be written as

$$u = \begin{cases} x & 0 < x < \xi \\ A(\xi)u_1(x) + B(\xi)u_2(x) & \xi < x < 1 \end{cases}$$

And our goal is to determine $A(\xi), B(\xi)$. In the above u_1, u_2 are the 2 independent solution to the homogenous equation $-x^2u'' - xu' + u = 0$

We start by finding u_1, u_2. We try solution $u = x^m$ and substitute this into the above homogenous equation, we obtain the characteristic equation $m^2 = 1$, hence $m = \pm 1$ the 2 solution are $\boxed{u_1 = x \text{ and } u_2 = x^{-1}}$. Hence our fundamental solution now looks like

$$u = \begin{cases} x & 0 < x < \xi \\ A(\xi)x + B(\xi)x^{-1} & \xi < x < 1 \end{cases}$$

Now consider the test function ϕ, hence

$$\begin{aligned}
(Lu,\phi) &= \left(-x^2u''-xu'+u,\phi\right) \\
&= \left(-x^2u'',\phi\right)-\left(xu',\phi\right)+(u,\phi) && \text{linearity of distribution} \\
&= \left(u'',-x^2\phi\right)-\left(u',x\phi\right)+(u,\phi) && \text{property of distribution} \\
&= \left(u,\left(-x^2\phi\right)''\right)+\left(u,(x\phi)'\right)+(u,\phi) && \text{property of distribution} \\
&= \left(u,\left(-x^2\phi\right)''+(x\phi)'+\phi\right) \\
&= (u,L^*\phi)
\end{aligned}$$

Where $L^*\phi=\left(-x^2\phi\right)''+(x\phi)'+\phi$

Hence expanding the differentiation in the above and simplifying we obtain

$$\boxed{(Lu,\phi)=\left(u,-3x\phi'-x^2\phi''\right)}$$

Now take $Lu=\delta_\xi$, i.e. put a point source as input, then we are looking for $(Lu,\phi)=\phi(\xi)$ from the properties of delta function. In other words, we are looking for

$$(Lu,\phi)=\int_0^1 u\left(-3x\phi'-x^2\phi''\right)dx=\phi(\xi)$$

Hence

$$\begin{aligned}
\phi(\xi) &= \int_0^\xi u_1\left(-3x\phi'-x^2\phi''\right)dx+\int_\xi^1 u_2\left(-3x\phi'-x^2\phi''\right)dx \\
&= \int_0^\xi x\left(-3x\phi'-x^2\phi''\right)dx+\int_\xi^1\left(A(\xi)x+B(\xi)x^{-1}\right)\left(-3x\phi'-x^2\phi''\right)dx \qquad (1)
\end{aligned}$$

Looking at the first integral, and perform integration by parts. In these calculations we note that

$$\phi(0)=\phi(1)=\phi'(0)=\phi(1)=0$$

Hence

$$\begin{aligned}
\int_0^\xi x\left(-3x\phi'-x^2\phi''\right)dx &= \int_0^\xi -3x^2\phi'-x^3\phi''dx \\
&= \int_0^\xi -3x^2\phi'dx-\int_0^\xi x^3\phi''dx \\
&= -3\left(\left[x^2\phi\right]_0^\xi-\int_0^\xi 2x\phi dx\right)-\left(\left[x^3\phi'\right]_0^\xi-\int_0^\xi 3x^2\phi'dx\right) \\
&= -3\left(\left[\xi^2\phi(\xi)\right]-2\int_0^\xi x\phi dx\right)-\left(\left[\xi^3\phi'(\xi)\right]-3\int_0^\xi x^2\phi'dx\right) \\
&= -3\xi^2\phi(\xi)+6\int_0^\xi x\phi dx-\xi^3\phi'(\xi)+3\int_0^\xi x^2\phi'dx
\end{aligned}$$

Now do integration by part on the last integral above

$$\begin{aligned}
\int_0^{\xi} x\left(-3x\phi' - x^2\phi''\right)dx &= -3\xi^2\phi(\xi) + 6\int_0^{\xi} x\phi dx - \xi^3\phi'(\xi) + 3\left(\left[x^2\phi\right]_0^{\xi} - \int_0^{\xi} 2x\phi dx\right) \\
&= -3\xi^2\phi(\xi) + 6\int_0^{\xi} x\phi dx - \xi^3\phi'(\xi) + 3\xi^2\phi(\xi) - 6\int_0^{\xi} x\phi dx \\
&= -\xi^3\phi'(\xi) \qquad (2)
\end{aligned}$$

Now looking at (1) above, we now do integration by part on $\int_{\xi}^{1}\left(Ax+Bx^{-1}\right)\left(-3x\phi' - x^2\phi''\right)dx$

$$\int_{\xi}^{1}\left(Ax+Bx^{-1}\right)\left(-3x\phi' - x^2\phi''\right)dx = \int_{\xi}^{1} Ax\left(-3x\phi' - x^2\phi''\right)dx + \int_{\xi}^{1} Bx^{-1}\left(-3x\phi' - x^2\phi''\right)dx$$

Consider the first integral above in the RHS, we write

$$\begin{aligned}
\int_{\xi}^{1} Ax\left(-3x\phi' - x^2\phi''\right)dx &= A\int_{\xi}^{1} -3x^2\phi' - x^3\phi'' dx \\
&= A\left(\int_{\xi}^{1} -3x^2\phi' dx - \int_{\xi}^{1} x^3\phi'' dx\right) \\
&= A\left(3\left(\left[-x^2\phi\right]_{\xi}^{1} - \int_{\xi}^{1} -2x\phi dx\right) - \left(\left[x^3\phi'\right] - \int_{\xi}^{1} 3x^2\phi' dx\right)\right) \\
&= A\left(3\left(\left[-\phi(1) + \xi^2\phi(\xi)\right] + 2\int_{\xi}^{1} x\phi dx\right) - \left(\left[\phi'(1) - \xi^3\phi'(\xi)\right] - 3\int_{\xi}^{1} x^2\phi' dx\right)\right) \\
&= A\left(3\left(\xi^2\phi(\xi) + 2\int_{\xi}^{1} x\phi dx\right) - \left(-\xi^3\phi'(\xi) - 3\int_{\xi}^{1} x^2\phi' dx\right)\right) \\
&= A\left(3\xi^2\phi(\xi) + 6\int_{\xi}^{1} x\phi dx + \xi^3\phi'(\xi) + 3\int_{\xi}^{1} x^2\phi' dx\right)
\end{aligned}$$

Now do integration by part on the last term in the above line

$$\begin{aligned}
\int_{\xi}^{1} Ax\left(-3x\phi' - x^2\phi''\right)dx &= A\left(3\xi^2\phi(\xi) + 6\int_{\xi}^{1} x\phi dx - \xi^3\phi'(\xi) + 3\left(\left[x^2\phi\right]_{\xi}^{1} - \int_{\xi}^{1} 2x\phi dx\right)\right) \\
&= A\left(3\xi^2\phi(\xi) + 6\int_{\xi}^{1} x\phi dx - \xi^3\phi'(\xi) + 3\left(-\xi^2\phi(\xi) - 2\int_{\xi}^{1} x\phi dx\right)\right) \\
&= A\left(3\xi^2\phi(\xi) + 6\int_{\xi}^{1} x\phi dx - \xi^3\phi'(\xi) - 3\xi^2\phi(\xi) - 6\int_{\xi}^{1} x\phi dx\right) \\
&= A\xi^3\phi'(\xi) \qquad (3)
\end{aligned}$$

Now we do integration by parts on $\int_{\xi}^{1} Bx^{-1}\left(-3x\phi' - x^2\phi''\right)dx$

$$
\begin{aligned}
\int_{\xi}^{1} Bx^{-1}\left(-3x\phi' - x^2\phi''\right)dx &= B\int_{\xi}^{1} -3\phi' - x\phi'' dx \\
&= -B\left(\int_{\xi}^{1} 3\phi' dx + \int_{\xi}^{1} x\phi'' dx\right) \\
&= -B\left(3\left[\phi\right]_{\xi}^{1} + \left(\left[x\phi'\right]_{\xi}^{1} - \int_{\xi}^{1}\phi' dx\right)\right) \\
&= -B\left(-3\phi\left(\xi\right) + \left(-\xi\phi'\left(\xi\right) - \left[\phi\right]_{\xi}^{1}\right)\right) \\
&= -B\left(-3\phi\left(\xi\right) - \xi\phi'\left(\xi\right) + \phi\left(\xi\right)\right) \\
&= 2B\phi\left(\xi\right) + B\xi\phi'\left(\xi\right) \qquad (5)
\end{aligned}
$$

Hence, from (2),(3),(4),(5), we have

$$
\begin{aligned}
\phi\left(\xi\right) &= \int_{0}^{\xi} x\left(-3x\phi' - x^2\phi''\right)dx + \int_{\xi}^{1}\left(A\left(\xi\right)x + B\left(\xi\right)x^{-1}\right)\left(-3x\phi' - x^2\phi''\right)dx \\
\phi\left(\xi\right) &= -\xi^3\phi'\left(\xi\right) + A\xi^3\phi'\left(\xi\right) + 2B\phi\left(\xi\right) + B\xi\phi'\left(\xi\right)
\end{aligned}
$$

or

$$
\phi\left(\xi\right) = \phi\left(\xi\right)\left(2B\right) + \phi'\left(\xi\right)\left[-\xi^3 + A\xi^3 + B\xi\right] \qquad (6)
$$

By looking at the coefficients on $\phi\left(\xi\right)$, and compare, we see that $2B = 1$ or

$$
B = \frac{1}{2}
$$

We can now use the continuity condition at $x = \xi$ and write
$u_1 = u_2$ at $x = \xi$, hence

$$
\begin{aligned}
\xi &= Au_1 + Bu_2 \\
&= A\xi + B\xi^{-1} \\
\xi &= A\xi + \frac{1}{2}\xi^{-1} \\
2\xi^2 &= 2A\xi^2 + 1
\end{aligned}
$$

Hence

$$
A = \frac{2\xi^2 - 1}{2\xi^2}
$$

Therefor the fundamental solution is

$$u = \begin{cases} x & 0 < x < \xi \\ Ax + Bx^{-1} & \xi < x < 1 \end{cases}$$

$$= \begin{cases} x & 0 < x < \xi \\ \frac{2\xi^2 - 1}{2\xi^2} x + \frac{1}{2x} & \xi < x < 1 \end{cases}$$

Here is a plot for few values of ξ

```
In[50]:= u[x_, ξ_] := x /; x < ξ

         u[x_, ξ_] := (2 ξ^2 - 1)/(2 ξ^2) x + 1/(2 x) /; x > ξ

         u[x_, ξ_] := 1 /; x == ξ

         p = Table[Plot[u[x, ξ], {x, 0, 1}, PlotLabel → "ξ=" <> ToString[ξ], ImageSize
```

Out[53]=