HW 12 Mathematics 503, analytical part, July 26, 2007

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1 Problem

Let V be the space of continuously differentiable functions y such that $y(0)=0, y(1)=0$. On this space consider the functional

$$J(y)=\int_0^1 \left[y'(x)\right]^2+q\left[y(x)\right]^2-2fy(x)\ \ dx$$

where $q>0$ and f are given constants.

(a) Show that J achieves a minimum at y iff

$$I=\int_0^1 y'(x)\,\phi'(x)+q\,y(x)\,\phi(x)-f\,\phi(x)\ \ dx=0$$

for all $\phi(x)\in V$

(b)Show that if $y\in V$ is twice continuously differentiable and satisfies this optimality condition then y satisfies the differential equation $-y''(x)+qy(x)=f, u(0)=0, y(1)=0$

(c)Conversely show that if $y\in V$ is twice continuously differentiable, and satisfies the differential equation above, then y satisfies the optimality condition of part (a)

2 Solution

2.1 part(a)

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Let $A=\left\{y\in C^1[0,1], y(0)=y(1)=0\right\}$, and let $A_y=\left\{\phi\in C^1[0,1], \phi(0)=\phi(1)=0\right\}$. The set A is called the set of admissible functions (from which the function which will minimize the functional will be found), and A_y is called the set of admissible directions. We require also that $y+t\phi\in A$ where t is some small scalar.

Since this is an IFF, then we need to show the forward and the backward case. Start with the *forwards case*:

Given: $J(y)=\int_0^1 y'^2+qy^2-2fy\ \ dx$show that

$$\boxed{\text{if } I=\int_0^1 y'\phi'+q\,y\phi-f\,\phi\ \ dx=0 \text{ then } J(y) \text{ is minimum}}$$

Consider

$$\begin{aligned} J(y+\phi) &= \int_0^1 \left[(y+\phi)'\right]^2 + q\left[(y+\phi)\right]^2 - 2f(y+\phi) \; dx \\ &= \int_0^1 \left(y'^2 + \phi'^2 + 2y'\phi'\right) + q\left(y^2+\phi^2+2y\phi\right) - 2fy - 2f\phi \; dx \\ &= \int_0^1 y'^2 + qy^2 - 2fy \, dx + \int_0^1 \phi'^2 + q\phi^2 \; dx + \int_0^1 2y'\phi' + 2qy\phi - 2f\phi \, dx \\ &= J(y) + Q + \int_0^1 2y'\phi' + 2qy\phi - 2f\phi \, dx \end{aligned}$$

Where $Q = \int_0^1 \phi'^2 + q\phi^2 \; dx \geq 0$ since $q > 0$ and the other functions are squared. Hence this implies from the above that if $\int_0^1 2y'\phi' + 2qy\phi - 2f\phi \, dx = 0$ then y is a minimizer of $J(y)$. In other words

$$J'(y;\phi) = \int_0^1 y'\phi' + qy\phi - f\phi \, dx = 0$$

(This is because any change from y along any of the admissible directions ϕ will result in a functional $J(y+\phi)$ which is larger than it was at $J(y)$).

Now to show the *backward* case:

If $J'(y;\phi) = \int_0^1 y'\phi' + qy\phi - f\phi \, dx \neq 0$ then $J(y)$ is not a minimum.

Assume for some $\phi = \phi_0$ we have $J'(y;\phi_0) < 0$

$$\begin{aligned} J(y+t\phi_0) &= \int_0^1 \left[(y+t\phi_0)'\right]^2 + q\left[(y+t\phi_0)\right]^2 - 2f(y+t\phi_0) \; dx \\ &= \int_0^1 \left(y'^2 + t^2\phi_0'^2 + 2ty'\phi_0'\right) + q\left(y^2 + t^2\phi_0^2 + 2ty\phi\right) - 2fy - 2tf\phi_0 \; dx \\ &= \int_0^1 y'^2 + qy^2 - 2fy \, dx + \int_0^1 t^2\phi_0'^2 + qt^2\phi_0^2 \; dx + \int_0^1 2ty'\phi_0' + 2tqy\phi_0 - 2tf\phi_0 \, dx \\ &= J(y) + t^2 Q + t\int_0^1 2y'\phi_0' + 2qy\phi_0 - 2f\phi_0 \, dx \end{aligned}$$

Where $Q = \int_0^1 \phi_0'^2 + q\phi_0^2 \geq 0$ .Hence we have

$$J(y+t\phi_0) = J(y) + t\left[2\int_0^1 y'\phi_0' + qy\phi_0 - f\phi_0 \, dx + \; tQ\right]$$

Now, no matter how large Q is, we can make t small enough so that tQ is smaller than the absolute value of $|2J'(y;\phi_0)|$. But since $J'(y;\phi_0) < 0$, then $\left[2\int_0^1 y'\phi_0' + qy\phi_0 - f\phi_0 \, dx + \; tQ\right]$ will be a *negative quantity*. Hence, since $t\times$negative quantity is also negative quantity, then we conclude that

$$J(y+t\phi_0) < J(y)$$

Hence $\boxed{y \text{ is not a minimizer of } J(y)}$. So no matter which y we hope it is our minimum, we can find an admissible direction ϕ_0 such that if move very slightly away from this y in this admissible direction, we find that $J(y+t\phi_0)$ is smaller (this will always be the case if $J'(y;\phi_0) < 0$)

2.2 Part (b)

Given $y \in C^2[0,1]$, and $J'(y;\phi) = 0$ for all admissible directions ϕ. Show that y satisfies the differential equation $-y''(x) + qy(x) = f, u(0) = 0, y(1) = 0$

Since $J'(y;\phi) = 0$, in other words $\int_0^1 y'\phi' + q\ y\phi - f\ \phi\ \ dx = 0$. Then now we do integration by parts.

$$\int_0^1 \overbrace{y'}^{"u"}\overbrace{\phi' dx}^{"dv"} = [y'\phi]_0^1 - \int_0^1 y''\phi dx$$
$$\int_0^1 y'\phi' dx = -\int_0^1 y''\phi dx$$

Since $\phi(0) = \phi(1) = 0$.Now substitute the above back into $\int_0^1 y'\phi' + q\ y\phi - f\ \phi\ \ dx$ and take ϕ as common factor, we obtain

$$\int_0^1 \left(-y'' + qy - f\ \right)\phi\ dx = 0$$

Now we apply the fundamental theory of variational calculus (which Lemma 3.13 is special case) and argue that since ϕ is arbitrary admissible direction, then for the above to be zero for every ϕ, we must have

$$-y'' + qy - f\ = 0$$

or

$$\boxed{-y'' + qy = f}$$

with $y(0) = y(1) = 0$

2.3 Part (c)

Given $y \in C^2[0,1]$ and $-y'' + qy = f, y(0) = y(1) = 0$ show that $\int_0^1 y'\phi' + q\ y\phi - f\ \phi\ \ dx = 0$

Let $\phi \in A_d$. Now

$$\frac{d}{dx}(y'\phi) = y''\phi + y'\phi'$$
$$\int_0^1 \frac{d}{dx}(y'\phi)\, dx = \int_0^1 y''\phi + y'\phi'\ dx$$
$$[y'\phi]_0^1 = \int_0^1 y''\phi + y'\phi'\ dx$$

But $[y'\phi]_0^1 = 0$ since $\phi = 0$ at both ends. Hence the above becomes

$$0 = \int_0^1 y''\phi + y'\phi'\ dx$$

But $y'' = -f + qy$ since y satisfies the differential equation. Hence the above becomes

$$
\begin{aligned}
0 &= \int_0^1 (-f + qy)\,\phi + y'\phi'\;dx \\
&= \int_0^1 y'\phi' + qy\phi - f\phi\;dx
\end{aligned}
$$

Which is the optimality condition (weak form) of part (a) we are asked to show.