HW 10 Mathematics 503, Mathematical Modeling, CSUF , July 17, 2007

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June 15, 2014

Contents

1 Problem 4 page 225 section 4.1

problem:

Consider SLP $-y'' = \lambda y, \;\; 0 < x < 1$ with B.V. $y(0) + y'(0) = 0, y(1) = 0$

is $\lambda = 0$ an eigenvalue? are there negative eigenvalues? show that there are infinitely many positive eigenvalues by finding an equation whose roots are those eigenvalues and show graphically that there are infinity many root

answer:

The SLP has the form $-(p(x)y')' + (q(x) - \lambda)y = 0$ or $-(p(x)y')' + q(x)y = \lambda y$ for $a < x < b$, where $p(x)$ not zero function and does not change sign over the interval, hence we can assume it to be positive. If we compare this form to the given problem we see that $\boxed{p(x) = 1 \text{ and } q(x) = 0}$

Assume $\lambda = 0$, hence the ODE become $-y'' = 0$ which has the solution $y = Ax + B$ for some constants A, B. Now lets see is this solution can satisfy the B,V, given.

$y(1) = 0 \to A + B = 0$, and $y(0) + y'(0) = 0 \to A = 0$, hence since $A = 0$, then $B = 0$ hence the only solution is $y(x) = 0$. Hence for non-trivial solution $\boxed{\lambda \neq 0}$

Now let us assume $\lambda < 0$. Assume $y = Ae^{mx}$, hence the characteristic equation is $-m^2 = \lambda$ or $m^2 = -\lambda$, but since $\lambda < 0$, then m is a real quantity. Let $-\lambda = \beta^2$ where β is some non zero real constant, hence we have $\boxed{m = \pm\beta}$ and so the solution is

$$y = c_1 e^{\beta x} + c_2 e^{-\beta x}$$

Let see is this solution will satisfy the B.V. $y(1) = 0 \to 0 = c_1 e^{\beta} + c_2 e^{-\beta}$, and $y(0) + y'(0) = 0 \to c_2 + c_1 = 0$, hence $c_1 = -c_2$, and we have $-c_2 e^{\beta} + c_2 e^{-\beta} = 0$, hence $c_2 \left(e^{-\beta} - e^{\beta}\right) = 0$, or but $e^{-\beta} - e^{\beta} \neq 0$ (it is zero only if $\beta = 0$ but we have that $\beta > 0$) then this means that $c_2 = 0$. But this means that $c_1 = 0$, which then means that the solution is again $y(x) = 0$. Therefore, for non-trivial solution, $\boxed{\lambda \text{ can not be negative.}}$

$\boxed{\text{Then then only choice left is for } \lambda > 0}$. (We do not have to check for this, since we know that λ does not change sign) but for an exercise, let us verify it any way. As above, we obtain $m^2 = -\lambda$ but since $\lambda > 0$ then solution will now contain complex exponential since $m = \pm i\sqrt{\lambda}$, then solution is (by writing$\sqrt{\lambda} = \beta$)

$$y(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x)$$

Verify B,V, The first one leads to

$$\begin{aligned} y(1) &= 0 \\ 0 &= c_1 \cos\beta + c_2 \sin\beta \end{aligned} \tag{1}$$

and the second one leads to, since $y'(x) = -c_1\beta \sin\beta x + c_2\beta \cos\beta x$, we obtain

$$\begin{aligned} y(0) + y'(0) &= 0 \\ c_1 + c_2\beta &= 0 \end{aligned}$$

or $\boxed{c_1 = -\beta c_2}$, now substitute this in the first initial condition (1) we obtain

$$\begin{aligned} 0 &= -\beta c_2 \cos\beta + c_2 \sin\beta \\ 0 &= c_2 (\sin\beta - \beta\cos\beta) \end{aligned} \tag{2}$$

But if $c_2 = 0$ this will lead to $c_1 = 0$ also and to a trivial solution. Hence we need to consider $\sin\beta - \beta\cos\beta = 0$ or roots of

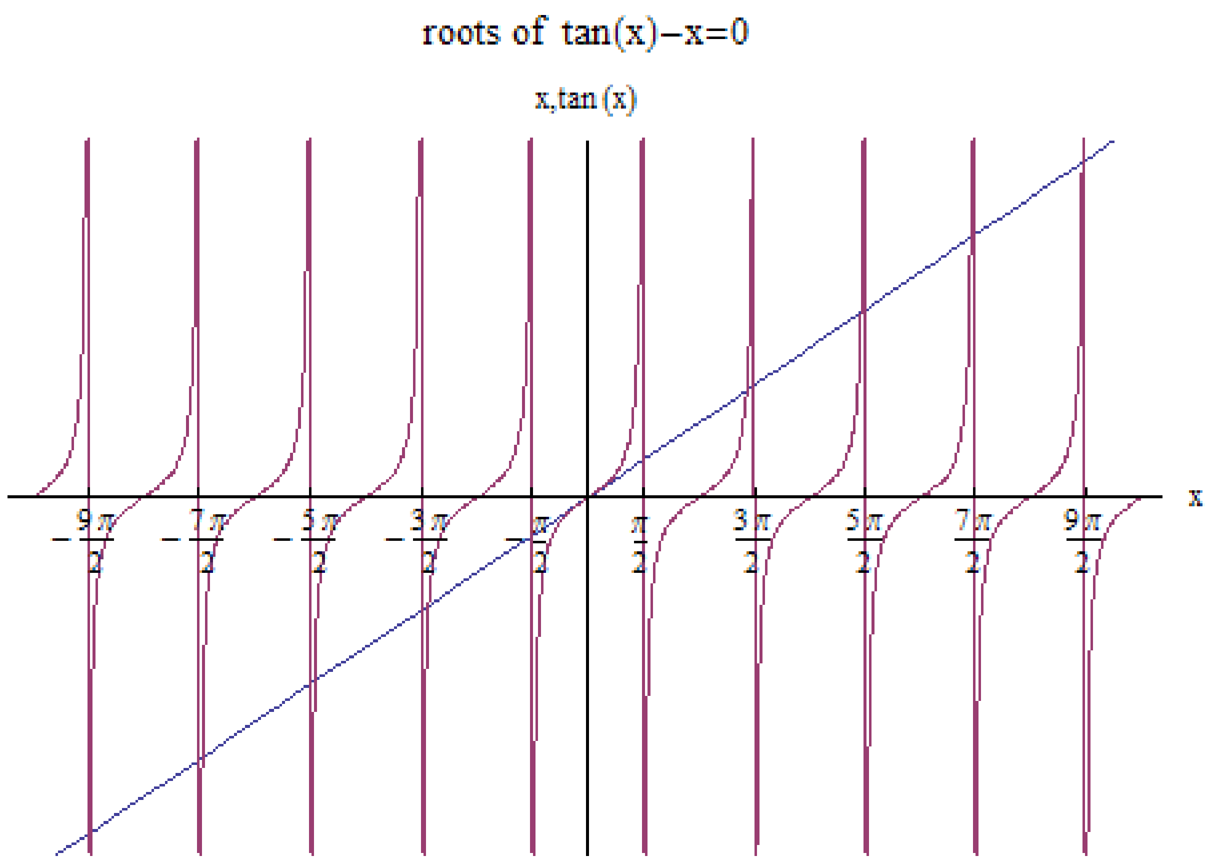
$$\boxed{\tan\beta - \beta = 0}$$

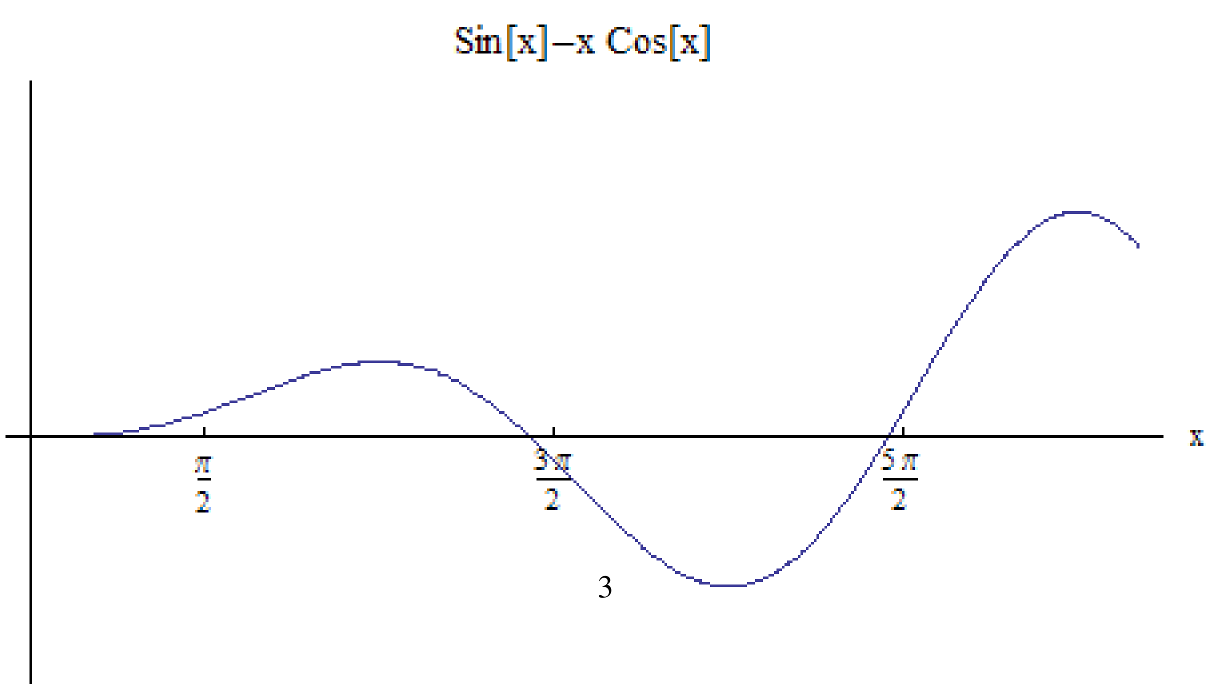
The roots are the intersection of $\tan(x)$ with the line x, graphically we see the roots occur *close to* multiplies of $\frac{\pi}{2}$

Or we can just plot the function $\sin\beta - \beta\cos\beta$

In[45]:=
```
Remove["Global`*"]
Plot[{x, Tan[x]}, {x, -5 Pi, 5 Pi}, AxesLabel → {"x", "x,tan
 PlotLabel → "roots of tan(x)-x=0", Ticks → {Table[i, {i, -
 PlotRange → {Automatic, {-15, 15}}]
```

Out[46]=





To find the roots, use a numerical root finder (Newton's method), here are the first 10 positive roots (we do not pick the zero root, since $\lambda \neq 0$)

```
In[120]:= r = z /. Table[FindRoot[Sin[z] - z Cos[z] == 0, {z, x}]
```

```
Out[120]//TableForm=
4.49341
7.72525
10.9041
14.0662
17.2208
20.3713
23.5195
26.6661
29.8116
```

Hence the square of the above is the list of the eigenvalues. Here are first few

```
In[123]:= r^2 // TableForm
```

```
Out[123]//TableForm=
20.1907
59.6795
118.9
197.858
296.554
414.99
553.165
711.078
888.731
```

Hence the eigenfunctions are

$$v_n = \cos\left(\sqrt{\lambda_n} x\right)$$

and

$$u_n = \sin\left(\sqrt{\lambda_n} x\right)$$

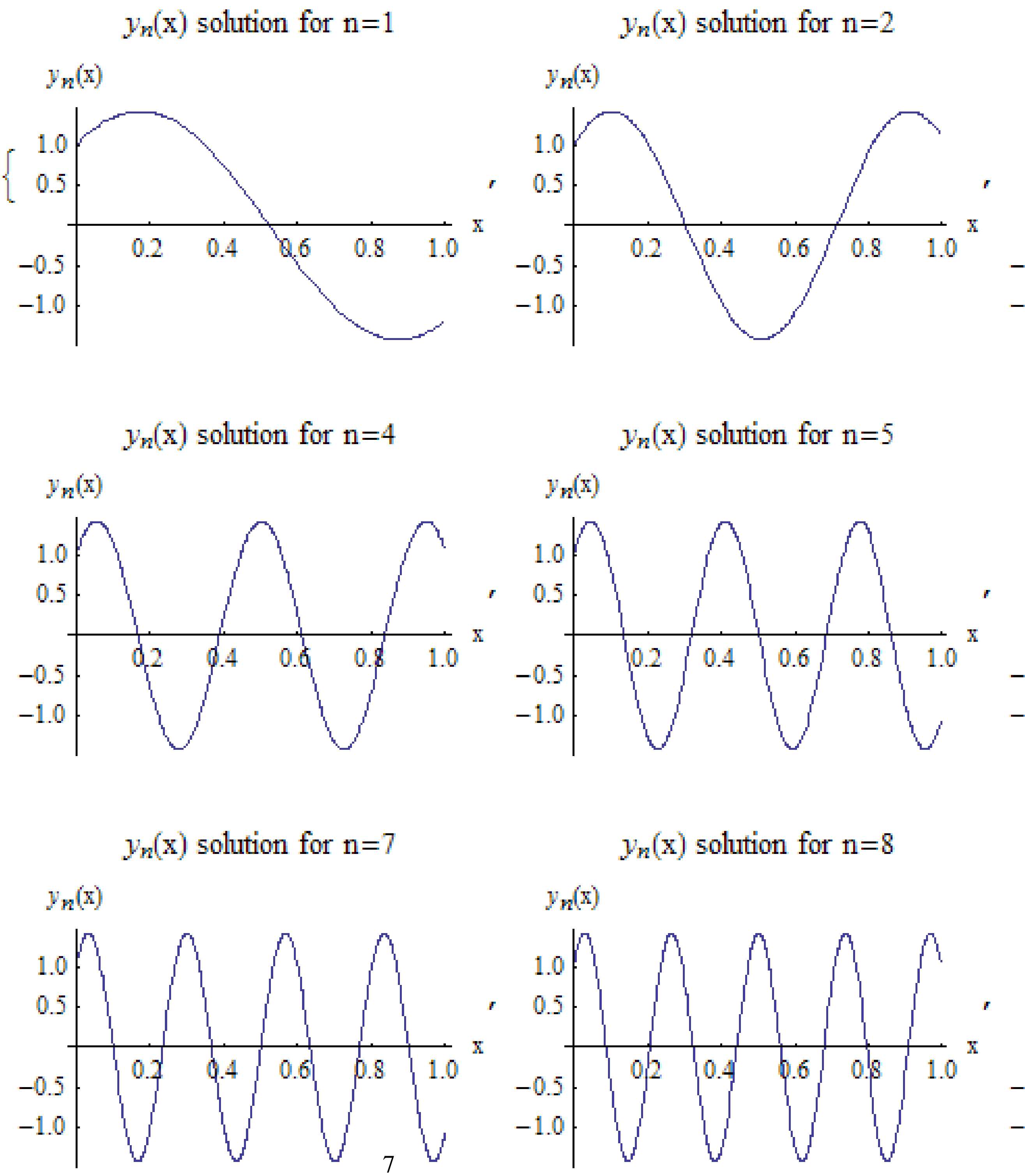
for $n = 1,2,3,\cdots$ where $\sqrt{\lambda_n}$ is the root of $\tan\sqrt{\lambda_n} - \sqrt{\lambda_n} = 0$, and the first few λ_n are shown above.

$$\boxed{y_n(x) = \cos\left(\sqrt{\lambda_n}x\right) + \sin\left(\sqrt{\lambda_n}x\right) \qquad n = 1,2,3,\cdots}$$

Here is a plot of few solutions for $n = 1 \cdots 9$

```
In[219]:= r = z /. Table[FindRoot[Sin[z] - z Cos[z] == 0, {z, x}], {x, 3
           y[x_, n_] := Cos[r[[n]] x] + Sin[r[[n]] x]
           Table[Plot[y[x, n], {x, 0, 1}, AxesLabel → {"x", "yn(x)"},
            {n, 1, Length[r]}]
```

Out[221]= {



2 Problem 8 page 225 section 4.1

problem:

Find eigenvalues and eigenfunctions for the problem $-y'' - 2by' = \lambda y, \quad 0 < x < 1 \ , b > 0$, and $y(0) = y(1) = 0$

answer:

The SLP has the form $-(p(x)y')' + q(x)y = \lambda y$ for $a < x < b$, where $p(x)$ not zero function and does not change sign over the interval, hence we can assume it to be positive. If we compare this form to the given problem we see that $\boxed{p(x) = 1 \text{ and } q(x) = -2b}$ and since $b > 0$ then $q(x)$ is always negative over this range

Assume $\lambda = 0$, hence the ODE become $-y'' - 2by' = 0$ which has the characteristic equation $-m^2 - 2bm = 0$ or $-m - 2b = 0$, hence $m = 2b$, then the solution is $y = c_1 x e^{2bx} + c_2 e^{2bx}$. Now from $y(0) = 0 \to c_2 = 0$, and so $y = c_1 x e^{2bc}$, Now from $y(1) = 0 \to 0 = c_1 e^{2b}$, but $e^{2b} \neq 0$ hence $c_1 = 0$, hence we obtain trivial solution $y = 0$, hence for non-trivial solution $\boxed{\lambda \neq 0}$.

Now $-y'' - 2by' = \lambda y$, and the characteristic equation is $m^2 + 2bm + \lambda = 0$, hence $m = \frac{-2(b) \pm \sqrt{4b^2 - 4\lambda}}{2}$, hence $\boxed{m = -b \pm \sqrt{b^2 - \lambda}}$. There are 3 cases: $b^2 - \lambda < 0$ and $b^2 - \lambda = 0$ and $b^2 - \lambda > 0$.

When $b^2 - \lambda > 0$, we have m will be real. Hence the solution will be of the form $y = c_1 e^{mx} + c_2 e^{-mx}$, where m is real. Now let see if we can satisfy the boundary conditions. From $y(0) = 0 \to c_1 + c_2 = 0$, and from $y(1) = 0 \to 0 = c_1 e^m + c_2 e^{-m}$, hence $0 = c_1 (e^m - e^{-m})$, but this means $c_1 = 0$ since m is not zero. This leads to $c_2 = 0$ which leads to trivial solution $y = 0$. Therefore $\boxed{b^2 - \lambda > 0 \text{ is not possible choice}}$.

When $b^2 - \lambda = 0$, hence $m = -b$, then the solution is $y = c_1 x e^{-bx} + c_2 e^{-bx}$, and by similar argument as above for the case of $\lambda = 0$, we conclude that $\boxed{\text{it is not possible to have } b^2 - \lambda = 0}$

Hence $b^2 - \lambda < 0$ or $\boxed{\lambda > b^2}$. In other words, λ is positive and must be greater than b^2. Let $b^2 - \lambda = -k^2$ for k real and nonzero. Hence

$$m = -b \pm ik$$

and the solution is

$$y(x) = e^{-bx}(c_1 \cos kx + c_2 \sin kx)$$

at $y(0) = 0 \to \boxed{0 = c_1}$ and $y(1) = 0 \to \boxed{0 = e^{-b} c_2 \sin k}$

Hence for non-trivial solution,

$$\sin k = 0$$

or $k = n\pi$ or $k^2 = n^2\pi^2$. But $\boxed{k^2 = \lambda - b^2}$ Hence $\lambda_n - b^2 = n^2\pi^2$. Now since $\lambda > b^2$ we can eliminate that $n = 0$ case. Then we have

$$\boxed{\lambda_n = b^2 + n^2\pi^2 \qquad n = 1, 2, 3, \cdots}$$

Hence $\lambda_1 = b^2 + \pi^2, \lambda_2 = b^2 + 4\pi^2\ , \cdots$

So the eigenfunctions are

$$u_n = \sin k_n x$$

where $k_n = \sqrt{\lambda_n - b^2}$

So the $y_n(x)$ solution is

$$y(x) = e^{-bx} u_n \qquad n = 1, 2, 3, \cdots$$

$$\boxed{y_n(x) = e^{-bx} \sin\left(\sqrt{\lambda_n - b^2}\, x\right)}$$

3 Problem 3 page 225 section 4.1 (extra)

problem:

Find eigenvalues and eigenfunctions for the problem with periodic boundary conditions $-y'' = \lambda y$, $0 < x < L$ and $y(0) = y(L), y'(0) = y'(L)$

answer:

The SLP has the form $-(p(x)y')' + q(x)y = \lambda y$ for $a < x < b$, where $p(x)$ not zero function and does not change sign over the interval, hence we can assume it to be positive. If we compare this form to the given problem we see that $\boxed{p(x) = 1 \text{ and } q(x) = 0}$

Assume $\lambda = 0$, hence we have $y'' = 0$ or $y(x) = Ax + B$. Now to satisfy $y(0) = y(L)$ we must have $B = AL + B$ which implies $A = 0$, hence $y(x) = B$. Now this solution does satisfy $y'(0) = y'(L)$ since $y'(0) = 0$ and $y'(L) = 0$ hence $\boxed{\lambda = 0}$ is an eigenvalue.

Now Assume $\lambda < 0$. Hence $y'' + \lambda y = 0$ and characteristic equation is $m^2 + \lambda = 0$ or $m^2 = -\lambda$, since $\lambda < 0$, then $-\lambda$ is positive, hence this leads to solution of $y = c_1 e^{mx} + c_2 e^{-mx}$ where m is real. Now to satisfy $y(0) = y(L)$ we must have

$$c_1 + c_2 = c_1 e^{mL} + c_2 e^{-mL} \tag{1}$$

and to satisfy $y'(0) = y'(L)$ we must have, since $y'(x) = c_1 m e^{mx} - c_2 m e^{-mx}$ that

$$c_1 m - c_2 m = c_1 m e^{mL} - c_2 m e^{-mL}$$

Since $\lambda \neq 0$ in this case, then $m \neq 0$ so we can divide by m and obtain

$$c_1 - c_2 = c_1 e^{mL} - c_2 e^{-mL} \tag{2}$$

add (1)+(2) we have

$2c_1 = 2c_1 e^{mL}$ or $e^{mL} = 1$ hence $\boxed{mL = 0}$ or $m = 0$ which contradicts our assumption that $\lambda \neq 0$.

So $\boxed{\lambda < 0 \text{ is not possible.}}$

$\boxed{\text{Now assume } \lambda > 0}$, Hence $y'' + \lambda y = 0$ and characteristic equation is $m^2 + \lambda = 0$ or $m^2 = -\lambda$, since $\lambda > 0$, then m is complex,, hence $m = \pm i\sqrt{\lambda}$ and this leads to solution of (by letting $\beta = \sqrt{\lambda}$)

$$\boxed{y = c_1 \sin \beta x + c_2 \cos \beta x}$$

Now to satisfy $y(0) = y(L)$ we must have

$$\begin{aligned} c_2 &= c_1 \sin \beta L + c_2 \cos \beta L \\ c_2 (1 - \cos \beta L) &= c_1 \sin \beta L \\ c_1 &= c_2 \frac{(1 - \cos \beta L)}{\sin \beta L} \end{aligned} \tag{3}$$

and to satisfy $y'(0) = y'(L)$ we must have, since

$$y'(x) = c_1\beta\cos\beta x - c_2\beta\sin\beta x$$

that

$$c_1\beta = c_1\beta\cos\beta L - c_2\beta\sin\beta L \tag{4}$$

Substitute (3) into (4) we have

$$c_2\frac{(1-\cos\beta L)}{\sin\beta L}\beta = c_2\frac{(1-\cos\beta L)}{\sin\beta L}\beta\cos\beta L - c_2\beta\sin\beta L$$

$$c_2\left(\frac{(1-\cos\beta L)}{\sin\beta L}\sqrt{\lambda} - \frac{(1-\cos\beta L)}{\sin\beta L}\beta\cos\beta L + \beta\sin\beta L\right) = 0$$

Since $\sqrt{\lambda} \neq 0$ the above becomes

$$c_2\left((1-\cos\beta L) - (1-\cos\beta L)\cos\beta L + \sin^2\beta L\right) = 0$$

Now $c_2 \neq 0$ else this makes $c_1 = 0$ also and we obtain trivial solution. Hence we must have

$$\begin{aligned}
(1-\cos\beta L) - (1-\cos\beta L)\cos\beta L + \sin^2\beta L &= 0\\
(1-\cos\beta L) - \left(\cos\beta L - \cos^2\beta L\right) + \sin^2\beta L &= 0\\
1-\cos\beta L - \cos\beta L + \cos^2\beta L + \sin^2\beta L &= 0\\
1-\cos\beta L - \cos\beta L + \overbrace{\left(\cos^2\beta L + \sin^2\beta L\right)}^{1} &= 0\\
2 - 2\cos\beta L &= 0
\end{aligned}$$

Hence

$$\cos\beta L = 1$$

or

$$\beta L = 2n\pi \qquad n = 1,2,3,\ldots$$

Hence

$$\lambda_n = \left(\frac{2n\pi}{L}\right)^2 \qquad n = 1,2,3,\ldots$$

Hence the eigenfunctions are $v_n(x) = \sin\beta_n x$ and $u_n = \cos\beta_n x$

For $\lambda_0 = 0$, $v_1(x) = 0$ and $u_n = 1 \rightarrow \boxed{y_0(x) = 1}$

For $\lambda_1 = 1,2,3,\cdots \rightarrow v_1(x) = \sin\frac{2n\pi}{L}x$ and $u_n = \cos\frac{2n\pi}{L}x \rightarrow \boxed{y_n(x) = c_1\sin\frac{2n\pi}{L}x + c_2\cos\frac{2n\pi}{L}x}$

this is a plot of few eigenfunctions v_n, u_n and the complete solution $y_n = u_n + v_n$ for first few eigenvalues

```
L = 1;
λ[n_] := (2 n π / L)^2;
v[x_, n_] := Sin[√λ[n] x]
u[x_, n_] := Cos[√λ[n] x]
p1 = Table[v[x, n], {n, 1, 3}];
p2 = Table[u[x, n], {n, 1, 3}];
t = Table[λ[n], {n, 1, 4}] // N;
t2 = Flatten[Append[{1}, t]] // N;
s1 = Plot[p1, {x, 0, L}, AxesLabel → {"x", "yn(x)"},
   PlotLabel → "First 4 Sin eigenfunctions -y''=λy with
s2 = Plot[p2, {x, 0, L}, AxesLabel → {"x", "yn(x)"},
   PlotLabel → "First 4 Cos eigenfunctions -y''=λy with
GraphicsRow[{s1, s2}]
p3 = Table[v[x, n] + u[x, n], {n, 1, 3}];
Plot[{1, p3}, {x, 0, L}, AxesLabel → {"x", "yn(x)"},
 PlotLabel → "First 4 yn(x) solutions to -y''=λy with pe
```

Out[183]=