EE 409, Linear systems. California State University, Fullerton. Spring 2010

Nasser M. Abbasi

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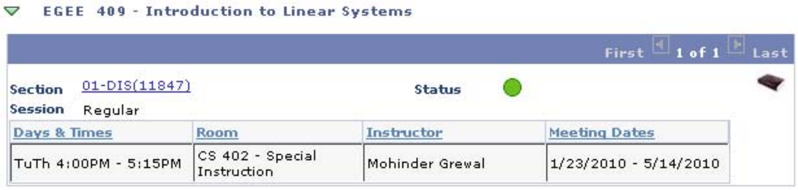
Contents

Chapter 1

Introduction

I took this course in spring 2010 to help me review linear systems again.

Good useful course and Professor Grewal was a nice and good instructor. He worked many examples in the class which was very useful.



EGEE 409 - Introduction to Linear Systems

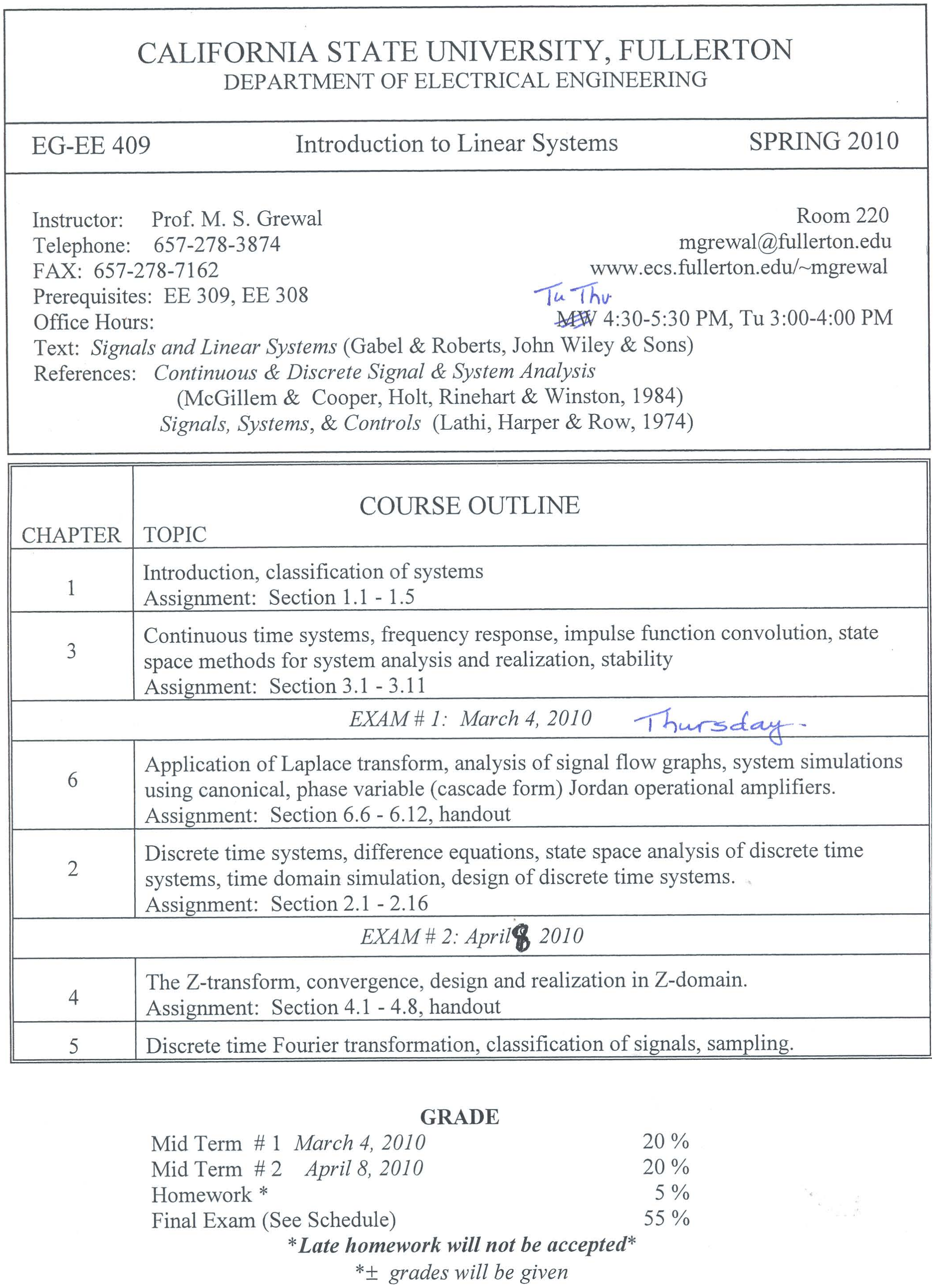
Section 01-DIS(11847) **Status** ●

Session Regular

Days & Times	Room	Instructor	Meeting Dates
TuTh 4:00PM - 5:15PM	CS 402 - Special Instruction	Mohinder Grewal	1/23/2010 - 5/14/2010

Figure 1.1: class schedule

Figure 1.2: Text book. Signals and Linear Systems 3rd edition by Robert A. Gabel and Richard Roberts



CALIFORNIA STATE UNIVERSITY, FULLERTON
DEPARTMENT OF ELECTRICAL ENGINEERING

EG-EE 409	Introduction to Linear Systems	SPRING 2010

Instructor: Prof. M. S. Grewal — Room 220
Telephone: 657-278-3874 — mgrewal@fullerton.edu
FAX: 657-278-7162 — www.ecs.fullerton.edu/~mgrewal
Prerequisites: EE 309, EE 308
Office Hours: Tu Thu ~~MW~~ 4:30-5:30 PM, Tu 3:00-4:00 PM
Text: *Signals and Linear Systems* (Gabel & Roberts, John Wiley & Sons)
References: *Continuous & Discrete Signal & System Analysis*
(McGillem & Cooper, Holt, Rinehart & Winston, 1984)
Signals, Systems, & Controls (Lathi, Harper & Row, 1974)

COURSE OUTLINE

CHAPTER	TOPIC
1	Introduction, classification of systems Assignment: Section 1.1 - 1.5
3	Continuous time systems, frequency response, impulse function convolution, state space methods for system analysis and realization, stability Assignment: Section 3.1 - 3.11
	EXAM # 1: March 4, 2010 Thursday.
6	Application of Laplace transform, analysis of signal flow graphs, system simulations using canonical, phase variable (cascade form) Jordan operational amplifiers. Assignment: Section 6.6 - 6.12, handout
2	Discrete time systems, difference equations, state space analysis of discrete time systems, time domain simulation, design of discrete time systems. Assignment: Section 2.1 - 2.16
	EXAM # 2: April 8 2010
4	The Z-transform, convergence, design and realization in Z-domain. Assignment: Section 4.1 - 4.8, handout
5	Discrete time Fourier transformation, classification of signals, sampling.

GRADE

Mid Term # 1 *March 4, 2010*	20 %
Mid Term # 2 *April 8, 2010*	20 %
Homework *	5 %
Final Exam (See Schedule)	55 %

Late homework will not be accepted

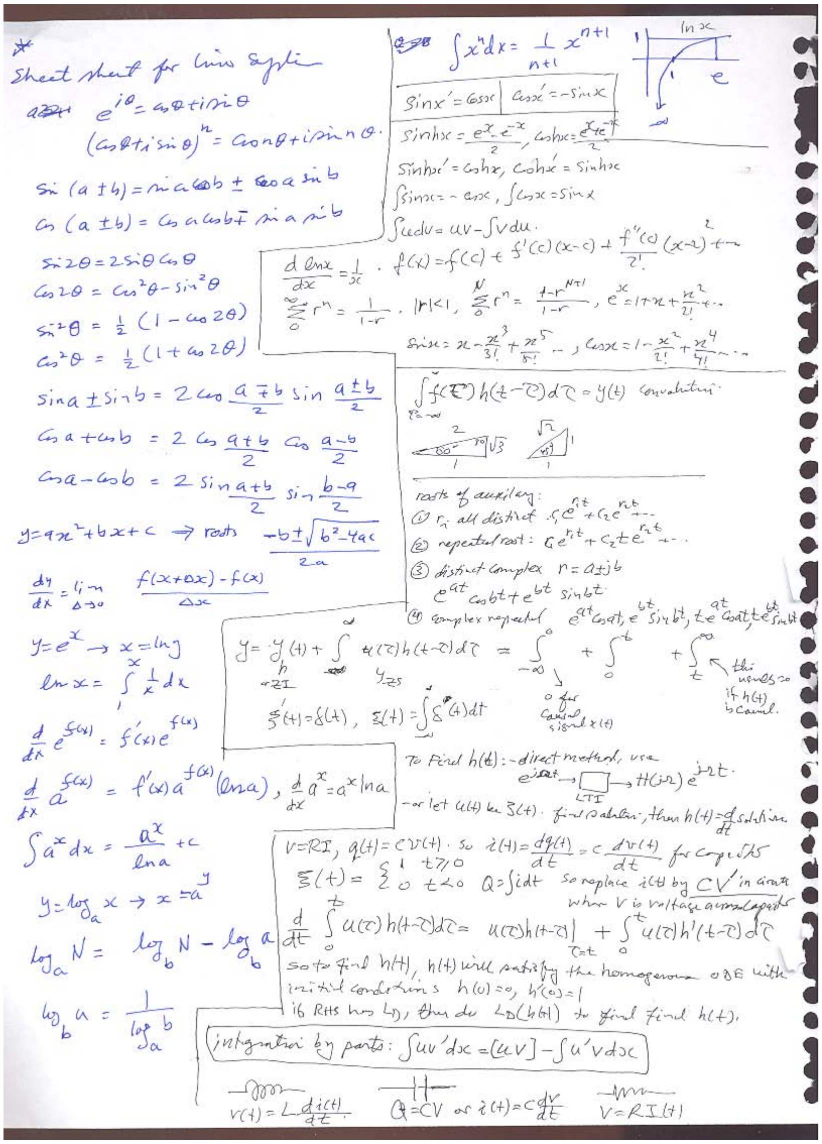
**± grades will be given*

Figure 1.3: syllabus

Chapter 2

cheat sheets

2.1 First mideterm



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Sheet sheet for Linr Systm

$e^{i\theta} = \cos\theta + i\sin\theta$

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

$\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b$

$\cos(a \pm b) = \cos a\cos b \mp \sin a\sin b$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$\sin a \pm \sin b = 2\cos\frac{a \mp b}{2}\sin\frac{a \pm b}{2}$

$\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}$

$\cos a - \cos b = 2\sin\frac{a+b}{2}\sin\frac{b-a}{2}$

$y = ax^2 + bx + c \Rightarrow$ roots $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\frac{dy}{dx} = \lim_{\Delta \to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x}$

$y = e^x \to x = \ln y$

$\ln x = \int_1^x \frac{1}{x}dx$

$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$

$\frac{d}{dx}a^{f(x)} = f'(x)a^{f(x)}(\ln a)$, $\frac{d}{dx}a^x = a^x\ln a$

$\int a^x dx = \frac{a^x}{\ln a} + c$

$y = \log_a x \to x = a^y$

$\log_a N = \log_b N - \log_b a$

$\log_b a = \frac{1}{\log_a b}$

$\int x^n dx = \frac{1}{n+1}x^{n+1}$

$\sin x' = \cos x$ | $\cos x' = -\sin x$

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$

$\sinh x' = \cosh x$, $\cosh x' = \sinh x$

$\int \sin x = -\cos x$, $\int \cos x = \sin x$

$\int u\,dv = uv - \int v\,du$.

$\frac{d\ln x}{dx} = \frac{1}{x}$. $f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$

$\sum_0^\infty r^n = \frac{1}{1-r}$. $|r| < 1$, $\sum_0^N r^n = \frac{1-r^{N+1}}{1-r}$, $e^x = 1 + x + \frac{x^2}{2!} + \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$

$\int_{\tau=-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = y(t)$ convolution.

roots of auxilary:

① r_i all distinct $c_1e^{r_1t} + c_2e^{r_2t} + \cdots$

② repeated root: $c_1e^{r_1t} + c_2te^{r_2t} + \cdots$

③ distinct complex $r = a \pm jb$

$e^{at}\cos bt + e^{bt}\sin bt$

④ complex repeated $e^{at}\cos at, e^{bt}\sin bt, te^{at}\cos at, te^{bt}\sin bt$

$y = \underset{ZI}{y_h(t)} + \underset{y_{ZS}}{\int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau} = \int_{-\infty}^{0} + \int_0^t + \int_t^\infty$

0 for causal signal x(t)

this vanishes if h(t) is causal.

$\xi'(t) = \delta(t)$, $\xi(t) = \int_{-\infty}^{t}\delta(t)dt$

To find h(t): - direct method, use

$e^{j\Omega t} \to \underset{LTI}{\boxed{\;}} \to H(j\Omega)e^{j\Omega t}$.

- or let u(t) be ξ(t). find solution, then $h(t) = \frac{d}{dt}$ solution

$V = RI$, $q(t) = Cv(t)$. so $i(t) = \frac{dq(t)}{dt} = C\frac{dv(t)}{dt}$ for capacitors

$\xi(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$ $Q = \int i\,dt$ So replace i(t) by $\underline{CV'}$ in circuit when V is voltage across capacitor

$\frac{d}{dt}\int_0^t u(\tau)h(t-\tau)d\tau = u(\tau)h(t-\tau)\Big|_{\tau=t} + \int_0^t u(\tau)h'(t-\tau)d\tau$

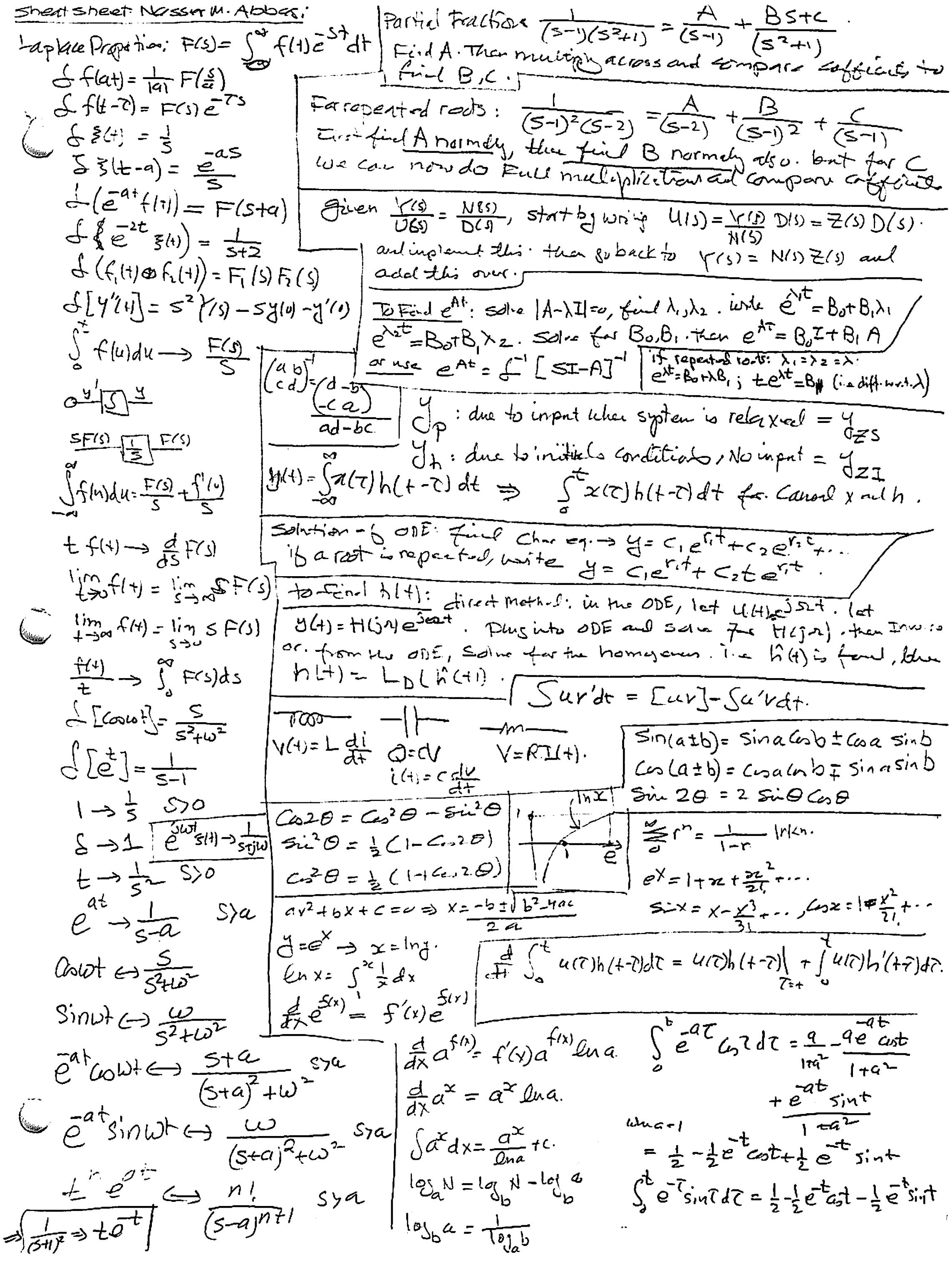
so to find h(t), h(t) will satisfy the homogeneous ODE with initial conditions $h(0) = 0$, $h'(0) = 1$

if RHS has L_D, then do $L_D(h(t))$ to find find h(t).

integration by parts: $\int uv'dx = [uv] - \int u'v\,dx$

$v(t) = L\frac{di(t)}{dt}$. $Q = CV$ or $i(t) = C\frac{dv}{dt}$ $V = RI(t)$

2.2 Second midterm



Sheat sheet: Nasser M. Abbasi:

Laplace Properties: $F(s) = \int_0^\infty f(t) e^{-st} dt$

$\mathcal{L} f(at) = \frac{1}{|a|} F(\frac{s}{a})$

$\mathcal{L} f(t-\tau) = F(s) e^{-\tau s}$

$\mathcal{L} \xi(t) = \frac{1}{s}$

$\mathcal{L} \xi(t-a) = \frac{e^{-as}}{s}$

$\mathcal{L}(e^{-at} f(t)) = F(s+a)$

$\mathcal{L}\{e^{-2t} \xi(t)\} = \frac{1}{s+2}$

$\mathcal{L}(f_1(t) \circledast f_2(t)) = F_1(s) F_2(s)$

$\mathcal{L}[y''(t)] = s^2 Y(s) - s y(0) - y'(0)$

$\int_0^t f(u) du \longrightarrow \frac{F(s)}{s}$

$\circ \xrightarrow{y'} \boxed{\int} \xrightarrow{y}$

$\xrightarrow{sF(s)} \boxed{\frac{1}{s}} \xrightarrow{F(s)}$

$\int_{-\infty}^{t} f(u) du = \frac{F(s)}{s} + \frac{f'(0)}{s}$

$t f(t) \longrightarrow \frac{d}{ds} F(s)$

$\lim_{t\to 0} f(t) = \lim_{s\to\infty} s F(s)$

$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s F(s)$

$\frac{f(t)}{t} \longrightarrow \int_0^\infty F(s) ds$

$\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2}$

$\mathcal{L}[e^{t}] = \frac{1}{s-1}$

$1 \rightarrow \frac{1}{s} \quad s>0$

$\delta \rightarrow 1 \qquad \boxed{e^{j\omega t} \xi(t) \rightarrow \frac{1}{s+j\omega}}$

$t \rightarrow \frac{1}{s^2} \quad s>0$

$e^{at} \rightarrow \frac{1}{s-a} \quad s>a$

$\cos\omega t \leftrightarrow \frac{s}{s^2+\omega^2}$

$\sin\omega t \leftrightarrow \frac{\omega}{s^2+\omega^2}$

$e^{-at} \cos\omega t \leftrightarrow \frac{s+a}{(s+a)^2+\omega^2} \quad s>a$

$e^{-at} \sin\omega t \leftrightarrow \frac{\omega}{(s+a)^2+\omega^2} \quad s>a$

$t^n e^{at} \leftrightarrow \frac{n!}{(s-a)^{n+1}} \quad s>a$

$\Rightarrow \boxed{\frac{1}{(s+1)^2} \Rightarrow t e^{-t}}$

Partial Fraction $\frac{1}{(s-1)(s^2+1)} = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+1)}$.
Find A. Then multiply across and compare coefficients to find B, C.

For repeated roots: $\frac{1}{(s-1)^2(s-2)} = \frac{A}{(s-2)} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)}$
First find A normally, then find B normally also. but for C we can now do full multiplication and compare coefficients

given $\frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)}$, start by writing $U(s) = \frac{Y(s)}{N(s)} D(s) = Z(s) D(s)$. and implement this. then go back to $Y(s) = N(s) Z(s)$ and add this over.

To Find e^{At}: solve $|A - \lambda I| = 0$, find λ_1, λ_2. write $e^{\lambda_1 t} = B_0 + B_1 \lambda_1$
$e^{\lambda_2 t} = B_0 + B_1 \lambda_2$. Solve for B_0, B_1. then $e^{At} = B_0 I + B_1 A$
or use $e^{At} = \mathcal{L}^{-1}[sI - A]^{-1}$

if repeated roots: $\lambda_1 = \lambda_2 = \lambda$.
$e^{\lambda t} = B_0 + \lambda B_1$; $t e^{\lambda t} = B_1$ (i.e. diff. w.r.t. λ)

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}{ad-bc}$

y_p : due to input when system is relaxed $= y_{ZS}$
y_h : due to initial conditions, No input $= y_{ZI}$

$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) dt \Rightarrow \int_0^t x(\tau) h(t-\tau) dt$ for causal x and h.

Solution of ODE: find char eq. $\rightarrow y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \dots$
if a root is repeated, write $y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$.

to find h(t): direct method: in the ODE, let $u(t) = e^{j\Omega t}$. let $y(t) = H(j\Omega) e^{j\Omega t}$. Plug into ODE and solve for $H(j\Omega)$. then Inverse
or. from the ODE, solve for the homogeneous. i.e. $\hat{h}(t)$ is found, then $h(t) = L_D(\hat{h}(t))$.

$\int u v' dt = [uv] - \int u' v dt$.

$V(t) = L \frac{di}{dt}$ $\qquad Q = CV$, $i(t) = C \frac{dV}{dt}$ $\qquad V = R I(t)$.

$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$
$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$
$\sin 2\theta = 2 \sin\theta \cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$
$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

ln x
1
1
e

$\sum_0^\infty r^n = \frac{1}{1-r} \quad |r| < n$.

$e^x = 1 + x + \frac{x^2}{2!} + \dots$

$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\sin x = x - \frac{x^3}{3!} + \dots$, $\cos x = 1 - \frac{x^2}{2!} + \dots$

$y = e^x \rightarrow x = \ln y$.
$\ln x = \int_1^x \frac{1}{x} dx$
$\frac{d}{dx} e^{f(x)} = f'(x) e^{f(x)}$

$\frac{d}{dt} \int_0^t u(\tau) h(t-\tau) d\tau = u(\tau) h(t-\tau) \Big|_{\tau=t} + \int_0^t u(\tau) h'(t-\tau) d\tau$.

$\frac{d}{dx} a^{f(x)} = f'(x) a^{f(x)} \ln a$.

$\frac{d}{dx} a^x = a^x \ln a$.

$\int a^x dx = \frac{a^x}{\ln a} + c$.

$\log_a N = \log_b N - \log_b a$

$\log_b a = \frac{1}{\log_a b}$

$\int_0^t e^{-a\tau} \cos\tau \, d\tau = \frac{a}{1+a^2} - \frac{a e^{-at} \cos t}{1+a^2} + \frac{e^{-at} \sin t}{1+a^2}$

when a=1
$= \frac{1}{2} - \frac{1}{2} e^{-t} \cos t + \frac{1}{2} e^{-t} \sin t$

$\int_0^t e^{-\tau} \sin\tau \, d\tau = \frac{1}{2} - \frac{1}{2} e^{-t} \cos t - \frac{1}{2} e^{-t} \sin t$

$\dot{X} - AX = Bu$

$X_h(t) = e^{A(t-t_0)} X(t_0)$ $\qquad X_p(t) = \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)\,d\tau.$

so $X(t) = \underbrace{e^{A(t-t_0)} X(t_0)} + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)\,d\tau.$ when $t_0 = 0 \Rightarrow X(t) = e^{At} X(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau$

$y(t) = CX + Du = \underbrace{Ce^{A(t)} X(0)}_{y_h \text{ or } y_{ZI}} + \int_0^t \underbrace{Ce^{A(t-\tau)} Bu(\tau)\,d\tau + Du(t)}_{y_{ZS} \text{ or } y_p \text{ or steady state.}}$

$\boxed{h(t) = Ce^{At}B + D\delta(t) \qquad t \geq 0}$ ← by setting $X(0) = 0$ above and compare to $\int h(t-\tau)u(\tau)\,d\tau$.

h(t) from state space!

Note: to find y(t) use: $\int_0^t Ce^{A\tau} Bu(\tau)\,d\tau$

Methods to find e^{At} / eigenvalues (spectral): $e^{\lambda_i t} = B_0 + B_1\lambda_i$ for repeated roots: $\boxed{e^{\lambda_1 t} = B_0 + B_1\lambda_1, \quad te^{\lambda_1 t} = B_0}$

$\left.\begin{aligned} \dot{X} &= AX + Bu \\ y &= CX + Du \end{aligned}\right\}$ assume $u(t) = e^{j\omega t} \Rightarrow y(t) = H(j\omega)e^{j\omega t}$. and $x(t) = X(j\omega)e^{j\omega t}$.

sub into, solve for $X(j\omega)$ which will be $(Ij\omega - A)^{-1}B$. , sub into second equation.

$H(j\omega)e^{j\omega t} = C(Ij\omega - A)^{-1}B + De^{j\omega t}$ $\Rightarrow \boxed{H(j\omega) = C(Ij\omega - A)^{-1}B + D}$

frequency response

$y'' + y = \sin t \Rightarrow (D^2+1)(D^2+1)y = 0 \Rightarrow y = y_p + y_h$, then use $y = y_p + y_h$ for IC to find c_1, c_2.

plug here, find c_3, c_4.

$y'' + y = (D+1)\sin t \Rightarrow$ first find L_A which make $L_A[\sin t] = 0 \Rightarrow$

$(L_A)(D^2+1)y = 0 \rightarrow$ find $y = y_p + y_h$, find c_3, c_4, then use $y = y_p + y_h$ with IC conditions

plug here!

---

to find h(t) directly from ODE: if $y'' + \dots + y(t) = L_D[f(t)]$

① solve for $\hat{h}(t)$ from $y'' + \dots + y(t) = 0$, with IC $\hat{h}(0) = 0, \hat{h}'(0) = 1$

② once $\hat{h}(t)$ is found, find $h(t) = L_D[\hat{h}(t)]$.. note $\boxed{\xi'(t) = \delta(t)}$

$[\sin t + e^t ..]\,\xi(t)$ $\qquad$ then $\int_0^\infty h(t)e^{-j\omega t}\,dt$

~~for~~ to find $H(j\omega)$: if we know h(t), OR use direct method: plug $u = e^{j\Omega t}$, and $y = H(j\Omega)e^{j\Omega t}$ into the ODE, then solve for $H(j\Omega)$.

H(s) = system transfer function.

$H(s)|_{s=j\omega}$ = system frequency response ; steady state response of the system due to sinusoidal input. (IC = 0)

Poles of H(s) gives stability of steady state response.

or $H(j\omega) = C(Ij\omega - A)^{-1}B + D$

$H(s) = \frac{N(s)}{D(s)} \rightarrow \frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)}$

$\Rightarrow U(s) = Y(s)D(s) \cdot N(s) = Z(s)N(s)$ ← implement

$U(s) \leftarrow \rightarrow Z(s)$.

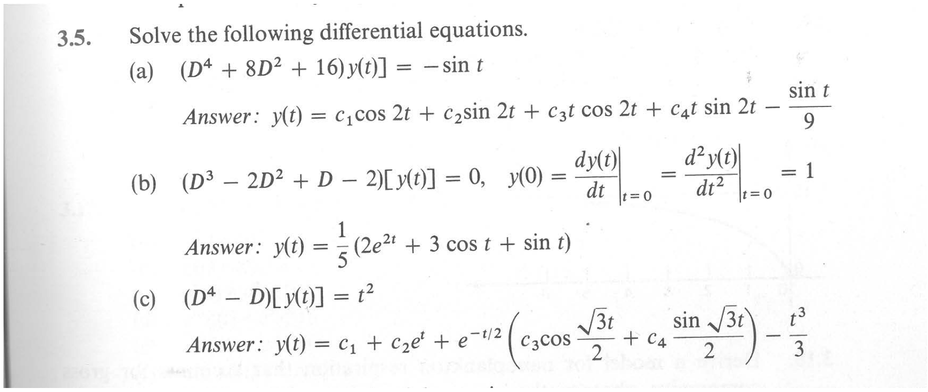
Chapter 3

HWs

3.1 HW 1

Date due and handed in Feb. 11,2010

3.1.1 Problem 3.5



3.5. Solve the following differential equations.

(a) $(D^4 + 8D^2 + 16)y(t)] = -\sin t$

Answer: $y(t) = c_1\cos 2t + c_2\sin 2t + c_3t\cos 2t + c_4t\sin 2t - \frac{\sin t}{9}$

(b) $(D^3 - 2D^2 + D - 2)[y(t)] = 0, \quad y(0) = \left.\frac{dy(t)}{dt}\right|_{t=0} = \left.\frac{d^2y(t)}{dt^2}\right|_{t=0} = 1$

Answer: $y(t) = \frac{1}{5}(2e^{2t} + 3\cos t + \sin t)$

(c) $(D^4 - D)[y(t)] = t^2$

Answer: $y(t) = c_1 + c_2e^t + e^{-t/2}\left(c_3\cos\frac{\sqrt{3}t}{2} + c_4\frac{\sin\sqrt{3}t}{2}\right) - \frac{t^3}{3}$

Figure 3.1: Problem description

Part a

Let $L \equiv D^4 + 8D^2 + 16$ and let $L_A \equiv D^2 + 1$. Since[1] $L_A[-\sin t] = 0$, then the differential equation can be written as

$$
\begin{aligned}
L_A[L[y(t)]] &= 0\\
\left(D^2+1\right)\left(D^4+8D^2+16\right) &= 0\\
\left(D^2+1\right)\left(D^2+4\right)\left(D^2+4\right) &= 0
\end{aligned}
$$

---

[1] $L_A[-\sin t] = \left(D^2+1\right)(-\sin t) = (D(D(-\sin t)) - \sin t) = (D(-\cos t) - \sin t) = (\sin t - \sin t) = 0$

Hence the characteristic equation is

$$\left(r^2+1\right)\left(r^2+4\right)\left(r^2+4\right)=0$$

And the roots from the particular solution are $r_1=j$ and $r_2=-j$ and the roots from the homogeneous solution are $\pm 2j$ and $\pm 2j$, which we call $r_3=2j, r_4=-2j$ and $r_5=2j$ and $r_6=-2j$. Hence

$$y_p(t)=c_1e^{-r_1t}+c_2e^{-r_2t}$$

and

$$y_h(t)=c_3e^{-r_3t}+c_4e^{-r_4t}+c_5te^{-r_5t}+c_6te^{-r_6t}$$

Hence

$$\begin{aligned}y_p(t)&=c_1e^{-jt}+c_2e^{jt}\\&=c_1(\cos t-j\sin t)+c_2(\cos t+j\sin t)\\&=(c_1+c_2)\cos t+(jc_2-jc_1)\sin t\\&=C_1\cos t+C_2\sin t\end{aligned}$$

Where $C_1=(c_1+c_2)$ and $C_2=(jc_2-jc_1)$

and

$$\begin{aligned}y_h(t)&=c_3e^{-2jt}+c_4e^{2jt}+c_5te^{-2jt}+c_6te^{2jt}\\&=c_3(\cos 2t-j\sin 2t)+c_4(\cos 2t+j\sin 2t)\\&+c_5t(\cos 2t-j\sin 2t)+c_6t(\cos 2t+j\sin 2t)\\&=(c_3+c_4)\cos 2t+(-jc_3+jc_4)\sin 2t+(c_5+c_6)t\cos 2t+(-jc_5+jc_6)t\sin 2t\\&=C_3\cos 2t+C_4\sin 2t+C_5t\cos 2t+C_6t\sin 2t\end{aligned}$$

Where $C_3=(c_3+c_4), C_4=(-jc_3+jc_4), C_5=(c_5+c_6), C_6=(-jc_5+jc_6)$

Hence we have

$$y(t)=\overbrace{C_1\cos t+C_2\sin t}^{y_p}+\overbrace{C_3\cos 2t+C_4\sin 2t+C_5t\cos 2t+C_6t\sin 2t}^{y_h}\tag{1}$$

To determine C_1 and C_2, we insert $y_p(t)$ into the ODE and obtain

$$\begin{aligned}\left(D^4+8D^2+16\right)y_p(t)&=-\sin t\\\left(D^4+8D^2+16\right)(C_1\cos t+C_2\sin t)&=-\sin t\\C_1\left(D^4+8D^2+16\right)\cos t+C_2\left(D^4+8D^2+16\right)\sin t&=-\sin t\end{aligned}\tag{2}$$

But $D^4(\cos t)=D^3(-\sin t)=D^2(-\cos t)=D(\sin t)=\cos t$ and $D^2(\cos t)=D(-\sin t)=-\cos t$ and $D^4(\sin t)=D^3(\cos t)=D^2(-\sin t)=D(-\cos t)=\sin t$ and $D^2(\sin t)=D(\cos t)=-\sin t$, hence (2) becomes

$$\begin{aligned}C_1(\cos t-8\cos t+16\cos t)+C_2(\sin t-8\sin t+16\sin t)&=-\sin t\\(C_1-8C_1+16C_1)\cos t+(C_2-8C_2+16C_2)\sin t&=-\sin t\end{aligned}$$

Hence by comparing coefficients, we see that

$$\begin{aligned} C_2 - 8C_2 + 16C_2 &= -1 \\ C_1 - 8C_1 + 16C_1 &= 0 \end{aligned}$$

Or

$$\begin{aligned} 9C_2 &= -1 \\ 9C_1 &= 0 \end{aligned}$$

Hence $C_2 = \frac{-1}{9}$ and $C_1 = 0$, therefore the particular solution is

$$\begin{aligned} y_p(t) &= C_1 \cos t + C_2 \sin t \\ &= \frac{-1}{9} \sin t \end{aligned}$$

Substitute the above into (1), we obtain

$$y(t) = \frac{-\sin t}{9} + C_3 \cos 2t + C_4 \sin 2t + C_5 t \cos 2t + C_6 t \sin 2t$$

Which is what we are required to show. Book uses different names for the constants I used. This can be easily changed: Let $C_3 = C_1$, Let $C_4 = C_2$, Let $C_5 = C_3$ and let $C_6 = C_4$, the above can be written as

$$y(t) = C_1 \cos 2t + C_2 \sin 2t + C_3 t \cos 2t + C_4 t \sin 2t - \frac{\sin t}{9}$$

Part b

We need to solve $\left(D^3 - 2D^2 + D - 2\right) y(t) = 0$ subject to the initial conditions $y(0) = y'(0) = y''(0) = 1$. The characteristic equation is

$$r^3 - 2r^2 + r - 2 = 0$$

By trial and error, we see that

$$\begin{aligned} (r-2)(r-j)(r+j) &= (r-2)\left(r^2+1\right) \\ &= r^3 - 2r^2 + r - 2 \end{aligned}$$

Therefore, the roots are $r_1 = 2, r_2 = j, r_3 = -j$, hence the solution can be written as

$$\begin{aligned} y(t) &= c_1 e^{r_1 t} + c_2 e^{r_2 t} + c_3 e^{r_3 t} \\ &= c_1 e^{2t} + c_2 e^{jt} + c_3 e^{-jt} \\ &= c_1 e^{2t} + c_2 (\cos t + j \sin t) + c_3 (\cos t - j \sin t) \\ &= c_1 e^{2t} + (c_2 + c_3) \cos t + (jc_2 - jc_3) \sin t \end{aligned}$$

Let $c_2 + c_3 = C_2$ and let $jc_2 - jc_3 = C_3$, the above can be written as

$$y(t) = C_1 e^2 t + C_2 \cos t + C_3 \sin t \tag{1}$$

Now to find the constants C_i we apply the boundary conditions. The first boundary condition $y(0) = 1$ yields

$$y(0) = 1 = C_1 + C_2 \tag{2}$$

Now

$$y'(t) = 2C_1 e^{2t} - C_2 \sin t + C_3 \cos t$$

And the second boundary condition $y'(0) = 1$ yields

$$y'(0) = 1 = 2C_1 + C_3 \tag{3}$$

and

$$y''(t) = 4C_1 e^{2t} - C_2 \cos - C_3 \sin t$$

and the third boundary condition $y''(0) = 1$ yields

$$y''(0) = 1 = 4C_1 - C_2 \tag{4}$$

So we have 3 equations to solve for C_1, C_2, C_3. Add (2) and (4), we obtain $2 = 5C_1$, hence

$$C_1 = \frac{2}{5}$$

Hence from (2) we obtain $C_2 = 1 - \frac{2}{5}$

$$C_2 = \frac{3}{5}$$

and from (3) we obtain

$C_3 = 1 - 2C_1 = 1 - \frac{4}{5}$, hence

$$C_3 = \frac{1}{5}$$

Hence the solution is from (1) is found to be

$$\begin{aligned} y(t) &= C_1 e^{2t} + C_2 \cos t + C_3 \sin t \\ &= \frac{2}{5} e^{2t} + \frac{3}{5} \cos t + \frac{1}{5} \sin t \\ &= \frac{1}{5} \left(2e^{2t} + 3 \cos t + \sin t \right) \end{aligned}$$

Which is the answer we are asked to show.

Part(c)

The ODE is

$$\left(D^4 - D\right) y(t) = t^2$$

Hence $L \equiv D^4 - D$ and $L_A = D^3$ since $D^3\left(t^2\right) = D^2(2t) = D(2) = 0$, then the above ODE can be written as

$$D^3\left(D^4 - D\right) y(t) = 0$$

And the characteristic equation is

$$\begin{aligned} r^3\left(r^4 - r\right) &= 0 \\ r^3 r\left(r^3 - 1\right) &= 0 \end{aligned}$$

Hence, for the roots that are related to the particular solution are $r_1 = r_2 = r_3 = 0$.

And the roots that are related to the homogenous solution are $r_4 = 0$ (notice now that this root is repeated 4 times now), and the roots of $\left(r^3 - 1\right) = 0$ which are the cubic roots of unity and can be found as follows

$$\begin{aligned} r^3 &= 1 \\ r^3 &= e^{2\pi j} \\ r &= e^{\frac{2\pi}{3}j} \end{aligned}$$

Hence the 3 roots of unity are $1, e^{\frac{2\pi}{3}j}, e^{\frac{4\pi}{3}j}$, therefore the first root of unity 1, and the second root of unity is $e^{\frac{2\pi}{3}j} = \cos\left(\frac{2}{3}\pi\right) + j\sin\left(\frac{2}{3}\pi\right) = -\frac{1}{2} + j\frac{1}{2}\sqrt{3}$and the third root of unity is $e^{\frac{4\pi}{3}j} = \cos\left(\frac{4}{3}\pi\right) + j\sin\left(\frac{4}{3}\pi\right) = -\frac{1}{2} - j\frac{1}{2}\sqrt{3}$

Hence $r_5 = 1$, $r_6 = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$, $r_7 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$, in otherwords, the solution is

$$y(t) = \overbrace{c_1 e^{r_1 t} + c_2 t e^{r_2 t} + c_3 t^2 e^{r_3 t}}^{y_p(t)} + \overbrace{c_4 t^3 e^{r_4 t} + c_5 e^{r_5 t} + c_6 e^{r_6 t} + c_7 e^{r_7 t}}^{y_h(t)}$$

We now substitute the values of r_i we found and obtain

$$\begin{aligned} y(t) &= \overbrace{c_1 + c_2 t + c_3 t^2}^{y_p(t)} + \overbrace{c_4 t^3 + c_5 e^t + c_6 e^{\left(-\frac{1}{2}+j\frac{1}{2}\sqrt{3}\right)t} + c_7 e^{\left(-\frac{1}{2}-j\frac{1}{2}\sqrt{3}\right)t}}^{y_h(t)} \\ &= c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^t + c_6 e^{-\frac{1}{2}t} e^{j\frac{\sqrt{3}}{2}t} + c_7 e^{-\frac{1}{2}t} e^{-j\frac{\sqrt{3}}{2}t} \\ &= c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^t + e^{-\frac{1}{2}t}\left(c_6 e^{j\frac{\sqrt{3}}{2}t} + c_7 e^{-j\frac{\sqrt{3}}{2}t}\right) \\ &= c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^t + e^{-\frac{1}{2}t}\left(c_6\left[\cos\frac{\sqrt{3}}{2}t + j\sin\frac{\sqrt{3}}{2}t\right] + c_7\left[\cos\frac{\sqrt{3}}{2}t - j\sin\frac{\sqrt{3}}{2}t\right]\right) \end{aligned}$$

Hence

$$y(t) = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^t + e^{-\frac{1}{2}t}\left(\left[c_6 + c_7\right]\cos\frac{\sqrt{3}}{2}t + \left[jc_6 - jc_7\right]\sin\frac{\sqrt{3}}{2}t\right)$$

Let $[c_6 + c_7] = C_6$ and let $jc_6 - jc_7 = C_7$ the above becomes

$$y(t) = \overbrace{c_1 + c_2 t + c_3 t^2}^{y_p(t)} + \overbrace{c_4 t^3 + c_5 e^t + e^{-\frac{t}{2}} \left(C_6 \cos \frac{\sqrt{3}}{2} t + C_7 \sin \frac{\sqrt{3}}{2} t \right)}^{y_h(t)} \tag{1}$$

Now plug $y_p(t)$ back in the original ODE we obtain

$$\begin{aligned} \left(D^4 - D\right) y_p(t) &= t^2 \\ \left(D^4 - D\right) \left(c_1 + c_2 t + c_3 t^2\right) &= t^2 \\ D^4 \left(c_1 + c_2 t + c_3 t^2\right) - D \left(c_1 + c_2 t + c_3 t^2\right) &= t^2 \\ D^3 (c_2 + 2c_3 t) - (c_2 + 2c_3 t) &= t^2 \\ D^2 (2c_3) - (c_2 + 2c_3 t) &= t^2 \\ -(c_2 + 2c_3 t) &= t^2 \end{aligned}$$

Hence we see that $c_2 = 0$ and $c_3 = 0$, then (1) simplifies to

$$y(t) = c_1 + c_4 t^3 + c_5 e^t + e^{-\frac{t}{2}} \left(C_6 \cos \frac{\sqrt{3}}{2} t + C_7 \sin \frac{\sqrt{3}}{2} t \right) \tag{2}$$

To find c_4 , we substitute $y(t)$ found above, into the ode, hence

$$\begin{aligned} \left(D^4 - D\right) y(t) &= t^2 \\ \left(D^4 - D\right) \left[c_1 + c_4 t^3 + c_5 e^t + e^{-\frac{t}{2}} \left(C_6 \cos \frac{\sqrt{3}}{2} t + C_7 \sin \frac{\sqrt{3}}{2} t \right) \right] &= 0 \end{aligned}$$

Now, since we only care about finding c_4, we can just apply D on that, hence

$$\begin{aligned} D^4 \left[\cdots + c_4 t^3 + \cdots \right] - D \left[\cdots + c_4 t^3 + \cdots \right] &= t^2 \\ D^3 \left[\cdots + 3c_4 t^2 + \cdots \right] - \left[\cdots + 3c_4 t^2 + \cdots \right] &= t^2 \\ D^2 [\cdots + 6c_4 t + \cdots] - \left[\cdots + 3c_4 t^2 + \cdots \right] &= t^2 \\ D [\cdots + 6c_4 + \cdots] - \left[\cdots + 3c_4 t^2 + \cdots \right] &= t^2 \\ -\left[\cdots + 3c_4 t^2 + \cdots \right] &= t^2 \end{aligned}$$

By comparing coefficients, we see that $c_4 = -\frac{1}{3}$ then (1) becomes

$$y(t) = c_1 + c_5 e^t + e^{-\frac{t}{2}} \left(C_6 \cos \frac{\sqrt{3}}{2} t + C_7 \sin \frac{\sqrt{3}}{2} t \right) - \frac{1}{3} t^3$$

Which is what we are asked to show.

3.2 HW 2

and HW3 combinedand HW3 combinedand HW3 combined and HW3 combined

Date due and handed in Feb. 18,2010

3.2.1 Problem 3.8

Find the impulse response of the following systems defined by the following differential equations. Verify your answer

Part a

$\left(D^2+7D+12\right)y(t)=u(t)$

Answer

The impulse reponse $h(t)$ satisfies the homogenouse part of the differential equation under the initial conditions $h(0)=0, h'(0)=1$.

Hence we solve the following

$$\left(D^2+7D+12\right)h(t)=0 \tag{1}$$

The charateristic equation is $r^2+7r+12=0$ or $(r+4)(r+3)=0$, hence

$$h(t)=\left(c_1e^{-3t}+c_2e^{-4t}\right)\xi(t) \tag{2}$$

Where $\xi(t)$ is the unit step function. Now find c_1 and c_2 from initial conditions

$$h(0)=0=c+c_2 \tag{3}$$

and

$$\begin{aligned} h'(t) &= \left(-3c_1e^{-3t}-4c_2e^{-4t}\right)\xi(t)+\left(c_1e^{-3t}+c_2e^{-4t}\right)\delta(t)\\ h'(0) &= 1=(-3c_1-4c_2)+(c_1+c_2)\\ 1 &= -2c_1-3c_2 \end{aligned} \tag{4}$$

From (3) and (4), we solve for c_1, c_2

$$\begin{aligned} c_1 &= 1\\ c_2 &= -1 \end{aligned}$$

Hence $h(t)$ from (2) becomes

$$h(t)=(e^-3t-e^-4t)\,\xi(t) \tag{5}$$

Now we verify this solution (note that $\xi'(t)=\delta(t)$)

$$\begin{aligned} h'(t) &= \left(-3e^{-3t}+4e^{-4t}\right)\xi(t)+\left(e^{-3t}-e^{-4t}\right)\delta(t)\\ h'(t) &= \left(-3e^{-3t}+4e^{-4t}\right)\xi(t) \end{aligned} \tag{6}$$

And

$$\begin{aligned} h''(t) &= \left(9e^{-3t}-16e^{-4t}\right)\xi(t)+\left(-3e^{-3t}+4e^{-4t}\right)\delta(t)\\ &= \left(9e^{-3t}-16e^{-4t}\right)\xi(t)+(-3+4)\,\delta(t)\\ &= \left(9e^{-3t}-16e^{-4t}\right)\xi(t)+\delta(t) \end{aligned} \tag{7}$$

Substitute (5),(6) and (7) into LHS of (1) we obtain

$$
\begin{aligned}
\left(D^2+7D+12\right)h(t) &= h''(t)+7h'(t)+12h(t)\\
&= \left(9e^{-3t}-16e^{-4t}\right)\xi(t)+\delta(t)+\\
&\quad 7\left(-3e^{-3t}+4e^{-4t}\right)\xi(t)+\\
&\quad 12\left(e^{-3t}-e^{-4t}\right)\xi(t)\\
&= \left(9e^{-3t}-16e^{-4t}-21e^{-3t}+28e^{-4t}+12e^{-3t}-12e^{-4t}\right)\xi(t)+\delta(t)\\
&= \left[(9-21+12)e^{-3t}+(-16+28-12)e^{-4t}\right]\xi(t)+\delta(t)\\
&= \delta(t)
\end{aligned}
$$

Hence we see that when the input is $\delta(t)$, then the solution is $h(t)$, which is the definition of $h(t)$. Hence the solution is verified.

Part d

$\left(D^3+6D^2+12D+8\right)y(t)=u(t)$

Answer

The impulse reponse $h(t)$ satisfies the homogenouse part of the differential equation under the initial conditions $h(0)=0, h'(0)=0, h''(0)=1$

Hence we solve the following

$$\left(D^3+6D^2+12D+8\right)h(t)=0 \tag{1}$$

The charateristic equation is $r^3+6r^2+12r+8=0$ or $(r+2)(r+2)(r+2)=0$, hence

$$h(t)=\left(c_1e^{-2t}+c_2te^{-2t}+c_3t^2e^{-2t}\right)\xi(t) \tag{2}$$

Now we find unknown $c's$. We start from $h(0)=0$ and obtain

$$h(0)=0=c_1$$

Hence the solution becomes

$$
\begin{aligned}
h(t) &= \left(c_2te^{-2t}+c_3t^2e^{-2t}\right)\xi(t)\\
h'(t) &= \left(c_2t\left(-2e^{-2t}\right)+c_2e^{-2t}+c_3t^2\left(-2e^{-2t}\right)+2c_3te^{-2t}\right)\xi(t)+\left(c_2te^{-2t}+c_3t^2e^{-2t}\right)\delta(t)\\
&= \left(-2c_2te^{-2t}+c_2e^{-2t}-2c_3t^2e^{-2t}+2c_3te^{-2t}\right)\xi(t)
\end{aligned}
$$

And from $h'(0)=0$ we obtain

$$0=c_2$$

Hence the solution becomes

$$\begin{aligned}
h(t) &= \left(c_3 t^2 e^{-2t}\right) \xi(t) \\
h'(t) &= \left(2c_3 t e^{-2t} - 2c_3 t^2 e^{-2t}\right) \xi(t) + \left(c_3 t^2 e^{-2t}\right) \delta(t) \\
&= \left(2c_3 t e^{-2t} - 2c_3 t^2 e^{-2t}\right) \xi(t) \\
h^{''}(t) &= \left(2c_3 e^{-2t} - 4c_3 t e^{-2t} - 4c_3 t e^{-2t} + 4c_3 t^2 e^{-2t}\right) \xi(t) + \left(2c_3 t e^{-2t} - 2c_3 t^2 e^{-2t}\right) \delta(t) \\
&= \left(2c_3 e^{-2t} - 4c_3 t e^{-2t} - 4c_3 t e^{-2t} + 4c_3 t^2 e^{-2t}\right) \xi(t)
\end{aligned}$$

And from $h''(0) = 1$ we find that

$$\begin{aligned}
h'' &= 1 = 2c_3 \\
c_3 &= \frac{1}{2}
\end{aligned}$$

Hence the final solution is

$$h(t) = \left(\frac{1}{2} t^2 e^{-} 2t\right) \xi(t)$$

To verify, we need to evaluate $h'''(t) + 6h''(t) + 12h'(t) + 8h(t)$ and see if we obtain $\delta(t)$ as the result.

$$\begin{aligned}
h'(t) &= \left(t e^{-2t} - t^2 e^{-2t}\right) \xi(t) + \left(\frac{1}{2} t^2 e^{-2t}\right) \delta(t) \\
&= \left(t e^{-2t} - t^2 e^{-2t}\right) \xi(t)
\end{aligned}$$

And

$$\begin{aligned}
h''(t) &= \left(e^{-2t} - 2t e^{-2t} - 2t e^{-2t} + 2t^2 e^{-2t}\right) \xi(t) + \left(t e^{-2t} - t^2 e^{-2t}\right) \delta(t) \\
&= \left(e^{-2t} - 4t e^{-2t} + 2t^2 e^{-2t}\right) \xi(t)
\end{aligned}$$

And

$$\begin{aligned}
h'''(t) &= \left(-2e^{-2t} - 4e^{-2t} + 8t e^{-2t} + 4t e^{-2t} - 4t^2 e^{-2t}\right) \xi(t) + \left(e^{-2t} - 4t e^{-2t} + 2t^2 e^{-2t}\right) \delta(t) \\
&= \left(-6e^{-2t} + 12t e^{-2t} - 4t^2 e^{-2t}\right) \xi(t) + \delta(t)
\end{aligned}$$

Therefore, $LHS = h'''(t) + 6h''(t) + 12h'(t) + 8h(t)$ becomes

$$\begin{aligned}
LHS = & \left(-6e^{-2t} + 12t e^{-2t} - 4t^2 e^{-2t}\right) \xi(t) + \delta(t) \\
& + 6\left(\left(e^{-2t} - 4t e^{-2t} + 2t^2 e^{-2t}\right) \xi(t)\right) \\
& + 12\left(\left(t e^{-2t} - t^2 e^{-2t}\right) \xi(t)\right) \\
& + 8\left(\left(\frac{1}{2} t^2 e^{-2t}\right) \xi(t)\right) \\
\\
& = e^{-2t}(-6 + 6) + t e^{-2t}(12 - 24 + 12) + t^2 e^{-2t}(-4 + 12 - 12 + 4) + \delta(t) \\
& = \delta(t)
\end{aligned}$$

Hence we see that when the input is $\delta(t)$, then the solution is $h(t)$, which is the definition of $h(t)$. Hence the solution is verified

Part e

$$\left(D^3+6D^2+12D+8\right)y\left(t\right)=\left(D-1\right)u\left(t\right)$$

Note: There is a typo in the textbook. The problem as shown in the text had the number 4 in the above equation when it should be 6. I confirmend this with our course instructor. I am sloving the correct version of the problem statment as shown above.

We start by finding the impluse response for the system $\left(D^3+6D^2+12D+8\right)y\left(t\right)=u\left(t\right)$, which we call $\hat{h}\left(t\right)$, then find the required impulse reponse using

$$h\left(t\right)=\left(D-1\right)\hat{h}\left(t\right)$$

However, the impulse reponse of the above was found in part (d), and it is

$$\hat{h}\left(t\right)=\left(\frac{1}{2}t^2e^{-2t}\right)\xi\left(t\right)$$

Therefore the required reponse reponse is

$$\begin{aligned}h\left(t\right)&=\left(D-1\right)\left(\frac{1}{2}t^2e^{-2t}\right)\xi\left(t\right)\\&=\left(te^{-2t}-t^2e^{-2t}\right)\xi\left(t\right)+\left(\frac{1}{2}t^2e^{-2t}\right)\delta\left(t\right)-\left(\frac{1}{2}t^2e^{-2t}\right)\xi\left(t\right)\\&=\left(te^{-2t}-\frac{3}{2}t^2e^{-2t}\right)\xi\left(t\right)\end{aligned}$$

Therefore

$$h\left(t\right)=\left(te^{^-}2t-\frac{3}{2}t^2e^{^-}2t\right)\xi\left(t\right)$$

Now we need to verify this solution.

$$\begin{aligned}h'\left(t\right)&=\left(e^{-2t}-2te^{-2t}-3te^{-2t}+3t^2e^{-2t}\right)\xi\left(t\right)+\left(te^{-2t}-\frac{3}{2}t^2e^{-2t}\right)\delta\left(t\right)\\&=\left(e^{-2t}-5te^{-2t}+3t^2e^{-2t}\right)\xi\left(t\right)\end{aligned}$$

And

$$\begin{aligned}h''\left(t\right)&=\left(-2e^{-2t}-5e^{-2t}+10te^{-2t}+6te^{-2t}-6t^2e^{-2t}\right)\xi\left(t\right)+\left(e^{-2t}-5te^{-2t}+3t^2e^{-2t}\right)\delta\left(t\right)\\&=\left(-7e^{-2t}+16te^{-2t}-6t^2e^{-2t}\right)\xi\left(t\right)+\delta\left(t\right)\end{aligned}$$

And

$$\begin{aligned}h'''\left(t\right)&=\left(14e^{-2t}+16e^{-2t}-32te^{-2t}-12te^{-2t}+12t^2e^{-2t}\right)\xi\left(t\right)+\left(-7e^{-2t}+16te^{-2t}-6t^2e^{-2t}\right)\delta\left(t\right)+\delta'\left(t\right)\\&=\left(30e^{-2t}-44te^{-2t}+12t^2e^{-2t}\right)\xi\left(t\right)-7\delta\left(t\right)+\delta'\left(t\right)\end{aligned}$$

Now using the above, we evaluate the LHS of the ODE, we obtain

$$
\begin{aligned}
LHS &= \left(D^3 + 6D^2 + 12D + 8\right) h(t) \\
&= h'''(t) + 6h''(t) + 12h'(t) + 8h(t) \\
\\
&= \left(30e^{-2t} - 44te^{-2t} + 12t^2e^{-2t}\right) \xi(t) - 7\delta(t) + \delta'(t) \\
&+ 6\left[\left(-7e^{-2t} + 16te^{-2t} - 6t^2e^{-2t}\right) \xi(t) + \delta(t)\right] \\
&+ 12\left[\left(e^{-2t} - 5te^{-2t} + 3t^2e^{-2t}\right) \xi(t)\right] \\
&+ 8\left[\left(te^{-2t} - \frac{3}{2}t^2e^{-2t}\right) \xi(t)\right] \\
\\
&= e^{-2t}(30 - 42 + 12)\,\xi(t) \\
&+ te^{-2t}(-44 + 96 - 60 + 8)\,\xi(t) \\
&+ t^2e^{-2t}(12 - 36 + 36 - 12)\,\xi(t) \\
&- \delta(t) + \delta'(t) \\
\\
&= e^{-2t}(0) + te^{-2t}(0) + t^2e^{-2t}(0) - \delta(t) + \delta'(t) \\
\\
&= \delta'(t) - \delta(t)
\end{aligned}
$$

But the RHS is $(D-1)\,\delta(t)$ which is $\delta'(t) - \delta(t)$. Hence LHS=RHS, hence verified.

and HW3 combined

3.3 HW 4

Date due and handed in March 18,2010

3.3.1 Problem 3.23 (a)

Write the state variable equation for the following

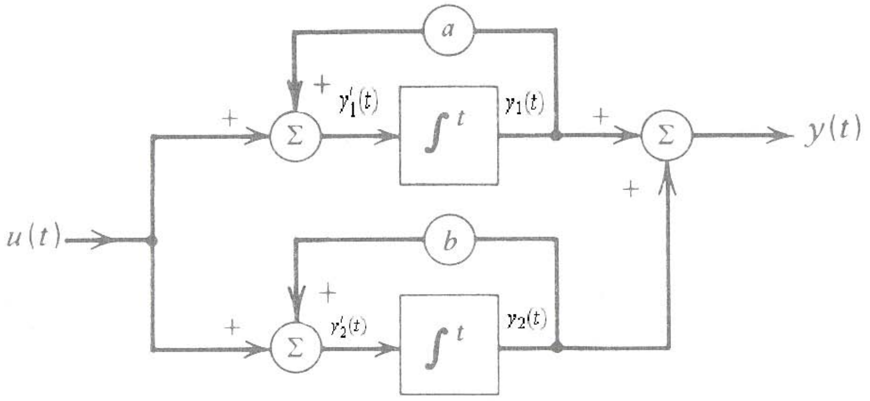


Figure 3.2: System description

Solution

Let $x_1(t)$ and $x_2(t)$ be the state variables. Hence from the diagram we see the following

$$x_1'(t) = ax_1(t) + u(t)$$
$$x_2'(t) = bx_2(t) + u(t)$$

And

$$y(t) = x_1(t) + x_2(t)$$

Hence

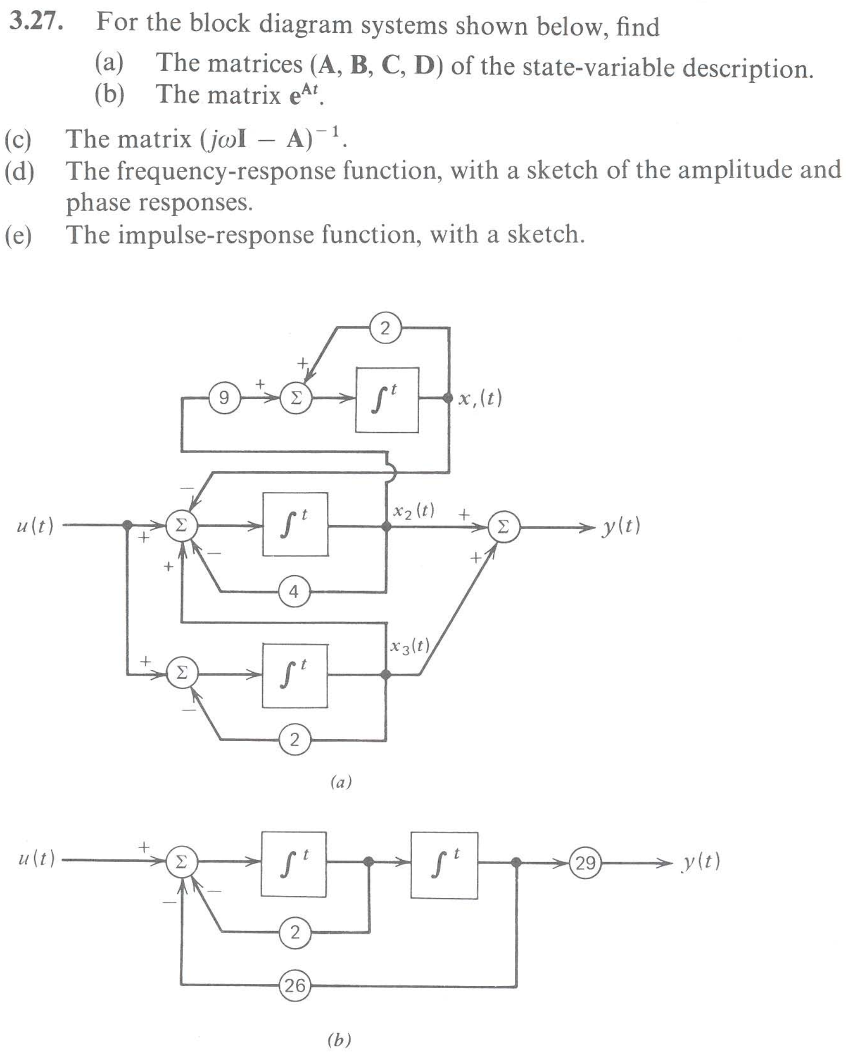
$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \overbrace{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}}^{A} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \overbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}^{B} u(t)$$

$$y(t) = \overbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}^{C} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

3.4 HW 5

Date due and handed in March 18,2010

3.4.1 Problem 3.23 (a)



3.27. For the block diagram systems shown below, find

(a) The matrices (**A**, **B**, **C**, **D**) of the state-variable description.

(b) The matrix $\mathbf{e}^{\mathbf{A}t}$.

(c) The matrix $(j\omega\mathbf{I} - \mathbf{A})^{-1}$.

(d) The frequency-response function, with a sketch of the amplitude and phase responses.

(e) The impulse-response function, with a sketch.

Figure 3.3: Problem description

Part(a)

Labeling the output from the branches as follows

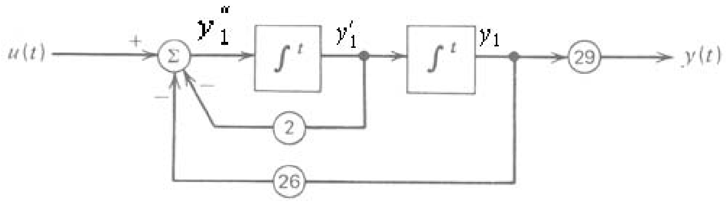


Figure 3.4: Problem description part(a) labeled

Then the differential equation becomes

$$y_1'' = u - 2y_1' - 26y_1$$

While the output equation become

$$y = 29y_1$$

Let $x_1 = y_1$

$$\left.\begin{array}{c} x_1 = y_1 \\ x_2 = y_1' \end{array}\right\} \rightarrow \left.\begin{array}{c} x_1' = y_1' = x_2 \\ x_2' = y_1'' = u - 2y_1' - 26y_1 = u - 2x_2 - 26x_1 \end{array}\right\}$$

Hence

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -26 & -2 \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \overbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{B} u(t)$$

$$y(t) = \overbrace{\begin{pmatrix} 29 & 0 \end{pmatrix}}^{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \overbrace{(0)}^{D} u(t)$$

Part b

To find e^{At} use the eigenvalue approach. Find find $|A - \lambda I|$

$$|A - \lambda I| = \left| \begin{pmatrix} 0 & 1 \\ -26 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \left| \begin{pmatrix} -\lambda & 1 \\ -26 & -2-\lambda \end{pmatrix} \right| = -\lambda(-2-\lambda) + 26$$

Now solve $-\lambda(-2-\lambda) + 26 = 0$ or $\lambda^2 + 2\lambda + 26 = 0$, which has solutions

$$\lambda_1 = -1 + 5j$$
$$\lambda_2 = -1 - 5j$$

Hence we have the following 2 equations to solve for β_0 and β_1

$$e^{\lambda_1 t} = \beta_0 + \lambda_1 \beta_1$$
$$e^{\lambda_2 t} = \beta_0 + \lambda_2 \beta_1$$

Solving we find

$$
\begin{aligned}
\beta_0 &= e^{-t}\left(\cos 5t + \frac{1}{5}\sin 5t\right)\\
\beta_1 &= \frac{1}{5}e^{-t}\sin 5t
\end{aligned}
$$

Hence

$$
\begin{aligned}
e^{At} &= \beta_0 + \beta_1 A\\
&= e^{-t}\left(\cos 5t + \frac{1}{5}\sin 5t\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{1}{5}e^{-t}\sin 5t\begin{pmatrix}0 & 1\\ -26 & -2\end{pmatrix}\\
&= e^{-t}\begin{pmatrix}\cos 5t + \frac{1}{5}\sin 5t & \frac{1}{5}\sin 5t\\ \frac{-26}{5}\sin 5t & \cos 5t - \frac{1}{5}\sin 5t\end{pmatrix}
\end{aligned}
$$

Part c

To find matrix $(j\omega I - A)^{-1}$

$$
\begin{aligned}
j\omega I - A = j\omega\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}0 & 1\\ -26 & -2\end{pmatrix}\\
= \begin{pmatrix}j\omega & 0\\ 0 & j\omega\end{pmatrix} - \begin{pmatrix}0 & 1\\ -26 & -2\end{pmatrix}\\
= \begin{pmatrix}j\omega & -1\\ 26 & j\omega + 2\end{pmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\begin{pmatrix}j\omega & -1\\ 26 & j\omega + 2\end{pmatrix}^{-1} &= \frac{\begin{pmatrix}j\omega + 2 & 1\\ -26 & j\omega\end{pmatrix}}{(j\omega)(j\omega + 2) + 26} = \frac{\begin{pmatrix}j\omega + 2 & 1\\ -26 & j\omega\end{pmatrix}}{-\omega^2 + 2j\omega + 26}\\
&= \frac{1}{-\omega^2 + 2j\omega + 26}\begin{pmatrix}j\omega + 2 & 1\\ -26 & j\omega\end{pmatrix}
\end{aligned}
$$

Part d

To find the frequency response function. Assuming zero initial conditions, from equation 3.10.4 in the book

$$\begin{aligned} H(j\omega) &= C(j\omega I - A)^{-1} B \\ &= \begin{pmatrix} 29 & 0 \end{pmatrix} \frac{1}{-\omega^2 + 2j\omega + 26} \begin{pmatrix} j\omega + 2 & 1 \\ -26 & j\omega \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{-\omega^2 + 2j\omega + 26} \begin{pmatrix} 29 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ j\omega \end{pmatrix} \\ &= \frac{29}{-\omega^2 + 2j\omega + 26} \end{aligned}$$

Hence

$$|H(j\omega)| = \frac{29}{|-\omega^2 + 2j\omega + 26|} = \frac{29}{\sqrt{(26 - \omega^2)^2 + 4\omega^2}}$$

And phase is

$$\begin{aligned} \arg(H(j\omega)) &= \arg(29) - \arg\left(-\omega^2 + 2j\omega + 26\right) \\ &= -\tan^{-1} \frac{2\omega}{26 - \omega^2} \end{aligned}$$

Part e

The state solution is

$$x(t) = \int_0^t e^{A\tau} B u(\tau)\, d\tau$$

and

$$y(t) = Cx(t) = \int_0^t C e^{A\tau} B u(\tau)\, d\tau$$

Hence, let $u(\tau) = \delta(t)$, then

$$\begin{aligned} h(t) &= Ce^{At}B \\ &= \begin{pmatrix} 29 & 0 \end{pmatrix} e^{-t} \begin{pmatrix} \cos 5t + \frac{1}{5}\sin 5t & \frac{1}{5}\sin 5t \\ \frac{-26}{5}\sin 5t & \cos 5t - \frac{1}{5}\sin 5t \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= e^{-t} \begin{pmatrix} 29 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{5}\sin 5t \\ \cos 5t - \frac{1}{5}\sin 5t \end{pmatrix} \\ &= e^{-t} \left(\frac{29}{5} \sin 5t \right) \xi(t) \end{aligned}$$

3.5 HW 6

Date due and handed in April 6,2010

3.5.1 Problem 3.25

Write state variable description of the following 2 systems. For what values of k will the system be stable?

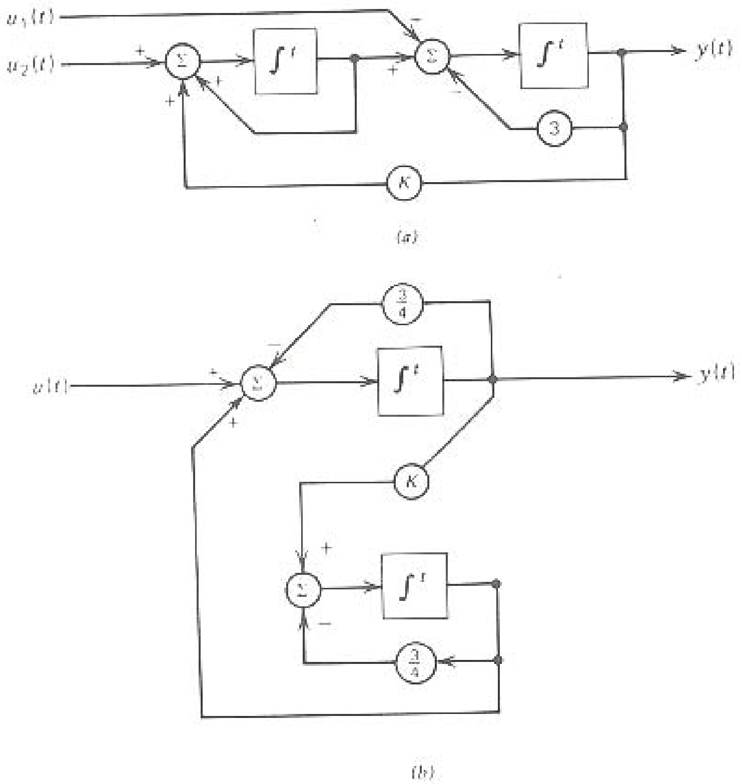


Figure 3.5: Problem description

part(a)

This system has 2 integrators, hence it is of order 2. Hence we need 2 state variables. Assign a state variable as the output of each integrator

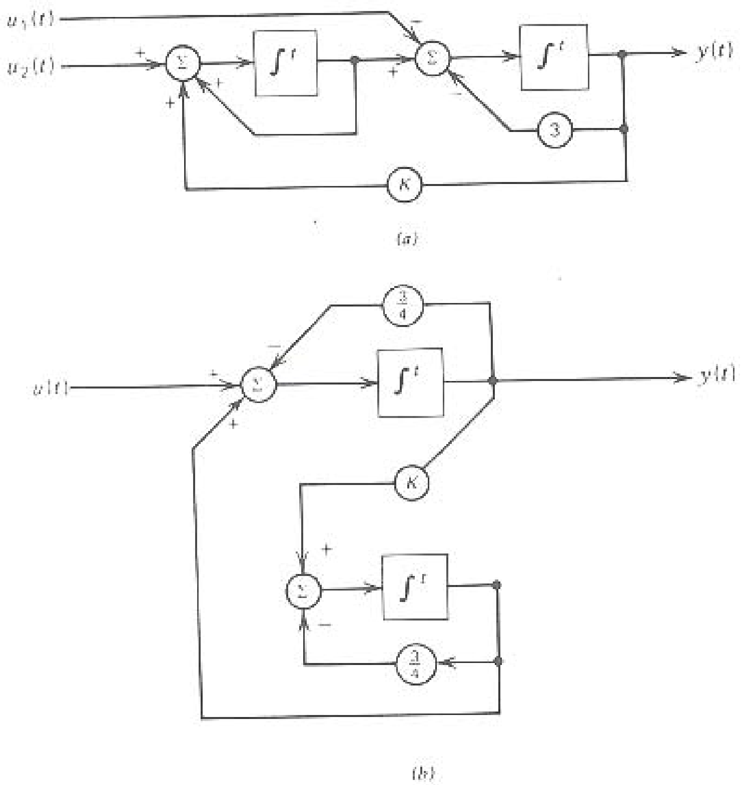


Figure 3.6: part(a) system with labels

Hence

$$x_1' = -3x_1 + u_1 + x_2$$
$$x_2' = x_2 + kx_1 + u_2$$

and $y = x_1$,Hence

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \overbrace{\begin{pmatrix} -3 & 1 \\ k & 1 \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^{B} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$y = \overbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}^{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

To find what values of k the system is stable, the eigenvalues of the A matrix are found and the K range which makes these values negative is the range of value needed.

$$|A - \lambda I| = \left| \begin{pmatrix} -3-\lambda & 1 \\ k & 1-\lambda \end{pmatrix} \right| = (1-\lambda)(-3-\lambda) - k$$

Hence the characteristic equation is

$$\lambda^2 + 2\lambda - k - 3 = 0$$

and the roots are

$$\begin{aligned}\lambda_1 &= -1 + \sqrt{k+4}\\ \lambda_2 &= -1 - \sqrt{k+4}\end{aligned}$$

consider λ_1. For this root to be stable, then $\sqrt{k+4} < 1$ or $k < -3$

consider λ_2. This root is stable for any value of k since when $k + 4 < 0$ then it is stable since real part is already negative, and when $k + 4 > 0$ then it is stable also.

Hence we conclude that the system is stable for $k < -3$

To find the ODE:

From $x_1' = -3x_1 + u_1 + x_2$ we obtain $x_1'' = -3x_1' + u_1' + x_2'$. Substitute the value of x_2' from above, we obtain $x_1'' = -3x_1' + u_1' + x_2 + kx_1 + u_2$, but $x_2 = .x_1' + 3x_1 - u_1$, hence

$$\begin{aligned}x_1'' &= -3x_1' + u_1' + x_1' + 3x_1 - u_1 + kx_1 + u_2\\ &= -2x_1' + x_1(3+k) - u_1 + u_1' + u_2\end{aligned}$$

since $x_1 = y$ we obtain

$$y'' = -2y' + y(3+k) - u_1 + u_1' + u_2$$

Part(b)

This system has 2 integrators, hence it is of order 2. Hence we need 2 state variables. Assign a state variable as the output of each integrator

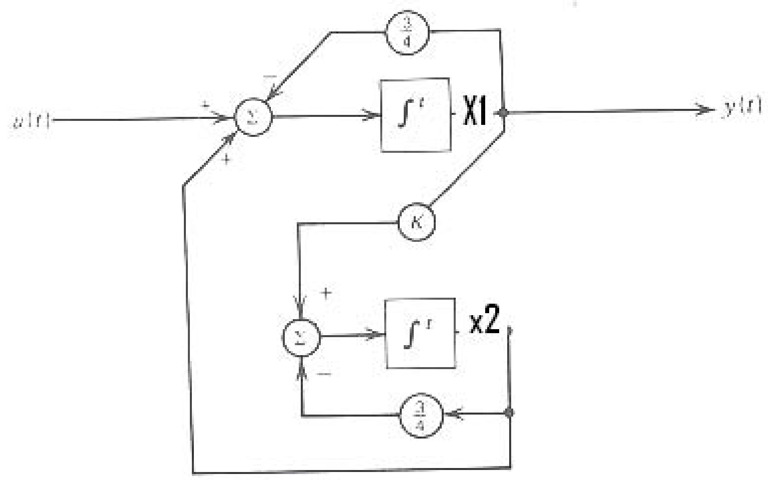


Figure 3.7: Part(b) system

Hence

$$\begin{aligned} x_1' &= -\frac{3}{4}x_1 + u_1 + x_2 \\ x_2' &= -\frac{3}{4}x_2 + kx_1 \end{aligned}$$

and $y = x_1$,Hence

$$\begin{aligned} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= \overbrace{\begin{pmatrix} -\frac{3}{4} & 1 \\ k & -\frac{3}{4} \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \overbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}^{B} u_1 \\ y &= \overbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}^{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

To find what values of k the system is stable, the eigenvalues of the A matrix are found and the K range which makes these values negative is the range of value needed.

$$|A - \lambda I| = \left| \begin{pmatrix} -\frac{3}{4} - \lambda & 1 \\ k & -\frac{3}{4} - \lambda \end{pmatrix} \right| = \left(-\frac{3}{4} - \lambda\right)\left(-\frac{3}{4} - \lambda\right) - k$$

Hence the characteristic equation is

$$\lambda^2 + \frac{3}{2}\lambda - k + \frac{9}{16} = 0$$

and the roots are

$$\begin{aligned} \lambda_1 &= -\frac{3}{4} - \sqrt{k} \\ \lambda_2 &= -\frac{3}{4} + \sqrt{k} \end{aligned}$$

For λ_1, all values of k will result in stable root. For λ_2, $\sqrt{k} < \frac{3}{4}$ or $k < \frac{9}{16}$ or $k < 0.5625$

Hence $k < \frac{9}{16}$ or $k < 0.5625$ is the range of k for stability.

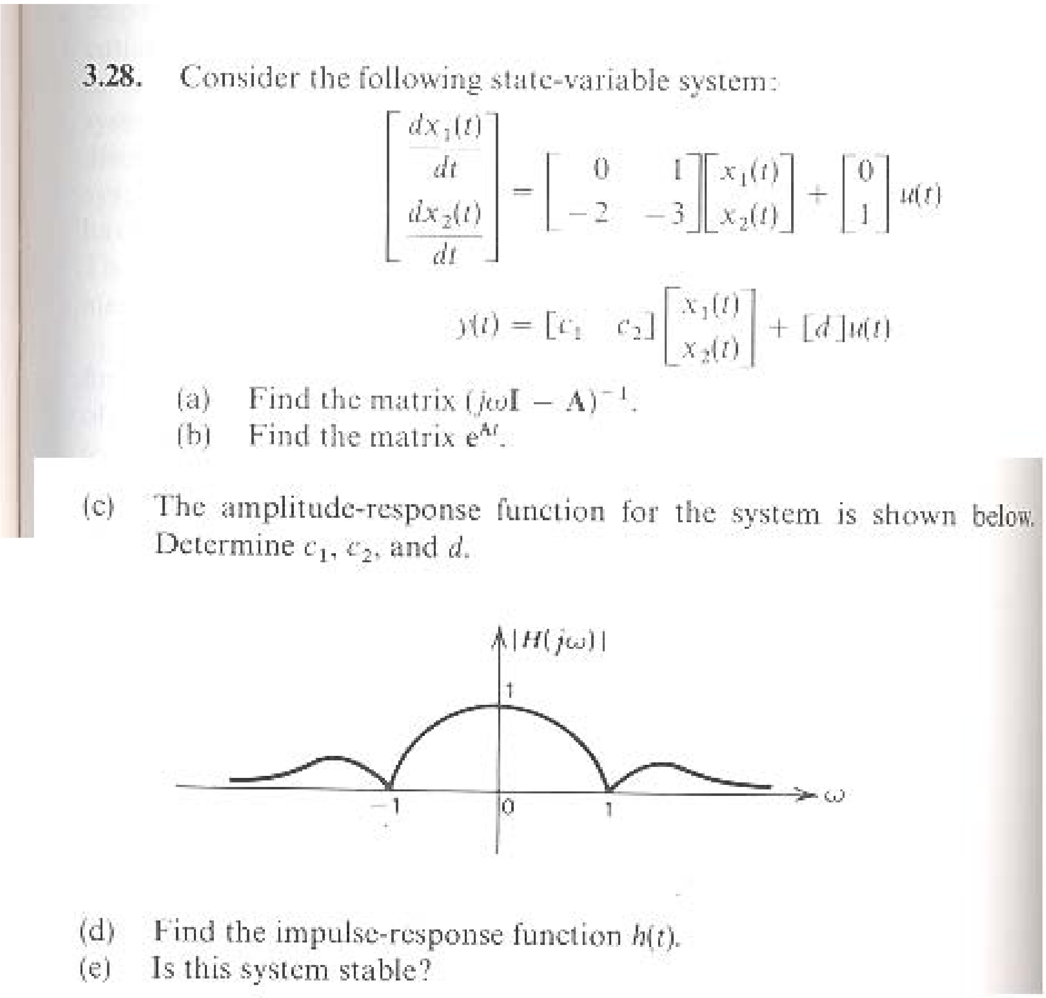
To find the ODE: From $x_1' = -\frac{3}{4}x_1 + u_1 + x_2$, we obtain $x_1'' = -\frac{3}{4}x_1' + u_1' + x_2'$ Substitute the value of x_2' from above, we obtain $x_1'' = -\frac{3}{4}x_1' + u_1' - \frac{3}{4}x_2 + kx_1$ but $x_2 = x_1' + \frac{3}{4}x_1 - u_1$, hence

$$\begin{aligned} x_1'' &= -\frac{3}{4}x_1' + u_1' - \frac{3}{4}\left(x_1' + \frac{3}{4}x_1 - u_1\right) + kx_1 \\ &= -\frac{3}{4}x_1' + u_1' - \frac{3}{4}x_1' - \frac{9}{16}x_1 + \frac{3}{4}u_1 + kx_1 \\ &= -\frac{3}{2}x_1' + x_1\left(k - \frac{9}{16}\right) + u_1' + \frac{3}{4}u_1 \end{aligned}$$

since $x_1 = y$ we obtain

$$y'' + \frac{3}{2}y' - y\left(k - \frac{9}{16}\right) = u_1' + \frac{3}{4}u_1$$

3.5.2 Problem 2



3.28. Consider the following state-variable system:

$$\begin{bmatrix} \dfrac{dx_1(t)}{dt} \\ \dfrac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [c_1 \quad c_2] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [d]u(t)$$

(a) Find the matrix $(j\omega\mathbf{I} - \mathbf{A})^{-1}$.

(b) Find the matrix $\mathbf{e}^{\mathbf{A}t}$.

(c) The amplitude-response function for the system is shown below. Determine c_1, c_2, and d.

(d) Find the impulse-response function $h(t)$.

(e) Is this system stable?

Figure 3.8: Problem description

part(a)

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \overbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{B} u_1$$

$$y = \overbrace{\begin{pmatrix} c_1 & c_2 \end{pmatrix}}^{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + [d] \ u_1$$

$$(j\omega I - A) = j\omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} j\omega & -1 \\ 2 & j\omega + 3 \end{pmatrix}$$

Hence

$$\begin{aligned}(j\omega I - A)^{-1} &= \begin{pmatrix} j\omega & -1 \\ 2 & j\omega+3 \end{pmatrix}^{-1} = \frac{1}{(j\omega)(j\omega+3)+2}\begin{pmatrix} j\omega+3 & 1 \\ -2 & j\omega \end{pmatrix} \\ &= \frac{1}{-\omega^2+3j\omega+2}\begin{pmatrix} j\omega+3 & 1 \\ -2 & j\omega \end{pmatrix}\end{aligned}$$

part(b)

To find e^{At} use the eigenvalue method.

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2$$

Hence the roots of $\lambda^2 + 3\lambda + 2 = 0$ are found to be $\lambda_1 = -1$ and $\lambda_2 = -2$. Hence the 2 equations to solve are

$$\begin{aligned} e^{\lambda_1 t} &= \beta_0 + \beta_1 \lambda_1 \\ e^{\lambda_2 t} &= \beta_0 + \beta_1 \lambda_2 \end{aligned}$$

or

$$\begin{aligned} e^{-t} &= \beta_0 - \beta_1 \\ e^{-2t} &= \beta_0 - 2\beta_1 \end{aligned}$$

Solving we obtain

$$\begin{aligned} \beta_0 &= 2e^{-t} - e^{-2t} \\ \beta_1 &= e^{-t} - e^{-2t} \end{aligned}$$

Hence

$$\begin{aligned} e^{At} &= \beta_0 I + \beta_1 A \\ &= \left(2e^{-t} - e^{-2t}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(e^{-t} - e^{-2t}\right)\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \\ &= \end{aligned}$$

Hence

$$e^{At} = \begin{pmatrix} \left(2e^{-t} - e^{-2t}\right) & \left(e^{-t} - e^{-2t}\right) \\ -2\left(e^{-t} - e^{-2t}\right) & -e^{-t} + 2e^{-2t} \end{pmatrix}$$

part (c)

First need to find $H(j\omega)$. We start from the system equations

$$x' = Ax + Bu \tag{1}$$

$$y = Cx + Du \tag{2}$$

Let $u = e^{j\omega t}$, hence the state particular solution is

$$x_p(t) = X(j\omega)\, e^{j\omega t} \tag{3}$$

And

$$y_p(t) = H(j\omega)\, e^{j\omega t} \tag{4}$$

From (1) and (3) , we obtain

$$\begin{aligned} j\omega X(j\omega)\, e^{j\omega t} &= AX(j\omega)\, e^{j\omega t} + Be^{j\omega t} \\ j\omega X(j\omega) &= AX(j\omega) + B \\ (j\omega I - A)\, X(j\omega) &= B \\ X(j\omega) &= (j\omega I - A)^{-1} B \end{aligned} \tag{5}$$

and from (2) and (4) we obtain

$$\begin{aligned} H(j\omega)\, e^{j\omega t} &= CX(j\omega)\, e^{j\omega t} + De^{j\omega t} \\ H(j\omega) &= CX(j\omega) + D \end{aligned}$$

Substitute (5) into the above

$$H(j\omega) = C(j\omega I - A)^{-1} B + D$$

From part(a) we found $(j\omega I - A)^{-1}$, hence the above becomes

$$\begin{aligned} H(j\omega) &= \begin{pmatrix} c_1 & c_2 \end{pmatrix} \frac{1}{-\omega^2 + 3j\omega + 2} \begin{pmatrix} j\omega + 3 & 1 \\ -2 & j\omega \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + d \\ &= \frac{1}{-\omega^2 + 3j\omega + 2} \begin{pmatrix} (j\omega + 3)\, c_1 - 2c_2 & c_1 + c_2 j\omega \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + d \\ &= \frac{(c_1 + c_2 j\omega)}{-\omega^2 + 3j\omega + 2} + d \\ &= \frac{(c_1 + c_2 j\omega) + d\left(-\omega^2 + 3j\omega + 2\right)}{-\omega^2 + 3j\omega + 2} \\ &= \frac{\left(c_1 + 2d - d\omega^2\right) + j\,(c_2\omega + 3d\omega)}{(-\omega^2 + 2) + 3j\omega} \end{aligned}$$

Hence

$$\begin{aligned} |H(j\omega)|^2 &= \frac{\left(c_1 + 2d - d\omega^2\right)^2 + (c_2\omega + 3d\omega)^2}{\left(-\omega^2 + 2\right)^2 + 9\omega^2} \\ &= \frac{d^2\omega^4 + 5d^2\omega^2 + 4d^2 - 2d\omega^2 c_1 + 6d\omega^2 c_2 + 4dc_1 + \omega^2 c_2^2 + c_1^2}{\omega^4 + 5\omega^2 + 4} \end{aligned}$$

Now, from diagram, at $\omega = 0$ we have $|H(j\omega)|^2 = 1$, hence

$$1 = d^2 + dc_1 + \frac{1}{4}c_1^2 \tag{6}$$

And at $\omega = 1$ we have $|H(j\omega)|^2 = 0$ hence

$$0 = \frac{10d^2 + 2dc_1 + 6dc_2 + c_2^2 + c_1^2}{10}$$

Or

$$0 = 10d^2 + 2dc_1 + 6dc_2 + c_2^2 + c_1^2 \tag{7}$$

And at $\omega = -1$ we have $|H(j\omega)|^2 = 0$ but this will not add new equation. So need to look at the limit as $\omega \rightarrow \infty$

$$|H(j\omega)|^2 = \frac{d^2 + \frac{5d^2}{\omega^2} + \frac{4d^2}{\omega^4} - \frac{2dc_1}{\omega^2} + \frac{6dc_2}{\omega^2} + \frac{4dc_1}{\omega^4} + \frac{c_2^2}{\omega^2} + \frac{c_1^2}{\omega^4}}{1 + \frac{5}{\omega^2} + \frac{4}{\omega^4}}$$

Hence we see that as $\omega \rightarrow \infty$, $|H(j\omega)|^2 \rightarrow d^2$, hence $d = 0$ since $|H(j\omega)| \rightarrow 0$ in the limit. So now we know d, we have 2 equations and 2 unknowns to solve for from (6) and (7). Re write (6) and (7) again by setting $d = 0$ we obtain

$$1 = \frac{1}{4}c_1^2 \tag{6}$$

$$0 = c_2^2 + c_1^2 \tag{7}$$

Hence $c_1 = 2$ and $c_2 = 2j$ therefore, the system now looks like

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \overbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{B} u_1$$

$$y = \overbrace{\begin{pmatrix} 2 & 2j \end{pmatrix}}^{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Part(d)

To find $h(t)$, Let the input be $\delta(t)$, and find $y(t)$. From the system equation

$$y_p(t) = \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau$$

Let $u(\tau) = \delta(t)$, so the above becomes

$$\begin{aligned} h(t) &= \int_{t_0}^{t} Ce^{A(t-\tau)}B\delta(\tau)\,d\tau \\ &= Ce^{A(t)}B \qquad t \geq 0 \end{aligned}$$

But we found $e^{A(t)}$ in part (b), hence

$$\begin{aligned} h(t) &= \begin{pmatrix} 2 & 2j \end{pmatrix} \begin{pmatrix} \left(2e^{-t} - e^{-2t}\right) & \left(e^{-t} - e^{-2t}\right) \\ -2\left(e^{-t} - e^{-2t}\right) & -e^{-t} + 2e^{-2t} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= 2e^{-t} - 2e^{-2t} - 2j\left(e^{-t} - 2e^{-2t}\right) \end{aligned}$$

part(e)

To check for stability

$$|A - \lambda I| = \left| \begin{pmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix} \right| = (-\lambda)(-3-\lambda) + 2$$

Hence

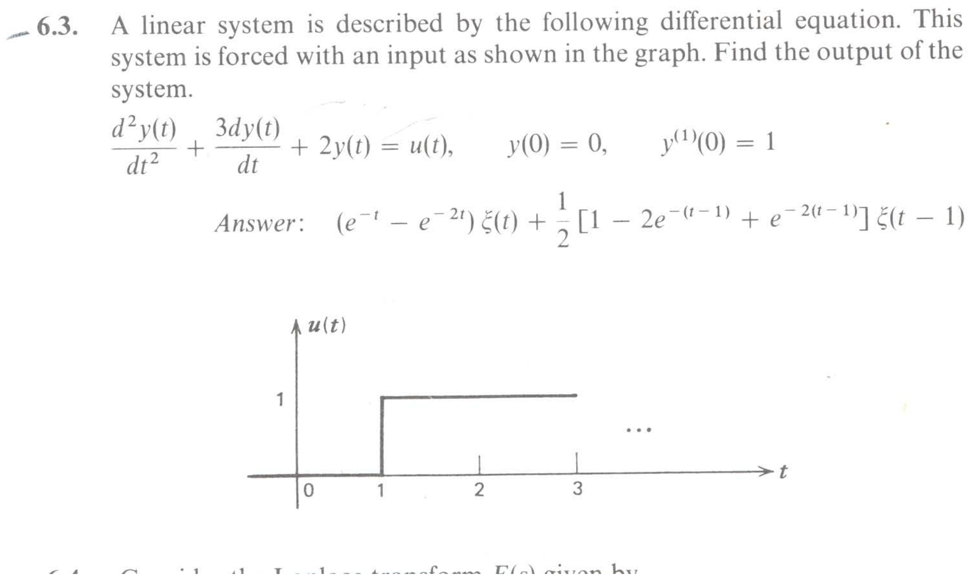
$$\lambda^2 + 3\lambda + 2 = 0$$

The roots are $-1, -2$ and since they are both negative, hence the system is stable.

3.6 HW 7

Date due and handed in April 13,2010

3.6.1 Problem 3.25



6.3. A linear system is described by the following differential equation. This system is forced with an input as shown in the graph. Find the output of the system.

$$\frac{d^2y(t)}{dt^2} + \frac{3dy(t)}{dt} + 2y(t) = u(t), \qquad y(0) = 0, \qquad y^{(1)}(0) = 1$$

Answer: $(e^{-t} - e^{-2t})\,\xi(t) + \frac{1}{2}[1 - 2e^{-(t-1)} + e^{-2(t-1)}]\,\xi(t-1)$

Figure 3.9: Problem description

$$y''(t) + 3y'(t) + 2y(t) = u(t)$$

Using the Laplace approach. First we note that the input is a delayed step input, hence $u(t) = \xi(t-1)$ where $\xi(t)$ is the unit step function. Laplace transform of a delayed unit step is $\int\limits_0^\infty \xi(t-1)\, e^{-st} dt = \int\limits_1^\infty e^{-st} dt = \frac{\left[e^{-st}\right]_1^\infty}{-s} = \frac{e^{-s}}{s}$

Applying the Laplace transformation on the ODE gives

$$\begin{aligned} s^2 Y(s) - sy(0) - y'(0) + 3sY(s) - y(0) + 2Y(s) &= \frac{e^{-s}}{s} \\ s^2 Y(s) - 1 + 3sY(s) + 2Y(s) &= \frac{e^{-s}}{s} \\ Y(s)\left(s^2 + 3s + 2\right) - 1 &= \frac{e^{-s}}{s} \\ Y(s) &= \frac{1}{s^2+3s+2} + \frac{e^{-s}}{s(s^2+3s+2)} \qquad (1) \end{aligned}$$

Considering the first term on the RHS of (1), calling it $Y_1(s) = \frac{1}{s^2+3s+2}$, and using partial fractions gives

$$\begin{aligned} Y_1(s) &= \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \\ A &= \lim_{s->-1} \frac{1}{(s+2)} = 1 \\ B &= \lim_{s->-2} \frac{1}{(s+1)} = -1 \end{aligned}$$

Hence

$$Y_1(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

Considering the second term on the RHS, calling it $Y_2(s) = \frac{e^{-s}}{s(s^2+3s+2)}$, and using partial fractions gives

$$\frac{Y_2(s)}{e^{-s}} = \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\begin{aligned} A &= \lim_{s->0} \frac{1}{(s+1)(s+2)} = \frac{1}{2} \\ B &= \lim_{s->-1} \frac{1}{s(s+2)} = -1 \\ C &= \lim_{s->-2} \frac{1}{s(s+1)} = \frac{1}{2} \end{aligned}$$

Hence

$$\frac{Y_2(s)}{e^{-s}} = \frac{1}{2}\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\frac{1}{s+2}$$

Therefore

$$
\begin{aligned}
Y(s) &= Y_1(s) + Y_2(s) \\
&= \left(\frac{1}{s+1} - \frac{1}{s+2}\right) + \left(\frac{1}{2}\frac{e^{-s}}{s} - \frac{e^{-s}}{s+1} + \frac{1}{2}\frac{e^{-s}}{s+2}\right)
\end{aligned} \tag{2}
$$

The effect of e^{-as} is to cause a time delay when finding the inverse Laplace transform.

$$
e^{-as}F(s) \rightarrow f(t-a)\,\xi(t-a)
$$

Now, taking the inverse Laplace transform of (2) gives the solution

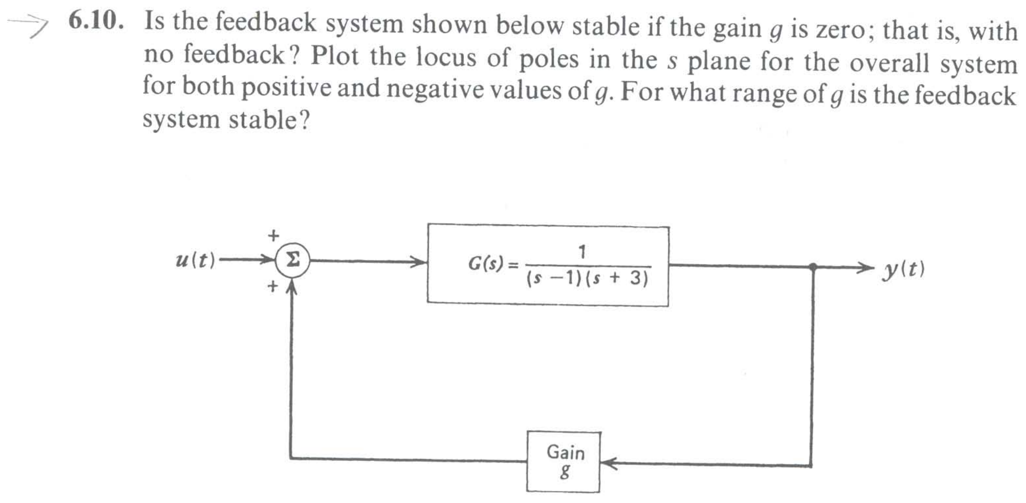
$$
\begin{aligned}
y(t) &= e^{-t}\xi(t) - e^{-2t}\xi(t) + \frac{1}{2}\xi(t-1) - e^{-(t-1)}\xi(t-1) + \frac{1}{2}e^{-2(t-1)}\xi(t-1) \\
&= \left(e^{-t} - e^{-2t}\right)\xi(t) + \frac{1}{2}\left(1 - 2e^{-(t-1)} + e^{-2(t-1)}\right)\xi(t-1)
\end{aligned}
$$

3.7 HW 8

and HW9 combinedand HW9 combinedand HW9 combined and HW9 combined

Date due and handed in April 29,2010

3.7.1 Problem 1 (problem 6.10 in text)



6.10. Is the feedback system shown below stable if the gain g is zero; that is, with no feedback? Plot the locus of poles in the s plane for the overall system for both positive and negative values of g. For what range of g is the feedback system stable?

Figure 3.10: Problem description

Let $E(s)$ be the Laplace transform of the error signal, then we write

$$E(s) = u(s) + g\,y(s) \tag{1}$$
$$y(s) = E(s)\,G(s) \tag{2}$$

Substitute (1) into (2)

$$\begin{aligned} y(s) &= (u(s) + gy(s))\,G(s) \\ &= u(s)\,G(s) + gy(s)\,G(s) \\ y(s)\,[1 - gG(s)] &= u(s)\,G(s) \\ H(s) &= \frac{y(s)}{u(s)} = \frac{G(s)}{1 - gG(s)} \end{aligned}$$

But $G(s) = \frac{1}{(s-1)(s+3)}$, hence the above becomes

$$H(s) = \frac{1}{(s-1)(s+3) - g}$$

Pole of $H(s)$ is when denominator is zero. When $g = 0$, then the poles are $s = 1$ and $s = -3$. Since one of poles is in the RHS plane (pole $s = 1$), then the system is unstable when $g = 0$.

In other words, system stability is determined by the plant stability itself. Since the plant itself is unstable, then the overall system is unstable.

positive feedback

We found from the above what $H(s)$ is.

$$H(s) = \frac{1}{(s-1)(s+3) - g} = \frac{1}{s^2 + 2s - (3 + g)}$$

The roots of the denominator of $H(s)$ are

$$s_{1,2} = \frac{-b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4ac} = -1 \pm \frac{1}{2}\sqrt{4 + 4(3 + g)} = -1 \pm \sqrt{4 + g}$$

Hence

$$\begin{aligned} s_1 &= -1 + \sqrt{4 + g} \\ s_2 &= -1 - \sqrt{4 + g} \end{aligned}$$

For s_1 to be stable, then $\sqrt{4 + g} < 1$ or $4 + g < 1$ or $g < -3$. For s_2, it is always stable for any value of g.

negative feedback

When using negative feedback, the overall system transfer function will come out to be

$$H(s) = \frac{1}{(s-1)(s+3) + g} = \frac{1}{s^2 + 2s + (g - 3)}$$

Hence the roots of the denominator of $H(s)$ are

$$s_{1,2} = \frac{-b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4ac} = -1 \pm \frac{1}{2}\sqrt{4 - 4(g-3)} = -1 \pm \sqrt{4-g}$$

Hence

$$\begin{aligned} s_1 &= -1 + \sqrt{4-g} \\ s_2 &= -1 - \sqrt{4-g} \end{aligned}$$

For s_1 to be stable, then $\sqrt{4-g} < 1$ or $4 - g < 1$ or $g > 3$. For s_2, it is always stable for any value of g.

<u>Conclusion</u>: For positive feedback, system is stable for $g < -3$ and for negative feedback, system is stable for $g > 3$

3.7.2 Problem 2 (problem 2.2 part (c) in textbook)

Solve the following difference equation

$$y(k+2) + y(k) = \sin k \qquad k \geq 0 \tag{1}$$

$L_A = \left(1 - e^{j}S^{-1}\right)\left(1 - e^{-j}S^{-1}\right)$, hence

$$\begin{aligned} L_A\left[S^2 + 1\right] y(k) &= 0 \\ \left(1 - e^{j}S^{-1}\right)\left(1 - e^{-j}S^{-1}\right)\left[S^2 + 1\right] y(k) &= 0 \end{aligned}$$

The roots for $y_p(k)$ are $r_3 = e^{j}$ and $r_4 = e^{-j}$, hence $y_p(k) = c_3 e^{jk} + c_4 e^{-jk}$. Substituting this into (1) gives

$$c_3 e^{j(k+2)} + c_4 e^{-j(k+2)} + c_3 e^{jk} + c_4 e^{-jk} = \sin k$$

But $\sin k = \frac{e^{jk} - e^{-jk}}{2j}$ hence

$$\begin{aligned} c_3 e^{j(k+2)} + c_4 e^{-j(k+2)} + c_3 e^{jk} + c_4 e^{-jk} &= \frac{e^{jk} - e^{-jk}}{2j} \\ c_3 e^{jk} e^{2j} + c_4 e^{-jk} e^{-2j} + c_3 e^{jk} + c_4 e^{-jk} &= \frac{1}{2j} e^{jk} - \frac{1}{2j} e^{-jk} \\ e^{jk}\left(c_3 e^{2j} + c_3\right) + e^{-jk}\left(c_4 e^{-2j} + c_4\right) &= \frac{1}{2j} e^{jk} - \frac{1}{2j} e^{-jk} \end{aligned}$$

Hence

$$\begin{aligned} \left(c_3 e^{2j} + c_3\right) &= \frac{1}{2j} \\ \left(c_4 e^{-2j} + c_4\right) &= -\frac{1}{2j} \end{aligned}$$

or

$$\begin{aligned} c_3\left(1 + e^{2j}\right) &= \frac{1}{2j} \\ c_4\left(1 + e^{-2j}\right) &= -\frac{1}{2j} \end{aligned}$$

or

$$\begin{aligned} c_3 &= \frac{-j}{2\left(1+e^{2j}\right)} \\ c_4 &= \frac{j}{2\left(1+e^{-2j}\right)} \end{aligned}$$

Hence since $y_p(k) = c_3 e^{jk} + c_4 e^{-jk}$ we now obtain

$$y_p(k) = \frac{-je^{jk}}{2\left(1+e^{2j}\right)} + \frac{je^{-jk}}{2\left(1+e^{-2j}\right)}$$

Therefore

$$y(k) = y_p(k) + y_h(k)$$

But $y_h(k)$ has the auxiliary equation $r^2+1=0$, hence roots are $r=\pm j$ hence $y_h(k) = c_1 j^k - c_2 j^k$ hence

$$\begin{aligned} y(k) &= y_p(k) + y_h(k) \\ &= \frac{-je^{jk}}{2\left(1+e^{2j}\right)} + \frac{je^{-jk}}{2\left(1+e^{-2j}\right)} + c_1 j^k - c_2 j^k \end{aligned}$$

To find c_1 and c_2 we need initial conditions, which is not given. So we stop here. Hence

$$y(k) = \frac{j}{2}\left(\frac{e^{-jk}}{1+e^{-2j}} - \frac{e^{jk}}{1+e^{2j}}\right) + j^k\left(c_1 - c_2\right)$$

This can be simplified to

$$\begin{aligned} y(k) &= \frac{j}{2}\left(\frac{e^{-jk}\left(1+e^{2j}\right) - e^{jk}\left(1+e^{-2j}\right)}{\left(1+e^{-2j}\right)\left(1+e^{2j}\right)}\right) + j^k\left(c_1-c_2\right) \\ &= \frac{j}{2}\left(\frac{e^{-jk} + e^{j(2-k)} - e^{jk} - e^{-j(2-k)}}{2+2\cos 2}\right) + j^k\left(c_1-c_2\right) \\ &= \frac{j}{2}\left(\frac{\left(e^{-jk} - e^{jk}\right) + \left(e^{j(2-k)} - e^{-j(2-k)}\right)}{2+2\cos 2}\right) + j^k\left(c_1-c_2\right) \\ &= \frac{j}{2}\left(\frac{-2j\sin k + 2j\sin(2-k)}{2+2\cos 2}\right) + j^k\left(c_1-c_2\right) \\ &= \frac{j}{2}\left(\frac{-2j\sin k - 2j\sin(k-2)}{2+2\cos 2}\right) + j^k\left(c_1-c_2\right) \\ &= \frac{-1}{2}\left(\frac{-2\sin k - 2\sin(k-2)}{2+2\cos 2}\right) + j^k\left(c_1-c_2\right) \end{aligned}$$

Hence

$$y(k) = \frac{1}{2}\left(\frac{\sin k + \sin(k-2)}{1+\cos 2}\right) + j^k\left(c_1-c_2\right)$$

3.7.3 check what is wrong version of solution and delete

Let $E(s)$ be the Laplace transform of the error signal, then we write

$$E(s) = u(s) + g \times y(s) \tag{1}$$
$$y(s) = E(s)G(s) \tag{2}$$

Substitute (1) into (2)

$$\begin{aligned} y(s) &= (u(s) + gy(s))G(s) \\ &= u(s)G(s) + gy(s)G(s) \\ y(s)[1 - gG(s)] &= u(s)G(s) \\ H(s) &= \frac{y(s)}{u(s)} = \frac{G(s)}{1 - gG(s)} \end{aligned}$$

But $G(s) = \frac{1}{(s-1)(s+3)}$, hence the above becomes

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Pole of $H(s)$ is when denominator is zero. When $g = 0$, then the poles are $s = 1$ and $s = -3$. Since one of poles is in the RHS plane (pole $s = 1$), then the system is unstable when $g = 0$.

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Hence

$$\begin{aligned} s_1 &= -1 + \sqrt{4+g} \\ s_2 &= -1 - \sqrt{4+g} \end{aligned}$$

For s_1 to be stable, then $\sqrt{4+g} < 1$ or $4 + g < 1$ or $g < -3$. For s_2, it is always stable for any value of g.

negative feedback

When using negative feedback, the overall system transfer function will come out to be

$$H(s) = \frac{1}{(s-1)(s+3)+g} = \frac{1}{s^2+2s+(g-3)}$$

Hence the roots of the denominator of $H(s)$ are

$$s_{1,2} = \frac{-b}{2} \pm \frac{1}{2}\sqrt{b^2-4ac} = -1 \pm \frac{1}{2}\sqrt{4-4(g-3)} = -1 \pm \sqrt{4-g}$$

Hence

$$\begin{aligned} s_1 &= -1+\sqrt{4-g} \\ s_2 &= -1-\sqrt{4-g} \end{aligned}$$

For s_1 to be stable, then $\sqrt{4-g} < 1$ or $4-g < 1$ or $g > 3$. For s_2, it is always stable for any value of g.

Conclusion: For positive feedback, system is stable for $g < -3$ and for negative feedback, system is stable for $g > 3$

Problem 2 (problem 2.2 part (c) in textbook)

Solve the following difference equation

$$y(k+2)+y(k) = \sin k \qquad k \geq 0 \tag{1}$$

$L_A = \left(1-e^jS^{-1}\right)\left(1-e^{-j}S^{-1}\right)$, hence

$$\begin{aligned} L_A\left[S^2+1\right]y(k) &= 0 \\ \left(1-e^jS^{-1}\right)\left(1-e^{-j}S^{-1}\right)\left[S^2+1\right]y(k) &= 0 \end{aligned}$$

The roots for $y_p(k)$ are $r_3 = e^j$ and $r_4 = e^{-j}$, hence $y_p(k) = c_3e^{jk} + c_4e^{-jk}$ Substituting this into (1) gives

$$c_3e^{j(k+2)} + c_4e^{-j(k+2)} + c_3e^{jk} + c_4e^{-jk} = \sin k$$

But $\sin k = \frac{e^{jk}-e^{-jk}}{2j}$ hence

$$\begin{aligned} c_3e^{j(k+2)} + c_4e^{-j(k+2)} + c_3e^{jk} + c_4e^{-jk} &= \frac{e^{jk}-e^{-jk}}{2j} \\ c_3e^{jk}e^{2j} + c_4e^{-jk}e^{-2j} + c_3e^{jk} + c_4e^{-jk} &= \frac{1}{2j}e^{jk} - \frac{1}{2j}e^{-jk} \\ e^{jk}\left(c_3e^{2j}+c_3\right) + e^{-jk}\left(c_4e^{-2j}+c_4\right) &= \frac{1}{2j}e^{jk} - \frac{1}{2j}e^{-jk} \end{aligned}$$

Hence

$$\begin{aligned} \left(c_3e^{2j}+c_3\right) &= \frac{1}{2j} \\ \left(c_4e^{-2j}+c_4\right) &= -\frac{1}{2j} \end{aligned}$$

Or

$$\begin{aligned} c_3\left(1+e^{2j}\right) &= \frac{1}{2j} \\ c_4\left(1+e^{-2j}\right) &= -\frac{1}{2j} \end{aligned}$$

Or

$$\begin{aligned} c_3 &= \frac{-j}{2\left(1+e^{2j}\right)} \\ c_4 &= \frac{j}{2\left(1+e^{-2j}\right)} \end{aligned}$$

Hence since $y_p(k) = c_3 e^{jk} + c_4 e^{-jk}$ then

$$y_p(k) = \frac{-je^{jk}}{2\left(1+e^{2j}\right)} + \frac{je^{-jk}}{2\left(1+e^{-2j}\right)}$$

Therefore

$$y(k) = y_p(k) + y_h(k)$$

But $y_h(k)$ has the auxiliary equation $r^2+1=0$, hence roots are $r=\pm j$ hence $y_h(k) = c_1 j^k - c_2 j^k$ and

$$\begin{aligned} y(k) &= y_p(k) + y_h(k) \\ &= \frac{-je^{jk}}{2\left(1+e^{2j}\right)} + \frac{je^{-jk}}{2\left(1+e^{-2j}\right)} + c_1 j^k - c_2 j^k \end{aligned}$$

To find c_1 and c_2 we need initial conditions, which is not given. So we stop here.

Using initial conditions. Assuming zero initial conditions, we have at $k=0$ that $y(0)=0$, hence

$$\begin{aligned} 0 &= \frac{-j}{2\left(1+e^{2j}\right)} + \frac{j}{2\left(1+e^{-2j}\right)} + c_1 - c_2 \\ &= \frac{1}{2}\frac{-j\left(1+e^{-2j}\right)+j\left(1+e^{2j}\right)}{\left(1+e^{2j}\right)\left(1+e^{-2j}\right)} + c_1 - c_2 \\ 0 &= \frac{1}{2}\frac{-je^{-2j}+je^{2j}}{\left(2+e^{-2j}+e^{2j}\right)} + c_1 - c_2 \\ 0 &= \frac{1}{2}\frac{2\sin 2}{\left(2+2\cos 2\right)} + c_1 - c_2 \\ 0 &= \frac{1}{2}\frac{\sin 2}{1+\cos 2} + c_1 - c_2 \end{aligned}$$

Therefore

$$c_1 - c_2 = \frac{-1}{2}\frac{\sin 2}{1+\cos 2} \tag{2}$$

Now at $k=1$, $y(k)=0$, hence from $y(k) = \frac{-je^{jk}}{2(1+e^{2j})} + \frac{je^{-jk}}{2(1+e^{-2j})} + c_1 j^k - c_2 j^k$ we obtain

$$\begin{aligned}0 &= \frac{-je^{j}}{2\left(1+e^{2j}\right)}+\frac{je^{-j}}{2\left(1+e^{-2j}\right)}+c_1 j-c_2 j\\ &= \frac{1}{2}\left(\frac{-e^{j}}{\left(1+e^{2j}\right)}+\frac{e^{-j}}{\left(1+e^{-2j}\right)}\right)+c_1-c_2\\ &= \frac{1}{2}\frac{\left(-e^{j}-e^{-j}\right)+\left(e^{-j}+e^{j}\right)}{\left(1+e^{2j}\right)\left(1+e^{-2j}\right)}+c_1-c_2\\ &= \frac{1}{2}\frac{0}{2+e^{-2j}+e^{2j}}+c_1-c_2\end{aligned}$$

Hence

$$c_1 = c_2 \tag{3}$$

(2)+(3) gives

$$\begin{aligned}2c_1 &= \frac{1}{2}\frac{-\sin 2}{1+\cos 2}\\ c_1 &= \frac{-1}{4}\frac{\sin 2}{1+\cos 2}\end{aligned}$$

And

$$c_2 = \frac{1}{4}\frac{\sin 2}{1+\cos 2}$$

Hence the final solution is

$$\begin{aligned}y\left(k\right) &= \frac{-je^{jk}}{2\left(1+e^{2j}\right)}+\frac{je^{-jk}}{2\left(1+e^{-2j}\right)}+c_1 j^{k}-c_2 j^{k}\\ &= \frac{-je^{jk}}{2\left(1+e^{2j}\right)}+\frac{je^{-jk}}{2\left(1+e^{-2j}\right)}-\frac{1}{4}\frac{j^{k}\sin 2}{1+\cos 2}-\frac{1}{4}\frac{j^{k}\sin 2}{1+\cos 2}\end{aligned}$$

and HW9 combined

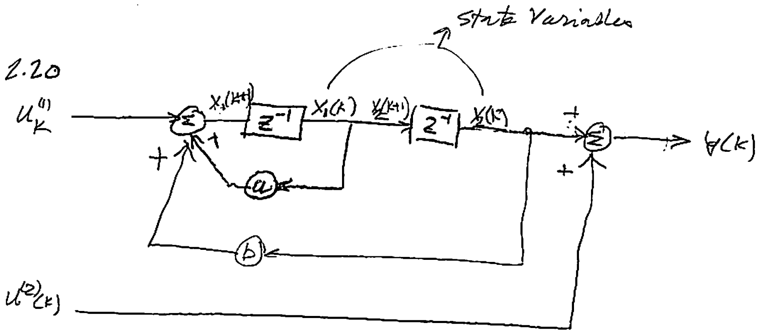
3.8 HW 10

HW#10
EE 409

Nasser M. Abbasi
CSUF
Spring 2010

①

2.20



From diagram we observe the following:

$$x_1(k+1) = a\, x_1(k) + b\, x_2(k) + u^{(1)}(k)$$

$$x_2(k+1) = x_1(k)$$

$$y(k) = x_2(k) + u^{(2)}(k)$$

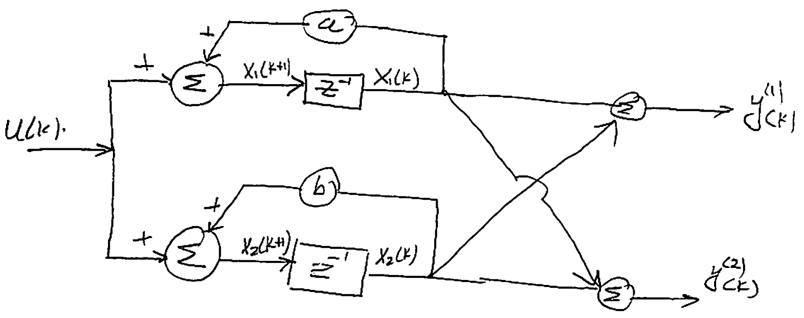
hence

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \overbrace{\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}}^{A} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}^{B} \begin{pmatrix} u^{(1)}(k) \\ u^{(2)}(k) \end{pmatrix}$$

$$y(k) = \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{D} \begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix}$$

$\xrightarrow{\quad}$ next

②



from the diagram :

$$x_1(k+1) = a\,x_1(k) + u(k)$$
$$x_2(k+1) = b\,x_2(k) + u(k)$$
$$y^{(1)}(k) = x_1(k) + x_2(k)$$
$$y^{(2)}(k) = x_1(k) + x_2(k)$$

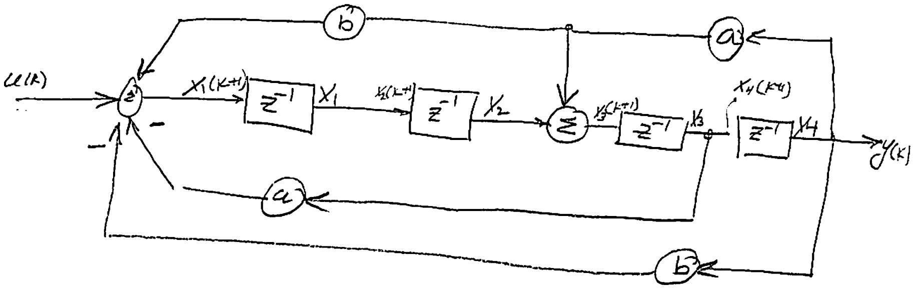
Hence

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \overbrace{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}}^{A} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \overbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}^{B} u(k)$$

$$\begin{pmatrix} y^{(1)}(k) \\ y^{(2)}(k) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \underbrace{[0]}_{D} u(k)$$

$\xrightarrow{}$ next

③



From diagram . 4 state Variables:

$$X_1(K+1) = a\,b\,X_4(k) - a X_3(k) - b X_4(k) + u(k)$$

$$X_2(K+1) = X_1(k)$$

$$X_3(K+1) = a X_4(k) + X_2(k)$$

$$X_4(K+1) = X_3(k)$$

$$y(k) = X_4(k)$$

Hence

$$\begin{pmatrix} X_1(K+1) \\ X_2(K+1) \\ X_3(K+1) \\ X_4(K+1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -a & b(a-1) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X_1(k) \\ X_2(k) \\ X_3(k) \\ X_4(k) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1(k) \\ X_2(k) \\ X_3(k) \\ X_4(k) \end{pmatrix} + [0]\; u(K)$$

2.23 e

Find A^K for $A = \begin{pmatrix} 3/4 & -1/2 \\ \frac{-15}{32} & \frac{1}{2} \end{pmatrix}$.

$$|A-\lambda I| = 0 \rightarrow \begin{vmatrix} \frac{3}{4}-\lambda & -1/2 \\ \frac{-15}{32} & \frac{1}{2}-\lambda \end{vmatrix} = 0$$

so $\left(\frac{3}{4}-\lambda\right)\left(\frac{1}{2}-\lambda\right) - \left(\frac{1}{2}\right)\left(\frac{15}{32}\right) = 0 \Rightarrow \boxed{\lambda^2 - \frac{5}{4}\lambda + \frac{9}{64} = 0}$

$\Rightarrow \lambda = \frac{5}{8} \pm \frac{1}{2} \Rightarrow \boxed{\lambda_1 = \frac{9}{8}, \quad \lambda_2 = \frac{1}{8}}$

hence $\boxed{\begin{matrix} \lambda_1^K = B_0 + B_1\lambda_1 \\ \lambda_2^K = B_0 + B_1\lambda_2 \end{matrix}} \Rightarrow \begin{matrix} \left(\frac{9}{8}\right)^k = B_0 + \frac{9}{8}B_1 \\ \left(\frac{1}{8}\right)^K = B_0 + \frac{1}{8}B_1 \end{matrix}$

Solving for B_0, B_1 results in

$$\boxed{\begin{matrix} B_0 = \frac{-1}{8^{k+1}}\left(9^k - 9\right) \\ B_1 = \frac{1}{8}\left(9^k - 1\right) \end{matrix}}$$

hence $A^K = B_0 I + B_1 A$

$$A^K = \frac{-(9^k-9)}{8^{k+1}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{(9^k-1)}{8}\begin{pmatrix} \frac{3}{4} & -1/2 \\ \frac{-15}{32} & \frac{1}{2} \end{pmatrix}$$

$\longrightarrow$

Simplifying :

$$A^k = \begin{pmatrix} 2^{-1-3k}(3+9^k) & -2^{-2-3k}(-11+3^{1+2k}) \\ 2^{-5-3k}(87-23(9^k)) & 2^{-2-3k}(7+9^k) \end{pmatrix}$$

3.9 extra problem. verification of class problem

3.9.1 Problem:

Given $y(k+2)+\frac{1}{2}y(k)=\frac{1}{4}u(k+2)-\frac{1}{4}u(k)$ find the frequency transfer function $H\left(e^{j\omega}\right)$

Answer

I will use the Z transform as it is a little faster. Let $Y(z)$ be the Z transform of $y(k)$ and let $U(z)$ be the Z transform of $u(k)$, we obtain from the above

$$z^2Y(z)+\frac{1}{2}Y(z)=\frac{1}{4}z^2U(z)-\frac{1}{4}U(z)$$

Hence

$$H(z)=\frac{Y(z)}{U(z)}=\frac{\frac{1}{4}z^2-\frac{1}{4}}{z^2+\frac{1}{2}}=\frac{1}{4}\frac{-1+z^2}{\frac{1}{2}+z^2}$$

Since the DTFT $H(z)$ at the unit circle, then let $z=e^{j\omega}$ in the above we obtain

$$\begin{aligned}
H\left(e^{j\omega}\right) &= \frac{1}{4}\left(\frac{-1+e^{2j\omega}}{\frac{1}{2}+e^{2j\omega}}\right)\\
&= \frac{1}{4}\left(\frac{-1+e^{2j\omega}}{\frac{1}{2}+e^{2j\omega}}\right)\left(\frac{\frac{1}{2}+e^{-2j\omega}}{\frac{1}{2}+e^{-2j\omega}}\right)\\
&= \frac{1}{4}\left(\frac{-\frac{1}{2}-e^{-2j\omega}+\frac{1}{2}e^{2j\omega}+1}{\frac{1}{4}+\frac{1}{2}e^{-2j\omega}+\frac{1}{2}e^{2j\omega}+1}\right)\\
&= \frac{1}{4}\left(\frac{\frac{1}{2}-(\cos 2\omega-j\sin 2\omega)+\frac{1}{2}(\cos 2\omega+j\sin 2\omega)}{\frac{5}{4}+\cos 2\omega}\right)\\
&= \frac{1}{4}\left(\frac{\frac{1}{2}-\frac{1}{2}\cos 2\omega+\frac{3}{2}j\sin 2\omega}{\frac{5}{4}+\cos 2\omega}\right)
\end{aligned}$$

Hence

$$H\left(e^{j}\omega\right)=\frac{\left(\frac{1}{2}-\frac{1}{2}\cos 2\omega\right)+j\left(\frac{3}{2}\sin 2\omega\right)}{5+4\cos 2\omega}$$

Hence

$$\begin{aligned}
\left|H\left(e^{j\omega}\right)\right|^2 &= \frac{\left(\frac{1}{2}-\frac{1}{2}\cos 2\omega\right)^2+\left(\frac{3}{2}\sin 2\omega\right)^2}{(5+4\cos 2\omega)^2}\\
&= \frac{\left(\frac{1}{4}+\frac{1}{4}\cos^2 2\omega-\frac{1}{4}\cos 2\omega\right)+\left(\frac{9}{4}\sin^2 2\omega\right)}{(5+4\cos 2\omega)^2}\\
&= \frac{\frac{1}{4}+\frac{1}{4}\cos^2 2\omega-\frac{1}{4}\cos 2\omega+\frac{9}{4}\sin^2 2\omega}{(5+4\cos 2\omega)^2}\\
&= \frac{\sin^2\omega}{5+4\cos 2\omega}
\end{aligned}$$

And

$$\arg\left(H\left(e^{j\omega}\right)\right)=\arctan\left(\frac{3}{\tan(\omega)}\right)$$

Please note, for the final 2 lines calculation above, I wanted to obtain the most simple result, so I used Mathematica to simplify.

Here is a plot of the magnitude and phase frequency response from Mathematica: (this is a bandpass filter).

```
In[•]:=
h = 1/4 (1 + Cos[2 w]^2 - Cos[2 w] + 9 Sin[2 w]^2)/(5 + 4 Cos[2 w]);
Plot[Sqrt[Sin[w]^2/(5 + 4 Cos[2 w])], {w, -Pi, Pi}, Ticks → {{-Pi, -Pi / 2, 0, Pi / 2, Pi}, Automatic},
 PlotLabel → Text@Style["|H(e^jω|", 12], AxesLabel → {ω, None}]
Out[•]=
```

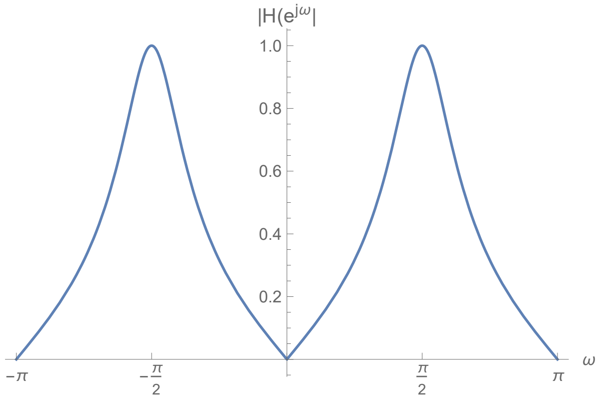


Figure 3.11: First plot

```
In[•]:= Plot[ArcTan[3/Tan[w]], {w, -Pi, Pi}, Ticks → {{-Pi, -Pi / 2, 0, Pi / 2, Pi}, {-Pi, Pi}},
  PlotLabel → Text@Style["Arg(H(e^jω)", 12], AxesLabel → {ω, None},
  PlotRange → {{-Pi, Pi}, {-Pi, Pi}}]
Out[•]=
```

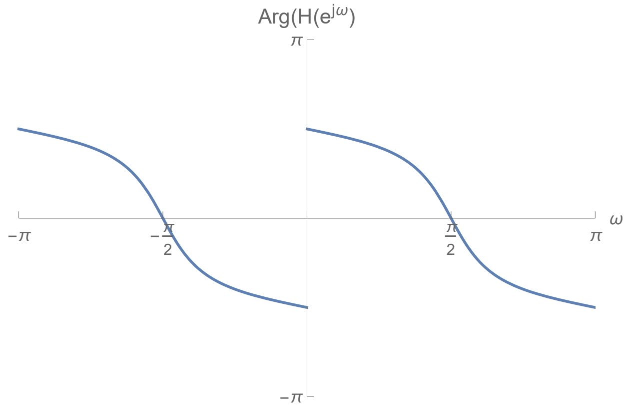
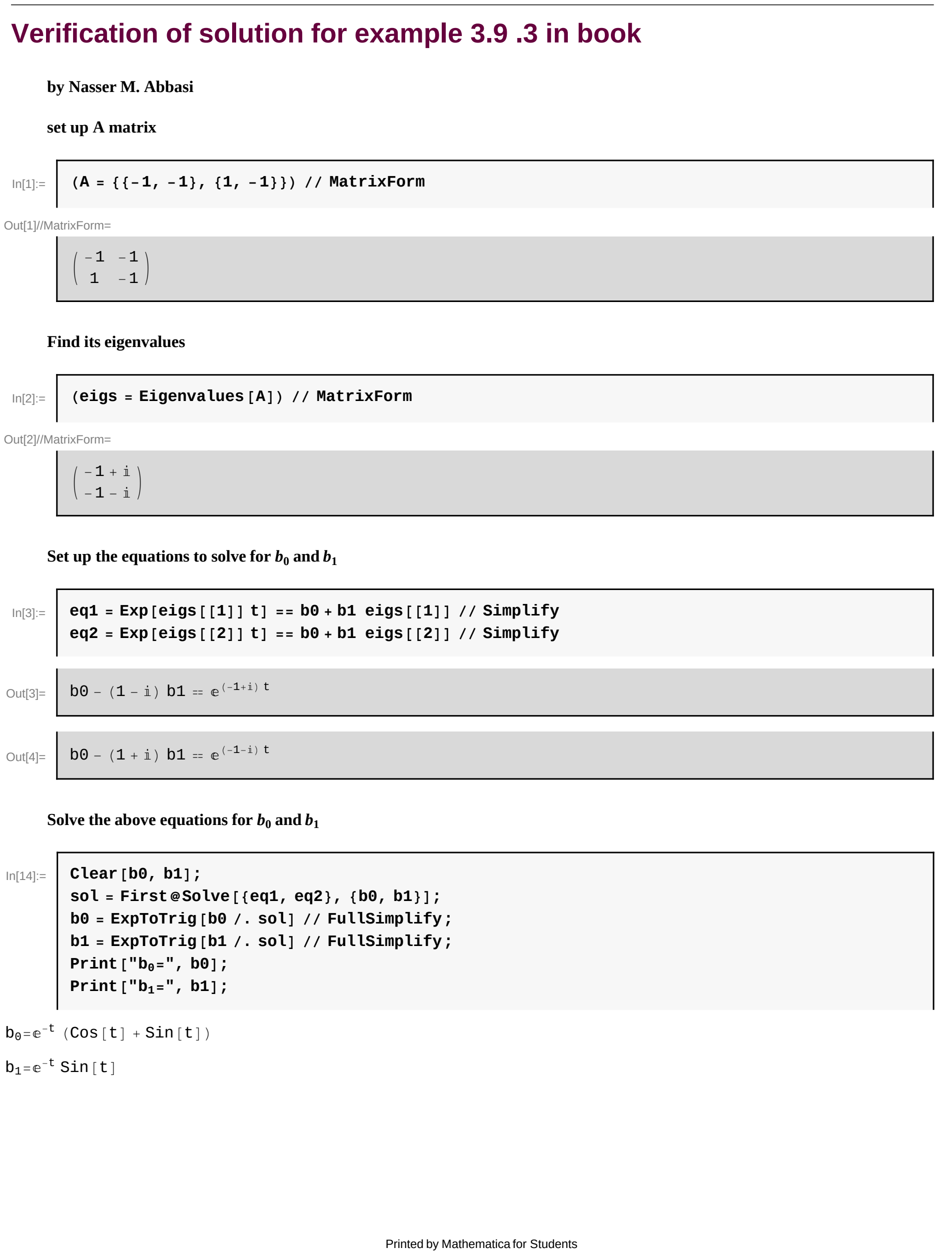


Figure 3.12: second plot

3.10 Verification of example 3.9.3 in book



Verification of solution for example 3.9 .3 in book

by Nasser M. Abbasi

set up A matrix

```
In[1]:= (A = {{-1, -1}, {1, -1}}) // MatrixForm
```

Out[1]//MatrixForm=

$$\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

Find its eigenvalues

```
In[2]:= (eigs = Eigenvalues[A]) // MatrixForm
```

Out[2]//MatrixForm=

$$\begin{pmatrix} -1 + i \\ -1 - i \end{pmatrix}$$

Set up the equations to solve for b_0 and b_1

```
In[3]:= eq1 = Exp[eigs[[1]] t] == b0 + b1 eigs[[1]] // Simplify
        eq2 = Exp[eigs[[2]] t] == b0 + b1 eigs[[2]] // Simplify
```

Out[3]= $b0 - (1 - i)\, b1 == e^{(-1+i)\, t}$

Out[4]= $b0 - (1 + i)\, b1 == e^{(-1-i)\, t}$

Solve the above equations for b_0 and b_1

```
In[14]:= Clear[b0, b1];
         sol = First@Solve[{eq1, eq2}, {b0, b1}];
         b0 = ExpToTrig[b0 /. sol] // FullSimplify;
         b1 = ExpToTrig[b1 /. sol] // FullSimplify;
         Print["b₀=", b0];
         Print["b₁=", b1];
```

$b_0 = e^{-t}\,(\text{Cos}[t] + \text{Sin}[t])$

$b_1 = e^{-t}\,\text{Sin}[t]$

Printed by Mathematica for Students

Now display e^{At}

```
(b0 * IdentityMatrix[2] + b1 A) // FullSimplify // MatrixForm
```

Out[21]//MatrixForm=

```
( ⅇ^-t Cos[t]  -ⅇ^-t Sin[t] )
( ⅇ^-t Sin[t]   ⅇ^-t Cos[t] )
```

Redo the solution, but change the b0 and b1 order, we obtain the solution given in class

Now display e^{At}

In[39]:=

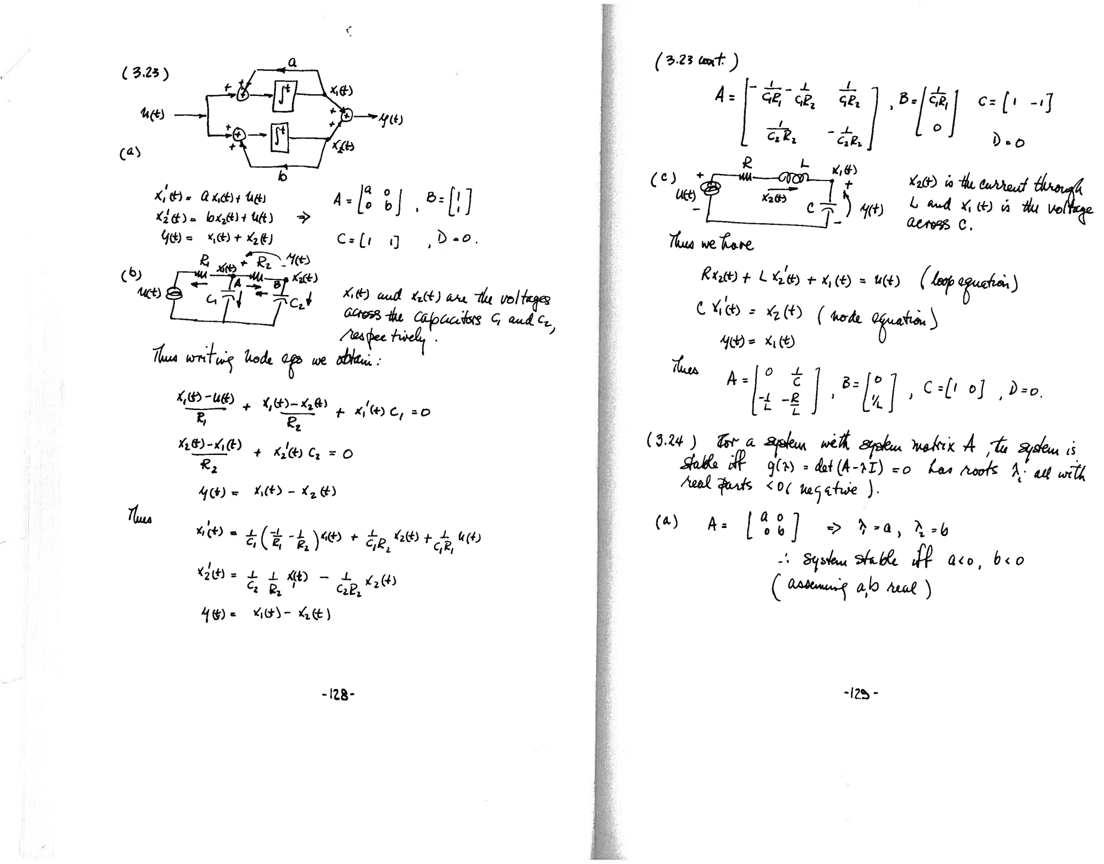
```
(b1 * IdentityMatrix[2] + b0 A) // FullSimplify // MatrixForm
```

Out[39]//MatrixForm=

```
(       -ⅇ^-t Cos[t]         -ⅇ^-t (Cos[t] + Sin[t]) )
( ⅇ^-t (Cos[t] + Sin[t])          -ⅇ^-t Cos[t]        )
```

3.11 Key solutions to some problems

3.11.1 HW 4,5 and 6 key



(3.23)

(a)

$x_1'(t) = a x_1(t) + u(t)$

$x_2'(t) = b x_2(t) + u(t)$ $\Rightarrow$ $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$y(t) = x_1(t) + x_2(t)$ $\quad C = [1 \quad 1]$, $D = 0$.

(b)

$x_1(t)$ and $x_2(t)$ are the voltages across the capacitors C_1 and C_2, respectively.

Thus writing node eqs we obtain:

$$\frac{x_1(t) - u(t)}{R_1} + \frac{x_1(t) - x_2(t)}{R_2} + x_1'(t) C_1 = 0$$

$$\frac{x_2(t) - x_1(t)}{R_2} + x_2'(t) C_2 = 0$$

$$y(t) = x_1(t) - x_2(t)$$

Thus

$$x_1'(t) = \frac{1}{C_1}\left(\frac{-1}{R_1} - \frac{1}{R_2}\right) x_1(t) + \frac{1}{C_1 R_2} x_2(t) + \frac{1}{C_1 R_1} u(t)$$

$$x_2'(t) = \frac{1}{C_2} \frac{1}{R_2} x_1(t) - \frac{1}{C_2 R_2} x_2(t)$$

$$y(t) = x_1(t) - x_2(t)$$

(3.23 cont.)

$$A = \begin{bmatrix} -\frac{1}{C_1R_1} - \frac{1}{C_1R_2} & \frac{1}{C_1R_2} \\ \frac{1}{C_2R_2} & -\frac{1}{C_2R_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{C_1R_1} \\ 0 \end{bmatrix} \quad C = [1 \quad -1] \quad D = 0$$

(c)

$x_2(t)$ is the current through L and $x_1(t)$ is the voltage across C.

Thus we have

$R x_2(t) + L x_2'(t) + x_1(t) = u(t)$ (loop equation)

$C x_1'(t) = x_2(t)$ (node equation)

$y(t) = x_1(t)$

Thus $A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1/L \end{bmatrix}$, $C = [1 \quad 0]$, $D = 0$.

(3.24) For a system with system matrix A, the system is stable iff $g(\lambda) = \det(A - \lambda I) = 0$ has roots λ_i all with real parts < 0 (negative).

(a) $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \Rightarrow \lambda_1 = a, \lambda_2 = b$

$\therefore$ system stable iff $a < 0$, $b < 0$

(assuming a, b real)

(3.27 cont.)

$$x_1'(t) = 2x_1(t) + 9x_2(t)$$
$$x_2'(t) = -x_1(t) - 4x_2(t) + u(t)$$
$$x_3'(t) = -2x_3(t) + u(t)$$

$$A = \begin{bmatrix} 2 & 9 & 0 \\ -1 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad C = [0 \ 1 \ 1], \quad D = 0$$

$$g(\lambda) = \det(A - \lambda I) = 0 \Rightarrow \lambda_1 = -2, \lambda_2 = -1, \lambda_3 = -1.$$

Because we have one repeated root A is of the form:

$$A = \lambda_1 E_1 + \lambda_2 E_2 + N_2$$

There are several methods one can use to find E_1, E_2, N_2. We shall use a method which is not covered in the text but is straightforward and useful to know.

Consider the function of a matrix $f(A) = (sI - A)^{-1}$. Then we have that, from (2.85),

$$f(A) = f(\lambda_1) E_1 + f(\lambda_2) E_2 + f'(\lambda_2) N_2$$

where $f(\lambda_i) = \frac{1}{s - \lambda_i} \Rightarrow f'(\lambda_i) = \frac{1}{(s-\lambda_i)^2}$

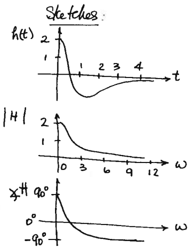
Thus

$$(sI - A)^{-1} = \frac{1}{s-\lambda_1} E_1 + \frac{1}{s-\lambda_2} E_2 + \frac{1}{(s-\lambda_2)^2} N_2$$

Now $(sI - A)^{-1} = \begin{bmatrix} s-2 & -9 & 0 \\ 1 & s+4 & 0 \\ 0 & 0 & s+2 \end{bmatrix}^{-1}$

(3.27 cont.)

Sketches:



(b) choose $x_1(t)$ as the output of right most integrator and $x_2(t)$ as the output of the left most integrator. Then we have

$$A = \begin{bmatrix} 0 & 1 \\ -26 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [29 \ \ 0], \quad D = 0$$

Again compute $(sI-A)^{-1}$ and expand in partial fractions. The "coefficients" of the partial fraction terms are matrices, E_1 & E_2.

$$g(\lambda) = \det(A - \lambda I) = 0 \Rightarrow \lambda_1 = -1-5j, \ \lambda_2 = -1+5j$$

$$(sI-A)^{-1} = \begin{bmatrix} s & -1 \\ 26 & s+2 \end{bmatrix}^{-1} = \frac{1}{s^2+2s+26} \begin{bmatrix} s+2 & 1 \\ -26 & s \end{bmatrix}$$

$$= \frac{1}{s-(1+5j)} \begin{bmatrix} \frac{5-j}{10} & \frac{-j}{10} \\ \frac{26j}{10} & \frac{5+j}{10} \end{bmatrix} + \frac{1}{s-(1-5j)} \begin{bmatrix} \frac{5+j}{10} & j/10 \\ -\frac{26j}{10} & \frac{5-j}{10} \end{bmatrix}$$

$$= \frac{1}{s-\lambda_1} E_1 + \frac{1}{s-\lambda_2} E_2 .$$

(3.27 cont)

Note: $E_1 = E_2^*$, $E_1 + E_2 = I$, $E_1E_2 = 0$

Thus
$$e^{At} = e^{\lambda_1 t}E_1 + e^{\lambda_2 t}E_2 = e^{-t}\begin{bmatrix} \cos 5t + \frac{\sin 5t}{5} & \frac{\sin 5t}{5} \\ -\frac{26\sin 5t}{5} & \cos 5t - \frac{\sin 5t}{5} \end{bmatrix}$$

Impulse response:

$$h(t) = Ce^{At}B = [29 \;\; 0]\, e^{At}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{29e^{-t}\sin 5t}{5}\;\xi(t)$$

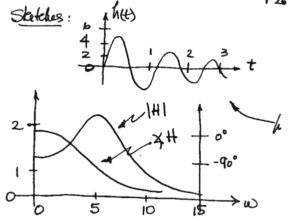
Transfer function:

$$H(j\omega) = D + C(j\omega I - A)^{-1}B = \frac{29}{(j\omega)^2 + 2j\omega + 26}$$

$$\begin{cases} \text{poles: } 1 \pm 5j \\ \text{zeros: } 2 \text{ at } \infty \end{cases}$$

Frequency response: $|H(j\omega)| = \dfrac{29}{\sqrt{26-\omega^2+4\omega^2}}$

Sketches:



t	$\frac{29}{5}e^{-t}\sin 5t$
.1	2.52
.2	3.99
.3	4.28
.4	3.54
.5	2.11
.7	-1.01
1.0	-2.05
1.2	-.488
1.4	.94
1.6	1.16

etc.

(3.28) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [c_1, c_2]$ $D = 0$

Note: In the first printing there is an error in the sign of the a_{22} entry in A. It should be -3, not 3. If 3 is used the eigen values are 1 and 2 implying the system is unstable. If -3 is used, we obtain

$$g(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = 3\lambda + \lambda^2 + 2 = 0$$

$\Rightarrow \lambda_1 = -1$, $\lambda_2 = -2$.

Then $A = -E_1 - 2E_2$ where we can obtain E_1 & E_2 via

$$(sI - A)^{-1} = \frac{1}{s+1}E_1 + \frac{1}{s+2}E_2 = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

or $E_1 = I - E_2$ and $E_2 = -(I + A) = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$

$\Rightarrow E_1 = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$. Thus

$$e^{At} = e^{-t}\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} + e^{-2t}\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$$

Now
$$\begin{aligned} H(j\omega) &= D + C(j\omega I - A)^{-1}B \\ &= D + [c_1 \;\; c_2]\begin{bmatrix} 1 \\ j\omega \end{bmatrix}\frac{1}{(j\omega+1)(j\omega+2)} \\ &= \frac{d\left((j\omega)^2 + 3j\omega + 2\right) + c_1 + c_2 j\omega}{(j\omega+1)(j\omega+2)} \qquad (1) \end{aligned}$$

(3.28 cont.)

From the graph we have a DC gain of unity and two zeros at $\omega = \pm 1$. Thus

$$H(j\omega) = \frac{2((j\omega)^2+1)}{(j\omega+1)(j\omega+2)} \qquad (2)$$

At $j\omega=0$, $H(0)=1$ giving the DC gain. The term $(j\omega)^2+1$ gives us the zeros at $\omega = \pm 1$. Equating coefficients of like powers of $(j\omega)$ in (1) and (2) in the numerators gives:

$$\left.\begin{aligned} (j\omega)^0 &: \ 2d + c_1 = 2 \\ (j\omega)^1 &: \ 3d + c_2 = 0 \\ (j\omega)^2 &: \ d = 2 \end{aligned}\right\} \Rightarrow \begin{aligned} d &= 2 \\ c_1 &= -2 \\ c_2 &= -6 \end{aligned}$$

(3.29) There are (at least) three possible approaches:

(a) Use state variable methods or classical methods to solve for $w(t)$ in

$$b_n \frac{d^n}{dt^n} w(t) + \cdots + b_1 \frac{dw(t)}{dt} + b_0 w(t) = u(t)$$

Then operate on $w(t)$ to obtain $y(t)$ using superposition

$$y(t) = L_D[w(t)] = [a_0 + a_1 D + \cdots + a_m D^m]\, w(t)$$

$$= a_0 w(t) + a_1 \frac{d}{dt} w(t) + \cdots + a_m \frac{d^m}{dt^m} w(t)$$

(3.29 cont)

(b) If $m \le n$, we can integrate both sides of the given equation n times to obtain an equation for $y(t)$ in terms of integrals of $y(t)$ and $u(t)$, with no derivatives present:

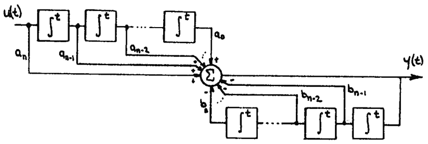
$$b_n y(t) + b_{n-1} \int^t y(t')\,dt' + \cdots + b_0 \int^t\!\!\int^{t_{n-1}}\!\!\cdots\!\int^{t_2} y(t_1)\,dt_1 \ldots dt$$

$$= a_0 \int^t\!\!\int^{t_{n-1}}\!\!\cdots\!\int^{t_2} u(t_1)\,dt_1 \ldots dt_n + \cdots + a_m \underbrace{\int^t\!\!\cdots\!\int^{t_2}}_{n-m \text{ fold}} u(t_1)\,dt_1 \cdots dt$$

This system is sketched below, assuming that $n=m$ and with b_n normalized to 1 by dividing through the equation by b_n. If $m<n$, the coefficients a_k below with $k>m$ will be zero.

$$[1 + b_{n-1}D^{-1} + \cdots + b_1 D^{-n+1} + b_0 D^{-n}]\,y(t)$$

$$= [a_0 D^{-n} + a_1 D^{-n+1} + \cdots + a_n]\,u(t)$$



This system can now be solved using state variable methods. Note that the A matrix has dimension $2n \times 2n$.

(c) An equivalent n-integral system is shown in the block diagram below. Again, state variables may be used to solve for the output $y(t)$, here with only an $n \times n$ A-matrix. Using Laplace transforms, one can readily establish the equivalence of these two block diagrams.

(3.24 cont)

$$A = \begin{bmatrix} -\frac{1}{C_1R_1} - \frac{1}{C_2R_2} & \frac{1}{C_1R_2} \\ \frac{1}{C_2R_2} & -\frac{1}{C_2R_2} \end{bmatrix}$$

$$g(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} a-\lambda & \frac{1}{C_1R_2} \\ \frac{1}{C_2R_2} & b-\lambda \end{bmatrix} = (a-\lambda)(b-\lambda) + c = 0$$

Where $a = -\frac{1}{C_1R_1} - \frac{1}{C_2R_2}$, $b = -\frac{1}{C_2R_2}$, $c = -\frac{1}{C_1C_2R_1R_2}$

and we know that a, b, c are all < 0.

Now $g(\lambda) = \lambda^2 - (a+b)\lambda + ab + c = 0$

Further let $ab + c = \beta$. Then $\beta = \left(-\frac{1}{C_2R_2}\right)\left(-\frac{1}{C_1R_1} - \frac{1}{C_2R_2}\right) - \frac{1}{C_1C_2R_1R_2}$

$$= \left(\frac{1}{C_2R_2}\right)^2 > 0$$

Let $-(a+b) = \alpha$; then $\alpha > 0$.

Thus

$$\lambda_1, \lambda_2 = -\frac{\alpha}{2} \pm \frac{\sqrt{\alpha^2 - 4\beta}}{2}$$

But $\beta > 0 \Rightarrow -4\beta < 0 \Rightarrow (\alpha^2 - 4\beta)^{1/2} < \alpha$

$\therefore$ System is always stable for any values of R_1, R_2, C_1, C_2 which are > 0. (This result is, of course, clear from the structure of the circuit.)

(c)

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -R/L \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ -\beta & -\gamma \end{bmatrix} \text{ with } \alpha, \beta, \gamma > 0.$$

(3.24 cont)

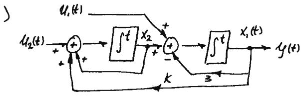
$$g(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} -\lambda & \alpha \\ -\beta & -\lambda-\gamma \end{pmatrix} = \lambda^2 + \lambda\gamma + \alpha\beta = 0$$

$$\therefore \lambda_1, \lambda_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\alpha\beta}}{2}$$

But $\alpha\beta > 0 \Rightarrow -4\alpha\beta < 0 \Rightarrow (\gamma^2 - 4\alpha\beta)^{1/2} < \gamma$

$\therefore$ This system is always stable for any values of $R, L, C > 0$.

(3.25)



(a)

$$x_1'(t) = 3x_1(t) + x_2(t) + u_1(t)$$
$$x_2'(t) = kx_1(t) + x_2(t) + u_2(t)$$
$$y(t) = x_1(t)$$

$$A = \begin{bmatrix} 3 & 1 \\ k & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For stability the real part of the eigenvalues of A must be negative.

$$g(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 1 \\ k & 1-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + (3-k) = 0$$

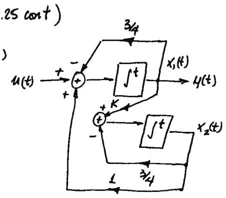
$$\lambda_1, \lambda_2 = \frac{4 \pm \sqrt{16 - 4(3-k)}}{2} = 2 \pm \sqrt{1+k}$$

$$\left.\begin{aligned} \lambda_1 &= 2 + \sqrt{1+k} \\ \lambda_2 &= 2 - \sqrt{1+k} \end{aligned}\right\} \Rightarrow$$ λ_1 can never have a real part <

System is never stable.

(3.25 cont)

(b)



$$\left.\begin{aligned} x_1'(t) &= -\tfrac{3}{4}x_1(t) + x_2(t) + u(t) \\ x_2'(t) &= K\,x_1(t) - \tfrac{3}{4}x_2(t) \end{aligned}\right\} \quad A = \begin{bmatrix} -3/4 & 1 \\ K & -3/4 \end{bmatrix}$$

$$g(\lambda) = \det\begin{pmatrix} -3/4-\lambda & 1 \\ k & -3/4-\lambda \end{pmatrix} = \lambda^2 + \frac{3}{2}\lambda + \frac{9}{16} - k = 0$$

$$\lambda_1, \lambda_2 = \frac{-3/2 \pm \sqrt{(3/2)^2 - 4\left(\frac{9}{16}-k\right)}}{2} = -3/4 \pm \sqrt{k}$$

$$\left.\begin{aligned} \lambda_1 &= -3/4 + \sqrt{k} \\ \lambda_2 &= -3/4 - \sqrt{k} \end{aligned}\right\} \quad \therefore \text{ For stability } \sqrt{k} - 3/4 < 0$$

$$\Rightarrow K < \frac{9}{16}.$$

(3.26) In general, we have:

$$A = \sum_{i=1}^{n} \lambda_i E_i \quad \& \quad f(A) = \sum_{i=1}^{n} f(\lambda_i) E_i \quad \text{for distinct } \lambda_i.$$

Thus $f(A) = e^{At} = \sum_{i=1}^{n} e^{\lambda_i t} E_i$

where E_i can be obtained via: $E_1 = \dfrac{A - \lambda_2 I}{\lambda_1 - \lambda_2}$, $E_2 = \dfrac{A - \lambda_1 I}{\lambda_2 - \lambda_1}$

(for $n = 2$)

(3.26 cont)

(a) $A = \begin{bmatrix} 3/4 & 0 \\ 1/2 & 1/2 \end{bmatrix}$ $g(\lambda) = \det(A - \lambda I) = 0 \Rightarrow \lambda_1 = 1/2,\ \lambda_2 = 3/4$

Thus $E_1 = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ (Note: $E_1 + E_2 = I$)

$$\therefore e^{At} = e^{t/2}\begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} + e^{\frac{3}{4}t}\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} e^{3t/4} & 0 \\ 2e^{3t/4} - 2e^{t/2} & e^{t/2} \end{bmatrix}.$$

(b) $A = \begin{bmatrix} 1/2 & 1/4 \\ \frac{1}{16} & 1/2 \end{bmatrix}$, $g(\lambda) = 0 \Rightarrow \lambda_1 = 3/8,\ \lambda_2 = 5/8$

$E_1 = \begin{bmatrix} 1/2 & -1 \\ -1/4 & 1/2 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1/2 & 1 \\ 1/4 & 1/2 \end{bmatrix}$

$$\therefore e^{At} = \begin{bmatrix} \frac{1}{2}\left(e^{5t/8} + e^{5t/8}\right) & e^{\frac{5t}{8}} - e^{3t/8} \\ 1/4\left(e^{5t/8} - e^{3t/8}\right) & \frac{1}{2}\left(e^{3t/8} + e^{5t/8}\right) \end{bmatrix}$$

(c) $A = \begin{bmatrix} 1/2 & 1/4 \\ 1 & 1/2 \end{bmatrix}$, $g(\lambda) = 0 \Rightarrow \lambda_1 = 0,\ \lambda_2 = 1$

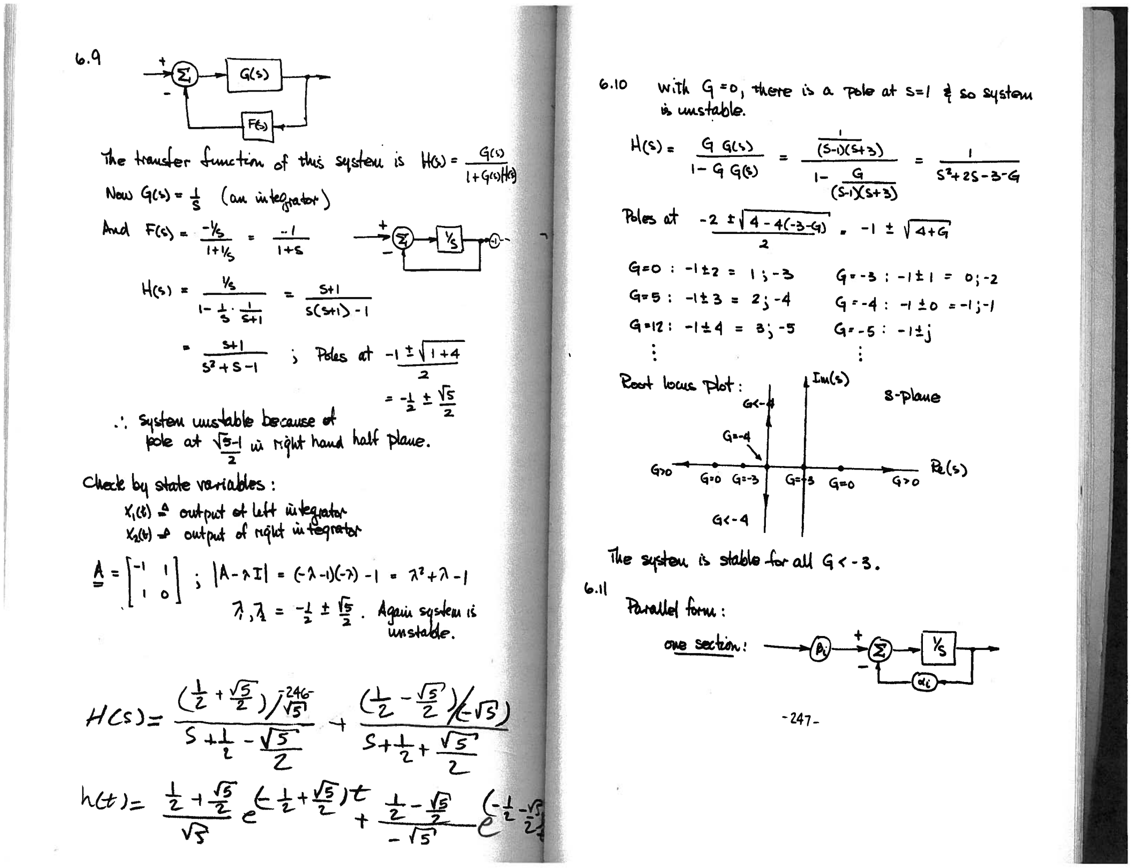
Thus $E_1 = \begin{bmatrix} 1/2 & 1/4 \\ -1 & 1/2 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1/2 & -1/4 \\ 1 & 1/2 \end{bmatrix}$

$$\therefore e^{At} = \begin{bmatrix} \frac{1}{2}(1 + e^t) & 1/4(1 - e^t) \\ e^t - 1 & \frac{1}{2}(1 + e^t) \end{bmatrix}$$

(d) $A = \begin{bmatrix} 1/2 & 0 \\ 1 & 1/2 \end{bmatrix}$, $g(\lambda) = 0 \Rightarrow \lambda_1 = 1/2,\ \lambda_2 = 1/2$

In the case of repeated roots $A = \lambda E_1 + N_1$

3.11.2 HW 8 and 9 key



6.9

The transfer function of this system is $H(s) = \frac{G(s)}{1+G(s)H(s)}$

Now $G(s) = \frac{1}{s}$ (an integrator)

And $F(s) = \frac{-1/s}{1+1/s} = \frac{-1}{1+s}$

$$H(s) = \frac{1/s}{1-\frac{1}{s}\cdot\frac{1}{s+1}} = \frac{s+1}{s(s+1)-1}$$

$$= \frac{s+1}{s^2+s-1} \text{ ; Poles at } \frac{-1 \pm \sqrt{1+4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

∴ system unstable because of pole at $\frac{\sqrt{5}-1}{2}$ in right hand half plane.

Check by state variables:

$x_1(t) \triangleq$ output of left integrator

$x_2(t) \triangleq$ output of right integrator

$$\underline{\underline{A}} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} ; \ |A - \lambda I| = (-\lambda-1)(-\lambda) - 1 = \lambda^2 + \lambda - 1$$

$\lambda_1, \lambda_2 = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$. Again system is unstable.

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$$H(s) = \frac{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)/\sqrt{5}}{s+\frac{1}{2}-\frac{\sqrt{5}}{2}} + \frac{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)/(-\sqrt{5})}{s+\frac{1}{2}+\frac{\sqrt{5}}{2}}$$

$$h(t) = \frac{\frac{1}{2}+\frac{\sqrt{5}}{2}}{\sqrt{5}} e^{\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)t} + \frac{\frac{1}{2}-\frac{\sqrt{5}}{2}}{-\sqrt{5}} e^{\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right.}$$

6.10 With $G = 0$, there is a pole at $s = 1$ & so system is unstable.

$$H(s) = \frac{G\,G(s)}{1-G\,G(s)} = \frac{\frac{1}{(s-1)(s+3)}}{1-\frac{G}{(s-1)(s+3)}} = \frac{1}{s^2+2s-3-G}$$

Poles at $\frac{-2 \pm \sqrt{4-4(-3-G)}}{2} = -1 \pm \sqrt{4+G}$

$G=0 : -1 \pm 2 = 1 ; -3$

$G=5 : -1 \pm 3 = 2 ; -4$

$G=12 : -1 \pm 4 = 3 ; -5$

⋮

$G=-3 : -1 \pm 1 = 0 ; -2$

$G=-4 : -1 \pm 0 = -1 ; -1$

$G=-5 : -1 \pm j$

⋮

Root locus plot:

The system is stable for all $G < -3$.

6.11

Parallel form:

one section:

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chapter 2

#2.1 (a) $e^{bk} = (e^b)^k$; $\underline{\underline{L_A = s - e^b}}$

(b) $B\sinh ak = \frac{B}{2}(e^{ak}+e^{-ak})$; $L_A = \underline{\underline{(s-e^a)(s-e^{-a})}}$

$= \underline{\underline{s^2 - 2\sinh a\, s + 1}}$

(c) $k^2 a^k + Ae^{bk}$; $L_A = \underline{\underline{(s-a)^3 (s-e^b)}}$

(d) $ka^k + A\sin bk$; $L_A = \underline{\underline{(s-a)^2 (s-e^{jb})(s-e^{-jb})}}$

#2.2 (a) $y_{k+2} + 7y_{k+1} + 12y_k = 0$

Char eqn: $r^2+7r+12 = 0$

ie $(r+4)(r+3)=0$ $\begin{cases} r_1 = -3 \\ r_2 = -4 \end{cases}$

$\underline{\underline{y_k = c_1(-4)^k + c_2(-3)^k}}$

(b) $y_{k+2} + 2y_{k+1} + 2y_k = 0$

Char eqn: $r^2+2r+2 = 0$ $\begin{cases} r_1 = 1+\sqrt{1-2} = 1+j \\ r_2 = 1-j = r_1^* \end{cases}$

$y_k = \underline{\underline{c_1(1+j)^k + c_2(1-j)^k}}$

(4 equivalent forms)

$= \underline{\underline{c_1\left(\sqrt{2}e^{j\pi/4}\right)^k + c_2\left(\sqrt{2}e^{-j\pi/4}\right)^k}}$

$= \underline{\underline{c_1 2^{k/2} e^{j\pi k/4} + c_2 2^{k/2} e^{-j\pi k/4}}}$

$= \underline{\underline{\hat{c}_1 2^{k/2}\cos \pi k/4 + \hat{c}_2 2^{k/2} \sin \pi k/4}}$

#2.2 (c) $y_{k+2} + y_k = \sin k$

Since $\sin k = \frac{e^{jk}+e^{-jk}}{j2}$,

Choose $L_A = (s-e^j)(s-e^{-j})$

Then $(s-e^j)(s-e^{-j})(s^2+1) = 0$

The characteristic equation is

$(r-e^j)(r-e^{-j})(r-e^{j\pi/2})(r-e^{-j\pi/2}) = 0$

(where $e^{\pm j\pi/2} = \pm j$)

Thus $y_k = c_1 \cos \frac{\pi k}{2} + c_2 \sin \frac{\pi k}{2}$

$+ c_3 \cos k + c_4 \sin k$

The constants c_3 and c_4 are found from

$(s^2+1)(c_3\cos k + c_4 \sin k) \equiv$

ie $c_3[\cos(k+2) + \cos k] + c_4[\sin(k+2) + \sin k] \equiv$

$c_3 \cos k \cos 2 - c_3 \sin k \sin 2 + c_3 \cos k$

$+ c_4 \sin k \cos 2 - \cos k \sin 2 + c_4 \sin k \equiv$

ie $\cos k\,(c_3\cos 2 + c_3 - c_4 \sin 2)$

$+ \sin k\,(-c_3 \sin 2 + c_4 \cos 2 + c_4) \equiv$

Thus $c_3(1+\cos 2) - c_4 \sin 2 = 0$

$c_3(-\sin 2) + c_4(1+\cos 2) = 1$

#2.2 (c) (cont)

$$c_3 = \frac{\begin{vmatrix} 0 & -\sin 2 \\ 1 & 1+\cos 2 \end{vmatrix}}{\begin{vmatrix} 1+\cos 2 & -\sin 2 \\ -\sin 2 & 1+\cos 2 \end{vmatrix}} = \frac{\sin 2}{1+2\cos 2+\cos^2 2-\sin^2 2}$$

$$= \frac{\sin 2}{2\cos 2(1+\cos 2)}$$

$$= \frac{\tan 2}{2(1+\cos 2)}$$

$$c_4 = \frac{\begin{vmatrix} 1+\cos 2 & 0 \\ -\sin 2 & 1 \end{vmatrix}}{2\cos 2(1+\cos 2)} = \frac{1}{2\cos 2}$$

soln:

$$\underline{\underline{y_k = c_1 \cos\frac{\pi k}{2} + c_2 \sin\frac{\pi k}{2} + \frac{\tan 2}{2(1+\cos 2)}\cos k + \frac{\sin k}{2\cos 2}}}$$

(d) $y_{k+2} - \frac{5}{2} y_{k+1} + y_k = 1$

use $L_A = S-1$ to annihilate the constant

Thus $(S^2 - \frac{5}{2}S+1)(S-1) = 0$

Char eqn $(r^2 - \frac{5}{2}r+1)(r-1) = 0$

$$(r-\tfrac{1}{2})(r-2)(r-1) = 0 \quad \left\{ \begin{array}{l} r_1 = 1/2 \\ r_2 = 2 \\ r_3 = 1 \end{array} \right.$$

$$y_k = c_1(1/2)^k + c_2(2)^k + c_3$$

2.2(d) (cont)

c_3 is found from $L[y_k] = 1$:

$$(S^2 - \tfrac{5}{2}S+1)\,c_3 = c_3(1-\tfrac{5}{2}+1) = c_3(-1/2) \equiv 1$$

$$\Rightarrow c_3 = -2$$

Thus $y_k = c_1(1/2)^k + c_2 2^k - 2$

with $y_0 = y_1 = 0$

$$\left.\begin{array}{l} y_0 = c_1 + c_2 - 2 \equiv 0 \\ y_1 = \frac{1}{2}c_1 + 2c_2 - 2 \equiv 0 \end{array}\right\} \begin{array}{l} c_1 = 4/3 \\ c_2 = 2/3 \end{array}$$

soln: $\underline{\underline{y_k = \frac{4}{3}(1/2)^k + \frac{2}{3}2^k - 2}}$

*2.3 $y_{k+2} - 2\tau y_{k+1} + y_k = 0$

Find solutions for above as τ varies. the auxiliary equation is: $r^2 - 2\tau r + 1 = 0$ with roots $r_1, r_2 = \tau \pm \sqrt{\tau^2-1}$.

(a) $\tau < -1$: The roots are distinct and real

$$y_k = c_1\left(\tau + \sqrt{\tau^2-1}\right)^k + c_2\left(\tau - \sqrt{\tau^2-1}\right)^k$$

(b) $\tau = -1$: The roots are repeated : $r_1 = r_2 = -1$

$$y_k = c_1(-1)^k + c_2 k(-1)^k$$

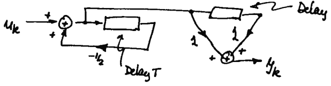
$$= (c_1 + c_2 k)(-1)^k$$

(2.12 cont.)

by convolving the sequences

$$\{1, 0, 1/2, 0, 1/4, 0, 1/8, 0, \cdots\} * \{1, 1, 0, 0, \cdots\}$$

We can preform this convolution by convolving $\{1, 1/2, 1/4, 1/8, \cdots\}$ with $\{1, 1, 0, 0, \cdots\}$ in which the "clock" of the sequence $\{1, 1, 0, 0, \cdots\}$ runs twice the rate as the clock of the sequence $(1/2)^k$. Thus a system is:



#2.13

$$y_{k+2} - y_{k+1} + \tfrac{1}{4} y_k = u_k$$

$$r^2 - r + 1/4 = 0 \Rightarrow r_1, r_2 = 1/2 \Rightarrow h_k = c_1(1/2)^k + c_2 k (1/2)^k$$

Initial Conditions: $h_{k+2} - h_{k+1} + 1/4\, h_k = \delta_k$

$\therefore h_0 = 0,\ h_1 = 0,\ h_2 = 1,\ h_3 = \delta_1 + h_2 - 1/4\, h_1 = 1$

Using h_2 and h_3 (h_0 and h_1 are special cases) we have

$$\left.\begin{aligned} h_2 &= c_1(1/2)^2 + c_2(2)(1/2)^2 = 1/4\, c_1 + 1/2\, c_2 = 1 \\ h_3 &= c_1(1/2)^3 + c_2(3)(1/2)^3 = 1/8\, c_1 + \tfrac{3}{8} c_2 = 1 \end{aligned}\right\} \begin{aligned} c_1 &= -4 \\ c_2 &= 4 \end{aligned}$$

$$\therefore h_k = \begin{cases} -4(1/2)^k + 4k(1/2)^k, & k \geq 2 \\ 0, & k < 2 \end{cases} \longleftarrow$$

(2.13 cont.)

check: $(S^2 - S + 1/4)\, h_k = \left[4(k+2-1)1/4(1/2)^k\right] \xi_{k+1}$

$$- 4\left[(k+1-1)(1/2)(1/2)^k\right] \xi_k + \left[(k-1)(1/2)^k\right] \xi_{k-1}$$

$$= 0 \cdot \delta_{k+1} + 1 \cdot \delta_k + 0 \cdot \delta_k + (k+1-2k+k-1)(1/2)^k \xi_{k-1}$$

$= \delta_k$ Recall: $\xi_{k-a} = \begin{cases} 1, & k \geq a \\ 0, & k < a \end{cases}$.

(b) $(S^2 - 1/4)\, y_k = u_k$

From $r^2 - 1/4 = (r - 1/2)(r + 1/2) = 0$

$$r_1 = 1/2, \quad r_2 = -1/2$$

Thus $h_k = c_1(1/2)^k + c_2(-1/2)^k$

The initial conditions are:

$$h_{k+2} - 1/4\, h_k = \delta_k \Rightarrow \begin{cases} h_0 = 0 \\ h_1 = 0 \\ h_2 = 1 \end{cases}$$

Thus $\left.\begin{aligned} c_1(1/2) + c_2(-1/2) &= 0 \\ c_1(1/4) + c_2(1/4) &= 1 \end{aligned}\right\} c_1 = c_2 = 2$

And so, $h_k = \begin{cases} \left[1 + (-1)^k\right](1/2)^{k-1}, & k \geq 1 \\ 0, & k < 1 \end{cases}$

(2.13 cont.)

check: $(S^2-\frac{1}{4})h_k = \left[1+(-1)^{k+2}\right](\frac{1}{2})^{k+1}\xi_{k+1}$

$$-\left[1+(-1)^k\right](\tfrac{1}{2})^{k-1}\xi_{k-1}$$

$$= 0\cdot\delta_{k+1} + 1\cdot\delta_k + 0 \quad , k \geq 1$$

$$= \delta_k .$$

(c) $y_k = u_k + 3y_{k-1} - 3y_{k-2} + y_{k-3}$

From $r^3-3r^2+3r-1 = 0$

$(r-1)^3 = 0 \Rightarrow r_1 = r_2 = r_3 = 1$

$\Rightarrow h_k = c_1 + c_2 k + c_3 k^2$

with initial conditions: $h_{-2} = 0$, $h_{-1} = 0$, $h_0 = 1$

Thus
$$\left.\begin{array}{l} c_1 - 2c_2 + 4c_3 = 0 \\ c_1 - c_2 + c_3 = 0 \\ c_1 = 1 \end{array}\right\} \Rightarrow c_1 = 1, c_2 = 3/2, c_3 = 1/2$$

$$\Rightarrow h_k = 1 + \frac{3}{2}k + \frac{1}{2}k^2, \quad k \geq 0 \quad (\text{also for } -1,-2)$$

check: $(1-3S^{-1}+3S^{-2}-S^{-3})h_k = (1+\frac{3}{2}k+\frac{1}{2}k^2)\xi_{k+2}$

$$+\left[-3-\frac{9}{2}(k-1)-\frac{3}{2}(k-1)^2\right]\xi_{k+1}$$

(cont)

(2.13 cont)

$$+\left[+3+\frac{9}{2}(k-2)+\frac{3}{2}(k-2)^2\right]\xi_k$$

$$+\left[-1-\frac{3}{2}(k-3)-\frac{1}{2}(k-3)^2\right]\xi_{k-1}$$

Thus $(1-3S^{-1}+3S^{-2}-S^{-3})h_k =$

$$\left[0\cdot\delta_{k-2} + 0\cdot\delta_{k-1} + \delta_k + (1+\frac{3}{2}k+\frac{1}{2}k^2)\right]\xi_{k-1}$$

$$+\left[0\cdot\delta_{k-1} + 0\cdot\delta_k + (-3-\frac{9}{2}k+\frac{9}{2}-\frac{3}{2}k^2+3k-\frac{3}{2})\right]\xi_{k-1}$$

$$+\left[0\cdot\delta_k + (3+\frac{9}{2}k-9+\frac{3}{2}k^2-6k+6)\right]\xi_{k-1}$$

$$+\left[-1-\frac{3}{2}k+\frac{9}{2}-\frac{1}{2}k^2+3k-\frac{9}{2}\right]\xi_{k-1}$$

$$= \delta_k + \Big\{(1-3+\frac{9}{2}-\frac{3}{2}+3-9+6-1+\frac{9}{2}-\frac{9}{2})$$

$$+k(\frac{3}{2}-\frac{9}{2}+3+\frac{9}{2}-6-\frac{3}{2}+3)$$

$$+k^2(\frac{1}{2}-\frac{3}{2}+\frac{3}{2}-\frac{1}{2})\Big\}$$

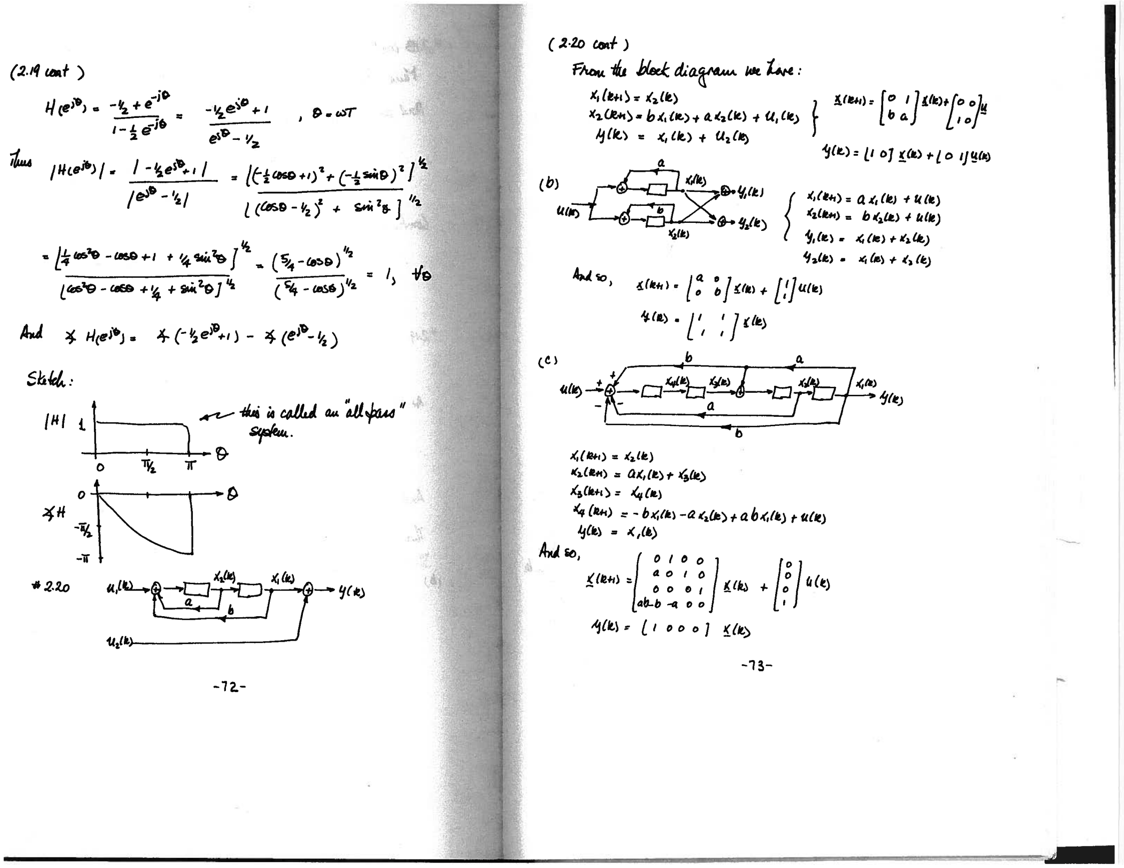
$$= \delta_k$$

(d) $(1-3S^{-1}+3S^{-2}-S^{-3})y_k = S^{-3}u_k$

From (c) $h_k = L_0(\hat{h}_k) = S^{-3}(1+\frac{3}{2}k+\frac{1}{2}k^2) = (1-\frac{3}{2}k+\frac{1}{2}k^2)\cdot$

ξ_{k-3}

3.11.3 HW 10 key



(2.19 cont)

$$H(e^{j\theta}) = \frac{-\frac{1}{2}+e^{-j\theta}}{1-\frac{1}{2}e^{-j\theta}} = \frac{-\frac{1}{2}e^{j\theta}+1}{e^{j\theta}-\frac{1}{2}}, \quad \theta = \omega T$$

Thus
$$|H(e^{j\theta})| = \frac{|-\frac{1}{2}e^{j\theta}+1|}{|e^{j\theta}-\frac{1}{2}|} = \frac{\left[(-\frac{1}{2}\cos\theta+1)^2+(-\frac{1}{2}\sin\theta)^2\right]^{1/2}}{\left[(\cos\theta-\frac{1}{2})^2+\sin^2\theta\right]^{1/2}}$$

$$= \frac{\left[\frac{1}{4}\cos^2\theta-\cos\theta+1+\frac{1}{4}\sin^2\theta\right]^{1/2}}{\left[\cos^2\theta-\cos\theta+\frac{1}{4}+\sin^2\theta\right]^{1/2}} = \frac{(\frac{5}{4}-\cos\theta)^{1/2}}{(\frac{5}{4}-\cos\theta)^{1/2}} = 1, \quad \forall\theta$$

And
$$\measuredangle H(e^{j\theta}) = \measuredangle(-\tfrac{1}{2}e^{j\theta}+1) - \measuredangle(e^{j\theta}-\tfrac{1}{2})$$

Sketch:

this is called an "all pass" system.

#2.20

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(2.20 cont)

From the block diagram we have:

$$\left.\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= b x_1(k) + a x_2(k) + u_1(k) \\ y(k) &= x_1(k) + u_2(k) \end{aligned}\right\} \quad \underline{x}(k+1) = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}\underline{x}(k) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\underline{u}$$

$$y(k) = [1 \; 0]\,\underline{x}(k) + [0 \; 1]\,\underline{u}(k)$$

(b)

$$\left\{\begin{aligned} x_1(k+1) &= a x_1(k) + u(k) \\ x_2(k+1) &= b x_2(k) + u(k) \\ y_1(k) &= x_1(k) + x_2(k) \\ y_2(k) &= x_1(k) + x_2(k) \end{aligned}\right.$$

And so,
$$\underline{x}(k+1) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\underline{x}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(k)$$

$$y(k) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\underline{x}(k)$$

(c)

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = a x_1(k) + x_3(k)$$
$$x_3(k+1) = x_4(k)$$
$$x_4(k+1) = -b x_1(k) - a x_2(k) + ab x_1(k) + u(k)$$
$$y(k) = x_1(k)$$

And so,
$$\underline{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ ab-b & -a & 0 & 0 \end{bmatrix}\underline{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}u(k)$$

$$y(k) = [1 \; 0 \; 0 \; 0]\,\underline{x}(k)$$

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(2.23 cont)

Thus $E_1 = \begin{bmatrix} \frac{a_{11}-\alpha+\beta}{2\beta} & \frac{a_{12}}{2\beta} \\ \frac{a_{21}}{2\beta} & \frac{a_{22}-\alpha+\beta}{2\beta} \end{bmatrix}$, $E_2 = \begin{bmatrix} \frac{a_{11}-\alpha-\beta}{-2\beta} & \frac{a_{12}}{-2\beta} \\ \frac{a_{21}}{-2\beta} & \frac{a_{22}-\alpha-\beta}{-2\beta} \end{bmatrix}$

Thus $A^k = (\alpha+\beta)^k E_1 + (\alpha-\beta)^k E_2$ (in general)

(a) $A = \begin{bmatrix} 3/4 & 0 \\ 1/2 & 1/2 \end{bmatrix}$, $g(\lambda) = \det \begin{bmatrix} 3/4-\lambda & 0 \\ 1/2 & 1/2-\lambda \end{bmatrix} = 0 \Rightarrow \lambda_1 = 3/4$, $\lambda_2 = 1/2$

$A = 3/4\, E_1 + 1/2\, E_2$ $\quad E_1 = \frac{A-\lambda_2 I}{\lambda_1-\lambda_2} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ $\quad E_2 = \frac{A-\lambda_1 I}{\lambda_2-\lambda_1} = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$

$\therefore A^k = (3/4)^k \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + (1/2)^k \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1/2 & 1/4 \\ 1/16 & 1/2 \end{bmatrix}$, $g(\lambda) = \det \begin{bmatrix} 1/2-\lambda & 1/4 \\ \frac{1}{16} & \frac{1}{2}-\lambda \end{bmatrix} = 0 \Rightarrow \lambda_1 = \frac{1}{2} + \frac{1}{\sqrt{48}}$
$\lambda_2 = \frac{1}{2} - \frac{1}{\sqrt{48}}$

Let $\alpha = 1/2$, $\beta = \frac{1}{\sqrt{48}}$

Using general result at top of page we have:

$$A^k = \left(\frac{1}{2}+\frac{1}{\sqrt{48}}\right)^k \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \frac{\sqrt{3}}{4} & 1 \end{bmatrix} + \left(\frac{1}{2}-\frac{1}{\sqrt{48}}\right)^k \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ -\frac{\sqrt{3}}{4} & 1 \end{bmatrix}$$

(c) $A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 1/2 \end{bmatrix}$, $g(\lambda) = \det \begin{bmatrix} 1/2-\lambda & 1/2 \\ 1 & 1/2-\lambda \end{bmatrix} = 0 \Rightarrow \lambda_1 = \frac{1}{2} + \frac{\sqrt{2}}{2}$
$\lambda_2 = \frac{1}{2} - \frac{\sqrt{2}}{2}$

$\Rightarrow \alpha = 1/2 \quad \beta = \frac{\sqrt{2}}{2}$

then $A^k = \left(\frac{1}{2}+\frac{\sqrt{2}}{2}\right)^k \begin{bmatrix} 1/2 & \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1/2 \end{bmatrix} + \left(\frac{1}{2}-\frac{\sqrt{2}}{2}\right)^k \begin{bmatrix} 1/2 & \frac{-1}{2\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 1/2 \end{bmatrix}$

(2.23 cont)

(d) $A = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1/2 \end{bmatrix}$, $g(\lambda) = \det \begin{bmatrix} 1/2-\lambda & 0 \\ 1/2 & 1/2-\lambda \end{bmatrix} = 0 \Rightarrow \lambda_1 = \lambda_2 = 1/2$

$A = \lambda_1 E_1 + N_1$; $A^k = \lambda_1^k E_1 + k(\lambda_1)^{k-1} N_1$

$A^0 = I = E_1$; $A = \frac{1}{2}I + N_1$, $N_1 = A - 1/2\, I = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}$

$\therefore A^k = \left(\frac{1}{2}\right)^k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k\left(\frac{1}{2}\right)^{k-1} \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}$.

(e) $A = \begin{bmatrix} 3/4 & -1/2 \\ -15/32 & 1/2 \end{bmatrix}$, $g(\lambda) = \det(A-\lambda I) = 0 \Rightarrow \lambda_1 = 9/8$, $\lambda_2 = 1/8$

$E_1 = \frac{A-\lambda_2 I}{\lambda_1-\lambda_2} = 1/2 \begin{bmatrix} 5/4 & -1 \\ \frac{15}{16} & 3/4 \end{bmatrix}$, $E_2 = \frac{1}{2}\begin{bmatrix} 3/8 & 1 \\ \frac{15}{16} & 5/4 \end{bmatrix}$

$\therefore A^k = \left(\frac{9}{8}\right)^k E_1 + \left(\frac{1}{8}\right)^k E_2$.

$I = E_1 + E_2$
$A = \lambda_1 E_1 + \lambda_2 E_2$

#2.24 $\quad x(n+1) = 3x(n) + 5y(n) + 2z(n)$
$y(n+1) = x(n) - y(n) + z(n)$
$z(n+1) = 2x(n) + y(n) + 3z(n)$

then $\underline{v}(n+1) = \begin{bmatrix} 3 & 5 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \underline{v}(n)$ with $\underline{v}(n) = \begin{bmatrix} x(n) \\ y(n) \\ z(n) \end{bmatrix}$

Our solution is thus $\underline{v}(n) = \underline{A}^n \underline{v}(0)$. We need A^n.

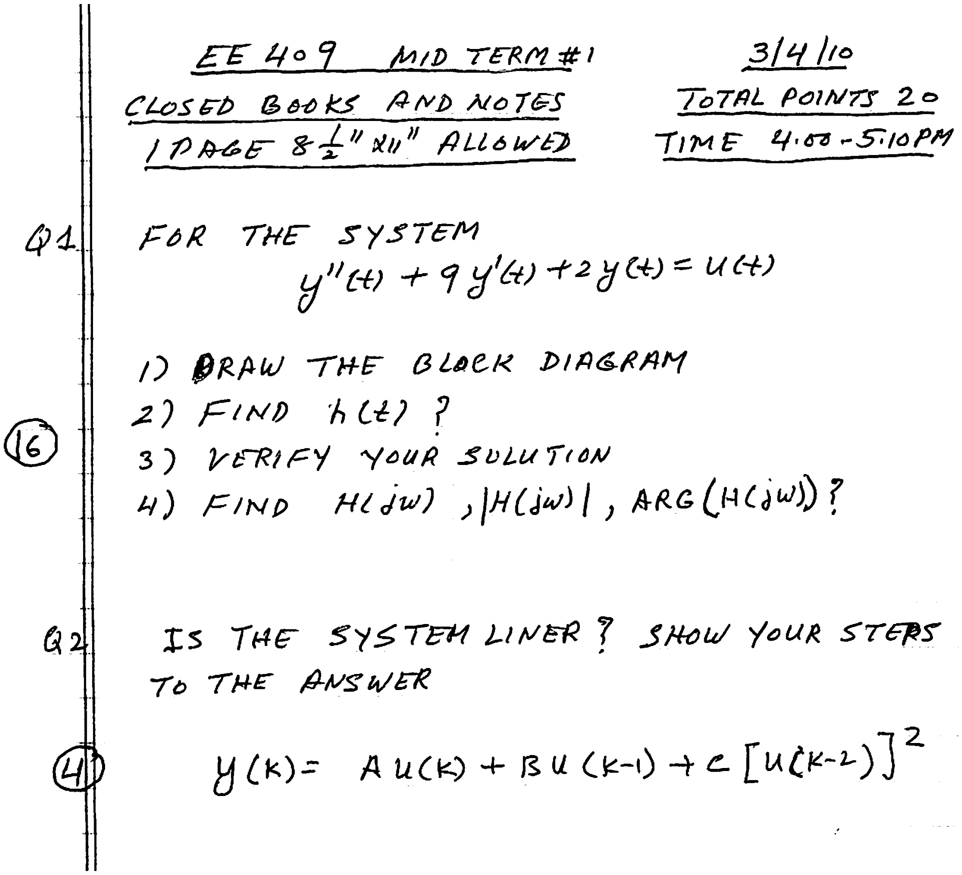
$g(\lambda) = \det[A - \lambda I] = \lambda^3 - 5\lambda^2 - 7\lambda + 11 = 0 \Rightarrow \lambda_1 = 1$
$\lambda_2 = 2 + \sqrt{15}$
$\lambda_3 = 2 - \sqrt{15}$

Chapter 4

Exams

4.1 First exam

4.1.1 Questions



EE 409 MID TERM #1 3/4/10

CLOSED BOOKS AND NOTES TOTAL POINTS 20

1 PAGE $8\frac{1}{2}$" x11" ALLOWED TIME 4.00-5.10PM

Q1. FOR THE SYSTEM

$$y''(t) + 9y'(t) + 2y(t) = u(t)$$

⑯

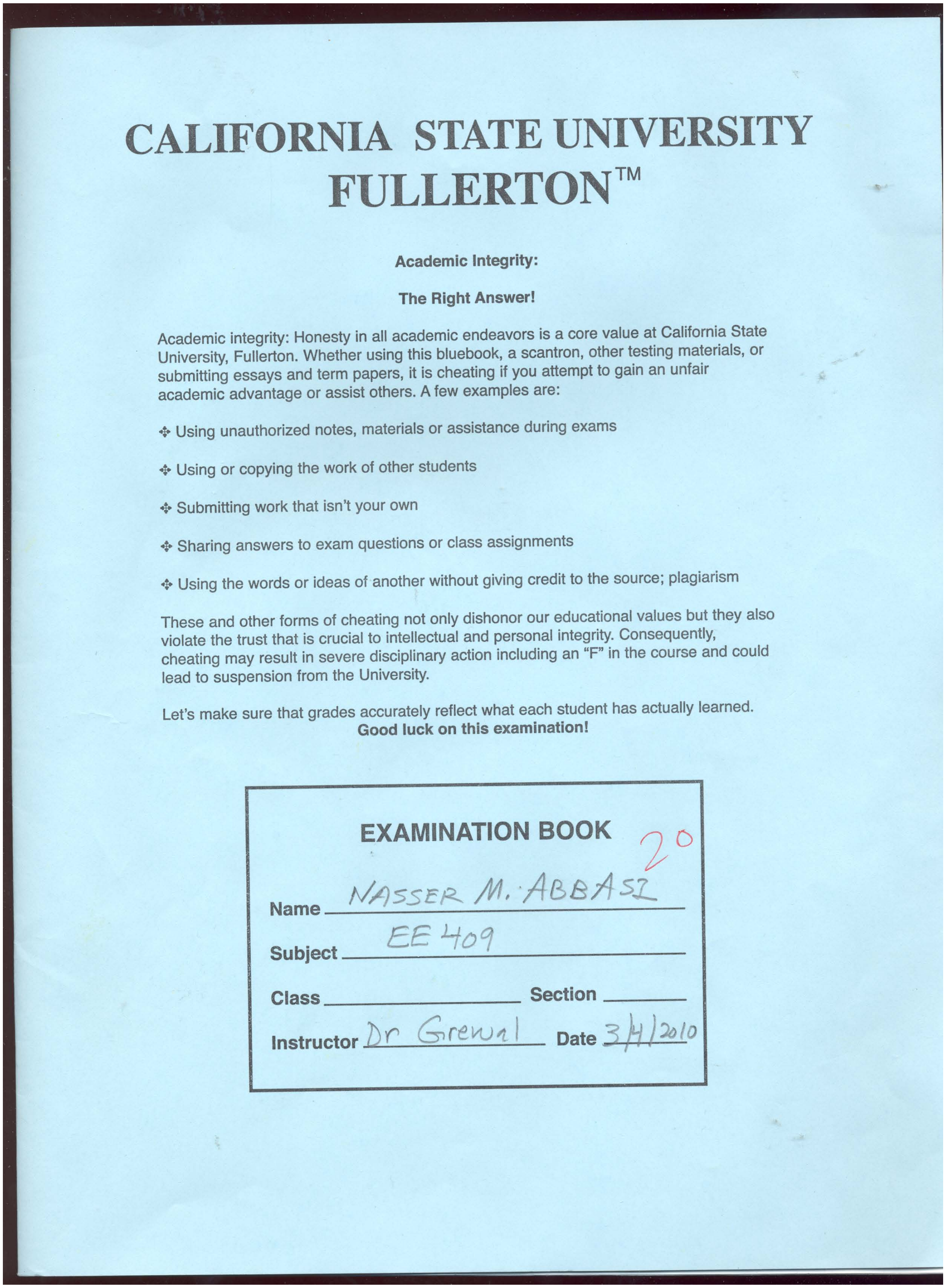
1) DRAW THE BLOCK DIAGRAM
2) FIND $h(t)$?
3) VERIFY YOUR SOLUTION
4) FIND $H(jw)$, $|H(jw)|$, $ARG(H(jw))$?

Q2 IS THE SYSTEM LINER ? SHOW YOUR STEPS TO THE ANSWER

④

$$y(k) = A u(k) + B u(k-1) + c\left[u(k-2)\right]^2$$

4.1.2 my solution



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Good luck on this examination!

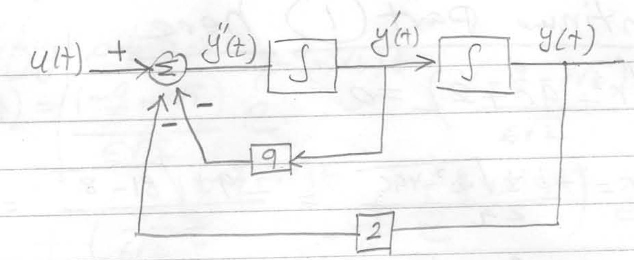
EXAMINATION BOOK

20

Name NASSER M. ABBASI
Subject EE 409
Class ______ **Section** ______
Instructor Dr Grewal **Date** 3/4/2010

Q1

①



$$y'' = -9y'(t) - 2y(t) + u(t)$$

② to find h(t): solve the homogenous D.E.

$$y''(t) + 9y'(t) + 2y(t) = 0$$

with IC $y(0)=0$, $y'(0)=1$.

hence $(D^2+9D+2)\ y(t) = 0$

so char eq $\boxed{r^2+9r+2=0}$ $\longrightarrow$ please see Next page.

Part ④ $\Rightarrow$ method: let $u = e^{j\omega t}$.

so $y(t) = H(j\omega)e^{j\omega t}$

now substitute into ODE, we obtain

$$\left(H(j\omega)e^{j\omega t}\right)'' + 9\left(H(j\omega)e^{j\omega t}\right)' + 2\left(H(j\omega)e^{j\omega t}\right) = e^{j\omega t}$$

$$\left(j\omega H(j\omega)e^{j\omega t}\right)' + 9\left(j\omega H(j\omega)e^{j\omega t}\right) + 2H(j\omega)e^{j\omega t} = e^{j\omega t}$$

$$(j\omega)^2 H(j\omega) + 9j\omega H(j\omega) + 2H(j\omega) = 1$$

$$H(j\omega)\left[-\omega^2 + 9j\omega + 2\right] = 1$$

so $\boxed{H(j\omega) = \dfrac{1}{2-\omega^2+9j\omega}}$

continue part ① here

$(r^2 + 9r + 2) = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-9 \pm \sqrt{81 - 8}}{2} = -\frac{9}{2} \pm \sqrt{\frac{72}{4}}$$

$$= -\frac{9}{2} \pm \sqrt{18} = -\frac{9}{2} \pm 3\sqrt{2}$$

so $\boxed{r_1 = -\frac{9}{2} + 3\sqrt{2}}$ and $\boxed{r_2 = -\frac{9}{2} - 3\sqrt{2}}$

so $h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$$= c_1 e^{-\frac{9}{2}t} e^{3\sqrt{2}t} + c_2 e^{-\frac{9}{2}t} e^{-3\sqrt{2}t}.$$

now find c_1 and c_2

$h(0) = 0 \Rightarrow \boxed{0 = c_1 + c_2}$

$$h'(t) = c_1\left(-\frac{9}{2} + 3\sqrt{2}\right) e^{\left(-\frac{9}{2} + 3\sqrt{2}\right)t} + c_2\left(-\frac{9}{2} - 3\sqrt{2}\right) e^{\left(-\frac{9}{2} - 3\sqrt{2}\right)}$$

$h'(0) = 1 \Rightarrow c_1\left(-\frac{9}{2} + 3\sqrt{2}\right) + c_2\left(-\frac{9}{2} - 3\sqrt{2}\right)$

or $\boxed{1 = c_1\left(-\frac{9}{2} + 3\sqrt{2}\right) + c_2\left(-\frac{9}{2} - 3\sqrt{2}\right)}$ ✓

but $c_1 = -c_2$ so the above is

$$1 = -c_2\left(-\frac{9}{2} + 3\sqrt{2}\right) + c_2\left(-\frac{9}{2} - 3\sqrt{2}\right)$$

$$1 = \frac{9}{2}c_2 - 3\sqrt{2}\,c_2 - \frac{9}{2}c_2 - 3\sqrt{2}\,c_2$$

$1 = -6\sqrt{2}\,c_2$ so $\boxed{c_2 = -\frac{1}{6\sqrt{2}}}$

so $\boxed{c_1 = \frac{1}{6\sqrt{2}}}$

$$\text{so } h(t) = \left(\frac{1}{6\sqrt{2}} e^{\left(-\frac{9}{2} + 3\sqrt{2}\right)t} - \frac{1}{6\sqrt{2}} e^{\left(-\frac{9}{2} - 3\sqrt{2}\right)t}\right) \xi(t)$$

③ to verify solution.

$$h'(t) = \left(\frac{\left(-\frac{9}{2}+3\sqrt{2}\right)}{6\sqrt{2}} e^{\left(-\frac{9}{2}+3\sqrt{2}\right)t} - \frac{\left(-\frac{9}{2}-3\sqrt{2}\right)}{6\sqrt{2}} e^{\left(-\frac{9}{2}-3\sqrt{2}\right)t} \right) \xi(t)$$

$$+ \overset{=0}{\cancel{\left(\frac{1}{6\sqrt{2}} e^{\left(-\frac{9}{2}+3\sqrt{2}\right)t} - \frac{1}{6\sqrt{2}} e^{\left(-\frac{9}{2}-3\sqrt{2}\right)t} \right)}} \delta(t)$$

$$\text{so } h'(t) = \left(\frac{\left(-\frac{9}{2}+3\sqrt{2}\right)}{6\sqrt{2}} e^{\left(-\frac{9}{2}+3\sqrt{2}\right)t} - \frac{\left(-\frac{9}{2}-3\sqrt{2}\right)}{6\sqrt{2}} e^{\left(-\frac{9}{2}-3\sqrt{2}\right)t} \right) \xi(t)$$

$$h''(t) = \left(\frac{\left(-\frac{9}{2}+3\sqrt{2}\right)^2}{6\sqrt{2}} e^{\left(-\frac{9}{2}+3\sqrt{2}\right)t} - \frac{\left(-\frac{9}{2}-3\sqrt{2}\right)^2}{6\sqrt{2}} e^{\left(-\frac{9}{2}-3\sqrt{2}\right)t} \right) \xi(t)$$

$$+ \left(\frac{\left(-\frac{9}{2}+3\sqrt{2}\right)}{6\sqrt{2}} e^{\left(-\frac{9}{2}+3\sqrt{2}\right)t} - \frac{\left(-\frac{9}{2}-3\sqrt{2}\right)}{6\sqrt{2}} e^{\left(-\frac{9}{2}-3\sqrt{2}\right)t} \right) \delta(t)$$

$$h''(t) = \frac{\left(-\frac{9}{2}+3\sqrt{2}\right)^2}{6\sqrt{2}} e^{\left(-\frac{9}{2}+3\sqrt{2}\right)t} - \frac{\left(-\frac{9}{2}-3\sqrt{2}\right)^2}{6\sqrt{2}} e^{\left(-\frac{9}{2}-3\sqrt{2}\right)t} \Big) \xi(t)$$

$$+ \underbrace{\frac{1}{6\sqrt{2}} \left(-\frac{9}{2}+3\sqrt{2} - \left(-\frac{9}{2}-3\sqrt{2}\right) \right)}_{=1 \text{ !! as we want}} \delta(t)$$

$$\text{so } h''(t) = \left[\frac{\left(-\frac{9}{2}+3\sqrt{2}\right)^2}{6\sqrt{2}} e^{\left(-\frac{9}{2}+3\sqrt{2}\right)t} - \frac{\left(-\frac{9}{2}-3\sqrt{2}\right)^2}{6\sqrt{2}} e^{\left(-\frac{9}{2}-3\sqrt{2}\right)t} \right] \xi(t)$$

$$+ \delta(t).$$

Plug into ODE $\Rightarrow$

$$LHS = h''(t) + 9h'(t) + 2h(t)$$

next $\longrightarrow$

$$LHS = \left[\frac{\left(-\frac{9}{2}+3\sqrt{2}\right)^2}{6\sqrt{2}} e^{\left(-\frac{9}{2}+3\sqrt{2}\right)t} - \frac{\left(-\frac{9}{2}-3\sqrt{2}\right)^2}{6\sqrt{2}} e^{\left(-\frac{9}{2}-3\sqrt{2}\right)t}\right]\xi(t)$$

$$+\delta(t)$$

$$+9\left[\frac{\left(-\frac{9}{2}+3\sqrt{2}\right)}{6\sqrt{2}} e^{\left(-\frac{9}{2}+3\sqrt{2}\right)t} - \frac{\left(-\frac{9}{2}-3\sqrt{2}\right)}{6\sqrt{2}} e^{\left(-\frac{9}{2}-3\sqrt{2}\right)t}\right]\xi(t)$$

$$+2\left[\frac{1}{6\sqrt{2}} e^{\left(-\frac{9}{2}+3\sqrt{2}\right)t} - \frac{1}{6\sqrt{2}} e^{\left(-\frac{9}{2}-3\sqrt{2}\right)t}\right]\xi(t)$$

simplifying gives $LHS = \delta(t)$

hence verified.

since $h(t)$ is defined as the solution to the ODE when the input is $\delta(t)$, i.e RHS is $\delta(t)$, then we verified it.

Part ④ continue here from first page.

④ Find $H(j\omega)$, $|H(j\omega)|$, $\text{Arg}(H(j\omega))$

from 1st page, I found $H(j\omega)$

$$\boxed{H(j\omega) = \frac{1}{2-\omega^2+9j\omega}}$$

From Here

so $|H(j\omega)|^2 = \dfrac{1}{(2-\omega^2)^2+(9\omega)^2}$

so $|H(j\omega)| = \sqrt{\dfrac{1}{(2-\omega^2)^2+(9\omega)^2}}$

✓

$$\text{Arg}(H(j\omega)) = -\tan^{-1}\left(\frac{9\omega}{2-\omega^2}\right)$$

→

Q2

$$y_1(k) = A\,u_1(k) + B\,u_1(k-1) + C[u_1(k-2)]^2$$
$$y_2(k) = A\,u_2(k) + B\,u_2(k-1) + C[u_2(k-2)]^2$$

so

① $$\alpha y_1(k) + \beta y_2(k) = A[\alpha u_1(k) + \beta u_2(k)] + B[\alpha u_1(k-1) + \beta u_2(k-1)] + C[\alpha (u_1(k-2))^2 + \beta (u_2(k-2))^2]$$

different power

let $u_3 = \alpha u_1 + \beta u_2$.

$$\text{So } y_3(k) = A[\alpha u_1(k) + \beta u_2(k)] + B[\alpha u_1(k-1) + \beta u_2(k-1)] + C[(\alpha u_1(k-2) + \beta u_2(k-2))]^2$$

extra term.

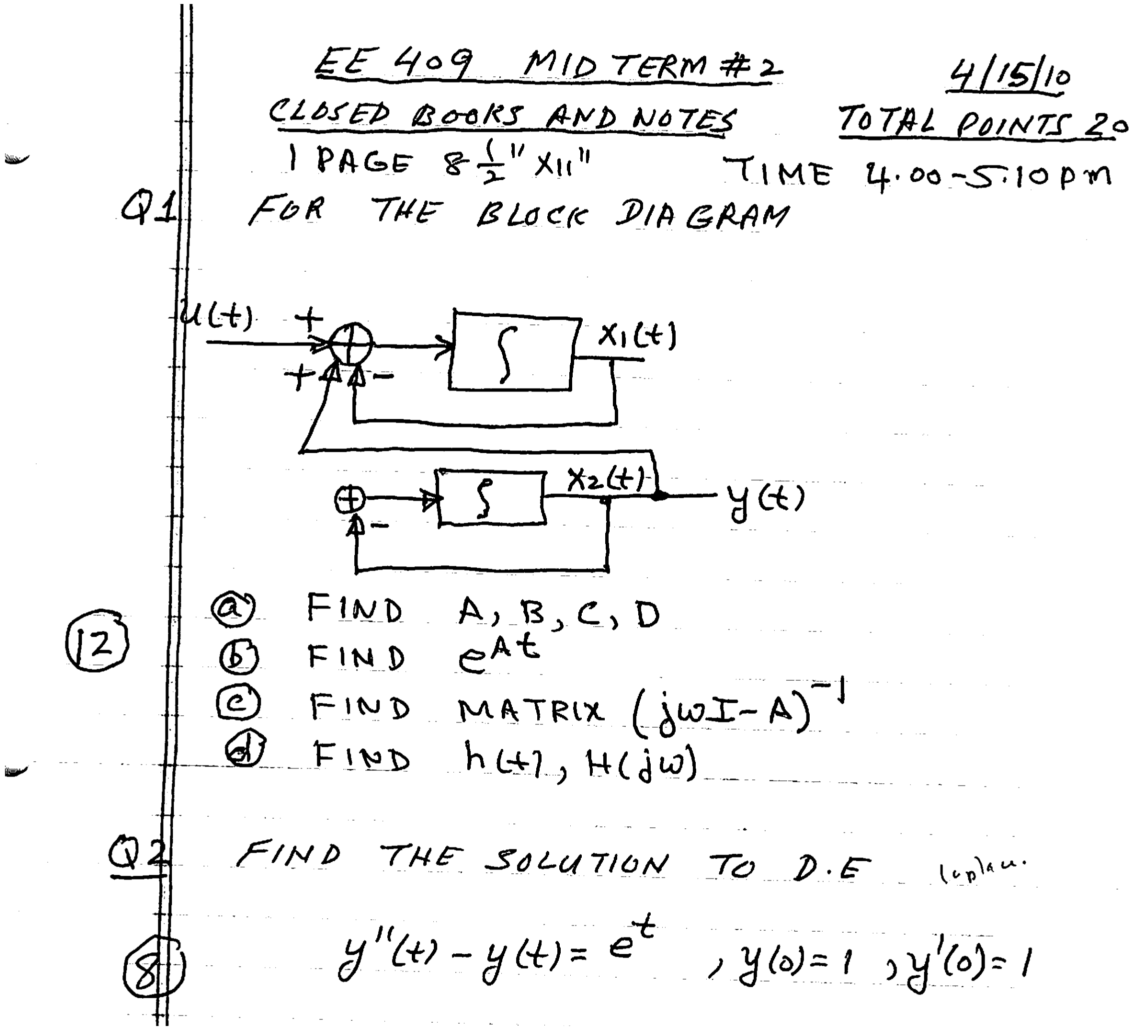
② $$\text{or } y_3(k) = A[\alpha u_1(k) + \beta u_2(k)] + B[\alpha u_1(k-1) + \beta u_2(k-1)] + C[\alpha^2 (u_1(k-2))^2 + \beta^2 (u_2(k-2))^2 + 2\alpha\beta\, u_1(k-2)u_2(k-2)]$$

Compare ① and ② $\Rightarrow$ Not same.

$\Rightarrow$ Not linear.

4.2 Second exam

4.2.1 Questions



EE 409 MID TERM #2 4/15/10

CLOSED BOOKS AND NOTES TOTAL POINTS 20

1 PAGE $8\frac{1}{2}$" x11" TIME 4.00-5.10pm

Q1 FOR THE BLOCK DIAGRAM

(12)

(a) FIND A, B, C, D

(b) FIND e^{At}

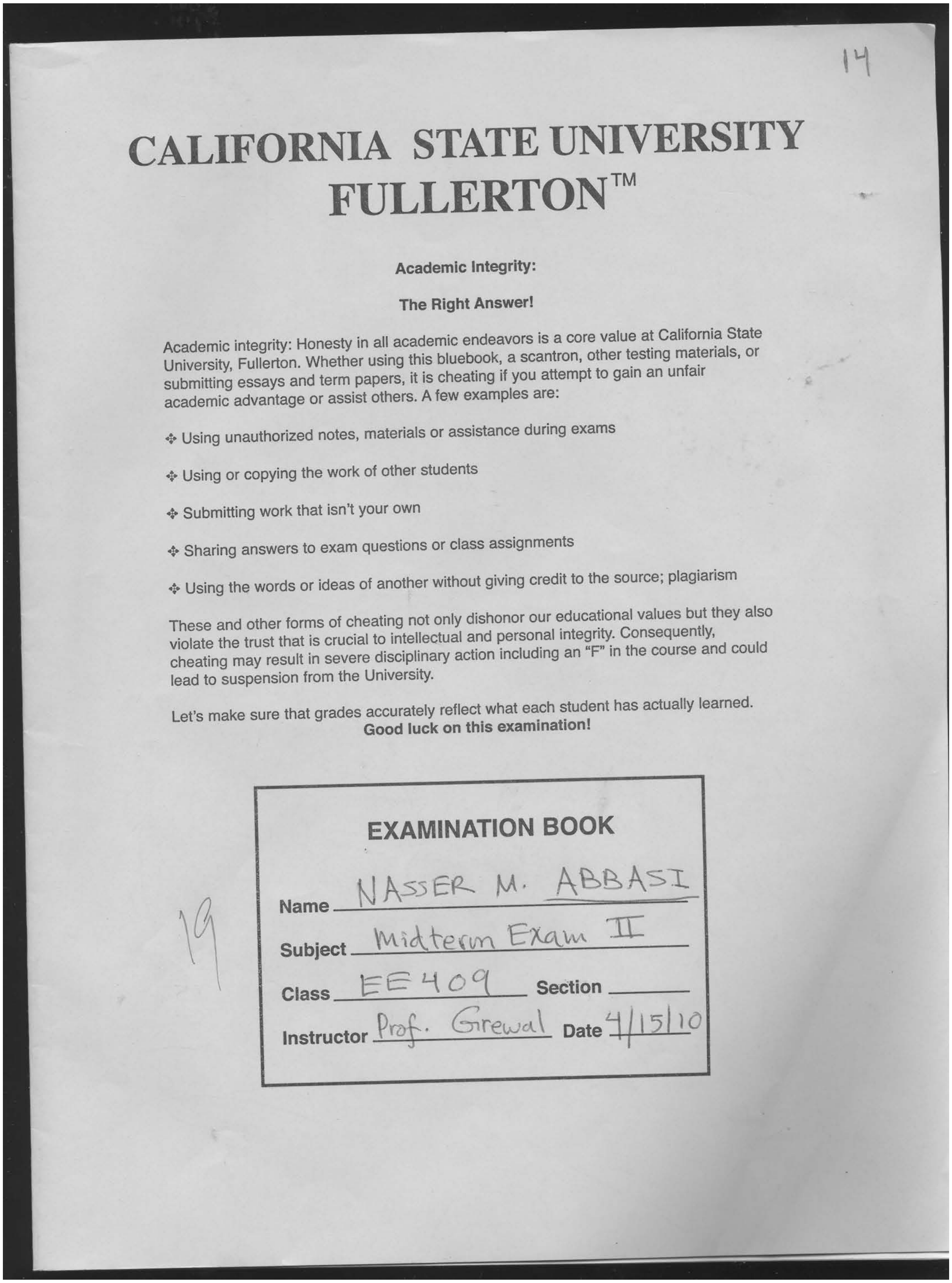
(c) FIND MATRIX $(j\omega I - A)^{-1}$

(d) FIND $h(t)$, $H(j\omega)$

Q2 FIND THE SOLUTION TO D.E laplace

(8) $y''(t) - y(t) = e^{t}$, $y(0) = 1$, $y'(0) = 1$

4.2.2 my solution



14

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EXAMINATION BOOK

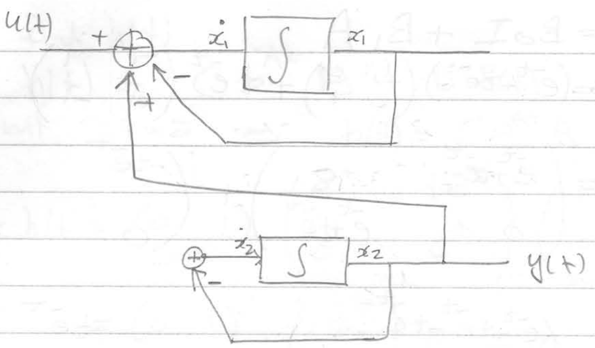
19

Name NASSER M. ABBASI
Subject Midterm Exam II
Class EE409 **Section**
Instructor Prof. Grewal **Date** 4/15/10

$-\lambda = 1$

$\boxed{\lambda = -1}$

Q1



(a) $\dot{x}_1 = u(t) - x_1 + x_2$

$\dot{x}_2 = -x_2$

$y(t) = x_2$

so $$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \overbrace{\begin{pmatrix}-1 & +1\\ 0 & -1\end{pmatrix}}^{A}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \overbrace{\begin{pmatrix}1\\ 0\end{pmatrix}}^{B} u$$

$$y = \underbrace{\begin{pmatrix}0 & 1\end{pmatrix}}_{C}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \underbrace{[0]}_{D} u$$

(b) to find e^{At}, first find eigenvalues of A.

$$|A - \lambda I| = \begin{vmatrix}-1-\lambda & 1\\ 0 & -1-\lambda\end{vmatrix} = (-1-\lambda)(-1-\lambda) = 0$$

so $(-1-\lambda)^2 = 0$ i.e roots are $-\lambda - 1 = 0$ or $\boxed{\lambda = -1}$ repeated roots

so $\boxed{\lambda_1 = -1,\ \lambda_2 = -1}$

so $e^{\lambda_1 t} = B_0 + B_1\lambda_1$

derivative w.r.t. λ → and $te^{\lambda_1 t} = B_1$

→ $e^{-t} = B_0 - B_1$

$te^{-t} = B_1$

$B_0 = e^{-t} + te^{-t}$

$B_1 = te^{-t}$

so $\boxed{B_0 = e^{-t} + te^{-t},\quad B_1 = te^{-t}}$

→

so $e^{At} = B_0 I + B_1 A$

$$= (e^{-t}+te^{-t})\begin{pmatrix}1&0\\0&1\end{pmatrix} + te^{-t}\begin{pmatrix}-1&1\\0&-1\end{pmatrix}$$

$$= \begin{pmatrix} e^{-t}+te^{-t} & 0\\ 0 & e^{-t}+te^{-t}\end{pmatrix} + \begin{pmatrix}-te^{-t} & te^{-t}\\ 0 & -te^{-t}\end{pmatrix}$$

$$= \begin{pmatrix} e^{-t}+te^{-t}-te^{-t} & te^{-t}\\ 0 & e^{-t}+te^{-t}-te^{-t}\end{pmatrix}$$

$$= \boxed{\begin{pmatrix} e^{-t} & te^{-t}\\ 0 & e^{-t}\end{pmatrix}}$$

ⓒ Find $(J\omega I - A)^{-1}$ let $\Delta = (J\omega I - A)$

$$\Delta = J\omega\begin{pmatrix}1&0\\0&1\end{pmatrix} - \begin{pmatrix}-1&1\\0&-1\end{pmatrix} = \begin{pmatrix}J\omega&0\\0&J\omega\end{pmatrix} - \begin{pmatrix}-1&1\\0&-1\end{pmatrix}$$

$$= \begin{pmatrix}J\omega+1 & -1\\ 0 & J\omega+1\end{pmatrix}$$

so $\Delta^{-1} = \begin{pmatrix}J\omega+1 & 1\\ 0 & J\omega+1\end{pmatrix}\dfrac{1}{(J\omega+1)^2}$

$$= \boxed{\frac{\begin{pmatrix}J\omega+1 & 1\\ 0 & J\omega+1\end{pmatrix}}{-\omega^2+2J\omega+1}}$$

ⓓ find h(t)

$$h(t) = C e^{At} B + D\,\delta(t) \qquad t \geq 0.$$

but $[D] = 0 \Rightarrow h(t) = Ce^{At}B \qquad t \geq 0.$

$$\text{so } h(t) = \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{1\times2} \underbrace{\begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix}}_{2\times2} \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{2\times1}$$

$$= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} = 0 \qquad \underline{\underline{t>0}}$$

$$H(jw) = C\,(Ijw - A)^{-1}\,B$$

$$= \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{\begin{pmatrix} jw+1 & 1 \\ 0 & jw+1 \end{pmatrix}}{2jw - w^2 + 1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

✓

$$= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} jw+1 \\ 0 \end{pmatrix} \frac{1}{2jw + w^2 + 1} = 0.$$

Q2 $y''(t) + y(t) = e^{t}$ $y(0) = 1$, $y'(0) = 1$

using Laplace Method:

$$s^2 Y(s) - s y(0) - y'(0) + Y(s) = \frac{1}{s-1}$$

$$s^2 Y(s) - s - 1 + Y(s) = \frac{1}{s-1}$$

$$Y(s)\left[s^2+1\right] - s - 1 = \frac{1}{s-1}$$

$$Y(s) = \frac{s}{s^2+1} + \frac{1}{s^2+1} + \boxed{\frac{1}{(s-1)(s^2+1)}}$$

Consider this term for now:

$$\frac{1}{(s-1)(s^2+1)} = \frac{A}{(s-1)} + \frac{Bs+C}{s^2+1}$$

$$A = \lim_{s \to 1} \frac{1}{s^2+1} = \boxed{\frac{1}{2}}$$

So $$\frac{1}{(s-1)(s^2+1)} = \frac{\frac{1}{2}}{(s-1)} + \frac{Bs+C}{s^2+1}$$

So $$1 = \tfrac{1}{2}(s^2+1) + (Bs+C)(s-1)$$

$$1 = \tfrac{1}{2}s^2 + \tfrac{1}{2} + Bs^2 - Bs + Cs - C$$

$$1 = s^2\left[\tfrac{1}{2}+B\right] + s\left[C-B\right] + \tfrac{1}{2} - C$$

So $$\left.\begin{aligned} 1 &= \tfrac{1}{2} - C \\ 0 &= \tfrac{1}{2} + B \\ 0 &= C - B \end{aligned}\right\} \quad \boxed{\begin{aligned} C &= -\tfrac{1}{2} \\ B &= -\tfrac{1}{2} \end{aligned}}$$

So $$\frac{1}{(s-1)(s^2+1)} = \frac{\frac{1}{2}}{(s-1)} + \frac{-\frac{1}{2}s - \frac{1}{2}}{s^2+1}$$

$$= \boxed{\frac{\frac{1}{2}}{(s-1)} - \frac{1}{2}\,\frac{s}{s^2+1} - \frac{1}{2}\,\frac{1}{s^2+1}} \longrightarrow$$

so $$\boxed{Y(s) = \frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{s}{s^2+1} - \frac{1}{2}\frac{1}{s^2+1}}$$

$$\boxed{Y(s) = \frac{1}{2}\frac{1}{s^2+1} + \frac{1}{2}\frac{1}{s-1}}$$

now apply Laplace transform.

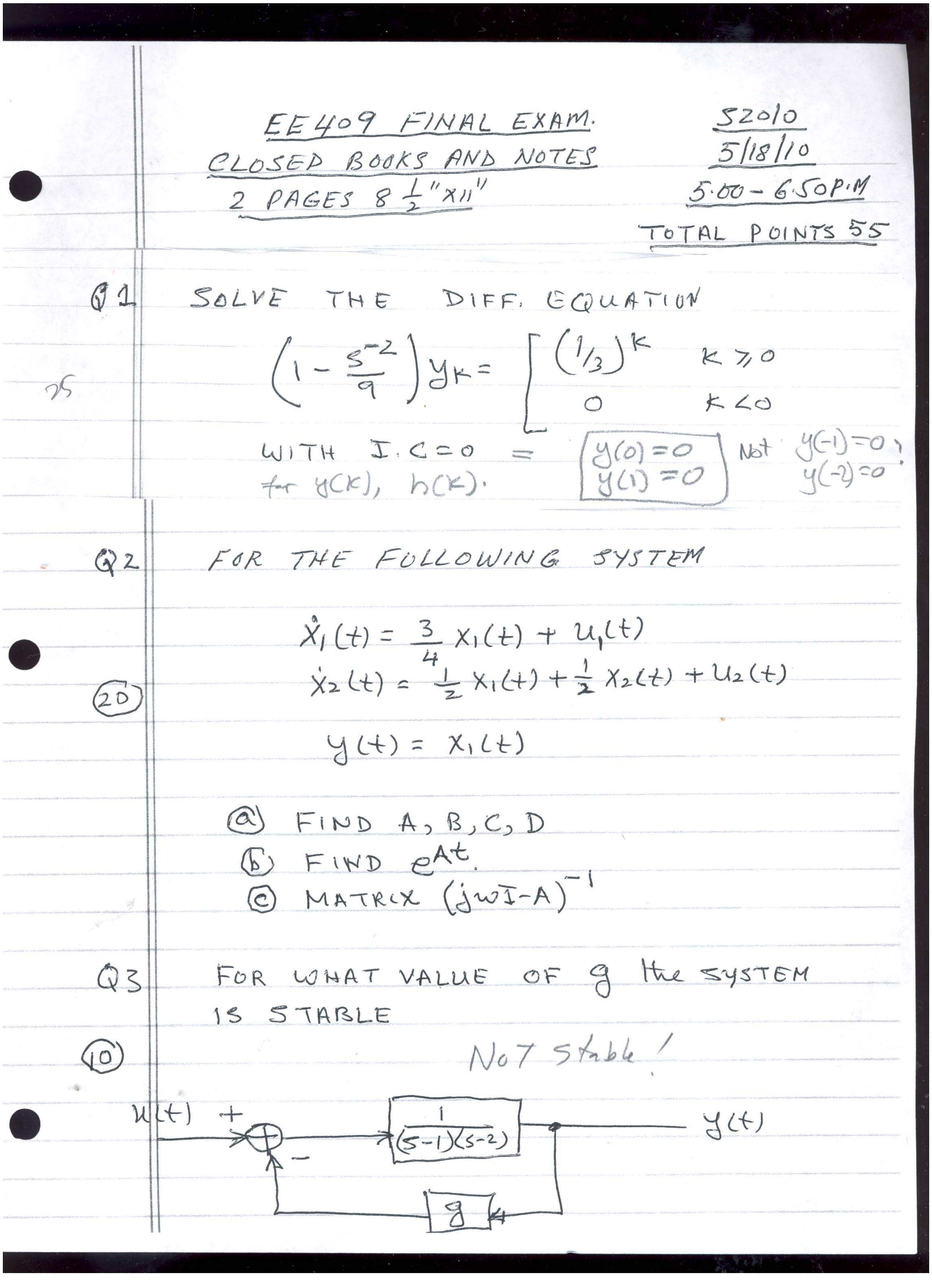
$$y(t) = \left(\frac{1}{2}\sin t + \frac{1}{2}e^{t}\right)\xi(t)$$

$$= \boxed{\frac{1}{2}\left(\sin t + e^{t}\right)\xi(t)}$$

instable!
due to this

4.3 Final exam

4.3.1 Questions



EE409 FINAL EXAM.
CLOSED BOOKS AND NOTES
2 PAGES $8\frac{1}{2}'' \times 11''$

S2010
5/18/10
5.00 - 6.50 P.M
TOTAL POINTS 55

Q1 SOLVE THE DIFF. EQUATION

25

$$\left(1 - \frac{S^{-2}}{9}\right) y_K = \begin{cases} (1/3)^K & K \geq 0 \\ 0 & K < 0 \end{cases}$$

WITH I.C=0 = $\boxed{\begin{matrix} y(0)=0 \\ y(1)=0 \end{matrix}}$ Not $y(-1)=0$! $y(-2)=0$

for $y(K)$, $h(K)$.

Q2 FOR THE FOLLOWING SYSTEM

⑳

$$\dot{x}_1(t) = \frac{3}{4} x_1(t) + u_1(t)$$
$$\dot{x}_2(t) = \frac{1}{2} x_1(t) + \frac{1}{2} x_2(t) + u_2(t)$$
$$y(t) = x_1(t)$$

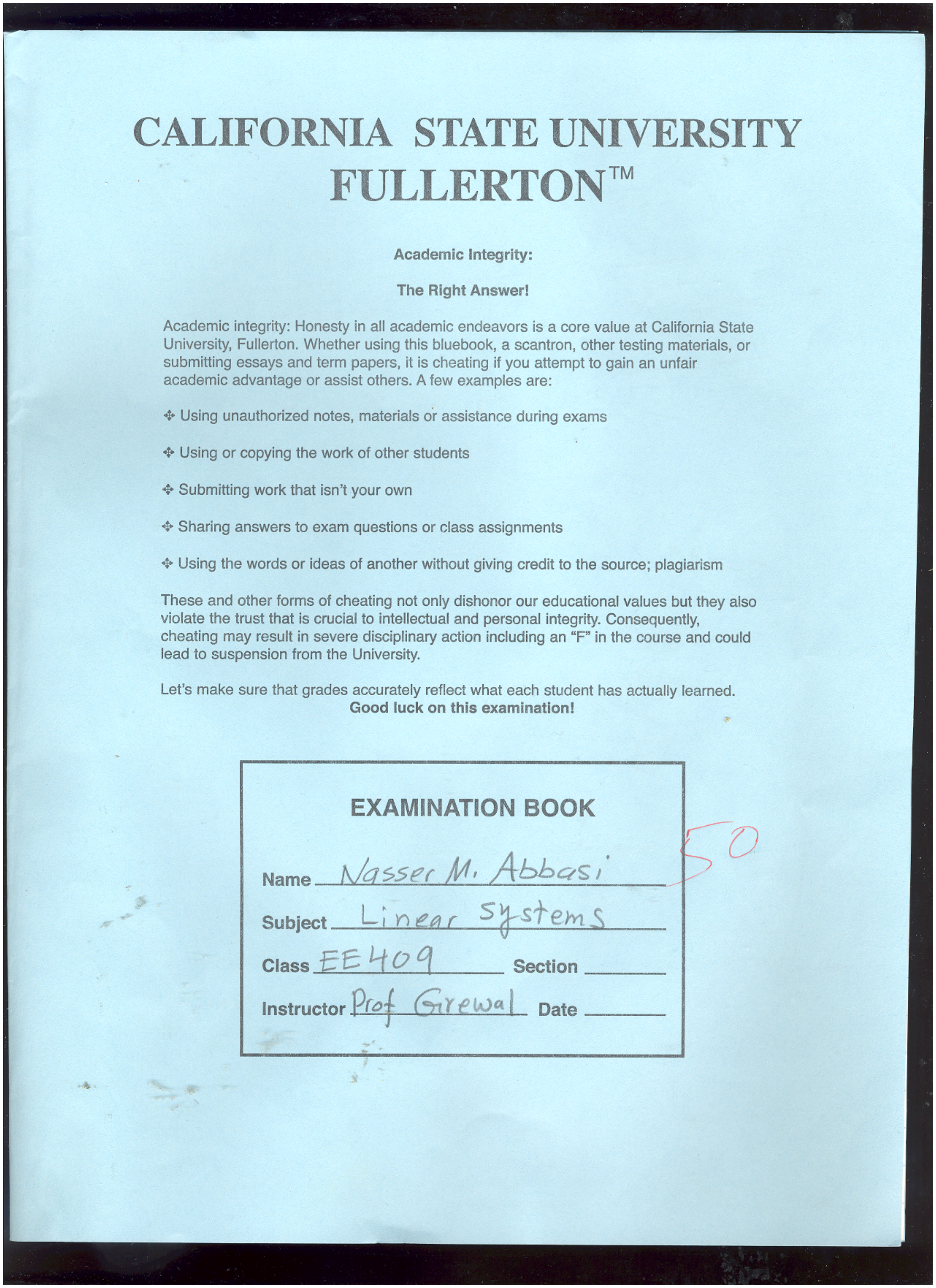
ⓐ FIND A, B, C, D
ⓑ FIND e^{At}.
ⓒ MATRIX $(j\omega I - A)^{-1}$

Q3 FOR WHAT VALUE OF g the SYSTEM IS STABLE

⑩

Not stable!

4.3.2 my solution



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EXAMINATION BOOK

50

Name Nasser M. Abbasi
Subject Linear systems
Class EE409 **Section** ______
Instructor Prof Grewal **Date** ______

① Solve $\left(1-\frac{S^{-2}}{9}\right) y(k) = \left(\frac{1}{3}\right)^{k}$ $\quad k \geq 0$

with IC = 0

find y(k) and h(k).

$y_h(k)$ is solution to $\left(1-\frac{S^{-2}}{9}\right) y(k) = 0$.

the char. equation is $r^2 - \frac{1}{9} = 0$

so roots are $\boxed{r = \pm\frac{1}{3}}$

hence $\boxed{y_h(k) = C_1\left(\frac{1}{3}\right)^k + C_2\left(-\frac{1}{3}\right)^k}$

$y_p(k)$ is solution due to forcing function $\left(\frac{1}{3}\right)^k$.

from table, solution is $y_p(k) = C_3\left(\frac{1}{3}\right)^k$. but since this is double root, it becomes

$$\boxed{y_p(k) = C_3\, k\left(\frac{1}{3}\right)^k}$$

hence total solution is

$$y(k) = y_h(k) + y_p(k)$$

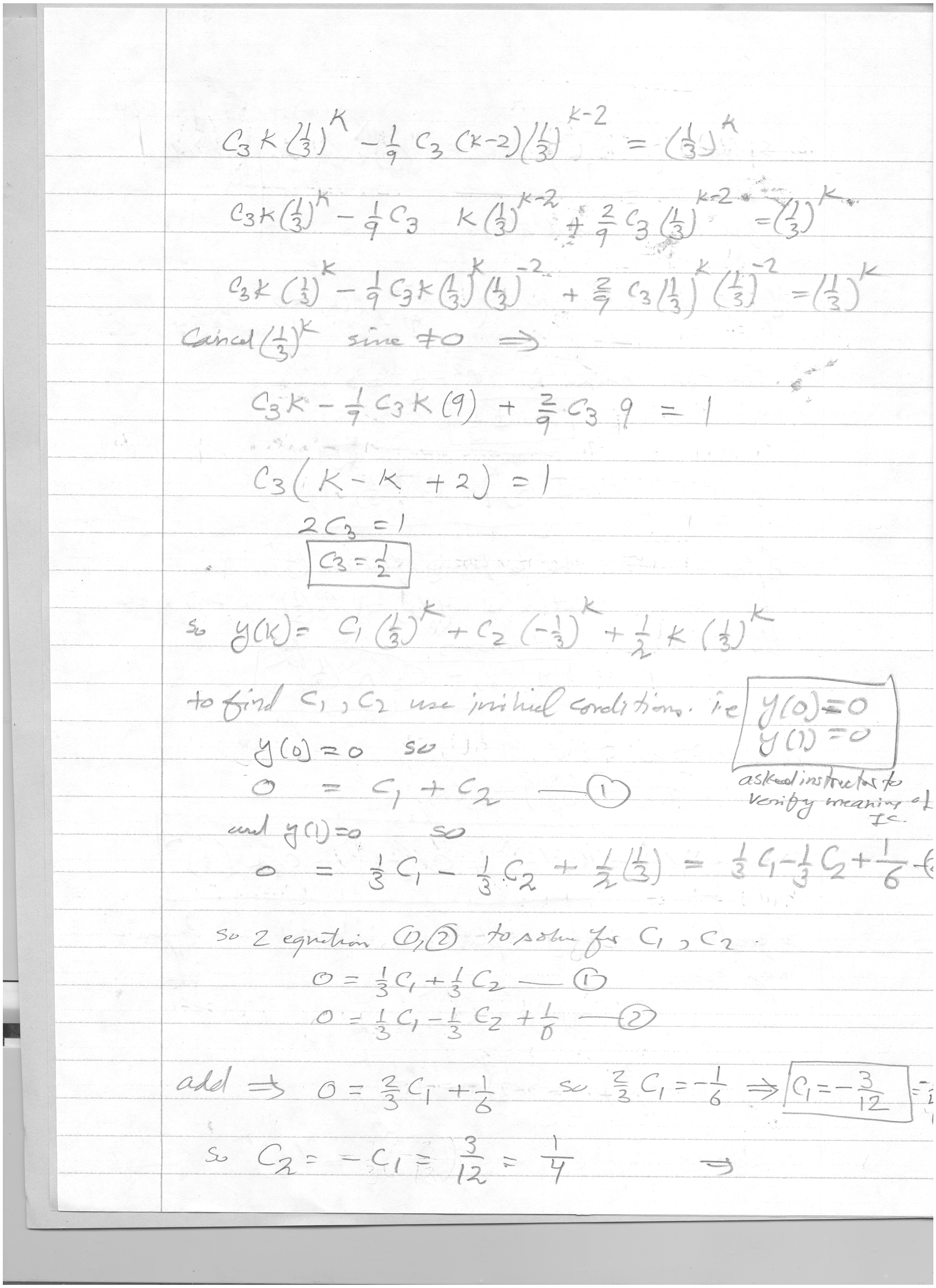
$$\boxed{y(k) = C_1\left(\frac{1}{3}\right)^k + C_2\left(-\frac{1}{3}\right)^k + C_3\, k\left(\frac{1}{3}\right)^k}$$

to find C_3, we plug $y_p(k)$ back into original difference equation

so $\left(1-\frac{S^{-2}}{9}\right) y_p(k) = \left(\frac{1}{3}\right)^k$

$$\left(1-\frac{S^{-2}}{9}\right) C_3\, k\left(\frac{1}{3}\right)^k = \left(\frac{1}{3}\right)^k$$

$$C_3 k\left(\frac{1}{3}\right)^k - \frac{1}{9} C_3 (k-2)\left(\frac{1}{3}\right)^{k-2} = \left(\frac{1}{3}\right)^k \longrightarrow$$



$$C_3 k \left(\tfrac{1}{3}\right)^k - \tfrac{1}{9} C_3 (k-2) \left(\tfrac{1}{3}\right)^{k-2} = \left(\tfrac{1}{3}\right)^k$$

$$C_3 k \left(\tfrac{1}{3}\right)^k - \tfrac{1}{9} C_3 \; k \left(\tfrac{1}{3}\right)^{k-2} + \tfrac{2}{9} C_3 \left(\tfrac{1}{3}\right)^{k-2} = \left(\tfrac{1}{3}\right)^k$$

$$C_3 k \left(\tfrac{1}{3}\right)^k - \tfrac{1}{9} C_3 k \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{3}\right)^{-2} + \tfrac{2}{9} C_3 \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{3}\right)^{-2} = \left(\tfrac{1}{3}\right)^k$$

Cancel $\left(\tfrac{1}{3}\right)^k$ since $\neq 0$ $\Rightarrow$

$$C_3 k - \tfrac{1}{9} C_3 k (9) + \tfrac{2}{9} C_3 \, 9 = 1$$

$$C_3 (k - k + 2) = 1$$

$$2 C_3 = 1$$

$$\boxed{C_3 = \tfrac{1}{2}}$$

so $y(k) = C_1 \left(\tfrac{1}{3}\right)^k + C_2 \left(-\tfrac{1}{3}\right)^k + \tfrac{1}{2} k \left(\tfrac{1}{3}\right)^k$

to find C_1, C_2 use initial conditions. i.e $\boxed{\begin{array}{l} y(0)=0 \\ y(1)=0 \end{array}}$

asked instructor to verify meaning of IC.

$y(0)=0$ so

$0 = C_1 + C_2$ — ①

and $y(1)=0$ so

$0 = \tfrac{1}{3} C_1 - \tfrac{1}{3} C_2 + \tfrac{1}{2}\left(\tfrac{1}{3}\right) = \tfrac{1}{3} C_1 - \tfrac{1}{3} C_2 + \tfrac{1}{6}$ —

so 2 equation ①, ② to solve for C_1, C_2.

$0 = \tfrac{1}{3} C_1 + \tfrac{1}{3} C_2$ — ①

$0 = \tfrac{1}{3} C_1 - \tfrac{1}{3} C_2 + \tfrac{1}{6}$ — ②

add $\Rightarrow$ $0 = \tfrac{2}{3} C_1 + \tfrac{1}{6}$ so $\tfrac{2}{3} C_1 = -\tfrac{1}{6}$ $\Rightarrow$ $\boxed{C_1 = -\tfrac{3}{12}}$ =

so $C_2 = -C_1 = \tfrac{3}{12} = \tfrac{1}{4}$ $\Rightarrow$

hence $\boxed{y(k) = -\frac{1}{4}\left(\frac{1}{3}\right)^{k} + \frac{1}{4}\left(-\frac{1}{3}\right)^{k} + \frac{1}{2}\,k\left(\frac{1}{3}\right)^{k}}$ $k \geq 0$

zero otherwise

now to find $h(k)$.

let input be $\delta(k)$, hence

$$\left(1 - \frac{s^{-2}}{9}\right) h(k) = \delta(k) \qquad k \geq 0.$$

$$h(k) - \frac{1}{9}\,h(k-2) = \delta(k).$$

Solution for homogeneous part was already found to be

$$\boxed{h(k) = C_1\left(\frac{1}{3}\right)^{k} + C_2\left(-\frac{1}{3}\right)^{k}} \quad \text{——} \; ①$$

at $k=0$ difference equation becomes

$$h(0) - \frac{1}{9}\,\overset{=0}{h(-2)} = \delta(0) = 1$$

$$\boxed{h(0) = 1}$$

at $k=1$

$$h(1) - \frac{1}{9}\,\overset{=0}{h(-1)} = \delta(1) = 0$$

$$\boxed{h(1) = 0}$$

$h_k = LD\,\hat{h}_k$

-5

hence the equation to solve are (from ①)

$h(0) = 1 = C_1 + C_2$ (when $k=0$)

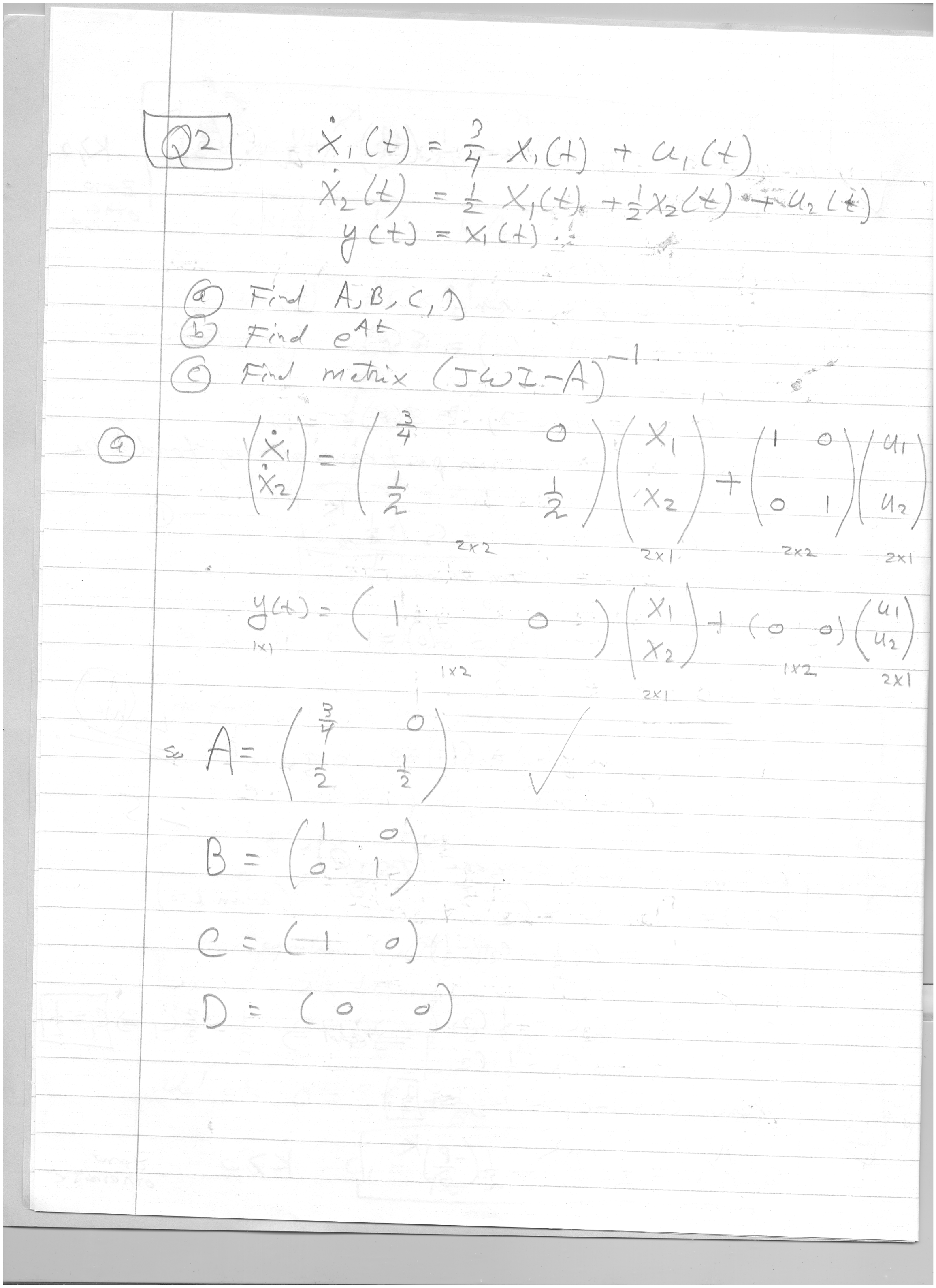
$h(1) = 0 = C_1\frac{1}{3} + C_2\left(-\frac{1}{3}\right)$

or

$$\left.\begin{aligned} \frac{1}{3} &= \frac{1}{3}C_1 + \frac{1}{3}C_2 \\ 0 &= \frac{1}{3}C_1 - \frac{1}{3}C_2 \end{aligned}\right\} \Rightarrow \text{add} \Rightarrow \frac{1}{3} = \frac{2}{3}C_1 \Rightarrow \boxed{C_1 = \frac{1}{2}}$$

hence $C_2 = 1 - C_1 = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$

so $\boxed{\hat{h}(k) = \frac{1}{2}\left(\frac{1}{3}\right)^{k} + \frac{1}{2}\left(-\frac{1}{3}\right)^{k}}$ $k \geq 0$, zero otherwise



$\boxed{Q2}$

$$\dot{x}_1(t) = \frac{3}{4}x_1(t) + u_1(t)$$
$$\dot{x}_2(t) = \frac{1}{2}x_1(t) + \frac{1}{2}x_2(t) + u_2(t)$$
$$y(t) = x_1(t)$$

ⓐ Find A, B, C, D

ⓑ Find e^{At}

ⓒ Find matrix $(j\omega I - A)^{-1}$

ⓐ

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \underset{2\times2}{\begin{pmatrix}\frac{3}{4} & 0\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}} \underset{2\times1}{\begin{pmatrix}x_1\\ x_2\end{pmatrix}} + \underset{2\times2}{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} \underset{2\times1}{\begin{pmatrix}u_1\\ u_2\end{pmatrix}}$$

$$\underset{1\times1}{y(t)} = \underset{1\times2}{\begin{pmatrix}1 & 0\end{pmatrix}} \underset{2\times1}{\begin{pmatrix}x_1\\ x_2\end{pmatrix}} + \underset{1\times2}{\begin{pmatrix}0 & 0\end{pmatrix}} \underset{2\times1}{\begin{pmatrix}u_1\\ u_2\end{pmatrix}}$$

so $A = \begin{pmatrix}\frac{3}{4} & 0\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$ ✓

$B = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$C = \begin{pmatrix}1 & 0\end{pmatrix}$

$D = \begin{pmatrix}0 & 0\end{pmatrix}$

ⓑ to find e^{At}, first find eigenvalues of A.

$$|A-\lambda I|=0$$

$$\begin{vmatrix} 3/4-\lambda & 0 \\ \frac{1}{2} & \frac{1}{2}-\lambda \end{vmatrix} = 0 \Rightarrow \left(\frac{3}{4}-\lambda\right)\left(\frac{1}{2}-\lambda\right)=0$$

so $\boxed{\lambda_1 = \frac{3}{4} \quad , \quad \lambda_2 = \frac{1}{2}}$

so

$$\left.\begin{aligned} e^{\lambda_1 t} &= B_0 + B_1\lambda_1 \\ e^{\lambda_2 t} &= B_0 + B_1\lambda_2 \end{aligned}\right\} \Rightarrow \begin{aligned} e^{\frac{3}{4}t} &= B_0 + \tfrac{3}{4}B_1 \quad \text{—①} \\ e^{\frac{1}{2}t} &= B_0 + \tfrac{1}{2}B_1 \quad \text{—②} \end{aligned}$$

② − ① ⇒

$$e^{\frac{1}{2}t} - e^{\frac{3}{4}t} = \frac{1}{2}B_1 - \frac{3}{4}B_1$$

$$e^{\frac{1}{2}t} - e^{\frac{3}{4}t} = -\frac{1}{4}B_1$$

so $\boxed{B_1 = 4\left(e^{\frac{3}{4}t} - e^{\frac{1}{2}t}\right)}$

so from ① we find B_0:

$$B_0 = e^{\frac{3}{4}t} - \frac{3}{4}\left(4\left(e^{\frac{3}{4}t} - e^{\frac{1}{2}t}\right)\right)$$

$$= e^{\frac{3}{4}t} - 3e^{\frac{3}{4}t} + 3e^{\frac{1}{2}t}$$

$$\boxed{B_0 = -2e^{\frac{3}{4}t} + 3e^{\frac{1}{2}t}}$$

so $e^{At} = B_0 I + B_1 A$

$$= \left(-2e^{\frac{3}{4}t} + 3e^{\frac{t}{2}}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 4\left(e^{\frac{3t}{4}} - e^{\frac{t}{2}}\right)\begin{pmatrix} \frac{3}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

→

$$e^{At} = \begin{pmatrix} -2e^{\frac{3}{4}t}+3e^{\frac{t}{2}} & 0 \\ 0 & -2e^{\frac{3}{4}t}+3e^{\frac{t}{2}} \end{pmatrix} + \begin{pmatrix} 3(e^{\frac{3}{4}t}-e^{\frac{t}{2}}) & 0 \\ 2(e^{\frac{3}{4}t}-e^{\frac{t}{2}}) & 2(e^{\frac{3}{4}t}-e^{\frac{t}{2}}) \end{pmatrix}$$

$$\boxed{e^{At} = \begin{pmatrix} e^{\frac{3}{4}t} & 0 \\ 2(e^{\frac{3}{4}t}-e^{\frac{t}{2}}) & e^{\frac{t}{2}} \end{pmatrix}} \checkmark$$

ⓒ

$$j\omega I - A = \begin{pmatrix} j\omega & 0 \\ 0 & j\omega \end{pmatrix} - \begin{pmatrix} \frac{3}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} j\omega-\frac{3}{4} & 0 \\ -\frac{1}{2} & j\omega-\frac{1}{2} \end{pmatrix}$$

$$|j\omega I-A| = \left(j\omega-\tfrac{3}{4}\right)\left(j\omega-\tfrac{1}{2}\right) = -\omega^2-\tfrac{5}{4}j\omega+\tfrac{3}{8}$$

$$\boxed{|j\omega I-A| = \left(\tfrac{3}{8}-\omega^2\right)+j\left(-\tfrac{5}{4}\omega\right)} \checkmark$$

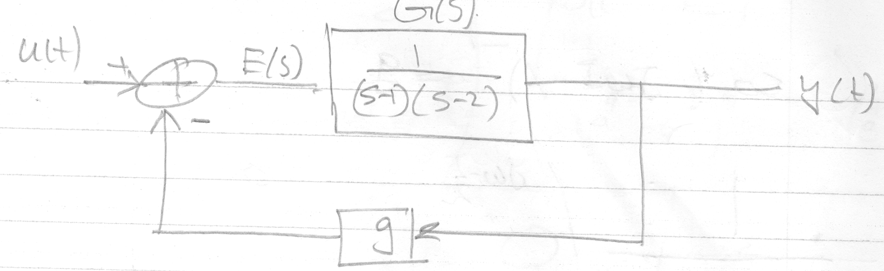
so $(j\omega I - A)^{-1} =$

$$\frac{1}{\left(\frac{3}{8} - \omega^2\right) + j\left(\frac{-5}{4}\omega\right)} \begin{pmatrix} j\omega - \frac{1}{2} & 0 \\ \frac{1}{2} & j\omega - \frac{3}{4} \end{pmatrix}$$

$$\boxed{\text{or}} \quad \frac{1}{\left(j\omega - \frac{3}{4}\right)\left(j\omega - \frac{1}{2}\right)} \begin{pmatrix} j\omega - \frac{1}{2} & 0 \\ \frac{1}{2} & j\omega - \frac{3}{4} \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} \frac{1}{\left(j\omega - \frac{3}{4}\right)} & 0 \\ \frac{1/2}{\left(j\omega - \frac{3}{4}\right)\left(j\omega - \frac{1}{2}\right)} & \frac{1}{\left(j\omega - \frac{1}{2}\right)} \end{pmatrix}}$$

$\boxed{Q3}$



$$\boxed{\begin{aligned} E(s) &= u(s) - g\,y(s) \quad \text{—— ①} \\ y(s) &= E(s)\,G(s) \quad \text{—— ②} \end{aligned}}$$

where $G(s) = \frac{1}{(s-1)(s-2)}$

Sub ① into ② we obtain.

$$y(s) = [u(s) - g\,y(s)]\,G(s)$$

$$y(s) = u(s)\,G(s) - g\,y(s)\,G(s)$$

$$y(s)[1 + g\,G(s)] = u(s)\,G(s)$$

$$\boxed{\text{so } \frac{y(s)}{u(s)} = H(s) = \frac{G(s)}{1 + g\,G(s)}} \quad \text{—— ③}$$

for stability, we need to find the roots of the denominator of ③ and see for what values of g these roots are < 0.

Char. equation is $1 + g\,G(s) = 0$

$$1 + g\frac{1}{(s-1)(s-2)} = 0$$

or. $(s-1)(s-2)+g=0$

hence $s^2-3s+(2+g)=0$

hence $s=-\frac{b}{2}\pm\frac{1}{2}\sqrt{b^2-4ac}=\frac{3}{2}\pm\frac{1}{2}\sqrt{9-4(2+g)}$

or $s=\frac{3}{2}\pm\frac{1}{2}\sqrt{9-8-4g}=\frac{3}{2}\pm\frac{1}{2}\sqrt{1-4g}$

So $$\boxed{\begin{aligned}s_1&=\frac{3}{2}+\sqrt{\frac{1}{4}-g}\\ s_2&=\frac{3}{2}-\sqrt{\frac{1}{4}-g}\end{aligned}}$$

both poles must be <0 for stability. Consider each pole at a time.

$\underline{\underline{s_1}}$

this is NOT stable for any $\overset{\text{real}}{g}$ value.
If $\frac{1}{4}-g<0$, then we have $Re(s_1)=\frac{3}{2}>0$
and if $\frac{1}{4}-g>0$, then we also have $Re(s_1)>\frac{3}{2}$

✓

So pole s_1 is always unstable!

So No value of g will make this system stable
Since both poles must be stable.

This can be also shown using Routh table:

problem!! $\Rightarrow$

s^2	1	$2+g$
s^1	-3	0
s^0	$-3(2+g)$	

↑ this column must all be >0

Assume g is real. not possible to find such value.

$\Rightarrow$ we see that the first column can't be made all >0 due to presence of "-3" term.